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Introduction to Shafarevich Conjecture for Projective Calabi-Yau Varieties

Second Version

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Abstract

In this notes,we introduce some interesting and important topics in algebraic geometry: Shafarevich conjecture over function field,moduli space of polarized projective Calabi-Yau manifolds and the analogue Shafarevich Conjecture of families of Calabi-Yau manifolds. The author taught some parts of the notes in a short course of the 2003 summer school held in Center of Mathematical Sciences at Zhejiang University.

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1. Introduction to Shafarevich Conjecture

1.1. **Shafarevich conjecture over function field.** At the 1962 ICM in Stockholm, Shafarevich [26] conjectured: *"There exists only a finite number of fields of algebraic functions $K/\mathbb{C}$ of a given genus $g \geq 1$, the critical prime divisors of which belong to a given finite set S."*

Let C be a smooth projective curve of genus $g(C)$ over an algebraically closed field k of characteristic 0 and $S \subset C$ a finite subset. C and S will be fixed. A family of curves is called isotrivial,if any two general fibers are isomorphic. We can reformulate the conjecture:

Shafarevich Conjecture: Let (C, S) be fixed and $q \geq 2$ an integer.

(I) There exist only finitely many isomorphism class of non-isotrivial families of curves of genus q over C which have at most singular fibers over S.

(II) If $2g(C) - 2 + \#S \leq 0$,then there exist no such families.

In one unpublished work, Shafarevich proved his conjecture in the setting of hyperelliptic curves.The conjecture was confirmed by Parshin for the case of $S = \emptyset$,by Arakelov in general.

A *deformation* of a family $f : \mathcal{X} \to C$ with the fixed base C is a family $F : \mathfrak{X} \to C \times T$ such that for some $t_0 \in T$,

$$\begin{array}{ccc} \mathcal{X} \simeq \mathfrak{X}_{t_0} & \longrightarrow & \mathfrak{X} \\ \Big\downarrow f & & \Big\downarrow F \\ C \simeq C \times t_0 & \longrightarrow & C \times T \end{array}$$

We say that two families $\mathcal{X}_1 \to C$ and $\mathcal{X}_2 \to C$, have the same *deformation type* if they can be deformed into each other,i.e.if there exists a family $\mathfrak{X} \to C \times T$ such that for some $t_1, t_2 \in T$, $(\mathfrak{X}_{t_i} \to C \times t_i) \simeq (\mathcal{X}_i \to C)$ for $i = 1, 2$.

In order to prove that there are only finitely many non-isotrivial families,one can proceed the following way.

(a) To prove that there are only finitely many deformation types, i.e.(**Boundedness**)

(b) To prove that the family does not admit no-trivial deformations, i.e.(**Rigidity**).

If (a) and (b) are right,every deformation type contains only one family and since there are only finitely many deformation types,this proves the original statements.

A family $(f : \mathcal{X} \to C)$ naturally corresponds to a map $\eta_f : C \setminus S \to \mathcal{M}_q$,and since C is a smooth curve,that induces a morphism $\overline{\eta}_f : C \to \overline{\mathcal{M}}_q$(Here $\mathcal{M}_q$ the coarse moduli space for smooth curve of genus q and $\overline{\mathcal{M}}_q$ is for stable curve.So shown by Mumford $\overline{\mathcal{M}}_q$ is a projective and $\mathcal{M}_q \subset \overline{\mathcal{M}}_q$ as a open subscheme) Hence parameterizing families translates to parameterizing these morphisms which can be characterized by their graphs. The graph $\Gamma_{\overline{\eta}_f}$ of such $\overline{\eta}_f$ is a curve contained in $C \times \overline{\mathcal{M}}_q$ such that the first projection maps it isomorphically onto C.Therefore the problem is translated to look for a parametrization in the Hilbert scheme of $C \times \overline{\mathcal{M}}_q$.The Hilbert scheme is an infinite union of schemes of finite type,the components corresponding to the different Hilbert polynomials represent the deformation types of the families. One should prove the parameterizing scheme is of finite type,i.e.only finite Hilbert polynomials can actually occur.

In their original proofs of the conjecture,Paršin [2] and Arakelov [1] reformulate the conjecture in the following:

A family $f : \mathcal{X} \to B$ is called **isotrivial** if $\mathcal{X}_a \simeq \mathcal{X}_b$ for general points $a, b \in B(\mathbb{C})$.

Conjecture 1.1. Fixing (C, S),let $q \geq 2$ be an integer.

- (**B**) Non-isotrivial families of curves of genus q with singular locus S are parameterized by $\mathbb{T}$, a scheme of finite type .(`Boundedness`)
- (**R**) All deformations of the non-isotrivial family is trivial,i.e.
$$\dim \mathbb{T} = 0.(\texttt{Rigidity})$$
- (**H**) No non-isotrivial families of curves of genus q exist if $2g(C) - 2 + \#S \leq 0$,i.e.$\mathbb{T} \neq \emptyset \Rightarrow 2g(C) - 2 + \#S > 0$.(`Hyperbolicity`)
- (**WB**) For an non-isotrivial family $f : X \to C$,deg $f_*\omega^m_{X/C}$ is bounded above in term of $g(C), \#S, g(X_{gen}), m$.In particular, the bound is independent of f.(`Weak Boundedness`)

In this case, because all graphs are isomorphic to C, the Hilbert polynomial is determined by the first term which is the $\deg \eta_f^* \mathcal{L}$ for a fixed ample line on $\overline{\mathcal{M}_q}$. To shown boundedness is just to show $\deg \eta_f^* \mathcal{L}$ is bounded. Due to Mumford 's works (one can refer theorem 3.9),it is sufficient to show (**WB**).Furthermore,the property (**WB**) will imply both (**B**) and (**H**) when the fibers are curves:

Theorem 1.2. *Fixing* (C, S)*,let* $q \geq 2$ *be an integer.*

- (I) *(*Bedulev − Viehweg[3]*)* (**WB**) ⇒ (**B**).
- (II) *(*Kovács[14] *)* (**WB**) ⇒ (**H**).

One will obtains rigidity in curve case by the well known **de Franchis** theorem which is a key lemma in Faltings's proof of Mordell Conjecture.

Exercise 1.3. $f : X \to C$ is a smooth family of curves where C is a smooth projective curve with genus $g(C) = 0$ or 1.Then this family must be isotrivial.

Hint: Using the Torelli theorem which one can find in Griffiths-Harrris's textbook.

1.2. **The case of higher dimensional fibers.** Let the base field is always complex number field $\mathbb{C}$. Define $Sh(C, S, K)$ to be the set of all equivalent classes of non-isotrivial family $\{f : \mathcal{X} \to C\}$ such that $\mathcal{X}_b$ is a smooth projective variety with 'type' K for any $b \in C \setminus S$. Two such families are equivalent if they are isomorphic over $C - S$.

The general Shafarevich type problem is:

For which type K of varieties and data (C, S),is $Sh(C, S, K)$ finite ?

Example 1.4. [8]Faltings has dealt with the case where the fibers are Abelian varieties,and he formulates a Hodge theoretic condition for the fiber space to be rigid. This condition is always called Deligne-Faltings $(*)$ condition:

A smooth family $f : \mathcal{X}_0 \to C \setminus S$ *of Abelian variety satisfies* $(*)$ *if any anti-symmetric endomorphism* σ *of* $\mathbb{V}_{\mathbb{Z}} = R^1 f_* \mathbb{Z}$ *defines an endomorphism of* $\mathcal{X}_0$*(so* σ *is of type* $(0, 0)$*).*

Let Q be the sympletic bilinear form on $R^1 f_* \mathbb{Z}$.By global Torelli theorem Q is induced by a polarization of the Abelian scheme.It is shown by Faltings that the

$(*)$-condition is equivalent to

$$\mathrm{End}^Q(\mathbb{V}) \otimes \mathbb{C} = (\mathrm{End}^Q(\mathbb{V}) \otimes \mathbb{C})^{0,0}.$$

On the other hand,the Zariski tangent space of the moduli space of the Abelian schemes over $C - S$ with a fixed polarization type is isomorphism to

$$(\mathrm{End}^Q(\mathbb{V}) \otimes \mathbb{C})^{-1,1}.$$

Thus one obtains the rigidity.

Example 1.5. [24] If a family is not rigid,then the corresponding VHS is surely nonrigid. By utilizing differential geometric aspects of the period map and the associated metrics on the period domain,Peters extend Faltings's result to general polarized variation of Hodge Structure of arbitrary weight

A polarized variation of Hodge Structure underlying $\mathbb{V}_{\mathbb{Z}}$ is rigid if and only if $(\mathrm{End}^Q(\mathbb{V}_{\mathbb{Q}}) \otimes \mathbb{C})^{-1,1} = 0$.

Example 1.6 (Jost-Yau). Using techniques from harmonic maps Jost and Yau analyzed $Sh(C, E, Z)$ for a large class of varieties. See [10]. They gave differential geometric proofs of above theorems.Their paper provided analytic methods to solve rigidity problems, and also gave a powerful tool to analyze the Higgs bundles with singular Hermitian metric.

The deformation of the family can be reduced to deformation of the corresponding period map.It is interesting to study the case that the period domain is Hermitian symmetric space.

Example 1.7 (Mok Ngaiming). Considering those arithmetic varieties arising as moduli spaces for certain polarized Abelian varieties,Mok [20] proves a finiteness theorem for the Mordell-Weil group (i.e., the group of holomorphic sections) of the associated universal Abelian variety.

Let Ω/Γ be a quotient of a bounded symmetric domain by a discrete properly discontinuous subgroup Γ in $\mathrm{Aut}(\Omega)$, Mok and Eyssidieux show that for any immersed compact complex submanifold $S \hookrightarrow \Omega/\Gamma$, assume that the tangent subspaces are generic in some algebro-geometric sense,S can be locally approximated by a unique isomorphism class of totally geodesic complex submanifolds of Ω.

Example 1.8 (No rigid families of Abelian varieties)**.** Faltings constructed examples showing that $Sh(C, E, Z)$ is infinite for Abelian varieties of dimension ≥ 8. See [8]. Saito and Zucker extended the construction of Faltings to the setting when Z is an algebraic polarized K3 surface. They were able to classify all cases when the set $Sh(C, E, Z)$ is infinite.

For boundedness of above examples,one may refer to the works of [8],[22],[**?**].

It should be pointed out they were not considering polarized families.

In the case of fibers are curve, we assume the condition $q = g(\mathcal{X}_{gen}) \geq 2$ which is equivalent to the condition that the canonical line bundle of generic fiber $K_{\mathcal{X}_{gen}}$ is ample.

The role of the genus is played by the Hilbert polynomial, thus fixing $g(\mathcal{X}_{gen})$ can be replaced by fixing $h_{K_{\mathcal{X}_{gen}}}$, the Hilbert polynomial of $K_{\mathcal{X}_{gen}}$. And over $C - S$,the family $f : \mathcal{X} \to C$ should be smooth canonical polarized family .

In higher dimension case,instead of fixing $g(\mathcal{X}_{gen})$, it is reasonable to we think about the polarized projective variety (X, L) such that L is ample line bundle on X and $\chi(X, L^\nu)$ is a fixed Hilbert polynomial $h(\nu)$.

As pointed out in following section,by Matsusaka-Mumford Big theorem L can be regard as a very ample line bundle such that $N = h(1) - 1$ is independent on L,there is an embedding

$$j(X) : X \hookrightarrow \mathbb{P}^N.$$

Let C be a fixed nonsingular projective curve and S be fixed finite points on C.Let $f : \mathcal{X} \to C$ be a family which is asked to satisfied the following condition:

Over $C \setminus S$, $(f : (\mathcal{X}, \mathcal{L}) \to C \setminus S)$ is a polarized family such that $\mathcal{L}_b = \mathcal{L}|_{\mathcal{X}_b}$ ample on $\mathcal{X}_b$ and

$$\chi(\mathcal{X}_b, \mathcal{L}_b^\nu) = h(\nu)$$

the fixed Hilbert polynomial for any $b \in C \setminus S$.

With the polarization condition, we define $Sh(C, S, K)$ to be the set of all equivalent classes of non-isotrivial family $\{f : \mathcal{X} \to C\}$ as above such that $\mathcal{X}_b$ is a smooth projective variety with 'type' K for any $b \in C \setminus S$. Two such families are equivalent if they are isomorphic over $C - S$ as polarized families.

One can formulate the problem into:

Conjecture 1.9 (**Higher dimensional Shafarevich conjecture).** Fixing (C, S) and the Hilbert polynomial $h(\nu)$.

- (**B**) The elements of $Sh(C, S, K)$ are parameterized by $\mathbb{T}$,a scheme of finite type over $\mathbb{C}$.
- (**R**) $\dim \mathbb{T} = 0$.
- (**H**) $\mathbb{T} \neq \emptyset \Rightarrow 2g(C) - 2 + \#S > 0$

The important works of Viehweg on Moduli problem together with so called Arakelov-Yau inequality guarantee **the Boundedness of Analogue Shafarevich conjecture for the family of polarized of Calabi-Yau manifolds.**

One also can refer to the paper [18] for the proof by using the well known Schwarz-Yau lemma and Bishop compactness.

In higher dimensional fibers case,one also have the weak boundedness conjecture:

(**WB**) For a family $(f : X \to C) \in Sh(C, S, K)$,$\deg f_* \omega^m_{X/C}$ is bounded above in term of $g(C), \#S, h, m$.In particular, the bound is independent of f.

These theorem and proposition together with so called Arakelov-Yau-Schwarz inequality guarantee the Boundedness of Analogue Shafarevich conjecture for the family of polarized of Calabi-Yau manifolds.

(**WB**) $\Rightarrow$ (**H**) is still true for the canonically polarized families.But it is an enigma whether (**WB**) implies (**B**) in higher dimensional fibers case.

Example 1.10. (New important results).

(M-K-Z) Migliorni,Kovác and Zhang Qi proved that any family of minimal algebraic surfaces of general type over a curve of genus g and m singular points such that $2g(C) - 2 + \#S \leq 0$ is isotrivial. See [13], [19], [40] and [3]. Oguiso and Viehweg[23] proved the same results for families of elliptic surfaces.

(B-V) Bedulev and Viehweg [3] have proved the boundedness for families of algebraic surfaces of general type over a fixed algebraic curve, the weak boundedness for family of canonically polarized varieties.

(V-Z) Recently Viehweg and Zuo have obtained very important results [32][33] [34]: Brody hyperbolicity was proved for the moduli space of canonically polarized complex manifolds. They proved the boundedness for Sh(C,E,Z) for arbitrary Z, with ω_Z semi-ample. They also established that the automorphism group of moduli stacks of polarized manifolds is finite. The rigidity property for the generic family of polarized manifolds has been proved too.

(L-T-Y-Z) In their preprint"The Analogue of Shafarevich's Conjecture for Some CY Manifolds", Liu,K.,Todorov,A.,Yau,S.T.,Zuo,K give a simple and readable proof of the boundednes(Their idea is to use Schwarz-Yau lemma and Bishop compactness).They spend most chapters to deal with rigidity by the idea of using Yukawa coupling.

Remark: A complex analytic space $\mathcal{N}$ is called *Brody hyperbolic* if every holomorphic map $\mathbb{C} \to \mathcal{N}$ is constant.

Certainly,we have an algebraic version:*algebraic hyperbolic*.The essential fact is that if the moduli space is algebraic hyperbolic,then **(H)** in Shafarevich conjecture will hold. It is the motivation for us to study the Brody hyperbolic.

Here are the new process on the conjecture:

Notations 1.11 (**Key Observation by Viehweg and Zuo**[36])**.** The rigid property for the generic family of polarized manifolds has been proved by Viehweg and Zuo. Now they point out that the conjecture will fail for general condition though most families of Calabi-Yau manifolds are rigid. They construct some important counterexamples and show a principle that there always exist a product of the moduli space of hypersurface of degree d in $\mathbb{P}^n$ embedded into the moduli space of hypersurface of degree d in $\mathbb{P}^N$ where $N > n$.

So the first key step for Shafarevich conjecture is to find a more fine condition.

Question : Can we find the necessary and sufficient condition? It is also interesting to classify the non-rigid families of Calabi-Yau manifolds.

Example 1.12 (Zhang [41][42])**.** In his Thesis,Zhang proves

(1) Lefschetz pencils of Calabi-Yau manifolds of odd dimension are rigid.The proof depends on the special properties of Lefschetz pencils. As Deligne showed in his proof of Weil conjecture I,the VHS of Lefschetz pencils can be decomposed into two sub VHSs: One is invariant space and another is vanishing cycles space which is absolute irreducible under the action of fundamental group $\pi_1(\mathbb{P}^1 \backslash S)$ where S is set of singular values(this is essential the Kazdan-Magulis theorem).

According to this observation, Zhang shows that for arbitrary given nontrivial pencil of CY manifolds,the vanishing cycles space is not empty and the pieces of $(n, 0)$-type and $(0, n)$-type of the VHS are both in the vanishing cycles space. But on the other hand,assuming the family is not rigid,one can obtain a flat non zero $(-1, 1)$ type endomorphism σ of the VHS. It makes the local system splitting such that $(n,0)$ and $(0,n)$ are in different factors by local Torelli theorem. Therefore,it is a contradiction.

(2) Zhang obtains a general result on so called strong degenerate families(**not only for families of Calabi-Yau manifolds**)which certainly are of "general" type.We call a family over any closed Riemann surface strong degenerate if it has a singular fiber such that every component is dominated by projective space $\mathbb{P}^n$. The author shows this family must be rigid. He uses the similar trick in (1) to get a nonzero endomorphism σ.By the properties of Higgs bundles,he identifies this σ to be a monodromy-invariant section of the VHS of the self-product family by Künneth formula.Finally, using the properties of the strong degenerate singular fiber,he shows the endomorphism σ is zero so that the family must be rigid.As a corollary,he obtains a weak Arakelov theorem of high dimensional version.

(3) Following the recent works of **Liu-Todorov-Yau-Zuo** and **Viehweg-Zuo**, Zhang gives another proof of a criterion of rigidity by using the technique of Higgs bundles: A Calabi-Yau family with nonzero Yukawa coupling should be rigid. As an application of this criterion,with the arguments of **Schmid** and **Simpson** on the residues of vector bundles over singularizes,the author shows that families of CY manifolds admitting a degeneration with maximal unipotent monodromy must be rigid.

1.3. **Appendix: Arithmetic version of Shafarevich conjecture.** Let $(R,\mathfrak{m})$ be a DVR, $F=\mathrm{Frac}(R)$,and C a smooth projective curve over F.C is said to have **good reduction over** R if there exists a smooth projective variety B over $\mathrm{Spec}(R)$ such that

$$\begin{array}{ccccc} C & \xrightarrow{\simeq} & B_F & \longrightarrow & B \\ & & \downarrow & & \downarrow \\ & & \mathrm{Spec}(F) & \longrightarrow & \mathrm{Spec}(R) \end{array}.$$

Therefore,let R be a Dedekind ring,$F=\mathrm{Frac}(R)$,and C a smooth projective curve over F. C has **good reduction at the closed point** $\mathfrak{m}\in\mathrm{Spec}(R)$ if it has a good reduction over $R_{\mathfrak{m}}$.

Conjecture 1.13 (**Shafarevich**)**.** Let $q\geq 2$ be an integer,F a number field,$R\subset F$ the ring of integers of F,and $\Delta\subset\mathrm{Spec}R$ a finite set.Then there exists only finitely many smooth projective curves over F of genus q that have good reduction outside Δ.

The Shafarevich's conjecture was inspired by the well-known **Hermit** theorem in algebraic number theory :

the number of extensions k'/k of a given degree whose critical prime divisors belong to a given finite set S is finite.

Therefore,we can get **the function field version** of the conjecture is :

Let $q\geq 2$ be an integer,$F=K(B)$ the function field of B,R the subring of F such that $B-\Delta=\mathrm{Spec}(R)$ a number field,$R\subset F$ the ring of integers of F.Then there exists only finitely many smooth projective non-isotrivial curves over F of genus q that have good reduction over all closed points of $\mathrm{Spec}(R)$.

The arithmetic version was confirmed by Faltings in 80s and he used it to prove the

Conjecture 1.14 (Mordell). Let F be a number field and C a smooth projective curve of genus $q \geq 2$ defined over F. Then $C(F)$ is finite.

2. The Modern Proof of Original Shafarevich Conjecture

Here we will give a sharp proof due to the Viehweg and Zuo.

2.1. **Moduli of Curves.** Recall that a reduced projective curve C is called stable, if the singularities of C are normal crossings(it is locally isomorphic to the singularity of the plane curve $xy = 0$), and if ω_C is ample.

Theorem 2.1 ((Mumford [21])). *For $g \geq 2$, define*

$$\overline{\mathfrak{M}}_g(\mathbb{C}) = \{ \text{ stable curves of genus } g, \text{ defined over } \mathbb{C}\}/\cong .$$

Then there exists a projective coarse moduli scheme $\overline{\mathcal{M}}_g$ for $\overline{\mathfrak{M}}_g$, of dimension $3g-3$. i.e. a variety $\overline{\mathcal{M}}_g$ and a natural bijection $\overline{\mathfrak{M}}_g(\mathbb{C}) \cong \overline{\mathcal{M}}_g(\mathbb{C})$ where $\overline{\mathcal{M}}_g(\mathbb{C})$ denotes the $\mathbb{C}$-valued points of $\overline{\mathcal{M}}_g$.

Let

$$\mathfrak{M}_g(\mathbb{C}) = \{ \text{ smooth curves of genus } g, \text{ defined over } \mathbb{C}\}/\cong .$$

Similar, there is a quasi-projective coarse moduli scheme $\mathcal{M}_g$ for $\mathfrak{M}_g$ and $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ as Zariski open set.

Remarks :

1. Stable curve means *Deligne-Mumford stable curve*,the definition is equivalent to a reduced irreducible projective curve with only ordinary doubles as singularities and only finitely many automorphisms. The restriction on the number of automorphisms thus means that every smooth rational component of a stable curve intersects the remaining components in at least three points.
2. The moduli scheme $\overline{\mathcal{M}}_g$ is normal, connected and reduced.The subscheme $\mathcal{M}_g$, corresponding to non-singular curves of genus g, is open in $\overline{\mathcal{M}}_g$. As in GIT,one can use "level μ-structures"to shown that the moduli schemes $\overline{\mathcal{M}}_g$ have finite coverings $\phi : \overline{\mathcal{M}}_g^{\Gamma} \to \overline{\mathcal{M}}_g$ which carry a universal family
$$g : \mathfrak{X}^{\Gamma} \to \overline{\mathcal{M}}_g^{\Gamma}.$$
3. We will give the precise definition of a coarse moduli scheme late. Let us just explain that the "natural" meaning of coarse moduli space :
   For each flat family $g : \mathcal{X} \to Z$, whose fibers $g^{-1}(z)$ belong to $\overline{\mathfrak{M}}_g(\mathbb{C})$, the induced map $Z(\mathbb{C}) \to \overline{\mathcal{M}}_g(\mathbb{C})$ comes from a morphism of schemes $\phi : Z \to \overline{\mathcal{M}}_g$.(Hence for every family $(g : \mathcal{X} \to Z)$ of stable curves of genus of g,there exists a morphism $\eta_g : Z \to \overline{\mathcal{M}}_g$ such that for all $z \in Z$,$\eta_g(z) = [\mathcal{X}_z]$.)

It follows from the construction of $\overline{\mathcal{M}}_g$, that for all $\nu > 0$ and for some $p \gg \nu$ there exists an invertible sheaf $\lambda_\nu^{(p)}$, such that for all families $g : \mathcal{X} \to Z$,

$$\det(g_*\omega_{\mathcal{X}/Z}^{\nu})^p = \phi^*(\lambda_\nu^{(p)}).$$

Proposition 2.2 ((Mumford [21])). *For ν, μ and p sufficiently large and divisible, for*

$$\alpha = (2g-2) \cdot \nu - (g-1) \quad \text{and} \quad \beta = (2g-2) \cdot \nu \cdot \mu - (g-1)$$

the sheaf $\lambda_{\nu \cdot \mu}^{(p)^{\alpha}} \otimes \lambda_{\nu}^{(p)^{-\beta \cdot \mu}}$ is ample.

Definition 2.3. $\mathcal{E}$ be a locally free sheaf on a curve Y,We say $\mathcal{E}$ *numerically effective*(nef) if $\mathcal{O}_{\mathbb{P}(\mathcal{E})(1)}$ is nef line bundle on projective space $\mathbb{P}(\mathcal{E})$.Similiarly we get the definition of ample vector bundle.

Remark: [31]Let $\mathcal{H}$ be an ample line bundle on Y,U Zariski open set of Y.

- It is not difficult to show that $\mathcal{E}$ is nef if and only if for all finite covering $\pi : C \to Y$ and for all invertible quotients $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{N}$,one has $\deg \mathcal{N} \geq 0$.
- It is shown by Hartshone that $\mathcal{E}$ is ample(with respect to U) if and only if for some $\eta > 0$ there exists a morphism
$$\bigoplus \mathcal{H} \longrightarrow S^\eta(\mathcal{E}),$$
surjective (over U).
- It is natural that we have the definition **weakly positive:**
$\mathcal{E}$ is weakly positive over U,if for all $\alpha > 0$ the sheaf $S^\alpha(\mathcal{E}) \otimes \mathcal{H}$ is ample with respect to U.
- One can use this properties to give similar definition on a quasi-projective variety Z.Ther is a little modification: $\mathcal{E}$ is called nef on Z, if for all morphisms $\pi : C \to Z$, from a curve C to Z, and for all invertible quotients $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{N}$ one has $\deg\ \mathcal{N} \geq 0$.

Exercise 2.4. Given $d \in \mathbb{N}$, assume that for all $\delta \in \mathbb{N} - \{0\}$, there exists a covering $\tau : Y' \to Y$ of degree δ such that $\tau^*\mathcal{E} \otimes \mathcal{H}$ is nef, for one $\mathcal{H}$ of degree d(hence for all invertible sheaves of degree d). Then $\mathcal{E}$ is nef.

2.2. **Rigidity property.**

Conjecture 2.5 (Viehweg)**.** On $U = Z - S$, for $\varphi : U \to \mathcal{M}_g$ induced by a family

(a) φ generically finite $\overset{?}{\Longrightarrow}$ $\Omega^1_Z(\log S)$ weakly positive?
(b) φ generically finite $\overset{?}{\Longrightarrow}$ $\det(\Omega^1_Z(\log S))$ big ?

Viehweg-Zuo's Theorem[33] For $\eta \gg 1$ there is an invertible subsheaf
$$\mathcal{L} \subset S^\eta(\Omega^1_Z(\log S))$$
with $\kappa(\mathcal{L}) \geq \dim(\varphi(Z))$. Therefore,one has Conj.(a) $\Longrightarrow$ Conj.(b).

Theorem 2.6 (Zuo[44])**.** *Conjecture (a) holds true if the fibres of $\mathcal{X}_0 \to U$ satisfy local Torelli theorem.*

Because local Torelli theorem always holds for curve case,the conjecture (a) holds for families of algebraic curves(Originally,this fact is bellowed to Mumford in his works on Moduli space of curves.)

Corollary 2.7. *Zuo's theorem implies the* **Rigidity** *property of the Shafarevich conjecture.*

Proof. Assume a non-isotrivial family over curve Y is not rigid then there exists curve T_0 and smooth extended family of curves
$$\mathcal{X}_0 \longrightarrow Z_0 = T_0 \times Y_0,$$
and the induced morphism $\phi : Z_0 \to M_g$ is generically finite.

In fact, it is impossible by Zuo's theorem. Otherwise, replacing Y_0 and T_0 by some covering, one would find for compactifications T and Y of T_0 and Y_0 the sheaf

$$\Omega^1_{T\times Y}(\log(T\times(Y-Y_0)+(T-T_0)\times Y)) = pr_1^*\Omega^1_T(\log(T-T_0))\oplus pr_2^*\Omega^1_Y(\log(Y-Y_0))$$

to be ample over some dense open subset. Obviously this cannot be true: Choose a generic quasi-projective curve $Y_0\times t$,still denote it Y_0, then
$\Omega^1_{T\times Y}(\log(T\times(Y-Y_0)+(T-T_0)\times Y))|_{Y_0}$ will be ample over Y_0.But it is impossible because

$$(pr_1^*\Omega^1_T(\log(T-T_0))\bigoplus\Box)|_{Y_0} = \oplus\mathcal{O}_{Y_0}\bigoplus\Box|_{Y_0}.$$

□

Remarks :

1. That the canonical morphism is generically finite is equivalent to that there are no isotrivial subfamilies including a general fiber.
2. Using the argument in Chapter 4, we will have a nonzero section
$$\mathcal{O}_{T_0}\to\phi^*(\Omega^\vee_{T_0}),$$
then we get a contradiction.

2.3. **Boundedness.** Here I would like to give a proof by using the arguments of Kefeng Liu in [15],[16].

2.3.1. *Teichmüler space $\mathcal{T}_g$ and period map.* Fix a compact Riemann surface R_0 of genus g.Consider a pair (R,H) of a compact Riemann surface R and a homotopy class H of orientation-preserving homeomorphisms from R_0 to R.Define an equivalence relation
sim:

$$(R,H)\sim(R',H')\Longleftrightarrow H'\circ H \text{ contains a biholomorphic map from } R \text{ to } R'.$$

Let $\mathcal{T}_g := \{(R,H)\}/\sim$,the classic Teichmüler theorem tell us that $\mathcal{T}_g$ has a complex structure and is a smooth complex ball. Fix a symplectic basic

$$\{\alpha_1,\cdots,\alpha_g,\beta_1,\cdots,\beta_g\}$$

of $H_1(R_0,\mathbb{Z})$.Then,for (R,H) the set

$$\{H(\alpha_1),\cdots,H(\alpha_g),H(\beta_1),\cdots,H(\beta_g)\}$$

is a symplectic basis of $H_1(R,\mathbb{Z})$.We choose a basis $\{\omega_1,\cdots,\omega_g\}$ of holomorphic 1-form on R such that

$$\int_{H(\beta_i)}\omega_j=\delta_{ij},\quad 1\le i,j\le g.$$

Then these integral determine uniquely on the equivalence class a point

$$(\int_{H(\alpha_i)}\omega_j)$$

of Siegel upper half space $\mathfrak{D}_g$. Exactly,there is a holomorphic map

$$\tau:\quad \mathcal{T}_g\longrightarrow\mathfrak{D}_g,$$
$$[(R,H)]\longmapsto(\int_{H(\alpha)}\omega_j).$$

The standard Teichmüler theory tell us there is a modular group $\mathbf{Mod}_g$ which is a subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$ such that

$$\mathcal{M}_g = \mathbf{Mod}_g \backslash \mathcal{T}_g.$$

Thus τ descends to a holomorphic map

$$j : \mathcal{M}_g \to \mathcal{A}_g = \mathrm{Sp}(2g,\mathbb{Z})\backslash \mathfrak{D}_g,$$

and $\dim_{\mathbb{C}} \mathcal{A}_g = g(g+1)/2$.

Geometry on $\mathcal{T}_g$: For a compact Riemann surface of genus $g \geq 2$,we know $H^2(R,\Theta_R) = 0$ and $\dim_{\mathbb{C}} H^1(R,\Theta_R) = 3g-3$.This implies that the base space T of the Kuranishi family $\mathcal{U} \to T$ is a smooth complex manifolds with dimension $3g-3$ and the Kuranishi family is universal at each point of T because $H^0(R,\Theta_R) = 0$.Moreover,there is a complex analytic family

$$\pi : \mathcal{R} \to \mathcal{T}_g$$

of the compact Riemann surface of genus g and that it is universal at every point of $\mathcal{T}_g$.However there is no analytic family over $\mathcal{M}_g$,this is why we call it coarse moduli space.We have

Torelli Theorem

(1) $\tau : \mathcal{T}_g \to \mathfrak{D}_g$ is locally injective,
(2) $j : \mathcal{M}_g \to \mathcal{A}_g = \mathrm{Sp}(2g,\mathbb{Z})\backslash \mathfrak{D}_g$ is injective.

2.3.2. *Weil-Petersson metric.* Let $\pi : \mathcal{X} \to \mathcal{T}_g$ be the universal family over the Teichmüller space. The Poincaré metric on each fiber patches together to give a smooth metric on $\Omega_{\mathcal{X}/\mathcal{T}_g}$. Then $\pi_*\Omega^{\otimes 2}_{\mathcal{X}/\mathcal{T}_g}$, the push-down of $\Omega^{\otimes 2}_{\mathcal{X}/\mathcal{T}_g}$, is the cotangent bundle of $\mathcal{T}_g$. Recall that for any point $s \in \mathcal{T}_g$

$$R^0\pi_*\Omega^{\otimes 2}_{\mathcal{X}/\mathcal{T}_g}|_s = H^0(\mathcal{X}_s, \Omega^{\otimes 2}_{\mathcal{X}_s}) \otimes k(s)$$

where $\mathcal{X}_s = \pi^{-1}(s)$.

There exists a natural inner product on $\pi_*\Omega^{\otimes 2}_{\mathcal{X}/\mathcal{T}_g}$ induced from the Poincaré metric on each fiber.

$$G_{WP}(s)(\mu_1,\mu_2) = \int_{\mathcal{X}_s} <\mu_1,\mu_2>_{\rho(s)}$$

where $\mu_1,\mu_2 \in H^0(\mathcal{X}_s,\Omega^{\otimes 2}_{\mathcal{X}_s})$ and $\rho(s)$ is Poincáre metric on $\mathcal{X}_s$. This inner product induces the Weil-Petersson metric on $\pi_*\Omega^{\otimes 2}_{\mathcal{X}/\mathcal{T}_g}$.Moreover,shown by Wolpert [37]

$$\pi_* c_1^2(\omega_{\mathcal{X}/\mathcal{T}_g}) = \frac{1}{2\pi^2}\omega_{\mathrm{WP}}. \tag{2.7.1}$$

where ω_{WP} is the Kähler form of G_{WP}.So Weil-Petersson metric is Kähler metric,furthermore it is not complete.

Properties 2.8. Important facts:

(1) Shown by Alforhs the sectional curvature of G_{WP} is bounded from above by
$$-\frac{1}{2\pi(g-1)}.$$
(2) There is a obvious fact that the Weil-Petersson metric is invariant under the modular group $\mathbf{Mod_g}$,thus the equation 2.7.1 still hold for the universal family over $\mathcal{M}_g^{level}$.

(3) It can be shown that,as currents the equation 2.7.1 holds on $\overline{\mathcal{M}}_g^{level}$.

2.3.3. *Weak Boundedness.* Given a family $f : \mathfrak{X} \to Y$ over projective curve,we always assume it is stable family(!we can do it) and the restricted family $f : \mathfrak{X}_0 \to Y_0 = Y - S$ is smooth. Moreover we can assume the image of moduli map is in the fine moduli space $\mathcal{M}_g^{level}$) with level structure(we can get it by Galois covering).The family is uniquely induced from the universal family $\mathcal{U} \to \mathcal{M}_g^{level}$,then the Weil-Petersson metric can descend to $\mathcal{M}_g^{level}$.

Remark: $\mathcal{M}_g^{level}$ will be of $\Gamma\backslash\mathcal{T}_g$ where Γ is subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ with a level structure.

We have the moduli map $g : Y_0 \to \mathcal{M}_g^{level}$ and

$$(f : \mathfrak{X}_0 \to Y_0) \cong \mathcal{U} \times_g Y_0 \to Y_0.$$

we have the intersection(Calculated by Liu,K.F in [15] [16]).

$$(\omega^2_{\mathfrak{X}/Y}) = \int_{\mathfrak{X}} c_1^2(\omega_{\mathfrak{X}/Y}) = \int_Y f_* c_1^2(\omega_{\mathfrak{X}/Y})$$

By the current property in 2.8,

$$\int_Y f_*(c_1^2(\omega_{\mathfrak{X}/Y})) = \int_{Y_0} f_* c_1^2(\omega_{\mathfrak{X}_0/Y_0}) = \frac{1}{2\pi^2}\int_{Y_0} g^*\omega_{WP}.$$

But

$$\int_{Y_0} g^*\omega_{WP} < 2\pi(g-1)\int_{Y_0}\omega_P = 2\pi(g-1)(2g(Y) - 2 + \#S)$$

where ω_P is the Poincaré metric on Y_0.The last step is Gauss-Bonnet and the inequality is the following :

Theorem 2.9 (Schwarz-Yau Lemma)**.** *Let R be a compact Riemann surface ($g > 1$)with curvature -1 or an affine Riemann surface of hyperbolic.Let N be Hermitain manifolds with holomorphic sectional curvature strictly bounded above by negative constant $-K$.Then for any non-constant holomorphic map f from R to N,one has*

$$f^*\omega_N < \frac{1}{K}\omega_R$$

where ω_R, ω_N denote respectively the Kähler form of R, N.

Using a inequality of Xiao Gang

$$\deg f_*\omega_{\mathfrak{X}/Y} \leq (\frac{g}{4g-4})(\omega^2_{\mathfrak{X}/Y})$$

we get **(WB)**.By the theorem of Bedulev-Viehweg [3],(**WB**) $\Rightarrow$ (**B**).

Remark: Exactly,Deligne proved that

$$\deg f_*\omega_{\mathcal{X}/Y} \leq \frac{g}{2}(2g(Y) - 2 + \#S).$$

Jost and Zuo have a smart proof the weak boundedness.Moreever,Zuo has a series works on the powerful Arakelove-Yau inequality in case of higher dimensional fiber. Using Miyaoka-Yau inequality,Tan S.L proved the Beauville's conjecture(In the papers of Liu which I mention above,he also solved the problem by the method I show),then got a more fine estimate: for semistable fiberation with $g \geq 2$

$$\deg f_*\omega_{\mathcal{X}/Y} < \frac{g}{2}(2g(Y) - 2 + \#S).$$

3. Moduli Properties of Families of Calabi-Yau Manifolds

3.1. **Family of polarized algebraic manifolds.** In last chapter of the thesis,the author will study the rigid problem of the family of polarized variety. Why should we study the polarized family ?

Generally,for technical reason, we should endow Moduli stack of the geometric object with the given type a nice algebraic structure such that every family of the given type is induced geometrically from the it(Universal property). In fact, in most cases the fine moduli scheme (which other family is uniquely induced from) does not exist.Fortunately,Viehweg has shown that the coarse moduli space of polarized varieties with semi-simple canonical line bundle and given Hilbert polynomial exists and is quasi-projective. As an application,one obtains the quasi-projective moduli space for $K3$ surfaces,Abelian varieties and Calabi-Yau manifolds.With these results and recent works of Zuo Kang on Arakelov-Yau's inequalities,one can obtain the boundedness of the Shafarevich conjecture for Calabi-Yau manifolds. On the shoulder of Viehweg and Zuo's working, it is meaningful to consider the rigid problem of Shafarevich conjecture.

Now, Let the ground field be complex field $\mathbb{C}$.

Definition 3.1. (Moduli Problem [31]**)**

1. The objects of a **moduli problem** of polarized schemes will be a class $\mathfrak{F}(\mathbb{C})$, consisting of isomorphism classes of certain pairs $(W, \mathcal{H})$, with:
   a) W is a connected equidimensional projective scheme over $\mathbb{C}$.
   b) $\mathcal{H}$ is an ample invertible sheaf on Γ or, as we will say, a **polarization** of W
2. For a scheme Y a **family of objects** in $\mathfrak{F}(\mathbb{C})$ will be a pair $(f : X \to Y, \mathcal{L})$ which satisfies
   a) f is a flat proper morphism of schemes,
   b) $\mathcal{L}$ is invertible on X,
   c) $(f^{-1}(y), \mathcal{L}|_{f^{-1}(y)}) \in \mathfrak{F}(\mathbb{C})$, for all $y \in Y$,
   d) some additional properties, depending on the moduli problem one is interested in.
3. If $(f : X \to Y, \mathcal{L})$ and $(f' : X' \to Y, \mathcal{L}')$ are two families of objects in $\mathfrak{F}(\mathbb{C})$ we write $(f, \mathcal{L}) \sim (f', \mathcal{L}')$ if there exists a Y-isomorphism $\tau : X \to X'$, an invertible sheaf $\mathcal{F}$ on Y and an isomorphism $\tau^*\mathcal{L}' \cong \mathcal{L} \otimes f^*\mathcal{F}$. If one has $X = X'$ and $f = f'$ one writes $\mathcal{L} \sim \mathcal{L}'$ if $\mathcal{L}' \cong \mathcal{L} \otimes f^*\mathcal{F}$.
4. If Y is a scheme over $\mathbb{C}$ we define the **Moduli Functor**

$$\mathfrak{F} : (Sch/\mathbb{C})^o \longrightarrow (Sets)$$

   by

$$\mathfrak{F}(Y) = \{(f : X \to Y, \mathcal{L});\ (f, \mathcal{L}) \text{ a family of objects in } \mathfrak{F}(\mathbb{C})\}/\sim .$$

   The natural functor transform for any $\tau : Y' \longrightarrow Y$ with $Y, Y' \in (Sch/\mathbb{C})$ is

$$\mathfrak{F}(\tau) : \mathfrak{F}(Y) \longrightarrow \mathfrak{F}(Y')$$

   by sending $(f : X \to Y, \mathcal{L})$ in $\mathfrak{F}(Y)$ to $(pr_2 : X \times_Y Y' \to Y', pr_1^*\mathcal{L}) \in \mathfrak{F}(Y')$.

Let us introduce a coarser equivalence relation on $\mathfrak{F}(\mathbb{C}) = \mathfrak{F}(\mathrm{Spec}(\mathbb{C}))$ and on $\mathfrak{F}(Y)$, which sometimes replaces "$\sim$".

Definition 3.2. Let $(f : X \to Y, \mathcal{L})$ and $(f : X' \to Y, \mathcal{L}')$ be elements of $\mathfrak{F}(Y)$. Then $(f, \mathcal{L}) \equiv (f', \mathcal{L}')$ if there exists an Y-isomorphism $\tau : X \to X'$ such that the sheaves $\mathcal{L}|_{f^{-1}(y)}$ and $\tau^*\mathcal{L}'|_{f^{-1}(y)}$ are numerically equivalent for all $y \in Y$. By definition this means that for all curves C in X, for which $f(C)$ is a point, one has $\deg(\mathcal{L} \otimes \tau^*\mathcal{L}'^{-1}|_C) = 0$.

Then one defines functors $\mathfrak{PF}$ from the category of $\mathbb{C}$-schemes to the category of sets by choosing

1. On objects: For a scheme Y defined over $\mathbb{C}$ one takes for $\mathfrak{PF}(Y) = \mathfrak{F}(Y)/\equiv$ the set defined as in 3.1, 4).
2. On morphisms: For $\tau : Y' \to Y$ one defines

$$\mathfrak{PF}(\tau) : \mathfrak{PF}(Y) \to \mathfrak{PF}(Y')$$

as the map obtained by pullback of families.

We will call $\mathfrak{F}$ the moduli functor of the moduli problem $\mathfrak{F}(\mathbb{C})$ and $\mathfrak{PF}$ the moduli functor of **polarized schemes in $\mathfrak{F}(\mathbb{C})$, up to numerical equivalence**.

If $\mathfrak{F}'(\mathbb{C})$ is a subset of $\mathfrak{F}(\mathbb{C})$ for some moduli functor $\mathfrak{F}$ then one obtains a new functor by choosing

$$\mathfrak{F}'(Y) = \{(f : X \to Y, \mathcal{L}) \in \mathfrak{F}(Y);\ f^{-1}(y) \in \mathfrak{F}'(\mathbb{C}) \text{ for all } y \in Y\}.$$

We will call $\mathfrak{F}'$ a **sub-moduli functor** of $\mathfrak{F}$.

In order to make sense of the definition of 3.1,one should make precise the pairs in 1) and the additional properties in 2).Thus,we have

Definition 3.3. (Moduli problem on polarized manifolds)

1. **Polarized manifolds:** $\mathfrak{M}'$ is the moduli functor such that

$$\mathfrak{M}'(\mathbb{C}) = \{(W, \mathcal{H});\ W \text{ a projective manifold, } \mathcal{H} \text{ ample invertible on } W\}/\sim$$

and defines again $\mathfrak{M}'(Y)$ to be the set of pairs $(f : X \to Y, \mathcal{L})$, with f a flat morphism and with $\mathcal{L}$ an invertible sheaf on X, whose fibres all belong to $\mathfrak{M}'(\mathbb{C})$.

2. **Polarized manifolds with a semi-ample canonical sheaf:**
$\mathfrak{M}$ is the moduli functor given by

$$\mathfrak{M}(\mathbb{C}) = \{(W, \mathcal{H});\ \Gamma \text{ a projective manifold, } \mathcal{H} \text{ ample} \\ \text{invertible and } \omega_W \text{ semi-ample } \}/\sim$$

and, for a scheme Y, by defining $\mathfrak{M}(Y)$ to be the subset of $\mathfrak{M}'(Y)$, consisting of pairs $(f : X \to Y, \mathcal{L})$, whose fibres are all in $\mathfrak{M}(\mathbb{C})$. We write $\mathfrak{P}$ instead of $\mathfrak{PM}$ for the moduli functor, up to numerical equivalence.

Remark 3.4. Let $h(T) \in \mathbb{Q}[T]$ be a polynomial with $h(\mathbb{Z}) \subset \mathbb{Z}$,then for the moduli functor $\mathfrak{F}$ as one of above functors,one defines $\mathfrak{F}_h$ by

$$\mathfrak{F}_h(Y) = \{(f : X \to Y, \mathcal{L}) \in \mathfrak{F}(Y); h(\nu) = \chi(\mathcal{L}^\nu|_{f^{-1}(y)}) \text{for all } \nu \text{ and all } y \in Y\ \}.$$

Thus for $(\Gamma, \mathcal{H}) \in \mathfrak{F}(\mathbb{C})$,$h(T)$ is the Hilbert polynomial of $\mathcal{H}$.Moreover,

$$\mathfrak{F}(Y) = \dot{\bigcup_h} \mathfrak{F}_h(Y).$$

If the Moduli functor can be represented(i.e $\mathfrak{F} = \mathrm{Hom}(-, \mathcal{M})$ for some scheme $\mathcal{M}$), then we say the Moduli scheme exists.Geometrically,it just means whether the universal family exists.

Definition 3.5 (Moduli Space)**.** Mumford's definition in GIT [21]

(A) A **fine** moduli scheme for $\mathfrak{F}$ consist of a scheme $\mathcal{M}$ and a *universal family* $(g : \mathfrak{X} \to M, \mathcal{L}) \in \mathfrak{F}(\mathcal{M})$.Here *universal* means,that for all $(f : X \to Y, \mathcal{H}) \in \mathfrak{F}(Y)$,there is a unique morphism $\tau : Y \to \mathcal{M}$ with

$$(f : X \to Y, \mathcal{H}) \cong \mathfrak{F}(\tau)(g, \mathcal{L}) = (\mathfrak{X} \times_{\mathcal{M}} Y \xrightarrow{pr_2} Y, pr_1^*\mathcal{L}).$$

(B) A **coarse** moduli scheme for $\mathfrak{F}$ is a scheme $\mathcal{M}$ together with a bijection $\mu : \mathfrak{F}(\mathbb{C}) \leftrightarrow \mathcal{M}(\mathbb{C})$ such that for any family $(f : X \to Y, \mathcal{L}) \in \mathfrak{F}(Y)$:
   (1) The induced map of set $\phi : Y(\mathbb{C}) \to \mathcal{M}(\mathbb{C})$ comes from a morphism

$$Y \to \mathcal{M}.$$

   (1) Given a scheme $N/\mathbb{C}$ and a natural transformation

$$\Phi : \mathfrak{F} \to \mathrm{Hom}(-, N),$$

   the map of set $\Phi(\mathbb{C}) \circ \mu^{-1} : \mathcal{M}(\mathbb{C}) \to N(\mathbb{C})$ comes form a morphism.

Exercise 3.6. It is easy to check the coarse moduli space is unique in the following meaning: For any two coarse moduli space $(\mathcal{M}_i, \mu_i)$ $i = 1, 2$ of the functor $\mathfrak{F}$, there is an isomorphism $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\Psi \circ \mu_1 = \mu_2$.

Theorem 3.7 (Viehweg,E.[31])**.** The Existence of Coarse Moduli

1. *Given a polynomial $h \in \mathbb{Q}[T_1, T_2]$ of degree n such that with $h(\mathbb{Z} \times \mathbb{Z}) \subset \mathbb{Z}$, Then,there exists a coarse quasi-projective moduli scheme $\mathcal{M}_h$ for the sub-moduli functor $\mathfrak{M}_h$(same for $\mathfrak{P}\mathfrak{M}_h$),of polarized manifolds $(W, \mathcal{H}) \in \mathfrak{M}(\mathbb{C})$,with*

$$h(\alpha, \beta) = \chi(\mathcal{H}^\alpha \otimes \omega_W^\beta) \text{ for all} \alpha, \beta \in \mathbb{N}$$

   Furthermore,the moduli functor $\mathfrak{M}_h$(same for $\mathfrak{P}_h$) is bounded by Matsusaka Big theorem.

2. *Given a polynomial h of degree n such that $h(\mathbb{Z}) \subset \mathbb{Z}$,there exists a quasi-projective scheme $\mathcal{M}_h$ of finite type over $\mathbb{C}$ for the set*

$$\{(X, L); \omega_X \text{ semi-ample}, L \text{ ample on } X \text{ and } \chi(X, L^\nu) = h(\nu) \text{for all } \nu\}/\sim .$$

   Furthermore,$\mathcal{M}_h$ is bounded by Matsusaka Big theorem.

   As in item (I), the quasi-projective coarse moduli space for $\mathfrak{P}\mathfrak{M}_h$ exists and is bounded too.

Here $\mathcal{L}$ semi-ample means that if for some $\mu > 0$ the sheaf $\mathcal{L}^\mu$ is generated by global sections.

Corollary 3.8. *In particular,adding the condition as canonical line bundle trivial(the deformation invariant condition),one may get the coarse quasi-projective moduli space for $K3$ surfaces, Calabi-Yau manifolds and Abelian variety.Also these Moduli functor are bounded.*

Proposition 3.9 (Mumford-Viehweg [31])**.** *Let the family $(g : \mathcal{X} \to Y)$ be induced by the morphism $\phi : Y \to \mathcal{M}_h$.*

(I) *Assume all $\Gamma \in \mathcal{M}_h(\mathbb{C})$ are canonical polarization manifolds.For $\eta \geq 2$ with $h(\eta) > 0$,there exists some $p > o$ and an ample invertible sheaf $\lambda_\eta^{(p)}$ on $\mathcal{M}_h$ such that*

$$\phi^* \lambda_\eta^{(p)} = \det(g_* \omega_{\mathcal{X}/Y}^\eta)^p$$

(II) *Assume for some $\delta > 0$,one has $\omega_\Gamma^\delta = \mathcal{O}_\Gamma$ all manifolds $\Gamma \in \mathcal{M}_h(\mathbb{C})$ (for example,K3 surfaces,Calabi-Yau manifolds,Abelain varieties,etc). There exists some $p > o$ and an ample invertible sheaf $\lambda_\eta^{(p)}$ on $\mathcal{M}_h$ such that*

$$\phi^* \lambda^{(p)} = g_* \omega_{\mathcal{X}/Y}^{\delta \cdot p}$$

Theorem 3.10 (Matsusaka-Mumford Big Theorem)**.** *Assume the ground field is always zero.Let $\mathfrak{M}$ be the set of isomorphism of polarized smooth projective varieties with a fixed Hilbert polynomial h. Then $\mathfrak{M}$ is bounded,i.e, $\exists$ an integer m_0 independent of the choice of $(X, \mathcal{H}) \in \mathfrak{M}$ such that $m_0\mathcal{H}$ is very ample for $m \geq m_0$ on X and in fact one can also suppose that $H^j(X, m\mathcal{H}) = 0$ for $j > 0$ and $m \geq m_0$.Hence,the complete linear system $|m_0\mathcal{H}|$ gives a closed immersion:*

$$i : X \hookrightarrow \mathbb{P}^N$$

with $N = h(m_0) - 1$.In particular the Hilbert polynomial of $i(X)$ is $h_1(m) = h(mm_0)$ and hence independent of $(X, \mathcal{H}) \in \mathfrak{M}$.

Therefore, geometrically,the families we study are

$$f : \mathfrak{X} \to M$$

where $\mathfrak{X}$ and M are algebraic manifolds defined over $\mathbb{C}$ and f is surjective morphism with following conditions:
(a) f is a smooth morphism with every closed fiber $f^{-1}(t)$ a connected and reduced smooth projective variety.
(b) f is a projective morphism i.e. there exists a projective space $\mathbb{P}^N$ such that the diagram is commutative:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\ i\ } & \mathbb{P}^N \times M \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & M & \end{array}$$

(c) the varieties $\mathfrak{X}$ and M and every closed fiber $\mathfrak{X}_t = f^{-1}(t)$ are connected. Moreover,let $\omega_t = \mathcal{O}_{\mathbb{P}^N}(1)|_{\mathfrak{X}_t}$ then $(\mathfrak{X}_t, \omega_t)$ is a polarized algebraic variety such that the embedding $i_t : \mathfrak{X}_t \hookrightarrow \mathbb{P}^N$ is determined by the very ample line bundle ω_t,thus ω_t is also a Kähler structure of $\mathfrak{X}_t$.

Remark: We can assume the conditions (b) and (c) according to the Matsusaka-Mumford Big theorem.Because of the flatness of f,the dimension of the closed fiber is a constant.

3.2. **Deformation theory and Torelli theorem.**

3.2.1. *Deformation theory.* Let $f : \mathfrak{X} \to M$ be a smooth family of complex manifolds over a complex manifold M,Let $t_0 \in M$ such that $X = X_{t_0} = f^{-1}(s_0)$.Such a family is called a **deformation** of the complex manifold X.

There exists an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-homomorphisms

$$0 \longrightarrow T_{\mathfrak{X}/M} \longrightarrow T_{\mathfrak{X}} \longrightarrow f^* T_M \longrightarrow 0$$

Form the exact sequence one obtains a long exact sequence

$$\cdots \longrightarrow f_*T_{\mathfrak{X}} \longrightarrow f_*f^*T_M = T_M \xrightarrow{\kappa} R^1f_*T_{\mathfrak{X}/M} \longrightarrow R^1f_*T_{\mathfrak{X}} \longrightarrow \cdots$$

Then the $\mathcal{O}_M$-homomorphism

$$\kappa : T_M \longrightarrow R^1f_*T_{\mathfrak{X}/M}$$

is just the Kodaira-Spencer map,i.e. κ is in the class $H^0(M, R^1f_*T_{\mathfrak{X}/M} \otimes \Omega^1_M)$.At any point $t \in M$,

$$\kappa(t) : T_{M,t} \longrightarrow H^1(\mathfrak{X}_t, T_{\mathfrak{X}_t}).$$

Let $\mathfrak{M}$ be the moduli space of the polarized algebraic variety (X, L),while D is the classifying space of the polarized Hodge Structure of weight n associated to (X, L),there is a natural mapping

$$\phi : \mathfrak{M} \to D/G_{\mathbb{Z}}$$

which is an extended variation of Hodge structures.

Intuitionally,the Kuranishi base (refer to the Kodaira's textbook)is the maximal deformation space of X which is contained in Euclidean space $H^1(X, \Theta_X)$. In generalization, The Kuranishi family always exists for each complex manifold X, and if $\dim_{\mathbb{C}} H^0(X_s, \Theta_{X_s})$ keeps constant in the neighborhood of t_0,the family is universal (refer to Kuranishi's works). Given a orientation $\eta \in H^1(X, \Theta_X)$,whether X can be deformed along η is dependent on the corresponding $[\eta, \eta] \in H^2(X, \Theta_X)$. The set of all $[\eta, \eta]$ in $H^2(X, \Theta_X)$ is called **obstruction class**.Thus if the obstruction class vanish,the Kuranish base is a smooth complex manifold of dimension $\dim_{\mathbb{C}} H^1(X, \Theta_X)$. In particular,for the example when $H^2(X, \Theta_X) = 0$.

Bogomolov,Todorov,Tian, Ran and Kawamata proved that

Theorem 3.11 (BTT)**.** *Let X be a compact Kähler manifold with trivial canonical bundle,then the Kuranishi family is universal, and the base is a smooth open set of dimension* $\dim_{\mathbb{C}} H^1(X, \Theta_X)$*. In particular,the result holds for X Calabi-Yau manifold.*

Remark 3.12. Actually, the family is universal at every point of the base and K-S mapping is an isomorphism at every point of the base.

3.2.2. *Torelli theorem.* What's the meaning of Torelli theorem? Roughly specking,the question is equivalent to that for a compact Kähler manifold, **whether the Hodge Structure determines the complex structure.**

Suppose a compact Kähler manifold X admits a universal Kuranishi family

$$f : (\mathfrak{X}, X) \to (S, 0)$$

with a nonsingular base S which is isomorphic to the open set in $H^1(X, \Theta_X)$. Certainly,this may not be polarized family. There is a well-defined holomorphic map (period map)

$$\lambda = \prod \lambda^p : S \to \prod \mathrm{Gr}(h^p, H)$$

The **infinitesimal Torelli theorem** is just to ask whether $(d\lambda)_0$ is injective.Now one think about the Kuranishi family of the polarized variety (X, L) where X is projective manifold and L an ample very bundle.

Consider the submanifold $S_{c_1(L)} \subset S$,the deformation space determined by $c_1(L)$, the restriction of f over $S_{c_1(L)}$ is the universal Kuranishi family of the polarized algebraic variety (X, L) (i.e any deformation $\phi : (\mathcal{Y}, X) \to (T, 0)$ of polarized variety (X, L) is locally obtained from f by a unique base change $\pi : (T, 0) \to (S_{c_1(L)}, 0)$). We also have

$$T_{S_{c_1(L)},0} = H^1(X, \Theta)_{c_1(L)} := \mathrm{Ker}(H^1(X, \Theta_X) \xrightarrow{\wedge c_1(L)} H^2(X, \mathcal{O}_X))$$

Suppose Φ is the period mapping for f over $S_{c_1(L)}$ and Φ_T is period mapping for T,Thus the diagram is commutative(locally,then in this case $\Gamma = \{1\}$)

$$\begin{array}{ccc} T & \xrightarrow{\pi} & S_{c_1(L)} \\ & \searrow^{\Phi_T} \quad \swarrow_{\Phi} & \\ & D & \end{array}$$

It is said that **infinitesimal Torelli theorem** holds for a polarized algebraic variety (X, L) if Φ is local embedding. The condition is equivalent to that $d\Phi$ and $d\lambda$ is injective on holomorphic tangent space $\Theta_{S_{c_1(L)},0}$.

On the other way,the bilinear $\Theta_X \times \Omega_X^{n-1}$ defines a paring

$$H^1(X, \Theta_X) \times H^{n-p,p} \to H^{n-p-1,p+1}$$

It gives a homomorphism,

$$\mu : H^1(X, \Theta_X) \to \bigoplus \mathrm{Hom}(H^{n-p,p}, H^{n-p-1,p+1})$$

Thus,there is a homomorphism

$$\mu_0 : H^1(X, \Theta_X)_{c_1(L)} \to \bigoplus \mathrm{Hom}(P^{n-p,p}, P^{n-p-1,p+1})$$

Theorem 3.13 (Griffiths)**.**

$$(d\lambda)_0 = \mu_0 \circ \rho$$

where $\rho = \kappa(0)$. κ is the Kodaira-Spencer map for the family f over $S_{c_1(L)}$.In the case ρ is injective ,then the infinitesimal Torelli theorem will follows from the injectivity of μ_0.

When X is algebraic manifold with trivial canonical line bundle, the first piece of μ

$$H^1(X, \Theta_X) \to \mathrm{Hom}(H^0(X, \Omega_X^n), H^1(X, \Omega_X^n))$$

is naturally injective.

Corollary 3.14. *If (X, L) is algebraic manifold with trivial canonical line bundle,then the infinitesimal Torelli theorem holds for the Kuranishi family . In particular,it holds for X Calabi-Yau manifold.*

Definition 3.15 (Local Torelli)**.** we say **Local Torelli Theorem** holds for (X, L) if the differential $d\phi$ is an injection from tangent space $T_{[X]}$ into $T_{\phi([X])}$ while $[X] \in \mathfrak{M}$ a closed point corresponds to (X, L).(For example, Calabi-Yau manifolds,etc)

Remarks 3.16. Infinitesimal Torelli theorem $\Rightarrow$ Local Torelli theorem. But the converse is not true.For example,the family of hyperelliptic curves.

3.2.3. *The Weil-Petersson metric on the moduli space.* Let

$$f : \mathfrak{X} \to \mathfrak{M}$$

be a maximal subfamily of the Kuranishi family of Calabi-Yau manifolds with a fixed polarization $[\omega]$.

By the BTT theorem,the Kuranishi space of $\mathfrak{X}_t$ is unobstructed and the Kodaira-Spencer map

$$\rho_t : T_{\mathfrak{M},t} \to H^1(\mathfrak{X}_t, T_{\mathfrak{X}_t})_\omega \cong H^{0,1}_{\overline{\partial}}(T_{\mathfrak{X}_t})_\omega$$

is injective everywhere along $\mathfrak{M}$.According to Yau's solution to Calabi's Conjecture,there is an unique Kähler Einstein(Ricci flat) metric $g(t)$ on X_t in the given polarization $[\omega(t)]$. Then $g(t)$ induces a metric on $\Lambda^{0,1}(T_{\mathfrak{X}_t})$,so one can define Weil-Petersson metric G_{WP} on $\mathfrak{M}$:

for any $v, w \in T_{\mathfrak{M},t}$

$$G_{WP}(v, w) := \int_{\mathfrak{X}_t} < \rho_t(v), \rho_t(w) >_{g(t)}$$

Let $\Omega(t) \in \Gamma(\mathfrak{X}_t, \wedge^n \Omega^1_{\mathfrak{X}_t})) = \Gamma(\mathfrak{X}_t, \mathcal{O}_{\mathfrak{X}_t})$ be a flat holomorphic n-form on $\mathfrak{X}_t$ with respect to the K-E metric $g(t)$, it had been shown by Tian and Todorov (cf.[29] [30])

$$G_{WP}(v, w) = \frac{Q(C(i(v)\Omega(t)), \overline{i(w)\Omega(t)})}{Q(C(\Omega(t)), \overline{\Omega(t)})} = -\frac{\int_{\mathfrak{X}_t} i(v)\Omega(t) \wedge \overline{i(w)\Omega(t)}}{\int_{\mathfrak{X}_t} \Omega(t) \wedge \overline{\Omega(t)}}$$

where the morphism

$$H^1(\mathfrak{X}_t, T_{\mathfrak{X}_t})_\omega \to \mathrm{Hom}(P^{n,0}, P^{n-1,1}) \cong P^{n-1,1}(\mathfrak{X}_t)$$

via $v \mapsto i(v)\Omega(t)$ is an isomorphism.

Here Q is the flat inner product on the VHS $P^n f_*(\mathbb{Q})$.In fact,because $T_{\mathfrak{M}}$ is mapped to $P^{n-1,1}$ isomorphically,the Weil-Petersson metric G_{WP} is induced form the Hodge metric h (i.e. $h(\cdot, \cdot) := Q(C\cdot, \bar{\cdot})$) on the n-th piece of the polarized VHS associated to $P^n f_*(\mathbb{Q})$,hence G_{WP} is Riemannian metric on M.

Furthermore, the Kähler form of the metric G_{WP} is

(WP) $$\omega_{WP}(t) = -\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log h = \frac{\sqrt{-1}}{2} Ric_h(H^{n,0}(\mathfrak{X}_t))$$

In particular,ω_{WP} is independent of the polarization due to $H^{n,0} = P^{n,0}$.

3.3. **Appendix: Fundamental results of Calabi-Yau manifolds.** The following is the well-known theorem of Yau Shing-Tung :

Theorem 3.17 (Calabi-Yau Theorem[38])**.** *For any compact Kähler Manifold X of complex dimension n with $K_X = \wedge^n \Omega^1_X = \mathcal{O}_X$ (i.e. $c_1(X) = 0$), then there is an unique Ricci flat metric on M(ie. $\exists \mid$ Kähler metric g_{ij} such that $R(g)_{ij} = 0$).*

Definition 3.18 (Equivalence Definition of **Calabi-Yau Manifold**)**.** Let X be a compact manifold.

(a) Originally, X admits a Riemannian metric with global holonomy group

$$0 \neq H^* \subseteq \mathrm{SU}(n).$$

(b) X is a compact Kähler manifold with trivial canonical bundle.

Now, we give a sketch description the equivalence of the definitions.

As well known,SU(n) is the subgroup of $U(n)$ preserving an alternate complex n-form on $\mathbb{C}^n$.Thus, a compact manifold X with holonomy $H^* \neq 0$ contained in SU(n) is really a Kähler manifold (of complex dimension m) with a non-zero parallel form ω_X of type $(n, 0)$(under the Levi-Civita connection D).By standard Riemannian geometry,one has

$$d = \sum \varpi^i \wedge D_{X_i}$$

where $\{X_i\}$ is moving frame of tangent bundle and $\{\varpi^i\}$ is the dual frame in cotangent bundle,So $0 = d(\omega) = \overline{\partial}(\omega) + \partial(\omega)$.Thus the non-zero $(n, 0)$- parallel form ω is $\overline{\partial}$-closed,i.e. holomorphic.

Let Θ_X be holomorphic tangent sheaf of X and $\Omega^1_X = (\Theta_X)^*$, the canonical bundle $K_X := \wedge^n \Omega^1_X$ is flat holomorphic line bundle and ω_X is the global nonzero holomorphic section of K_X.Therefore,the canonical line bundle K_X is trivial. In other words,by Calabi-Yau theorem there is unique Kähler metric on M such that the Ricci curvature (which for a Kähler manifold is just the curvature of K_X) is zero.Thus the holonomy group H^* is in SU(n).

Therefore, by Calabi-Yau theorem,the compact manifold X admits a metric with holonomy contained in SU(n) if and only if X is **Calabi-Yau** manifold. As shown above,let X be a Calabi-Yau manifold of n dimension, then

$$\Theta_X \cong \Omega^{n-1}_X.$$

By the short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}^*_X \to 0,$$

one will obtain

$$\cdots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to Pic(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to \cdots.$$

Because X is Calabi-Yau, so for $0 < i < n$(using the following proposition 3.19),

$$h^i(X, \mathcal{O}_X) = h^{0,i}(X) = h^{i,0}(X) = h^0(X, \Omega^i_X) = h^{n,n-i}(X) = 0$$

Thus

$$0 \to Pic(X) \to H^2(X, \mathbb{Z})$$

Especially, by Lefschetz-$(1, 1)$ Theorem

$$Pic(X) = H^{1,1}_{\mathbb{Z}} =: H^{1,1} \bigcap H^2(X, \mathbb{Z})$$

Proposition 3.19. *Let X be a compact Kähler manifold with dimension n .*

(I) *If X has dimension $n \geq 3$, with holonomy group $H = \mathrm{SU}(n)$,then*

$$H^0(X, \Omega^p_X) = 0$$

for $0 < p < n$.Moreover, X is a projective variety.

(II) *Let Kähler class be the set of the class in $H^2(X, \mathbb{Z})$ which forms Kähler metrics,denote it $\mathcal{K}$.It is obvious the Kähler cone $\mathcal{K} \otimes \mathbb{R}$ is open in*

$$H^{1,1}_{\mathbb{R}} := H^{1,1} \bigcap H^2(X, \mathbb{R}).$$

Thus, any compact Kähler manifold X with $H^0(X, \Omega^2_X) = 0$ is projective.

Proof of 3.19. **Sketch :**

(I) Because the global holomony group $H* = \mathrm{SU}(n)$, $K_X = \wedge^n \Omega_X^1 \simeq \mathcal{O}_X$.Thus one has a Ricci-flat Kälher metric by 3.17.

Let τ be any holomorphic tensor filed(covariant or contravariant), by Bochner formula

$$\Delta(\|\tau\|^2) = \|\nabla\tau\|^2 + Ric(\tau^*, \tau^*) = \|\nabla\tau\|^2$$

So τ is parallel.

Let $x \in X$ and $V = T_x(X)$,by Borchner formula, $H^0(X, \Omega_X^p)$ can be identified with $\mathrm{SU}(n)$ invariant subspace of $\bigwedge^p V^*$. Because $\mathrm{SU}(V)$ acts irreducibly on $\bigwedge^p V^*$, the invariant space is zero unless $p = 0$ or $p = n$.
($H^0(X, \Omega_X^p) \subsetneqq \bigwedge^p V^*$ for $0 < p < n$,because the connection is not zero)

(II) Now,$H^2(X, \mathbb{C}) = H^{1,1}$ and $H^{1,1}_{\mathbb{R}} = H^2(X, \mathbb{R})$,so Kähler cone is open in $H^2(X, \mathbb{R})$.Therefore, the Kähler cone contains an positive integral class $[\omega]$ and by

$$Pic(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) = 0,$$

this $[\omega]$ corresponds a L in $Pic(X)$. The well-known Kodaira embedding theorem asserts :

A line bundle must be ample when its Chern Class is in Kähler class.

Thus, L is an ample line bundle,i.e. X is a projective manifold.

- (Continue I) From (II),when X is a Calabi-Yau manifold ,then

$$H^0(X, \Omega_X^2) = 0.$$

So X is projective . □

In this Lecture, when we consider Calabi-Yau manifold, we always mean **the projective Calabi-Yau manifold**,i.e.the manifold has **Holomony Group**

$$H^* = \mathrm{SU}(n).$$

Example 3.20 (Complete intersection)**.** Let $X \subset \mathbb{P}^{n+k}$ be a varieties defined by $F_1 = F_2 = \cdots = F_k = 0$ and $\deg F_i = d_i$.For generic choice of F_i X is smooth manifold of dimension n. By adjunction formula,the canonical line bundle

$$K_X = \mathcal{O}_X(\sum_1^k d_i - n - k - 1)$$

is trivial if $\sum di = n + k + 1$.By Lefschetz's hyperplane theorem, these manifolds satisfy

$$H^0(\Omega_X^i) = 0 \ \text{ for } 0 < i < n.$$

Thus one obtains Calabi-Yau manifold of complete intersection type.

Example 3.21. Taking a double cover of $\mathbb{P}^3$ branched over 8 planes in general position,blowing up along the 28 singular lines,then one will obtain a Calabi-Yau threedfold.These works are started by S.T.Yau, and in a recent processing works of Lian,Todorov and Yau,they want to show the moduli space of CY threefolds of this type is

$$(G \setminus \mathrm{SU}(3,3))/\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(3))$$

(It really bellowed to a conjecture of Dolgachev.I) a Shimura variety which can be embedded into the moduli stack of Abelian varieties.

4. Rigidity on Families of Calabi-Yau manifolds

4.1. **Endomorphism of Higgs bundles over product varieties.** Let $\mathbb{V}$ be an arbitrary polarized $\mathbb{R}$-VHS over $S^0 \times T^0$ such that local monodromies around the divisor at infinity are quasi unipotent.(it shown by Landman,Katz and Borel that it is true for polarized $\mathbb{Z}$-VHS). Extending the associated Higgs bundle E to the infinity and one gets the quasi canonical extension Higgs bundle :

$$\theta : \overline{E} \to \overline{E} \otimes \Omega^1_{S\times T}(\log D)$$

One can obtain a meaningful endomorphism σ on $\mathbb{V}|_{\mathrm{S}_t}$ (cf. Jost-Yau Theorem in [10] and another proof of Zuo in [44]):

Hint: *Consider the two projections $p_S : S \times T \to S$ and $p_T : S \times T \to T$, One has then*

$$\Omega^1_{S\times T}(\log D) = p_S^*\Omega^1_S(\log D_S) \oplus p_T^*\Omega^1_T(\log D_T),$$

and the Higgs map

$$\theta : p_S^*\Theta_S(-\log D_S) \oplus p_T^*\Theta_T(-\log D_T) \to End(\overline{E}), \tag{4.0.1}$$

is really the sheaf map of the differential of the extended period map.

The restriction of the Higgs map to S_t, is

$$\theta|_{S_t} : (p_S^*\Theta_S(-\log D_S) \oplus p_T^*\Theta_T(-\log D_T))|_{S_t} \to End(\overline{E})|_{S_t}. \tag{4.0.2}$$

Note that

$$p_T^*\Theta_T(-\log D_T)|_{S_t} \simeq \bigoplus^l \mathcal{O}_{S_t},$$

where l is dimension of T,let $1_T \in p_T^*\Theta_T(-\log D_T)|_{S_t}$ be any constant section, then one obtains an endomorphism

$$\sigma := \theta|_{S_t}(1_T) : E|_{S_t^0} \to E|_{S_t^0}. \tag{4.0.3}$$

As σ is coming form Higgs filed, it must be of $(-1,1)$ type and over $\mathbb{C}$.Moreover σ is a morphism of Higgs sheaf,i.e the diagram

$$\begin{array}{ccc} E|_{S_t^0} & \xrightarrow{\theta_{S_t^0}} & E|_{S_t^0} \otimes \Omega^1_{S_t^0} \\ \downarrow{\sigma} & & \downarrow{\sigma\otimes id} \\ E|_{S_t^0} & \xrightarrow{\theta_{S_t^0}} & E|_{S_t^0} \otimes \Omega^1_{S_t^0} \end{array}$$

is commutative because of $\theta_{S_t^0} \wedge \theta_{S_t^0} = 0$ where

$$\theta_{S_t} : \Theta_{S_t}(-\log D_{S_t}) \hookrightarrow (p_S^*\Theta_S(-\log D_S) \oplus p_T^*\Theta_T(-\log D_T))|_{S_t} \to End(\overline{E})|_{S_t}$$

the Higgs map of $E|_{S_t}$.

Actually,it is shown by Zuo [44] that the image of the map 4.0.2 is contained in the kernel of the induced Higgs map on $\mathrm{End}(E)$. Therefore, the image is a Higgs subsheaf with the trivial Higgs field. The Higgs poly-stability of $\mathrm{End}(E)|_{S_t}$ implies any section in this subsheaf is flat. One sees also this flat section is of Hodge type $(-1, 1)$.

For a weight n $\mathbb{R}$-VHS $\mathbb{V}_\mathbb{R}$, we always have a non degenerate pairing $\mathbb{V}_\mathbb{R} \times \mathbb{V}_\mathbb{R} \to \mathbb{R}(-n)$,then we get $(\mathbb{V}_\mathbb{R})^\vee = \mathbb{V}_\mathbb{R}(n)$,a VHS by shifted $(-n, -n)$.

Similarly,when Higgs bundle E carries $\mathbb{R}$-structure,then the dual Higgs bundle is

$$(E^\vee = \bigoplus_{p+q=n} E^{\vee -p,-q}, \theta_\vee)$$

where $E^{\vee -p,-q} = (E^{p,q})^\vee = E^{q,p}$, $\theta_\vee^{-p,-q} = -\theta^{q,p}$ and

$$\theta_\vee^{-p,-q} : E^{\vee -p,-q} \to E^{\vee -p-1,-q+1} \otimes \Omega^1_M$$

Thus we get the Higgs bundle

$$(\mathrm{End}(E) = \bigoplus_{r+s=0} \mathrm{End}(E)^{r,s}, \theta^{end})$$

with

$$\theta^{end}(u \otimes v^\vee) = \theta(v) \otimes v^\vee + u \otimes \theta_\vee(v^\vee). \tag{4.0.4}$$

Proposition 4.1. *(Proposition 2.1 in* [44]*) Denote* $(\mathrm{End}(E), \theta^{end})$ *the system Hodge bundles corresponding to the polarized VHS on* $\mathrm{End}(\mathbb{V}_\mathbb{R})$ *which is induced by the polarized VHS on* $\mathbb{V}_\mathbb{R}$*.Then*

$$\theta^{end}(d\phi(T_{\overline{M}})(-\log D_\infty)) = 0$$

Using the generalized Uhlenbeck-Yau-Donaldson-Simpson correspondence on higher dimensional quasi-projective manifolds(Jost-Zuo [11][43]),we obtain

Theorem 4.2 (Zhang [41],[42])**.** *Let M be quasi-projective manifold with good completion.Assume the Higgs bundle $(E, \overline{\partial}, \theta)$ comes form tamed harmonic bundle (V, H, ∇). Let e be a holomorphic section of $(E, \overline{\partial})$,then $\theta(e) = 0$ if and only if e is a flat section of (V, ∇).*

Remark: If M is compact Kähler manifold,the statement has already been shown by Simpson.

Exercise 4.3. The endomorphism σ obtained above is a flat $(-1, 1)$-type section of $\mathrm{End}(E)|_{S_t^0}$.Therefore, σ can be seen as a endomorphism of local system $\mathbb{C}$,i.e.

$$\sigma : \mathbb{V}_\mathbb{C}|_{S_t^0} \to \mathbb{V}_\mathbb{C}|_{S_t^0}. \tag{4.3.1}$$

Remark: *Naturally, we have a question: What's the conditions of family such that σ is defined over $\mathbb{R}$,more over $\mathbb{Q}$?*

4.2. **Rigid criterion for families of Calabi-Yau manifolds.** Here we should declare

The infinitesimal Torelli theorem holds for all manifolds we study in the following sections.

So, there is a natural condition for a smooth family $f : \mathcal{X} \to M$ of n dimensional projective manifolds not to be trivial:

(∗∗) The differential of the period map for $P^n f_*(\mathbb{C})$ is injective at some points of M.

The meaning of the condition:
We have $\dim \mathrm{Image}(\eta) = \dim C$ where η is the induced modular map $C \to \mathfrak{M}$.

We say f is rigid, if there exists no non-trivial deformation over a non-singular quasi-projective curve T^0.

Here a deformation of f over T^0, with $0 \in T^0$ a base point, is a smooth projective morphism

$$g : \mathfrak{X} \to M \times T^0$$

for which there exists a commutative diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\simeq} & g^{-1}(M \times \{0\}) & \xrightarrow{\subset} & \mathfrak{X} \\ f\downarrow & & \downarrow & & \downarrow g \\ M & \xrightarrow{\simeq} & M \times \{0\} & \xrightarrow{\subset} & M \times T^0 \end{array} .$$

Observation: Let $f : \mathcal{X} \to M$ be a smooth polarized family of n dimensional projective manifolds. Following the infinitesimal Torelli theorem of CY manifolds, if the deformation g of f is non-trivial,then the period map of the extended family g is not degenerate along T^0-direction for some points of $M \times \{0\}$.

At all, we obtain a criterion for rigidity from the previous subsection 4.1.

Theorem 4.4 (Criterion for Rigidity [44],[42])**.** *Let $f : \mathcal{X} \to M$ be a smooth polarized family of n dimensional satisfying the condition $(**)$,if the family f is not rigid,then there is a non-zero flat section σ of* $\mathrm{End}(P^n f_*(\mathbb{C}))^{-1,1}$*. And this endomorphism comes from a flat section of $\sigma \in$* $\mathrm{End}(E)^{-1,1}$ *where (E, θ) be a Higgs bundle induced from $P^n f_*(\mathbb{Q})$. Moreover, the Zariski tangent space of deformation space of this family f is into*

$$\mathrm{End}(P^n f_*(\mathbb{C}))^{-1,1}.$$

One can also refer to the papers of Jost-Yau [10]and Faltings [8], to get the similar criterion on Families of Abelian Varieties. With the same method,it is not difficult to generalize this result to nonrigid polarized VHS.

4.3. **Some applications of the criterion.**

4.3.1. *Rigidity of Lefeschetz pencils of Calabi-Yau manifolds.*

Theorem 4.5 (Zhang [41] [42])**.** *Let* $\overline{f} : \mathfrak{X} \to \mathbb{P}^1$ *be Lefschetz pencil such that* $f : \mathfrak{X}_0 \to \mathbb{P}^1 \setminus S$ *is smooth and smooth fibers are Calabi-Yau n-folds. Assume f satisfies the condition (∗∗),then n must be odd.*

The local system of vanishing cycles space $\mathbb{V}_{\mathbb{Q}}$ *will be in* $P^n f_*(\mathbb{Q})$ *because* $\mathbb{V}$ *is absolute irreducible with* $(\mathbb{V}_{\mathbb{C}})^{n,0}, (\mathbb{V}_{\mathbb{C}})^{0,n} \in P^n f_*(\mathbb{C})$*.Actually,from the Lefschetz decomposition*

$$R^n f_*(\mathbb{Q}) = \oplus_k L^k P^{n-2k} f_*(\mathbb{Q})$$

where $P^{n-2k} f_*(\mathbb{Q}) = \ker(L^{k+1} : R^{n-k} f_*(\mathbb{Q}) \to R^{n+k+2} f_*(\mathbb{Q}))$ *with the polarization L,we have the decomposition*

$$P^n f_*(\mathbb{Q}) = \mathbb{V} \oplus (P^n f_*(\mathbb{Q}))^{\pi_1(\mathbb{P}^1 \setminus S)}.$$

Remarks 4.6. Notes on the theorem:

(I) The Lefschetz pencil is just the set of hypersurfaces over a complex line,so the singular fiber only has unique simple singularity. Lefschetz pencils have very good properties.It should be noted that most pencils are Lefschetz pencils.One can refer to Deligne's paper for the strict definition.

(II) The proof the theorem is dependent on the moduli properties of Calabi-Yau manifolds and the **Kazhdan-Margulis Theorem**[6] which is essential in Deligne's proof of Weil Conjecture I :

The image $\pi_1(\mathbb{P}^1 \setminus S)$ *in* $\mathrm{Sp}(V_{\mathbb{C}}, (\ ,\ ))$ *is open.*

Theorem 4.7 (Zhang [41],[42])**.** *The above families must be rigid.*

Proof. Let $C_0 = \mathbb{P}^1 \setminus S$.Assume the statement is not true, we have the nontrivial extension family

$$\begin{array}{ccc} \mathcal{X} \setminus f^{-1}(S) & \xrightarrow{\subset} & \mathfrak{X} \\ f\downarrow & & \downarrow g \\ C_0 \times \{0\} & \xrightarrow{\subset} & C_0 \times T^0 \end{array} .$$

where T^0 is a smooth quasi-projective curve.

The local system $P^n f_*(\mathbb{C})$ defines a polarized VHS,then we have a associated Higgs bundle (E, θ) .

Lemma 4.8. *There is a splitting of the Higgs bundle*

$$(E, \theta) = \mathrm{Ker}(\sigma) \bigoplus (\mathrm{Ker}(\sigma))^{\perp}$$

for the flat $(-1, 1)$*-type non zero endomorphism* $\sigma : E \to E$*. The statement is also true when* C_0 *is replaced by* M *a higher dimensional quasi-projective variety.*

proof of the lemma. First, the non-degenerate polarized $\mathbf{Q}$ and σ are flat and defined over $\mathbb{C}$. Thus,it is not difficult to obtain the splitting of the local system over $\mathbb{C}$

$$P^n f_*(\mathbb{C}) = \mathrm{Ker}(\sigma) \oplus (\mathrm{Ker}(\sigma))^{\perp}$$

which is compatible with the polarization $\mathbf{Q}$. $(\mathrm{Ker}(\sigma))^{\perp}$ is orthogonal component of $\mathrm{Ker}(\sigma)$ in $P^n f_*(\mathbb{C})$. It is a special case of Deligne's complete reducibility. Then,

$$P^n f_*(\mathbb{C}) \otimes \mathcal{O}_C = \mathrm{Ker}(\sigma) \oplus (\mathrm{Ker}(\sigma))^{\perp}$$

because σ is also a $\mathcal{O}_C$-linear map. Restricting the Hodge filtration of VHS to these sub local system and taking the grading of the Hodge filtration,one obtains a decomposition of the Higgs bundles.Exactly,we obtain a splitting of *Complex Variation of Hodge Structure*[27][28]. It is a example of generalized **Uhlenbeck-Yau-Donaldson-Simpson** correspondence theorem. □

Back to the proof of the theorem:
$E^{0,n} \in \mathrm{Ker}(\sigma)$ is always true along M,but as non-triviality of the deformation of the family,at one point s_0 of $\mathbb{P}^1 \setminus S$, $\sigma : E^{n,0}|_{C_0} \to E^{n-1,1}|_{C_0}$ is injective(local Torelli theorem),thus at s_0,

$$E^{n,0} \nsubseteq \mathrm{Ker}(\sigma) \text{ and } E^{0,n} \subset \mathrm{Ker}(\sigma).$$

As for Lefschetz pencil,we have already shown that the fundamental representation into space of the vanishing cycles

$$\rho : \pi_1(\mathbb{P}^1 \setminus S) \to \mathrm{Aut}(\mathbb{V}_{\mathbb{C},s_0})$$

is irreducible and $E^{n,0}$ and $E^{0,n}$ are all in $\mathbb{V}_{\mathbb{C}}(4.5)$. It is a contradiction. □

Comments 4.9. The proof of this theorem depends heavily on the structure of Lefschetz pencils,Hard Lefschetz Theorem and properties of Moduli space of Calabi-Yau manifolds. It seems that we can generalize this statement to $l-adic$ cohomology.We also know Lefschetz pencil is just algebraic geometry version of Morse theory. Can we find the Physics model of this structure,or can we find the Witten-style proof of this theorem as Laumon has reproved the Weil conjecture by replacing the technique of Lefschetz pencil of Deligne with the method of deformation in Witten's Morse theory.

Definition 4.10. Let $\mathbb{V}$ be a local system of $\mathbb{Q}$-vector space on a quasi-projective manifold S with monodromy representation

$$\rho : \pi_1(S, s_0) \to \mathrm{GL}(V),\ V := \mathbb{V}_{s_0}.$$

1. The monodromy group $\pi_1(S, s_0)^{mon}$ is denoted by the Zariski closure of the smallest algebraic subgroup of $\mathrm{GL}(V)$ containing the monodromy representation $\rho(\pi_1(S, s_0))$.
2. If V carries a nondegenerate bilinear form Q which is symmetric or anti-symmetric and which is preserved by the monodromy group. We call the monodromy is *big* if the connected component of origin $\pi_1(S, s_0)^{mon}_0$ acts irreducibly on $\mathbb{V}_{\mathbb{C}}$.

Theorem 4.11. *Let $\pi : \mathfrak{X} \to Y$ a non-isotrivial smooth family of CY manifold of dimension n over the quasi-projective manifold Y and $(R^n\pi_*\mathbb{Q})_{prim}$ carries a nature polarized VHS. Suppose that there is a sub $\mathbb{Q}$-VHS $\mathbb{V}$ with big monodromy and the first Hodge piece is in $\mathbb{V}_{\mathbb{C}}$.Then the family $\pi : \mathfrak{X} \to Y$ must be rigid.*

4.3.2. *General result on rigidity of families with strong degenerate.* We assume all families in this subsection satisfy the $(**)$ condition in section 4.2.

Definition 4.12. Let $f : \mathcal{X} \to C$ be a family over a smooth projective curve C, and $\{c_0, \cdots, c_k\}$ are singular values of f.Assume f satisfies the following condition:

a) $X_i = f^{-1}(c_i) = X_{i1} + \cdots + X_{ir_i}$ is a reduced fiber of a union of transversally crossing smooth divisors for all i i.e. f is a semi-stable family;
b) the cohomology of every smooth component of X_0 has type (p, p).

We say this family has strong degenerate at the singular fiber $X_0 = f^{-1}(c_0)$

Remark: The conception of strong degenerate comes from the literature of Hodge Conjecture on Abelian variety.

Using the Künneth formula,we will get the following result immediately.

Corollary 4.13. *If the family $f : \mathcal{X} \to C$ has strong degenerate property at c_0,then the product family*

$$\pi : \mathcal{Y} \triangleq \mathcal{X} \times_C \mathcal{X} \to C$$

is also strong degenerate at c_0.

Let M be a quasi-projective variety,(E, θ) be the Higgs bundle with positive Hermitian metric H over M.We always assume (E, θ) has $\mathbb{R}$-structure.

Exercise 4.14. By Künneth's formula,as a vector space

$$\mathrm{End}(E)^{-1,1} \subset R^{n+1}\pi_*(\Omega^{n-1}_{\mathcal{Y}/M}) \tag{4.14.1}$$

Theorem 4.15 (Zhang [41],[42])**.** *Strong degenerate families(***not only for families of Calabi-Yau manifolds***) over any algebraic curve must be rigid.*

Proof. $f : \overline{\mathcal{X}} \to C$ is a strong degenerate family with

$$\{c_0, \cdots, c_k\} = C - C_0$$

the singular values of f and the fiber is a projective variety of dimension n. Let

$$f : \mathcal{X} = f^{-1}(C_0) \to C_0$$

be the restricted smooth family.Define the product family

$$\pi : \overline{\mathcal{Y}} \triangleq \overline{\mathcal{X}} \times_C \overline{\mathcal{X}} \to C,$$

as shown in the above corollary,it is strong degenerate.Let $\mathcal{Y} := \pi_1^{-1}(C_0) = \mathcal{X} \times_{C_0} \mathcal{X}$,then we have the restricted smooth family

$$\pi : \mathcal{Y} \to C_0$$

If $f : \mathcal{X} \to C_0$ is not rigid, We will get a nonzero flat (-1,1) morphism

$$\sigma : P^n f_*(\mathbb{C}) \to P^n f_*(\mathbb{C})$$

Moreover, shown in the above lemma

$$\sigma \in (R^{n+1}\pi_*(\Omega^{n-1}_{\mathcal{Y}/C_0}))^{\pi_1(C_0)}.$$

Following Deligne's Hodge theory [5],we know the commutative diagram

$$\begin{array}{ccc} H^n(\overline{\mathcal{Y}}, \mathbb{C}) & \xrightarrow{\quad i^* \quad} & H^n(\mathcal{Y}, \mathbb{C}) \\ & \searrow^{\overline{i}_t^*} \quad {}^{i_t^*}\swarrow & \\ & H^n(\mathcal{Y}_t, \mathbb{C})^{\pi_1(C_0,t)} & \end{array}$$

where $i_t : \mathcal{Y}_t \hookrightarrow \mathcal{Y}$, $\overline{i}_t : \mathcal{Y}_t \hookrightarrow \overline{\mathcal{Y}}$ are natural embedding;and

$$H^n(\mathcal{Y}_t, \mathbb{C})^{\pi_1(C_0,t)} \cong H^0(C_0, R^n\pi_*(\mathbb{C}))$$

is an isomorphism of Hodge Structure. As $\overline{i}_t^*$ is surjective Hodge morphism for every $t \in C_0$, we have the following restriction maps induced by $\mathcal{Y}_t \subset \overline{\mathcal{Y}}$,

$$r_t^{p,q} : H^q(\overline{\mathcal{Y}}, \Omega^p_{\overline{\mathcal{Y}}}) \hookrightarrow H^n(\overline{\mathcal{Y}}, \mathbb{C}) \xrightarrow{\overline{i}_t^*} H^n(\mathcal{Y}_t, \mathbb{C}) \to H^q(\mathcal{Y}_t, \Omega^p_{\mathcal{Y}_t}) \tag{4.15.1}$$

where $p+q=n$.

From the restriction maps, we have

Claim 4.16. The component of type (p,q) of the group $H^n(\mathcal{Y}_t,\mathbb{C})^{\pi_1(C_0,t)}$ is the image of $H^q(\overline{\mathcal{Y}},\Omega^p_{\overline{\mathcal{Y}}})$ under $r_t^{p,q}$.

For each nonsingular dimensional n reduced algebraic cycle $B\subset\overline{\mathcal{Y}}$ and each $\alpha\in H^q(\overline{\mathcal{Y}},\Omega^p_{\overline{\mathcal{Y}}})$,we consider the integral

$$\int_B \alpha\wedge\overline{\alpha}=\int_B \alpha|_B\wedge\overline{\alpha}|_B$$

then

$$\int_B \alpha\wedge\overline{\alpha}=0\Leftrightarrow \alpha|_B=0$$

WLOG, f is strong degenerate at c_0, so is π. Denote

$$Y_0=D_1+\cdots+D_k$$

Fixing p,q,if $H^{p,q}(D_j)=0$ for each smooth component D_j of Y_0,then

$$\alpha|_{D_j}=0,\forall j$$

So

$$\int_{Y_0}\alpha\wedge\overline{\alpha}=0.$$

The fiber $\mathcal{Y}_t$ is homological equivalent to the singular Y_0 for any $t\in C_0$, and the form $\alpha\wedge\overline{\alpha}$ is closed,therefore

$$0=\int_{Y_0}(\alpha\wedge\overline{\alpha})|_{Y_0}=\int_{\mathcal{Y}_t}(\alpha\wedge\overline{\alpha})|_{\mathcal{Y}_t}$$

We obtain $\alpha|_{\mathcal{Y}_t}=0$,i.e. the restriction map

$$r_t^{p,q}:H^q(\overline{\mathcal{Y}},\Omega^p_{\overline{\mathcal{Y}}})\to H^q(\mathcal{Y}_t,\Omega^p_{\mathcal{Y}_t})$$

is zero map.

Now, π is a family strong degenerate at c_0, the cohomology of every smooth component of Y_0 has type only (p,p).Especially,

$$H^{n-1,n+1}(D_j)=0$$

for all j.Therefore,the restriction map

$$r_t^{n-1,n+1}:H^{n+1}(\overline{\mathcal{Y}},\Omega^{n-1}_{\overline{\mathcal{Y}}})\to H^{n+1}(\mathcal{Y}_t,\Omega^{n-1}_{\mathcal{Y}_t})$$

is zero map,i.e

$$(R^{n+1}\pi_*(\Omega^{n-1}_{\mathcal{Y}/C_0}))^{\pi_1(C_0)}=0.$$

It is a contradiction.So the family f is rigid. □

In particular, form the theorem we get immediately

Corollary 4.17 (Weak Arakelov's Theorem)**.** *Let $f:\mathcal{X}\to C$ be a semi-stable family over a smooth projective curve C, X_0 is the singular fiber*

$$X_0=f^{-1}(c_0)=D_1+\cdots+D_r.$$

If every D_i is dominated by $\mathbb{P}^n$,then this family is rigid.

Remarks 4.18. If this family is a fibration of curves,then we obtain a similar result of rigid part of Shafarevich conjecture over function field.Though it is a weak result,a family admitting strong degenerate are in general case.

Example 4.19. All one parameter family in $\mathbb{P}^n$ of type

$$F(X_0, \cdots, X_n) + t\prod_{i=1}^{d} X_{\tau(i)} = 0$$

are rigid, where F is smooth homogenous polynomial with degree d.

4.3.3. *Rigidity and Yukawa coupling.* In this subsection, we show the relation between the rigidity of Calabi-Yau manifolds and Yukawa coupling, then we give applications.

• **n-iterated endomorphism of σ and Yukama coupling**

It is better to understand more and more about the endomorphism σ which has deep background in string theory.

Anyway,there is n-iterated operator

$$\sigma^n = \underbrace{\sigma \circ \cdots \circ \sigma}_{n}$$

on E, and we know $\sigma^k \equiv 0$ if $k >> 0$.Furthermore, the following is held for Calabi-Yau manifolds.

Proposition 4.20. *Let $f : \mathcal{X} \to M$ be a smooth family of polarized n dimensional Calabi-Yau manifolds satisfying the condition $(**)$.If f is not a rigid family,then the n-iterated endomorphism σ must be zero.*

Proof. Assume the statement is not true,we have a non zero flat endomorphism σ^n which is in fact a global holomorphic section of $(L^*)^{\otimes 2}$ (under the holomorphic structure $\overline{\partial}_E$ of the associated Higgs bundle)where

$$L := R^0 f_* \Omega^n_{\mathcal{X}/M}.$$

Therefore $(L^*)^{\otimes 2}$ is trivial. But it is impossible for the following reason:

Let $\pi : \mathfrak{X} \to \mathfrak{M}$ be a maximal subfamily of the Kuranishi family of Calabi-Yau manifolds with a fixed polarization $[\omega]$,by the BTT theorem,the Kuranishi space of $\mathfrak{X}_t$ is unobstructed and the Kodaira-Spencer map

$$\rho(t) : T_{\mathfrak{M},t} \to H^1(\mathfrak{X}_t, T_{\mathfrak{X}_t})$$

is injective everywhere along $\mathfrak{M}$.Thus, we have the commutative diagram,

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Psi} & \mathfrak{X} \\ \downarrow f & & \downarrow \pi \\ M & \xrightarrow{\psi} & \mathfrak{M} \end{array}$$

By base change formula,

$$\psi^* R^0 \pi_*(\Omega^n_{\mathfrak{X}/\mathfrak{M}}) = R^0 f_*(\Psi^* \Omega^n_{\mathfrak{X}/\mathfrak{M}}) = R^0 f_* \Omega^n_{\mathcal{X}/M}$$

Denote

$$L := R^0 f_* \Omega^n_{\mathcal{X}/M}.$$

We have

$$\int_M (c_1(L))^{\dim M} = \int_M (c_1(\psi^* R^0 \pi_*(\Omega^n_{\mathfrak{X}/\mathfrak{M}})))^{\dim M} = \int_M (\psi^* c_1(R^0 \pi_*(\Omega^n_{\mathfrak{X}/\mathfrak{M}})))^{\dim M}$$

Due to the works of Todorev and Tian, one has the Weil-Petersson metric on $\mathfrak{M}$ and the formula (cf.[29],[30])

$$\omega_{WP}(t) = -\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log H = \frac{\sqrt{-1}}{2} Ric_H(H^{n,0}(\mathfrak{X}_t))$$

Here H is the Hodge metric on the polarized VHS $P^n f_*(\mathbb{C})$. Thus

$$\int_M (c_1(L))^{\dim M} = \int_{\psi(M)} (c_1(R^0\pi_*(\Omega^n_{\mathfrak{X}/\mathfrak{M}})))^{\dim M} = \int_{\psi(M)} VOL(\omega_{WP}) > 0,$$

Because ψ does not degenerate.

□

Definition 4.21. Let $f : \mathcal{X}_0 \to M$ be smooth family of n-dimensional CY manifolds over a quasi-projective manifold M, (E, θ) is the Higgs bundle of $P^n f_*(\mathbb{C})$.

Yukawa coupling is just the n-iterated Higgs field

$$\theta^n : E \to E \otimes S^n \Omega^1_M$$

which has deep background in string theory.Maybe you will find the definition is different from other literature,but they essentially are compatible.

As the Higgs bundle (E, θ) can be splitting into

$$(E, \theta) = (\bigoplus_{p+q=n} E^{p,q}, \bigoplus \theta^{p,q})$$

with

$$\theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_M$$

Because of $\theta \wedge \theta = 0$,

$$\theta^{n,0} \circ \cdots \circ \theta^{0,n} : E^{n,0} \longrightarrow E^{0,n} \otimes \bigotimes^n \Omega^1_M$$

can factor through

$$\theta^n : E^{n,0} \longrightarrow E^{0,n} \otimes S^n \Omega^1_M.$$

So we have this definition.WLOG,the Yukama coupling can be written as

$$\theta^n : S^n \Theta_M \to \mathcal{H}om(E^{n,0}, E^{0,n}) = ((R^0 f_* \Omega^n_{\mathcal{X}_0/M})^*)^{\otimes 2}$$

It is very interesting to understand to relations among these morphisms

Let M be the quasi-projective smooth curve $C_0 = C \setminus S$ with $S = \{s_1, \cdots, s_k\}$. Using local Torelli theorem and above proposition we obtain :

Exercise 4.22 (Viehweg-Zuo & L-T-Y-Z)**.** Let $f : \mathcal{X}_0 \to C_0$ be a smooth family of n-dimensional Calabi-Yau manifolds satisfying the condition $(**)$. If the Yukawa coupling is not zero at some point,then the family f must be rigid.

Hint *1.Using the properties of Higgs bundle and locally Torelli theorem for Calabi-Yau.*

2.Certainly,one can prove this criterion directly by the same method in proof of proposition 4.20.

3.One can obtain the result right away by Viehweg-Zuo's criterion 4.24

Example 4.23 (Liu-Todorov-Yau-Zuo)**.** Let F,H be homogenous polynomial of degree $n+2$ such that F defines a nonsingular hypersurface in $\mathbb{P}^{n+1}$ and H is not in the Jacobian ideal of F.Let us think about a special family $F(t) = F + tH$ for $t \in \mathbb{P}^1$.If the family satisfied the condition that

$$H^n \textit{ is not in the Jacobian ideal of } F + \mu H \textit{ for some } \mu \in \mathbb{P}^1,$$

as shown in the paper [18],the Yukawa coupling at the point $\mu \in \mathbb{P}^1$ is not zero.Thus this family will be rigid.

We should give a picture to this description,though Higgs field is essential the Kodaira-Spencer map which determines totally the deformation of the manifold,if we deform every manifold of the family along same direction, we just get a deformation of the family.

Remark on the proposition 4.22: Let $(E, \overline{\partial}, \theta)$ be the analytic Higgs bundle over a quasi projective variety M such that $D_\infty = \overline{M} \setminus M$ normal crossing.If Higgs map θ has regular singularities at D,E has a canonical extension over $\overline{M}$,by GAGA principal $\overline{E}$ becomes a algebraic vector bundle under holomorphic structure $\overline{\partial}$. Restricting to M, we get an algebraic Higgs bundle (E, θ),then θ is really a algebraic map,so that the Yukawa coupling is algebraic θ^n is a global section of $((R^0 f_*(\Omega_{\mathcal{X}/C_0}))^*)^{\otimes 2} \otimes S^n \Omega_{C_0}$. Denote

$$\mathcal{Z} = \{s \in C_0 \mid \theta_s^n = 0\},$$

then $\mathcal{Z}$ is either a set of finite points or C_0.The criterion says that the family will be rigid when $\mathcal{Z}$ is of finite points.And $\mathcal{Z} = C_0$ when the family is non-rigid, but the converse is not necessary true.

A readable proof from view of differential geometric can be found in preprint of Liu-Todorev-Yau-Zuo [18].Actually,the author got idea of the proposition form a manuscript of Kang Zuo and this paper.Communication by email, Liu,K.F and Todorov again pointed out the relation between σ^n and Yukawa coupling from the view of solution of Picard-Fuch equation,and that one zero implies another zero too.

The proposition 4.22 is really a special case of the Viehweg-Zuo's original works,they deal with more general cases: not only Calabi-Yau manifolds but also any projective manifolds with semi-ample canonical line bundle(include CY);or of general type [33, Corollary 6.5].The proof can also be found in their another paper[35, corollary 8.4].

I would like to introduce this general criterion and show the application:

Criterion 4.24. [**Viehweg-Zuo**] Let $\mathfrak{M}_h$ be the coarse moduli space of polarized n-folds with semi-ample canonical line bundle ω; or of general type.Assume $\mathfrak{M}_h$ has a nice compactification and carries a universal family $\pi : \mathfrak{X} \to \mathfrak{M}$(In the real situation,one needs to work on stacks).Let $k_{\mathfrak{M}}$ be the largest integer such that $k_{\mathfrak{M}}$-times iterate Kodaira-Spencer class of this deformation complex on the moduli stack $\mathfrak{M}$ is not zero.Certainly, $1 \leq k_{\mathfrak{M}} \leq$ dimension of the fibres.

If $k_{\mathfrak{M}}$-times iterated K-S class for a family $f : \mathcal{X} \to Y$ is not zero,then the family f must be rigid.

Remark: Here we always assume the flat family $f : \mathcal{X} \to Y$ satisfies the "good "completion : $\mathcal{X}$,Y are projective manifolds,$S = Y - U$ is a reduced normal crossing divisor such that the restricted family $f : V \to U$ is smooth morphism and $\Delta = f^*S = \mathcal{X} \setminus U$ is a also a normal crossing divisor(if Δ is reduced,we get a semi-stable family).

We have Deligne quasi-canonical extension of the VHS $R^n f_* \mathbb{Q}_V$(i.e. that the real part of the eigenvalues of the residues around the components of S lies in $[0, 1)$), the reader can see a simple example of canonical extension soon),take grading of the filtration,so get a extension Higgs bundle

$$(\bigoplus_{p+q=n} E^{p,q}, \bigoplus \theta^{p,q})$$

where

$$\theta^{p,q} : E^{p,q} \to E^{p-1,p+1} \otimes \Omega^1_Y(\log S).$$

Generally, we still call the n-iterated Higgs map

$$\theta^n : E^{n,0} \longrightarrow E^{0,n} \otimes S^n \Omega^1_Y(\log S)$$

be **Yukawa** coupling.By abusing the notation,sometimes we regard the coupling as

$$\theta^n : S^n(T_Y(-\log S)) \longrightarrow E^{0,n} \otimes (E^{n,0})^\vee.$$

When the local monodromies are all unipotent around the components of S(for example,if f is semi-stable the condition will hold),it happens that (cf.[9])

$$E^{p,q} = R^q f_* \Omega^p_{\mathcal{X}/Y}(\log \Delta).$$

One finds for $\Delta = f^*S$

$$\Omega^n_{\mathcal{X}/Y}(\log \Delta) = \omega_{\mathcal{X}/Y}(\Delta_{\text{red}} - \Delta) = \omega_{X/Y}. \tag{4.24.1}$$

The Yukawa coupling is

$$\theta^n : R^0 f_* \Omega^n_{\mathcal{X}/Y}(\log \Delta) \longrightarrow R^n f_* \mathcal{O}_{\mathcal{X}} \otimes S^n \Omega^1_Y(\log S),$$

so it can be regarded as

$$\theta^n : S^n T^1_Y(-\log S) \longrightarrow R^n f_* \mathcal{O}_{\mathcal{X}} \otimes (f_* \omega_{\mathcal{X}/Y})^{-1}.$$

The Criterion 4.24 implies the following :

Corollary 4.25. *$f : \mathcal{X} \to Y$ is a family as above and every smooth fiber is of n-dim projective manifolds with semi-ample canonical line bundle or of general type.*

If the Yukawa coupling

$$E^{n,o} \xrightarrow{\theta^n} E^{0,n} \otimes S^n \Omega^1_Y(\log S)$$

is not zero,then the family $f : \mathcal{X} \to Y$ is rigid.

Proof of the corollary 4.25: We give a proof in situation of the monodromies are unipotent(otherwise one can use the Kawamata covering trick to reduce the problem to the case [33]).

$$0 \longrightarrow f^*\Omega^1_Y(\log S) \longrightarrow \Omega^1_{\mathcal{X}}(\log \Delta) \longrightarrow \Omega^1_{\mathcal{X}/Y}(\log \Delta) \longrightarrow 0 \tag{4.25.1}$$

The wedge product sequences

$$0 \longrightarrow f^*\Omega^1_Y(\log S) \otimes \Omega^{p-1}_{\mathcal{X}/Y}(\log \Delta) \longrightarrow \mathfrak{gr}(\Omega^p_{\mathcal{X}}(\log \Delta)) \longrightarrow \Omega^p_{\mathcal{X}/Y}(\log \Delta) \longrightarrow 0, \tag{4.25.2}$$

where

$$\mathfrak{gr}(\Omega^p_{\mathcal{X}}(\log \Delta)) = \Omega^p_{\mathcal{X}}(\log \Delta)/f^*\Omega^2_Y(\log S) \otimes \Omega^{p-2}_{\mathcal{X}/Y}(\log \Delta).$$

For the invertible sheaf $\mathcal{L} = \Omega^n_{\mathcal{X}/Y}(\log \Delta) = \omega_{\mathcal{X}/Y}$ we consider the sheaves

$$F^{p,q} := R^q f_*(\Omega^p_{\mathcal{X}/Y}(\log \Delta) \otimes \mathcal{L}^{-1})$$

together with the edge morphisms

$$\tau_{p,q} : F^{p,q} \longrightarrow F^{p-1,q+1} \otimes \Omega^1_Y(\log S),$$

induced by the exact sequence (4.25.2), tensored with $\mathcal{L}^{-1}$. As explained in [33], Proof of 4.4 iii), $F^{p,q} = R^q f_*(\bigwedge^{n-p} T_{\mathcal{X}/Y}(-\log \Delta))$ can be regarded as the deformation complex of $f : \mathcal{X} \to Y$.Moreever over U the edge morphisms $\tau_{p,q}$ can also be obtained in the following way. Consider the exact sequence

$$0 \longrightarrow T_{V/U} \longrightarrow T_V \longrightarrow f^*T_U \longrightarrow 0,$$

and the induced wedge product sequences

$$0 \longrightarrow \bigwedge^{n-p+1} T_{V/U} \longrightarrow \tilde{T}_V^{n-p+1} \longrightarrow \bigwedge^{n-p} T_{V/U} \otimes f^*T_U \longrightarrow 0,$$

where $\tilde{T}_V^{n-p+1}$ is a subsheaf of $\bigwedge^{n-p+1} T_V$. One finds edge morphisms

$$\tau^{\vee}_{p,q} : (R^q f_*(\bigwedge^{n-p} T_{V/U})) \otimes T_U \longrightarrow R^{q+1} f_*(\bigwedge^{n-p+1} T_{V/U}).$$

Restricted to $\eta \in U$ those are just the wedge product with the Kodaira-Spencer class. Moreover, tensoring with Ω^1_U one gets back $\tau_{p,q}|_U$.

Because of $R^q f_*(\bigwedge^{n-p} T_{\mathcal{X}/Y}(-\log \Delta)) = R^q f_*(\Omega^p_{\mathcal{X}/Y}(\log \Delta) \otimes \omega^{-1}_{\mathcal{X}/Y})$,we also call

$$\tau_{n-q,q} : F^{p,q} \longrightarrow F^{p-1,q+1} \otimes \Omega^1_Y(\log S)$$

the **log Kodaira-Spencer map** and

$$\tau^m : F^{n,0} = \mathcal{O}_Y \xrightarrow{\tau_{n,0}} F^{n-1,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\tau_{n-1,1}} F^{n-2,2} \otimes S^2(\Omega^1_Y(\log S)) \longrightarrow \cdots \xrightarrow{\tau_{n-m+1,m-1}} F^{n-m,m} \otimes S^m(\Omega^1_Y(\log S))$$

m**-times iterated K-S class.**

One has a factor map

$$\begin{array}{ccc} & S^n(T_Y(-\log S)) & \\ {\scriptstyle \tau^n(f)} \swarrow & & \searrow {\scriptstyle \theta^n(f)} \\ R^n f_*(\omega_{\mathcal{X}/Y}^{-1}) & \longrightarrow & R^n f_* \mathcal{O}_{\mathcal{X}} \otimes (f_* \omega_{\mathcal{X}/Y})^{-1} \end{array}$$

Hence

$$\theta^n(f) \neq 0 \Longrightarrow \tau^n(f) \neq 0 \Longrightarrow \tau^n_{\mathfrak{M}} \neq 0 \Longrightarrow k_{\mathfrak{M}} = n.$$

By the Viehweg-Zuo criterion 4.24, the family f is rigid.

□

Remark : Except rank$E^{n,0} = 1$,it seems difficult to get a proof only using Variation of Hodge Structure.

• **Local study of families admitting maximal unipotent degenerate**

Let $f : \mathcal{X} \to C$ be family smooth over C_0, $(\mathcal{V}, \nabla)$ be integrable algebraic vector come from the polarized VHS of smooth part of the family. $(\mathcal{V}, \nabla)$ admits quasi-unipotent monodromy,thus we have Deligne's quasi-unipotent extension $(\overline{\mathcal{V}}, \nabla)$,

$$\overline{\nabla} : \overline{\mathcal{V}} \to \overline{\mathcal{V}} \otimes \Omega^1_C(\log S),$$

This algebraic vector bundle corresponds to a regular filtered Higgs bundle $(E, \theta)_\alpha$.

Locally, It is explained as following:

Restricting the family $f : \mathcal{X} \to C$ to the unit disk Δ,We think about the local degenerated family

$$f : \mathcal{X}_{loc} \to (\Delta, t)$$

Now,WLOG we assume

$$f' = f|_{\mathcal{X}^*_{loc}} : \mathcal{X}^*_{loc} \to (\Delta^*, t)$$

is smooth and the local monodromy is unipotent,i.e.

(4.25.3) $\qquad (T-1)^{k+1} = 0$ and $(T-1)^k \neq 0$ for a fixed integer $k,\ 0 \leq k \leq n$.

If $k = n$, T is the **maximal unipotent monodromy**. Let

$$N = \log T = \sum_{j=1}^{k} (1/j)(-1)^j (T - \mathrm{Id})^j,$$

we have $N^k \neq 0$ and $N^{k+1} = 0$.

In this simple case,let the $\Phi : \Delta^* \to D/\Gamma$ be the period map corresponding to the local VHS $\mathbb{V}_{\mathbb{C}} = P^n f'_*(\mathbb{C})$. Before Schmid's Nilpotent orbit theorem,Deligne also showed that the holomorphic map

$$\Psi(t) = \exp(\frac{-\log t}{2\pi\sqrt{-1}} N)\Phi(t)$$

can extend cross zero.Therefore we have the Deligne canonical extension $(\overline{\mathcal{V}}, \overline{\nabla})$ where $\overline{\mathcal{V}}$ is generated holomorphically by

$$\widetilde{v}(t) = \exp(\frac{-\log t}{2\pi\sqrt{-1}} N) \cdot v$$

for all $v \in P^n f'_*(\mathbb{C})$ and $\widetilde{v}(te^{2\pi\sqrt{-1}}) = \widetilde{v}(t)$. However, as we have shown in the previous chapter, $\overline{\mathcal{V}}|_0$ can be represented by $\mathcal{V}_U$ for a small neighborhood(in complex topology) U of 0.[27] Thus the monodromy T can be extended naturally to an endomorphism on $\overline{E}$ and we have the restriction and we have

$$T_0 : \mathcal{V}|_0 \longrightarrow \mathcal{V}|_0$$

T_0 has same property 4.25.3 as T.

Definition 4.26 (Residues of $(\overline{\mathcal{V}}, \overline{\nabla})$)**.** For an integrable logarithmic connection

$$\overline{\nabla} : \overline{\mathcal{V}} \longrightarrow \overline{\mathcal{V}} \otimes \Omega^1_\Delta(\log\ 0)$$

we have the composed map

$$(\mathrm{id}_{\overline{\mathcal{V}}} \otimes R_0) \circ \overline{\nabla} : \overline{\mathcal{V}} \xrightarrow{\overline{\nabla}} \overline{\mathcal{V}} \otimes \Omega^1_\Delta(\log\ 0) \xrightarrow{\mathrm{id}_{\overline{\mathcal{V}}} \otimes R_0} \overline{\mathcal{V}}|_0\ ,$$

where $R_0 : \Omega^1_\Delta(\log\ 0) \longrightarrow \mathbb{C}$ is defined by $R_0(f dt/t) = f(0)$.

This composed map is zero at $t\overline{\mathcal{V}}$,then it induces a residue map along 0

$$\mathrm{Res}_0(\overline{\nabla}) : \overline{\mathcal{V}}|_0 \longrightarrow \overline{\mathcal{V}}|_0\ .$$

Remark 4.27. Generally,let $\overline{M}\backslash M = D$ be a reduced smooth connected divisor.For any integrable logarithmic connection

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_{\overline{M}}(\log\ D)$$

we will get

$$(id_{\mathcal{F}} \otimes R_0) \circ \nabla : \mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes \Omega^1_{\overline{M}}(\log\ D) \xrightarrow{id_{\mathcal{F}} \otimes R_0} \mathcal{O}_D \otimes \mathcal{F}\ ,$$

$(id_{\mathcal{F}} \otimes R_0) \circ \nabla$ is $\mathcal{O}_{\overline{M}}-$linear morphism and factors through

$$\mathcal{F} \xrightarrow{restr.} \mathcal{F} \otimes \mathcal{O}_D \xrightarrow{\mathrm{Res}_D(\nabla)} \mathcal{F} \otimes \mathcal{O}_D.$$

Thus $\mathrm{Res}_D(\nabla) \in \mathcal{E}nd(\mathcal{F}) \otimes \mathcal{O}_D$ and we have the short sequence

$$0 \longrightarrow \Omega^1_{\overline{M}} \otimes \mathcal{E}nd(\mathcal{F}) \longrightarrow \Omega^1_{\overline{M}}(\log D) \otimes \mathcal{E}nd(\mathcal{F}) \xrightarrow{R_D \otimes id} \mathcal{O}_D \otimes \mathcal{E}nd(\mathcal{F}) \longrightarrow 0.$$

Therefore by directly calculation, we have the

Exercise 4.28. If $(\mathcal{V}, \nabla)$ is induces from the VHS of the above local family,then

$$\mathrm{Res}_0(\overline{\nabla}) = \frac{-1}{2\pi\sqrt{-1}} N. \tag{4.28.1}$$

which is a particular case of Theorem II,3.11 in Deligne's book [4].

Definition 4.29 (Residues of Higgs map [27])**.** Let (V, H, D) is tamed harmonic bundle, it induces $(E, \theta)_\alpha$ a regular filtered Higgs system which includes (E, θ) a Higgs bundle over Δ^* and a system of decreasing filtered sheaves $E_{\alpha,0}$ which extend E across 0 and extend the Higgs map

$$\theta : E_{\alpha,0} \longrightarrow E_{\alpha,0} \otimes \Omega^1_\Delta(\log\ 0)$$

Here the coherent sheaf E_α is generated by $e \in E|_{\Delta^*}$ such that $|e(t)|_H \leq C|t|^{\alpha+\varepsilon}$ for every $\varepsilon > 0$.

Denote $\overline{E} = E_0 = \cup E_{\alpha,0}$

$$(\mathrm{id}_{\overline{E}} \otimes R_0) \circ \theta : \overline{E} \xrightarrow{\theta} \overline{E} \otimes \Omega^1_\Delta(\log\ 0) \xrightarrow{\mathrm{id}_{\overline{E}} \otimes R_0} \overline{E}|_0\ ,$$

this morphism induces

$$\mathrm{Res}_0(\theta) : \overline{E}|_0 \longrightarrow \overline{E}|_0.$$

Also, formally we have the short exact sequence:

$$0 \longrightarrow \Omega^1_\Delta \otimes \mathcal{E}nd(\overline{E}) \longrightarrow \Omega^1_\Delta(\log\ 0) \otimes \mathcal{E}nd(\overline{E}) \xrightarrow{R_0 \otimes id} \mathcal{E}nd(\overline{E})|_0 \longrightarrow 0.$$

Thus $\mathrm{Res}_0(\theta) = (R_0 \otimes \mathrm{id})(\theta)$.

Remark: In this case (E, θ) is weight n Hodge bundle, because of $\theta^{n+1} \equiv 0$, we have $(\mathrm{Res}_0(\theta))^{n+1} = 0$.Then $\mathrm{Res}_0(\theta)$ is automatically nilpotent.

Proposition 4.30 (Simpson[27],Schmid[25]). *The nilpotent parts of* $\mathrm{Res}_0(\overline{\nabla})$ *is isomorphic to the nilpotent part of* $\mathrm{Res}_0(\theta)$ *under* $\overline{E}|_0 \cong \overline{\mathcal{V}}|_0$.*In the case of Hodge bundle shown by Schmid,with the above lemma*

$$\mathrm{Res}_0(\theta) \cong \mathrm{Res}_0(\overline{\nabla}) \cong \frac{-1}{2\pi\sqrt{-1}} N,$$

where N *is the nilpotent part of the monodromy.*

Now, we come back to the Yukawa coupling.

Exercise 4.31. If the local family $f : \mathcal{X}_{loc} \to (\Delta, t)$ degenerate at 0 with maximal unipotent monodromy,then the Yukawa coupling is nonzero.

Therefore,as an application of the proposition 4.22,we have an interesting result which is implied in the paper of Liu-Todorov-Yau-Zuo(cf [18]) and the manuscript of Zuo Kang[45].

Theorem 4.32 (Zhang [41][42]). *Let* $f : \mathcal{X} \to C$ *be a family of* n*-dimensional Calabi-Yau manifolds satisfying the condition* $(**)$ *in the section 4.2.*

If the family f *admits a degeneration with maximal unipotent monodromy,then* f *must be rigid.*

Remark 4.33. We should point out the degenerations of Lefschetz pencil of CY manifolds are all of minimal unipotent monodromy. So, we will ask the question: Whether the CY family with minimal unipotent monodromy degeneration is rigid?

Example 4.34. In the recent paper of Lian-Todorov-Yau,the authors show the following type CY families have the degenerate of maximal unipotent monodromy (cf.[17]),thus they will be rigid by the theorem 4.32.

One parameter family of complete intersections of CY manifolds in $\mathbb{P}^{n+k}$ *for* $n \geq 4$ *and* $k \geq 1$ *defined by the following equations :*

$$G_{1,t} = tF_1 - \prod_{i=0}^{n_1} x_i = 0, .., G_{k,t} = tF_k - \prod_{j=n_1+..+n_{k-1}}^{n_k} x_j = 0, t \in \mathbb{P}^1$$

where the system $F_1 = .. = F_k = 0$ *defines a non singular CY manifolds,*

$$n_i = \deg F_i \geq 2 \text{ and } \sum n_i = n + k + 1$$

and x_i *are the standard homogeneous coordinates in* $\mathbb{P}^{n+k}$.

The condition $\sum n_i = n + k + 1$ *implies that the fibers* $\pi^{-1}(t) = X_t$ *for* $t \neq 0$ *are CY manifolds of complex dimension* n.

Example 4.35. As a special case of Lian-Todorov-Yau, Morrison.D showed the Yukawa coupling of the following family is not zero at 0.

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 - 5\lambda X_0 X_1 X_2 X_3 X_4 = 0$$

for $\lambda \in \mathbb{P}^1$ and $[X_0, X_1, X_2, X_3, X_4] \in \mathbb{P}^4$.

By the example 4.34,this family admits a degeneration with maximal unipotent monodromy at 0.One also can show this family rigid by the example 4.23,because $(X_0X_1X_2X_3X_4)^3$ is not in the Jacobian ideal of

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5.$$

Example 4.36. We should point out these two examples are both of strong degeneration(with surgery of semi-stable reduction).Thus they have to be rigid by theorem 4.15.Recently Viehweg-Zuo construct some meaningful families with maximal unipotent degeneration by covering trick (not only just by complete intersection)[36].

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