On the L^2 and L^1 Convergence Rates of Viscous Solutions of the Keyfitz-Kranzer System with Piecewise Smooth and Large BV Data

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Abstract

We study the zero-dissipation problem in L^2 and L^1 spaces of the Keyfitz-Kranzer system. When the solution of the inviscid problem is piecewise smooth and having finitely many noninteracting shocks with finite strength, there exists unique solution to the viscous problem which converges to the given inviscid solution away from shock discontinuities. Convergence rates are given in terms of ϵ the viscosity. The proof is given by a matched asymptotic analysis and a weighted elementary energy method.

Keywords: Keyfitz-Kranzer systems, shock waves, zero dissipation, shock profile .

1 Introduction

We consider the following two 2×2 systems

$$\begin{cases} u_t + (\phi(u,v)u)_x = 0, \\ v_t + (\phi(u,v)v)_x = 0, \qquad t > 0, x \in \mathbb{R}, \end{cases} \tag{1.1}$$

$$\begin{cases} u_t + (\phi(u,v)u)_x = \epsilon u_{xx}, \\ v_t + (\phi(u,v)v)_x = \epsilon v_{xx}, \qquad t > 0, x \in \mathbb{R}. \end{cases} \tag{1.2}$$

The system (1.1) is a special form of the Temple class system (see [2]), with one contact field and one line field. And the shock wave curves and rarefaction wave curves coincide. Such systems arise in the fields of elasticity theory (see [3],enhanced oil recovery and magnetohydrodynamics (see [4]), etc.

Let (r, θ) be the polar coordinates,

$$r(u,v) = \sqrt{u^2+v^2}, \quad \tan\theta = v/u.$$

In this paper we only consider the case when $\phi(u,v)=\phi(r)$. The following assumption on ϕ are consistent with physical considerations:

- (A_1) $\phi(r)\to+\infty$ as $r\to 0$ or $r\to+\infty$,
- (A_2) $\phi(r)>0$,
- (A_3) $\phi_r>0$,$\frac{\partial(r\phi)}{\partial r}>0$, $\frac{\partial^2(r\phi)}{\partial r^2}>0$,
- (A_4) ϕ is convex.

Then the system is strictly hyperbolic with two eigenvalues

$$\lambda_1=\phi, \qquad \lambda_2=\phi+r\phi_r=(r\phi)_r, \tag{1.3}$$

where λ_1 is linearly degenerate and λ_2 genuinely nonlinear; the corresponding eigenvectors are

$$\mathbf{r_1}=\mathbf{l_1}=(-\sin\theta,\cos\theta), \quad \mathbf{r_2}=\mathbf{l_2}=(\cos\theta,\sin\theta). \tag{1.4}$$

The well-posedness of the system (1.1) in our case has been studied by G.Q. Chen in [5],[6],[7] and he obtained some properties which are similar to those in the scalar conservation law. A very important approach to study the well-posedness of the hyperbolic system is the viscosity method. Ever since Goodman and Xin [1] studied the rates of convergence of the viscous approximate solutions for general strictly hyperbolic systems with weak shock initial data, we are interested in knowing for what kinds of systems we can get convergence results for bounded shock data, just like in the scalar case.

Our aim is to get the L^1 and L^2 convergence rates of the viscous solutions, given the inviscid solution $h(x,t)=(u,v)(x,t)$ which has finitely many noninteracting shocks of finite strength. The method we use is mainly motivated by [1]. But in the stability analysis, we use an initial weighted energy estimate. The result in the L^1 space is based on that in the L^2 space.

The proof consists of main parts. In the first part, we use the weighted asymptotic expansions to construct an approximate solution $A^\epsilon(x,t)$ of (1.2) without requiring that the shock is weak. The $A^\epsilon(x,t)$ is close to the given solution $h(x,t)$ for $\epsilon=0$ away from the shock. However, $A^\epsilon(x,t)$ has a smoothed viscous shock profile of width ϵ near the shock. The detailed construction of the approximate solution, is also crucial for our method, since we need to have estimates on the higher order correction. In the second part, we show a priori estimate on the difference between A^ϵ and the exact viscous solution h^ϵ. The crucial part is the estimate in an very thin initial layer $0\le t\le O(1)\epsilon$ obtained by using a weighted energy estimate. Here we choose the weight to be $t^{-\frac{2}{3}}$. Then the special feature of the two eigenvalues allows us to get the a priori estimates. In the last section, we get the L^1 estimate. Without loss of generality, we assume the given inviscid solution $h(x,t) = (u,v)(x,t)$ is a single-shock solution up to time T, that is

1. $h(x,t)$ is a distributional solution of the hyperbolic system (1.1) in the region $R^1 \times [0,T]$;
2. There is a smooth curve, the shock, $x = s(t)$, $0 \le t \le T$, so that $h(x,t)$ is sufficiently smooth at any point $x \ne s(t)$.
3. The limits
$$\partial_x^k h(s(t)-0,t) = \lim_{x \to s(t)^-} \partial_x^k h(x,t), \qquad \partial_x^k h(s(t)+0,t) = \lim_{x \to s(t)^+} \partial_x^k h(x,t),$$
exist and are finite for $t \le T$ and $k = 0,1,2,3,4$.
4. The Lax geometrical entropy condition is satisfied at $x = s(t)$, that is
$$\lambda_1(h(s(t)-0,t)) < \lambda_2(h(s(t)-0,t)), \tag{1.5}$$
$$\lambda_2(h(s(t)+0,t)) < \dot{s}(t) < \lambda_2(h(s(t)-0,t)). \tag{1.6}$$

Here we assume the discontinuity is of the second family. Here and in the following, we always use the notation $\dot{s} = \frac{ds(t)}{dt}$. We also assume that

$$\lambda_1^l < \dot{s}, \quad \text{and} \quad r(u,v) > r_*, \tag{1.7}$$

for some positive constant r_*. Now we state our main theorem

Theorem 1. *Suppose that $(u,v)(x,t)$ is a single-shock solution of (1.1)up to time $T > 0$. Under condition* (1.7), *if*

$$\sum_{1 \le \alpha \le 7} \int_0^T \left(\int_{-\infty}^{x=s(t)} + \int_{x=s(t)}^{\infty} \right) |\partial_x^\alpha (u,v)(x,t)|^2 dx dt < +\infty, \tag{1.8}$$
$$\int_{R^1} \big(|(u,v)_x(x,0)| + |(u,v)_{xx}(x,0)| \big) dx \le +\infty, \tag{1.9}$$

there exists positive constant ϵ_0, such that for any $\epsilon \in (0,\epsilon_0]$, there is a smooth solution $(u^\epsilon, v^\epsilon)(x,t)$ of (1.2), satisfying

$$(u_x^\epsilon, v_x^\epsilon) \in C^1([0,T], H^2). \tag{1.10}$$

Moreover, for any given $\beta \in (0,1)$,

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^1} |(u^\epsilon - u, v^\epsilon - v)(x,t)|^2 dx \le C_\beta \epsilon^\beta, \tag{1.11}$$

and

$$\sup_{0\le t\le T,|x-s(t)|\ge\epsilon^\beta} |(u^\epsilon - u, v^\epsilon - v)(x,t)| \le C\epsilon, \tag{1.12}$$

and

$$\sup_{0\le t\le T}\int_{\mathbb{R}^1} |(u^\epsilon - u, v^\epsilon - v)(x,t)|dx \le C_\beta\epsilon, \tag{1.13}$$

where C_β, C *are positive constants independent of* ϵ.

2 Approximate solutions

Suppose the exact solution to (1.2) is $h^\epsilon(x,t) = (u^\epsilon, v^\epsilon)(x,t)$. Following Goodman-Xin, in [1], we will use the formal Hilbert expansion and the shock expansion to construct an approximate solution to $h^\epsilon(x,t)$.

2.0.1 Outer expansion

Let $h_0(x,t) = (v_0, u_0)(x,t)$, $h_i(x,t) = (v_i, u_i)(x,t), i = 1, 2, \cdots\cdots$. In the domain away from the shock, we expand $h^\epsilon(x,t)$ formally in order of ϵ.

$$h^\epsilon(x,t) \sim h_0(x,t) + \epsilon h_1(x,t) + \epsilon^2 h_2(x,t) + \cdots\cdots,\ x \ne s(t). \tag{2.1}$$

Substituting (2.1) into (1.2) and comparing the coefficients of powers, we get, with $f(h) = f(u,v) = (\phi(r)u, \phi(r)v)^t$ (where F^t denotes the transport of F),

$$O(1): \qquad h_{0t} + f(h_0)_x = 0,, \tag{2.2}$$

$$O(1)\epsilon: \qquad h_{1t} + \big(f'(h_0)h_1\big)x = h_{0xx}, \tag{2.3}$$

$$O(1)\epsilon^2: \qquad h_{2t} + \big(f'(h_0)h_2\big)_x = h_{1xx} - \frac{1}{2}\big(f''(h_0)(h_1,h_1)\big)_x. \tag{2.4}$$

etc.

The outer functions, $h_0, h_1, \cdots\cdots$ are generally discontinuous at the shock, but smooth up to the shock. The leading term, h_0, is taken to be the single shock solution of (1.1), $(u,v)(x,t)$. Near the shock, $h^\epsilon(x,t)$ will be represented by a shock layer expansion

$$H^\epsilon(x,t) \sim H_0(\xi,t) + \epsilon H_1(\xi,t) + \cdots\cdots, \tag{2.5}$$

where $H_i = (U_i, V_i), i = 0, 1, 2, \cdots$, and

$$\xi = \frac{x - s(t)}{\epsilon} + \delta(t,\epsilon), \tag{2.6}$$

and $\delta(t,\epsilon)$ is the perturbation of the shock position to be determined later. We assume $\delta(t,\epsilon)$ has the form

$$\delta(t,\epsilon) = \delta_0(t) + \epsilon\delta_1(t) + \epsilon^2\delta_2(t) + \cdots\cdots. \tag{2.7}$$

Substituting (2.5)-(2.6)-(2.7) into (1.2) to obtain

$$O(1)\frac{1}{\epsilon}: \qquad H_{0\xi\xi} + \dot{s}H_{0\xi} - f(H_0)_\xi = 0, \tag{2.8}$$

$$O(1): \qquad H_{1\xi\xi} + \dot{s}H_{1\xi} - \big(f'(H_0)H_1\big)_\xi = \dot{\delta}_0(t)H_{0\xi} + H_{0t}, \tag{2.9}$$

$$O(1)\epsilon: \qquad H_{2\xi\xi} + \dot{s}H_{2\xi} - \big(f'(H_0)H_2\big)_\xi \tag{2.10}$$

$$= \dot{\delta}_0(t)H_{1\xi} + \dot{\delta}_1(t)H_{0\xi} + H_{1t} + \frac{1}{2}\big(f''(H_0)(H_1,H_2)\big)_\xi, \tag{2.11}$$

etc.

The inner expansion is supposed to hold in a small zone of width $O(\epsilon)$ around $x = s(t)$. The outer expansion and inner expansion are expected to agree with each other in the "matching zone", where $|\xi| \to +\infty$ and $|x - s(t)|$ is small. Using Taylor's expansion to express the outer solution in terms of ξ, we get the following "matching conditions" as $\xi \to \pm\infty$:

$$\begin{aligned}
&H_0(\xi,t) = h_0(s(t)\pm 0,t) + o(1).\\
&H_1(\xi,t) = h_1(s(t)\pm 0,t) + (\xi-\delta_0)\partial_x h_0(s(t)\pm 0,t) + o(1)\\
&H_2(\xi,t) = h_2(s(t)\pm 0,t) + (\xi-\delta_0)\partial_x h_1(s(t)\pm 0,t)\\
&\qquad\qquad - \delta_1\partial_x h_0(s(t)\pm 0,t) + \frac{1}{2}(\xi-\delta_0)^2\partial_x^2 h_0(s(t)\pm 0,t) + o(1).
\end{aligned} \tag{2.12}$$

etc. After we construct the various outer and inner functions, we can verify the algebraic growth of H_i as $\xi \to \pm\infty$.

2.1 Properties of the viscous shock profile

Since much of our construction depends on the properties of viscous shock profiles, we recall them as follows. Viscous shock profiles are the travelling wave solutions of (1.2)on the whole $\mathbb{R}^1$ of the form

$$(u,v)(x,t) = (U,V)(\xi) = H(\xi), \quad \xi = \frac{x-\sigma}{\epsilon},$$

which satisfies

$$-\sigma H' + f(H)' = H'', \tag{2.13}$$

and $(U,V)(\pm\infty) = (v_\pm, u_\pm)$, with

$$\begin{cases} \sigma(u_+ - u_-) = \big(\phi(r)u\big)_+ - \big(\phi(r)u\big)_-, \\ \sigma(v_+ - v_-) = \big(\phi(r)v\big)_+ - \big(\phi(r)v\big)_-, \end{cases} \tag{2.14}$$

where $' = d/d\xi$, σ denotes the shock speed.

The advantage of taking $\phi(u, v) = \phi(r)$ is that the behavior of the 2-waves of (1.1) can be studied independently. Across the 1-wave, the value of r is unchanged; across the 2-wave, the value of θ is constant. Therefore, the behavior of r and hence the behavior of 2-waves, can be described by the scalar conservation law

$$r_t + \big(r\phi(r)\big)_x = 0. \tag{2.15}$$

This can be justified because the jump condition and entropy conditions for (1.1) are consistent with that for (2.18), i.e. $\sigma(r_- - r_+) = \big(r\phi(r)\big)_- - \big(r\phi(r)\big)_+$.

Then the behavior of the viscous shock profile of the 2-wave is like that in the scalar equation

$$r_t + \big(r\phi(r)\big)_x = \epsilon r_{xx}, \quad r \to r_\pm, \quad \text{as} \quad \xi \to \pm\infty, \tag{2.16}$$

and $r(x, t) = R(\xi)$, $\theta(x, t) = \theta_- = \theta_+ = constant$. Integrate (2.16) once to give

$$R_\xi = \big(R\phi(R)\big) - \big(r\phi(r)\big)_- - \sigma(R - r_-). \tag{2.17}$$

Consequently, we get the following results adopted from that in the scalar case ,without requirements on the shock strength,

1. $\partial_\xi \lambda_2(R) < 0$, for all ξ,

2. $|\partial_\xi R| \le c'|r_- - r_+|$,

3. as $\xi \to -\infty$, $\quad R(\xi, r_-, \sigma) - r_- = O(1)|r_- - r_+|e^{-\alpha|\xi|}$,
$$\frac{\partial R}{\partial r_-} - 1 = O(1)e^{-\alpha|\xi|}, \qquad \frac{\partial R}{\partial \sigma} = O(1)e^{-\alpha|\xi|}$$
.

4. as $\xi \to +\infty$, $\quad R(\xi, r_-, \sigma) - r_+ = O(1)|r_- - r_+|e^{-\alpha|\xi|}$,
$$\frac{\partial R}{\partial r_-} - \frac{\partial r_+}{\partial r_-} = O(1)e^{-\alpha|\xi|}, \qquad \frac{\partial R}{\partial \sigma} - \frac{\partial r_-}{\partial \sigma} = O(1)e^{-\alpha|\xi|}.$$

We remark here that these estimates can be proved by estimating the linear systems of ordinary differential equations obtained by differentiating the equation (2.25) .

2.2 Constructions of the outer and inner solutions

We need to construct the outer and inner solutions order by order simultaneously, making use of the matching conditions. The leading order of outer solutions, h_0, is exactly the single shock solution given in Theorem 1. For any fixed t (viewed as a parameter), the leading order of inner solutions, $H_0(\xi, t)$ determined by (2.9) is just the viscous shock profile with end states $h_- = (u_-(t), v_-(t)) = h_0(s(t)-0, t)$, and $h_+ = (u_+(t), v_+(t)) = h_0(s(t)+0, t)$, and the shock speed $\sigma = \dot{s}(t)$. So we take

$$H_0(\xi, t) = (R, \theta)(\xi; h_-(t), \dot{s}(t)),$$

Since the shift can be absorbed by $\delta_0(t, \epsilon)$, we can take it to be zero. The next order terms h_1, H_1 and $\dot{\delta}_0(t)$ are determined together. Integrating the two equations of (2.9) over $[0, \xi)$, we have

$$H_{1\xi\xi} + \dot{s}H_{1\xi} - \big(f'(H_0)H_1\big)_\xi = \dot{\delta}_0(t)H_{0\xi} + \frac{\partial H_0}{\partial h_-}\dot{h}_- + \frac{\partial H_0}{\partial \dot{s}}\ddot{s}. \tag{2.18}$$

Set $H_1 = F_1 + D_1$, where D_1 is smooth and

$$D_1 = \begin{cases} \xi\partial_x h_0(s(t)-0, t), & \xi < -1, \\ \xi\partial_x h_0(s(t)+0, t), & \xi > 1. \end{cases} \tag{2.19}$$

Now using the identity

$$\dot{h}_{-,+} = \frac{d}{dt}h_0(s(t)\pm 0, t) = \big(\dot{s}I - f'\big(h_0(s(t)\pm 0, t)\big)\big)h_{0x}(s(t)\pm 0, t), \tag{2.20}$$

we compute that

$$F_{1\xi\xi} + \dot{s}F_{1\xi} - \big(f'(H_0)F_1\big)_\xi = \dot{\delta}_0(t)H_{0\xi} + g(\xi, t), \tag{2.21}$$

where $|g(\xi, t)| \le ce^{-\alpha|\xi|}$ for large $|\xi|$. Then defining $G(\xi, t) = \int_0^\xi g(\eta, t)d\eta$, we get

$$F_{1\xi} = \big(f'(H_0) - \dot{s}I\big)F_1 + \dot{\delta}_0 H_0 + G + \mathbf{c}(t), \tag{2.22}$$

for some constants of integration $c(t)$ in R^2, to be defined later. Now we are to determine F_1, δ_0 and $c(t)$. First we express F_1 in terms of the basis $\mathbf{r}_1(H_0), \mathbf{r}_2(H_0)$ of the right eigenvectors of $f'(H_0)$, where $\mathbf{r_1} = (-\sin\theta, \cos\theta)$, $\mathbf{r_2} = (\cos\theta, \sin\theta)$, $\theta = \theta(H_0)$. We also note that $\theta(h_0(s(t)\pm 0, t) = \theta(H_0(\xi, t))$, hence we can express $h_1(s(t)\pm o, t), \partial_x h_0(s(t)\pm 0, t)$ at $\mathbf{r_1}, \mathbf{r_2}$ too. Now we write

$$\begin{aligned} F_1 &= \alpha_1(\xi, t)\mathbf{r_1} + \alpha_\mathbf{2}(\xi, \mathbf{t})\mathbf{r_2}, \\ h_1(s(t)\pm 0, t) &= \beta_{1\pm}(t)\mathbf{r_1} + \beta_{\mathbf{2}\pm}(\mathbf{t})\mathbf{r_2}, \\ \partial_x h_0(s(t)\pm 0, t) &= \gamma_{1\pm}(t)\mathbf{r_1} + \gamma_{\mathbf{2}\pm}(\mathbf{t})\mathbf{r_2}. \end{aligned} \tag{2.23}$$

Taking the matching condition into account, we have

$$\alpha_j(\xi,t) = \beta_{j\pm}(t) - \delta_0\gamma_{j\pm} + o(1), \quad \text{as} \quad \xi \to \pm\infty, \quad \text{for} \quad j = 1,2. \tag{2.24}$$

So it can be easily seen that

$$\begin{aligned} &\alpha_{1\xi} + (\dot{s} - \lambda_1(H_0))\alpha_1 = \mathbf{r_1 G} + \mathbf{r_1 c(t)}, \\ &\alpha_{2\xi} + (\dot{s} - \lambda_2(H_0))\alpha_2 = \dot{\delta}_0 r(H_0) + \mathbf{r_2 G} + \mathbf{r_2 c(t)}, \end{aligned} \tag{2.25}$$

And we can solve the above equations and get unique solutions, as stated in the following lemma.

Lemma 1. *There exists a smooth solution,* $(\alpha_1,\alpha_2)(\xi,t)$*, to equations* (2.1)*, with the following property:*

$$\alpha_j(\xi,t) = \begin{cases} \left(\dot{s} - \lambda_j(h_-)\right)^{-1}\mathbf{r_j}[\mathbf{c(t)} + \mathbf{G}_- + \eta_{\mathbf{j}}\dot{\delta}\mathbf{r}_-] + \mathbf{O(1)e}^{-\alpha_\mathbf{0}|\xi|}, & , \quad \xi \to -\infty, \\ \left(\dot{s} - \lambda_2(h_+)\right)^{-1}\mathbf{r_2}[\mathbf{c(t)} + \mathbf{G}_+ + \eta_{\mathbf{j}}\dot{\delta}\mathbf{r}_+] + \mathbf{O(1)e}^{-\alpha_\mathbf{0}|\xi|} & \quad \xi \to +\infty. \end{cases} \tag{2.26}$$

for $j = 1,2, \eta_1 = 0, \eta_2 = 1, G_\pm = \lim_{\xi\to\pm\infty} G(\xi,t)$*, and* α_0 *is a positive constant.*

We omit the proof.

Taking (2.24) and (2.2) together, we have

$$\begin{aligned} \mathbf{r_1}[\mathbf{G}_- + \mathbf{c(t)}] &= \left(\beta_{1-} - \delta_0(t)\gamma_{1-}\right)(\dot{s} - \lambda_{1-}), \\ \dot{\delta}_0 r_- + \mathbf{r_2}[\mathbf{G}_- + \mathbf{c(t)}] &= \left(\beta_{2-} - \delta_0(t)\gamma_{2-}\right)(\dot{s} - \lambda_{2-}), \\ \mathbf{r_1}[\mathbf{G}_+ + \mathbf{c(t)}] &= \left(\beta_{1+} - \delta_0(t)\gamma_{1+}\right)(\dot{s} - \lambda_{1+}), \\ \dot{\delta}_0 r_+ + \mathbf{r_2}[\mathbf{G}_+ + \mathbf{c(t)}] &= \left(\beta_{2+} - \delta_0(t)\gamma_{2+}\right)(\dot{s} - \lambda_{2+}). \end{aligned} \tag{2.27}$$

So we can solve $\delta_0(t), \mathbf{r_1c(t)}, \mathbf{r_2c(t)}, \beta_{\mathbf{1}-}$ in terms of $\beta_{1+}, \beta_{2+}, \beta_{2-}$ from the above equations.

And in view of the linear initial-boundary problem (2.3) for $h_1(x,t)$, , and after taking up suitable initial values of $h_1(x,t)$ around $x = s(0)$, we can solve , by the theory of first order linear hyperbolic systems, $h_1(x,t)$ uniquely and furthermore have the following regularity assertion (see [8],[9]). Here we make use of the condition (1.7).

Proposition 1. $h_1(x,t)$*,* $H_1(\xi,t)$ *and* δ_0 *can be determined such that*

- $h_1(x,t)$ *and its derivatives are uniformly continuous up to* $x = s(t)$ *and*

$$\sum_{|\alpha|\le 5} \int_0^T \int_{x\neq s(t)} |\partial_x^\alpha h_1(x,t)|^2 dxdt < +\infty. \tag{2.28}$$

- $H_1(\xi,t)$ *is smooth and for some* $c_0 > 0$*,*

$$\begin{aligned} H_1(\xi,t) = h_1(s(t)\pm 0,t) + (\xi - \delta_0)\partial_x h_0(s(t)\pm 0,t) + O(1)\exp\{-c_0|\xi|\}, \\ \textit{as} \qquad \xi \to \pm\infty. \end{aligned} \tag{2.29}$$

It is clear that the above procedure can be carried out similarly to any order. In particular, we can construct $h_2, H_2, h_3, H_3, \delta_1$ and δ_2 and similar estimates hold for them.

2.3 Construction of the Approximate solution

Now we can construct a smooth approximate solution to (1.1) by patching the truncated outer and inner solutions in the previous discussion as in [1].

Set

$$I(x,t) = (H_0 + \epsilon H_1 + \epsilon^2 H_2)(\frac{x - s(t)}{\epsilon} + \delta_0 + \epsilon\delta_1 + \epsilon^2\delta_2, t), \tag{2.30}$$

and

$$O(x,t) = (h_0 + \epsilon h_1 + \epsilon^2 h_2)(x,t), \quad x \neq s(t). \tag{2.31}$$

Let $m(y) \in C_0^\infty(\mathbb{R}^1)$ such that $0 \leq m(y) \leq 1$ and

$$m(y) = \begin{cases} 1, & |y| \leq 1 \\ 0, & |y| \geq 2. \end{cases} \tag{2.32}$$

Choose $\gamma \in (\frac{2}{3}, 1)$ as a constant. Then we define the approximate solutions as

$$A^\epsilon(x,t) = m\big(\frac{x - s(t)}{\epsilon^\gamma}\big)I(x,t) + (1 - m\big(\frac{x - s(t)}{\epsilon^\gamma}\big))O(x,t) + d(x,t), \tag{2.33}$$

where $d(x,t) = (d_1, d_2)$ is a higher order correction to be determined later. Using the structure of various orders of outer and inner solutions, and the estimates in Proposition 1 , we can choose a suitable $d(x,t)$ such that

$$A_t^\epsilon + f(A^\epsilon)_x - \epsilon A_{xx}^\epsilon = \big(f(A^\epsilon) - f(A^\epsilon) - d\big)_x \tag{2.34}$$

and $d(x,0) = 0$. In the following we give the estimates on $d(x,t)$ but omit the proof which is exactly as that in [1].

Lemma 2. *We can find a smooth $d(x,t)$ satisfies* (2.34)*, and the following estimates*

$$\begin{array}{lll} ||\partial_x^l d(\cdot,t)||_{L_\infty} & \leq O(1)\epsilon^{(3-l)\gamma - 1/2} & \textit{for} \quad l = 0,1,2,3. \\ ||d(\cdot,t)||_{L_2(R^1)} & \leq O(1)\epsilon^{3\gamma - 1/2} & \textit{for} \quad \alpha \in (0, 1/2), \\ ||\partial_x^l d(\cdot,t)||_{L_2(R^1)} & \leq O(1)\epsilon^{(3-l+1/2)\gamma - 1/2} & \textit{for} \quad l = 1,2. \end{array} \tag{2.35}$$

Lemma 2 implies the following estimates on $A^\epsilon(x,t)$.

Lemma 3. *Let $A^\epsilon(x,t)$ be defined as in* (2.33)*. Then*

$$A^\epsilon(x,t) = \begin{cases} h_0(x,t) + O(1)\epsilon & \textit{if} \quad |x - s(t)| \geq \epsilon^\gamma, \\ H_0(\xi,t) + O(1)\epsilon^\gamma & \textit{if} \quad |x - s(t)| \leq 2\epsilon^\gamma. \end{cases} \tag{2.36}$$

For ϵ small, there exists positive constant r_{**} such that

$$r(A^\epsilon) > r_{**}. \tag{2.37}$$

And taking the coordinate transformation $y = \big(x - s(t)\big)/\epsilon, \tau = t/\epsilon$, we have

$$\frac{\partial A^\epsilon}{\partial y} = m\partial_\xi H_0 + \epsilon O(1), \qquad \frac{\partial A^\epsilon}{\partial \tau} = \epsilon O(1). \tag{2.38}$$

To be exact, we set

$$R = R(A^\epsilon), \quad \theta = \theta(A^\epsilon), \tag{2.39}$$

then

$$\frac{\partial R}{\partial y} = m\partial_\xi R(H_0) + \epsilon O(1), \quad \frac{\partial \theta}{\partial y} = \epsilon^2 O(1), \quad \frac{\partial (R,\theta)}{\partial \tau} = \epsilon O(1). \tag{2.40}$$

Proof: By construction, we have

$$A^\epsilon(x,t) = \begin{cases} O + d & \text{for} \quad |x - s(t)| \geq 2\epsilon^\gamma, \\ O + m(I - O) + d & \text{for} \quad \epsilon^\gamma < |x - s(t)| < 2\epsilon^\gamma, \\ I + d & \text{for} \quad |x - s(t)| \leq \epsilon^\gamma. \end{cases}$$

We also have $O(x,t) = h_0 + O(1)\epsilon$ on $|x - s(t)| > \epsilon$, $I(x,t) = H_0 + O(1)\epsilon^\gamma$ on $|x - s(t)| \leq \epsilon^\gamma$ which can be obtained by using (2.29) , and for $l = 0, 1, 2$, $\partial_x^l (I - O)(x,t) = O(1)\epsilon^{(3-l)\gamma}$ on $\{(x,t) : \epsilon^\gamma \leq |x - s(t)| \leq 2\epsilon^\gamma, t \in [0,T]\}$ which is verified by using the matching conditions with $O(1) = O(1)e^{-\alpha_0|\xi|}$. These, together with $(2.35)_1$, yield (2.36). And (2.37) is the direct consequence of (2.36). Similarly, one can show (2.38). Moreover, again by construction we have $\partial_y \theta(H_0) = 0$ which gives that $\partial_y \theta(A^\epsilon) = O(1)\epsilon^2$. This completes the proof. □

This finishes the construction of the formal approximation solution to (1.2).

3 Stability Analysis

Having the approximate solution $A^\epsilon(x,t)$ at hand, we now show that there exists an exact solution $h^\epsilon(x,t)$ to (1.2) that is close to $A^\epsilon(x,t)$. Here we let

$$h^\epsilon(x,0) = A^\epsilon(x,0), \quad \text{for each} \quad \epsilon. \tag{3.1}$$

Set

$$\tilde{w}(x,t) = h^\epsilon(x,t) - A^\epsilon(x,t), \tag{3.2}$$

$\tilde{w} = (\tilde{u}, \tilde{v})$, then $\tilde{w}(x,t)$ satisfies the error equation

$$\begin{gathered} \tilde{w}_t + \big(f'(A^\epsilon)\tilde{w}\big)_x + Q(A^\epsilon, \tilde{w})_x = \epsilon \tilde{w}_{xx} + \big(f(A^\epsilon - d) - f(A^\epsilon)\big)_x. \\ \tilde{w}(x,0) = 0, \end{gathered} \tag{3.3}$$

where $Q(A^\epsilon, \tilde{w}) = f(h^\epsilon) - f(A^\epsilon) - f'(A^\epsilon)\tilde{w}$ satisfying $|Q| \le O(1)|\tilde{w}|^2$ for small $\tilde{w}$. To exploit the fact that a shock satisfying the entropy condition is compressive, we need to integrate system (3.3) once. Thus we use the coordinate transformation

$$y = \frac{x - s(t)}{\epsilon}, \qquad \tau = \frac{t}{\epsilon}.$$

and set

$$\tilde{w}(x,t) = \epsilon w_x = w_y, \quad \text{for} \quad (x,t) \in \mathbb{R} \times [0,T]. \tag{3.4}$$

So $w(y,\tau)$ satisfies

$$\begin{gathered} w_\tau - \dot{s} w_y + f'(A^\epsilon) w_y = w_{yy} + q(A^\epsilon, d) - Q(A^\epsilon, w_y), \\ w(y,0) = 0, \end{gathered} \tag{3.5}$$

where $q(A^\epsilon, d) = f(A^\epsilon - d) - f(A^\epsilon)$ and $|q| \le O(1)|f'(A^\epsilon)d| \le O(1)|d|$. Our purpose is to show that for ϵ suitably small, (3.5) has a unique "small" smooth solution up to time T. This will follow from the following three lemmas.

Lemma 4. *(Local estimate) For each ϵ, the initial value problem* (3.5) *has a unique solution $w \in C^1\big([0,\tau_0] : H^2(R^1)\big)$ for some $\tau_0 \le 1/N$, with N sufficiently large and independent of ϵ, and*

$$e^{-N\tau}||w||^2_{H^1(R^1)} + \int_0^{\tau_0} e^{-N\tau}||w_y||^2_{H^1(R^1)} ds \le c\epsilon^{6\gamma+\alpha-2}, \tag{3.6}$$

where γ and α are defined in Section 2.3.

Lemma 5. *(A priori estimate) Suppose that the Cauchy problem* (3.5) *has a solution $w \in C^1\big([0,\tau_1] : H^2(R^1)\big)$ for some $\tau \in (0,T]$, and*

$$\sup_{[0,\tau_1]} ||w(\cdot,\tau)||_{L_\infty} \le c\epsilon \tag{3.7}$$

for some constant c independent of ϵ and τ. There exist positive constants ϵ_1, μ_1 and C, which are independent of ϵ and τ_1, such that if

$$\epsilon \in (0,\epsilon_1), \quad \sup_{[0,\tau_1]} ||w(\cdot,\tau)||_{H^2} \le \mu_1, \tag{3.8}$$

then

$$\sup_{[0,\tau_1]} ||w(\cdot,\tau)||^2_{H^2} + \int_0^{\tau_1} ||w_y(\cdot,\tau)||^2_{H^2} d\tau \le C\epsilon^{6\gamma+\alpha-4}. \tag{3.9}$$

Remark : The discovery of Lemma 4 is very important in the energy proof. Having the lemma, we need not impose restriction on the shock strength. In what follows, we use $H^l (l \ge 1)$ to denote the usual Sobolev's space with the norm $||\cdot||_l$ and $||\cdot|| = ||\cdot||_0$ denotes the usual L_2-norm. We also use c to denote any positive constant which is independent of ϵ, y and τ and use $O(1)$ to denote any positive bounded function.

3.1 Proof of Lemma 3.1

Since (3.5) is an initial-value problem for a uniformly parabolic system, the existence theory (local in time) and the uniqueness theory in the space $C^1\big([0,\tau_0];H^2(R^1)\big)$ is standard. Thus we can declare that the smooth solution w satisfies

$$\sup_{[0,\tau_0]}||w(\cdot,\tau)||_{H^2}\le\mu_2, \tag{3.10}$$

where μ_2 is small.

Step 1: Multiplying both sides of (3.5) by $e^{-N\tau}w$ and integrating over R^1, we obtain after integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\tau}e^{-N\tau}||w(\cdot,\tau)||^2+e^{-N\tau}||w_y(\cdot,\tau)||^2+Ne^{-N\tau}||w(\cdot,\tau)||^2\\&=e^{-N\tau}\int_{R^1}\Big\{-f'(A^\epsilon)ww_y+q(A^\epsilon,d)w-Q(A^\epsilon,w_y)w\Big\}dy.\end{aligned} \tag{3.11}$$

Each term on the right hand side above will be estimated separately. First,

$$\int_{R^1}f'(A^\epsilon)ww_ydy\le\frac{1}{3}||w_y(\cdot,\tau)||^2+c_1||w(\cdot,\tau)||^2. \tag{3.12}$$

Next,

$$\int_{R^1}q(A^\epsilon)wdy\le c||w(\cdot,\tau)||^2+c\epsilon^{-1}||d(\cdot,\epsilon\tau)||^2\le c_2||w(\cdot,\tau)||^2+c\epsilon^{6\gamma+\alpha-2}, \tag{3.13}$$

where we make use of Lemma 2. The third term,

$$\int_{R^1}Q(A^\epsilon,w_y)wdy\le c\int_{R^1}|ww_y^2|dy\le c_3||w||_{L_\infty}||w_y(\cdot,\tau)||^2. \tag{3.14}$$

Now choose N sufficiently large to insure that

$$c_1+c_2<N,\quad\text{and}\quad\frac{1}{3}+c_3\mu_2<\frac{1}{2}. \tag{3.15}$$

Collecting all the above estimates and integrating the resulting inequality with respect to τ, we have

$$e^{-N\tau}||w(\cdot,\tau)||^2+\int_0^\tau e^{-Ns}||w_y(\cdot,s)||^2ds\le c\frac{1}{N}\epsilon^{6\gamma+\alpha-2}. \tag{3.16}$$

Step 2: Now we are to get higher order estimates. Applying ∂_y^l to (3.5) for $l=1,2$, multiplying both sides of the resulting equation by $e^{-N\tau}\partial_y^lw$ and integrating over R^1, we compute that

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\tau}e^{-N\tau}||\partial_y^lw(\cdot,\tau)||^2+e^{-N\tau}||\partial_y^{l+1}w(\cdot,\tau)||^2+Ne^{-N\tau}||\partial_y^lw(\cdot,\tau)||^2\\&=e^{-N\tau}\int\partial_y^{l+1}w\cdot\partial_y^{l-1}\big\{-f'(A^\epsilon)w_y+q(A^\epsilon,d)w-Q(A^\epsilon,w_y)\big\}dy\end{aligned} \tag{3.17}$$

In the case $l = 1$, we have by the Cauchy inequality that the right hand side (3.17) can be estimated as

$$\begin{aligned}\int \partial_y^2 w(\cdot,\tau)[-f'(A^\epsilon)w_y + q(A^\epsilon, d) - Q(A^\epsilon, w_y)]dy \\ \leq \frac{1}{2}||\partial_y^2 w(\cdot,\tau)||^2 + c\int \big(|d|^2 + |w_y|^2 + |w_y|^4\big)dy \\ \leq \frac{1}{2}||\partial_y^2 w(\cdot,\tau)||^2 + c(1 + ||w_y||^2_{L_\infty})||w_y(\cdot,\tau)||^2 + c\epsilon^{6\gamma+\alpha-2},\end{aligned} \tag{3.18}$$

where $||w_y||_{L_\infty}$ is bounded because of (3.10) we use Lemma 2 . It follows that

$$\begin{aligned}\frac{d}{d\tau}e^{-N\tau}||w_y(\cdot,\tau)||^2 + e^{-N\tau}||\partial_y^2 w(\cdot,\tau)||^2 + Ne^{-N\tau}||w_y(\cdot,\tau)||^2 \\ \leq c\big(1+\mu_2^2)e^{-N\tau}||w_y(\cdot,\tau)||^2 + c\epsilon^{6\gamma+\alpha-2}e^{-N\tau}.\end{aligned} \tag{3.19}$$

Integrating this inequality with respect to τ, we obtain, by virtue of (3.16), that

$$e^{-N\tau}||w_y(\cdot,\tau)||^2 + \int_0^\tau e^{-Ns}||\partial_y^2 w(\cdot,s)||^2 ds \leq c\frac{1}{N}\epsilon^{6\gamma+\alpha-2}. \tag{3.20}$$

Similarly, for $l = 2$, we can estimate the right hand side (3.16) as follows

$$\begin{aligned}&\int \partial_y^3 w\cdot\{\big(-f'(A^\epsilon)w_y\big)_y + q(A^\epsilon,d)_y - Q(A^\epsilon,w_y)_y\}dy \\ &\leq \frac{1}{2}||\partial_y^3(\cdot,\tau)||^2 + c\int \big(|w_y|^2 + |\partial_y^2 w|^2 + |w_y|^4 + |w_y|^2|\partial_y^2 w|^2 + |d|^2 + |d_y|^2\big)dy \\ &\leq \frac{1}{2}||\partial_y^3 w||^2 + c(1+||w_y||^2_{L_\infty})(||w_y||^2 + ||\partial_y^2 w||^2) + c\int(|d|^2 + |d_y|^2)dy.\end{aligned}$$

Using Lemma 2 again, one gets

$$\begin{aligned}&\frac{d}{d\tau}e^{-N\tau}||\partial_y^2 w(\cdot,\tau)||^2 + e^{-N\tau}||\partial_y^3 w(\cdot,\tau)||^2 \\ &\leq ce^{-N\tau}\{(1+||w_y(\cdot,\tau)||^2 + ||\partial_y^2 w(\cdot,\tau)||^2) + c(\epsilon^{6\gamma+\alpha-2} + \epsilon^{5\gamma})\},\end{aligned}$$

provided $||w_y||_2$ is bounded. Then it follows from (3.16) and (3.20) that

$$e^{-N\tau}||\partial_y^2 w(\cdot,\tau)||^2 + \int_0^\tau e^{-Ns}||\partial_y^3 w(\cdot,s)||^2 ds \leq c\frac{1}{N}\epsilon^{6\gamma+\alpha-2}. \tag{3.21}$$

Combining (3.16), (3.20) and (3.21), we complete the proof of Lemma 4. □

Remark: The above proof is valid only when τ is very small.

3.2 Proof of Lemma 5

First we diagonalize the system (3.5). We take $\mathbf{L} = (\mathbf{l}_1, \mathbf{l}_2)^t(A^\epsilon)$, where $\mathbf{l}_1(A^\epsilon) = (-\sin\theta, \cos\theta)$, $\mathbf{l}_2(A^\epsilon) = (\cos\theta, \sin\theta)$, $\theta = \theta(A^\epsilon)$, and define

$$\mathbf{L}w := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1(A^\epsilon) & \\ & \lambda_2(A^\epsilon) \end{pmatrix}. \tag{3.22}$$

Then we have

$$\begin{aligned}&(\mathbf{L}w)_\tau - \dot{s}(\mathbf{L}w)_y + \Lambda(\mathbf{L}w)_y - (\mathbf{L}w)_{yy}\\ &= (\mathbf{L}_\tau + \Lambda\mathbf{L}_y - \mathbf{L}_{yy})w - 2\mathbf{L}_y w_y - \dot{s}\mathbf{L}_y w + \mathbf{L}q(A^\epsilon, d) + \mathbf{L}Q(A^\epsilon, w_y).\end{aligned} \tag{3.23}$$

Notice that

$$\begin{array}{l}\mathbf{L}_\tau = \mathbf{L}_\theta\theta_\tau,\\ \mathbf{L}_y = \mathbf{L}_\theta\theta_y,\end{array} \quad \mathbf{L}_\theta = \begin{pmatrix} -\mathbf{l}_2 \\ \mathbf{l}_1 \end{pmatrix}, \quad \mathbf{L}_{\theta\theta} = -\mathbf{L}, \quad \mathbf{L}_{yy} = -\mathbf{L}\theta_y^2 + \mathbf{L}_\theta\theta_{yy}, \tag{3.24}$$

$$\mathbf{L}_y w_y = \mathbf{L}_\theta w_y\theta_y, \quad \text{and} \quad \mathbf{L}_\theta w_y = (\mathbf{L}_\theta w)_y - \mathbf{L}_{\theta\theta} w\theta_y, \tag{3.25}$$

so (3.21) can be rewritten as

$$\begin{aligned}&\begin{pmatrix} w_{1\tau} + (\lambda_1 - \dot{s})w_{1y} - w_{1yy} \\ w_{2\tau} + (\lambda_2 - \dot{s})w_{2y} - w_{2yy} \end{pmatrix}\\ &= \begin{pmatrix} -w_2\theta_\tau + (\dot{s} - \lambda_1)w_2\theta_y + 2w_1\theta_y^2 + w_2\theta_{yy} + 2w_{2y}\theta_y \\ w_1\theta_\tau - (\dot{s} - \lambda_2)w_1\theta_y + 2w_2\theta_y^2 - w_1\theta_{yy} - 2w_{1y}\theta_y \end{pmatrix} + \mathbf{L}\cdot\big(q(A^\epsilon, d) + Q(A^\epsilon, w_y)\big).\end{aligned} \tag{3.26}$$

Step 1: (Basic estimate) Taking the inner product of both sides above with $(w_1, w_2)^t$ and integrating over R^1, we get after integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\tau}||(w_1, w_2)||^2 + ||(w_1, w_2)_y||^2 - \frac{1}{2}\int \big(\partial_y\lambda_1 w_1^2 + \partial_y\lambda_2 w_2^2\big)dy\\ &= \int \big\{(\lambda_2 - \lambda_1)w_1w_2\theta_y + 2(w_1^2 + w_2^2)\theta_y^2 + 2(w_1w_{2y} - w_2w_{1y})\theta_y + w\mathbf{L}q + w\mathbf{L}Q\big\}dy.\end{aligned} \tag{3.27}$$

Since $\partial_r(\lambda_1, \lambda_2) > 0$ by our assumption on $\phi(r)$, and $\partial_y R(H_0) < 0$ by our construction, one gets

$$\partial_y(\lambda_1, \lambda_2)(A^\epsilon) = \partial_r(\lambda_1, \lambda_2)\big(m(\epsilon^{1-\gamma}y)\partial_y R(H_0) + O(1)\epsilon\big). \tag{3.28}$$

Notice that $\partial_y\theta = O(1)\epsilon$ (see (2.2)), so we have

$$\begin{aligned}&\int \big\{(\lambda_2 - \lambda_1)w_1w_2\theta_y + 2(w_1^2 + w_2^2)\theta_y^2 + 2(w_1w_{2y} - w_2w_{1y})\theta_y\big\}dy\\ &\le c\epsilon||(w_1, w_2)||^2 + c\epsilon||(w_1, w_2)_y||^2.\end{aligned}$$

Now

$$\begin{aligned}\int w\mathbf{L}q(A^\epsilon, d)dy &\le \epsilon||(w_1, w_2)||^2 + c\epsilon^{-1}\int d^2(\epsilon y + \delta_0 + \epsilon\delta_1, \epsilon\tau)dy\\ &\le \epsilon||(w_1, w_2)||^2 + c\epsilon^{-2}||d(\cdot, \epsilon\tau)||^2\\ &\le \epsilon||(w_1, w_2)(\cdot, \tau)||^2 + c\epsilon^{6\gamma+\alpha-3}.\end{aligned}$$

Finally,

$$\int w\mathbf{L}Q(A^\epsilon, w_y)dy \leq c\int |w|w_y^2|dy \leq c\epsilon||w_y(\cdot,\tau)||^2$$

which is by the assumption that $||w||_{L_\infty} \leq c\epsilon$. Collecting all the estimates we have obtained thus far, we get

$$\begin{aligned}&\frac{d}{d\tau}||w(\cdot,\tau)||^2 + ||w_y(\cdot,\tau)||^2\\ &\leq -c\int m|\partial_y R(H_0)|w^2dy + c\epsilon||w(\cdot,\tau)||^2 + c\epsilon||w_y(\cdot,\tau)||^2 + c\epsilon^{6\gamma+\alpha-3}.\end{aligned}$$

Choosing ϵ suitably small, we have

$$\frac{d}{d\tau}||w(\cdot,\tau)||^2 + ||w_y(\cdot,\tau)||^2 \leq -c\int m|\partial_y R(H_0)|w^2dy + c\epsilon||w(\cdot,\tau)||^2 + c\epsilon^{6\gamma+\alpha-3}. \tag{3.29}$$

Applying a classical Gronwall-type inequality to (3.29) yields

$$||w(\cdot,\tau)||^2 + \int_0^\tau ||w_y(\cdot,s)||^2ds \leq c\epsilon^{6\gamma+\alpha-4} \quad \text{for all} \quad \tau\in[0,\tau_0]. \tag{3.30}$$

Here we used the fact that

$$\epsilon\int_0^\tau ||w(\cdot,\tau)||^2d\tau \leq c\epsilon^{6\gamma+\alpha-4} \quad \text{for all} \quad \tau\leq\tau_0.$$

Therefore, we have derived the desired L_2 energy estimate on w.

Step 2: To complete the proof of Lemma 5 , we need the higher order L_2 estimates on w. But the procedures are exactly the same as that **Step2** in the proof of Lemma 4. So we omit the proof here. In this section, we prove that

$$||\partial_y w(\cdot,\tau)||_1^2 + \int_0^\tau ||\partial_y w(\cdot,\tau)||_2^2 d\tau \leq c\epsilon^{6\gamma+\alpha-4} \tag{3.31}$$

for $\tau\in[0,\tau]$ and c is independent of τ_0 and ϵ. □

3.3 Proof of Theorem 1

To combine Lemma 4 and Lemma 5, we choose γ, α such that

$$6\gamma+\alpha-4\geq 2,$$

then from Lemma 4, one has Sobolev's inequality that

$$\begin{aligned}||w||_{L_\infty} &\leq ||w||^{1/2}\cdot||w_y||^{1/2}\\ &\leq ce^{-N\tau}\frac{1}{N}\epsilon^{(6\gamma+\alpha-2)/2} \leq c\epsilon^2 \leq c\epsilon \quad \text{for all} \quad \tau\in[0,1/N].\end{aligned}$$

So (3.7) is satisfied. Furthermore, when Lemma 5 is valid, again by Sobolev's inequality, one gets

$$||w||_{L_\infty} \leq ||w||^{1/2} \cdot ||w_y||^{1/2} \leq c\epsilon^{(6\gamma+\alpha-4)/2} \leq c\epsilon,$$

therefore, Lemma 5 can be carried out till T/ϵ. We conclude that

Proposition 2. *There exist positive constants ϵ_0 and c_0, independent of ϵ, such that if $0 < \epsilon \leq \epsilon_0$, then the Cauchy problem (3.5) has a unique smooth solution $w \in C^1\big([0, T/\epsilon]\big) : H^2(R^1)\big)$. And the following inequality holds:*

$$\sup_{[0,T/\epsilon]} ||w(\cdot,\tau)||_2^2 + \int_0^{T/\epsilon} ||w_y(\cdot,\tau)||_2^2 d\tau \leq c_0\epsilon^{6\gamma+\alpha-4}. \tag{3.32}$$

Consequently, we have from $(3.2,(3.4) and (3.32)$ that

$$\sup_{[0,T]} ||(h^\epsilon - A^\epsilon)(\cdot,t)||^2 = \epsilon \sup_{[0,T/\epsilon]} ||w_y(\cdot,\tau)||^2 \leq c_0\epsilon^{6\gamma+\alpha-3} \leq c_0\epsilon^3. \tag{3.33}$$

Next , using Sobolev's inequality, we have

$$\begin{aligned} ||(h^\epsilon - S^\epsilon)(\cdot,t)||_{L^\infty} &= ||w_y(\cdot,\tau)||_{L^\infty} \\ &\leq O(1)||w_y(\cdot,\tau)||^{\frac{1}{2}}||w_{yy}(\cdot,\tau)||^{\frac{1}{2}} \\ &\leq C\epsilon^{(6\gamma+\alpha-4)/2} \leq C\epsilon, \end{aligned}$$

which together with (2.36) gives (1.12).

Finally , we are to get the L^1 estimate. For this aim, we need several lemmas.

Lemma 6. *For the given single-shock solution $h(x,t) = (u,v)(x,t)$ to (1.1), we have*

$$\textstyle\int_0^T \fint_{R^1} + |(u,v)_{xx}(x,t)| dxdt \leq c, \tag{3.34}$$

for some constant c.

Proof: For simplicity of presentation, we take the polar coordinates (r,θ). Then the behavior of $h(x,t)$ can be described by

$$\begin{aligned} &r_t + \big(r\phi(r)\big)_x = 0, \\ &\theta_t + \phi(r)\theta_x = 0. \end{aligned} \tag{3.35}$$

Denote by $(r_0,\theta_0)(x)$ the initial value. By definition, there is no discontinuity on θ, no spontaneous shock on r, and only one initial shock on r for each t , $0 \leq t \leq T$. Therefore, for each (x,t) with $x \neq s(t)$, we can trace two characteristic lines backward to $t = 0$:

$$x(t) = \xi_1 + \lambda_2(r_0(\xi_1))t, \quad x(t) = \xi_2 + \int_0^t \phi\big(r(x(s))\big)ds, \tag{3.36}$$

where $\lambda_2 = \big(r\phi(r)\big)_r$ and

$$r_0(\xi_1) = r(x), \quad \theta_0(\xi_2) = \theta(x). \tag{3.37}$$

Differentiating the two equations in (3.36) with respect to ξ_1 and ξ_2 respectively , one has

$$\tfrac{\partial x}{\partial \xi_1} = 1 + \tfrac{d}{d\xi_1}\lambda_2(r_0(\xi_1))t, \quad \tfrac{\partial x}{\partial \xi_2} = 1 + \int_0^t \tfrac{\partial}{\partial \xi_2}\phi\big(r(x(s))\big)ds. \tag{3.38}$$

Since there is no other discontinuity,

$$0 < c_1 < 1 + \frac{d}{d\xi_1}\lambda_2(r_0(\xi_1))t < c_2, \quad 0 < c_3 < 1 + \int_0^t \frac{\partial}{\partial \xi_2}\phi\big(r(x(s))\big)ds < c_4. \tag{3.39}$$

Differentiating once again the two equations in (3.38) with respect to ξ_1 and ξ_2 respectively, we have

$$\tfrac{\partial^2 x}{\partial \xi_1^2} = \tfrac{d^2}{d\xi_1^2}\lambda_2(r_0(\xi_1))t, \quad \tfrac{\partial^2 x}{\partial \xi_2^2} = \int_0^t \big\{\tfrac{\partial^2 \phi}{\partial x^2}\big(\tfrac{\partial x}{\partial \xi_2}\big)^2 + \tfrac{\partial \phi}{\partial x}\tfrac{\partial x^2}{\partial \xi_2^2}\big\}ds \tag{3.40}$$

By the definition of single shock solution, ϕ_x andϕ_{xx} are bounded. This together with (3.39) yields that

$$|\tfrac{\partial^2 x}{\partial \xi_1^2}| \le c, \quad |\tfrac{\partial^2 x}{\partial \xi_2^2}| \le c, \tag{3.41}$$

for some constant c.

Differentiating both equations of (3.37) once with respect to ξ_1, ξ_2 respectively, we have

$$r_{0\xi_1} = r_x \frac{\partial x}{\partial \xi_1}, \quad \theta_{0\xi_2} = \theta_x \frac{\partial x}{\partial \xi_2}. \tag{3.42}$$

Continuing, differentiate the above two equations once again with respect to ξ_1, ξ_2, respectively, to give

$$r_{xx}\frac{\partial x}{\partial \xi_1} = r_{0\xi_1\xi_1}\big(\frac{\partial x}{\partial \xi_1}\big)^{-1} - r_{0\xi_1}\frac{\partial^2 x}{\partial \xi_1^2}\big(\frac{\partial x}{\partial \xi_1}\big)^{-2}, \tag{3.43}$$

$$\theta_{xx}\frac{\partial x}{\partial \xi_2} = \theta_{0\xi_2\xi_2}\big(\frac{\partial x}{\partial \xi_2}\big)^{-1} - \theta_{0\xi_2}\frac{\partial^2 x}{\partial \xi_2^2}\big(\frac{\partial x}{\partial \xi_2}\big)^{-2}. \tag{3.44}$$

Thus

$$\begin{aligned}\int_{R^1}|r_{xx}(x,t)|dx &= \int_{R^1}|r_{xx}\tfrac{\partial x}{\partial \xi_1}|d\xi_1 \\ &\le \int_{R^1}|r_{0\xi_1\xi_1}|\big(\tfrac{\partial x}{\partial \xi_1}\big)^{-1}d\xi_1 + \int_{R^1}|r_{0\xi_1}\tfrac{\partial^2 x}{\partial \xi_1^2}|\big(\tfrac{\partial x}{\partial \xi_1}\big)^{-2}d\xi_1 \\ &\le c,\end{aligned} \tag{3.45}$$

$$\begin{aligned}\int_{R^1}|\theta_{xx}(x,t)|dx &= \int_{R^1}|\theta_{xx}\tfrac{\partial x}{\partial \xi_2}|d\xi_2 \\ &\le \int_{R^1}|\theta_{0\xi_2\xi_2}|\big(\tfrac{\partial x}{\partial \xi_2}\big)^{-1}d\xi_2 + \int_{R^1}|\theta_{0\xi_2}\tfrac{\partial^2 x}{\partial \xi_2^2}|\big(\tfrac{\partial x}{\partial \xi_2}\big)^{-2}d\xi_2 \\ &\le c,\end{aligned} \tag{3.46}$$

where we have made use of the assumption (1.9). Hence

$$\int_0^T \int_{R^1} |(r_{xx}, \theta_{xx})(x,t)| dxdt \le c, \tag{3.47}$$

which is (3.34). $\square$

In the polar coordinates, equations (1.2) can be written as

$$r_t + \big(r\phi(r)\big)_x = \epsilon r_{xx} - \epsilon r \theta_x^2, \tag{3.48}$$

$$\theta_t + \phi(r)\theta_x = \epsilon\theta_{xx} - 2\epsilon \frac{1}{r} r_x \theta_x. \tag{3.49}$$

To derive the L^1 estimate, we construct another approximate solution $h^a(x,t)$ as

$$h^a(x,t) = h(x,t) + H_0(y; h_-, t) - J(y; h_-, h_+) \tag{3.50}$$

where $y = \frac{x - s(t)}{\epsilon}$ and J is the so-called Heaviside function defined by

$$J(y; h_-0, h_+) = \begin{cases} h_+ & \text{if} \quad y \ge 0, \\ h_- & \text{if} \quad y < 0. \end{cases} \tag{3.51}$$

. Then $h^a(x,t)$ is continuous . Define

$$\tilde{r}(x,t) = r^\epsilon(x,t) - r^a(x,t) = r(h^\epsilon) - r(h^a), \tag{3.52}$$

$$\tilde{\theta}(x,t) = \theta^\epsilon(x,t) - \theta^a(x,t) = \theta(h^\epsilon) - \theta(h^a), \tag{3.53}$$

then for $x \ne s(t)$, the error equations are

$$\tilde{r}_t + \big(r^\epsilon\phi(r^\epsilon) - r\phi(r)\big)_x = \epsilon\tilde{r}_{xx} + \epsilon r_{xx} - \epsilon r^\epsilon \theta_x^{\epsilon 2}, \tag{3.54}$$

$$\tilde{\theta}_t + \phi(r^\epsilon)\tilde{\theta}_x + \big(\phi(r^\epsilon) - \phi(r)\big)\theta_x = \epsilon\tilde{\theta}_{xx} + \epsilon\theta_{xx} - 2\epsilon\frac{1}{r^\epsilon} r_x^\epsilon \theta_x^\epsilon. \tag{3.55}$$

Lemma 7.

$$\sup_{[0,T]} ||(h^a - A^\epsilon(\cdot,t)||^2 \le c\epsilon, \tag{3.56}$$

for some constant c.

Proof:

$$h^a(x,t) - A^\epsilon(x,t) = (H_0 - J) + m(H_0 - h) - \epsilon(mH_1 + (1-m)h_1) \tag{3.57}$$

$$+ \epsilon^2(mH_2 + (1-m)h_2) - d \tag{3.58}$$

Note that in our case, the viscous travelling wave of the second family is actually like that in the scalar conservation laws with convex flux function. So we can apply the result obtained in [10], that

$$|H_0(y; h_-, h_+) - J(y; h_-, h_+)| \le (r_- - r_+)e^{-\alpha(r_- - r_+)|y|/2}. \tag{3.59}$$

where $\alpha = \min\{(r\phi(r))_{rr}\} > 0$. Thus

$$\int |H_0(y; h_-, t) - J(y; h_-, h_+)|^2 dx \le c\epsilon.$$

On the other hand, since $h(x,t)$ is left and right continuous at $x = s(t)$ for each $t \le T$, $\exists\ \epsilon_3$ such that if $\epsilon \le \epsilon_3$, then

$$|h(x,t) - h_-|, |h(x,t) - h_+| \le \epsilon^{1-\gamma}/2, \quad \text{for all} \quad |x - s(t)| \le 2\epsilon^{\gamma}. \tag{3.60}$$

Consequently, we have

$$\int |m(H_0(y,t) - h(x,t))|^2 dx \tag{3.61}$$

$$= \int m\big|\big(H_0(y,t) - J(y; h_-, h_+)\big) + \big(J(y; h_-, h_+) - h(x,t)\big)\big|^2 dx \le c\epsilon. \tag{3.62}$$

Then by the estimates on the functions H_1, H_2, h_1 h_2, d and (3.57), the inequality (3.56) follows. □

Lemma 8.

$$\sup_{[0,T]} \int |h^{\epsilon}(x,t) - h^{a}(x,t)| dx \le c\epsilon. \tag{3.63}$$

Proof: Multiplying both sides of (3.54) with $sign(\tilde{r})$ and integrating it over R^1, we obtain after integrating by parts that

$$\frac{d}{dt}\int |\tilde{r}| dx = \epsilon\big(a_j \tilde{r}_x(p_{j+1} - 0, t) - a_j \tilde{r}_x(p_j + 0, t)\big) \tag{3.64}$$

$$+ \epsilon \int r^{\epsilon} \theta_x^{\epsilon 2} dx + \epsilon \int |r_{xx}| dx, \tag{3.65}$$

where a_j is the sign of $\tilde{r}_x$ in (p_j, p_{j+1}). Since $\tilde{r}(p_j, t) = \tilde{r}(p_{j+1}, t) = 0$ and $a_j \tilde{r} \ge 0$ for $x \in (p_j, p_{j+1})$, we have $a_j \tilde{r}_x(p_j + 0, t) \ge 0$ and $a_j \tilde{r}_x(p_{j+1} - 0, t) \le 0$. By (2.37) , (3.32) and choosing ϵ small enough, $r^{\epsilon} > r^* > 0$ for some constant r^*. Then $\int_0^T \int \theta_x^2 dx dt \le c$. Using Lemma 6, we integrate (3.64) over $[0,T]$ to get

$$\sup_{[0,T]} \int |\tilde{r}(x,t)| dx \le c\epsilon. \tag{3.66}$$

Continuing, we multiply both sides of (3.55) with $sign(\tilde{\theta})$ and integrate the resultant equation over R^1 to get , after integrating by parts , that

$$\frac{d}{dt}\int |\tilde{\theta}(x,t)| dx \tag{3.67}$$

$$\le \int \phi(r^{\epsilon})_x |\tilde{\theta}| dx + c\int |\tilde{r}| \cdot |\theta_x| dx + \epsilon \int |\theta_{xx}| dx + \epsilon c \int (r_x^{\epsilon 2} + \theta_x^{\epsilon 2}) dx. \tag{3.68}$$

Now the first term on the right of (3.67) is estimated as

$$\begin{aligned}
&\int \phi(r^\epsilon)_x|\tilde\theta|dx\\
&\le \int \phi(r^\epsilon)-\phi(R)\big)_x|\theta^\epsilon-\theta(A^\epsilon)+\theta(A^\epsilon)-\theta(h^a)|dx+\int\phi(R)_x|\tilde\theta|dx\\
&\le \int \big(\phi(r^\epsilon)-\phi(R)\big)_x|\theta(A^\epsilon)-\theta(h^a)|dx+\int\phi'(R)\big(\tfrac{1}{\epsilon}mR(H_0)_y+c\big)|\tilde\theta|dx+c||\tilde w(\cdot,t)||_1^2\\
&\le c\int|\tilde\theta(x,t)|dx+c||\tilde w(\cdot,t)||_1^2+c||A^\epsilon-h^a||^2
\end{aligned}$$

where $R=r(A^\epsilon)$, $\tilde w=h^\epsilon-A^\epsilon$. Note that $\theta(h^a)_x$ is bounded. By (3.66),t he second term on the right is estimated as

$$\int(\phi(r^\epsilon)-\phi(r^a))\theta_x dx\le c\epsilon.$$

Collecting all the above estimates and using Lemma 6, one has

$$\frac{d}{dt}\int|\tilde\theta(x,t)|dx\le c\int|\tilde\theta(x,t)|dx+c\big(||\tilde w(\cdot,t)||_1^2\big)+c\epsilon \tag{3.69}$$

Applying Gronwall-type inequality, one obtains

$$\sup_{[0,T]}\int|\tilde\theta(x,t)|dx\le c\epsilon. \tag{3.70}$$

□

Finally, by (3.59), one easily has that

$$\int|h(x,t)-h^a(x,t)|dx\le c\epsilon. \tag{3.71}$$

This together with Lemmas 7 and 8, leads to (1.13). This completes the proof.

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References

[1] J. Goodman and Z.P. Xin, *Viscous limits for piecewise smooth solutions to systems of conservation laws* , Arch. Rational Mech. Anal. **121** (1992), pp.235–265.

[2] Temple, B. *Degenerate systems of conservation laws*, Cont. Math. **60**, (1987), pp.125-133.

[3] Keyfith, B. and Kranzer.*A system of nonstrictly hyperbolic conservation laws arising in elasticity theory*, Arch. Rat. Mech.Anal. **72**, (1980), pp.219-241.

[4] Pope, G.A., *Teh application of fractional flow theroty to enhanced oil recovery*, Soc. of Petrol.Eng. J.,**20**(1980), p. 191 .

[5] Chen, G.Q., *Hyperbolic systemso f conservaton laws with a symmetry*, Comm.Partial Diff. Eqs., **16**(1990),pp. 1461-1487.

[6] Chen, G.Q., *Propogation and cancellation of osillations for hyperbolic systems of conservation laws*, Comm. Pure Appl. Math. **44**(1991),pp. 121-139.

[7] Chen, g. Q., *The method of quasidecoupling for discontinuous solutions to conservation laws*, Arch. Ration Mech. Anal. **1221**(1991), pp.131-185.

[8] H.Kreiss, *Initial boundary value problems for hyperbolic systems*, Comm.Pure Appl.Math., **23**(1970),pp.277-295.

[9] J.Rauch, L_2 is continuable initial condition for Kreiss's mixed problem, Comm.Pure Appl.Math.,**25**(1972),pp.265-285.

[10] T. Tang and Z.H. Teng, *Viscosity methods for pieces smooth solutions to scalar conservation laws*, Math. Comp.,**66**(1997), pp. 495-526.