Quasineutral Limit of the Drift Diffusion Models for Semiconductors: The Case of General Sign-changing Doping Profile

Shu Wang [1]
College of Science, Beijing University of Technology
PingLeYuan100, ChaoyangDistrict, Beijing100022, P. R. China

Zhouping Xin[2]
The IMS and Department of Mathematics,
The Chinese University of Hong Kong, Shatin, N. T., Hong Kong

Peter A. Markowich[3]
Institut für Mathematik, Universität Wien,
Boltzmanngasse 9, A-1090 Wien, Austria

Abstract: In this paper the vanishing Debye length limit (space charge neutral limit) of bipolar time-dependent drift-diffusion models for semiconductors with p-n-junctions (i.e. with a fixed bipolar background charge) is studied in one space-dimension. For general sign-changing doping profiles, the quasineutral limit (zero-Debye-length limit) is justified rigorously in the spatial mean square norm uniformly in time. One main ingredient of our analysis is to construct a more accurate approximate solution, which takes into account of the effects of initial and boundary layers, by using multiple scaling matched asymptotic analysis. Another key point of the proof is to establish the structural stability of this approximate solution by an elaborate energy method which yields the uniform estimates with respect to the scaled Debye length.

Keywords: Quasineutral limit, drift-diffusion equations, multiple scaling asymptotic expansions, singular perturbation, classical energy methods, λ-weighted Liapunov-type functional

1991 AMS Classifications: 35B25, 35B40, 35K57

[1]Supported by the National Youth Natural Science Foundation (grant 10001034) of China, the FWF-Projekt "14876-MAT" "Fokker-Planck und Mittlere-Feld-Gleichungen" (Austria) , the Wittgenstein-Award 2000 of Peter A. Markowich, funded by the FWF(Austria), the Zheng Ge Ru Foundation, and Grants from RGC of HKSAR CUHK4040/02P.

[2]Supported by in parts by Grants from RGC of HKSAR CUHK4279/00P and CUHK4040/02P and Zheng Ge Ru Foundation

[3]Supported by the EU-funded network HYKE and the Wittgenstein-Award 2000 of Peter A. Markowich, funded by the Austrian Science Fund FWF.

Contents

1 Introduction

The scaled one-dimensional isothermal drift-diffusion model for semiconductors reads

$$n^\lambda_t = (n^\lambda_x - n^\lambda \Phi^\lambda_x)_x, 0 < x < 1, t > 0, \tag{1}$$
$$p^\lambda_t = (p^\lambda_x + p^\lambda \Phi^\lambda_x)_x, 0 < x < 1, t > 0, \tag{2}$$
$$\lambda^2 \Phi^\lambda_{xx} = n^\lambda - p^\lambda - D, 0 < x < 1, t > 0, \tag{3}$$
$$n^\lambda_x - n^\lambda \Phi^\lambda_x = p^\lambda_x + p^\lambda \Phi^\lambda_x = \Phi^\lambda_x = 0, x = 0, 1, \quad t > 0, \tag{4}$$
$$n^\lambda(t = 0) = n^\lambda_0, p^\lambda(t = 0) = p^\lambda_0, 0 \le x \le 1. \tag{5}$$

The variables $n^\lambda, p^\lambda, \Phi^\lambda$ are the electron density, the hole density and the electric potential, respectively. The constant λ is the scaled Debye length of the semiconductor device under consideration. $D = D(x)$ is the given function of space and models the doping profile(i.e., the preconcentration of electrons and holes). Because of the occurrence of p-n-junctions in realistic semiconductor devices, the doping profile $D(x)$ typically changes its sign.

In this paper, we assume that $D(x)$ is a smooth function.

Note that for the sake of simplicity we take insulating boundary conditions modelled by outward electric field and current density components.

A necessary solvability condition for the Poisson equation (3) subject to the Neumann boundary condition for the field in $(4)_3$ is global space charge neutrality,

$$\int_0^1 (n^\lambda - p^\lambda - D)dx = 0.$$

Since the total numbers of electrons and holes are conserved, it is sufficient to require the corresponding condition for the initial data:

$$\int_0^1 (n^\lambda_0 - p^\lambda_0 - D)dx = 0. \tag{6}$$

Usually semiconductor physics are concerned with large scale structures with respect to the Debye length λ (λ takes small values, typically $\lambda^2 \approx 10^{-7}$). For such scales, the semiconductor is almost electrically neutral, i.e. there is no space charge separation or electric field. This is so called quasineutrality assumption of semiconductors or plasma physics, which had been applied by W. Shockley [31] in the first theoretical studies of semiconductor devices in 1949, but also in other contexts such as the modelling of plasmas [32] and ionic membranes [29]. Under the assumption of space charge neutrality, i.e. $\lambda = 0$, we formally arrive at the following quasineutral drift-diffusion model

$$
\begin{aligned}
n_t &= (n_x + n\mathcal{E})_x, && (7)\\
p_t &= (p_x - p\mathcal{E})_x, && (8)\\
0 &= n - p - D, && (9)\\
\mathcal{E} &= -\Phi_x.
\end{aligned}
$$

This formal limit was obtained by Roosbroeck [28] in 1950. For further formal asymptotic analysis, see [25, 27, 20].

Generally speaking, it should be expected at least formally that $(n^\lambda, p^\lambda, -\Phi_x^\lambda) \to (n, p, \mathcal{E})$ as $\lambda \to 0$ in the interior of the interval $[0, 1]$ while it cannot be a priorily expected that all boundary and initial value conditions are maintained for the limit problem because of the singular perturbation character of the problem (the Poisson equation becomes an algebraic equation in the limit). However, by the conservation form of the continuity equations the property of zero fluxes through the boundary will prevail in the limit:

$$
(n_x + n\mathcal{E})(x = 0, 1) = 0, \qquad (p_x - p\mathcal{E})(x = 0, 1) = 0 \tag{10}
$$

while the boundary condition for the electric field E^λ does not.

Similarly, we can a priorily expect that quasineutral drift-diffusion models (7)-(9) is supplemented by the following initial data:

$$
n(t = 0) = n_0, p(t = 0) = p_0 \tag{11}
$$

satisfying locally initial time space charge neutrality

$$
n_0 - p_0 - D = 0. \tag{12}
$$

The aim of this paper is to justify rigorously the above formal limit for $O(1)-$time and sufficiently smooth solutions.

It is important to mention that the quasineutral limit is a well-known challenging and physically very complex modelling problem for (bipolar) fluid dynamic models and for kinetic models of semiconductors and plasmas. In both cases there exist only partial results. In particular, for time-dependent transport models, the limit $\lambda \to 0$ has been performed for the Vlasov-Poisson system by Brenier [2], Grenier [12, 13] and Masmoudi [21], for the Schrödinger-Poisson system by Puel [26] and Jüngel and Wang [16], for the drift-diffusion-Poisson system by Gasser et al [10, 11], Jüngel and Peng [15] and Schmeiser and Wang [30] under much more restrictive assumptions on the doping profile that used in this paper (no sign-changes of D are allowed), and for the Euler-Poisson system by Cordier and Grenier [5, 6], Cordier et al [4] and Wang [35], respectively. However, as already mentioned, all these results are restricted to *the special cases of doping profiles*, i.e., *either assuming that $D(x)$ is constant zero, or assuming that $D(x)$ does not change its sign.* But, p-n junctions are of great importance both in modern electronic applications and in understanding semiconductor devices since the p-n junction theory serves as foundation of the physics of semiconductor devices(see Sze[34]). For physically interesting doping profiles with p-n junctions, i.e., for the case where the doping profile can change its sign, there is no rigorous result available for time-dependent semiconductor models both for fluid dynamic models and for kinetic models up to now. Therefore, it is natural to study the quasineutral limit on the level of the drift-diffusion-Poisson models first. For stationary drift-diffusion-Poisson models, rigorous convergence results for p-n junction devices with contacts can be found in Markowich [19] and recent extensions were done by Caffarelli et al [3] and Dolbeault et al [7].

In this paper we consider the quasineutral limit of the time-dependent drift-diffusion model (1)-(6) for semiconductors with p-n junctions in the general case of physically relevant sign-changing doping profiles.

Our main result can be summarized as follows: The convergence of the drift diffusion models (1)-(5) to (7)-(11) is rigorously proven for general sign-changing and smooth doping profiles in one-space dimension case on time intervals, on which a smooth nonvacuum solution of the reduced problem (7)-(11) exists(The precise statement will be given in section 2).

We mention that one of the main difficulties in dealing with quasineutral limits is the oscillatory behavior of the electric field. Usually it is difficult to obtain uniform estimates on the electric field with respect to the Debye length λ due to a possible vacuum set of the density, in particular, the occurrence of the depletion region.

To overcome this difficulty, our main strategy is to construct a better uniformly valid approximation solution to (1)-(5) by using the matched asymptotic analysis methods, and then to show the asymptotic structural stability of the resulting approximate solution by energy methods. This approach is strongly motivated by the analysis of boundary layers in the fluid-dynamic limit of a nonlinear Boltzmann equation by Liu and Xin in [18] and viscous boundary layers by Xin in [36]. This ansatz deviates from the solutions to (7)-(11) slightly in a region away from the parabolic boundaries, as changes more rapidly in the electronic field in the parabolic boundaries (boundary layers and initial layers). Due to the special structure of the quasineutral drift-diffusion model (7)-(11), and the more or less explicit forms of the boundary layer functions and initial layer functions, the asymptotic ansatz can be estimated rather easily. Thus, the quasineutral limit problem is reduced to show the scaling structure stability of such as ansatz. To this end, one control the deviation of the solution to (1)-(5) from the ansatz by introducing the following two $\lambda-$weighted Liapunov-type functionals

$$\Gamma^\lambda(t) = \int_0^1 \left(|z_R^\lambda|^2 + |z_{R,x}^\lambda|^2 + |z_{R,t}^\lambda|^2 + \lambda^2(|E_R^\lambda|^2 + |E_{R,x}^\lambda|^2 + |E_{R,t}^\lambda|^2) + |E_R^\lambda|^2\right)dx$$

and

$$G^\lambda(t) = \int_0^1 \left(|z_{R,x}^\lambda|^2 + |z_{R,xt}^\lambda|^2 + |E_R^\lambda|^2 + |E_{R,t}^\lambda|^2 + \lambda^2(|E_{R,x}^\lambda|^2 + |E_{R,xt}^\lambda|^2)\right)dx,$$

where $z_R^\lambda = n_R^\lambda + p_R^\lambda$, $E_R^\lambda = -\Phi_{R,x}^\lambda$, and $(n_R^\lambda, p_R^\lambda, \Phi_R^\lambda)^T$ denotes the difference between the solution to (1)-(5) from the ansatz, see section 2 for details. By a careful energy method, we are able to prove the following entropy production integration inequality:

$$\begin{aligned} & \Gamma^\lambda(t) + \int_0^t G^\lambda(s)ds \\ \leq\ & M\Gamma^\lambda(t=0) + M\int_0^t (\Gamma^\lambda(s) + (\Gamma^\lambda(s))^\iota)ds \\ & +M\int_0^t \Gamma^\lambda(s)G^\lambda(s)ds + M\lambda^q, t \geq 0 \end{aligned}$$

for some $\iota > 1, q > 0$ and $M > 0$, independent of λ, which implies the desired convergence result of this paper.

Finally, we also mention that for drift-diffusion models there are many results on existence, uniqueness, large time asymptotic behavior, stability of stationary states and regularity of weak solutions etc., for example, see, [1, 8, 9, 14, 20, 22, 23, 24].

The plan of this paper is as follows. In Section 2 we reformulate our problem and state the main results of this paper. In Section 3 we give the formal asymptotic expansion. In section 4 we discuss the existence and regularity of solutions of quasineutral drift-diffusion models. In Section 5 we discuss the properties of the initial layer and boundary layer functions. Sections 6 is devoted to the energy estimates for the main theorems of this paper.

2 Reformulation of the Equations and Main Results

Introduce the new variables (z^λ, E^λ) by the following transformation

$$E^\lambda = -\Phi_x^\lambda, n^\lambda = \frac{z^\lambda + D - \lambda^2 E_x^\lambda}{2}, p^\lambda = \frac{z^\lambda - D + \lambda^2 E_x^\lambda}{2}(z^\lambda = n^\lambda + p^\lambda), \tag{13}$$

we can reduce the initial boundary value problem (1)-(6) to the following equivalent system

$$z_t^\lambda = (z_x^\lambda + DE^\lambda)_x - \lambda^2 (E^\lambda E_x^\lambda)_x, 0 \le x \le 1, t > 0, \tag{14}$$
$$\lambda^2 (E_t^\lambda - E_{xx}^\lambda) = -(D_x + z^\lambda E^\lambda), 0 \le x \le 1, t > 0, \tag{15}$$
$$z_x^\lambda = E^\lambda = 0, x = 0, 1, \quad t > 0, \tag{16}$$
$$z^\lambda (t = 0) = z_0^\lambda, E^\lambda (t = 0) = E_0^\lambda, 0 \le x \le 1. \tag{17}$$

Note that the equivalence between system (1)-(6)and system (14)-(17) is easy to be verified for classical solutions. Thus, we have

Proposition 1 *(***Existence and Uniqueness***) Assume that $(z_0^\lambda, E_0^\lambda) \in (C^2)^2$ satisfies the compatibility conditions*

$$z_{0,x}^\lambda = E_0^\lambda = 0, \quad -\lambda^2 E_{0,xx}^\lambda = -D_x, \ \text{at } x = 0, 1. \tag{18}$$

Then system (14)-(17) has a unique, global and classical solution $(z^\lambda, E^\lambda) \in C^{2,1}([0,1] \times [0, \infty))$.

Remark 1 The existence in Proposition 1 is obtained by the known existence results for (1)-(6), see, for example, [24, 9], and by the transformation (13) while uniqueness in Proposition 1 can be proved easily for H^1-solutions of (14)-(17).

Let us assume that the initial datum $(z_0^\lambda, E_0^\lambda)$ is taken to guarantee that boundary-initial consistency for the initial boundary value problem (14)-(17) for $\lambda > 0$ holds. In particular, the compatibility condition (18) is assumed and the initial datum $(z_0^\lambda, E_0^\lambda)$ is assumed to have an expansion of the form

$$\begin{aligned} (z_0^\lambda, E_0^\lambda)^T \;\; = \;\; & \Big(z_0^0(x) + \lambda \big(f(x) z_+^1 (\frac{x}{\lambda}) + g(x) z_-^1 (\frac{1-x}{\lambda}) \big) + \lambda z_{0R}^\lambda (x), \\ & E_0^0(x) + f(x) E_+^0 (\frac{x}{\lambda}) + g(x) E_-^0 (\frac{1-x}{\lambda}) + \lambda E_{0R}^\lambda (x) \Big)^T. \end{aligned} \tag{19}$$

To justify the rigorous quasineutral assumptions, we make the following "ansatz" for the approximate solution:

$$\begin{aligned} (z^\lambda, E^\lambda)_{app}^T \;\; = \;\; & \Big(\mathcal{Z}^0(x,t) + \sum_{i=0}^{2} \lambda^i \big(f(x) z_+^i (\xi, t) + g(x) z_-^i (\eta, t) + z_I^i (x, s) \big), \\ & \mathcal{E}^0(x,t) + f(x) E_+^0 (\xi, t) + g(x) E_-^0 (\eta, t) + E_I^0 (x, s) \Big)^T, \end{aligned} \tag{20}$$

where the inner function $(\mathcal{Z}^0, \mathcal{E}^0)^T$ is independent of λ; $z_+^i, E_+^0, z_-^i, E_-^0, i = 0, 1, 2$, are the left boundary layer functions near $x = 0$ and the right boundary layer functions near $x = 1$, respectively, and $z_I^i, i = 0, 1, 2, E_I^0$, are the initial time layer functions near $t = 0$. The cut-off functions $f(x)$ and $g(x)$ are smooth C^2 functions satisfying $f(0) = g(1) = 1$ and $f(1) = f'(1) = f''(1) = f'(0) = f''(0) = g(0) = g'(0) = g''(0) = g'(1) = g''(1) = 0$. Here we set $\xi = \frac{x}{\lambda}, \eta = \frac{1-x}{\lambda}$ and $s = \frac{t}{\lambda^2}$, corresponding physically to the dielectric relaxation time scale, and $(\cdot, \cdot)^T$ represents transposition. We will discuss in detail the construction of the inner, boundary layer and initial layer functions in the next section, however, we summarize the results here.

First, the inner function $(\mathcal{Z}^0, \mathcal{E}^0)^T$ is determined as a solution of the following initial boundary value problems for the transformed quasineutral drift diffusion equations:

$$\mathcal{Z}_t^0 = (\mathcal{Z}_x^0 + D\mathcal{E}^0)_x, 0 < x < 1, t > 0, \tag{21}$$

$$0 = -(D_x + \mathcal{Z}^0\mathcal{E}^0), 0 < x < 1, t > 0, \tag{22}$$

$$(\mathcal{Z}_x^0 + D\mathcal{E}^0)(x = 0, 1; t) = 0, t > 0, \tag{23}$$

$$\mathcal{Z}^0(t = 0) = z_0^0(x), 0 \leq x \leq 1. \tag{24}$$

The existence of the above inner problem is guaranteed by the following proposition:

Proposition 2 *Assume that $D \in C^{2(l+1)+1}$ and that $z_0^0 \in C^{2(l+1)}$ for some integer $l \geq 0$. Also assume that $z_0^0 \geq \delta_0 > 0$ satisfy the compatibility condition of order l for (21)-(24). Then there exist a $T_0 \in (0, +\infty]$ and a unique classical solution $(\mathcal{Z}^0, \mathcal{E}^0)$, well-defined on $[0, 1] \times [0, T_0]$, of (21)-(24) satisfying $\mathcal{Z}^0, \mathcal{E}^0 \in C^{2(l+1),l+1}([0, 1] \times [0, T_0])$ and $\mathcal{Z}^0(x, t) \geq \delta_1 > 0$ on $[0, 1] \times [0, T_0]$ for some positive constant δ_1. In particular, if $D \in C^\infty([0, 1])$ and $z_0^0 \in C^\infty([0, 1])$ satisfying the compatibility condition of any order, then $\mathcal{Z}^0, \mathcal{E}^0 \in C^\infty([0, 1] \times [0, T_0])$.*

Moreover, if δ_0 is suitably large, then $T_0 = \infty$.

Remark 2 By the transformation

$$n(x, t) = \frac{\mathcal{Z}^0(x, t) + D(x)}{2}, \quad p(x, t) = \frac{\mathcal{Z}^0(x, t) - D(x)}{2}, \quad \mathcal{E}(x, t) = \mathcal{E}^0(x, t)$$

it is easy to verify that the system (7)-(12) and the system (21)-(24) are equivalent. Thus, by Proposition 2, one obtains the existence of the classical non-vacuum solution of the quasineutral drift diffusion system (7)-(12). The uniform positivity of $z_0^0(x)$, together with (12), excludes singularities of the solution of the quasineutral drift-diffusion system (7)-(12). Indeed, if $z_0^0(x) = D(x)$, then (21)-(24) has a stationary solution

$$\mathcal{Z}^0(x,t) = D(x), \mathcal{E}^0 = -(lnD(x))_x.$$

In this case, the electric field $\mathcal{E}^0$ has a singularity in the vacuum set of the density $\mathcal{Z}^0$. In the present paper, thus the case of singular solutions of the quasineutral drift diffusion models is not allowed due to our assumption that $z_0^0 \geq \delta_0 > 0$. But the singular solution case is interesting and will be investigated in the future.

Remark 3 Proposition 2 can not hold true in the unipolar case. In fact, in the unipolar case, we must have $z_0^0(x) = D(x)$ or $z_0^0(x) = -D(x)$ due to the local quasineutrality assumption (12) of the initial data, which excludes the uniform positivity of $z_0^0(x)$ if the doping profile $D(x)$ has the zero roots. This comes back to the above vacuum singular solution case.

Next, the boundary layer functions $z_B^i, E_B^0, B = +/-, i = 0,1,2$, are governed by the following boundary value problems for the elliptic equations:

$$-E^0_{+,\xi\xi} = J^0_+, \quad -E^0_{-,\eta\eta} = J^0_-, 0 < \xi, \eta < \infty, t > 0, \tag{25}$$
$$E^0_+(\xi = 0, t) = -\mathcal{E}^0(x = 0, t), E^0_-(\eta = 0, t) = -\mathcal{E}^0(x = 1, t), t > 0, \tag{26}$$
$$E^0_+(\xi \to \infty, t) = E^0_-(\eta \to \infty, t) = 0, t > 0 \tag{27}$$

and

$$z^0_+ = z^0_- = z^2_+ = z^2_+ = 0, 0 < \xi, \eta < \infty, t > 0, \tag{28}$$
$$z^1_{+,\xi} + D(0)E^0_+ = 0, 0 < \xi, \eta < \infty, t > 0, \tag{29}$$
$$-z^1_{-,\eta} + D(1)E^0_- = 0, 0 < \xi, \eta < \infty, t > 0, \tag{30}$$
$$z^0_+(\xi \to \infty, t) = z^0_-(\eta \to \infty, t) = 0, t > 0, \tag{31}$$

where

$$J^0_+ = -\mathcal{Z}^0(0,t)E^0_+, J^0_- = -\mathcal{Z}^0(1,t)E^0_-. \tag{32}$$

Finally, the initial layer functions $z_I^i, i = 0,1,2, E_I^0$ are given by the following equations (IVPs)

$$E^0_{I,s} = J^0_I, s > 0, 0 < x < 1, \tag{33}$$
$$E^0_I(s = 0) = E^0_0(x) - \mathcal{E}^0(t = 0), 0 < x < 1, \tag{34}$$

and

$$z_I^0 = z_I^1 = 0, 0 < x < 1, s > 0, \tag{35}$$
$$z_{I,s}^2 = (DE_I^0)_x, 0 < x < 1, s > 0, \tag{36}$$
$$z_I^2(s=0) = 0, 0 < x < 1, \tag{37}$$

where

$$J_I^0 = -\mathcal{Z}^0(x,0)E_I^0. \tag{38}$$

It follows from the special structures of the boundary layer problem (25)-(31) and the initial layer problem (33)-(37) that the existence of solutions of these equations is immediate. We will solve these equations explicitly in section 3 and section 5.

Define the error term $(z_R^\lambda, E_R^\lambda)^T$ of the approximation solution (20) to (14)-(17) with the initial datum

$$\begin{aligned}
&(z_0^\lambda, E_0^\lambda)^T \\
= &\Big(z_0^0(x) + \lambda\big(f(x)z_+^1(\frac{x}{\lambda}, t=0) + g(x)z_-^1(\frac{1-x}{\lambda}, t=0)\big) + \lambda z_{0R}^\lambda(x), \\
&E_0^0(x) + f(x)E_+^0(\frac{x}{\lambda}, t=0) + g(x)E_-^0(\frac{1-x}{\lambda}, t=0) + \lambda E_{0R}^\lambda(x)\Big)^T
\end{aligned} \tag{39}$$

by

$$(z_R^\lambda(x,t), E_R^\lambda(x,t))^T \;=\; (z^\lambda, E^\lambda)^T - (z^\lambda, E^\lambda)_{app}^T. \tag{40}$$

Theorem 3 *Let $l \geq 1$ and all assumptions of Proposition 2 hold. Assume also that the initial data $(z_0^\lambda, E_0^\lambda)$ satisfies (39) with $E_0^0 \in C^{2(l+1)}([0,1])$,*

$$E_0^0(x)|_{x=0,1} = -\frac{D_x(x)}{z_0^0(x)}|_{x=0,1} (= \mathcal{E}^0(x=0,1; t=0)) \tag{41}$$

and

$$\|z_{0R}^\lambda(x)\|_{H^1} \leq M\sqrt{\lambda}, \quad \|z_{0R,xx}^\lambda(x)\|_{L_x^2} \leq M\lambda^{-\frac{1}{2}}, \tag{42}$$
$$\|\partial_x^j E_{0R}^\lambda(x)\|_{L_x^2} \leq M\lambda^{\frac{1}{2}-j}, j = 0,1,2. \tag{43}$$

Then, for any $T \in (0, T_0)$, where T_0 is given by Proposition 2, there exist positive constants M and $\lambda_0, \lambda_0 \ll 1$, such that, for any $\lambda \in (0, \lambda_0]$,

$$\sup_{0\leq t\leq T} \big(\|(z_R^\lambda, E_R^\lambda, z_{R,x}^\lambda, z_{R,t}^\lambda)\|_{L_x^2} + \lambda\|(E_R^\lambda, E_{R,x}^\lambda, E_{R,t}^\lambda)\|_{L_x^2}\big) \leq M\sqrt{\lambda^{1-\delta}} \tag{44}$$

for any δ *with* $0 < \delta < 1$.

In particular, if $(z_0^\lambda, E_0^\lambda)$ *satisfies (39) with* $(z_{0R}^\lambda, E_{0R}^\lambda) = (0,0)$, *then*

$$\sup_{0\le t\le T} \|(z^\lambda - \mathcal{Z}^0)(\cdot,t)\|_{L_x^\infty} \le M\sqrt{\lambda^{1-\delta}}.$$

Remark 4 The compatibility assumption (41) in Theorem 3 is important in our analysis. It guarantees that one can take the 'well-prepared' initial datum (39) instead of the general initial datum (19) and hence the "ansatz" (20) is appropriate in this case while, generally speaking, its breakdown will introduce an extra layer $W_{IB}(x,\xi,\eta,s)$ of mixing of fast time and fast space scales. The main strategy involved here can be applied to this case too. This will be done in the future.

It should also be noted that the assumptions (42) and (43) are just technical ones. In general, $(z_{0R}^\lambda, E_{0R}^\lambda)^T$ in (19) can be written as

$$(z_{0R}^\lambda, E_{0R}^\lambda)^T = (z_0^1(x), E_0^1(x))^T + (\tilde{z}_{0R}^\lambda, \tilde{E}_{0R}^\lambda)^T, \tag{45}$$

where

$$(\tilde{z}_{0R}^\lambda, \tilde{E}_{0R}^\lambda)^T = \lambda O(1), \tag{46}$$

here $O(1)$ is a smooth bounded function in $x, \frac{x}{\lambda}, \frac{1-x}{\lambda}$, so that the general assumptions on the initial data become

$$\|(z_{0R}^\lambda - z_0^1)(x)\|_{H^1} \le M\sqrt{\lambda}, \quad \|(z_{0R}^\lambda - z_0^1)_{xx}(x)\|_{L_x^2} \le M\lambda^{-\frac{1}{2}}, \tag{47}$$

$$\|\partial_x^j (E_{0R}^\lambda - E_0^1)(x)\|_{L_x^2} \le M\lambda^{\frac{1}{2}-j}, j = 0,1,2. \tag{48}$$

In this case, it turn out that an additional correction term $\lambda(z_0^1, E_0^1)$ and hence an extra initial layer term $(\lambda^3 z_I^3, \lambda E_I^1)$, caused by z_0^1, will appear in the solution. Thus, we have more general results as follows.

Theorem 4 *Under the assumptions of Theorem 3 with the assumptions (42) and (43) replaced by (47) and (48) with* $(z_0^1, E_0^1) \in C^3$, *we have that, for any* $T \in (0, T_0)$, *where* T_0 *is given by Proposition 2, there exist positive constants* M *and* $\lambda_0, \lambda_0 \ll 1$, *such that, for any* $\lambda \in (0, \lambda_0]$,

$$\sup_{0\le t\le T} \left(\|(\tilde{z}_R^\lambda, \tilde{E}_R^\lambda, \tilde{z}_{R,x}^\lambda, \tilde{z}_{R,t}^\lambda)\|_{L_x^2} + \lambda\|(\tilde{E}_R^\lambda, \tilde{E}_{R,x}^\lambda, \tilde{E}_{R,t}^\lambda)\|_{L_x^2}\right) \le M\sqrt{\lambda^{1-\delta}} \tag{49}$$

for any $\delta \in (0,1)$, *where*

$$\begin{aligned}(\tilde{z}_R^\lambda(x,t), \tilde{E}_R^\lambda(x,t))^T &= (z_R^\lambda(x,t), E_R^\lambda(x,t))^T - (\lambda z_0^1 + \lambda^3 z_I^3, \lambda(E_0^1 + E_I^1))^T \\ &= (z^\lambda, E^\lambda)^T - (z^\lambda, E^\lambda)^T_{app} - (\lambda z_0^1 + \lambda^3 z_I^3, \lambda(E_0^1 + E_I^1))^T.\end{aligned}$$

The initial layer functions z_I^3 and E_I^1 solve the following problems (IVPs)

$$E^1_{I,s} = -\mathcal{Z}^0(x,0)E^1_I - z^1_0(x)E^0_I, s>0, 0<x<1, \quad (50)$$
$$E^1_I(s=0) = 0, 0<x<1 \quad (51)$$

and

$$z^3_{I,s} = (DE^1_I)_x, 0<x<1, s>0, \quad (52)$$
$$z^3_I(s=0) = 0, 0<x<1. \quad (53)$$

Remark 5 It follows from Theorems 3 and 4 above that the approximation of vanishing space charge holds in the interior part of the parabolic domain, but cannot be valid uniformly up to the boundary in the case where the doping profile changes its sign.

Remark 6 Note that our smoothness assumption on the doping profile D excludes so called abrupt p-n junctions, where the doping profile has a jump discontinuity. Additional layer thus has to be introduced locally at abrupt junctions. This will be studied further in the future.

Remark 7 It should be noted in Theorem 3 and 4, the quasineutral limits justified rigorously only in spatial L^2-norm. In order to justify this limit in super-norm, a more accurate ansatz than (20) has to be constructed by using higher order corrections. This is left for the future.

3 Formal Asymptotic Expansion

In this section we derive the limit equation and the forms of the boundary layers and of the initial time layers by the multiple scaling asymptotic expansion of a singular perturbation with respect to the scaled Debye length.

Let us look for $W^\lambda = (z^\lambda, E^\lambda)^T$ of the following form

$$W^\lambda = \sum_{i=0}^N \lambda^i W^i(x, \frac{x}{\lambda}, \frac{1-x}{\lambda}, t, \frac{t}{\lambda^2}) + W^\lambda_R(x,t),$$

where λ and λ^2 are the lengths of the boundary layer and of the initial time layer respectively, and

$$W^i = W^i_{Inn}(x,t) + W^i_B(x,\xi,\eta,t) + W^i_I(x,s),$$

sum of an interior term W^i_{Inn}, of the boundary layer term W^i_B near $x=0$ and $x=1$ and of the initial time layer term W^i_I near $t=0$. Here we set $\xi=\frac{x}{\lambda},\eta=\frac{1-x}{\lambda}$, $s=\frac{t}{\lambda^2}$ and $W=(z,E)^T$.

For simplicity of presentation, we will carry out the constructions of boundary layers, $W^i_B=W^i_+(\xi,t)$, only near the left boundary, $x=0$, the parts at $x=1$ can be done similarly. Thus, we enforce

$$\lim_{\xi\to\infty} W_B(\xi,t)=0. \tag{54}$$

In this section, without explicitly writing out the scaled variables the functions marked by "Inn, B, I, BI" and "R" are ones with respect to (x,t), (ξ,t), (x,s), (x,ξ,t,s) and (x,t) respectively. In the following we denote (z_{Inn},E_{Inn}) by $(\mathcal{Z},\mathcal{E})$.

Our primary interests lie in the rigorous justification of the quasinuetral assumptions. Thus, we will ignore the higher corrections to the drift-diffusion equations. Hence, we impose the following decomposition for the solution (z^λ,E^λ) of (14)-(17):

$$\begin{aligned}(z^\lambda,E^\lambda)^T \;=\;& \big(\mathcal{Z}^0+z^0_B+z^0_I+\lambda(z^1_B+z^1_I)+\lambda^2(z^2_B+z^2_I)+z^\lambda_R(x,t),\\ &\mathcal{E}^0+E^0_B+E^0_I+E^\lambda_R(x,t)\big)^T.\end{aligned} \tag{55}$$

Thus, we obtain an approximation of the solution (z^λ,E^λ) of (14)-(17). The expansion (55) will satisfy the differential equations (14)-(15), the boundary condition (16) and the initial condition (17) for arbitrary 'well-prepared' initial data $(z^\lambda_0,E^\lambda_0)$ satisfying (39).

Inserting (55) into (14) and (15), by direct computations, one gets

$$\begin{aligned}&\mathcal{Z}^0_t+\sum_{i=0}^2\lambda^i z^i_{B,t}+\frac{1}{\lambda^2}z^0_{I,s}+\frac{1}{\lambda}z^1_{I,s}+z^2_{I,s}+z^\lambda_{R,t}\\ =\;&\Big[(z^\lambda_{R,x}+DE^\lambda_R)_x+(\mathcal{Z}^0_x+D\mathcal{E}^0)_x\\ &+\frac{1}{\lambda}\big(\frac{1}{\lambda}z^0_{B,\xi}+(z^1_{B,\xi}+D(0)E^0_B)+\lambda z^2_{B,\xi}+(D(\lambda\xi)-D(0))E^0_B\big)_\xi\\ &+\sum_{i=0}^2\lambda^i z^i_{I,xx}+(DE^0_I)_x\Big]-\lambda^2\Big[K^0_{Inn}+\tilde{K}_B+K^\lambda_I+\tilde{K}^\lambda_{IB}+\tilde{F}^\lambda_R\Big]_x\end{aligned} \tag{56}$$

and

$$\begin{aligned}&\lambda^2(\mathcal{E}^0_t-\mathcal{E}^0_{xx})+\big(E^0_{I,s}-\lambda^2E^0_{I,xx}\big)+\big(\lambda^2E^0_{B,t}-E^0_{B\xi\xi}\big)+\lambda^2(E^\lambda_{R,t}-E^\lambda_{R,xx})\\ =\;&J^0_{Inn}+(\tilde{J}^0_B+\tilde{J}^0_{BR})+(J^0_I+J^0_{IR})+\tilde{J}^0_{BI}+\sum_{i=1}^2\lambda^i\big(\tilde{J}^i_B+J^i_I+\tilde{J}^i_{BI}\big)+\tilde{G}^\lambda_R,\end{aligned} \tag{57}$$

where K^0_{Inn}, $\tilde{K}^i_B$, K^λ_I, $\tilde{K}^\lambda_{IB}$ and $\tilde{F}^\lambda_R$ are defined by the following:

$$\begin{aligned}
K^0_{Inn} &= \mathcal{E}^0\mathcal{E}^0_x, \\
\tilde{K}^\lambda_B &= (\mathcal{E}^0(0,t)+E^0_B)E^0_{B,\xi} + E^0_B\mathcal{E}^0_x(0,t) \\
&\quad +(\mathcal{E}^0(\lambda\xi,t)-\mathcal{E}^0(0,t))E^0_{B,\xi} + E^0_B(\mathcal{E}^0_x(\lambda\xi,t)-\mathcal{E}^0_x(0,t)), \\
K^\lambda_I &= \big(\mathcal{E}^0(x,\lambda^2 s)E^0_{I,x} + E^0_I(E^0_{Inn,x}(x,\lambda^2 s)+E^0_{I,x})\big), \\
\tilde{K}^\lambda_{IB} &= E^0_B E^0_{I,x} + E^0_I\frac{1}{\lambda}E^0_{B,\xi}, \\
\tilde{F}^\lambda_R &= (\mathcal{E}^0+E^0_B+E^0_I)E^\lambda_{R,x} + E^\lambda_R(\mathcal{E}^0_x + E^0_{B,\xi}\frac{1}{\lambda} + E^0_{I,x}) + E^\lambda_{R,x}),
\end{aligned}$$

and J^0_{Inn}, $\tilde{J}^0_B$, $\tilde{J}^0_{BR}$, J^0_I, J^0_{IR}, $\tilde{J}^0_{BI}$, $\tilde{J}^i_B$, J^i_I, $\tilde{J}^i_{BI}$, $i=1,2$ and $\tilde{G}^\lambda_R$ are defined by the following:

$$\begin{aligned}
J^0_{Inn} &= -(D_x + \mathcal{Z}^0\mathcal{E}^0), \\
\tilde{J}^0_B &= -(\mathcal{Z}^0(0,t)E^0_B + z^0_B(\mathcal{E}^0(0,t)+E^0_B)), \\
\tilde{J}^0_{BR} &= -\big((\mathcal{Z}^0-\mathcal{Z}^0(0,t))E^0_B + z^0_B(\mathcal{E}-\mathcal{E}^0(0,t))\big), \\
J^0_I &= -\big(\mathcal{Z}^0(x,0)E^0_I + z^0_I(\mathcal{E}^0(x,0)+E^0_I)\big), \\
J^0_{IR} &= -\big((\mathcal{Z}^0-\mathcal{Z}^0(x,0))E^0_I + z^0_I(\mathcal{E}^0-\mathcal{E}^0(x,0))\big), \\
\tilde{J}^0_{BI} &= -(z^0_B E^0_I + z^0_I E^0_B), \\
\tilde{J}^i_B &= -z^i_B(\mathcal{E}^0+E^0_B), i=1,2, \\
J^i_I &= -z^i_I(\mathcal{E}^0+E^0_I), i=1,2, \\
\tilde{J}^i_{BI} &= -z^i_B E^0_I + z^i_I E^0_B, i=1,2,
\end{aligned}$$

and

$$G^\lambda_R = -\big((\mathcal{E}^0+E^0_B+E^0_I)z^\lambda_R + (\mathcal{Z}^0+z^0_B+z^0_I+\sum_{i=1}^{2}\lambda^i(z^i_B+z^i_I))E^\lambda_R\big) - z^\lambda_R E^\lambda_R.$$

Similarly, inserting (55) into the boundary condition (16) yields an expansion at the boundary $x=0$. Since the boundary expansion is expected to correct well the boundary conditions of inner solutions to quasineutral drift diffusion equations, according to the expansion at the boundary $x=0$, we may impose the following boundary conditions

$$z^0_{B,\xi}(\xi=0;t)=0, \tag{58}$$

$$z^1_{B,\xi}(\xi=0;t)=-\mathcal{Z}^0_x(x=0;t), \tag{59}$$

$$z^2_{B,\xi}(\xi=0;t)=0, \tag{60}$$

$$E^0_B(\xi=0;t)=-\mathcal{E}^0(x=0;t). \tag{61}$$

Now we start to derive the equations of the inner solution $(\mathcal{Z}^0, \mathcal{E}^0)$, of the various orders of boundary layer and initial time layer functions in the above expansion (55) by comparing coefficients of $O(\lambda^k)$ of (56) and (57). At the leading order λ^{-2} of (56), one gets

$$z^0_{I,s}(x,s) = 0. \tag{62}$$

For z^0_I we take the initial data

$$z^0_I(s=0) = 0. \tag{63}$$

The only solution of (62) and (63) is given as

$$z^0_I(x,s) = 0, x \in [0,1], s \geq 0. \tag{64}$$

Similarly

$$z^0_{B,\xi\xi} = 0. \tag{65}$$

We also expect the decay condition at the infinity for z^0_B such that

$$z^0_B(\xi,t) \to 0 \text{ as } \xi \to \infty. \tag{66}$$

The only solution of (58), (65) and (66) is given as

$$z^0_B(\xi,t) = 0, \xi \geq 0, t \geq 0, \tag{67}$$

which explains partially that the Neumann boundary condition of the density does not produce the boundary layer at the leading order.

At the order λ^{-1} of (56), one gets

$$z^1_{I,s}(x,s) = 0, \text{ hence } z^1_I = 0, x \in [0,1], s \geq 0 \tag{68}$$

since $z^1_I(x,s=0) = 0$.

One also has from the order λ^{-1} of (56)

$$z^1_{B,\xi\xi} + D(0)E^0_{B,\xi} = 0. \tag{69}$$

As before, we impose the decay condition at the infinity such that

$$z^1_B(\xi,t) \to 0, E^0_B(\xi,t) \to 0 \text{ as } \xi \to \infty. \tag{70}$$

It follows from (69) and (70) that

$$z^1_{B,\xi} + D(0)E^0_B = 0. \tag{71}$$

Next, we determine the limit equations, which form a system satisfied by the first-order term $(\mathcal{Z}^0, \mathcal{E}^0)$ in the above expansion. At the order of λ^0 of (56) and (57), one gets

$$\begin{aligned} \mathcal{Z}^0_t &= (\mathcal{Z}^0_x + D\mathcal{E}^0)_x, 0 < x < 1, t > 0, && (72) \\ 0 &= -(D_x + \mathcal{Z}^0\mathcal{E}^0), 0 < x < 1, t > 0. && (73) \end{aligned}$$

This is nothing but the well-known quasineutral drift diffusion models, which can be formally obtained by setting λ equal to zero in (14)-(15), too. In the context of semiconductor device physics problems (72)-(73) is referred to as "space charge approximation".

Notice that (73) is a algebraic equation. If $\mathcal{Z}^0(x,t) \geq C_0 > 0$, then

$$\mathcal{E}^0(x,t) = -\frac{D_x(x)}{\mathcal{Z}^0(x,t)}.$$

Generally speaking, $\mathcal{E}^0(x,t)|_{x=0,1} \neq 0$, but $E^\lambda(x,t)|_{x=0,1} = 0$. Therefore, $\mathcal{E}^0(x,t)$ has to be supplemented by a boundary layer term there. Similarly, owe to the arbitrary of the initial data $E^\lambda_0(x)$, there is an initial layer. Furthermore, it should be clear that the boundary and initial layers are caused by the electric field.

Now we supplement the limit equations (72)-(73) by the appropriate boundary conditions. According to the condition conditions (59), (61) and the equation (71), one gets

$$\mathcal{Z}^0_x(x=0,t) = -z^1_{B,\xi}(\xi=0,t) = D(0)E^0_B(\xi=0) = -D(0)\mathcal{E}^0(x=0,t), t \geq 0,$$

i.e.

$$\mathcal{Z}^0_x + D(0)\mathcal{E}^0 = 0, x = 0, t \geq 0. \tag{74}$$

For the initial data of $\mathcal{Z}^0(x,t)$, we can take it as

$$\mathcal{Z}^0(x, t=0) = z^0_0(x) \geq \delta > 0, 0 \leq x \leq 1. \tag{75}$$

Here $z^0_0(x)$ is given by (19).

Finally, one also gets from the order λ^0 of (56) and (57)

$$\begin{aligned} E^0_{I,s}(x,s) &= J^0_I(x,s), 0 \leq x \leq 1, s \geq 0 && (76) \\ z^2_{I,s}(x,s) &= (D(x)E^0_I)_x, 0 \leq x \leq 1, s \geq 0, && (77) \end{aligned}$$

and

$$\begin{aligned} -E^0_{B,\xi\xi}(\xi,t) &= \tilde{J}^0_B(\xi,t), \xi > 0, t > 0. && (78) \\ z^2_{B,\xi\xi} &= 0, \xi > 0, t > 0, && (79) \end{aligned}$$

The initial data of E_I^0 can be taken as

$$E_I^0(x, s=0) = E_0^0(x) - \mathcal{E}^0(x, t=0), 0 \leq x \leq 1. \tag{80}$$

The only solution to (76) and (80) can be given explicitly by

$$E_I^0(x, s) = (E_0^0(x) - \mathcal{E}^0(x, t=0)) \exp(-z_0^0(x)s). \tag{81}$$

The initial data of z_I^2 is

$$z_I^2(x, s=0) = 0, 0 \leq x \leq 1. \tag{82}$$

The unique solution of (77) and (82), can be given, using (81), by

$$\begin{aligned} z_I^2(x, s) &= \int_0^s (D(x)E_I^0)_x ds \\ &= b(x) + (b_0(x) + b_1(x)s) \exp\{-z_0^0(x)s\}, 0 \leq x \leq 1, s \geq 0, \end{aligned} \tag{83}$$

where $b(x), b_0(x)$ and $b_1(x)$ depend only upon $D(x)$ and (z_0^0, E_0^0) and satisfy $b_0(x) = -b(x) \neq 0$.

For E_B^0, we impose the decay condition at infinity as

$$E_B^0(\xi, t) = 0, \text{ as } \xi \to \infty \tag{84}$$

and we also take the boundary condition at $\xi = 0$ as

$$E_B^0(\xi = 0, t) = -\mathcal{E}^0(x = 0, t), t > 0. \tag{85}$$

The unique solution of (78), (84) and (85) can be given by

$$E_B^0(\xi, t) = -\mathcal{E}^0(x = 0, t) \exp(-\sqrt{\mathcal{Z}^0(x = 0, t)}\xi). \tag{86}$$

For z_B^2, the decay condition at infinity is

$$z_B^2(\xi, t) = 0, \text{ as } \xi \to \infty. \tag{87}$$

Then the only solution of (79), (60) and (87) is given by

$$z_B^2(\xi, t) = 0, \xi > 0, t > 0. \tag{88}$$

Similarly, we can construct the boundary layer functions near $x = 1$ and hence deduce the similar boundary conditions of the inner solutions at $x = 1$.

4 Existence and Regularity of Solutions to Quasineutral Drift Diffusion Models

In this section we discuss existence and regularity of solutions to quasineutral drift diffusion models (21)-(24) and we will prove Proposition 2.

The Proof of Proposition 2 The proof is elementary. For completeness, we outline it here. First, it follows from (73) that $\mathcal{E}^0(x,t) = -\frac{D_x}{\mathcal{Z}^0(x,t)}$. Then the problem (72)-(75) is reduced to the following system

$$\mathcal{Z}^0_t = (\mathcal{Z}^0_x - \frac{DD_x}{\mathcal{Z}^0})_x, 0 < x < 1, t > 0, \tag{89}$$

$$(\mathcal{Z}^0_x - \frac{DD_x}{\mathcal{Z}^0})(x = 0, 1, t) = 0, t > 0, \tag{90}$$

$$\mathcal{Z}^0(t = 0) = z^0_0(x), 0 \le x \le 1. \tag{91}$$

For $z^0_0 \ge \delta_0 > 0$, the standard parabolic theory yields the desired local existence of classical positive solution $\mathcal{Z}^0$. This concludes the first part of proposition 2.

To prove the global existence of large classical solution for large initial data, one introduces the transformation

$$(\mathcal{Z}^0)^2 - D^2 = w. \tag{92}$$

Then it follows from the system (89)-(91) that w satisfies

$$w_t = w_{xx} - \frac{w_x^2 + 2DD_x w_x}{2(w + D^2)}, 0 < x < 1, t > 0 \tag{93}$$

$$w_x(x = 0, 1, t) = 0, t > 0 \tag{94}$$

$$w(t = 0) = w_0 = (z^0_0)^2 - D^2. \tag{95}$$

If $\delta_0 \ge \sqrt{D^2 + \delta_2}$ for some $\delta_2 > 0$, then $w_0 \ge \delta_2 > 0$.

By the standard parabolic theory [17], we know that there exists a unique, classical and global solution w for (93)-(95) satisfying $0 < \delta_2 \le w \in C^{2(l+1),l+1}([0,1] \times [0,T])$ for any $T > 0$. By transformation (92), we conclude the second part of Proposition 2. The proof of Proposition 2 is complete. $\square$

5 Properties of the Initial Time Layer and the Boundary Layer Functions

In this section we summarize the properties of the boundary layers and the initial time layer and discuss their decay rate, which will be useful for the energy estimates next section.

Using (86), one gets from (71) and (70) that

$$\begin{aligned} z_B^1(\xi,t) &= -D(0)\mathcal{E}^0(0,t)\int_\xi^\infty e^{-\sqrt{\mathcal{Z}^0(0,t)}y}dy \\ &= -\frac{D(0)\mathcal{E}^0(0,t)}{\sqrt{\mathcal{Z}^0(0,t)}}e^{-\sqrt{\mathcal{Z}^0(0,t)}\xi}. \end{aligned} \tag{96}$$

Thus, we obtained exact formulaes of all initial layer functions and the left boundary layer functions. In particular, we can determine the values of $z_B^1(\xi)$ and $E_B^0(\xi)$, depending only upon $D(0), z_0^0(0)$ and $\mathcal{E}^0(0,0) = E_0^0(0)$, which is given by

$$\begin{aligned} z_B^1(\xi) &= z_B^1(x,t=0) \\ &= -\frac{D(0)E_0^0(0)}{\sqrt{z_0^0(0)}}e^{-\sqrt{z_0^0(0)}\xi}, \xi > 0, \end{aligned} \tag{97}$$

$$E_B^0(\xi) = E_B^0(x,t=0) = -E_0^0(0)e^{-\sqrt{z_0^0(0)}\xi}, \xi > 0. \tag{98}$$

Similarly, the right boundary layer functions at $x = 1$, denoted by $z_-^i(\eta,t), i = 0,1,2, E_-^0(\eta,t)$, satisfy similar equations and have completely same properties as the left boundary layer functions $z_B^i(\xi,t), i = 0,1,2, E_B^0(\xi,t)$ at $x = 0$, denoted by $z_+^i(\xi,t), i = 0,1,2, E_+^0(\xi,t)$. We omit this.

We end this section by summarizing the properties of the boundary layer and initial layer functions:

Lemma 5 *(i)* $z_+^0 = z_-^0 = z_+^2 = z_-^2 = z_I^0 = z_I^1 = 0.$

(ii) Assume that the inner solution $(\mathcal{Z}^0, \mathcal{E}^0)$ *is suitably smooth. Then,*

(a) For any $T > 0$*, there exists a positive constant* M *independent of* λ *such that*

$$\|(\partial_t^{k_1}(\xi^{k_2}\partial_\xi^{k_3}(z_+^1, E_+^0), \eta^{k_4}\partial_\eta^{k_5}(z_-^1, E_-^0))\|_{L^\infty_{(x,t)}([0,1]\times[0,T])} \le M \tag{99}$$

and

$$\|(\partial_t^{k_1}(\xi^{k_2}\partial_\xi^{k_3}(z_+^1, E_+^0), \eta^{k_4}\partial_\eta^{k_5}(z_-^1, E_-^0))\|_{L_t^\infty([0,T];L_x^2([0,1]))} \le M\lambda^{\frac{1}{2}} \tag{100}$$

for any nonnegative integer $k_j, j = 0, \cdots, 5.$

(b) For any $T > 0$*, there exists a positive constant* M *independent of* λ *such that*

$$\|\partial_x^{k_6}(z_I^2, s^{k_7}(\partial_s^{k_8} z_I^2, \partial_s^{k_9} E_I^0))\|_{L^\infty_{(x,t)}([0,1]\times[0,T])} \le M \tag{101}$$

and

$$\|\partial_x^{k_{10}} s^{k_{11}}(\partial_s^{k_{12}} z_I^2, \partial_s^{k_{13}} E_I^0)\|_{L_t^2([0,T];L_x^\infty([0,1]))} \le M\lambda \tag{102}$$

for any nonnegative integer $k_j, j = 6, 7, 9, 10, 11, 13$ *and any positive integer* k_8, k_{12}.

6 Energy Estimates

In this section we investigate the asymptotic behavior of the solution to the problem (14)-(17) as $\lambda \to 0$ and we will also prove our main theorems 3 and 4 in this section. From now on, we may assume $0 < \lambda \le 1$.

6.1 The Proof of Theorem 3

In this subsection we prove Theorem 3 by a careful energy method based on the approximate solutions constructed in the previous section.

Let $\mathcal{Z}^0, \mathcal{E}^0, E_+^0, E_-^0, E_I^0, z_+^1, z_-^1, z_I^2$ be the functions constructed in the previous sections.

Let us assume

$$\begin{aligned} & (z^\lambda(x,t=0), E^\lambda(x,t=0))^T \\ = & \big(z_0^0(x) + \lambda(f(x)z_+^1(\frac{x}{\lambda},0) + g(x)z_-^1(\frac{1-x}{\lambda},0)) + \lambda z_{0R}^\lambda(x), \\ & E_0^0(x) + f(x)E_+^0(\frac{x}{\lambda},0) + g(x)E_-^0(\frac{1-x}{\lambda},0) + \lambda E_{0R}^\lambda(x)\big)^T, \end{aligned}$$

where $f(x)$ and $g(x)$ are two smooth C^2 cut-off functions satisfying that $f(0) = g(1) = 1$ and $f(1) = f'(1) = f''(1) = f'(0) = f''(0) = g(0) = g'(0) = g''(0) = g'(1) = g''(1) = 0$ and $(z_{0R}^\lambda, E_{0R}^\lambda)$ satisfies assumptions (42) and (43). In this case, one gets

$$(z_R^\lambda, E_R^\lambda)^T(x,t=0) = \lambda(z_{0R}^\lambda(x), E_{0R}^\lambda(x))^T.$$

Replacing $(z^\lambda, E^\lambda)^T$ by

$$\begin{aligned} (z^\lambda, E^\lambda)^T \;=\; & \Big(\mathcal{Z}^0 + \lambda(f(x)z_+^1 + g(x)z_-^1) + \lambda^2 z_I^2 + z_R^\lambda(x,t), \\ & \quad \mathcal{E}^0 + f(x)E_+^0 + g(x)E_-^0 + E_I^0 + E_R^\lambda(x,t)\Big)^T \end{aligned} \tag{103}$$

in the system (14)-(15), using the equations of the inner solutions, of the boundary layers, and of the initial layers, one gets

$$z_{R,t}^\lambda = H_x^\lambda + f^\lambda, 0 < x < 1, t > 0, \tag{104}$$

$$\lambda^2(E_{R,t}^\lambda - E_{R,xx}^\lambda) + \mathcal{Z}^0 E_R^\lambda = g^\lambda, 0 < x < 1, t > 0, \tag{105}$$

where

$$\begin{aligned} & H^\lambda = z_{R,x}^\lambda + DE_R^\lambda + H_{Inn} + H_B^\lambda + H_I^\lambda + H_{IB}^\lambda + H_R^\lambda, \\ & f^\lambda = -\lambda^1(f(x)z_{+,t}^1 + g(x)z_{-,t}^1), \quad g^\lambda = G_{Inn} + G_B^\lambda + G_I^\lambda + G_{IB}^\lambda + G_R^\lambda \end{aligned}$$

and H_{Inn} (G_{Inn}), H_B^λ (G_B^λ), H_I^λ (G_I^λ), H_{IB}^λ (G_{IB}^λ), H_R^λ (G_R^λ) represent the inner part, the boundary layer part, the initial layer part, the mixed boundary and initial layer part and the error parts involving nonlinearities, respectively, and are defined by the following:

$$H_{Inn}(x,t) = -\lambda^2 \mathcal{E}^0 \mathcal{E}_x^0.$$

$$\begin{aligned}
H_B^\lambda(x,t,\xi,\eta) &= \big((D(x)-D(0))f(x)E_+^0 + (D(x)-D(1))g(x)E_-^0\big) \\
&\quad +\lambda\Big(f'(x)z_+^1 + g'(x)z_-^1 - \mathcal{E}^0(f(x)E_{+,\xi}^0 - g(x)E_{-,\eta}^0) \\
&\quad +(f(x)E_+^0 + g(x)E_-^0)(f(x)E_{+,\xi}^0 - g(x)E_{-,\eta}^0)\Big) \\
&\quad +\lambda^2\Big(-\mathcal{E}^0(f'(x)E_+^0 + g'(x)E_-^0) \\
&\quad -(f(x)E_+^0 + g(x)E_-^0)(\mathcal{E}_x^0 + f'(x)E_+^0 + g'(x)E_-^0)\Big) \\
&= \Big((D(x)-D(0))f(x)E_+^i + (D(x)-D(1))g(x)E_-^i\Big) + \lambda\{\cdots\}_{HB}^R.
\end{aligned}$$

Noting that $\{\cdots\}_{HB}^R$ is the sums of the boundary layer functions $z_+^1, z_-^1, E_+^0, E_-^0, E_{+,\xi}^0$, $E_{-,\eta}^0$, $E_+^0E_+^0$, $E_-^0E_-^0$, $E_+^0E_-^0$, $E_+^0E_{+,\xi}^0$, $E_-^0E_{-,\eta}^0$, $E_+^0E_{-,\eta}^0$ and $E_-^0E_{+,\xi}^0$ with the coefficients consisting of $D(x), f(x), g(x), \mathcal{E}^0, f'(x), g'(x)$ and $\mathcal{E}_x^0$ and that $\{\cdots\}_{HB}^R$ does not depend upon the fast dielectric relaxation time scale and hence, by (99) and (100), we easily obtain that there exists a constant M, independent of λ, such that

$$\|\{\cdots\}_{HB}^R(t)\|_{L_x^2}^2 + \int_0^t \|\partial_t\{\cdots\}_{HB}^R(t)\|_{L_x^2}^2 dt \le M\lambda. \tag{106}$$

$$\begin{aligned}
H_I^\lambda(x,s) &= \lambda^2 z_{I,x}^2 - \lambda^2\big(\mathcal{E}^0E_I^0 + E_I^0(\mathcal{E}_x^0 + E_{I,x}^0)\big), \\
H_{IB}^\lambda(x,\xi,\eta,t,s) &= -\lambda\Big(E_I^0(f(x)E_{+,\xi}^0 - g(x)E_{-,\eta}^0)\Big) \\
&\quad -\lambda^2\Big(E_I^0(f'(x)E_+^0 + g'(x)E_-^0) + (f(x)E_+^0 + g(x)E_-^0)E_{I,x}^0\Big),
\end{aligned}$$

$$\begin{aligned}
H_R^\lambda &= -\lambda E_R^\lambda(f(x)E_{+,\xi}^0 - g(x)E_{-,\eta}^0) \\
&\quad -\lambda^2\Big((\mathcal{E}^0 + f(x)E_+^0 + g(x)E_-^0)E_{R,x}^\lambda + (\mathcal{E}_x^0 + f'(x)E_+^0 + g'(x)E_-^0)E_R^\lambda\Big) \\
&\quad -\lambda^2\big(E_I^0E_{R,x}^\lambda + E_{I,x}^0E_R^\lambda\big) - \lambda^2E_R^\lambda E_{R,x}^\lambda,
\end{aligned}$$

$$G_{Inn}(x,t) = -\lambda^2\big(\mathcal{E}_t^0 - \mathcal{E}_{xx}^0\big),$$

$$
\begin{aligned}
G_B^\lambda(x,\xi,\eta,t) &= \Big(-f(x)(\mathcal{Z}^0(x,t)-\mathcal{Z}^0(0,t))E_+^0 - g(x)(\mathcal{Z}^0(x,t)-\mathcal{Z}^0(1,t))E_-^0\Big)\\
&\quad +\lambda\{\cdots\}_{GB}^R.
\end{aligned}
$$

Here $\{\cdots\}_{GB}^R$ is the sums of the boundary layer functions $z_{+,t}^1$, $z_{-,t}^1$, E_+^0, E_-^0, $E_{+,\xi}^0$, $E_{-,\eta}^0$, $z_+^1E_+^0$, $z_-^1E_-^0$, $z_+^1E_-^0$ and $z_-^1E_+^0$ with the coefficients consisting of $f(x), g(x), \mathcal{E}^0$, $f'(x)$, $g'(x)$, $f''(x)$, $g''(x)$ and $\mathcal{Z}^0$. Like $\{\cdots\}_{HB}^R$, $\{\cdots\}_{GB}^R$ does not depend upon the fast dielectric relaxation time scale and hence it easily follows from (99) and (100) that there exists a constant M, independent of λ, such that

$$
\|\{\cdots\}_{GB}^R(t)\|_{L_x^2}^2 + \int_0^t \|\partial_t\{\cdots\}_{GB}^R(t)\|_{L_x^2}^2 dt \le M\lambda. \tag{107}
$$

$$
\begin{aligned}
G_I^\lambda &= (\mathcal{Z}^0-\mathcal{Z}^0(x,0))E_I^0 + \lambda^2 E_{I,xx}^0 + \lambda^2 z_I^2(\mathcal{E}^0+E_I^0),\\
G_{IB}^\lambda &= -\lambda(f(x)z_+^1+g(x)z_-^1)E_I^0 - \lambda^2 z_I^2(f(x)E_+^0+g(x)E_-^0),\\
G_R^\lambda &= -(\mathcal{E}^0+f(x)E_+^0+g(x)E_-^0+E_I^0)z_R^\lambda\\
&\quad -\lambda(f(x)z_+^1+g(x)z_-^1)E_R^\lambda - \lambda^2 z_I^2E_R^\lambda - z_R^\lambda E_R^\lambda.
\end{aligned}
$$

We now derive the boundary conditions for the error functions.

First, the assumption

$$
E_0^0(x=0,1) = -\frac{D_x(x=0,1)}{z_0^0(x=0,1)} = \mathcal{E}^0(x=0,1;t=0),
$$

together with the initial layer function (81), gives

$$
E_I^0(x=0,1;t)=0, t>0. \tag{108}
$$

Then it follows from (17), (61) and (108) that

$$
E_R^\lambda(x=0,1;t)=0, t>0. \tag{109}
$$

Next we claim that

$$
H^\lambda(x=0,1;t)=0, t>0. \tag{110}
$$

In fact, we can rewrite $H^\lambda(x,t)$ as

$$
\begin{aligned}
H^\lambda &= z_{R,x}^\lambda + DE_R^\lambda + H_{Inn} + H_B^\lambda + H_I^\lambda + H_{IB}^\lambda + H_R^\lambda\\
&= z_{R,x}^\lambda + \lambda(f'(x)z_+^1+g'(x)z_-^1) + \lambda^2 z_{I,x}^2 + f(x)(D(x)-D(0))E_+^0\\
&\quad +g(x)(D(x)-D(1))E_-^0 + DE_R^\lambda - \lambda^2E^\lambda E_x^\lambda.
\end{aligned} \tag{111}
$$

Then, by the definitions of cut-off functions $f(x)$ and $g(x)$, the boundary condition $E^\lambda(x=0,1;t)=0$ and (109), one gets from (111) that

$$H^\lambda(x=0,1;t) = (z^\lambda_{R,x} + \lambda^2 z_I^2)|_{x=0,1}. \tag{112}$$

Also, replacing z^λ by (103) in the boundary condition $z^\lambda_x(x=0,1;t)=0$ and using

$$z^1_{+,\xi}(\xi=0;t) = -\mathcal{Z}^0_x(x=0;t);\ z^1_{-,\eta}(\eta=0;t) = \mathcal{Z}^0_x(x=1;t), t>0,$$

one gets

$$(z^\lambda_{R,x} + \lambda^2 z_I^2)|_{x=0,1} = 0,$$

which, together with (112), gives (110).

Now we start the energy estimates. In the following, we use c_i, δ_i, ϵ and $M(\epsilon)$ or M to denote the constants which are independent of λ and may be different from one line to another line.

First we derive the basic energy estimates on $(z^\lambda_R, E^\lambda_R)$.

Lemma 6 *Under the assumptions of Theorem 3, we have*

$$\begin{aligned}
& \|z^\lambda_R(t)\|^2_{L^2_x} + \lambda^2\|E^\lambda_R(t)\|^2_{L^2_x} + \int_0^t \|(z^\lambda_{R,x}, E^\lambda_R)\|^2_{L^2_x} dt + \lambda^2 \int_0^t \|E^\lambda_{R,x}\|^2_{L^2_x} dt \\
\le\ & \|z^\lambda_R(t=0)\|^2_{L^2_x} + \lambda^2\|E^\lambda_R(t=0)\|^2_{L^2_x} \\
& + M\int_0^t \|z^\lambda_R\|^2_{L^2_x} dt + M\lambda^4 \int_0^t \|(E^\lambda_R, E^\lambda_{R,x})\|^2_{L^2_x}\|E^\lambda_{R,x}\|^2_{L^2_x} dt \\
& + M\int_0^t \|(z^\lambda_R, z^\lambda_{R,x})\|^2_{L^2_x}\|E^\lambda_R\|^2_{L^2_x} dt + M\lambda.
\end{aligned} \tag{113}$$

Proof of Lemma 6 Multiplying (104) by z^λ_R and integrating the resulting equation over $[0,1]$ with respect to x, by (110) and integrations by parts, one gets

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z^\lambda_R\|^2_{L^2_x} &= -\int_0^1 H^\lambda z^\lambda_{R,x} dx + \int_0^1 f^\lambda z^\lambda_R dx \\
&= -\int_0^1 (z^\lambda_{R,x} + DE^\lambda_R) z^\lambda_{R,x} dx + \int_0^1 f^\lambda z^\lambda_R dx \\
&\quad -\int_0^1 (H_{Inn} + H^\lambda_B + H^\lambda_I + H^\lambda_{IB} + H^\lambda_R) z^\lambda_{R,x} dx.
\end{aligned} \tag{114}$$

Now we estimate each term in the right hand side of (114).

First, by the Cauchy-Schwarz's inequality and the properties of the boundary layers, one gets

$$-\int_0^1 (z^\lambda_{R,x} + DE^\lambda_R) z^\lambda_{R,x} dx \le -\frac{1}{2}\|z^\lambda_{R,x}\|^2_{L^2_x} + M(\epsilon)\|E^\lambda_R\|^2_{L^2_x} \tag{115}$$

and

$$\int_0^1 f^\lambda z^\lambda_R dx \le M\|z^\lambda_R\|^2_{L^2_x} + M\|f^\lambda\|^2_{L^2_x} dx \le M\|z^\lambda_R\|^2_{L^2_x} + M\lambda^3. \tag{116}$$

Here we used $\|(z^1_{+,t}, z^1_{-,t})\|^2_{L^2_x} \le M\lambda$ due to (100).

Then, using the regularity of inner solutions, the properties (99) and (100) of boundary layer functions, the properties (101) and (102) of initial layer functions and the definitions of $H_{Inn}, H^\lambda_B, H^\lambda_I$ and H^λ_{IB}, one easily gets

$$\|H_{Inn}\|^2_{L^2_x} + \|H^\lambda_B\|^2_{L^2_x} + \|H^\lambda_I\|^2_{L^2_x} + \|H^\lambda_{IB}\|^2_{L^2_x} \le M\lambda.$$

This, combining with the Cauchy-Schwarz's inequality, yields

$$\begin{aligned} & -\int_0^1 (H_{Inn} + H^\lambda_B + H^\lambda_I + H^\lambda_{IB}) z^\lambda_{R,x} dx \\ \le\ & \epsilon\|z^\lambda_{R,x}\|^2_{L^2_x} + M(\epsilon)(\|H_{Inn}\|^2_{L^2_x} + \|H^\lambda_B\|^2_{L^2_x} + \|H^\lambda_I\|^2_{L^2_x} + \|H^\lambda_{IB}\|^2_{L^2_x}) \\ \le\ & \epsilon\|z^\lambda_{R,x}\|^2_{L^2_x} + M\lambda. \end{aligned} \tag{117}$$

Finally, for the nonlinear term, using $\mathcal{E}^0, \mathcal{E}^0_x \in C^0([0,1]\times[0,T])$, (99) and (101), one gets, with the aid of the Cauchy-Schwarz's inequality and Sobolev's Lemma, that

$$\begin{aligned} & \int_0^1 H^\lambda_R z^\lambda_{R,x} dx \\ \le\ & \epsilon\|z^\lambda_{R,x}\|^2_{L^2_x} + M(\epsilon)\|H^\lambda_R\|^2_{L^2_x} \\ \le\ & \epsilon\|z^\lambda_{R,x}\|^2_{L^2_x} + M\lambda^2\|E^\lambda_R\|^2_{L^2_x} + M\lambda^4\|E^\lambda_{R,x}\|^2_{L^2_x} + M\lambda^4\int_0^1 |E^\lambda_R E^\lambda_{R,x}|^2 dx \\ \le\ & \epsilon\|z^\lambda_{R,x}\|^2_{L^2_x} + M\lambda^2\|E^\lambda_R\|^2_{L^2_x} + M\lambda^4\|E^\lambda_{R,x}\|^2_{L^2_x} \\ & + M\lambda^4\|(E^\lambda_R, E^\lambda_{R,x})\|^2_{L^2_x}\|E^\lambda_{R,x}\|^2_{L^2_x}. \end{aligned} \tag{118}$$

Thus, combining (114) with (115)-(118) and taking ϵ small enough, one gets

$$\begin{aligned} \frac{d}{dt}\|z^\lambda_R\|^2_{L^2_x} + c_1\|z^\lambda_{R,x}\|^2_{L^2_x} \ \le\ & M\|(z^\lambda_R, E^\lambda_R)\|^2_{L^2_x} + M\lambda^4\|E^\lambda_{R,x}\|^2_{L^2_x} \\ & + M\lambda^4(\|(E^\lambda_R, E^\lambda_{R,x})\|^2_{L^2_x}\|E^\lambda_{R,x}\|^2_{L^2_x} + M\lambda. \end{aligned} \tag{119}$$

Integrating (119) with respect to t over $[0,t]$, one gets

$$\begin{aligned}
&\|z_R^\lambda(t)\|_{L_x^2}^2 + c_1\int_0^t \|z_{R,x}^\lambda\|_{L_x^2}^2 dt \\
\leq\ & \|z_R^\lambda(t=0)\|_{L_x^2}^2 + M\int_0^t \|(z_R^\lambda, E_R^\lambda)\|_{L_x^2}^2 dt + M\lambda^4\int_0^t \|E_{R,x}^\lambda\|_{L_x^2}^2 dt \\
&+M\lambda^4\int_0^t \|(E_R^\lambda, E_{R,x}^\lambda)\|_{L_x^2}^2\|E_{R,x}^\lambda\|_{L_x^2}^2 dt + M\lambda. \qquad (120)
\end{aligned}$$

Multiplying (105) by E_R^λ and integrating the resulting equation over $[0,1]$ with respect to x, by (109) and integrations by parts, one gets

$$\frac{\lambda^2}{2}\frac{d}{dt}\|E_R^\lambda\|_{L_x^2}^2 + \lambda^2\|E_{R,x}^\lambda\|_{L_x^2}^2 + \int_0^1 \mathcal{Z}^0|E_R^\lambda|^2 dx = \int_0^1 g^\lambda E_R^\lambda dx. \qquad (121)$$

By the Cauchy-Schwarz's inequality, we have

$$\int_0^1 g^\lambda E_R^\lambda dx \leq \epsilon\|E_R^\lambda\|_{L_x^2}^2 + M(\epsilon)\|g^\lambda\|_{L_x^2}^2.$$

On one hand, noting that

$$\begin{aligned}
&\int_0^1 |(\mathcal{Z}^0(x,t) - \mathcal{Z}^0(x,0))E_I^0(x,\frac{t}{\lambda^2})|^2 dx \\
=\ & \int_0^1 |\int_0^1 \partial_t\mathcal{Z}^0(x,t\theta)d\theta t \cdot E_I^0(x,\frac{t}{\lambda^2})|^2 dx \\
\leq\ & M\sup_{s\geq 0}(\max_{0\leq x\leq 1}|sE_I^0(x,s)|^2)\lambda^4 \\
\leq\ & M\lambda^4,
\end{aligned}$$

using $\mathcal{Z}^0, \mathcal{E}^0 \in C^{2,1}([0,1]\times[0,T])$, (99), (100), (101), (102) and the definitions of G_{Inn}, G_B^λ, G_I^λ and G_{IB}^λ, we have

$$\int_0^1 (|G_{Inn}|^2 + |G_B^\lambda|^2 + |G_I^\lambda|^2 + |G_{IB}^\lambda|^2)dx \leq M\lambda.$$

On the other hand, as in (118), we have, with the aid of Sobolev's lemma, that

$$\|G_R^\lambda\|_{L_x^2}^2 \leq M\|z_R^\lambda\|_{L_x^2}^2 + M\lambda^2\|E_R^\lambda\|_{L_x^2}^2 + M\|(z_R^\lambda, z_{R,x}^\lambda)\|_{L_x^2}^2\|E_R^\lambda\|_{L_x^2}^2.$$

Thus

$$\int_0^1 g^\lambda E_R^\lambda dx \leq \epsilon\|E_R^\lambda\|_{L_x^2}^2 + M\|z_R^\lambda\|_{L_x^2}^2 + M\lambda^2\|E_R^\lambda\|_{L_x^2}^2 + M\|(z_R^\lambda, z_{R,x}^\lambda)\|_{L_x^2}^2\|E_R^\lambda\|_{L_x^2}^2 + M\lambda \qquad (122)$$

Then, combining (121) with (122), using the positivity of $\mathcal{Z}^0$, taking ϵ small enough and then restricting λ to be small enough, one gets

$$\begin{aligned} & \lambda^2 \frac{d}{dt}\|E_R^\lambda\|_{L_x^2}^2 + \lambda^2\|E_{R,x}^\lambda\|_{L_x^2}^2 + c_2\|E_R^\lambda\|_{L_x^2}^2 \\ \leq\ & M\|z_R^\lambda\|_{L_x^2}^2 + M\|(z_R^\lambda, z_{R,x}^\lambda)\|_{L_x^2}^2\|E_R^\lambda\|_{L_x^2}^2 + M\lambda. \end{aligned} \tag{123}$$

Integrating (123) with respect to t, one gets

$$\begin{aligned} & \lambda^2\|E_R^\lambda(t)\|_{L_x^2}^2 + \lambda^2\int_0^t \|E_{R,x}^\lambda\|_{L_x^2}^2 dt + c_2\int_0^t \|E_R^\lambda\|_{L_x^2}^2 dt \\ \leq\ & \lambda^2\|E_R^\lambda(t=0)\|_{L_x^2}^2 + M\int_0^t \|z_R^\lambda\|_{L_x^2}^2 dt \\ & +M\int_0^t \|(z_R^\lambda, z_{R,x}^\lambda)\|_{L_x^2}^2\|E_R^\lambda\|_{L_x^2}^2 dt + M\lambda. \end{aligned} \tag{124}$$

The desired estimate (113) follows from (120) and (124). This completes the proof of Lemma 6. □

Next we show the estimates of the time tangential derivatives $\partial_t(z_R^\lambda, E_R^\lambda)$ of $(z_R^\lambda, E_R^\lambda)$.

Lemma 7 *Under the assumptions of Theorem 3, we have*

$$\begin{aligned} & \|z_{R,t}^\lambda(t)\|_{L_x^2}^2 + \lambda^2\|E_{R,t}^\lambda(t)\|_{L_x^2}^2 + \int_0^t \|(z_{R,xt}^\lambda, E_{R,t}^\lambda)\|_{L_x^2}^2 dt + \lambda^2\int_0^t \|E_{R,xt}^\lambda\|_{L_x^2}^2 dt \\ \leq\ & M(\|z_{R,t}^\lambda(t=0)\|_{L_x^2}^2 + \lambda^2\|E_{R,t}^\lambda(t=0)\|_{L_x^2}^2) \\ & +M\int_0^t \|(E_R^\lambda, z_R^\lambda, z_{R,t}^\lambda)\|_{L_x^2}^2 dt + M\lambda^2\int_0^t \|E_{R,t}^\lambda\|_{L_x^2}^2 dt + M\lambda^4\int_0^t \|(E_{R,x}^\lambda, E_{R,xt}^\lambda)\|_{L_x^2}^2 dt \\ & +M\lambda^4\int_0^t \big(\|(E_{R,t}^\lambda, E_{R,xt}^\lambda)\|_{L_x^2}^2\|E_{R,x}\|_{L_x^2}^2 + \|E_R^\lambda\|_{L_x^2}^2\|E_{R,xt}^\lambda\|_{L_x^2}^2\big) dt \\ & +M\int_0^t \big(\|(z_{R,t}^\lambda, z_{R,xt}^\lambda)\|_{L_x^2}^2\|E_R^\lambda\|_{L_x^2}^2 + \|(z_R^\lambda, z_{R,x}^\lambda)\|_{L_x^2}^2\|E_{R,t}^\lambda\|_{L_x^2}^2\big) dt + M\lambda. \end{aligned} \tag{125}$$

Proof of Lemma 7 Differentiating (104) with respect to t, multiplying the resulting equations by $z_{R,t}^\lambda$, then integrating it over $[0,1]\times[0,t]$ and noting that H_t also satisfies the same boundary condition as in (110), one gets by integrations by parts that

$$\begin{aligned}
\|z^\lambda_{R,t}(t)\|^2_{L^2_x} &= \|z^\lambda_{R,t}(t=0)\|^2_{L^2_x} + \int_0^t\int_0^1 f^\lambda_t z^\lambda_{R,t}dxdt - \int_0^t\int_0^1 (z^\lambda_{R,xt} + DE^\lambda_{R,t})z^\lambda_{R,xt}dxdt \\
&\quad - \int_0^t\int_0^1 H_{Inn,t}z^\lambda_{R,xt}dxdt - \int_0^t\int_0^1 H^\lambda_{B,t}z^\lambda_{R,xt}dxdt - \int_0^t\int_0^1 H^\lambda_{I,t}z^\lambda_{R,xt}dxdt \\
&\quad - \int_0^t\int_0^1 H^\lambda_{IB,t}z^\lambda_{R,xt}dxdt - \int_0^t\int_0^1 H^\lambda_{R,t}z^\lambda_{R,xt}dxdt. \qquad (126)
\end{aligned}$$

One needs to estimate the terms on the right above carefully.

First, it follows from the Cauchy-Schwartz's inequality that

$$\int_0^t\int_0^1 (z^\lambda_{R,xt} + DE^\lambda_{R,t})z^\lambda_{R,xt}dxdt \le -\frac{1}{2}\int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x}dt + M\int_0^t \|E^\lambda_{R,t}\|^2_{L^2_x}dt \qquad (127)$$

and

$$\int_0^t\int_0^1 f^\lambda_t z^\lambda_{R,t}dxdt \le \epsilon\int_0^t \|z^\lambda_{R,t}\|^2_{L^2_x}dt + M(\epsilon)\lambda^3 \qquad (128)$$

since f^λ does not depend upon the fast time scale. Here we also used $\mathcal{Z}^0_{tt}, \mathcal{E}^0_{tt} \in C^0([0,1]\times[0,T])$.

Similarly, since H_{Inn} and H^λ_B do not depend upon the fast time scale, so $H_{Inn,t}$ and $H^\lambda_{B,t}$ have the same structures as H_{Inn} and $H_{B,t}$, respectively. Hence, one obtains in a similar way as for (117), (using (106)), that

$$\int_0^t\int_0^1 H_{Inn,t}z^\lambda_{R,xt}dxdt \le \epsilon\int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x}dt + M\lambda^4 \qquad (129)$$

and

$$\int_0^t\int_0^1 H^\lambda_{B,t}z^\lambda_{R,xt}dxdt \le \epsilon\int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x}dt + M\lambda. \qquad (130)$$

Here we used the regularity of the inner solutions $\mathcal{E}^0, \mathcal{E}^0_x, \mathcal{E}^0_t, \mathcal{E}^0_{xt}, \mathcal{Z}^0, \mathcal{Z}^0_x, \mathcal{Z}^0_t, \mathcal{Z}^0_{xt} \in C^0([0,1]\times[0,T])$.

Owe to the strong singularity of time derivatives of the initial layers, we must estimate the integrals involving initial layers carefully.

First, by the Cauchy-Schwarz's inequality, we have

$$\int_0^t\int_0^1 H^\lambda_{I,t}z^\lambda_{R,xt}dxdt \le \epsilon\int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x}dt + M(\epsilon)\int_0^t\int_0^1 |H^\lambda_{I,t}|^2dxdt.$$

But, by (101) and (102), one gets

$$\begin{aligned}
&\int_0^t\int_0^1 |H^\lambda_{I,t}|^2 dxdt\\
=\ &\int_0^t\int_0^1 |z^2_{I,xs} - \lambda^2\big(\mathcal{E}^0_t(x,t)E^0_{I,x} + E^0_I\mathcal{E}^0_{xt}(x,t)\big)\\
&-\big(\mathcal{E}^0(x,t)E^0_{I,xs} + E^0_{I,s}(\mathcal{E}^0_x(x,t) + E^0_{I,x}) + E^0_I E^0_{I,xs}\big)|^2 dxdt\\
\le\ & M\int_0^t\int_0^1 \Big(|z^2_{I,xs}|^2 + |E^0_{I,x}|^2 + |E^0_I|^2 + |E^0_{I,xs}|^2 + |E^0_{I,s}|^2\Big)dxdt\\
\le\ & M\lambda^2.
\end{aligned}$$

Hence

$$\int_0^t\int_0^1 H^\lambda_{I,t} z^\lambda_{R,xt} dxdt \le \epsilon\int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x} + M\lambda^2. \tag{131}$$

Then, using the definition of H^λ_{IB}, we have

$$\begin{aligned}
&\int_0^t\int_0^1 H^\lambda_{IB,t} z^\lambda_{R,tx} dxdt\\
=\ &\int_0^t\int_0^1 -\lambda\Big(E^0_I(f(x)E^0_{+,\xi} - g(x)E^0_{-,\eta})\Big)_t z^\lambda_{R,tx} dxdt\\
&+\int_0^t\int_0^1 \lambda^2\partial_t\Big(\cdots\Big)^R_{HIB} z^\lambda_{R,xt} dxdt,
\end{aligned} \tag{132}$$

where $\Big(\cdots\Big)^R_{HIB}$ represents the remaining higher order term $O(\lambda^2)$ of H^λ_{IB}. By (101) and (102), one easily gets

$$\int_0^t\int_0^1 |\lambda^2\partial_t\Big(\cdots\Big)^R_{HIB}|^2 dxdt \le M\lambda^3,$$

which leads to

$$\begin{aligned}
&\int_0^t\int_0^1 \lambda^2\partial_t\Big(\cdots\Big)^R_{HIB} z^\lambda_{R,tx} dxdt\\
\le\ &\epsilon\int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x} dt + M(\epsilon)\int_0^t\int_0^1 |\lambda^2\partial_t\Big(\cdots\Big)^R_{HIB}|^2 dxdt\\
\le\ &\epsilon\int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x} dt + M\lambda^3.
\end{aligned} \tag{133}$$

It remains to control the first term on the right hand side of (132). Note that this singular integration is caused by the interactions between the boundary layer and the initial layer. So, to control it, we must use two-fold integrals in the time and space directions to cancel the oscillation of the electric field. Indeed, it can be treated as follows:

$$\begin{aligned}
& -\int_0^t\int_0^1 \lambda\Big(E_I^0(f(x)E^0_{+,\xi}-g(x)E^0_{-,\eta})\Big)_t z^\lambda_{R,tx}dxdt \\
= & -\frac{1}{\lambda}\int_0^t\int_0^1 \Big(E^0_{I,s}(f(x)E^0_{+,\xi}-g(x)E^0_{-,\eta})\Big)z^\lambda_{R,tx}dxdt \\
& -\int_0^t\int_0^1 \lambda\Big(E_I^0(f(x)E^0_{+,\xi t}-g(x)E^0_{-,\eta t})\Big)z^\lambda_{R,tx}dxdt \\
\leq & \ \epsilon\int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x}dt+\frac{1}{\lambda^2}\int_0^t\int_0^1 |E^0_{I,s}(f(x)E^0_{+,\xi}-g(x)E^0_{-,\eta})|^2dxdt+M\lambda^5 \\
\leq & \ \epsilon\int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x}dt+M\lambda,
\end{aligned} \tag{134}$$

where we have used

$$\begin{aligned}
& \frac{1}{\lambda^2}\int_0^t\int_0^1 |E^0_{I,s}(f(x)E^0_{+,\xi}+g(x)E^0_{-,\eta})|^2dxdt \\
\leq & \ \frac{M}{\lambda^2}\int_0^t\int_0^1 |E^0_{I,s}|^2(|E^0_{+,\xi}|^2+|E^0_{-,\eta}|^2)dxdt \\
\leq & \ \frac{M}{\lambda^2}\int_0^t \max_{0\leq x\leq 1}|E^0_{I,s}|^2dt\Big(\int_0^1 \max_{0\leq t\leq T}|E^0_{+,\xi}|^2dx+\int_0^1 \max_{0\leq t\leq T}|E^0_{+,\xi}|^2dx\Big) \\
\leq & \ M\lambda.
\end{aligned}$$

Combining (132) with (133) and (134), one gets

$$\int_0^t\int_0^1 H^\lambda_{IB,t}z^\lambda_{R,xt}dxdt \leq \int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x}dt+M\lambda. \tag{135}$$

Finally, we estimate the last integral on the right hand side of (126). We split it into five parts.

$$\int_0^t\int_0^1 H^\lambda_{R,t}z^\lambda_{R,xt}dxdt = I_1+I_2+I_3+I_4+I_5, \tag{136}$$

where

$$\begin{aligned}
I_1 \ = \ & -\lambda\int_0^t\int_0^1 \Big\{\big(f(x)E^0_{+,\xi}-g(x)E^0_{-,\eta}\big)E^\lambda_{R,t} \\
& +\big(f(x)E^0_{+,\xi t}-g(x)E^0_{-,\eta t}\big)E^\lambda_R\Big\}z^\lambda_{R,xt}dxdt,
\end{aligned}$$

$$
\begin{aligned}
I_2 \;=\; & -\lambda^2 \int_0^t \int_0^1 \Big\{ \big(\mathcal{E}^0 + f(x)E_+^0 + g(x)E_-^0\big) E_{R,xt}^\lambda \\
& + \big(\mathcal{E}_t^0 + f(x)E_{+,t}^0 + g(x)E_{-,t}^0\big) E_{R,x}^\lambda + \big(\mathcal{E}_x^0 + f'(x)E_+^0 + g'(x)E_-^0\big) E_{R,t}^\lambda \\
& + \big(\mathcal{E}_{xt}^0 + f'(x)E_{+,t}^0 + g'(x)E_{-,t}^0\big) E_R^\lambda \Big\} z_{R,xt}^\lambda \,dxdt,
\end{aligned}
$$

$$
I_3 = -\lambda^2 \int_0^t \int_0^1 (E_I^0 E_{R,xt}^\lambda + E_{I,x}^0 E_{R,t}^\lambda) z_{R,xt}^\lambda \,dxdt,
$$

$$
I_4 = -\lambda^2 \int_0^t \int_0^1 (E_{R,t}^\lambda E_{R,x}^\lambda + E_R^\lambda E_{R,xt}^\lambda) z_{R,xt}^\lambda \,dxdt,
$$

$$
I_5 = -\int_0^t \int_0^1 (E_{I,s}^0 E_{R,x}^\lambda + E_{I,xs}^0 E_R^\lambda) z_{R,xt}^\lambda \,dxdt
$$

Noting that there are an λ factor in the first term I_1 of (136) and an λ^2 factor in the second and third terms I_2, I_3 of (136), by the Cauchy-Schwarz's inequality, (99), (101) and the fact that $\mathcal{E}^0, \mathcal{E}_x^0, \mathcal{E}_t^0, \mathcal{E}_{xt}^0 \in C^0([0,1]\times[0,T])$, we have

$$
I_1 \le \epsilon \int_0^t \|z_{R,xt}^\lambda\|_{L_x^2}^2 dt + M(\epsilon)\lambda^2 \int_0^t \|(E_R^\lambda, E_{R,t}^\lambda)\|_{L_x^2}^2 dt, \tag{137}
$$

$$
I_2 \le \epsilon \int_0^t \|z_{R,xt}^\lambda\|_{L_x^2}^2 dt + M(\epsilon)\lambda^4 \int_0^t \|(E_R^\lambda, E_{R,x}^\lambda, E_{R,t}^\lambda, E_{R,xt}^\lambda)\|_{L_x^2}^2 dt \tag{138}
$$

and

$$
I_3 \le \epsilon \int_0^t \|z_{R,xt}^\lambda\|_{L_x^2}^2 dt + M(\epsilon)\lambda^4 \int_0^t \|(E_{R,t}^\lambda, E_{R,xt}^\lambda)\|_{L_x^2}^2 dt. \tag{139}
$$

For the nonlinear term I_4 of (136), one gets by Sobolev's lemma that

$$
\begin{aligned}
I_4 \;\le\; & \epsilon \int_0^t \|z_{R,xt}^\lambda\|_{L_x^2}^2 + M(\epsilon)\lambda^4 \int_0^t \int_0^1 (|E_{R,t}^\lambda E_{R,x}^\lambda|^2 + |E_R^\lambda E_{R,xt}^\lambda|^2) dxdt \\
\le\; & \epsilon \int_0^t \|z_{R,xt}^\lambda\|_{L_x^2}^2 + M\lambda^4 \int_0^t (\|E_{R,t}^\lambda\|_{L_x^\infty}^2 \|E_{R,x}^\lambda\|_{L_x^2}^2 + \|E_R^\lambda\|_{L_x^\infty}^2 \|E_{R,xt}^\lambda\|_{L_x^2}^2) dt \\
\le\; & \epsilon \int_0^t \|z_{R,xt}^\lambda\|_{L_x^2}^2 \\
& + M\lambda^4 \int_0^t \big(\|(E_{R,t}^\lambda, E_{R,xt}^\lambda)\|_{L_x^2}^2 \|E_{R,x}^\lambda\|_{L_x^2}^2 + \|E_R^\lambda\|_{L_x^2}^2 \|E_{R,xt}^\lambda\|_{L_x^2}^2) dt.
\end{aligned} \tag{140}
$$

It remains to estimate I_5. This is more difficult due to the lack of the uniform L^2 estimate of $E^\lambda_{R,x}$. This will be achieved by using the uniform boundedness on $\|s(E^0_{I,s}, E^0_{I,xs})\|_{L^\infty_{(x,t)}([0,1]\times[0,T])}$ and employing Hardy-Littlewood's inequality. In fact, by the Cauchy-Schwarz's inequality, one gets

$$
\begin{aligned}
I_5 \quad \leq \quad & \epsilon \int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x} dt + M(\epsilon) \int_0^t \int_0^1 |E^0_{I,s} E^\lambda_{R,x} + E^0_{I,xs} E^\lambda_R|^2 dxdt \\
= \quad & \epsilon \int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x} dt + M(\epsilon) \int_0^t \int_0^1 |t\big(E^0_{I,s} \frac{E^\lambda_{R,x} - E^\lambda_{R,x}(x,t=0)}{t} \\
& + E^0_{I,xs} \frac{E^\lambda_R - E^\lambda_R(x,0)}{t}\big) + \big(E^0_{I,s} E^\lambda_{R,x}(x,0) + E^0_{I,xs} E^\lambda_R(x,0)\big)|^2 dxdt \\
= \quad & \epsilon \int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x} dt + M(\epsilon) \int_0^t \int_0^1 |\lambda^2 s \big(E^0_{I,s} \frac{E^\lambda_{R,x} - E^\lambda_{R,x}(x,t=0)}{t} \\
& + E^0_{I,xs} \frac{E^\lambda_R - E^\lambda_R(x,0)}{t}\big) + \big(E^0_{I,s} E^\lambda_{R,x}(x,0) + E^0_{I,xs} E^\lambda_R(x,0)\big)|^2 dxdt \\
\leq \quad & \epsilon \int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x} dt \\
& + M\lambda^4 \max_{0\leq s\leq\infty} \max_{0\leq x\leq 1} (s|E^0_{I,s}| + s|E^0_{I,xs}|)^2 \int_0^1 \int_0^t (|\frac{E^\lambda_{R,x} - E^\lambda_{R,x}(x,0)}{t}|^2 \\
& + |\frac{E^\lambda_R - E^\lambda_R(x,0)}{t}|^2) dtdx + M \int_0^t \int_0^1 (|E^0_{I,s} E^\lambda_{R,x}(x,0)|^2 + |E^0_{I,xs} E^\lambda_R(x,0)|^2) dxdt \\
\leq \quad & \epsilon \int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x} dt + M\lambda^4 \int_0^1 (\int_0^t |E^\lambda_{R,xt}|^2 dt + \int_0^t |E^\lambda_{R,t}|^2 dt) dx + M\lambda \\
\leq \quad & \epsilon \int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x} dt + M\lambda^4 \int_0^t \|(E^\lambda_{R,xt}, E^\lambda_{R,t})\|^2_{L^2_x} dt + M\lambda. \qquad (141)
\end{aligned}
$$

Here we have used

$$
\begin{aligned}
& \int_0^t \int_0^1 (|E^0_{I,s} E^\lambda_{R,x}(x,0)|^2 + |E^0_{I,xs} E^\lambda_R(x,0)|^2) dxdt \\
\leq \quad & M\lambda^2 (\|E^\lambda_{0R,x}\|^2_{L^\infty_x} \int_0^t \int_0^1 |E^0_{I,s}|^2 dxdt + \|E^\lambda_{0R}\|^2_{L^\infty_x} \int_0^t \int_0^1 |E^0_{I,xs}|^2 dxdt) \\
\leq \quad & M\lambda^4 ((M\lambda^{\frac{1}{2}-2})^2 + (M\lambda^{\frac{1}{2}-1})^2) \\
\leq \quad & M\lambda,
\end{aligned}
$$

due to Sobolev's lemma, (102) and the assumption (43),

$$\begin{aligned}
& \max_{0\le s\le\infty}\max_{0\le x\le 1}(s|E^0_{I,s}| + s|E^0_{I,xs}|) \\
= \;& \max_{0\le t\le T}\max_{0\le x\le 1}\big(\frac{t}{\lambda^2}(|E^0_{I,s}(x,\frac{t}{\lambda^2})| + |E^0_{I,xs}(x,\frac{t}{\lambda^2})|)\big) \\
\le \;& M\max_{0\le t\le T}((\frac{t}{\lambda^2})e^{-\delta\frac{t}{\lambda^2}}) \\
\le \;& M
\end{aligned}$$

and the fact that $(E^\lambda_R - E^\lambda_R(t=0))(t=0) = 0$ and hence $(E^\lambda_{R,x} - E^\lambda_{R,x}(t=0))(t=0) = 0$, and applied Hardy-Littlewood's inequality to control $\int_0^t |\frac{E^\lambda_{R,x}-E^\lambda_{R,x}(t=0)}{t}|^2 dt$ and $\int_0^t |\frac{E^\lambda_R - E^\lambda_R(t=0)}{t}|^2 dt$ by $\int_0^t |E^\lambda_{R,xt}|^2 dt$ and $\int_0^t |E^\lambda_{R,t}|^2 dt$, respectively.

Combining (136) with (137)-(141), one gets

$$\begin{aligned}
& \int_0^t\int_0^1 H^\lambda_{R,t} z^\lambda_{R,xt}\,dxdt \\
\le \;& \epsilon\int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x}dt + M\lambda^2\int_0^t \|(E^\lambda_R, E^\lambda_{R,t})\|^2_{L^2_x}dt + M\lambda^4\int_0^t \|(E^\lambda_{R,x}, E^\lambda_{R,xt})\|^2_{L^2_x}dt \\
& + M\lambda^4\int_0^t \big(\|(E^\lambda_{R,t}, E^\lambda_{R,xt})\|^2_{L^2_x}\|E^\lambda_{R,x}\|^2_{L^2_x} + \|E^\lambda_R\|^2_{L^2_x}\|E^\lambda_{R,xt}\|^2_{L^2_x}\big)dt + M\lambda^2. \quad (142)
\end{aligned}$$

Therefore, putting (126) and estimates (127), (128), (129), (130), (131), (135) and (142), together, and taking ϵ small enough, one shows that

$$\begin{aligned}
& \|z^\lambda_{R,t}(t)\|^2_{L^2_x} + c_3\int_0^t \|z^\lambda_{R,xt}\|^2_{L^2_x}dt \\
\le \;& \|z^\lambda_{R,t}(t=0)\|^2_{L^2_x} + M\int_0^t \|(z^\lambda_{R,t}, E^\lambda_{R,t})\|^2_{L^2_x}dt \\
& + M\lambda^2\int_0^t \|E^\lambda_R\|^2_{L^2_x}dt + M\lambda^4\int_0^t \|(E^\lambda_{R,x}, E^\lambda_{R,xt})\|^2_{L^2_x}dt \\
& + M\lambda^4\int_0^t \big(\|(E^\lambda_{R,t}, E^\lambda_{R,xt})\|^2_{L^2_x}\|E_{R,x}\|^2_{L^2_x} + \|E^\lambda_R\|^2_{L^2_x}\|E^\lambda_{R,xt}\|^2_{L^2_x}\big)dt + M\lambda. \quad (143)
\end{aligned}$$

Note that $E^\lambda_{R,t}$ also satisfies the same boundary condition as in (109). Thus, differentiating (104) with respect to t, multiplying the resulting equations by $z^\lambda_{R,t}$ and then integrating it over $[0,1]\times[0,t]$, one gets by integrations by parts that

$$\begin{aligned}
& \frac{\lambda^2}{2}\|E^\lambda_{R,t}(t)\|^2_{L^2_x} + \lambda^2\int_0^t \|E^\lambda_{R,xt}\|^2_{L^2_x}dt + \int_0^t\int_0^1 \mathcal{Z}^0|E^\lambda_{R,t}|^2 dxdt \\
= \;& \frac{\lambda^2}{2}\|E^\lambda_{R,t}(t=0)\|^2_{L^2_x} - \int_0^t\int_0^1 \mathcal{Z}^0_t E^\lambda_R E^\lambda_{R,t}\,dxdt + \int_0^t\int_0^1 g^\lambda_t E^\lambda_{R,t}\,dxdt. \quad (144)
\end{aligned}$$

First, by the Cauchy-Schwarz's inequality, one gets

$$-\int_0^t\int_0^1 \mathcal{Z}_t^0 E_R^\lambda E_{R,t}^\lambda dxdt \le \epsilon\int_0^t \|E_{R,t}^\lambda\|_{L_x^2}^2 dt + M(\epsilon)\int_0^t \|E_R^\lambda\|_{L_x^2}^2 dt \tag{145}$$

and

$$\begin{aligned}
&\int_0^t\int_0^1 g_t^\lambda E_{R,t}^\lambda dxdt\\
\le\ & \epsilon\int_0^t \|E_{R,t}^\lambda\|_{L_x^2}^2 dt + M(\epsilon)\int_0^t \|g^\lambda\|_{L_x^2}^2 dt\\
\le\ & \epsilon\int_0^t \|E_{R,t}^\lambda\|_{L_x^2}^2 dt + M\big(\int_0^t\int_0^1 |G_{Inn,t}|^2 dxdt + \int_0^t\int_0^1 |G_{B,t}^\lambda|^2 dxdt\\
&+\int_0^t\int_0^1 |G_{I,t}^\lambda|^2 dxdt + \int_0^t\int_0^1 |G_{IB,t}^\lambda|^2 dxdt + \int_0^t\int_0^1 |G_{R,t}^\lambda|^2 dxdt\big).
\end{aligned} \tag{146}$$

Now we treat each term on the right hand side of (146).

Using the structures of the inner solutions $\mathcal{E}_{tt}^0$, $\mathcal{E}_{xxt}^0 \in C^0([0,1]\times[0,T])$, one can get

$$\int_0^t\int_0^1 |G_{Inn,t}|^2 dxdt \le M\lambda^4. \tag{147}$$

term of (146), Since $G_{B,t}^\lambda$ has the same structure as G_B^λ, one can estimate the third term of (146) by (107) as

$$\int_0^t\int_0^1 |G_{B,t}^\lambda|^2 dxdt \le M\lambda. \tag{148}$$

Using the definition of G_I^λ, we have

$$\begin{aligned}
&\int_0^t\int_0^1 |G_{I,t}^\lambda|^2 dxdt\\
\le\ & \int_0^t\int_0^1 |E_{I,xxs}^0|^2 dxdt + J_{IR}\\
\le\ & M\lambda^2 + J_{IR},
\end{aligned} \tag{149}$$

where

$$\begin{aligned}
J_{IR} =\ & \int_0^t\int_0^1 |z_{I,s}^2(\mathcal{E}^0 + E_I^0) + z_I^2 E_{I,s}^0|^2 dxdt + \int_0^t\int_0^1 \big(|\lambda^2 z_I^2 \mathcal{E}_t^0|^2 + |\mathcal{Z}_t^0 E_I^0|^2\big) dxdt\\
&+\int_0^t\int_0^1 |\lambda^{-2}(\mathcal{Z}^0 - \mathcal{Z}^0(x,0)) E_{I,s}^0|^2 dxdt.
\end{aligned}$$

Using $\|z_I^2\|_{L^\infty_{x,t}} \le M$ and $\|(z^2_{I,s}, E^0_I, E^0_{I,s})\|_{L^2_t(L^\infty_x)} \le M\lambda$, one gets

$$J_{IR} \;\le\; M\lambda^2 + \int_0^t\int_0^1 |\lambda^{-2}(\mathcal{Z}^0 - \mathcal{Z}^0(x,0))E^0_{I,s}|^2 dxdt. \tag{150}$$

To estimate the remaining singular term on the right hand side of (150), we will use the higher regularity of $\mathcal{Z}^0_t$. It will follow from the mean value theorem and (102) that

$$\begin{aligned}
&\int_0^t\int_0^1 |\lambda^{-2}(\mathcal{Z}^0 - \mathcal{Z}^0(x,0))E^0_{I,s}|^2 dxdt\\
=\;& \int_0^t\int_0^1 |\int_0^1 \mathcal{Z}^0_t(x,t\theta)d\theta \frac{t}{\lambda^2} E^0_{I,s}(x,\frac{t}{\lambda^2})|^2 dxdt\\
\le\;& M\int_0^t\int_0^1 (\frac{t}{\lambda^2})^2 |E^0_{I,s}(x,\frac{t}{\lambda^2})|^2 dxdt\\
\le\;& M\lambda^2.
\end{aligned} \tag{151}$$

Thus,

$$J_{IR} \le M\lambda^2. \tag{152}$$

And so,

$$\int_0^t\int_0^1 |G^\lambda_{I,t}|^2 dxdt \le M\lambda^2. \tag{153}$$

The fifth term of (146) can be treated as in (132) so that

$$\begin{aligned}
&\int_0^t\int_0^1 |G^\lambda_{IB,t}|^2 dxdt\\
=\;& \int_0^t\int_0^1 |\lambda\Big((f(x)z^1_+ + g(x)z^1_-)E^0_I\Big)_t + \lambda^2\partial_t\Big(\cdots\Big)^R_{GIB}|^2 dxdt\\
\le\;& \frac{M}{\lambda^2}\int_0^t\int_0^1 (|z^1_+E^0_{I,s}|^2 + |z^1_-E^0_{I,s}|^2)dxdt + M\lambda^2\\
\le\;& M\lambda.
\end{aligned} \tag{154}$$

To estimate the sixth term of (146), we split it into four parts.

$$\int_0^t\int_0^1 G^\lambda_{R,t}E^\lambda_{R,t} dxdt = I_6 + I_7 + I_8 + I_9, \tag{155}$$

where

$$
\begin{aligned}
I_6 \quad = \quad & -\int_0^t \int_0^1 \big\{ (\mathcal{E}_t^0 + f(x) E_{+,t}^0 + g(x) E_{-,t}^0) z_R^\lambda \\
& + (\mathcal{E}^0 + f(x) E_+^0 + g(x) E_-^0 + E_I^0) z_{R,t}^\lambda \\
& + (\mathcal{Z}_t^0 + \lambda (f(x) z_{+,t}^1 + g(x) z_{-,t}^1)) E_R^\lambda \big\} E_{R,t}^\lambda dx dt,
\end{aligned}
$$

$$
I_7 = -\int_0^t \int_0^1 (\mathcal{Z}^0 + \lambda (f(x) z_+^1 + g(x) z_-^1) + \lambda^2 z_I^2) E_{R,t}^\lambda E_{R,t}^\lambda dx dt,
$$

$$
I_8 = -\int_0^t \int_0^1 (z_{R,t}^\lambda E_R^\lambda + z_R^\lambda E_{R,t}^\lambda) E_{R,t}^\lambda dx dt,
$$

and

$$
I_9 = -\frac{1}{\lambda^2} \int_0^t \int_0^1 (E_{I,s}^0 z_R^\lambda + \lambda^2 z_{I,s}^2 E_R^\lambda) E_{R,t}^\lambda dx dt.
$$

First I_6, I_7 and I_8 are treated as in (137)-(140) so that

$$
I_6 + I_7 \le \epsilon \int_0^t \|E_{R,t}^\lambda\|_{L_x^2}^2 dt + M \int_0^t \|(z_R^\lambda, z_{R,t}^\lambda, E_R^\lambda)\|_{L_x^2}^2 dt + M\lambda^2 \int_0^t \|E_{R,t}^\lambda\|_{L_x^2}^2 dt \quad (156)
$$

and

$$
\begin{aligned}
I_8 \quad \le \quad & \epsilon \int_0^t \|E_{R,t}^\lambda\|_{L_x^2}^2 dt + M(\epsilon) \int_0^t \int_0^1 (|z_{R,t}^\lambda E_R^\lambda|^2 + |z_R^\lambda E_{R,t}^\lambda|^2) dx dt \\
\le \quad & \epsilon \int_0^t \|E_{R,t}^\lambda\|_{L_x^2}^2 dt + M(\epsilon) \int_0^t (\|z_{R,t}^\lambda\|_{L^\infty}^2 \|E_R^\lambda\|_{L_x^2}^2 + \|z_R^\lambda\|_{L_x^\infty}^2 \|E_{R,t}^\lambda\|_{L_x^2}^2) dt \\
\le \quad & \epsilon \int_0^t \|E_{R,t}^\lambda\|_{L_x^2}^2 dt \\
& + M \int_0^t \big(\|(z_{R,t}^\lambda, z_{R,xt}^\lambda)\|_{L_x^2}^2 \|E_R^\lambda\|_{L_x^2}^2 + \|(z_R^\lambda, z_{R,x}^\lambda)\|_{L_x^2}^2 \|E_{R,t}^\lambda\|_{L_x^2}^2 \big) dt. \qquad (157)
\end{aligned}
$$

Now we treat the most singular term I_9 by employing Hardy-Littlewood's inequality.

$$\begin{aligned}
I_9 &= -\frac{1}{\lambda^2}\int_0^t\int_0^1 (E^0_{I,s}z^\lambda_R + \lambda^2 z^2_{I,s}E^\lambda_R)E^\lambda_{R,t}dxdt\\
&= -\frac{1}{\lambda^2}\int_0^t\int_0^1 t(E^0_{I,s}\frac{z^\lambda_R - z^\lambda_R(t=0)}{t} + \lambda^2 z^2_{I,s}\frac{E^\lambda_R - E^\lambda_R(t=0)}{t})E^\lambda_{R,t}dxdt\\
&\quad -\frac{1}{\lambda^2}\int_0^t\int_0^1 (E^0_{I,s}z^\lambda_R(t=0) + \lambda^2 z^2_{I,s}E^\lambda_R(t=0))E^\lambda_{R,t}dxdt\\
&\le \int_0^t\int_0^1 \Big(\|sE^0_{I,s}\|_{L^\infty_{(x,t)}}|\frac{z^\lambda_R - z^\lambda_R(t=0)}{t}||E^\lambda_{R,t}| + \lambda^2\|sz^2_{I,s}\|_{L^\infty_{(x,t)}}|\frac{E^\lambda_R - E^\lambda_R(t=0)}{t}||E^\lambda_{R,t}|\Big)dxdt\\
&\quad +\frac{1}{\lambda^2}\int_0^t\int_0^1 |(E^0_{I,s}z^\lambda_R(t=0) + \lambda^2 z^2_{I,s}E^\lambda_R(t=0))E^\lambda_{R,t}|dxdt\\
&\le M\int_0^1 \|\frac{z^\lambda_R - z^\lambda_R(t=0)}{t}\|_{L^2_t}\|E^\lambda_{R,t}\|_{L^2_t}dx + M\lambda^2\int_0^1 \|\frac{E^\lambda_R - E^\lambda_R(t=0)}{t}\|_{L^2_t}\|E^\lambda_{R,t}\|_{L^2_t}dx\\
&\quad +\frac{\epsilon}{2}\int_0^t \|E^\lambda_{R,t}\|^2_{L^2_x}dt + \frac{M}{\lambda^4}\int_0^t\int_0^1 |E^0_{I,s}z^\lambda_R(t=0)|^2dxdt + M\int_0^t\int_0^1 |z^2_{I,s}E^\lambda_R(t=0)|^2dxdt\\
&\le \frac{\epsilon}{2}\int_0^t \|E^\lambda_{R,t}\|^2_{L^2_x}dt + M\int_0^1 \|z^\lambda_{R,t}\|_{L^2_t}\|E^\lambda_{R,t}\|_{L^2_t}dx + M\lambda^2\int_0^1 \|E^\lambda_{R,t}\|_{L^2_t}\|E^\lambda_{R,t}\|_{L^2_t}dx\\
&\quad +\frac{M}{\lambda^4}\|z^\lambda_R(t=0)\|^2_{L^\infty_x}\int_0^t\int_0^1 |E^0_{I,s}|^2dxdt + M\|E^\lambda_R(t=0)\|^2_{L^\infty_x}\int_0^t\int_0^1 |z^2_{I,s}|^2dxdt\\
&\le (\epsilon + M\lambda^2)\int_0^t \|E^\lambda_{R,t}\|^2_{L^2_x}dt + M(\epsilon)\int_0^t \|z^\lambda_{R,t}\|^2_{L^2_x}dt + M\lambda. \qquad (158)
\end{aligned}$$

Here we used the fact that $\|z^\lambda_R(t=0)\|_{L^\infty_x} = \lambda\|z^\lambda_{0R}\|_{L^\infty_x} \le M\lambda^{\frac{3}{2}}$ and $\|E^\lambda_R(t=0)\|_{L^\infty_x} = \lambda\|E^\lambda_{0R}\|_{L^\infty_x} \le M$.

Hence, combining (155) with (156)-(158), one gets

$$\begin{aligned}
&\int_0^t\int_0^1 G^\lambda_{R,t}E^\lambda_{R,t}dxdt\\
&\le (\epsilon + M\lambda^2)\int_0^t \|E^\lambda_{R,t}\|^2_{L^2_x}dt + M\int_0^t \|(z^\lambda_R, z^\lambda_{R,t}, E^\lambda_R)\|^2_{L^2_x}dt\\
&\quad +M\int_0^t \big(\|(z^\lambda_{R,t}, z^\lambda_{R,xt})\|^2_{L^2_x}\|E^\lambda_R\|^2_{L^2_x} + \|(z^\lambda_R, z^\lambda_{R,x})\|^2_{L^2_x}\|E^\lambda_{R,t}\|^2_{L^2_x}\big)dt. \qquad (159)
\end{aligned}$$

Thus, putting (146), (147), (148), (153), (154) and (159) together and taking ϵ small enough shows

$$
\begin{aligned}
&\int_0^t\int_0^1 g_t^\lambda E_{R,t}^\lambda dxdt\\
\le\ &(\epsilon+M\lambda^2)\int_0^t\|E_{R,t}^\lambda\|_{L_x^2}^2dt+M\int_0^t\|(z_R^\lambda,z_{R,t}^\lambda,E_R^\lambda)\|_{L_x^2}^2dt\\
&+M\int_0^t\big(\|(z_{R,t}^\lambda,z_{R,xt}^\lambda)\|_{L_x^2}^2\|E_R^\lambda\|_{L_x^2}^2+\|(z_R^\lambda,z_{R,x}^\lambda)\|_{L_x^2}^2\|E_{R,t}^\lambda\|_{L_x^2}^2\big)dt+M\lambda. \quad (160)
\end{aligned}
$$

Therefore, for ϵ small enough, (144), together with (145) and (160), gives

$$
\begin{aligned}
&\lambda^2\|E_{R,t}^\lambda(t)\|_{L_x^2}^2+\lambda^2\int_0^t\|E_{R,xt}^\lambda\|_{L_x^2}^2dt+c_4\int_0^t\|E_{R,t}^\lambda\|_{L_x^2}^2dt\\
\le\ &M\lambda^2\|E_{R,t}^\lambda(t=0)\|_{L_x^2}^2+M\lambda^2\int_0^t\|E_{R,t}^\lambda\|_{L_x^2}^2dt+M\int_0^t\|(z_R^\lambda,E_R^\lambda,z_{R,t}^\lambda)\|_{L_x^2}^2dt\\
&+M\int_0^t\big\|(z_{R,t}^\lambda,z_{R,xt}^\lambda)\|_{L_x^2}^2\|E_R^\lambda\|_{L_x^2}^2+\|(z_R^\lambda,z_{R,x}^\lambda)\|_{L_x^2}^2\|E_{R,t}^\lambda\|_{L_x^2}^2\big)dt+M\lambda. \quad (161)
\end{aligned}
$$

The desired estimate (125) follows from (143) and (161). This completes the proof of Lemma 7. □

Finally, we use the basic estimates and the time derivative estimates of $(z_R^\lambda,E_R^\lambda)$ to obtain these of the space derivatives $\partial_x(z_R^\lambda,E_R^\lambda)$ of $(z_R^\lambda,E_R^\lambda)$.

Lemma 8 *Under the assumptions of Theorem 3, we have*

$$
\begin{aligned}
&\|(z_{R,x}^\lambda,E_R^\lambda)\|_{L_x^2}^2+\lambda^2\|E_{R,x}^\lambda\|_{L_x^2}^2\\
\le\ &M\|(z_R^\lambda,z_{R,t}^\lambda)\|_{L_x^2}^2+M\lambda^2\|(E_R^\lambda,E_{R,t}^\lambda)\|_{L_x^2}^2+M\lambda^4\|E_{R,x}^\lambda\|_{L_x^2}^2\\
&+M\lambda^4\|(E_R^\lambda,E_{R,x}^\lambda)\|_{L_x^2}^2\|E_{R,x}^\lambda\|_{L_x^2}^2+M\|(z_R^\lambda,z_{R,x}^\lambda)\|_{L_x^2}^2\|E_R^\lambda\|_{L_x^2}^2+M\lambda. \quad (162)
\end{aligned}
$$

Proof of Lemma 8 It follows from (119) and the Cauchy-Schwartz's inequality that

$$
\begin{aligned}
c_1\|z_{R,x}^\lambda\|_{L_x^2}^2 \le\ &-\frac{d}{dt}\|z_R^\lambda\|_{L_x^2}^2+M\|(z_R^\lambda,E_R^\lambda)\|_{L_x^2}^2+M\lambda^4\|E_{R,x}^\lambda\|_{L_x^2}^2\\
&+M\lambda^4(\|(E_R^\lambda,E_{R,x}^\lambda)\|_{L_x^2}^2\|E_{R,x}^\lambda\|_{L_x^2}^2+M\lambda\\
\le\ &M\|(z_R^\lambda,z_{R,t}^\lambda,E_R^\lambda)\|_{L_x^2}^2+M\lambda^4\|E_{R,x}^\lambda\|_{L_x^2}^2\\
&+M\lambda^4\|(E_R^\lambda,E_{R,x}^\lambda)\|_{L_x^2}^2\|E_{R,x}^\lambda\|_{L_x^2}^2+M\lambda. \quad (163)
\end{aligned}
$$

Similarly, it follows from (123) and the Cauchy-Schwartz's inequality that

$$
\begin{aligned}
&\lambda^2\|E_{R,x}^\lambda\|_{L_x^2}^2+c_2\|E_R^\lambda\|_{L_x^2}^2\\
\le\ &-\lambda^2\frac{d}{dt}\|E_R^\lambda\|_{L_x^2}^2+M\|z_R^\lambda\|_{L_x^2}^2+M\|(z_R^\lambda,z_{R,x}^\lambda)\|_{L_x^2}^2\|E_R^\lambda\|_{L_x^2}^2+M\lambda\\
\le\ &M\lambda^2\|(E_R^\lambda,E_{R,t}^\lambda)\|_{L_x^2}^2+M\|z_R^\lambda\|_{L_x^2}^2+M\|(z_R^\lambda,z_{R,x}^\lambda)\|_{L_x^2}^2\|E_R^\lambda\|_{L_x^2}^2+M\lambda. \quad (164)
\end{aligned}
$$

The desired estimate (162) follows from (163) and (164). This completes the proof of Lemma 8. □

The End of the Proof of Theorem 3 Introduce the following λ-weighted functional for the remainder terms

$$\Gamma^\lambda(t) = \|(z_R^\lambda, z_{R,x}^\lambda, z_{R,t}^\lambda)\|_{L_x^2}^2 + \lambda^2\|(E_R^\lambda, E_{R,x}^\lambda, E_{R,t}^\lambda)\|_{L_x^2}^2 + \|E_R^\lambda\|_{L_x^2}^2. \tag{165}$$

Then it follows from (113) + (125) + δ(162), by taking δ small enough and then λ small enough, and hence absorbing $\delta\big(M\|(z_R^\lambda, z_{R,t}^\lambda)\|_{L_x^2}^2 + M\lambda^2\|(E_R^\lambda, E_{R,t}^\lambda)\|_{L_x^2}^2\big)$ and $M\delta\lambda^4\|E_{R,x}^\lambda\|_{L_x^2}^2 + M\lambda^4\int_0^t \|E_{R,xt}^\lambda\|_{L_x^2}^2 dt$ of the right hand side of (113)+(125)+δ(162) by $\|(z_R^\lambda, z_{R,t}^\lambda)\|_{L_x^2}^2 + \lambda^2\|(E_R^\lambda, E_{R,t}^\lambda)\|_{L_x^2}^2$ and $\delta\lambda^2\|E_{R,x}^\lambda\|_{L_x^2}^2 + \lambda^2\int_0^t \|E_{R,xt}^\lambda\|_{L_x^2}^2 dt$ of the left hand side of (113)+(125)+δ(162) and then performing a lengthy and direct computations, that

$$\begin{aligned} &\Gamma^\lambda(t) + \int_0^t \|(z_{R,x}^\lambda, z_{R,xt}^\lambda, E_R^\lambda, E_{R,t}^\lambda)\|_{L_x^2}^2 dt + \lambda^2 \int_0^t \|(E_{R,x}^\lambda, E_{R,xt}^\lambda)\|_{L_x^2}^2 dt \\ \leq\ & M\Gamma^\lambda(t=0) + M\int_0^t (\Gamma^\lambda(t) + (\Gamma^\lambda(t))^2) dt + M\lambda^2 \int_0^t \Gamma^\lambda(t)\|E_{R,xt}^\lambda\|_{L_x^2}^2 dt \\ &+M\int_0^t \Gamma^\lambda(t)\|(z_{R,xt}^\lambda, E_{R,t}^\lambda)\|_{L_x^2}^2 dt + M\lambda + M(\Gamma^\lambda(t))^2. \end{aligned} \tag{166}$$

We claim that, for any $T \in [0, T_{\max}), T_{\max} \leq \infty$, there exists an $\lambda_0 \ll 1$ such that, for any $\lambda \leq \lambda_0$, if $\Gamma^\lambda(t=0) \leq \tilde{M}\lambda^{\min\{\alpha,1\}}$ for some $\alpha > 0$, then

$$\Gamma^\lambda(t) \leq \tilde{M}\lambda^{\min\{\alpha,1\}-\delta} \tag{167}$$

holds for any $\delta \in (0, \min\{\alpha, 1\})$ and $0 \leq t \leq T$.

Otherwise, there exists $T \in [0, T_{\max}), T_{\max} \leq \infty,$, for any $\lambda_0 \ll 1$ such that, for some $\lambda \leq \lambda_0$,

$$\Gamma^\lambda(t_0^\lambda) > \tilde{M}\lambda^{\min\{\alpha,1\}-\delta}$$

holds for some $\delta \in (0, \min\{\alpha, 1\})$ and for some $0 < t_0^\lambda \leq T$.

Denote the first root of $\Gamma^\lambda(t) = \tilde{M}\lambda^{\min\{\alpha,1\}-\delta}$ in $[0, t_0^\lambda]$ by t_1^λ. Then we have

$$\Gamma^\lambda(t) \leq \tilde{M}\lambda^{\min\{\alpha,1\}-\delta}, \quad 0 < t \leq t_1^\lambda \leq t_0^\lambda \leq T, \quad \Gamma^\lambda(t_1^\lambda) = \tilde{M}\lambda^{\min\{\alpha,1\}-\delta} \tag{168}$$

Using (166) and (168), one gets

$$\begin{aligned}
&\Gamma^\lambda(t)+\int_0^t\|(z^\lambda_{R,x},z^\lambda_{R,xt},E^\lambda_R,E^\lambda_{R,t})\|^2_{L^2_x}dt+\lambda^2\int_0^t\|(E^\lambda_{R,x},E^\lambda_{R,xt})\|^2_{L^2_x}dt\\
\le\ & M\tilde{M}\lambda^{\min\{\alpha,1\}}+M\int_0^t(\Gamma^\lambda(t)+\tilde{M}\lambda^{\min\{\alpha,1\}-\delta}\Gamma^\lambda(t))dt+M\lambda^2\int_0^t\tilde{M}\lambda^{\min\{\alpha,1\}-\delta}\|E^\lambda_{R,xt}\|^2_{L^2_x}dt\\
&+M\int_0^t\tilde{M}\lambda^{\min\{\alpha,1\}-\delta}\|(z^\lambda_{R,xt},E^\lambda_{R,t})\|^2_{L^2_x}dt+M\lambda+M\tilde{M}\lambda^{\min\{\alpha,1\}-\delta}\Gamma^\lambda(t)\\
\le\ & M\tilde{M}\lambda^{\min\{\alpha,1\}}+2M\int_0^t\Gamma^\lambda(t)dt+\frac{\lambda^2}{2}\int_0^t\|E^\lambda_{R,xt}\|^2_{L^2_x}dt\\
&+\frac{1}{2}\int_0^t(\|z^\lambda_{R,xt}\|^2_{L^2_x}+\|E^\lambda_{R,t}\|^2_{L^2_x})dt+M\lambda+\frac{1}{2}\Gamma^\lambda(t) \qquad (169)
\end{aligned}$$

since $\lambda\le\lambda_0\ll 1$ and λ_0 can be chosen to satisfy that

$$\tilde{M}\lambda_0^{\min\{\alpha,1\}-\delta}\le 1,\ M\tilde{M}\lambda_0^{\min\{\alpha,1\}-\delta}\le\frac{1}{2}.$$

Hence, it follow from (169) that

$$\Gamma^\lambda(t)\le 2M\tilde{M}\lambda^{\min\{\alpha,1\}}+4M\int_0^t\Gamma^\lambda(t)dt+2M\lambda.$$

Grownwall's Lemma gives

$$\begin{aligned}
\Gamma^\lambda(t)\ &\le\ (4Me^{4MT}T+1)\max\{2M\tilde{M},2M\}\lambda^{\min\{\alpha,1\}}\\
&\le\ (4Me^{4MT}T+1)\max\{2M\tilde{M},2M\}\lambda^{\delta}\lambda^{\min\{\alpha,1\}-\delta}\\
&\le\ \frac{\tilde{M}}{2}\lambda^{\min\{\alpha,1\}-\delta}
\end{aligned}$$

which contradicts with (168). This proves our claim (167).

The rest is to prove that there exist a positive constant $\tilde{M}$ and an $\alpha>0$ such that

$$\Gamma^\lambda(t=0)\le\tilde{M}\lambda^\alpha. \qquad (170)$$

In fact, it follows from the assumptions (42) and (43) on the initial data and (165) that

$$\Gamma^\lambda(t=0)=\|z^\lambda_{R,t}(t=0)\|^2_{L^2_x}+\lambda^2\|E^\lambda_{R,t}(t=0)\|^2_{L^2_x}+M\lambda. \qquad (171)$$

First, note that (104) implies

$$\begin{aligned}\|z^\lambda_{R,t}(t=0)\|_{L^2_x} &\le \lambda\|z^\lambda_{0R,xx}\|_{L^2_x}+\lambda\|(DE^\lambda_{0R})_x\|_{L^2_x}\\ &\quad+\|(H_{Inn,x},H^\lambda_{B,x},H^\lambda_{I,x},H^\lambda_{IB,x},H^\lambda_{R,x})(t=0)\|_{L^2_x}+M\|f^\lambda(t=0)\|_{L^2_x}.\end{aligned}$$

The assumptions (42) and (43) lead to

$$\lambda\|z^\lambda_{0R,xx}\|_{L^2_x}+\lambda\|(DE^\lambda_{0R})_x\|_{L^2_x}\le M\sqrt{\lambda},$$

while the definitions of $H_{Inn}, H^\lambda_B, H^\lambda_I, H^\lambda_{IB}$ and f^λ yield that

$$\|(H_{Inn,x},H^\lambda_{I,x},H^\lambda_{IB,x})(t=0)\|_{L^2_x}\le M\sqrt{\lambda},$$

$$\|f^\lambda(t=0)\|_{L^2_x}\le M\lambda^{\frac{3}{2}}$$

and

$$\begin{aligned}&\|H^\lambda_{B,x}(t=0)\|_{L^2_x}\\ \le\;& M\sqrt{\lambda}+\|\big(((D(x)-D(0))\frac{1}{\lambda}f(x)E^0_{+,\xi}+(D(x)-D(1))\frac{1}{\lambda}g(x)E^0_{-,\eta}\big)(t=0)\|_{L^2_x}\\ =\;& M\sqrt{\lambda}+\|\big(\int_0^1 D_x(\theta x)d\theta\frac{x}{\lambda}f(x)E^0_{+,\xi}\\ &-\int_0^1 D_x(1-\theta(1-x))d\theta(x)\frac{1-x}{\lambda}g(x)E^0_{-,\eta}\big)(t=0)\|_{L^2_x}\\ \le\;& M\sqrt{\lambda}.\end{aligned}$$

Here we have used the mean value theorem and the estimates $\|(\xi E^0_{+,\xi},\eta E^0_{+,\eta})(t=0)\|_{L^2_x}\le M\sqrt{\lambda}$.

In addition, the definition of $H^\lambda_R(t=0)$ and the assumption (43) imply that

$$\begin{aligned}\|H^\lambda_{R,x}\|_{L^2_x} &\le M\lambda^2\|\lambda E^\lambda_{0R,xx}(x)\|_{L^2_x}+M\lambda\|\lambda E^\lambda_{0R,x}\|_{L^2_x}+M\|\lambda E^\lambda_{0R}(x)\|_{L^2_x}\\ &\quad+\lambda^2\|(\lambda^2E^\lambda_{0R}(x)E^\lambda_{0R,xx}(x),(\lambda E^\lambda_{0R,x}(x))^2)\|_{L^2_x}\\ &\le M\lambda^{\frac{3}{2}}.\end{aligned}$$

Hence,

$$\|z^\lambda_{R,t}(t=0)\|_{L^2_x}\le M\sqrt{\lambda}. \tag{172}$$

Next, (105) implies that

$$\begin{aligned}\|\lambda E^\lambda_{R,t}(t=0)\|_{L^2_x} &\le \lambda^2\|E^\lambda_{0R,xx}(x)\|_{L^2_x} + \|z^0_0(x)E^\lambda_{0R}(x)\|_{L^2_x} \\ &\quad + \|\frac{1}{\lambda}(G_{Inn}, G^\lambda_B, G^\lambda_I, G^\lambda_{IB}, G^\lambda_R)(t=0)\|_{L^2_x}.\end{aligned}$$

Note that the only singular term is $I_{10} = \|\frac{1}{\lambda}\big(-f(x)(\mathcal{Z}^0(x,t)-\mathcal{Z}^0(0,t))E^0_+ - g(x)(\mathcal{Z}^0(x,t)-\mathcal{Z}^0(1,t))E^0_-\big)(t=0)\|_{L^2_x}$, while the other terms are easily controlled by $M\sqrt{\lambda}$. By the mean value theorem, one gets

$$\begin{aligned} I_{10} &\le M\|\frac{1}{\lambda}(\mathcal{Z}^0(x,t)-\mathcal{Z}^0(0,t))E^0_+(t=0)\|_{L^2_x} + M\|\frac{1}{\lambda}(\mathcal{Z}^0(x,t)-\mathcal{Z}^0(1,t))E^0_-(t=0)\|_{L^2_x} \\ &= M\|\frac{x}{\lambda}\int_0^1 \mathcal{Z}^0_x(\theta x, 0)d\theta E^0_+(t=0)\|_{L^2_x} + M\|\frac{1-x}{\lambda}\int_0^1 \mathcal{Z}^0_x(1-\theta(1-x),0)d\theta E^0_-(t=0)\|_{L^2_x} \\ &\le M\|(\xi E^0_+, \eta E^0_-)(t=0)\|^2_{L^2_x} \\ &\le M\sqrt{\lambda}.\end{aligned}$$

This gives

$$\|\lambda E^\lambda_{R,t}(t=0)\|_{L^2_x} \le M\sqrt{\lambda}. \tag{173}$$

Notice that here we have used the assumptions

$$\|E^\lambda_{0R,xx}(x)\|_{L^2_x} \le M\lambda^{\frac{1}{2}-2}, \quad \|E^\lambda_{0R}(x)\|_{L^2_x} \le M\sqrt{\lambda}.$$

Thus, (171), together with (172) and (173), gives the desired result (170) with $\alpha = 1$.

By (167) with $\alpha = 1$, one gets (44). This completes the proof of Theorem 3. □

6.2 The Proof of Theorem 4

In this subsection we prove Theorem 4 by pointing out some necessary modifications of the proof of Theorem 3. We want to proceed as in the proof of Theorem 3.

Now we assume that (47) and (48) hold. In this case, we must consider the effect of the nonzero limit (z^1_0, E^1_0) of the error terms $(z^\lambda_{0R}, E^\lambda_{0R})$ of the initial data (39). In fact, $z^1_0(x)$ produces the extra initial layer functions (z^3_I, E^1_I), given by the solution to (50)-(53). Since (50)-(53) can be solved exactly, it is easy to see that (z^3_I, E^1_I) has the completely same properties as these of (z^2_I, E^0_I). Hence we choose the 'ansatz' as

$$\begin{aligned}(\tilde{z}^\lambda, \tilde{E}^\lambda)^T_{app} &= \Big(\mathcal{Z}^0 + \lambda(f(x)z^1_+ + g(x)z^1_-) + \lambda z^1_0(x) + \lambda^2 z^2_I + \lambda^3 z^3_I, \\ &\qquad \mathcal{E}^0 + f(x)E^0_+ + g(x)E^0_- + E^0_I + \lambda(E^1_0(x) + E^1_I)\Big)^T.\end{aligned}$$

Set

$$\left(\tilde{z}_R^\lambda(x,t), \tilde{E}_R^\lambda(x,t)\right)^T = (z^\lambda, E^\lambda)^T - \left(\tilde{z}^\lambda, \tilde{E}^\lambda\right)_{app}^T. \tag{174}$$

Then

$$\begin{aligned}\left(\tilde{z}_R^\lambda(x,t), \tilde{E}_R^\lambda(x,t)\right)^T(t=0) &= \lambda(z_{0R}^\lambda - z_0^1, E_{0R}^\lambda - E_0^1)^T \\ &= \lambda(\tilde{z}_{0R}^\lambda, \tilde{E}_{0R}^\lambda).\end{aligned}$$

By assumptions (47) and (48), one gets that $(\tilde{z}_{0R}^\lambda, \tilde{E}_{0R}^\lambda)$ satisfies assumptions (47) and (48). It remains to establish the energy estimates for the error function $\left(\tilde{z}_R^\lambda(x,t), \tilde{E}_R^\lambda(x,t)\right)^T$.

First, replacing $(z^\lambda, E^\lambda)^T$ by

$$(z^\lambda, E^\lambda)^T = \left(\tilde{z}^\lambda, \tilde{E}^\lambda\right)_{app}^T + \left(\tilde{z}_R^\lambda(x,t), \tilde{E}_R^\lambda(x,t)\right)^T$$

in the system (14)-(15), we obtain the equations (104) and (105) with $(z_R^\lambda, E_R^\lambda)$ replaced by $(\tilde{z}_R^\lambda(x,t), \tilde{E}_R^\lambda(x,t))^T$ and H, G replaced by $\tilde{H}, \tilde{G}$, where $\tilde{H}_B^\lambda = H_B^\lambda$, $\tilde{G}_B^\lambda = G_B^\lambda$ and $\tilde{H}_{Inn}$ ($\tilde{G}_{Inn}$), $\tilde{H}_I^\lambda$ ($\tilde{G}_I^\lambda$), $\tilde{H}_{IB}^\lambda$ ($\tilde{G}_{IB}^\lambda$), $\tilde{H}_R^\lambda$ ($\tilde{G}_R^\lambda$) are defined by the following:

$$\tilde{H}_{Inn}(x,t) = \lambda(z_{0x}^1(x) + D(x)E_0^1(x)) - \lambda^2\mathcal{E}^0\mathcal{E}_x^0 - \lambda^3\mathcal{E}^0 E_0^1(x),$$

$$\begin{aligned}\tilde{H}_I^\lambda(x,s) &= \lambda^2 z_{I,x}^2 + \lambda^3 z_{I,x}^3 \\ &\quad -\lambda^2\left(\mathcal{E}^0(E_{I,x}^0 + \lambda E_{I,x}^1) + (E_I^0 + \lambda E_I^1)(\mathcal{E}_x^0 + E_{I,x}^0 + \lambda(E_{0x}^1 + E_{I,x}^1))\right), \\ \tilde{H}_{IB}^\lambda(x,\xi,\eta,t,s) &= -\lambda\Big((E_I^0 + \lambda E_I^1)(f(x)E_{+,\xi}^0 - g(x)E_{-,\eta}^0)\Big) \\ &\quad -\lambda^2\Big((E_I^0 + \lambda E_I^1)(f'(x)E_+^0 + g'(x)E_-^0) + (f(x)E_+^0 + g(x)E_-^0)(E_{I,x}^0 + \lambda E_{I,x}^1)\Big),\end{aligned}$$

$$\begin{aligned}\tilde{H}_R^\lambda &= -\lambda\tilde{E^\lambda}_R(f(x)E_{+,\xi}^0 - g(x)E_{-,\eta}^0) \\ &\quad -\lambda^2\Big(((\mathcal{E}^0 + \lambda E_0^1) + f(x)E_+^0 + g(x)E_-^0)\tilde{E}_{R,x}^\lambda + ((\mathcal{E}_x^0 + \lambda E_{0x}^1) + f'(x)E_+^0 + g'(x)E_-^0)\tilde{E^\lambda}_R\Big) \\ &\quad -\lambda^2\left((E_I^0 + \lambda E_I^1)\tilde{E^\lambda}_{R,x} + (E_{I,x}^0 + \lambda E_{I,x}^1)\tilde{E^\lambda}_R\right) - \lambda^2\tilde{E^\lambda}_R\tilde{E^\lambda}_{R,x},\end{aligned}$$

$$\tilde{G}_{Inn}(x,t) = -\lambda^2\left(\mathcal{E}_t^0 - \mathcal{E}_{xx}^0 - \lambda E_{0xx}^1\right) - \lambda(\mathcal{Z}^0 E_0^1 + z_0^1\mathcal{E}^0) - \lambda^2 z_0^1 E_0^1,$$

$$\begin{aligned}\tilde{G}_I^\lambda &= -(\mathcal{Z}^0 - \mathcal{Z}^0(x,0))(E_I^0 + \lambda E_I^1) + \lambda^2 E_{I,xx}^0 + \lambda^3 E_{I,xx}^1 - \lambda^2 z_0^1 E_I^1 \\ &\quad -\lambda^2(z_I^2 + \lambda z_I^3)(\mathcal{E}^0 + \lambda E_0^1 + E_I^0 + \lambda E_I^1), \\ \tilde{G}_{IB}^\lambda &= -\lambda(f(x)z_+^1 + g(x)z_-^1)(E_I^0 + \lambda E_I^1) - \lambda^2(z_I^2 + \lambda z_I^3)(f(x)E_+^0 + g(x)E_-^0), \\ \tilde{G}_R^\lambda &= -(\mathcal{E}^0 + \lambda E_0^1 + f(x)E_+^0 + g(x)E_-^0 + E_I^0 + \lambda E_I^1)\tilde{z^\lambda}_R \\ &\quad -\lambda(f(x)z_+^1 + g(x)z_-^1 + z_0^1)\tilde{E^\lambda}_R - \lambda^2(z_I^2 + \lambda z_I^3)\tilde{E^\lambda}_R - \tilde{z^\lambda}_R\tilde{E^\lambda}_R.\end{aligned}$$

Next, we point out the difference at the boundary between $(\tilde{z}_R^\lambda, \tilde{E}_R^\lambda)$ and $(z_R^\lambda, E_R^\lambda)$. At present, $\tilde{E}_R^\lambda$ satisfies the nonhomogeneous boundary condition

$$(\tilde{E}_R^\lambda + \lambda E_0^1)(x = 0, 1; t) = 0, t > 0. \tag{175}$$

In fact, since $E_I^0(x = 0, 1; t) = 0$, it follows from the system (50)-(51) that

$$E_I^0(x = 0, 1; t) = 0, t > 0, \tag{176}$$

which gives with (50) and (51) that

$$E_I^1(x = 0, 1; t) = 0, t > 0. \tag{177}$$

Combining (174), the boundary condition $(16)_2$, (61), (176) and (177), one gets (175).

But $\tilde{H}^\lambda$ still satisfies the homogeneous boundary condition

$$\tilde{H}^\lambda(x = 0, 1; t) = 0, t > 0.$$

Thus

$$\tilde{H}_t^\lambda(x = 0, 1; t) = \tilde{E}_{R,t}^\lambda(x = 0, 1; t) = 0, t > 0$$

due to the fact that $E_0^1 = E_0^1(x)$ does not depend upon time t.

Finally, notice that $\tilde{H}(\tilde{G})$ are the sum of $H(G)$ and the extra higher order $O(\lambda)$ and hence has the completely similar structure as $H(G)$ and that the only term to be affected by nonhomogeneous boundary condition (175) is $-\lambda^2 \int_0^1 \tilde{E}_{R,xx}^\lambda \tilde{E}_R^\lambda dx$, which can be dealt with as follows:

$$\begin{aligned}
& -\lambda^2 \int_0^1 \tilde{E}_{R,xx}^\lambda \tilde{E}_R^\lambda dx \\
= & -\lambda^2 \int_0^1 \tilde{E}_{R,xx}^\lambda (\tilde{E}_R^\lambda + \lambda E_0^1) dx + \lambda^3 \int_0^1 \tilde{E}_{R,xx}^\lambda E_0^1 dx \\
= & \lambda^2 \int_0^1 |\tilde{E}_{R,x}^\lambda|^2 dx + \lambda^3 \int_0^1 \tilde{E}_{R,x}^\lambda E_{0x}^1 dx + \lambda \int_0^1 (\lambda^2 \tilde{E}_{R,t}^\lambda + \mathcal{Z}^0 \tilde{E}_R^\lambda - g^\lambda) E_0^1 dx \\
\geq & \frac{\lambda^2}{2} \int_0^1 |\tilde{E}_{R,x}^\lambda|^2 dx - M\lambda - M\lambda^5 \int_0^1 |\tilde{E}_{R,t}^\lambda|^2 dx - \lambda \int_0^1 (|\tilde{E}_R^\lambda|^2 + |g^\lambda|^2) dx \\
\geq & \frac{\lambda^2}{2} \int_0^1 |\tilde{E}_{R,x}^\lambda|^2 dx - M\lambda^5 \int_0^1 |\tilde{E}_{R,t}^\lambda|^2 dx - \lambda \int_0^1 |\tilde{E}_R^\lambda|^2 dx - M\lambda.
\end{aligned}$$

Here we had used the equation (105). Thus, we can proceed the energy method as in the previous proof of Theorem 3. These remarks conclude the proof of Theorem 4. The proof of Theorem 4 is complete. □

References

[1] A. Arnold, P. A. Markowich, G. Toscani, On large time asymptotics for drift-diffusion-Poisson systems, *Transport Theory Statist. Phys.*, **29** (2000), 571–581.

[2] Y. Brenier, Converence of the Vlasov-Poisson system to the incompressible Euler equations, *Commun. Partial Diff. Eqs.*, **25** (2000), 737-754.

[3] L. Carffarelli, J. Dolbeault, P. A. Markowich and C. Schmeiser, On Marxwellian equalibria of insulted semiconductors, *Interfaces Free Bound.*, **2** (2000), 331-339.

[4] S. Cordier, P. Degond, P. Markowich and C. Schmeiser, A travelling wave analysis of a hydrodynamical model of quasineutral plasma, *Asymptotic Anal.,* **11**(1995), 209-224.

[5] S. Cordier and E. Grenier, Quasineutral limit of an Euler-Poisson system arising from plasma physics, *Comm. PDE*, **23**(2000), 1099-1113.

[6] S. Cordier and E. Grenier, Quasineutral limit of two species Euler-Poisson system, *Proceedings of the Workshop "Recent Progress in the Mathematical Theory on Vlasov-Maxwell Equations"(Paris)*, pages 95-122, 1997.

[7] J. Dolbeault, P. A. Markowich and A. Unterreiter, On singular limits of mean-field equations, *Arch. Ration. Mech Anal.,* **158**(2001), 319-351.

[8] H. Gajewski, On existence, uniqueness and asymptotic behaviour of solutions of the basic equations for carrier transport in semiconductors, *ZAMM*, **65**(1985), 101-108.

[9] H. Gajewski and K. Gröger, On the basic equations for carrier transport in semiconductors, *J. Math. Anal. Appl.*, **113** (1986), 12-35.

[10] I. Gasser, L. Hsiao, P. Markowich and S. Wang, Quasineutral limit of a non-linear drift-diffusion model for semiconductor models, *J. Math. Anal. Appl.*, **268**(2002), 184-199.

[11] I. Gasser, C. D. Levermore, P. Markowich and C. Schmeiser, The initial time layer problem and the quasineutral limit in the semiconductor drift-diffusion model, *Europ. J. Appl. Math.*, **12** (2001), 497-512.

[12] E. Grenier, Oscillations in quasineutral plasmas, *Commun. Partial Diff. Eqs.*, **21** (1996), 363-394.

[13] E. Grenier, Defect measures of the Vlasov-Poisson system, *Commun. Partial Diff. Eqs.*, **20** (1995), 1189-1215.

[14] A. Jüngel, Qualitative behavior of solutions of a degenerate nonlinear drift-diffusion model for semiconductors, *Math. Models Methods Appl. Sci.*, **5** (1995), 497-518.

[15] A. Jüngel and Y.-J. Peng, A hierarchy of hydrodynamic models for plasmas: quasi-neutral limits in the drift-diffusion equations, *Asympt. Anal.*, **28** (2001), 49-73.

[16] A. Jüngel and S. Wang, Convergence of nonlinear Schrödinger-Poisson systems to the compressible Euler equations, *Commun. in Patial Diff. Eqns.*, **28**(2003), 1005-1022.

[17] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Uralceva, Linear and Quasi-linear Equations of Parabolic Type, Translations of Math. Monographs, A. M. S., 1968.

[18] J. G. Liu and Z. P. Xin, Boundary layer behavior in the fluid-dynamic limit for a nonlinear model Boltzmann equation, *Arch. Rat. Mech. Anal.*, **135** (1996), 61-105.

[19] P. A. Markowich, A singular perturbation analysis of the fundational semiconductor device equations, *SIAM J. Appl. Math.*, **44**(1984), 231-256.

[20] P. A. Markowich, C. Ringhofer and C. Schmeiser, Semiconductors Equations, Springer-Verlag, Vienna, New York, 1990.

[21] N. Masmoudi, From Vlasov-Poisson system to the incompressible Euler system, *Comm. Partial Differential Equations*, **26**(2001), 1913-1928.

[22] M. S. Mock, On equations describing steady-state carrier distributions in a semiconductor device, *Comm. Pure Appl. Math.*, **25**(1972), 781-792.

[23] M. S. Mock, An initial value problem from semiconductor device theory, *SIAM J. Math. Anal.,* **5**(1974), 697-612.

[24] M. S. Mock, *Analysis of Mathematical Models of Semiconductor Devices.* Boole Press, Dublin, 1983.

[25] C. P. Please, An analysis of semiconductor p-n junctions, *IMA J. Appl. Math.*, **28** (1982), 301-318.

[26] M. Puel, *Etudes Variationnelles et Asymptotiques des Problèmes de la Mécanique des Fluides et des Plasmas*, PhD thesis, Université Paris 6, 2000.

[27] C. Ringhofer, An asymptotic analysis of a transient p-n junction model, *SIAM J. Appl. Math.*, **47** (1987), 624-642.

[28] W. V. Roosbroeck, Theory of the flow of electrons and holes in Germanium and other semiconductors, *Bell Syst. Tech. J.,* **29** (1950), 560-607.

[29] I. Rubinstein, *Electro-Diffusion of Ions.* SIAM, Philadelphia, 1990.

[30] C. Schmeiser and S. Wang, Quasineutral limit of the drift-diffusion model for semiconductors with general initial data, *Math. Modeling Method Appl. Sci.*, **13**(2003), 463-470.

[31] W. Shockley, The theory of p-n junctions in semiconductors and p-n junction transistors, *Bell. Syst. Tech. J.* , **28**(1949), 435-489. (Reprinted in *Annals Math. Pura Appl.*, **166**(1987), 65-96.)

[32] A. Sitenko and V. Malnev, *Plasma Physics Theory.* Chapman & Hall, London, 1995.

[33] R. A. Smith, *Semiconductors*, 2nd ed., Cambridge University Press, Cambridge, 1978.

[34] S. M. Sze, *Physics of Semiconductor Devices*, Wiley-Interscience, New York, 1969.

[35] S. Wang, Quasineutral limit of Euler-Poisson system with and without viscosity, Accepted by *Commun. P. D. E.*, 2002.

[36] Z. P. Xin, Viscous boundary layers and their stability (I), *J. Part. Diff. Eqs.* , **11** (1998), 74-124.