COMPLEX MANIFOLDS WITH CERTAIN FAMILIES OF BIHOLOMORPHISMS

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Abstract. Given a family of biholomorphisms ϕ_t on a noncompact complex manifold M, we provide conditions, on ϕ_t, under which M is biholomorphic to $\mathbb{C}^n$. As an application, we generalize previous results in [1]. we prove that if (M^n, g) is a complete non-compact gradient Kähler-Ricci soliton with potential function f which is either steady with positive Ricci curvature, or expanding with non-negative Ricci curvature, and if the eigenvalues of the complex Hessian of f at the unique stationary point of the soliton satisfy some Diophantine conditions, then M is biholomorphic to $\mathbb{C}^n$. Hence in a certain sense almost all complete non-compact gradient Kähler-Ricci solitons with the above curvature conditions are biholomorphic to $\mathbb{C}^n$.

1. introduction

Let M^n be a complex manifold of complex dimension n. In this paper, we want to study conditions on M so that it will be biholomorphic to $\mathbb{C}^n$. In [1], it was proved that if M is a gradient Kähler-Ricci soliton of steady type or expanding type (see the definitions below) with positive or nonnegative Ricci curvature respectively, then M is biholomorphic to $\mathbb{C}^n$ provided that all eigenvalues of the complex Hessian of the potential function at the stationary point are equal. In this work, we want to generalize this result. Interestingly, it is not essential that M is a gradient Kähler-Ricci soliton provided there is a nice family of biholomorphisms defined on M. So let us first describe the family of biholomorphisms that we need.

We assume that there is a family of biholomorphisms ϕ_t, $0 \le t < \infty$ on M such that (i) ϕ_0 is the identity map; (ii) ϕ_t satisfies the semi-group

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property, i.e. $\phi_{t_1} \circ \phi_{t_2} = \phi_{t_1+t_2}$ for all $t_1,\ t_2 \geq 0$; (iii) ϕ_t has a fixed point p, i.e. $\phi_t(p) = p$ for all t; (iv) ϕ_t is shrinking to p in the sense that for any open set U of p and for any compact set W of M, there exists $T > 0$ such that $\phi_T(W) \subset U$; (v) in some coordinate neighborhood U of p with coordinates $z = (z^1, \dots, z^n)$ so that p corresponds to the origin, ϕ_t is given by $\phi_t(z_0) = z(t)$ for all $z_0 \in U$ where $z(t)$ satisfies

$$\begin{cases} \frac{dz^i}{dt} & = \hat{F}^i(z) \\ z^i(0) & = z_0^i \end{cases} \tag{1.1}$$

for $1 \leq i \leq n$, with some holomorphic functions $\hat{F}^i$, as long as the solution curve $z(t)$ remains in U.

We want to prove the following:

Theorem 1.1. *Let M^n be a complex manifold with a family of biholomorphisms ϕ_t satisfying the conditions mentioned above. Suppose $\hat{F}^i$ in (1.1) satisfies the following:*

(a) *For each i:*

$$\hat{F}^i = -\eta_i z^i + z^i G^i + F^i,\ 1 \leq i \leq n \tag{1.2}$$

where $\eta_n \geq \cdots \geq \eta_1 > 0$, F^i, G^i are holomorphic, $|G^i| = O(|z|)$, $|F^i| = O(|z|^{k_i})$ for some $k_i \geq 2$, F^i does not depend on z_i.

(b) *The following Diophantine condition is true. For each $i \geq 2$:*

$$\eta_i \neq \sum_{j=1}^{i-1} m_j \eta_j \tag{1.3}$$

for all sets of nonnegative integers $m_1, \dots, m_{i-1}$ with

$$\ell_i - 1 \geq \sum_{j=1}^{i-1} m_j \geq 2,$$

where $\ell_i \geq 2$ is the smallest integer such that $\ell_i \eta_1 > \eta_i$.

Then M is biholomorphic to $\mathbb{C}^n$.

In particular, if $2\eta_1 > \eta_i$, then there will be no restriction on η_i. Since $0 < \eta_1 \leq \cdots \leq \eta_n$, condition (b) is in fact equivalent to

$$\ell_i \neq \sum_{j \neq i} m_j \eta_j$$

for all i and for all sets of nonnegative integers $m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_n$ with $\ell_i - 1 \geq \sum_{j \neq i} m_j \geq 2$.

We may apply Theorem 1.1 to Kähler-Ricci solitons. Recall that a Kähler manifold $(M, g_{i\bar{j}}(x))$ is said to be a Kähler-Ricci soliton if there is a family of biholomorphisms ϕ_t given by a holomorphic vector field V, such that $g_{ij}(x,t) = \phi_t^*(g_{ij}(x))$ is a solution of the Kähler-Ricci flow:

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -2R_{i\bar{j}} - 2\rho g_{i\bar{j}} \tag{1.4}$$

for $0 \le t < \infty$ with initial data $g_{i\bar{j}}(x,t) = g_{i\bar{j}}(x)$, where $R_{i\bar{j}}$ denotes the Ricci tensor at time t and ρ is a constant. If $\rho = 0$, then the Kähler-Ricci soliton is said to be of steady type and if $\rho > 0$ then the Kähler-Ricci soliton is said to be of expanding type. We always assume that g is complete and M is noncompact.

Corollary 1.1. *Let (M^n, g) be a Kähler-Ricci soliton such that $R_{i\bar{j}} > 0$ if it is of steady type and $R_{i\bar{j}} \ge 0$ if it is of expanding type. Let V be the holomorphic vector field of the Kähler-Ricci soliton. Suppose $V(p) = 0$ for some p and suppose near p, in holomorphic local coordinates z^i around p with p being at the origin, $V = \sum_{i=1}^n \hat{F}^i \frac{\partial}{\partial z^i}$ such that $\hat{F}^i$'s satisfy conditions (a) and (b) in Theorem 1.1. Then M is biholomorphic to $\mathbb{C}^n$.*

Suppose (M, g) is a Kähler-Ricci soliton. If in addition, the holomorphic vector field is given by the gradient of a real valued function f, then it is called a gradient Kähler-Ricci soliton. Note that in this case, we have that

$$\begin{aligned} f_{i\bar{j}} &= R_{i\bar{j}} + 2\rho g_{i\bar{j}} \\ f_{ij} &= 0. \end{aligned} \tag{1.5}$$

If (M, g) is a gradient Kähler-Ricci soliton (of steady or expanding type) satisfying the curvature condition of Corollary 1.1, so that the scalar curvature attains its maximum at some point in case it is a steady gradient Kähler-Ricci soliton, then it is not hard to see that the flow has a unique fixed point p, see [1, 3]. Since $f_{i\bar{j}}$ can be unitarily diagonalized at p with real eigenvalues $0 < \eta_1 \le \eta_2 \le \dots \eta_n$, the gradient of f near p can be written as $\sum_{i=1}^n \hat{F}^i \frac{\partial}{\partial z^i}$ in some holomorphic local coordinates which is unitary at p such that $\hat{F}^i(z) = -\eta_i z^i + z^i G^i + F^i$, where G^i and F^i are holomorphic, with $|G^i| = O(|z|)$ and $|F^i| = O(|z|^{k_i})$. We have the following:

Corollary 1.2. *Let (M, g) be a complete non-compact gradient Kähler-Ricci soliton with potential f satisfying the curvature condition as in Corollary 1.1. In case it is a steady gradient Kähler-Ricci soliton, it is assumed that the scalar curvature attains its maximum at some point. In the above notations, suppose η_i's satisfy Diophantine conditions (b) in Theorem 1.1. Then M is biholomorphic to $\mathbb{C}^n$.*

Remark 1.1.

(i) By the corollary, if (M, g) is a Kähler-Ricci soliton satisfying the curvature assumptions in the corollary, and if η_i's satisfy

$$\eta_i \neq \sum_{j\neq i} m_j \eta_j$$

for all i and for sets of nonnegative integers m_j's with $\sum_{j\neq i} m_j \geq 2$, then M is biholomorphic to $\mathbb{C}^n$. Hence in a certain sense, almost all Kähler-Ricci soliton with satisfying the curvature assumptions in the corollary are biholomorphic to $\mathbb{C}^n$.

(ii) By the corollary, it is easy to see that if $2\eta_1 > \eta_n$, then M is biholomorphic to $\mathbb{C}^n$. In particular, if (M, g) is a gradient Kähler-Ricci soliton of expanding type with potential f satisfying $f_{i\bar{j}} = R_{i\bar{j}} + 2\rho g_{i\bar{j}}$, then M is biholomorphic to $\mathbb{C}^n$ if $0 \leq R_{i\bar{j}} \leq \rho g_{i\bar{j}}$.

(iii) In [1], it was proved that if $\eta_1 = \cdots = \eta_n$, then M is biholomorphic to $\mathbb{C}^n$. Hence Corollary 1.2 generalize the result in [1].

2. Analysis of the original flow and a corrected flow

In this section we shall restrict ourselves on flows defined on an open ball with center at the origin in $\mathbb{C}^n$. Denote such an open ball with radius r by $D(r)$. For $a > 0$, Consider the following flow on $D(a)$.

$$\frac{dz^i}{dt} = -V^i(z) = -\eta_i z^i + z^i G^i + F^i(z), \ 1 \leq i \leq n \tag{2.1}$$

where $0 < \eta_1 \leq \eta_2 \leq \cdots \leq \eta_n$, G^i and F^i are holomorphic functions satisfying condition (a) in Theorem 1.1.

Lemma 2.1. *Consider (2.1) in $D(a)$. Suppose G^i and F^i satisfy conditions (a) and (b) in Theorem 1.1. Then there is a biholomorphism $w = w(z)$ with $w(0) = 0$ near the origin such that $z(t)$ is a solution of (2.1) if and only if $w(z(t))$ is a solution of*

$$\frac{dw^i}{dt} = -\eta_i w^i + w^i \tilde{G}^i + \tilde{F}^i, \ 1 \leq i \leq n \tag{2.2}$$

such that $\tilde{G}^i$, $\tilde{F}^i$ are holomorphic in w. Moreover $|\tilde{G}^i| = O(|w|)$, $|\tilde{F}^i| = O(|w|^{\ell_i})$ for all i, and F^i does not depend on w_i. Here as before, ℓ_i is the smallest integer such that $\ell_i \eta_1 > \eta_i$.

Proof. Suppose that $k_i \geq \ell_i$ for all i, then there is nothing to be proved. Suppose there is i such that $\ell_i < k_i$. Recall that $\ell_i \geq 2$ is the smallest integer such that $\ell_i \eta_1 > \eta_i$. Hence $i > 1$. We want to prove that there is a biholomorphism w near the origin with $w(0) = 0$ such that $z(t)$ is a solution of (2.1) if and only if $w(z(t))$ is a solution of (2.2) such that $|\tilde{G}^i| = O(|w|)$, $|\tilde{F}^j| = O(|w|^{k_j})$ if $j \neq i$ and $|\tilde{F}^i| = O(|w|^{k_i+1})$ and $\tilde{F}^j$ does not depend on w^j for each j. It is easy to see that the lemma follows from this and induction.

For simplicity, let us assume that $i = n$ and let $k = k_n \geq 2$. The other cases are similar. Denote $\tilde{z} = (z^1, \ldots, z^{n-1})$. Since F^n is holomorphic and $|F^n| = O(|z|^k)$, $F^n = F^n(\tilde{z})$, we have

$$F^n(\tilde{z}) = \sum_{n=k}^{\infty} \sum_{|\alpha|=n} a_\alpha \tilde{z}^\alpha \tag{2.3}$$

where α is a multi-index so that if $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ is a set of nonnegative integers, then $\tilde{z}^\alpha = (z^1)^{\alpha_1} \cdots (z^{n-1})^{\alpha_{n-1}}$. Define the map $w = w(z)$ as follows: $w^i = z^i$, $1 \leq i \leq n-1$ and $w^n = z^n + \sum_{|\alpha|=k} b_\alpha \tilde{z}^\alpha$ where b_α are constants to be determined later. For $1 \leq i \leq n-1$,

$$\begin{aligned} \frac{dw^i}{dt} &= \frac{dw^i}{dt} \\ &= -\eta_i z^i + z^i G^i(z) + F^i \\ &= -\eta_i w^i + w^i G^i(z(w)) + \tilde{F}^i(z(w)). \end{aligned} \tag{2.4}$$

We can write $F^i(z(w)) = w^i H^i(w) + \tilde{F}^i(w)$ where $F^i(w)$ does not depend on w^i because we can express $F^i(w)$ as a power series. Let $\tilde{G}^i(w) = G^i(z(w)) + H^i(w)$. By the assumptions on G^i and F^i and the fact that $k_i \geq 2$, we have $|\tilde{G}^i| = O(|w|)$, $|\tilde{F}^i(w)| = O(|w|^{k_i})$.

(2.5)

$$
\begin{aligned}
\frac{dw^n}{dt}
&= \frac{dz^n}{dt} + \sum_{|\alpha|=k} b_\alpha \frac{d\tilde{z}^\alpha}{dt} \\
&= -\eta_n z^n + z^n G^n(z) + F^n(\tilde{z}) \\
&\quad + \sum_{\alpha=(\alpha_1,\ldots,\alpha_{n-1}),|\alpha|=k} b_\alpha \times \\
&\quad \times \left(\sum_{j=1}^{n-1} \alpha_j (z^1)^{\alpha_1} \ldots (z^{j-1})^{\alpha_{j-1}} \frac{dz^j}{dt} (z^j)^{\alpha_j - 1} (z^{j+1})^{\alpha_j} \ldots (z^{n-1})^{\alpha_{n-1}} \right) \\
&= -\eta_n w^n + w^n G^n(z) + (\eta_n - G^n(z(w))) \sum_{|\alpha|=k} b_\alpha \tilde{z}^\alpha + F^n(\tilde{z}) \\
&\quad + \sum_{\alpha=(\alpha_1,\ldots,\alpha_{n-1}),|\alpha|=k} b_\alpha \Bigg(\sum_{j=1}^{n-1} \alpha_j (z^1)^{\alpha_1} \ldots (z^{j-1})^{\alpha_{j-1}} \times \\
&\qquad \times \left(-\eta_j z^j + z^j G^j + F^j \right) (z^j)^{\alpha_j - 1} (z^{j+1})^{\alpha_j} \ldots (z^{n-1})^{\alpha_{n-1}} \Bigg) \\
&= -\eta_n w^n + w^n G^n(z(w)) + H(w) \\
&\quad + \sum_{\alpha=(\alpha_1,\ldots,\alpha_{n-1}),|\alpha|=k} \left[b_\alpha \left(\eta_n - \sum_{j=1}^{n-1} \alpha_j \eta_j \right) + a_\alpha \right] (z^1)^{\alpha_1} \ldots (z^{n-1})^{\alpha_{n-1}}
\end{aligned}
$$

where $|H(w)| = O(|w|^{k+1})$ by the assumptions on G^i, F^i and the fact that $k_i \geq 2$. Since $\alpha_1 + \ldots, \alpha_{n-1} = k$ and $\ell_n > k_n = k \geq 2$, by condition (b) in Theorem 1.1, we conclude that $\eta_n - \sum_{j=1}^{n-1} \alpha_j \eta_j \neq 0$ for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ so that $|\alpha| = k$. Hence for such an α we can find b_α such that

$$
b_\alpha \left(\eta_n - \sum_{j=1}^{n-1} \alpha_j \eta_j \right) + a_\alpha = 0. \tag{2.6}
$$

Hence for such choices of b_α, we have

$$
\frac{dw_n}{dt} = -\eta_n w^n + w^n G^n(z(w)) + H(w) \tag{2.7}
$$

where H is holomorphic and $|H| = O(|w|^{k+1})$. We can write $H(w) = w^n \tilde{H}(w) + \tilde{F}^n(w)$ where $\tilde{F}^n$ does not depend on w_n. Let $\tilde{G}^n = G^n + \tilde{H}(w)$, the result follows. □

It is easy to see that condition (b) in Theorem 1.1 is necessary for the Lemma 2.1. Consider the following example, see [4].

Example: Consider the system in $\mathbb{C}^2$

$$\begin{cases} \frac{dz^1}{dt} &= -z^1 \\ \frac{dz^2}{dt} &= -2z^2 + (z^1)^2. \end{cases} \tag{2.8}$$

Suppose there is a biholomorphism $w = w(z)$ with $w(0) = 0$ such that the above equation can be transformed to

$$\begin{cases} \frac{dw^1}{dt} &= -w^1 + H(w) \\ \frac{dw^2}{dt} &= -2w^2 + w^2 G(w) + F(w^1) \end{cases} \tag{2.9}$$

where H, G and F are holomorphic with $|G| = O(|w|)$ and $|F| = O(|w|^3)$. Suppose $w^1 = f(z^1, z^2)$ and $w^2 = g(z^1, z^2)$. In the following, derivatives of a function ϕ with respect to z^1 is denoted by ϕ_1 etc.

$$\begin{aligned} \frac{dw^2}{dt} &= g_1 \frac{dz^1}{dt} + g_2 \frac{dz^2}{dt} \\ &= -z^1 g_1 + \left(-2z^2 + (z^1)^2\right) g_2. \end{aligned} \tag{2.10}$$

Since

$$\frac{dw^2}{dt} = -2w^2 + w^2 G(w) + F(w^1),$$

we have

$$-z^1 g_1 + \left(-2z^2 + (z^1)^2\right) g_2 = -2g + gG + F.$$

Differentiate with respect to z^1, we have

$$-g_1 - z^1 g_{11} + 2z^1 g_2 + \left(-2z^2 + (z^1)^2\right) g_{21} = -2g_1 + gG_1 + g_1 G + F_1.$$

By the assumptions on F and G and the fact that $w(z)$ is a biholomorphism with $w(0) = 0$, we conclude from the above equality that $g_1(0,0) = 0$. Differentiate once more with respect to z^1, we have

$$\begin{aligned} -g_{11} - g_{11} - z^1 g_{111} + 2g_2 + 4z^1 g_{21} + \left(-2z^2 + (z^1)^2\right) g_{211} \\ = -2g_{11} + gG_{11} + 2g_1 G_1 + g_{11} G + F_{11}. \end{aligned} \tag{2.11}$$

By the assumptions on F and G and the fact that $g_1(0,0) = 0$, we conclude from the above equality that $g_2(0,0) = 0$. Hence there is no such biholomorphism w so that the conclusion of the lemma is true.

Remark 2.1. By a theorem of Poincareé, (see [4]), if

$$\eta_i \neq \sum_j m_j \eta_j,$$

for all i and for *all* sets of nonnegative integers $m_1, \ldots, m_n$ with $m_1 + \cdots + m_n > 1$, then (2.1) can actually be transformed through a biholomorphism into the system

$$\frac{dw^i}{dt} = -\eta_i w^i.$$

By Lemma 2.1, after a holomorphic change of coordinates, we may assume that $|F^i|(z) = O(|z|^{\ell_i})$ for all i in (1.1). From now on, we always assume this is true in $D(a)$.

Lemma 2.2. *There exists $0 < b < a$ such that if $z(t)$ is an integral curve of 2.1 with initial data $|z_0| < b$, then $|z(t)| < b$ for all $t \geq 0$ and there is a constant C independent of z_0 and t such that*

$$|z(t)| \leq C \exp(-\eta_1 t). \tag{2.12}$$

Proof. Let z_0 be such that $|z_0| < a$. If $z_0 = 0$, then $z(t) = 0$ for all t and the lemma is obviously true. Suppose $z_0 \neq 0$ then $z(t) \neq 0$ for all t. Let $H^i = z^i G^i + F^i$. By (2.1), we have that

$$\begin{aligned} \frac{d}{dt}|z|^2 &= -2\sum_i \eta_i |z^i|^2 + \sum_i \left(z^i \overline{H^i} + \bar{z}^i H^i \right) \\ &\leq -2\eta_1 |z|^2 + \sum_i \left(z^i \overline{H^i} + \bar{z}^i H^i \right). \end{aligned} \tag{2.13}$$

Since there exists a constant C_1 such that $|H^i(z)| \leq C_1 |z|^2$ in $D(a)$ for all i,

$$\frac{d}{dt}|z|^2 \leq -2\eta_1 |z|^2 + C_2 |z|^3 \tag{2.14}$$

for some constant C_2 depending only on H^i's as long as $z(t) \in D(a)$. From this it is easy to see that given $\epsilon > 0$ there exists $a > b > 0$ depending on F^i's and ϵ such that if $|z_0| < b$, then the integral curve $z(t)$ with initial data z_0 will satisfy

$$\frac{d}{dt}|z|^2 \leq (-2\eta_1 + 2\epsilon)|z|^2 \tag{2.15}$$

as long as $z(t) \in D(b)$. In particular, $z(t)$ will be in $D(b)$ for all t if ϵ is small enough. From this inequality, we have

$$|z|^2(t) \leq |z_0|^2 \exp[(-2\eta_1 + 2\epsilon)t] \leq b^2 \exp[(-2\eta_1 + 2\epsilon)t].$$

By (2.14)

$$\frac{d}{dt} \log |z|^2 \leq -2\eta_1 + C_2 b \exp[(-\eta_1 + \epsilon)t].$$

Integrating from 0 to t and exponentiate, we conclude that the lemma is true provided ϵ is small enough. $\square$

From now on we assume that a is chosen small enough so that the conclusions of Lemma 2.2 are true in $D(a)$. In particular, (2.1) has long time solution in $D(a)$ if the initial point is in $D(a)$. Moreover, for any $t>0$, the map ϕ_t given by the flow (2.1) is a biholomorphism from $D(a)$ onto its image. In fact, from the proof of the lemma, it is easy to see that for any $0<a'<a$, ϕ_t is a biholomorphism from $D(a')$ onto its image which is a subset of $D(a')$.

Lemma 2.3. *In the flow (2.1), suppose for each i, $|G^i|=O(|z|)$, $|F^i|=O(|z|^{\ell_i})$ where $\ell_i\geq 2$ is the smallest integer such that $\ell_i\eta_1>\eta_i$. Then*

$$|z^i(t)|\leq C\exp(-\eta_i t) \tag{2.16}$$

for some constant C independent of $z(t)$, t and i.

Proof. As in the proof of Lemma 2.2,

$$\begin{aligned}\frac{d}{dt}|z^i|^2 &\leq (-2\eta_i+C|z|)|z^i|^2+C|z|^{\ell_i}\,|z^i|\\ &\leq (-2\eta_i+C\exp(-\eta_1 t))\,|z^i|^2+C\exp(-\ell_i\eta_1 t)|z^i|.\end{aligned} \tag{2.17}$$

Here and below C always denote a constant independent of $z(t)$, t and i. For any $\epsilon>0$, we have

$$\begin{aligned}\frac{d}{dt}\left(|z^i|^2+\epsilon\right) &\leq (-2\eta_i+C\exp(-\eta_1 t))\left(|z^i|^2+\epsilon\right)\\ &\qquad + C\exp(-\ell_i\eta_1 t)\left(|z^i|^2+\epsilon\right)^{\frac12}+2\eta_i\epsilon.\end{aligned} \tag{2.18}$$

Hence

(2.19)

$$\begin{aligned}\frac{d}{dt}\left(|z^i|^2+\epsilon\right)^{\frac12} &\leq (-\eta_i+C\exp(-\eta_1 t))\left(|z^i|^2+\epsilon\right)^{\frac12}+C\exp(-\ell_i\eta_1 t)\\ &\qquad +\eta_i\epsilon\left(|z^i|^2+\epsilon\right)^{-\frac12}\\ &\leq (-\eta_i+C\exp(-\eta_1 t))\left(|z^i|^2+\epsilon\right)^{\frac12}+C\exp(-\ell_i\eta_1 t)\\ &\qquad +\eta_i\epsilon^{\frac12}\end{aligned}$$

Hence

$$\begin{aligned}&\frac{d}{dt}\left(\exp\left[\eta_i t-\int_0^t C\exp(-\eta_1 s)ds\right]\left(|z^i|^2+\epsilon\right)^{\frac12}\right)\\ &\quad\leq \exp\left[\eta_i t-\int_0^t C\exp(-\eta_1 s)ds\right]\left(C\exp(-\ell_i\eta_1 t)+\eta_i\epsilon^{\frac12}\right).\end{aligned} \tag{2.20}$$

Integrating from 0 to t and let $\epsilon \to 0$, we have

(2.21)
$$\exp\left[\eta_i t - \int_0^t C\exp(-\eta_1 s)ds\right]|z^i|(t)$$
$$\leq |z^i(0)| + \int_0^t \exp\left[\eta_i \tau - \int_0^\tau C\exp(-\eta_1 s)ds\right]\cdot C\exp(-\ell_i\eta_1\tau)\,d\tau.$$

Since $\eta_1 > 0$ and $\ell_i\eta_1 > \eta_i$, the lemma follows from the above inequality. □

In [1], it was proved that if the eigenvalues η_i of the Hessian of the potential are equal, then the Kähler-Ricci soliton with nonnegative or positive Ricci curvature depending it is expanding or steady is biholomorphic to $\mathbb{C}^n$. In the present situation,η_i may not be equal to each other, we introduce the following corrected flow to correct the difference between η_i:

$$\frac{dz^i}{dt} = -(\sum_{j\neq i}\eta_j)z^i. \tag{2.22}$$

We will denote the flow corresponding to (2.22) by ψ_t. Note that ψ_t is a biholomorphism from $D(a)$ onto its image which is a subset of $D(a)$ because $\eta_i < 0$ for all i.

Lemma 2.4. *for any $T > 0$ we have that*

$$(\varphi_T)_*\left(\sum_k a^k\frac{\partial}{\partial z^k}(0)\right) = \sum_k \exp(-\eta_k T)a^k\frac{\partial}{\partial z^k}(0). \tag{2.23}$$

Proof. See the proof of Theorem 2.1 in [1]. □

Lemma 2.5. *There exists $0 < a' < a$ and $C > 0$ such that for any $q \in \tilde{D} = D(a')$, $v \in T_q^{1,0}(\mathbb{C}^n)$ and $T > 0$ we have*

(2.24)
$$C^{-1}\exp(-\sum_j \eta_j T)\|v\| \leq \|(\psi_T\circ\phi_T)_*(v)\| \leq C\exp(-\sum_j\eta_j T)\|v\|$$

where the norm is taken with respect to the Euclidean metric on $D(a)$.

Proof. First note that ϕ_T and ψ_T are holomorphic and will map $D(a')$ into $D(a')$ for all $0 < a' < a$. Let $\Phi(z) = \psi_T\circ\phi_T(z) = (\Phi^1(z),\ldots,\Phi^n(z))$. By Lemma 2.3 for any $z \in D(a)$, $|\,(\phi_T(z))^i\,| \leq C_1\exp(-\eta_i T)$ for some constant C_1 independent of T, z and i. By (2.22)

$$(\psi_T\circ\phi_T(z))^i = \exp(-\sum_{j\neq i}\eta_j T)\,(\phi_T(z))^i\,.$$

Hence

$$|(\psi_T \circ \phi_T(z))^i| \le C_1 \exp(-\sum_j \eta_j T)$$

and so

$$|\psi_T \circ \phi_T(z)| \le C_2 \exp(-\sum_j \eta_j T)$$

for some constant C_2 independent of T and z. Thus for each $\Phi^i(z)$ we have

(1) $\Phi^i(z)$ is holomorphic on $D(a)$.
(2) $|\Phi^i(z)| \le C_2 \exp(-\sum_i \eta_i T)$ on $D(a)$.

Combining these with standard derivative estimates for holomorphic functions we have,

$$\|\Phi^i(z)\|_m \le C(m, n, C_2) \exp(-\sum_i \eta_i T) \tag{2.25}$$

where $\|\cdot\|_m$ is the standard C^m norm on $D(a/2)$, here and below $C(m, n, C_2)$ denotes a positive constant depending only on m, n and C_2, but it may vary from line to line.

Now consider the following metric on $D(a)$.

$$h_{i\bar{j}}(z) = \exp(2 \sum_l \eta_l T) \cdot \sum_k \frac{d\Phi^k}{dz^i}(z) \overline{\frac{d\Phi^k}{dz^j}}(z). \tag{2.26}$$

Then $h_{i\bar{j}}$ is just the local components for the pullback metric

$$\exp(2 \sum_l \eta_l T) \cdot \Phi^*(g_\epsilon) \tag{2.27}$$

on $D(a)$ where g_ϵ is the Euclidean metric on $D(a)$. Differentiating (2.26), we can see from (2.25) that for any i and j

$$\|h_{i\bar{j}}(z)\|_m \le C(m, n, C_2) \tag{2.28}$$

For some positive constants $C(m, n, C_2)$ depending only on m, n and C_2. But Lemma 2.4 and the definition of ψ_T, we have

$$h_{i\bar{j}}(0) = \delta_{i\bar{j}}. \tag{2.29}$$

Thus by (2.28) with $m = 1$ we may conclude that for some $0 < a' < a$, $h_{i\bar{j}}(z)$ is uniformly equivalent to $\delta_{i\bar{j}}$ in $\tilde{D} = D(a')$ by some factor C which is independent of T. The lemma now follows from the definition of $h_{i\bar{j}}(z)$. □

Corollary 2.1. *Let $\tilde{D}$ be as in Lemma 2.5, then there exists $C > 0$ such that for any $q \in \tilde{D}$, $v \in T_q^{1,0}(\mathbb{C}^n)$ and $t_2 \geq t_1 > 0$ we have*

$$
\begin{aligned}
&C^{-1} \exp\left(-\sum_j \eta_j t_1 - \sum_{j\neq 1} \eta_j (t_2 - t_1)\right) ||v|| \\
&\leq \|(\psi_{t_2} \circ \phi_{t_1})_*(v)\| \qquad (2.30)\\
&\leq C \exp\left(-\sum_j \eta_j t_1 - \sum_{j\neq n} \eta_j (t_2 - t_1)\right) ||v||
\end{aligned}
$$

where the norm is taken with respect to the Euclidean metric on $\tilde{D}$.

Proof. Since

$$(\psi_t)_*(\frac{\partial}{\partial z^i}) = \exp(-\sum_{j\neq i} \eta_j t)(\frac{\partial}{\partial z^i})$$

and $\eta_1 \leq \eta_i \leq \eta_n$ for all i, we have that

$$\exp(-\sum_{j\neq 1} \eta_j t)||v|| \leq ||(\psi_t)_*(v)|| \leq \exp(-\sum_{j\neq n} \eta_j t)\|v\| \qquad (2.31)$$

for any holomorphic tangent vector v, where the norm is taken with respect to the Euclidean metric. Note that $\psi_{t_2} \circ \phi_{t_1} = \psi_{t_2 - t_1} \circ \psi_{t_1} \circ \phi_{t_1}$, the lemma follows from Lemma 2.5 and the (2.31). □

3. CONSTRUCTING THE LIMIT METRIC

The idea of the proof of the Theorem 1.1 is rather simple. We want to construct a complete flat Kähler metric on M. Let M be a complex manifold satisfying the conditions in the theorem. Let p be a fixed point of the biholomorphisms ϕ_t. We identify the holomorphic coordinate neighborhood of p in the assumptions of the theorem with a ball $\tilde{D}$ in $\mathbb{C}^n$ with center at the origin so that p corresponds the origin. By the assumptions of the theorem and Lemma 2.1, we may assume that $\tilde{D}$ is small enough so that G^i and F^i satisfy $|G^i| = O(|z|)$, $|F^i| = O(|z|^{\ell_i})$ where $\ell_i \eta_1 > \eta_i$. Hence by the results in §2, we may assume that the conclusions of Lemmas 2.3–2.5 and Corollary 2.1 are true for ϕ_t and ψ_t in $\tilde{D}$.

For the rest of the paper, l and k will always represent positive integers and g_ϵ is the Euclidean metric on $\tilde{D}$.

Lemma 3.1. *For any ℓ and for any $v \in T_0^{1,0}(\mathbb{C}^n)$*

$$(\psi_\ell \circ \phi_\ell)_* (v) = \lambda_\ell v$$

where $\lambda_l = \exp[-2(\sum_j \eta_j)l]$.

Proof. By Lemma 2.4 and the definition of ψ_t we have that

$$
\begin{aligned}
(\psi_l\circ\phi_l)_*\left(\sum_i a^i\frac{\partial}{\partial z^i}(0)\right) &= (\psi_l)_*\left(\sum_i \exp(-\eta_i l)a^i\frac{\partial}{\partial z^i}(0)\right)\\
&= \exp(-\sum_j \eta_j l)\left(\sum_i a^i\frac{\partial}{\partial z^i}(0)\right).
\end{aligned}
\tag{3.1}
$$

This completes the proof of the lemma. □

Since $\frac{\partial}{\partial z^i}$'s are orthonormal at 0, with respect to $g_\epsilon(0)$, it is easy to see that λ_l be the unique eigenvalue of $(\psi_l\circ\phi_l)^*g_\epsilon(0)$ relative to $g_\epsilon(0)$.

For every l, let

$$D(l) := \phi_l^{-1}(\tilde{D}). \tag{3.2}$$

where ϕ_t is considered as a flow on M and we identify $U\subset M$ with $D(a)$. Note that by the semi-group property of ϕ_t, if $l\geq k$, then $\psi_l\circ\phi_l = \psi_l\circ\phi_{l-k}\circ\phi_k$ and so

$$\psi_l\circ\phi_l(D(k)) = \psi_l\circ\phi_{l-k}(\tilde{D})\subset\tilde{D}.$$

Hence $D(l)\supset D(k)$. We also have:

Lemma 3.2. *$D(l)$ exhausts M with l.*

Proof. This follows from property (iv) of ϕ_t mentioned in §1 and the above remark. □

Now we define the following metrics on $D(l)$

$$g_l := \lambda_l^{-1}(\psi_l\circ\phi_l)^*g_\epsilon \tag{3.3}$$

For any k, for $l\geq k$, define metrics $h_{l,k}$ on $\tilde{D}$

$$h_{l,k} := \lambda_l^{-1}\left(\psi_l\circ\phi_{l-k}\right)^* g_\epsilon. \tag{3.4}$$

Since $\phi_k(D(k))=\tilde{D}$, on $D(k)$ we have

$$g_l = \phi_k^*(h_{l,k}). \tag{3.5}$$

Lemma 3.3. *There exists a constant $C>0$ independent of k and l such that if $l\geq k$, then*

$$C^{-1}\exp(\eta_1 k)g_\epsilon \leq h_{l,k}\leq C\exp(\eta_n k)g_\epsilon \tag{3.6}$$

in $\tilde{D}$. Moreover, for any $m\geq 1$ and for any compact subset K of $\tilde{D}$, there is a constant $C_{m,k,K}$ which is independent of l such that for $l\geq k$ and $z\in\tilde{D}$,

$$\|(h_{l,k})_{i\bar{j}}(z)\|_m\leq C_{m,k,K} \tag{3.7}$$

where $\|\cdot\|_m$ is the standard C^m norm on $\tilde{D}$.

Proof. For fixed k and $l \geq k$, we simply denote $h_{l,k}$ by h_l. Let $q \in \tilde{D}$ and $v \in T^{1,0}_q(\mathbb{C}^n)$. By Corollary 2.1 with $t_1 = l - k$ and $t_2 = l$ we have that

$$
\begin{aligned}
&C_1^{-1}\exp(-\sum_j \eta_j(l-k) - \sum_{j\neq 1}\eta_j k)||v|| \\
&\qquad \leq ||(\psi_l \circ \phi_{l-k})_*(v)|| \\
&\qquad \leq C_1 \exp(-\sum_j \eta_j(l-k) - \sum_{j\neq n}\eta_j k)||v||
\end{aligned}
\tag{3.8}
$$

for some constant $C_1 > 0$ independent of l, k, q and v, where the norm is with respect to the Euclidean metric on $\tilde{D}$. Hence by Lemma 3.1

$$
C_1^{-1}\exp(\eta_1 k)||v|| \leq ||\lambda_l^{-\frac{1}{2}}(\psi_l \circ \phi_{l-k})_*(v)|| \leq C_1 \exp(\eta_n k)||v|| \tag{3.9}
$$

Thus (3.6) is true.

To prove the estimates of the C^m norm of $(h_l)_{i\bar{j}}$, let $\Phi = \psi_l \circ \phi_{l-k}$. Then by (3.4) we have

$$
(h_l)_{i\bar{j}}(z) = \lambda_l^{-1}\frac{d\Phi^a}{dz^i}(z)\frac{d\Phi^{\bar{a}}}{dz^{\bar{j}}}(z). \tag{3.10}
$$

For each $\Phi^a(z)$ we have

(1) $\Phi^a(z)$ is holomorphic on $\tilde{D}$.
(2) There exists C_k such that for any l and for any $z \in \tilde{D}$, $|\Phi^a(z)|^2 \leq C_k\lambda_l$

where (2) follows from (3.6). Combining these with standard derivative estimates for holomorphic functions we have, for any l and m,

$$
\|\Phi^a(z)\|^2_m \leq C_{m,k,K}\lambda_l \tag{3.11}
$$

in any compact set K of $\tilde{D}$, where $C_{m,k,K}$ is a constant independent on l. Using these estimates, (3.7) follows from differentiating (3.10). □

Lemma 3.4. *There exists a subsequence of g_l which converges uniformly in the C^∞ norm on compact sets of M to a Kähler and flat metric g_∞.*

Proof. By Lemma 3.3, for any k we can find a subsequence of g_l, which is equal to $\phi_k^*(h_{l,k})$, on $D(k)$ such that the subsequence converges uniformly in the C^∞ norm on compact subsets to a flat Kähler metric on $D(k)$, where flatness follows from the flatness of the Euclidean metric. Let $k = 1, 2, 3, \ldots$ and use a diagonal process, the lemma then follows from the fact that $D(k)$ exhaust M with k. □

4. PROOF OF THE MAIN THEOREM AND ITS COROLLARIES

Proof of Theorem 1.1. By Lemma 3.4, it remains to prove that g_∞ constructed in the lemma is complete and that M is simply connected. Let $\alpha(s)$ be a divergent path from the stationary point p. For any k, let $s_k = \inf\{s|\ \alpha(s) \notin D(k)\}$ and let $\alpha_k = \alpha|_{[0,s_k]}$. Then $\phi_k(\alpha_k)$ is a curve in $\tilde{D}$ from the origin to a point on the boundary of $\tilde{D}$. Suppose $\tilde{D}$ is the Euclidean ball of radius $a' > 0$ with center at the origin in $\mathbb{C}^n$. Let β_k be that part of $\phi_k(\alpha_k)$ from p to the boundary of $D(a'/2)$. By Lemma 3.3, there is a constant $C > 0$ independent of l and k such that the length $L_{l,k}$ with $l \geq k$ of β_k in the metric $h_{l,k}$ in (3.4) satisfies

$$L_{l,k} \geq C \exp(\eta_1 k)$$

for some positive constant C independent of l and k. By the definition of g_∞, we conclude that the length of α_k in the metric g_∞ is at least $C\exp(\eta_1 k)$. Hence the length of α in the metric g_∞ is infinite.

Since $D(l)$ exhaust M with l and each $D(l)$ is homeomorphic to the Euclidean ball $\tilde{D}$, it is easy to see that M is simply connected. This completes the proof of the main theorem. □

Proof of Corollary 1.1. Let ϕ_t be the biholomorphisms generated by the holomorphic vector field V. Since $V(p) = 0$, it is easy to see that properties (i)–(iii) in the assumptions of Theorem 1.1 are satisfied by ϕ_t. As for property (iv), for any $R > 0$, let $B(R)$ be the geodesic ball of radius R with center at p with respect to the metric $g(0)$. From the proof of Lemma 3.2 in [1], there exists $C_R > 0$ such that for any $q \in B(R)$ and for any $v \in T^{1,0}(M)$ at q,

$$||v||_{\phi_t^*(g)} \leq \exp(-C_R t)||v||_g.$$

Since $\phi_t(p) = p$, it is easy to see that given any open set $U \subset M$ containing p, we have $\phi_t(B(R)) \subset U$ provided t is large. By Theorem 1.1, the corollary follows. □

Proof of Corollary 1.2. Under the curvature conditions of the Kähler-Ricci soliton, there is a unique fixed point of the flow by Lemma 3.1 in [1]. The result then follows from Corollary 1.1. □

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