On the Self-Similar Solutions of the Magneto-hydro-dynamic Equations

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Abstract: In this paper, we show that, for the three dimensional incompressible magneto-hydrodynamic equations, there exists only trivial backward self-similar solution in $L^p(\mathbb{R}^3)$ for $p \geq 3$, under some smallness assumption on either the kinetic energy of the self-similar solution related to the velocity field, or the magnetic field. Second, we construct a class of global forward self-similar solutions to the three-dimensional MHD equations with initial data being homogeneous of degree -1 and belonging to the closure of the Schwartz test functions in $L^2_{loc,unif}(\mathbb{R}^3)$, as motivated by the work in [11].

Keywords: Magnetohydrodynamics equations, backward self-similar solutions, forward self-similar solutions

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1 Introduction

We will consider the question of the existence of self-similar solutions to the three dimensional initial value problem of incompressible magneto-hydrodynamics (MHD) equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{Re}\Delta\ u + (u\cdot\nabla)u - S(b\cdot\nabla)b + \nabla p = 0, \\ \dfrac{\partial b}{\partial t} - \dfrac{1}{Rm}\Delta\ b + (u\cdot\nabla)b - (b\cdot\nabla)u = 0, \\ \mathrm{div} u = 0, \quad \mathrm{div} b = 0 \end{cases} \tag{1.1}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x), \\ b(x,0) = b_0(x). \end{cases} \tag{1.2}$$

Here u, p and b are nondimensional qualities corresponding to the velocity of the fluid, its pressure and the magnetic field. The nondimensional number Re is the Reynolds number, Rm is the magnetic Reynolds and $S = M^2/(ReRm)$ with M being the Hartman number. For simplicity, let $Re = Rm = S = 1$.

As for the incompressible Navier-Stokes system, the incompressible MHD system (1.1) is dialation invariant in the sense that if if (u, b, p) is a solution to (1.1), then so is (u_r, b_r, p_r) defined by scaling

$$\begin{cases} u_r(x,t) \triangleq r u(rx, r^2 t), \\ b_r(x,t) \triangleq r b(rx, r^2 t), \\ p_r(x,t) \triangleq r^2 p(rx, r^2 t) \end{cases} \tag{1.3}$$

for each $r > 0$ and any $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$. It is well-known that solutions invariant under dialations are important for regularity and asymptotic behavior of general solutions to underlying nonlinear partial differential equations such as Navier-Stokes system, see [8]. Then it is of interests to study the existence of such dialation invariant solutions for the incompressible MHD system (1.1).

The first purpose of this paper is to study the question whether there exists a class of backward self-similar solutions of the magneto-hydrodynamic equations. Duvaut and Lions [7] constructed a class of global weak solutions, which is similar to the Leray-Hopf weak solutions to the Navier-Stokes equations, and a class of local strong solutions to initial boundary value problems of the magneto-hydrodynamic equations. However, it remains to know whether there is a solution which can develop singularities in finite time, even if all the given data, such as initial and boundary values, are sufficient smooth. Due to the scaling property, it is natural to look for the existence of backward self-similar solutions as an good examples constructing singular solutions. Thus, Leray [15] raised his famous question about the existence of backward self-similar solutions for the Navier-Stokes equations in 1934. And this question was answered by Nečas, Ružička and Šverák [16] in 1996. They showed that the only backward self-similar solution satisfying the global energy estimates is zero. Same results also were obtained by Tsai [21] in 1998, under very general assumptions, for example, if the solutions satisfy the local energy estimates. One of the main ingredients in their proof is the partial regularity theory established in [3].

For the incompressible magneto-hydrodynamic equations, although the theory of partial regularity as in [3] has been developed by the authors in [12], yet, it seems quite difficult to apply the ideas in [16] [21] to study the backward self-similar solutions to this case due to the presence of the magnetic field. In this paper, we will use the energy estimates to show that there is only trivial backward self-similar solution in $L^p(\mathbb{R}^3)$ for $p \geq 3$, under some smallness assumption on the kinetic energy of the backward self-similar solution. Namely, we will first show that, if $U \in L^r(\mathbb{R}^3)$ with $3 \leq r \leq \infty$ and $B \in L^p(\mathbb{R}^3)$ with $3 \leq p < \infty$ with $\|U\|_r$ small, then there is only trivial backward self-similar solution. This reveals that the velocity field should play a more important role in the regularity theory of the magneto-hydrodynamic equations than the magnetic field. And this also coincides with the partial regularity theory obtained in [12] and the regularity criteria obtained in [13]. At the same time, we will show that, if $U \in L^r(\mathbb{R}^3)$ with $3 < r < \infty$ and $B \in L^p(\mathbb{R}^3)$ with $p \in (6r/(r+1), 2r]$ with $\|B\|_p$ small, then there is only trivial backward self-similar solution also. This implies that there is only trivial backward self-similar solution to the Navier-Stokes equations under the small perturbation in some sense.

The second purpose of this paper is to construct a class of global forward self-similar solutions to the magnetohydrodynamics equations (1.1). For incompressible Navier-Stokes system, there are extensive literatures on the studies of forward self-similar solutions. It started with Giga and Miyakawa who showed the existence and uniqueness of global forward self-similar solution in the Morrey-type spaces of measures as the initial vorticity is small in some sense [10]. This is then followed by extensive studies on the existence and uniqueness of the forward self-similar solutions in a variety of spaces including the homogeneous Besov spaces under the assumption that the initial data is small in some sense by many people, see [1], [6], [9], etc., which was surveyed by Cannone in [4] [5] where an abstract framework is presented. In particular, Barraza [1] showed that there exists a unique self-similar solution in weak-$L^q(3 < q < \infty)$ provided the initial data belongs to weak-L^3 and is small in some sense. Recently, Grujić [11] constructed a global regular forward self-similar solutions emanating from arbitrary large initial data which is homogeneous of degree -1 and belongs to $L^2_{loc,unif}(\mathbb{R}^3) \cap L^3_w(\mathbb{R}^3)$(here $L^p_w(\mathbb{R}^3)$ denotes the weak-$L^p(\mathbb{R}^3)$ space). It should be noted that the forward self-similar solutions constructed in [10], [14], [6] and [1] are unique for small initial data in some sense, while Grujić's forward self-similar solution exists for any large initial data in $L^2_{loc,unif}(\mathbb{R}^3) \cap L^3_w(\mathbb{R}^3)$ and is smooth, but the uniqueness of such a solution is not known. The Grujić's idea is as follows: First, by the modified Navier-Stokes equations, one can show the existence of "partially self-similar" solution on $(0, T)$ with some positive T. This solution is also a suitable weak solution in the sense of Caffarelli, Kohn and Nirenberg [3]. Then by the partial regularity theory [3] and the "partial self-similarity", it can be shown that the set of possible singular points for this solution is empty, so the local "partial self-similar" solution is infinitely differentiable with respect to spatial variables and then to time variables. Finally, the spatial continuity and "partial self-similarity" imply that the local "partial self-similar" solution can be extended to be a global full self-similar solution. Motivated by the Grujić's work [11], we establish the existence of forward self-similar solutions to the incompressible magneto-hydrodynamics equations (1.1) under the assumptions that the initial data are homogeneous of degree -1 and belong to $L^2_{loc,unif}(\mathbb{R}^3)$. It should be noted that our results are new even for the incompressible Navier-Stokes system since we do not require that initial data in $L^3_w(\mathbb{R}^3)$, which is the assumption by Grujic in [11]. This is so due to that we can construct the approximate solutions by studying the solutions to a linearized magneto-hydrodynamic equations instead of the solutions to the modified Navier-Stokes equations as in [11]. The advantage of our arguments is that our approximate solutions remain invariant under the scaling. By generalizing the ideas and techniques in [14] constructing the suitable local square-integrable weak solutions for the Navier-Stokes equations, we then show the local existence of "partial self-similar" solutions to the incompressible MHD equations. The other parts of our analysis will be based on the partial regularity theory established in [12] and the regular criteria obtained in [13] for incompressible magnetohydrodynamics equations.

The paper is organized as follows: some basic concepts and mathematical preliminaries are introduced in section 2. Then in section 3, we show the non-existence of backward self-similar solutions. And finally we study the forward self-similar solutions in section 4.

2 Mathematical Preliminaries

First, we recall the notations of some function spaces. Let $L^p(\mathbb{R}^3), 1 \le p \le \infty$, represent the usual Lesbegue space of scalar functions as well as that of vector-valued functions with norm denoted by $\|\cdot\|_p$. Let $C^\infty_{0,\sigma}(\mathbb{R}^3)$ denote the set of all $\phi \in C^\infty_0(\mathbb{R}^3)$ (the set of all real vector-valued functions with compact support in $\mathbb{R}^3$) such that div $\phi = 0$. Let $L^p_\sigma(\mathbb{R}^3), 1 < p < \infty$, be the closure of $C^\infty_{0,\sigma}(\mathbb{R}^3)$ in $L^p(\mathbb{R}^3)$. $H^1(\mathbb{R}^3)$ denotes the usual L^2-Sobolev Space. Finally, for a given Banach space X with norm $\|\cdot\|_X$, we denote by $L^p(0,T;X), 1 \le p \le \infty$, the set of function $f(t)$ defined on $(0,T)$ with values in X such that $\int_0^T \|f(t)\|^p_X dt < \infty$. For $x \in \mathbb{R}^3$, we denote $B_r(x) = \{y \in \mathbb{R}^3, |y-x| < r\}$. For point $(x,t) \in \mathbb{R}^3 \times R^+$, the parabolic ball centered at point (x,t) with radius r will be denoted as $Q_r(x,t) = B_r(x) \times (t-r^2, t)$. Let $L^p_{loc,unif}(\mathbb{R}^3)$ be the space of uniformly locally square integrable vector fields with norm

$$\|f\|_{L^p_{loc,unif}(\mathbb{R}^3)} =: \sup_{x\in\mathbb{R}^3} \sup_{0<R<1} \Big(\int_{B_R(x)} |f(y)|^p dy\Big)^{\frac{1}{p}}$$

for $p \in [1,\infty)$. Set E_p be the closure of Schwartz test functions in $L^p_{loc,unif}(\mathbb{R}^3)$. So $f \in E_p (p < \infty)$ if and only if f is locally L^p integrable and f vanishes at ∞ in the sense of $\lim_{x_0\to\infty} \int_{|x-x_0|<1} |f(x)|^p dx = 0$.

Due to the scaling property (1.3), the backward self-similar solutions to (1.1) are of the forms

$$\begin{cases} u(x,t) = \dfrac{1}{\sqrt{2a(T-t)}} U(\dfrac{x}{\sqrt{2a(T-t)}}), \\ b(x,t) = \dfrac{1}{\sqrt{2a(T-t)}} B(\dfrac{x}{\sqrt{2a(T-t)}}), \\ p(x,t) = \dfrac{1}{2a(T-t)} P(\dfrac{x}{\sqrt{2a(T-t)}}). \end{cases} \tag{2.1}$$

Where $T > 0$ and $a > 0$. Thus $U = (U_1(y), U_2(y), U_3(y))$, $B = (B_1(y), B_2(y), B_3(y))$ and $P(y)$ are all defined in $\mathbb{R}^3$. The magneto-hydrodynamic equations require that

$$\begin{cases} -\Delta U + aU + a(y\cdot\nabla)U + (U\cdot\nabla)U - (B\cdot\nabla)B + \nabla P = 0, \\ -\Delta B + aB + a(y\cdot\nabla)B + (U\cdot\nabla)B - (B\cdot\nabla)U = 0, \\ \mathrm{div}U = 0, \quad \mathrm{div}B = 0. \end{cases} \tag{2.2}$$

It is obvious that a nonzero (U, B, P) would produce a solution (u, b, p) of (1.1) with a singularity point $(0, T)$.

For simplicity, assume $a = 1$. For general case, one can use the transform

$$(U(x), B(x)) \longmapsto \left(\sqrt{a}U((\sqrt{a})x), \sqrt{a}B((\sqrt{a})x)\right),$$

to reduce to the case of $a = 1$.

Definition 2.1: A pair (U, B) ia called a *weak solution* to (2.2) in $\mathbb{R}^3$, if
i) U and B belong locally to $H^1(\mathbb{R}^3)$,
ii) U and B are divergence free,
iii) for any $\phi = (\phi_1, \phi_2, \phi_3) \in C_0^\infty(\mathbb{R}^3)$ with $div\phi = 0$ and any $\psi = (\psi_1, \psi_2, \psi_3) \in C_0^\infty(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \Big(\nabla U\cdot\nabla\phi + U\phi + (y\cdot\nabla)U\cdot\phi + (U\cdot\nabla)U\cdot\phi - (B\cdot\nabla)B\cdot\phi\Big)dy = 0,$$

$$\int_{\mathbb{R}^3} \Big(\nabla B\cdot\nabla\psi + B\psi + (y\cdot\nabla)B\cdot\psi + (U\cdot\nabla)B\cdot\psi - (B\cdot\nabla)U\cdot\psi\Big)dy = 0.$$

Employing the regularity theory for the linear Stokes operator, one can show that every weak solution (U, B) of (2.2) is smooth, cf. [9]. It should be noted that the pressure P does not appear explicitly in the definition of weak solutions. But the first equation of (2.2) implies formally that

$$-\Delta P = \sum_{i,j=1}^{3} \frac{\partial^2}{\partial y_i \partial y_j}\Big(U_iU_j - B_iB_j\Big). \tag{2.3}$$

Obviously, P is unique up to an addition of a harmonic function. In this paper, let P belong to $L^p(\mathbb{R}^3)$ for some $p > 1$. Then one can solve equation (2.3) by the classical Riesz transformation R_j, that is,

$$P = R_iR_j\Big(U_iU_j - B_iB_j\Big). \tag{2.4}$$

By the L^p-boundedness of the Riesz transformation (cf. [18]), one has

$$\|P\|_r \le C\Big(\|U\|_{2r}^2 + \|B\|_{2r}^2\Big), \qquad 1 < r < \infty \tag{2.5}$$

and

$$\|\nabla P\|_r \le C\Big(\|U\cdot\nabla U\|_r + \|B\cdot\nabla B\|_r\Big), \qquad 1 < r < \infty \tag{2.6}$$

if the right terms are well-defined. Here $\|\cdot\|_r$ denotes the norm in $L^r(\mathbb{R}^3)$. By the classical regularity theory for the Laplance equation, P is smooth, as long as U and B are smooth.

Similar to the discussion of Lemma 3.1 in [16], one can show that (U, B, P) solves the equation (2.2) in the classical sense.

Lemma 2.1 *Let (U, B) be a weak solution of (2.2) in $L^p(\mathbb{R}^3)$ with $p \geq 3$, and P be defined in (2.4). Then (U, B, P) is smooth, and solve the equations (2.2).*

The proof is similar to that of Lemma 3.1 in [16].

Next we recall the definition of suitable weak solutions to the magneto-hydro-dynamics equations (1.1) given in [12].

Definition 2.2. The pair (u, b, p) is called a *suitable weak solution* of the magneto-hydro-dynamic equations (1.1) in an open set $D \subset \mathbb{R}^3 \times \mathbb{R}^+$, if

1) $p \in L^{5/3}(D)$ with $\int\int_D |p(x,t)|^{5/3} dxdt \leq C_1$, and for some positive constants C_2 and C_3,

$$\int_{D_t} (|u(x,t)|^2 + |b(x,t)|^2) dx \leq C_2, \qquad \int\int_D (|\nabla u(x,t)|^2 + |\nabla b(x,t)|^2) dxdt \leq C_3 \tag{2.7}$$

for almost every t such that $D_t = D \cap \{\Omega \times \{t\}\} \neq \emptyset$.

2) (u, b, p) satisfies (1.1) in the sense of distribution on D.

3) For each real-valued $\phi \in C_0^\infty(D)$ with $\phi \geq 0$, the following generalized energy inequality is valid:

$$\begin{aligned} & 2\int\int_D (|\nabla u(x,t)|^2 + |\nabla b(x,t)|^2)\phi dxdt \\ \leq & \int\int_D (|u(x,t)|^2 + |b(x,t)|^2)(\phi_t(x,t) + \Delta\phi(x,t)) dxdt \\ & + \int\int_D (u(x,t)\cdot\nabla\phi)(|u(x,t)|^2 + |b(x,t)|^2 + 2p(x,t)) dxdt \\ & \qquad -2\int\int_D (b\cdot\nabla\phi)(u\cdot b) dxdt. \end{aligned} \tag{2.8}$$

4) For any $\chi \in C_0^\infty(D)$, the equation

$$\frac{\partial b\chi}{\partial t} - \Delta(b\chi) = b(\frac{\partial \chi}{\partial t} - \Delta\chi) - 2\nabla\chi\cdot\nabla b - \chi(u\cdot\nabla)b + \chi b\cdot\nabla u \tag{2.9}$$

holds in the sense of distribution. Now we recall one of the main results on partial regularity given in [12], which will be used later.

Theorem 2.1. *There exists an absolute constant ε with the following property. Let (u, b, p) be a suitable weak solution to (1.1), suppose further that, for some $r_0 > 0$,*

$$\frac{1}{r}\int\int_{Q_r(x_0,t_0)} |\nabla u(x,t)|^2 dxdt \leq \varepsilon \qquad \textit{for all } 0 < r \leq r_0,$$

and

$$\sup_{0<r\leq r_0} \frac{1}{r^3}\int\int_{Q_r(x_0,t_0)} |b(x,t)|^2 dxdt < \infty,$$

then, there is a $r_1 \le r_0$, such that

$$\sup_{Q_{r/2}(x_0,t_0)} (|\nabla u(x,t)| + |\nabla b(x,t)|) \le Cr^{-2} \tag{2.10}$$

for all $r \le r_1$. This implies that the one- dimensional Hausdorff measure of the set of possible singular points of u and b is zero, following the arguments given in [3]. Where a point $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$ is called singular if (u,b) is unbounded in any neighborhood of point (x,t); otherwise, if (u,b) is locally bounded in some neighborhood of point (x,t), then (x,t) is called a regular point.

See Theorem 2.3 and remarks 2 in [12]. Following the arguments [13], we can establish the local regularity criteria.

Theorem 2.2. *Let (u,b) be a weak solution in some open region $\Omega \times (t_1,t_2)$ with $\Omega \subset \mathbb{R}^3$ and $0 < t_1 < t_2$. If $u \in L^p(t_1,t_2,L^q(\Omega))$ for $1/p+3/2q \le 1/2$ and $q > 3$, then (u,b) is of class C^∞ with respect to space variables, and each derivative is bounded on compact subdomains of $\Omega \times (t_1,t_2)$.*

The proof is similar to that of Theorem 3 in [13].

3 Backward Self-similar Solutions

In this section, we present and prove our main result on backward self-similar solutions to (1.1).

Theorem 3.1 *Let (U,B) be a weak solution of (2.2) with $U \in L^r(\mathbb{R}^3)$ for $3 \le r \le \infty$ and $B \in L^p(\mathbb{R}^3)$ for $3 \le p < \infty$. Then if*

$$\|U\|_r < \begin{cases} c_0^{-1}; & r = 3; \\ \left(\dfrac{p-3}{p(p-1)}\right)^{\frac{r-3}{2r}} c_0^{-\frac{3}{r}} & r > 3 \text{ and } p > 3, \end{cases}$$

with $c_0 = \sqrt[3]{4}/\sqrt[3]{\pi}\sqrt{3}$, then

$$U = B = 0, \quad P = 0 \tag{3.1}$$

if $r < \infty$ and $p > 3$ or $r = p = 3$;

$$U = constant; \quad B = 0, \quad P = constant \tag{3.2}$$

if $r = \infty$ and $p > 3$.

Proof First we consider the case of $p > 3$. We multiply both sides of the second equation of (2.2) by $|B|^{p-2}B$, then integrate over $\mathbb{R}^3$ to get, with the help of integrations by parts, that

$$\|B\|_p^p - \frac{3}{p}\|B\|_p^p + \int_{\mathbb{R}^3} \nabla B \cdot \nabla(|B|^{p-2}B)dy = \int_{\mathbb{R}^3} (B \cdot \nabla U) \cdot |B|^{p-2}Bdy.$$

Thus

$$\frac{p-3}{p}\|B\|_p^p + (p-1)\int_{\mathbb{R}^3} |B|^{p-2}|\nabla B|^2 \, dx \le (p-1)\int_{\mathbb{R}^3} |U||B|^{p-1}|\nabla B| \, dx. \tag{3.3}$$

By the Hölder's inequality, the right hand term of (3.3) can be estimated as

$$I =: (p-1)\int_{\mathbb{R}^3} |U||B|^{p-1}|\nabla B|\ dx \le (p-1)\|U\|_r \|B\|^{\frac{p}{2}}_{\frac{rp}{r-2}} \Big(\int_{\mathbb{R}^3} |B|^{p-2}|\nabla B|^2\ dx\Big)^{\frac{1}{2}}.$$

By the interpolation inequality,

$$\|B\|_{\frac{rp}{r-2}} \le \|B\|_p^{\frac{r-3}{r}} \|B\|_{3p}^{\frac{3}{r}} = \|B\|_p^{\frac{r-3}{r}} \||B|^{\frac{p}{2}}\|_6^{\frac{6}{rp}}.$$

By the Sobolev inequality (see [19])

$$\|f\|_6 \le c_0 \|\nabla f\|_2 \tag{3.4}$$

with $c_0 = \sqrt[3]{4}/\sqrt[3]{\pi}\sqrt{3}$, one has

$$\|B\|_{\frac{rp}{r-2}} \le c_0^{\frac{6}{rp}} \|B\|_p^{\frac{r-3}{r}} \Big(\int_{\mathbb{R}^3} |B|^{p-2}|\nabla B|^2\ dx\Big)^{\frac{3}{rp}}.$$

Therefore,

$$\begin{aligned} & \frac{p-3}{p}\|B\|_p^p + (p-1)\int_{\mathbb{R}^3} |B|^{p-2}|\nabla B|^2\ dx \\ \le \quad & (p-1)c_0^{\frac{3}{r}} \|U\|_r \|B\|_p^{\frac{p(r-3)}{2r}} \Big(\int_{\mathbb{R}^3} |B|^{p-2}|\nabla B|^2\ dx\Big)^{\frac{r+3}{2r}}. \end{aligned} \tag{3.5}$$

If $r = 3$, taking

$$\|U\|_r \le c_0^{-1},$$

then (3.5) give us that $\|B\|_p = 0$, i.e., $B = 0$ a.e. in $\mathbb{R}^3$; While if $r > 3$, by the Young's inequality,

$$I \le (p-1)\int_{\mathbb{R}^3} |B|^{p-2}|\nabla B|^2\ dx + (p-1)c_0^{\frac{6}{r-3}} \|U\|_r^{\frac{2r}{r-3}} \|B\|_p^p.$$

So taking

$$\|U\|_r \le \Big(\frac{p-3}{p(p-1)}\Big)^{\frac{r-3}{2r}} c_0^{-\frac{3}{r}},$$

then (3.5) implies that $B = 0$ a.e. in $\mathbb{R}^3$.

In the case of $p = r = 3$, taking in (3.5)

$$\|U\|_3 < c_0^{-1},$$

then

$$\int_{\mathbb{R}^3} |B||\nabla B|^2\ dx = 0$$

which and $B \in L^3(\mathbb{R}^3)$ imply that $B = 0$ a.e. in $\mathbb{R}^3$.

In all the cases considered above, $B = 0$ a.e. in $\mathbb{R}^3$, then the equations is only about (U, P). By the results obtained in [21] for the incompressible Navier-Stokes equations, we have

$$U = P = 0,$$

if $r < \infty$, and

$$U = \text{constant}, \quad P = \text{constant}$$

if $r = \infty$. Then we complete the proof. ■

Next we turn to the problem when the magnetic is small in some sense. Namely, we have

Theorem 3.2 *Let (U, B) be a solution to (2.2) with $U \in L^r(\mathbb{R}^3)$ and $B \in L^p(\mathbb{R}^3)$ for some $3 < r < \infty$ and $p \in (6r/(r+1), 2r]$. Then there is a constant $\varepsilon = \varepsilon(r, p)$ such that if*

$$\|B\|_p < \varepsilon(r, p)$$

then

$$U = B = P = 0. \tag{3.6}$$

Proof We view the first system in (2.2) for unknown vector U with nonhomogeneous term $(B \cdot \nabla)B$. Since $B \in L^p_\sigma(\mathbb{R}^3)$, we choose that $B^k \in C^\infty_{0,\sigma}(\mathbb{R}^3)$ such that B^k converge strongly to B in $L^p_\sigma(\mathbb{R}^3)$ and

$$\|B^k\|_p \leq \|B\|_p, \qquad \forall k \geq 0. \tag{3.7}$$

Now we linearize the convection $(U \cdot \nabla U)$ to construct the approximate solutions as follows:

$$\begin{cases} -\Delta U^0 + U^0 + (y \cdot \nabla)U^0 + \nabla P^0 = (B^0 \cdot \nabla)B^0, \\ \text{div } U^0 = 0 \end{cases} \tag{3.8}$$

and

$$\begin{cases} -\Delta U^k + U^k + (y \cdot \nabla)U^k + (U^{k-1} \cdot \nabla)U^k + \nabla P^k = (B^k \cdot \nabla)B^k, \\ \text{div } U^k = 0 \end{cases} \tag{3.9}$$

for $k \geq 1$. Note that

$$(B^k \cdot \nabla)B^k = \nabla\big(B^k \otimes B^k\big) \in C^\infty_{0,\sigma}(\mathbb{R}^3)$$

since $B^k \in C^\infty_{0,\sigma}(\mathbb{R}^3)$. By the theory on steady Stokes equations (see section 2 in chapter IV in [9]), there is a unique solution U^0 to (3.8) with $U^0, \nabla U^0 \in L^q(\mathbb{R}^3)$ for any $q \geq 1$. By induction, there is a unique solution U^k to (3.9) with $U^k, \nabla U^k \in L^q(\mathbb{R}^3)$ for any $q \geq 1$ also.

In the following, we only need to establish the uniform estimates for approximate solutions U^k. For $r > 3$, we multiply both sides of (3.9) by $|U^k|^{r-2}U^k$, and integrate over $\mathbb{R}^3$ to get that

$$\begin{aligned} & \frac{r-3}{r}\int_{\mathbb{R}^3} |U^k(y)|^r \, dy + (r-1)\int_{\mathbb{R}^3} |U^k|^{r-2}|\nabla U^k|^2 \, dy \\ \leq \;& (r-1)\int_{\mathbb{R}^3} |B^k|^2|U^k|^{r-2}|\nabla U^k| \, dy + (r-1)\int_{\mathbb{R}^3} |P^k||U^k|^{r-2}|\nabla U^k| \, dy \\ =: \;& I_1 + I_2. \end{aligned} \tag{3.10}$$

It follows from equations (3.9), the pressure P^k obeys that

$$-\Delta P^k = \sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}\Big(u_i^{k-1}u_j^k - B_i^k B_j^k\Big) =: P_1 + P_2$$

with

$$\begin{aligned} P_1 &= \frac{1}{4\pi}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} \frac{U_i^{k-1}(y)U_j^k(y)}{|x-y|}\ dy \\ P_2 &= -\frac{1}{4\pi}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} \frac{B_i^k(y)B_j^k(y)}{|x-y|}\ dy. \end{aligned} \tag{3.11}$$

By the Calderón-Zygmund theory on singular integral, we have

$$\begin{cases} \|P_1\|_{(r+2)/2} \le c_1(\frac{r+2}{2})\|U^{k-1}\|_{r+2}\|U^k\|_{r+2}, \\ \|P_2\|_{2q} \le c_1(2q)\|B^k\|_{4q}^2 \end{cases} \tag{3.12}$$

for $q > 1$, with

$$c_1(l) = \begin{cases} (\dfrac{4l^2}{l-2})^{\frac{l-1}{l}}\Big(2^5 + 4J + 2^3 3^{\frac{3}{2}}\Big)^{\frac{l-2}{2l}}, & \text{as } \ l > 2; \\ 1, & \text{as } \ l = 2, \\ \Big(\dfrac{2l}{l-1} + \dfrac{4l}{2-l}\Big)^{\frac{1}{l}}\Big(2^5 + 4J + 2^3 3^{\frac{3}{2}}\Big)^{\frac{2-l}{2l}}, & \text{as } \ 1 < l < 2 \end{cases}$$

and

$$J = \sup_{\xi\neq 0, 1\le i,j\le 3} \frac{1}{4\pi}\int_{|x|\ge 2|\xi|}\Big|\partial_{ij}^2\frac{1}{|x-\xi|} - \partial_{ij}^2\frac{1}{|x|}\Big|\ dx < \infty.$$

Now we estimate I_1. For $q \in (3r/2(r+1), r/2]$, by Hölder inequality, we have

$$I_1 \le (r-1)M_k^{\frac{1}{2}}\|B^k\|_{4q}^2\|U^k\|_{q(r-2)/(q-1)}^{\frac{r-2}{2}}$$

with

$$M_k := \int_{\mathbb{R}^3} |U^k(y)|^{r-2}|\nabla U^k(y)|^2\ dy.$$

By the interpolation inequality and Sobolev inequality (3.4), we deduce that

$$\|U^k\|_{q(r-2)/(q-1)} \le \|U^k\|_r^{\frac{2qr+2q-3r}{2q(r-2)}}\|U^k\|_{3r}^{\frac{3(r-2q)}{2q(r-2)}} \le c_0^{\frac{3(r-2q)}{rq(r-2)}}\|U^k\|_r^{\frac{2qr+2q-3r}{2q(r-2)}}M_k^{\frac{3(r-2q)}{2qr(r-2)}}.$$

Thus,

$$I_1 \le (r-1)c_0^{\frac{3(r-2q)}{2qr}}\|B^k\|_{4q}^2\|U^k\|_r^{\frac{2qr+2q-3r}{4q}}M_k^{\frac{3r+2qr-6q}{4qr}}.$$

By the Young's inequality, we deduce that

$$I_1 \le \frac{r-1}{4}\frac{3r+2qr-6q}{4qr}M_k + (r-1)4^{\frac{3r+2qr-6q}{2qr+6q-3r}}c_0^{\frac{6(r-2q)}{2qr+6q-3r}}\|B^k\|_{4q}^{\frac{8qr}{2qr+6q-3r}}\|U^k\|_r^{\frac{r(2qr+2q-3r)}{2qr+6q-3r}}.$$

Let $p=4q$. By (3.7), we obtain that

$$I_1 \le \frac{r-1}{4}M_k + (r-1)4^{\frac{6r+pr-3p}{pr+3p-6r}} c_0^{\frac{6(2r-p)}{pr+3p-6r}} \|B\|_p^{\frac{4pr}{pr+3p-6r}} \|U^k\|_r^{\frac{r(pr+p-6r)}{pr+3p-6r}}. \tag{3.13}$$

Applying (3.12), one can estimate I_2 as

$$\begin{aligned} I_2 &\le (r-1)\Big(\|P_1\|_{2q}\|U^k\|_{q(r-2)/(q-1)}M_k^{\frac12} + \|P_2\|_{(r+2)/2}\|U^k\|_{r+2}^{\frac{r-2}{2}}M_k^{\frac12}\Big) \\ &\le (r-1)\Big(c_1(2q)\|B^k\|_p^2\|U^k\|_{q(r-2)/(q-1)}M_k^{\frac12} + c_1(\frac{r+2}{2})\|U^{k-1}\|_{r+2}\|U^k\|_{r+2}^{\frac r2}M_k^{\frac12}\Big) \\ &:= I_{21}+I_{22}. \end{aligned}$$

Similar to the estimate of I_1, one has for I_{21} that

$$\begin{aligned} I_{21} \le\ & \frac{r-1}{4}M_k \\ & +(r-1)4^{\frac{6r+pr-3p}{pr+3p-6r}} c_0^{\frac{6(2r-p)}{pr+3p-6r}} \Big(c_1(\frac p2)\Big)^{\frac{2pr}{pr+3p-6r}} \|B\|_p^{\frac{4pr}{pr+3p-6r}} \|U^k\|_r^{\frac{r(pr+p-6r)}{pr+3p-6r}}. \end{aligned}$$

It follows from the interpolation and Sobolev inequalities (3.4)that

$$\|U^k\|_{r+2} \le \|U^k\|_r^{\frac{r-1}{r+2}} \|U^k\|_{3r}^{\frac{3}{r+2}} \le c_0^{\frac{6}{r(r+2)}} \|U^k\|_r^{\frac{r-1}{r+2}} M_k^{\frac{3}{r(r+2)}}.$$

Then I_{22} can be estimated as

$$\begin{aligned} I_{22} &\le (r-1)c_1(\frac{r+2}{2})c_0^{\frac3r}\|U^{k-1}\|_r^{\frac{r-1}{r+2}} M_{k-1}^{\frac{3}{r(r+2)}} \|U^k\|_2^{\frac{r(r-1)}{2(r+2)}} M_k^{\frac{r+5}{2(r+2)}} \\ &\le \frac{r-1}{4}M_k + (r-1)4^{\frac{r+5}{r-1}}\Big(c_1(\frac{r+2}{2}\Big)^{\frac{2(r+2)}{r-1}} c_0^{\frac{6(r+2)}{r(r-1)}} \|U^{k-1}\|_r^2 M_{k-1}^{\frac{6}{r(r-1)}} \|U^k\|_r^r. \end{aligned}$$

Therefore we obtain the estimate on I_2 as

$$\begin{aligned} I_2 \le\ & \frac{r-1}{2}M_k + (r-1)\Big\{4^{\frac{r+5}{r-1}}\Big(c_1(\frac{r+2}{2}\Big)^{\frac{2(r+2)}{r-1}} c_0^{\frac{6(r+2)}{r(r-1)}} \|U^{k-1}\|_r^2 M_{k-1}^{\frac{6}{r(r-1)}} \|U^k\|_r^r \\ & +4^{\frac{6r+pr-3p}{pr+3p-6r}} c_0^{\frac{6(2r-p)}{pr+3p-6r}} \Big(c_1(\frac p2)\Big)^{\frac{2pr}{pr+3p-6r}} \|B\|_p^{\frac{4pr}{pr+3p-6r}} \|U^k\|_r^{\frac{r(pr+p-6r)}{pr+3p-6r}} \Big\}. \end{aligned} \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.10), we deduce that

$$\begin{aligned} & \min\{\frac{r-3}{r}, \frac{r-1}{4}\}\Big(\|U^k\|_r^r + M_k\Big) \\ \le\ & (r-1)4^{\frac{6r+pr-3p}{pr+3p-6r}} c_0^{\frac{6(2r-p)}{pr+3p-6r}} \Big\{\Big(c_1(\frac p2)\Big)^{\frac{2pr}{pr+3p-6r}} + 1\Big\} \|B\|_p^{\frac{4pr}{pr+3p-6r}} \|U^k\|_r^{\frac{r(pr+p-6r)}{pr+3p-6r}} \\ & +(r-1)4^{\frac{r+5}{r-1}}\Big(c_1(\frac{r+2}{2}\Big)^{\frac{2(r+2)}{r-1}} c_0^{\frac{6(r+2)}{r(r-1)}} \|U^{k-1}\|_r^2 M_{k-1}^{\frac{6}{r(r-1)}} \|U^k\|_r^r. \end{aligned} \tag{3.15}$$

Let

$$\begin{cases} c_2(r,p) := (r-1)\Big(\min\{\frac{r-3}{r}, \frac{r-1}{4}\}\Big)^{-1} 4^{\frac{6r+pr-3p}{pr+3p-6r}} c_0^{\frac{6(2r-p)}{pr+3p-6r}} \Big\{\Big(c_1(\frac{p}{2})\Big)^{\frac{2pr}{pr+3p-6r}} + 1\Big\}, \\ c_3(r,p) := (r-1)\Big(\min\{\frac{r-3}{r}, \frac{r-1}{4}\}\Big)^{-1} 4^{\frac{r+5}{r-1}} \Big(c_1(\frac{r+2}{2}\Big)^{\frac{2(r+2)}{r-1}} c_0^{\frac{6(r+2)}{r(r-1)}}. \end{cases}$$

By a similar argument, we can show that

$$\|U^0\|_r^r + M_0 \le c_2(r,p)\|B\|_p^{\frac{4pr}{pr+3p-6r}} \|U^0\|_r^{\frac{r(pr+p-6r)}{pr+3p-6r}}$$

which shows that

$$\|U^0\|_r^r + M_0 \le \Big(c_2(r,p)\Big)^{\frac{pr+3p-6r}{2p}} \|B\|_p^{2r}. \tag{3.16}$$

Therefore, by induction, we can show that

$$\|U^k\|_r^r + M_k \le \Big(2c_2(r,p)\Big)^{\frac{pr+3p-6r}{2p}} \|B\|_p^{2r} \tag{3.17}$$

holds uniformly for $k \ge 0$, provided that

$$c_3(r,p)\Big(2c_2(r,p)\Big)^{\frac{(r+2)(rp+3p-6r)}{rp(r-1)}} \|B\|_p^{\frac{4(r+2)}{r-1}} \le \frac{1}{2}. \tag{3.18}$$

The uniform estimate (3.17) implies that there is a subsequence (denoted still by U^k) and two functions U' and U such that

$$\begin{cases} |U^k|^{\frac{r-2}{2}} U^k \rightharpoonup U' \quad \text{weakly in} \quad L^2(\mathbb{R}^3), \\ |U^k|^{\frac{r-2}{2}} U^k \rightharpoonup U' \quad \text{weakly in} \quad H^1(\mathbb{R}^3), \\ |U^k|^{\frac{r-2}{2}} U^k \to U' \quad \text{strongly in} \quad L^2_{loc}(\mathbb{R}^3), \\ U^k \rightharpoonup U \quad \text{weakly in} \quad L^r(\mathbb{R}^3) \end{cases} \tag{3.19}$$

as $k \to \infty$. By the Tartar's inequality

$$(|a|^s a - |b|^s b)(a-b) \ge 2^{-s}|a-b|^{s+2} \qquad s > 0,$$

we have for any bounded domain Ω,

$$2^{\frac{2-r}{2}} \int_\Omega |U^k - U^m|^{\frac{r+2}{2}}\, dx \le \int_\Omega \Big(|U^k|^{\frac{r-2}{2}} U^k - |U^m|^{\frac{r-2}{2}} U^m\Big)\Big(U^k - U^m\Big)\, dx. \tag{3.20}$$

It is obvious that $|U^k|^{(r-2)/2}U^k$ is a Cauchy sequence in $L_{loc}^{r/(r-1)}(\mathbb{R}^3)$, which and the last one of (3.19) imply that U^k converges to U for almost every point in $\mathbb{R}^3$. For any $\phi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} U' \cdot \phi\, dx = \lim_{k\to\infty} \int_{\mathbb{R}^3} |U^k|^{\frac{r-2}{2}} U^k \cdot \phi\, dx = \int_{\mathbb{R}^3} |U|^{r-2} U \cdot \phi\, dx \tag{3.21}$$

by the first relation of (3.19) and Lebesgue dominated convergence theorem. By the arbitrariness of ϕ, we deduce that

$$U' = |U|^{\frac{r-2}{2}}U \qquad \text{a. e.} \quad \mathbb{R}^3.$$

Now it is routine to show that U is a solution of the first equation of (2.2) by passing the limit $k \to \infty$, provided B satisfies (3.18). Moreover

$$\|U\|_r^r + \int_{\mathbb{R}^3} |U|^{r-2}|\nabla U|^2 \, dx \le \Big(2c_2(r,p)\Big)^{\frac{pr+3p-6r}{2p}} \|B\|_p^{2r}. \tag{3.22}$$

The estimate also tell us that the solution U is unique as long as B satisfies (3.18). Taking $\varepsilon = \varepsilon(r,p)$

$$\varepsilon := \min\Big\{ \Big(2c_2(r,p)\Big)^{-\frac{pr+3p-6r}{4rp}} \Big(\frac{p-3}{p(p-1)}\Big)^{\frac{r-3}{4r}} c_0^{-\frac{3}{2r}}, \quad \Big(2c_3(r,p)\Big)^{-\frac{r-1}{4(4+2)}} \Big(2c_2(r,p)\Big)^{-\frac{rp+3p-6r}{4pr}} \Big\},$$

then we have

$$\|U\|_r \le \Big(\frac{p-3}{p(p-1)}\Big)^{\frac{r-3}{2r}} c_0^{-\frac{3}{r}}$$

provided

$$\|B\|_p < \varepsilon.$$

Therefore, the result of Theorem 3.2 follows from Theorem 3.1. ■

4 Forward Self-similar Solutions

In this section, we will construct a global forward self-similar solution to the incompressible magneto-hydrodynamics equations (1.1). It should be noted that the usual energy method cannot yield nontrivial forward self-similar solutions for the incompressible magneto-hydrodynamics equations (1.1), as which was first pointed out in the case of the Navier-Stokes equations in [10]. This fact can be shown by following the arguments in [10]. Indeed, suppose (U, B) is a self-similar solution of the incompressible magneto-hydrodynamics equations (1.1), which is the scaling invariant, i.e.

$$\lambda U(\lambda x, \lambda^2 t) = U(x,t), \quad \lambda B(\lambda x, \lambda^2 t) = B(x,t), \quad \forall \lambda > 0$$

and satisfies the energy inequality

$$\|U(t)\|_2^2 + \|B(t)\|_2^2 + 2\int_s^t \Big(\|\nabla u(\tau)\|_2^2 + \|\nabla B(\tau)\|_2^2\Big) d\tau \le \|U(s)\|_2^2 + \|B(s)\|_2^2$$

for some $s > 0$ and all $t \ge s$. By the scaling law, it is easy to verify that $(U(\cdot,t), B(\cdot,t))$ belong to $H^1(\mathbb{R}^3)$, $U(x,t) = (s/t)^{\frac{1}{2}}U((s/t)^{\frac{1}{2}}x, s)$ and $B(x,t) = (s/t)^{\frac{1}{2}}B((s/t)^{\frac{1}{2}}x, s)$ for all

$t \geq s$. Let $y = (s/t)^{\frac{1}{2}}x$, and $V(y) = (s/t)^{\frac{1}{2}}U((s/t)^{\frac{1}{2}}x, s)$, $W(y) = (s/t)^{\frac{1}{2}}B((s/t)^{\frac{1}{2}}x, s)$. Then for some $p = p(y)$,

$$-\Delta V - (1/2s)(1 + y \cdot \nabla)V + (V \cdot \nabla)V - (W \cdot \nabla)W = -\nabla p,$$
$$-\Delta W - (1/2s)(1 + y \cdot \nabla)W + (V \cdot \nabla)W - (W \cdot \nabla)V = 0,$$
$$\text{div } V = 0, \quad \text{div } W = 0.$$

Now we multiply the first equation by V, the second equation by W, then add the resulting equations and integrate over $\mathbb{R}^3$ to get, with the help of third equation, that

$$(1/4s)(\|V\|_2^2 + \|W\|_2^2) + (\|\nabla V\|_2^2 + \|\nabla W\|_2^2) = 0$$

which implies that $V = W = 0$.

The above arguments show that, in order to construct non-trivial forward self-similar solutions to MHD equations, one should introduce some spaces of non-square summable functions. Motivated by the studies on self-similar solution for the incompressible Navier-Stokes equations (cf. [10], [11], [14] and references therein), we will study the existence of forward self-similar solutions in the space of uniformly locally square integrable vector fields. Our main results are:

Theorem 4.1 *Let $u_0, b_0 \in E_2$ be divergence free and homogeneous of degree -1. Then there exists a global forward self-similar solution (u, b) to (1.1) on $\mathbb{R}^3 \times (0, \infty)$, satisfying:*
1) $u, b \in C^\infty(\mathbb{R}^3 \times (0, \infty))$.
2) $\lim_{t \to 0^+} (\|u(t) - u_0\|_{L^2_{loc,unif}(\mathbb{R}^3)} + \|b(t) - b_0\|_{L^2_{loc,unif}(\mathbb{R}^3)}) = 0$.
3) For any $t \in (0, \infty)$,

$$\|u(t)\|_\infty = \|u(1)\|_\infty t^{-\frac{1}{2}}, \qquad \|b(t)\|_\infty = \|b(1)\|_\infty t^{-\frac{1}{2}}.$$

Remarks

1 The forward self-similar solution in Theorem 4.1 is a classical solution to (1.1) on $\mathbb{R}^3 \times \mathbb{R}^+$. However, the uniqueness of such a solution is open.

2 Following the arguments in [14], [6], [1], we also can show that there exists a unique forward self-similar solution when the initial data (u_0, b_0) belongs to some homogeneous Besov space $\dot{B}^{\frac{3}{p}-1}_{p,\infty}(\mathbb{R}^3)$ for some $p \in [1, \infty]$ or weak-L^3 space respectively and is suitable small in some sense. In the present case, $L^2_{loc,unif}(\mathbb{R}^3)$ coincides with the Morrey's space $M^2_2(\mathbb{R}^3)$. And $L^3_w(\mathbb{R}^3) \subset \dot{B}^{\frac{3}{p}-1}_{p,\infty}(\mathbb{R}^3)$ for $p > 3$ and $\dot{B}^{\frac{3}{p}-1}_{\infty,\infty}(\mathbb{R}^3) \subset \dot{B}^{-\frac{3}{2}}_{\infty,\infty}(\mathbb{R}^3)$ for any $p \in [1, \infty]$. See [14] and [2].

3 For the incompressible Navier-Stokes equations, Grujic [11] recently construct a class of global forward self-similar solutions as initial data in $L^2_{loc,unif}(\mathbb{R}^3) \cap L^3_w(\mathbb{R}^3)$. The special case of Theorem 4.1, with $b_0 = b(t) \equiv 0$, yields the existence of a class of global smooth forward self-similar solutions for the three-dimensional incompressible Navier-Stokes equations with initial data $u_0 \in L^2_{loc,unif}(\mathbb{R}^3)$, which improves the results obtained by Grujic [11].

4 The fact that initial data is homogeneous of degree -1 implies that it possess a $1/|x|$-type singularity at origin. Direct calculations show that $L^2_{loc,unif}(\mathbb{R}^3)$ may contain the vector possessing a $1/|x|$-type singularity at origin. On $\mathbb{R}^3$, typical elementary examples of such vectors are given by any linear combinations of the vector fields

$$(0, -\frac{x_3}{|x|^2}, \frac{x_2}{|x|^2}), \quad (\frac{x_3}{|x|^2}, 0, -\frac{x_1}{|x|^2}), \quad (-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}, 0)$$

which are homogeneous of degree -1 and divergence free, see [10], [20].

Proof: We will construct the approximate solutions by using the solutions to the linearized MHD equations, which remain invariant under the scaling. Borrowing and generalizing the ideas and techniques in [14] constructing the suitable local square-integrable weak solutions for Navier-Stokes equations, we then show the local existence of "partial self-similar" solutions. The details are carried out in the following steps.

Step 1: Approximate solutions

We linearize the magneto-hydrodynamic equations (1.1) to construct the approximate solutions as follows:

$$\begin{cases} \dfrac{\partial u^0}{\partial t} - \Delta u^0 + \nabla p^0 = 0, \\ \dfrac{\partial b^0}{\partial t} - \Delta b^0 + \nabla q^0 = 0, \\ \text{div } u^0 = 0, \quad \text{div } b^0 = 0, \\ (u^0(x,0), b^0(x,0)) = (u_0(x), b_0(x)) \end{cases} \tag{4.1}$$

and for any $k \geq 1$

$$\begin{cases} \dfrac{\partial u^k}{\partial t} - \Delta u^k + (u^{k-1} \cdot \nabla)u^k - (b^{k-1} \cdot \nabla)b^k + \nabla p^k = 0, \\ \dfrac{\partial b^k}{\partial t} - \Delta b^k + (u^{k-1} \cdot \nabla)b^k - (b^{k-1} \cdot \nabla)u^k + \nabla q^k = 0, \\ \text{div } u^k = 0, \quad \text{div } b^k = 0, \\ (u^k(x,0), b^k(x,0)) = (u_0(x), b_0(x)). \end{cases} \tag{4.2}$$

By Proposition A1 in Appendix, there is a unique solution $(u^k, b^k) \in C^\infty(\mathbb{R}^3 \times (0,\infty))$ to (4.2) such that

$$u^k, \ b^k \in L^\infty(0,T; L^2_{loc,unif}(\mathbb{R}^3)), \qquad \nabla u^k, \ \nabla b^k \in L^2(0,T; L^2_{loc,unif}(\mathbb{R}^3))$$

for any $T > 0$, as long as $(u^{k-1}, b^{k-1}) \in C^\infty(\mathbb{R}^3 \times (0,\infty))$ and

$$(u^{k-1}, b^{k-1}) \in L^\infty(0,T; L^2_{loc,unif}(\mathbb{R}^3)) \cap L^2(0,T; L^2_{loc,unif}(\mathbb{R}^3)).$$

It is easy to see that $(u^0, b^0, p^0, q^0) \in C^\infty(\mathbb{R}^3 \times (0,\infty))$ and

$$(u^0, b^0) \in L^\infty(0,T; L^2_{loc,unif}(\mathbb{R}^3)) \cap L^2(0,T; L^2_{loc,unif}(\mathbb{R}^3)).$$

Then by induction, (u^k, b^k, p^k, q^k) are well defined for all $k \geq 0$, $(u^k, b^k, p^k, q^k) \in C^\infty(\mathbb{R}^3 \times (0, \infty)$, and

$$(u^k, b^k) \in L^\infty(0, T; L^2_{loc,unif}(\mathbb{R}^3)) \cap L^2(0, T; L^2_{loc,unif}(\mathbb{R}^3)).$$

Step 2: Estimates on the pressure

In order to establish the uniform estimates on the approximate solutions, we first introduce some cut-off functions and notations. Let

$$\phi_0 \in C_0^\infty(\mathbb{R}^3) \quad \text{with} \ \ \phi_0 \geq 0 \ \ \text{and} \ \ \sum_{k \in \mathbb{Z}^3} \phi_0(x-k) = 1.$$

Define $\mathcal{B} = \{\phi(x) =: \phi_0(x - x_0) : \ x_0 \in \mathbb{R}^3\}$. Then (see P_{342} in [14])

$$\|f\|_{L^2_{loc,unif}(\mathbb{R}^3)} \ \text{ is equivalent to } \quad \sup_{\phi \in \mathcal{B}} \|f\phi\|_2.$$

We fix ω_0 and $\psi_0 \in C_0^\infty(\mathbb{R}^3)$ so that ω_0 is identically equal to 1 in the neighborhood of the support of ϕ_0 and similarly, ψ_0 is identically equal to 1 in the neighborhood of the support of ω_0. Then for any $\phi \in \mathcal{B}$, $\phi = \phi_0(x - x_\phi)$, define $\psi(x) = \psi_0(x - x_\phi)$. Let

$$\alpha_k(t) = \sup_{\phi \in \mathcal{B}} \big(\|u^k(\cdot, t)\phi(\cdot)\|_2^2 + \|b^k(\cdot, t)\phi(\cdot)\|_2^2\big),$$

$$\beta_k(t) = \sup_{\phi \in \mathcal{B}} \int_0^t \big(\|\phi(\cdot)\nabla u^k(\cdot, \tau)\|_2^2 + \|\phi(\cdot)\nabla b^k(\cdot, \tau)\|_2^2\big) d\tau,$$

and

$$\beta_k^\eta(t) = \sup_{\phi \in \mathcal{B}} \int_\eta^t \big(\|\phi(\cdot)\nabla u^k(\cdot, \tau)\|_2^2 + \|\phi(\cdot)\nabla b^k(\cdot, \tau)\|_2^2\big) d\tau.$$

Since $u^k, b^k \in C^\infty(\mathbb{R}^3 \times [\eta, T])$ for any $0 < \eta < T < \infty$, then

$$\sup_{\eta < t < T} \alpha_k(t) < \infty \quad \text{ and } \quad \sup_{\eta < t < T} \beta_k^\eta(t) < \infty.$$

In the following, we show that, for all $\phi \in \mathcal{B}$ ($\phi = \phi_0(x - x_0)$ for some $x_0 \in \mathbb{R}^3$), there is a function $p_\phi^k(t)$ so that for all interval $I = (t_0, t_1)$ with $0 < t_0 < t_1 < \infty$,

$$\begin{aligned}
&\Big(\int_{t_0}^{t_1}\int_{\mathbb{R}^3} \big|p^k(x,t) - p_\phi^k(t)\big|^{\frac{3}{2}} \phi(x) \ dxdt\Big)^{\frac{2}{3}} \\
&\qquad \leq C\Big(\|u^{k-1}\|_{L^6(I, L^2_{loc,unif}(\mathbb{R}^3))}\|u^k\|_{L^2(I, L^2_{loc,unif}(\mathbb{R}^3))} \\
&\qquad\qquad + \|b^{k-1}\|_{L^6(I, L^2_{loc,unif}(\mathbb{R}^3))}\|b^k\|_{L^2(I, L^2_{loc,unif}(\mathbb{R}^3))} \\
&\qquad\qquad + \|\psi u^{k-1}\|_{L^6(I, L^2(\mathbb{R}^3))}\|\psi u^k\|_{L^2(I, L^6(\mathbb{R}^3))} \\
&\qquad\qquad + \|\psi b^{k-1}\|_{L^6(I, L^2(\mathbb{R}^3))}\|\psi b^k\|_{L^2(I, L^6(\mathbb{R}^3))}\Big)
\end{aligned} \tag{4.3}$$

with C independent of ϕ, t_0, t_1 and k.

For this purpose, we note that the pressure p^k obeys the equations

$$-\Delta p^k = \sum_{ij=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}\Big(u_i^{k-1}u_j^k - b_i^{k-1}b_j^k\Big).$$

Let $\Gamma(x)$ be the fundamental solution of the Laplace's equation in $\mathbb{R}^3$. Then

$$p^k = \frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathbb{R}^3}\Gamma(x-y)\Big(u_i^{k-1}u_j^k - b_i^{k-1}b_j^k\Big)(y)\ dy.$$

Then

$$p^k(x,t) - p^k_\phi(t) = p^k_1(x,t) + p^k_2(x,t)$$

with

$$\begin{aligned}
p_1^k(x,t) &= \frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathbb{R}^3}\Gamma(x-y)\Big\{\psi_0^2(y-x_0)\big(u_i^{k-1}u_j^k - b_i^{k-1}b_j^k\big)(y)\Big\}dy,\\
p_2^k(x,t) &= \int_{\mathbb{R}^3}\Big(\frac{\partial^2}{\partial x_i \partial x_j}\Gamma(x-y) - \frac{\partial^2}{\partial x_i \partial x_j}\Gamma(x_0-y)\Big)\\
&\qquad\qquad\times\Big\{\big(1-\psi_0^2(y-x_0)\big)\big(u_i^{k-1}u_j^k - b_i^{k-1}b_j^k\big)(y)\Big\}dy.
\end{aligned}$$

It is obvious that there is positive distance between the support of ϕ and the complement to the support of ψ_0^2. So

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^3}\Big|p_2^k(x,t)\Big|^{\frac{3}{2}}\phi(x)dx\Big)^{\frac{2}{3}} \le \Big(\int_{supp\ \phi}\Big|p_2^k(x,t)\Big|^{\frac{3}{2}}dx\Big)^{\frac{2}{3}}\\
\le\ & C\Big(\|u^{k-1}\|_{L^2_{loc,unif}(\mathbb{R}^3)}\|u^k\|_{L^2_{loc,unif}(\mathbb{R}^3)} + \|b^{k-1}\|_{L^2_{loc,unif}(\mathbb{R}^3)}\|b^k\|_{L^2_{loc,unif}(\mathbb{R}^3)}\Big).
\end{aligned}$$

Thus by Hölder inequality, it follows that

$$\begin{aligned}
&\Big(\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\Big|p_2^k(x,t)\Big|^{\frac{3}{2}}\phi(x)dxdt\Big)^{\frac{2}{3}}\\
\le\ & C\Big(\|u^{k-1}\|_{L^6(I,L^2_{loc,unif}(\mathbb{R}^3))}\|u^k\|_{L^2(I,L^2_{loc,unif}(\mathbb{R}^3))}\\
&\qquad\qquad + \|b^{k-1}\|_{L^6(I,L^2_{loc,unif}(\mathbb{R}^3))}\|b^k\|_{L^2(I,L^2_{loc,unif}(\mathbb{R}^3))}\Big).
\end{aligned}$$

As for p_1^k, by the Calderòn-Zygmund theory on singular integrals, we have

$$\begin{aligned}
\|p_1^k\|_{3/2} &\le C\Big(\|\psi^2 u^{k-1}u^k\|_{3/2} + \|\psi^2 b^{k-1}b^k\|_{3/2}\Big)\\
&\le C\Big(\|\psi u^{k-1}\|_2\|\psi u^k\|_6 + \|\psi b^{k-1}\|_2\|\psi b^k\|_6\Big).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\Big(\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\Big|p_1^k(x,t)\Big|^{\frac{3}{2}}\phi(x)dxdt\Big)^{\frac{2}{3}}\\
\le\ & C\Big(\|\psi u^{k-1}\|_{L^6(I,L^2(\mathbb{R}^3))}\|\psi u^k\|_{L^2(I,L^6(\mathbb{R}^3))}\\
&\qquad\qquad\qquad +\|\psi b^{k-1}\|_{L^6(I,L^2(\mathbb{R}^3))}\|\psi b^k\|_{L^2(I,L^6(\mathbb{R}^3))}\Big).
\end{aligned}
$$

Thus we show (4.3).

Similarly for q^k, for all $\phi\in\mathcal{B}$ ($\phi=\phi_0(x-x_0)$ with some $x_0\in\mathbb{R}^3$), there is a function $q_\phi^k(t)$ so that for all interval $I=(t_0,t_1)$ with $0<t_0<t_1<\infty$,

$$
\begin{aligned}
&\Big(\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\big|q^k(x,t)-q_\phi^k(t)\big|^{\frac{3}{2}}\phi(x)\ dxdt\Big)^{\frac{2}{3}}\\
&\quad\le C\Big(\|u^{k-1}\|_{L^6(I,L^2_{loc,unif}(\mathbb{R}^3))}\|b^k\|_{L^2(I,L^2_{loc,unif}(\mathbb{R}^3))}\\
&\qquad\qquad +\|b^{k-1}\|_{L^6(I,L^2_{loc,unif}(\mathbb{R}^3))}\|u^k\|_{L^2(I,L^2_{loc,unif}(\mathbb{R}^3))}\\
&\qquad\qquad +\|\psi u^{k-1}\|_{L^6(I,L^2(\mathbb{R}^3))}\|\psi b^k\|_{L^2(I,L^6(\mathbb{R}^3))}\\
&\qquad\qquad +\|\psi b^{k-1}\|_{L^6(I,L^2(\mathbb{R}^3))}\|\psi u^k\|_{L^2(I,L^6(\mathbb{R}^3))}\Big)
\end{aligned}\tag{4.4}
$$

with C independent of ϕ, t_0, t_1 and k.

Step 3: Local energy estimates.

Since $(u^k,b^k,p^k,q^k)\in C^\infty(\mathbb{R}^3\times(0,\infty))$, we have

$$
\begin{aligned}
&\partial_t\big(|u^k|^2+|b^k|^2\big)+2\big(|\nabla u^k|^2+|\nabla b^k|^2\big)-\Delta\big(|u^k|^2+|b^k|^2\big)\\
&=-\mathrm{div}\ \Big(\big(|u^k|^2+|b^k|^2\big)u^{k-1}-2\big(u^k\cdot b^k\big)b^k+2p^ku^k+2q^kb^k\Big).
\end{aligned}\tag{4.5}
$$

Multiplying both sides of (4.5) by ϕ^2, and integrating over $\mathbb{R}^3\times(\eta,t)$ with $0<\eta<t<\infty$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\Big(|u^k(x,t)|^2+|b^k(x,t)|^2\Big)\phi^2(x)dx+2\int_\eta^t\int_{\mathbb{R}^3}\Big(|\nabla u^k(x,\tau)|^2+|\nabla b^k(x,\tau)|^2\Big)\phi^2(x)dxd\tau\\
=\ &\int_{\mathbb{R}^3}\Big(|u^k(x,\eta)|^2+|b^k(x,\eta)|^2\Big)\phi^2(x)dx+\int_\eta^t\int_{\mathbb{R}^3}\Big(|u^k(x,\tau)|^2+|b^k(x,\tau)|^2\Big)\Delta\big(\phi^2(x)\big)dxd\tau\\
&+\int_\eta^t\int_{\mathbb{R}^3}\Big\{\big(|u^k(x,\tau)|^2+|b^k(x,\tau)|^2\big)u^{k-1}\cdot\nabla\big(\phi^2(x)\big)-2\big(u^k\cdot b^k\big)b^{k-1}\cdot\nabla\phi^2(x)\Big\}dxd\tau\\
&+2\int_\eta^t\int_{\mathbb{R}^3}\Big\{\Big(p^k-p_\phi^k\Big)u^k\cdot\phi^2(x)+\Big(q^k-q_\phi^k\Big)b^k\cdot\nabla\phi^2(x)\Big\}dxd\tau.
\end{aligned}\tag{4.6}
$$

Here we used the facts that

$$
\mathrm{div}\ \Big(p^ku^k\Big)=u^k\cdot\nabla p^k=u^k\cdot\nabla\Big(p^k-p_\phi^k\Big)=\mathrm{div}\ \Big(u^k\big(p^k-p_\phi^k\big)\Big)
$$

and similarly

$$\text{div}\,\Big(q^k b^k\Big) = \text{div}\,\Big(b^k\big(q^k - q^k_\phi\big)\Big).$$

It is obvious for all $t \in (\eta, \infty)$ and all $\phi \in \mathcal{B}$,

$$\Big|\int_\eta^t\!\!\int_{\mathbb{R}^3} \Big(|u^k(x,\tau)|^2 + |b^k(x,\tau)|^2\Big)\Delta\big(\phi^2(x)\big)dxd\tau\Big| \le C\int_\eta^t \alpha_k(\tau)\ d\tau. \tag{4.7}$$

Let $\psi_0(x)$ be equal to 1 on the support of ϕ_0. For $\phi(x) = \phi_0(x - x_0)$ with some $x_0 \in \mathbb{R}^3$, let $\psi(x) = \psi_0(x - x_0)$. Then

$$\begin{aligned}
&\int_\eta^t\!\!\int_{\mathbb{R}^3} \Big\{\big(|u^k(x,\tau)|^2 + |b^k(x,\tau)|^2\big)u^{k-1}\cdot\nabla\big(\phi^2(x)\big) - 2\big(u^k\cdot b^k\big)b^{k-1}\cdot\nabla\phi^2(x)\Big\}dxd\tau \\
\le\ & C\int_\eta^t\!\!\int_{\mathbb{R}^3} \Big(|\psi u^k|^2 + |\psi b^k|^2\Big)\Big(|\psi u^{k-1}| + |\psi b^{k-1}|\Big)\ dxd\tau \\
\le\ & C\int_\eta^t \Big(\|\psi u^k\|_4^2 + \|\psi b^k\|_4^2\Big)\Big(\|\psi u^{k-1}\|_2 + \|\psi b^{k-1}\|_2\Big)\ d\tau \\
\le\ & C\int_\eta^t \Big(\|\psi u^k\|_2^{\frac12}\|\nabla\big(\psi u^k\big)\|_2^{\frac32} + \|\psi b^k\|_2^{\frac12}\|\nabla\big(\psi b^k\big)\|_2^{\frac32}\Big)\Big(\|\psi u^{k-1}\|_2 + \|\psi b^{k-1}\|_2\Big)\ d\tau \\
\le\ & C\Big\{\int_\eta^t \Big(\|\psi u^k\|_2^2 + \|\psi b^k\|_2^2\Big)\Big(\|\psi u^{k-1}\|_2^4 + \|\psi b^{k-1}\|_2^4\Big)\ d\tau\Big\}^{\frac14} \\
&\times\Big\{\int_\eta^t \Big(\|\nabla\big(\psi u^k\big)\|_2^2 + \|\nabla\big(\psi b^k\big)\|_2^2\Big)\ d\tau\Big\}^{\frac34}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Big(|\psi_0(x-x_0)|^2 + |\nabla\psi_0(x-x_0)|^2\Big)\Big(|u^k|^2 + |b^k|^2\Big)\ dx \le C\alpha_k(t), \\
&\int_\eta^t\!\!\int_{\mathbb{R}^3} \big|\psi_0(x-x_0)\big|^2\Big(|\nabla u^k|^2 + |\nabla b^k|^2\Big)\ dxd\tau \le C\beta_k^\eta(t).
\end{aligned} \tag{4.8}$$

then we show that, for all $t \in (\eta, \infty)$ and all $\phi \in \mathcal{B}$,

$$\begin{aligned}
&\Big|\int_\eta^t\!\!\int_{\mathbb{R}^3} \Big\{\big(|u^k(x,\tau)|^2 + |b^k(x,\tau)|^2\big)u^{k-1}\cdot\nabla\big(\phi^2(x)\big) - 2\big(u^k\cdot b^k\big)b^{k-1}\cdot\nabla\phi^2(x)\Big\}dxd\tau\Big| \\
\le\ & C\Big\{\int_\eta^t \alpha_k(\tau)\big(\alpha_{k-1}(\tau)\big)^2\ d\tau\Big\}^{\frac14}\Big(\beta_k^\eta(t) + \int_\eta^t \alpha_k(\tau)\ d\tau\Big)^{\frac34} \\
\le\ & C\Big\{\int_\eta^t \big(\alpha_{k-1}(\tau)\big)^3 d\tau\Big\}^{\frac16}\Big\{\int_\eta^t \big(\alpha_k(\tau)\big)^3 d\tau\Big\}^{\frac1{12}}\Big(\beta_k^\eta + \int_\eta^t \alpha_k(\tau)d\tau\Big)^{\frac34}.
\end{aligned} \tag{4.9}$$

Next we turn to terms on the pressure. By Hölder inequality, we get that

$$\begin{aligned}
&\Big|2\int_\eta^t\!\!\int_{\mathbb{R}^3} \Big\{\Big(p^k - p^k_\phi\Big)u^k\cdot\phi^2(x)\Big\}dxd\tau\Big| \\
\le\ & C\Big\{\int_\eta^t\!\!\int_{\mathbb{R}^3} \big|p^k - p^k_\phi\big|^{\frac32}\omega_0(x-x_0)\ dxd\tau\Big\}^{\frac23}\Big\{\int_\eta^t\!\!\int_{\mathbb{R}^3} |u^k|^3\omega_0(x-x_0)\ dxd\tau\Big\}^{\frac13}.
\end{aligned}$$

Applying (4.3), by the Sobolev inequality and (4.8), we have

$$
\begin{aligned}
&\Big\{\int_\eta^t\!\!\int_{\mathbb{R}^3}|p^k-p^k_\phi|^{\frac32}\omega_0(x-x_0)\ dxd\tau\Big\}^{\frac23}\\
\le\ & C\Big\{\Big(\int_\eta^t\big(\alpha_{k-1}(\tau)\big)^3d\tau\Big)^{\frac16}\Big(\int_\eta^t\alpha_k(\tau)d\tau\Big)^{\frac12}\\
&\qquad+\Big(\int_\eta^t\big(\alpha_{k-1}(\tau)\big)^3d\tau\Big)^{\frac16}\Big(\beta_k^\eta(t)+\int_\eta^t\alpha_k(\tau)d\tau\Big)^{\frac12}\Big\}.
\end{aligned}
$$

By the Gagliardo-Nirenberg inequality and (4.8) again, we deduce

$$
\begin{aligned}
\Big\{\int_\eta^t\!\!\int_{\mathbb{R}^3}|u^k|^3\omega_0(x-x_0)\ dxd\tau\Big\}^{\frac13}\le\ & \Big\{\int_\eta^t\!\!\int_{\mathbb{R}^3}|u^k|^3\psi_0^3(x-x_0)\ dxd\tau\Big\}^{\frac13}\\
\le\ & C\Big\{\int_\eta^t\|\psi_0u^k\|_2^{\frac32}\|\nabla(\psi_0u^k)\|_2^{\frac32}d\tau\Big\}^{\frac13}\\
\le\ & C\Big\{\int_\eta^t\|\psi_0u^k\|_2^6d\tau\Big\}^{\frac1{12}}\Big\{\int_\eta^t\|\nabla(\psi_0u^k)\|_2^2d\tau\Big\}^{\frac14}\\
\le\ & C\Big\{\int_\eta^t\big(\alpha_k(\tau)\big)^3d\tau\Big\}^{\frac1{12}}\Big(\beta_k^\eta(t)+\int_\eta^t\alpha_k(\tau)d\tau\Big)^{\frac14}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\Big|2\int_\eta^t\!\!\int_{\mathbb{R}^3}\Big\{\Big(p^k-p^k_\phi\Big)u^k\cdot\phi^2(x)\Big\}dxd\tau\Big|\\
\le\ & C\Big\{\int_\eta^t\big(\alpha_{k-1}(\tau)\big)^3d\tau\Big\}^{\frac16}\Big\{\int_\eta^t\big(\alpha_k(\tau)\big)^3d\tau\Big\}^{\frac1{12}}\Big(\beta_k^\eta(t)+\int_\eta^t\alpha_k(\tau)d\tau\Big)^{\frac34}. \qquad (4.10)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\Big|2\int_\eta^t\!\!\int_{\mathbb{R}^3}\Big\{\Big(q^k-q^k_\phi\Big)b^k\cdot\phi^2(x)\Big\}dxd\tau\Big|\\
\le\ & C\Big\{\int_\eta^t\big(\alpha_{k-1}(\tau)\big)^3d\tau\Big\}^{\frac16}\Big\{\int_\eta^t\big(\alpha_k(\tau)\big)^3d\tau\Big\}^{\frac1{12}}\Big(\beta_k^\eta(t)+\int_\eta^t\alpha_k(\tau)d\tau\Big)^{\frac34}. \qquad (4.11)
\end{aligned}
$$

Therefore, substituting (4.7), (4.9) - (4.11) into (4.6), it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\Big(|u^k(x,t)|^2+|b^k(x,t)|^2\Big)\phi^2(x)dx+2\int_\eta^t\!\!\int_{\mathbb{R}^3}\Big(|\nabla u^k|^2+|\nabla b^k|^2\Big)\phi^2(x)dxd\tau\\
\le\ & \alpha_k(\eta)+C\int_\eta^t\alpha_k(\tau)d\tau\\
&\qquad+C\Big\{\int_\eta^t\big(\alpha_{k-1}(\tau)\big)^3d\tau\Big\}^{\frac16}\Big\{\int_\eta^t\big(\alpha_k(\tau)\big)^3d\tau\Big\}^{\frac1{12}}\Big(\beta_k^\eta(t)+\int_\eta^t\alpha_k(\tau)d\tau\Big)^{\frac34}.
\end{aligned}
$$

By taking the supremum on $\phi\in\mathcal{B}$ on the terms at left hand side above shows that

$$\alpha_k(t)+2\beta_k^\eta(t)\le\alpha_k(\eta)+C\int_\eta^t\alpha_k(\tau)d\tau$$
$$+C\Big\{\int_\eta^t\big(\alpha_{k-1}(\tau)\big)^3d\tau\Big\}^{\frac{1}{6}}\Big\{\int_\eta^t\big(\alpha_k(\tau)\big)^3d\tau\Big\}^{\frac{1}{12}}\Big(\beta_k^\eta(t)+\int_\eta^t\alpha_k(\tau)d\tau\Big)^{\frac{3}{4}}$$

for any $t\in(\eta,\infty)$. By the Young's inequality, we obtain that, for any $t\in(\eta,\infty)$, the inequality

$$\begin{aligned}\alpha_k(t)+2\beta_k^\eta(t)\le\ &\alpha_k(\eta)+C_1\int_\eta^t\alpha_k(\tau)d\tau\\ &+C_2\Big\{\int_\eta^t\big(\alpha_{k-1}(\tau)\big)^3d\tau\Big\}^{\frac{2}{3}}\Big\{\int_\eta^t\big(\alpha_k(\tau)\big)^3d\tau\Big\}^{\frac{1}{3}}\end{aligned}\tag{4.12}$$

holds uniformly for $k\ge0$ with positive constant C independent of t and η. Similar but simply calculations show that

$$\alpha_0(t)+2\beta_0(t)\le\alpha_0(0)+C_1\int_0^t\alpha_k(\tau)d\tau$$

which yields that

$$\alpha_0(t)\in L^\infty(0,T)$$

for any $T>0$. By induction, we can show that there is $T_0=T_0(\alpha_{k-1})$, so that for $0<T\le T_0$, $\alpha_k(t)\in L^\infty(0,T)$. So $\alpha_k(t)$ is bounded in some neighborhood of $t=0$. Let $\eta\to0$ in (4.12), we get that

$$\alpha_k(t)+2\beta_k(t)\le\alpha+C_1\int_0^t\alpha_k(\tau)d\tau+C_2\Big\{\int_0^t\big(\alpha_{k-1}(\tau)\big)^3d\tau\Big\}^{\frac{2}{3}}\Big\{\int_0^t\big(\alpha_k(\tau)\big)^3d\tau\Big\}^{\frac{1}{3}}\tag{4.13}$$

holds uniformly for $k\ge0$ with positive constant C independent of t. Here $\alpha=\alpha_k(0)$.

An induction argument implies us that

$$\alpha_k(t)+2\beta_k(t)\le\alpha\qquad\text{for any}\quad t\in[0,T^*]\tag{4.14}$$

provided

$$T^*\Big(2C_1+8C_2\alpha^2\Big)=1.$$

Step 4: Local existence of suitable weak solutions

From now on, we assume that ϕ is any given one in $C_0^\infty(\mathbb{R}^3)$ with $\phi(x)\ge0$ (note that in general $\phi\notin\mathcal{B}$). Then there are finite points $k_i\in\mathbb{Z}$, $i=1,\cdots I$, such that $\phi(x)=\sum_{i=1}^I\phi(x)\phi_0(x-k_i)$. Based on the uniform estimate (4.14), it follows from (4.3) and (4.4), that

$$\Big(\int_0^{T^*}\int_{\mathbb{R}^3}\big|p^k(x,t)-p_\phi^k(t)\big|^{\frac{3}{2}}\phi(x)\ dxdt\Big)^{\frac{2}{3}}+\Big(\int_0^{T^*}\int_{\mathbb{R}^3}\big|q^k(x,t)-q_\phi^k(t)\big|^{\frac{3}{2}}\phi(x)\ dxdt\Big)^{\frac{2}{3}}\le CT^*\alpha\tag{4.15}$$

uniformly for $k \geq 0$. Therefore, (4.14) and (4.15) imply that there is a subsequence of (u^k, b^k, p^k, q^k), denoted by themselves, such that there is weak limit (u, b, p, q) with

$$(u, b) \in L^\infty(0, T^*; L^2_{loc,unif}(\mathbb{R}^3)), \quad (p, q) \in L^{3/2}(0, T^*; L^{3/2}_{loc}(\mathbb{R}^3)),$$

and $\nabla u,\ \nabla b \in L^2(0, T^*; L^2_{loc,unif}(\mathbb{R}^3))$. Moreover,

$$(u^k,\ b^k) \rightharpoonup (u,\ b) \quad \text{weak} - \text{star in} \ \ L^\infty(0, T^*; L^2_{loc,unif}(\mathbb{R}^3)), \tag{4.16}$$

$$(\nabla u^k,\ \nabla b^k) \rightharpoonup (\nabla u,\ \nabla b) \quad \text{weakly in} \ \ L^2(0, T^*; L^2_{loc,unif}(\mathbb{R}^3)), \tag{4.17}$$

and

$$(p^k,\ q^k) \rightharpoonup (p,\ q) \quad \text{weakly in} \ \ L^{\frac{3}{2}}(0, T^*; L^{\frac{3}{2}}_{loc}(\mathbb{R}^3)) \tag{4.18}$$

as $k \to \infty$.

In order to show the strong convergence in $L^p(0, T^*; L^2_{loc}(\mathbb{R}^3))$, we need the following Friederichs inequality (see Lemma II.4.2 [9]): Let Q be a cube in $\mathbb{R}^3$, then for any $\varepsilon > 0$, there exist $K(\varepsilon, Q) \in \mathbb{N}$ functions $\omega_i \in L^\infty(Q)$, $i = 1, \cdots, K$ such that

$$\int_0^T \|\omega(t)\|^2_{L^2(Q)} dt \leq \sum_{i=1}^K \int_0^T \int_Q \omega(x,t)\omega_i(x,t) dxdt + \varepsilon \int_0^T \int_Q \left|\nabla \omega(x,t)\right|^2_{L^2(Q)} dxdt.$$

This inequality, together with (4.16) and (4.17), implies that

$$\lim_{k\to\infty} \int_0^{T^*} \int_Q \Big(\big|u^k(x,t) - u(x,t)\big|^2 + \big|b^k(x,t) - b(x,t)\big|^2 \Big) dxdt = 0. \tag{4.19}$$

By Sobolev embedding theorem, we also have

$$\lim_{k\to\infty} \int_0^{T^*} \int_Q \Big(\big|u^k(x,t) - u(x,t)\big|^r + \big|b^k(x,t) - b(x,t)\big|^r \Big) dxdt = 0 \quad \text{for any} \ \ r \in [2, 6). \tag{4.20}$$

Now we will show that q is a constant a.e. in $\mathbb{R}^3 \times (0, T^*)$. To this end, we note that

$$-\Delta q^k = \text{div} \ \Big(u^{k-1} \cdot \nabla b^k - b^{k-1} \cdot \nabla u^k \Big).$$

By (4.17) and (4.20), $u^{k-1} \cdot \nabla b^k - b^{k-1} \cdot \nabla u^k$ converges weakly to $(u \cdot \nabla b - b \cdot \nabla u)$ in $L^r(0, T^*; L^r_{loc}(\mathbb{R}^3))$ for any $r \in [1, 3/2)$ as $k \to \infty$. Note that

$$\text{div} \ \Big(u \cdot \nabla b - b \cdot \nabla u \Big) = \text{div} \ \Big(\text{curl} \ \big(u \times b \big) \Big) = 0.$$

Passing the limit $k \to \infty$, we deduce that

$$-\Delta \ q = 0$$

holds in the sense of distribution. In view of the previous arguments,

$$\begin{aligned} q^k_\phi(t) &= \int_{\mathbb{R}^3} \frac{\partial^2}{\partial x_i \partial x_j}\Gamma(x_0-y)\Big\{\big(1-\psi_0^2(y-x_0)\big)\big(u_i^{k-1}b_j^k - b_i^{k-1}u_j^k\big)(y)\Big\}dy \\ &= \int_{\mathbb{R}^3} \frac{\partial^2}{\partial x_i \partial x_j}\Gamma(x_0-y)\Big\{\big(1-\psi_0^2(y-x_0)\big)\big(u_i^{k-1}b_j^k - b_j^{k-1}u_i^k\big)(y)\Big\}dy \end{aligned}$$

which implies that $\lim_{k\to\infty} q^k_\phi(t) = 0$ for any x_0 in the support of ϕ. This and (4.15) imply that $q \in L^{3/2}(0,T^*; L^{3/2}_{loc,unif}(\mathbb{R}^3))$. By the mean value property of harmonic functions, q is bounded, therefore, q must be a constant. Without loss of generality, let $q = 0$.

By standard arguments, we can show that (u, b, p) is a weak solutions to (1.1). Here we omit the details. In the following, we show 2) of Theorem 4.1. From (4.14), we have

$$\|u(t)\|^2_{L^2_{loc,unif}(\mathbb{R}^3)} + \|b(t)\|^2_{L^2_{loc,unif}(\mathbb{R}^3)} \le \alpha, \quad \forall t \in [0,T^*]. \tag{4.21}$$

By the arguments presented in chap.14 and 32 in [14], it is easy to deduce from the fact that (u, b) is a weak solution to (1.1) that

$$(u(t),\ b(t)) \rightharpoonup (u_0,\ b_0) \quad \text{weakly in } \ L^2_{loc,unif}(\mathbb{R}^3)$$

as $t \to 0$. This and (4.21) give us that

$$\lim_{t\to 0^+}(\|u(t)-u_0\|_{L^2_{loc,unif}(\mathbb{R}^3)} + \|b(t)-b_0\|_{L^2_{loc,unif}(\mathbb{R}^3)}) = 0.$$

Now we verifies that the local energy inequality is valid for (u, b, p). Since

$$(u^k, b^k, p^k, q^k) \in C^\infty(\mathbb{R}^3 \times (0,\infty)),$$

a simple calculation shows that (u^k, b^k, p^k, q^k) satisfies the localized energy inequality: For each real-valued $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^+)$ with $\phi \ge 0$,

$$\begin{aligned} & 2\int_0^\infty\int_{\mathbb{R}^3}(|\nabla u^k(x,t)|^2 + |\nabla b^k(x,t)|^2)\phi dxdt \\ \le & \int_0^\infty\int_{\mathbb{R}^3}(|u^k(x,t)|^2 + |b^k(x,t)|^2)(\phi_t(x,t) + \Delta\phi(x,t))dxdt \\ & + \int_0^\infty\int_{\mathbb{R}^3}(u^{k-1}(x,t)\cdot\nabla\phi)(|u^k(x,t)|^2 + |b^k(x,t)|^2)dxdt \\ & \quad +2\int_0^\infty\int_{\mathbb{R}^3}\Big(p^k(u^k\cdot\phi) + q^k(b^k\cdot\phi) - (b^{k-1}\cdot\nabla\phi)(u^k\cdot b^k)\Big)dxdt \end{aligned} \tag{4.22}$$

and for any $\chi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^+)$, the equation

$$\frac{\partial b^k\chi}{\partial t} - \Delta(b^k\chi) = b^k(\frac{\partial\chi}{\partial t} - \Delta\chi) - 2\nabla\chi\cdot\nabla b^k - \chi(u^{k-1}\cdot\nabla)b^k + \chi b^{k-1}\cdot\nabla u^k \tag{4.23}$$

holds in the sense of distribution.

Applying (4.16) - (4.20), we deduce, by passing the limit $k \to \infty$, that for each real-valued $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^+)$ with $\phi \geq 0$, the generalized energy inequality is valid:

$$\begin{aligned}
& 2\int_0^\infty \int_{\mathbb{R}^3} (|\nabla u(x,t)|^2 + |\nabla b(x,t)|^2)\phi dxdt \\
\leq & \int_0^\infty \int_{\mathbb{R}^3} (|u(x,t)|^2 + |b(x,t)|^2)(\phi_t(x,t) + \Delta\phi(x,t))dxdt \\
& + \int_0^\infty \int_{\mathbb{R}^3} (u(x,t)\cdot\nabla\phi)(|u(x,t)|^2 + |b(x,t)|^2 + 2p(x,t))dxdt \\
& \qquad -2\int_0^\infty \int_{\mathbb{R}^3} (b\cdot\nabla\phi)(u\cdot b)dxdt
\end{aligned} \tag{4.24}$$

and for any $\chi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^+)$, the equation

$$\frac{\partial b\chi}{\partial t} - \Delta(b\chi) = b(\frac{\partial \chi}{\partial t} - \Delta\chi) - 2\nabla\chi\cdot\nabla b - \chi(u\cdot\nabla)b + \chi b\cdot\nabla u \tag{4.25}$$

holds in the sense of distribution for weak solution (u, b, p). Therefore, (u, b, p) is a suitable weak solution to (1.1) in the sense of definition 2.2.

Step 5: Regularity of the suitable weak solution

Since the initial data (u_0, b_0) is homogeneous of degree -1, direct calculation implies that

$$\lambda u^k(\lambda x, \lambda^2 t) = u^k(x,t), \qquad \lambda b^k(\lambda x, \lambda^2 t) = b^k(x,t)$$

for any $t \in (0, T^*)$ and $\lambda \in (0, 1]$, since the solution to (4.2) is unique. By the convergence obtained in step 4, for any $\phi \in C_0^\infty(\mathbb{R}^3 \times (0, T^*))$, we have

$$\begin{aligned}
\int_0^{T^*}\int_{\mathbb{R}^3} u(x,t)\phi(x,t)dxdt \quad & \overset{k\to\infty}{\longleftarrow} \quad \int_0^{T^*}\int_{\mathbb{R}^3} u^k(x,t)\phi(x,t)dxdt \\
& = \quad \int_0^{T^*}\int_{\mathbb{R}^3} \lambda u^k(\lambda x, \lambda^2 t)\phi(x,t)dxdt \\
& = \quad \int_0^{\lambda^2 T^*}\int_{\mathbb{R}^3} \lambda^{-4} u^k(y,\tau)\phi(\lambda^{-1}y, \lambda^{-2}\tau)dyd\tau \\
& \overset{k\to\infty}{\longrightarrow} \quad \int_0^{\lambda^2 T^*}\int_{\mathbb{R}^3} \lambda^{-4} u(y,\tau)\phi(\lambda^{-1}y, \lambda^{-2}\tau)dyd\tau \\
& = \quad \int_0^{T^*}\int_{\mathbb{R}^3} \lambda u(\lambda x, \lambda^2 t)\phi(x,t)dxdt.
\end{aligned}$$

Therefore, we deduce that

$$\int_0^{T^*}\int_{\mathbb{R}^3} u(x,t)\phi(x,t)dxdt = \int_0^{T^*}\int_{\mathbb{R}^3} \lambda u(\lambda x, \lambda^2 t)\phi(x,t)dxdt$$

for any $\phi \in C_0^\infty(\mathbb{R}^3 \times (0, T^*))$. It follows from a density argument that

$$\lambda u(\lambda x, \lambda^2 t) = u(x,t) \qquad \text{for any} \ \ t \in [0, T^*] \ \ \text{and} \ \ \lambda \in (0,1].$$

Similarly,

$$\lambda b(\lambda x, \lambda^2 t) = b(x,t) \qquad \text{for any} \ \ t \in [0, T^*] \ \ \text{and} \ \ \lambda \in (0,1].$$

In the following we show that the set of possible singular points is empty by the partial regularity theory Theorem 2.1 established in [12]. Suppose that $(x_0, t_0) \in \mathbb{R}^3 \times (0, T_0)$ is singular point for u for any $T_0 \le T^*$. Then for some $r \in (0, \sqrt{t_0})$,

$$\|u\|_{L^\infty(Q_r(x_0,t_0))} = \infty.$$

Then for any fixed $\lambda \in (0,1]$, $u(\lambda x, \lambda^2 t) = \lambda^{-1} u(x,t)$ for any $t \in (t_0 - r^2, t_0)$ and almost every point $x \in \mathbb{R}^3$. Thus we have

$$\|u\|_{L^\infty(Q_{\lambda r}(\lambda x_0, \lambda^2 t_0))} = \lambda^{-1} \|u\|_{L^\infty(Q_r(x_0,t_0))} = \infty.$$

Therefore, we have shown that $(\lambda x_0, \lambda^2 t_0)$ is a singular point of u for any $\lambda \in (0,1]$, provided (x_0, t_0) is a singular point for u. This implies that the one-dimensional Hausdorff measure of the set of possible singular points of u will be positive, which contradicts with Theorem 2.1. Similarly, we can show that the set of possible singular points of b is empty.

By the definition of regular point, (u,b) is locally bounded at each point $(x,t) \in \mathbb{R}^3 \times (0, T_0)$. Apply the regular criteria Theorem 2.2 obtained in [13], (u,b) is infinitely differentiable with respect to space variables, and each spatial derivatives is bounded on the compact subdomain of some neighborhood of each point.

From the spatial continuity for (u,b), we have that, for any given $\lambda > 0$, $\lambda u(\lambda x, \lambda^2 t) = u(x,t)$ for every $t \in (0, \min\{T_0, T_0/\lambda^2\})$ and for any $x \in \mathbb{R}^3$. By the self-similarity, we can write $(u(x,t), b(x,t)) = \frac{1}{\sqrt{t}}(U_T(\frac{x}{\sqrt{t}}), B_T(\frac{x}{\sqrt{t}}))$ for any $(x,t) \in \mathbb{R}^3 \times (0, T_0)$, where $(U_T, B_T)(y) = \sqrt{T/2}(u,b)((\sqrt{T/2})y, T/2)$. Thus, the fact that $(u,b) \in C_x^\infty$ implies that $(U_T, B_T) \in C^\infty$, which implies that $(u,b) \in C_t^\infty$. From the equation on pressure, it is easy to show the pressure also is infinitely differentiable with respect to spatial and time variables.

Therefore, u, b, and p belong to $C^\infty(\mathbb{R}^3 \times (0, T_0))$, and for any $\lambda \in (0,1]$, $(x,t) \in \mathbb{R}^3 \times (0, T_0)$,

$$u(x,t) = \lambda u(\lambda x, \lambda^2 t), \qquad b(x,t) = \lambda b(\lambda x, \lambda^2 t), \qquad p(x,t) = \lambda^2 p(\lambda x, \lambda^2 t).$$

Step 6: Extension to a global solution.

Finally we extend (u, b, p) to be globally in time by the self-similarity. In fact, let

$$(u(x,t), b(x,t)) =: (\frac{T_0}{2t})^{\frac{1}{2}} \Big(u\big((\frac{T_0}{2t})^{\frac{1}{2}} x, \frac{T_0}{2} \big), b\big((\frac{T_0}{2t})^{\frac{1}{2}} x, \frac{T_0}{2} \big) \Big)$$

and

$$p(x,t) =: \frac{T_0}{2t} p\Big((\frac{T_0}{2t})^{\frac{1}{2}} x, \frac{T_0}{2} \Big)$$

for any $t \ge T_0$, since the right terms are well-defined for any $x \in \mathbb{R}^3$. It is obvious that the global solution obtained by extension remains invariant under scaling, and belongs to $C^\infty(\mathbb{R}^3 \times \mathbb{R}^+)$. Hence it is a classical solution and solves the equations (1.1) pointwisely. The final result of Theorem 4.1 follows easily from the scaling law. So the proof of Theorem 4.1 is completed. ■

Appendix

In this appendix, we present the results on the existence and uniqueness for solutions to the linearized magnetohydrodynamics equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (v\cdot\nabla)u - (a\cdot\nabla)b + \nabla p = 0, \\ \dfrac{\partial b}{\partial t} - \Delta b + (v\cdot\nabla)b - (a\cdot\nabla)u + \nabla q = 0, \\ \text{div } u = 0, \quad \text{div } b = 0, \\ (u(x,0), b(x,0)) = (u_0(x), b_0(x)) \end{cases} \tag{A1}$$

in which v and a are given vectors with

$$v,\ a \in L^\infty(0,T; L^2_{loc,unif}(\mathbb{R}^3) \qquad \text{and} \qquad \nabla v,\ \nabla a \in L^2(0,T; L^2_{loc,unif}(\mathbb{R}^3)) \tag{A2}$$

for any $T>0$. Let

$$\gamma(t) =: \|v(t)\|_{L^2_{loc,unif}(\mathbb{R}^3)} + \|a(t)\|_{L^2_{loc,unif}(\mathbb{R}^3)}$$

and

$$\alpha' =: \|u_0\|^2_{L^2_{loc,unif}(\mathbb{R}^3)} + \|b_0\|^2_{L^2_{loc,unif}(\mathbb{R}^3)}.$$

Proposition A1: *Let $u_0, b_0 \in E_2$ be divergence free and $v,\ a \in C^\infty(\mathbb{R}^3\times(0,\infty))$ satisfy (A2). Then there exists a unique solution $(u,b,p,q) \in C^\infty(\mathbb{R}^3\times(0,\infty))$ to (A1) such that*

$$\begin{aligned} \|u(t)\|^2_{L^2_{loc,unif}(\mathbb{R}^3)} + \|b(t)\|^2_{L^2_{loc,unif}(\mathbb{R}^3)} + \int_0^t \Big(\|\nabla u(\tau)\|^2_{L^2_{loc,unif}(\mathbb{R}^3)} + \|\nabla b(\tau)\|^2_{L^2_{loc,unif}(\mathbb{R}^3)}\Big)\, d\tau \\ \le \alpha' \exp\Big\{C\Big(1+(\sup_{\tau\in[0,T]}\gamma(\tau))^4\Big)t\Big\} \end{aligned} \tag{A3}$$

for all $t\in[0,T]$ with any positive T. Furthermore,

$$\lim_{t\to 0^+}(\|u(t)-u_0\|_{L^2_{loc,unif}(\mathbb{R}^3)} + \|b(t)-b_0\|_{L^2_{loc,unif}(\mathbb{R}^3)}) = 0.$$

Proof Let $\{(u_0^m, b_0^m)\}_{m=1}^\infty \subset C_0^\infty(\mathbb{R}^3)$ such that

$$\|(u_0-u_0^m,\ b_0-b_0^m)\|_{L^2_{loc,unif}(\mathbb{R}^3)} = 0$$

and for any $m\ge 1$

$$\|u_0^m\|_{L^2_{loc,unif}(\mathbb{R}^3)} \le \|u_0\|_{L^2_{loc,unif}(\mathbb{R}^3)}, \qquad \|b_0^m\|_{L^2_{loc,unif}(\mathbb{R}^3)} \le \|b_0\|_{L^2_{loc,unif}(\mathbb{R}^3)}. \tag{A4}$$

Consider the solutions of the following equations

$$\begin{cases} \dfrac{\partial u^m}{\partial t} - \Delta u^m + (v\cdot\nabla)u^m - (a\cdot\nabla)b^m + \nabla p^m = 0, \\ \dfrac{\partial b^m}{\partial t} - \Delta b^m + (v\cdot\nabla)b^m - (a\cdot\nabla)u^m + \nabla q^m = 0, \\ \text{div } u^m = 0, \quad \text{div } b^m = 0, \\ (u^m(x,0), b^m(x,0)) = (u_0^m(x), b_0^m(x)) \end{cases} \tag{A5}$$

for all $m \geq 1$. It is known that there is a unique solution $(u^m, b^m) \in C^\infty(\mathbb{R}^3 \times (0,\infty))$ such that for any $t \geq 0$

$$\|u^m(t)\|_2^2 + \|b^m(t)\|_2^2 + 2\int_0^t \big(\|\nabla u^m(\tau)\|_2^2 + \|b^m(\tau)\|_2^2\big)d\tau \leq \|u_0^m\|_2^2 + \|b_0^m\|_2^2$$

for any $m \geq 1$. Thus the qualities

$$\begin{aligned} \alpha'_m(t) &= \sup_{\phi\in\mathcal{B}} \big(\|u^m(\cdot,t)\phi(\cdot)\|_2^2 + \|b^m(\cdot,t)\phi(\cdot)\|_2^2\big) \\ &= \|u^m(t)\|^2_{L^2_{loc,unif}(\mathbb{R}^3)} + \|b^m(t)\|^2_{L^2_{loc,unif}(\mathbb{R}^3)}, \\ \beta'_m(t) &= \sup_{\phi\in\mathcal{B}} \int_0^t \big(\|\phi(\cdot)\nabla u^m(\cdot,\tau)\|_2^2 + \|\phi(\cdot)\nabla b^m(\cdot,\tau)\|_2^2\big)d\tau \\ &= \int_0^t \big(\|\nabla u^m(\tau)\|^2_{L^2_{loc,unif}(\mathbb{R}^3)} + \|\nabla b^m(\tau)\|^2_{L^2_{loc,unif}(\mathbb{R}^3)}\big)d\tau \end{aligned}$$

are well-defined for each $m \geq 1$. Similar to the derivation of (4.3) and (4.4), for all $\phi \in \mathcal{B}$, there are functions $p_\phi^m(t)$ and $q_\phi^m(t)$ so that for any $t \in [0,T]$,

$$\Big(\int_{\mathbb{R}^3} \big|p^m(x,t) - p_\phi^m(t)\big|^{\frac{3}{2}} \phi(x)\ dx\Big)^{\frac{2}{3}} \leq C\gamma(t)\Big((\alpha'_m(t))^{\frac{1}{2}} + \big(\|\psi u^m\|_6 + \|\psi b^m\|_6\big)\Big) \tag{A6}$$

and

$$\Big(\int_{\mathbb{R}^3} \big|q^m(x,t) - q_\phi^m(t)\big|^{\frac{3}{2}} \phi(x)\ dx\Big)^{\frac{2}{3}} \leq C\gamma(t)\Big((\alpha'_m(t))^{\frac{1}{2}} + \big(\|\psi u^m\|_6 + \|\psi b^m\|_6\big)\Big) \tag{A7}$$

with C independent of ϕ, t and m.

Next, we establish the uniform energy estimates. For any $\phi \in \mathcal{B}$, multiplying the first equations of (A5) by $u^m\phi^2$, the second equations of (A5) by $b^m\phi^2$, then integrating over $\mathbb{R}^3 \times (0,t)$, one can deduce that

$$\begin{aligned} &\int_{\mathbb{R}^3} \Big(|u^m|^2 + |b^m|^2\Big)\phi^2(x)dx + 2\int_0^t\!\!\int_{\mathbb{R}^3} \Big(|\nabla u^m(x,\tau)|^2 + |\nabla b^m(x,\tau)|^2\Big)\phi^2(x)dxd\tau \\ = &\int_{\mathbb{R}^3} \Big(|u^m(x,0)|^2 + |b^m(x,0)|^2\Big)\phi^2(x)dx + \int_0^t\!\!\int_{\mathbb{R}^3} \Big(|u^m|^2 + |b^m|^2\Big)\Delta\big(\phi^2(x)\big)dxd\tau \\ &+ \int_0^t\!\!\int_{\mathbb{R}^3} \Big\{\big(|u^m(x,\tau)|^2 + |b^m(x,\tau)|^2\big)v\cdot\nabla\big(\phi^2(x)\big) - 2\big(u^m\cdot b^m\big)a\cdot\nabla\phi^2(x)\Big\}dxd\tau \\ &+2\int_0^t\!\!\int_{\mathbb{R}^3} \Big\{\Big(p^m - p_\phi^m\Big)u^m\cdot\phi^2(x) + \Big(q^m - q_\phi^m\Big)b^m\cdot\nabla\phi^2(x)\Big\}dxd\tau. \end{aligned} \tag{A8}$$

Here one has used the facts that

$$\mathrm{div}\,\Big(p^m u^m\Big) = u^m\cdot\nabla p^m = u^m\cdot\nabla\Big(p^m - p^m_\phi\Big) = \mathrm{div}\,\Big(u^m\big(p^m - p^m_\phi\big)\Big)$$

and similarly

$$\mathrm{div}\,\Big(q^m b^m\Big) = \mathrm{div}\,\Big(b^m\big(q^m - q^m_\phi\big)\Big).$$

Next we estimate the each terms of (A8). It is obvious for all $t\in(0,\infty)$ and all $\phi\in\mathcal{B}$,

$$\Big|\int_0^t\!\!\int_{\mathbb{R}^3}\Big(|u^m(x,\tau)|^2+|b^m(x,\tau)|^2\Big)\Delta\big(\phi^2(x)\big)dxd\tau\Big| \le C\int_0^t \alpha_m'(\tau)\;d\tau. \qquad (A9)$$

Let $\psi_0(x)$ be equal to 1 on the support of ϕ_0. For $\phi(x)=\phi_0(x-x_0)$ with some $x_0\in\mathbb{R}^3$, let $\psi(x)=\psi_0(x-x_0)$. Applying (4.8), similar to the derivation of (4.9), we have

$$\begin{aligned}
&\int_0^t\!\!\int_{\mathbb{R}^3}\Big\{\big(|u^m(x,\tau)|^2+|b^m(x,\tau)|^2\big)v\cdot\nabla\big(\phi^2(x)\big) - 2\big(u^m\cdot b^m\big)a\cdot\nabla\phi^2(x)\Big\}dxd\tau\\
\le\;& C\int_0^t\!\!\int_{\mathbb{R}^3}\Big(|\psi u^m|^2+|\psi b^m|^2\Big)\Big(|\psi v|+|\psi a|\Big)\;dxd\tau\\
\le\;& C\int_0^t\Big(\|\psi u^m\|_4^2+\|\psi b^m\|_4^2\Big)\gamma(\tau)d\tau\\
\le\;& C\sup_{\tau\in[0,T]}\gamma(\tau)\Big\{\int_0^t\alpha_m'(\tau)\;d\tau\Big\}^{\frac14}\Big(\beta_m'(t)+\int_0^t\alpha_m'(\tau)\;d\tau\Big)^{\frac34}\\
\le\;& \frac12\beta_m'(t)+C\Big(1+(\sup_{\tau\in[0,T]}\gamma(\tau))^4\Big)\int_0^t\alpha_m'(\tau)\;d\tau. \qquad (A10)
\end{aligned}$$

By the Gagliardo-Nirenberg inequality again, we deduce

$$\begin{aligned}
\Big\{\int_{\mathbb{R}^3}|u^m|^3\omega_0(x-x_0)\;dx\Big\}^{\frac13} \le\;& \Big\{\int_{\mathbb{R}^3}|u^m|^3\psi_0^3(x-x_0)\;dx\Big\}^{\frac13}\\
\le\;& C\|\psi u^m\|_2^{\frac12}\|\nabla(\psi u^m)\|_2^{\frac12}
\end{aligned}$$

Therefore, by (4.8) and (A6), we obtain

$$\begin{aligned}
&\Big|2\int_0^t\!\!\int_{\mathbb{R}^3}\Big\{\Big(p^m-p^m_\phi\Big)u^m\cdot\phi^2(x)\Big\}dxd\tau\Big|\\
\le\;& \int_0^t\Big(\int_{\mathbb{R}^3}\big|p^m(x,\tau)-p^m_\phi(\tau)\big|^{\frac32}\phi(x)\;dx\Big)^{\frac32}\Big(\int_{\mathbb{R}^3}|u^m|^3\phi(x)\;dx\Big)^{\frac13}d\tau\\
\le\;& C\int_0^t\gamma(\tau)\Big((\alpha_m'(\tau))^{\frac12}+\big(\|\psi u^m\|_6+\|\psi b^m\|_6\big)\Big)\|\psi u^m\|_2^{\frac12}\|\nabla(\psi u^m)\|_2^{\frac12}\;d\tau\\
\le\;& C\sup_{\tau\in[0,T]}\gamma(\tau)\Big\{\Big(\int_0^t\alpha_m'(\tau)\;d\tau\Big)^{\frac34}\beta_m'(t)^{\frac14}\\
&\qquad+(\beta_m'(t))^{\frac14}\Big(\int_0^t\alpha_m'(\tau)\;d\tau\Big)^{\frac14}\Big(\beta_m'(t)+\int_0^t\alpha_m'(\tau)\;d\tau\Big)^{\frac12}\Big\}\\
\le\;& \frac14\beta_m'(t)+C\Big(1+(\sup_{\tau\in[0,T]}\gamma(\tau))^4\Big)\int_0^t\alpha_m'(\tau)\;d\tau.
\end{aligned}$$

Similarly,

$$\begin{aligned} &\Big|2\int_0^t\int_{\mathbb{R}^3}\Big\{\Big(q^m - q_\phi^m\Big)b^m\cdot\phi^2(x)\Big\}dxd\tau\Big| \\ \leq\ & \frac{1}{4}\beta_m'(t) + C\Big(1 + (\sup_{\tau\in[0,T]}\gamma(\tau))^4\Big)\int_0^t \alpha_m'(\tau)\ d\tau. \end{aligned}$$

Substituting above estimates into (A8), we obtain that

$$\begin{aligned} \alpha_m'(t) + \beta_m'(t) \leq\ & \alpha_m'(0) + C\Big(1 + (\sup_{\tau\in[0,T]}\gamma(\tau))^4\Big)\int_0^t \alpha_m'(\tau)\ d\tau \\ \leq\ & \alpha' + C\Big(1 + (\sup_{\tau\in[0,T]}\gamma(\tau))^4\Big)\int_0^t \alpha_m'(\tau)\ d\tau \qquad (A11) \end{aligned}$$

holds uniformly for $m \geq 1$ with positive constant C independent of t. By the Gronwall's inequality, we deduce that

$$\alpha_m'(t) \leq \alpha' \exp\Big\{C\Big(1 + (\sup_{\tau\in[0,T]}\gamma(\tau))^4\Big)t\Big\}$$

for any $t \in [0, T]$. This and (A11) give us that

$$\alpha_m'(t) + \beta_m'(t) \leq \alpha' \exp\Big\{C\Big(1 + (\sup_{\tau\in[0,T]}\gamma(\tau))^4\Big)t\Big\} \qquad (A12)$$

for any $t \in [0, T]$.

By standard arguments, there is a weak limit (u, b) of the subsequences of $\{(u^m, b^m)\}_{m=1}^{\infty}$, which is a weak solution to (A1) and satisfies (A3). Since v and a are C^∞, we can further show that $(u, b) \in C^\infty(\mathbb{R}^3 \times (0, \infty)$ by the arguments showing the regularity of weak solutions to magnetohydrodynamics equations, as in [13]. Since (A1) is linear in (u, b), so the uniqueness of the solution follows from the estimate (A3). Similar to the Step 4 in the proof of Theorem 4.1, we can show that

$$\lim_{t\to 0^+}(\|u(t) - u_0\|_{L^2_{loc,unif}(\mathbb{R}^3)} + \|b(t) - b_0\|_{L^2_{loc,unif}(\mathbb{R}^3)}) = 0.$$

Then we complete the proof of the Proposition A1. ■

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