Partial Regularity of Suitable Weak Solutions to the Incompressible Magnetohydrodynamic Equations

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Abstract: In this paper, we study the local behavior of the solutions to the 3-dimensional magnetohydrodynamic equations. We are interested in both the uniform gradient estimates for smooth solutions and regularity of weak solutions. It is shown that, in some neighborhood of (x_0, t_0), the gradients of the velocity field u and the magnetic field B are locally uniformly bounded in L^∞ norm as long as that either the scaled local L^2- norm of the gradient or the scaled local total energy of the velocity field is small, and the scaled local total energy of the magnetic field is uniformly bounded. These estimates indicate that the velocity field plays a more dominant role than that of the magnetic field in the regularity theory. As an immediately corollary we can derive an estimates of Hausdorff dimension on the possible singular set of a suitable weak solution as in the case of pure fluid. Various partial regularity results are obtained as consequences of our blow-up estimates.

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1 Introduction

We are concerned with the uniform gradient estimations and the partial regularity of weak solutions to the three dimensional viscous incompressible magneto-hydrodynamics (MHD) equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{Re}\Delta u + (u\cdot\nabla)u - S(B\cdot\nabla)B + \nabla(p + \dfrac{S}{2}|B|^2) = 0, \\ \dfrac{\partial B}{\partial t} - \dfrac{1}{Rm}\Delta B + (u\cdot\nabla)B - (B\cdot\nabla)u = 0, \\ \mathrm{div} u = 0, \quad \mathrm{div} B = 0 \end{cases} \tag{1.1}$$

with the homogeneous boundary conditions and the following initial conditions

$$\begin{cases} u(x,0) = u_0(x), \\ B_0(x,0) = B_0(x). \end{cases} \tag{1.2}$$

Here u, p, and B are nondimensional quantities corresponding to the velocity of the fluid, its pressure, and the magnetic field. The nondimensional number Re is the Reynolds number, Rm is the magnetic Reynolds number and $S = M^2/(ReRm)$ with M being the Hartman number. For simplicity of writing, let $Re = Rm = S = 1$, and p denotes term $p + S|B|^2/2$.

There have been extensive mathematical studies on the solutions to MHD equations (1.1). In particular, Duvaut and Lions [4] constructed a global weak solution and the local strong solution to the initial boundary value problem, and properties of such solutions have been examined by Sermange and Temam in [12]. Furthermore, some sufficient conditions for smoothness were presented for the weak solutions to the MHD equations in [3]. However, in the case that the spatial dimension is three, a large gap remains between the regularity available in the existence results and additional regularity required in the sufficient conditions to guarantee the smoothness of weak solutions. In the absence of the magnetic fields, (1.1) is reduced to the three dimensional incompressible Navier-Stokes equations, this gap has been narrowed by the works of Scheffer [11], Caffarelli, Kohn and Nirenberg[1], Tian and Xin [15], see also [8], [2] and [13], and a deeper understanding has been achieved. In particular, some local partial regularity results and Hausdorff dimension estimates on the possible singular set have been obtained for a class of suitable weak solutions defined and constructed in [1], and the local regularity theorems [15] showed that there is an absolute constant ε such that the following statement is true: for any suitable weak solution u of Navier-Stokes equations, if one of the following conditions holds

1) Either $r^{-1}\iint_{Q_r(x_0,t_0)}|\nabla u|^2dxdt$ or $\sup_{t_0-r^2\le t<t_0} r^{-1}\int_{B_r(x_0)}|u(x,t)|^2dx$ is uniformly bounded and the scaled local energy $\sup_{r\le r_0} r^{-3}\iint_{Q_r(x_0,t_0)}|u(x,t)|^2dxdt<\varepsilon$,

2) $\sup_{r\le r_0} r^{-1}\iint_{Q_r(x_0,t_0)}|\text{curl } u|^2dxdt<\varepsilon$ or $\sup_{r\le r_0} r^{-1}\iint_{Q_r(x_0,t_0)}|\nabla u|^2dxdt<\varepsilon$,

3) $\sup_{r\le r_0} r^{-2}\iint_{Q_r(x_0,t_0)}|u(x,t)|^3dxdt<\varepsilon$,

for some $r_0>0$, then u is regular in some neighborhood of point (x_0,t_0). Here $B_r(x_0)$ is a ball with radius r and center at x_0, while $Q_r(x_0.t_0)$ denotes the parabolic ball with radius r and center at (x_0,t_0). These results imply that, for any suitable weak solutions, the possible singularity set has one-dimensional Hausdorff measure zero and the uniform gradient estimations also yield the possible pattens of singularity if they exist. The principal tools in this theory are the so-called generalized energy inequality and a scaling argument.

The main purpose of this paper is to study the effect of the presence of the magnetic field and to establish a theory of partial regularity for the weak solutions to the three dimensional incompressible magneto-hydrodynamic equations. The important characteristic of the magneto-hydrodynamics is the induction effect, which brings about the strong coupling of the magnetic field and velocity field. Therefore, the magneto-hydrodynamic equations are not only much complex, but the main estimates depend strongly on each other for the magnetic field and velocity field. This coupling has important effects in our discussion later. However, in view of the sufficient conditions for the regularity obtained in [3] and the numerical simulations in [10], the velocity field should play a more prominent role in the regularity theory of the magneto-hydrodynamic equations than the magnetic field. Some experiment also revealed this phenominon, see [7]. One of the main objectives of this paper is to confirm this for the local theory of partial regularity, i.e., we don't required the smallness of nondimensional quantities related to magnetic field for the regularity of suitable weak solutions. One of the main difficulties lies in the estimates about the nondimensional quantities involving the magnetic field. Due to technical difficulties in our analysis, we were not able to establish the local theory of partial regularity, for the weak solutions to the magneto-hydrodynamic equations, without any a priori assumptions on the nondimensional quantities of magnetic field. However, we will establish the local theory of partial regularity under much weaker conditions about the magnetic field than that of velocity field. In fact, we obtain the local partial regularity results under the assumption about the velocity field, which is same as that of the incompressible Navier-Stokes equations in the absence of the magnetic field, and the boundedness assumption of some scaled nondimensional quantities of magnetic field. As in the treatment of the incompressible Navier-Stokes equations, the bases of our analysis is the generalized energy inequality. To this end, one also needs the concept of suitable weak solutions. So we first introduce and construct the suitable weak solutions to the incompressible magneto-hydrodynamic equations. Then, by iteration, we derive some basic estimates on the boundedness of some important scaled quantities involving both the velocity field and the magnetic field, with the help of the various assumptions that some scaled quantities of the velocity field are small and some scaled quantities of the magnetic field are bounded. Further, we can get some dimensionless estimates on the pressure. Making using of these estimates and the generalized energy inequality, we obtain the refined estimates that some non-dimensional quantities involving the magnetic field are in fact small. These estimates, together with the smallness assumptions on some scaled quantities of velocity field, yield the local theory of partial regularity for the suitable weak solutions to the incompressible magneto-hydrodynamic equations, by a similar discussion as that in [15]. It should be noted that the iteration will be used many times, and some ideas and techniques will be borrowed and generalized from [15].

Furthermore, we establish the further regularity results for solutions to the magneto-hydrodynamic equations with additional hypothese on the given initial data, as doing for the incompressible Navier-Stokes equations in [1]. Following the discussion in section 8 in [1], we show that the solution is regular in the region $\{(x,t) \mid |x|^2 t > N_1\}$ with an absolute constant N_1 as the initial data decaying sufficiently rapidly, in a sense, at ∞, or in the region $\{(x,t) \mid |x|^2 < N_2 t\}$ with an absolute constant N_2 as the initial data is not too singular, in some sense, at the origin. These results are the direct extensions of the corresponding results on incompressible Navier-Stokes equations in [1] to the magneto-hydrodynamic equations.

The rest of the paper is organized as follows. The main results are started in section 2. In section 3, we define and construct the suitable weak solutions to the magneto-hydrodynamic equations. The estimates of some important scaled quantities will be given in section 4. And the boundedness and smallness of some scaled quantities of the magnetic field and the pressure will be obtained in section 5 and section 6 respectively. Then we will prove our main theorems in section 7. Some extensions and consequences will be presented in the last section.

We conclude this introduction by listing some notations used in the rest of the paper.

Let Ω be one of the following domains in R^3,

(Ω1) R^3,

(Ω2) a bounded domain in R^3,

(Ω3) a halfspace in R^3,

(Ω4) an exterior domain in R^3.

Then let $L^p(\Omega), 1 \le p \le +\infty$, represent the usual Lesbegue space of scalar functions as well as that of vector-valued functions with norm denoted by $\|\cdot\|_p$. Let $C_{0,\sigma}^\infty(\Omega)$ denote the set of all C^∞ real vector-valued functions $\phi = (\phi_1, \phi_2, \phi_3)$ with compact support in Ω, such that $\mathrm{div}\phi = 0$. $\overset{o}{J}{}^p(\Omega)$, $1 \le p < \infty$, is the closure of $C_{0,\sigma}^\infty(\Omega)$ with respect to $\|\cdot\|_p$. $W^{s,p}(\Omega)$ denotes the usual Sobolev Space. Finally, given a Banach space X with norm $\|\cdot\|_X$, we denote by $L^p(0,T;X), 1 \le p \le +\infty$, the set of function $f(t)$ defined on $(0,T)$ with values in X such that $\int_0^T \|f(t)\|_X^p dt < +\infty$. For $x \in \Omega$, we set $B_r(x) = \{y \in \Omega, |y-x| < r\}$. For point $(x,t) \in \Omega \times R^+$, the parabolic ball centered at point (x,t) with radius r will be denoted as $Q_r(x,t) = B_r(x) \times (t-r^2, t)$. In the case of no confusion, we will skip the center of the ball from the notation and simply write by B_r or Q_r. At last, by symbol C, we denote a generic constant whose value is unessential to our analysis, and it may change from line to line.

2 The Main Results

In this section, we present our main results in this paper. To the end, we first introduce the definition of suitable weak solutions and the notations of some scaled dimensionless quantities.

Definition. The triplet (u, B, p) is called a suitable weak solution of the magneto-hydrodynamic equations (1.1) in an open set $D \subset \Omega \times R^+$, if

1) $p \in L^{5/3}(D)$ with $\iint_D |p(x,t)|^{5/3} dxdt \le C_1$, and

$$\int_{D_t} (|u(x,t)|^2 + |B(x,t)|^2) dx \le C_2, \qquad \iint_D (|\nabla u(x,t)|^2 + |\nabla B(x,t)|^2) dxdt \le C_3 \tag{2.1}$$

for almost every t such that $D_t = D \cap \{\Omega \times \{t\}\} \ne \emptyset$, where C_1, C_2 and C_3 are some positive constants.

2) (u, B, p) satisfies (1.1) in the sense of distribution on D.

3) For each real-valued $\phi \in C_0^\infty(D)$ with $\phi \ge 0$, the following generalized energy inequality is valid:

$$\begin{aligned} & 2\iint_D (|\nabla u(x,t)|^2 + |\nabla B(x,t)|^2)\phi dxdt \\ \le & \iint_D (|u(x,t)|^2 + |B(x,t)|^2)(\phi_t(x,t) + \Delta\phi(x,t)) dxdt \\ & + \iint_D (u(x,t)\cdot\nabla\phi)(|u(x,t)|^2 + |B(x,t)|^2 + 2p(x,t)) dxdt \\ & \qquad -2\iint_D (B\cdot\nabla\phi)(u\cdot B) dxdt. \end{aligned} \tag{2.2}$$

4) For any $\chi \in C_0^\infty(D)$, the equation

$$\frac{\partial B\chi}{\partial t} - \Delta(B\chi) = B(\frac{\partial \chi}{\partial t} - \Delta\chi) - 2\nabla\chi \cdot \nabla B - \chi(u\cdot\nabla)B + \chi B\cdot\nabla u \tag{2.3}$$

holds in the sense of distribution.

For a given solution (u, B, p) to the magneto-hydrodynamic equations, the scaled total energy, the scaled vorticity and other scaled quantities, which will be used later, are defined to be the following dimensionless quantities

$$\begin{cases} E(r) \equiv \sup\limits_{t_0-r^2\le t<t_0} \frac{1}{r}\int_{B_r(x_0)} |u(x,t)|^2 dx, \\ E_p(r) \equiv \frac{1}{r^{5-p}} \iint_{Q_r(x_0,t_0)} |u(x,t)|^p dxdt, \\ E_*(r) \equiv \frac{1}{r} \iint_{Q_r(x_0,t_0)} |\nabla u(x,t)|^2 dxdt, \\ W(r) \equiv \frac{1}{r} \iint_{Q_r(x_0,t_0)} |\mathrm{curl} u(x,t)|^2 dxdt, \end{cases} \tag{2.4}$$

for the velocity field u and

$$\begin{cases} F(r) \equiv \sup\limits_{t_0-r^2\le t<t_0} \frac{1}{r}\int_{B_r(x_0)} |B(x,t)|^2 dx, \\ F_p(r) \equiv \frac{1}{r^{5-p}} \iint_{Q_r(x_0,t_0)} |B(x,t)|^p dxdt, \\ F_*(r) \equiv \frac{1}{r} \iint_{Q_r(x_0,t_0)} |\nabla B(x,t)|^2 dxdt, \\ P_p(r) \equiv \frac{1}{r^{5-2p}} \iint_{Q_r(x_0,t_0)} |p(x,t)|^p dxdt. \end{cases} \tag{2.5}$$

for the magnetic field B and the pressure p. Here $2 \le p \le 10/3$. Now the main results in this paper can be stated as follows.

Theorem 2.1. There exists an absolute constant ε with the following property. Let (u, B, p) be a suitable weak solution to (1.1) and (1.2), suppose further that, for some $r_0 > 0$,

1) Either $\sup\limits_{0<r\le r_0} (E(r) + F(r)) < +\infty$ or $\sup\limits_{0<r\le r_0} (E_*(r) + F_2(r)) < +\infty$,

2) $E_2(r) \le \varepsilon$ for all $0 < r \le r_0$,

then there exists a positive constant r_1 with $r_1 \le r_0$ such that

$$\sup_{Q_{r/2}(x_0,t_0)} (|\nabla u(x,t)| + |\nabla B(x,t)|) \le Cr^{-2} \tag{2.6}$$

for all $r \le r_1$.

Theorem 2.2. There exists an absolute constant ε with the following property. Let (u, B, p) be a suitable weak solution to (1.1) and (1.2), suppose further that, for some $r_0 > 0$, any one of the following three conditions is satisfied

1) For some p satisfying $3 \le p \le 10/3$, $\sup\limits_{0<r\le r_0} E_p(r) \le \varepsilon$ and $\sup\limits_{0<r\le r_0} F_{2p/(p-1)}(r) < +\infty$,
2) For some p satisfying $5/2 \le p < 3$, $\sup\limits_{0<r\le r_0} E_p(r) \le \varepsilon$ and $\sup\limits_{0<r\le r_0} F_3(r) < +\infty$,
3) For some p satisfying $5/2 < p \le 3$, $\sup\limits_{0<r\le r_0} E_p(r) \le \varepsilon$ and $\sup\limits_{0<r\le r_0} F(r) < +\infty$,

then

$$\sup_{Q_{r/2}(x_0,t_0)} (|\nabla u(x,t)| + |\nabla B(x,t)|) \le Cr^{-2} \tag{2.7}$$

for all $r \le r_1$ with $r_1 \le r_0$.

Theorem 2.3. There exists an absolute constant ε with the following property. Let (u, B, p) be a suitable weak solution to (1.1) and (1.2), suppose further that, for some $r_0 > 0$,

1) $E_*(r) \le \varepsilon$ for all $0 < r \le r_0$,
2) $\sup\limits_{0<r\le r_0} F_2(r) < +\infty$,

then, there is a $r_1 \le r_0$, such that

$$\sup_{Q_{r/2}(x_0,t_0)} (|\nabla u(x,t)| + |\nabla B(x,t)|) \le Cr^{-2} \tag{2.8}$$

for all $r \le r_1$.

Remarks:

1. For the incompressible Navier-Stokes equations, it has been shown that if there is an absolute constant $\varepsilon > 0$, such that, for any suitable weak solution (u, p), if any one of the following conditions holds, for all $0 < r \le r_0$ with some $r_0 > 0$: 1) $E(r) < +\infty$ or $E_*(r) < +\infty$ and $E_2(r) \le \varepsilon$, 2) $W(r) \le \varepsilon$, 3) $E_3(r) \le \varepsilon$, then u is regular on Q_{r_1} for some $r_1 \le r_0$. cf. [1], [15], [8]. Here our assumptions on velocity field are similar.

2. Similar to the discussion in [1], Theorem 2.3 implies that the one- dimension Hausdorff measure of the set of possible singular points of u and B is zero.

3. In Theorem 2.2, the restriction "$p \ge 5/2$" is due to the fact: in view of Lemma 4.2 later, p must be larger than 5/2, if one want to use $E_p(r)$ and $E_*(r)$ to control the quantity $E_3(r)$. Otherwise, the boundedness of $E(r)$ or $E_*(r)$ is necessary for the same purpose, as in the case when $E_2(r) \le \varepsilon$.

4. In view of the discussion in [15], the assumption on $E_*(r)$ can be replaced by the same assumption on $W(r)$.

5. In Theorem 2.1 - 2.3, the assumptions, which hold for all $0 < r \leq r_0$, can be weakened by that the assumptions hold only for a sequences $\{r_m\}$ satisfying: 1) $0 < r_{m+1} < r_m \leq c_0 r_{m+1}$ for each $m \in N$ with some positive constant c_0, and 2) $\lim_{m\to\infty} r_m = 0$.

6. It should be clear from the statements in Theorem 2.1 - 2.3 that our partial regularity theory requires much weaker conditions on the magnetic field.

If the solution decays sufficiently rapid at ∞, above results imply that

Theorem 2.4. Let u_0 and B_0 belong to $\overset{o}{J}{}^2(R^3)$. Then there is an absolute constant N_0 such that the suitable weak solutions is regular when $t \geq N_0(\|u_0\|_2^2 + \|B_0\|_2^2)^{3/2}$. Moreover, if $|x|^{1/2}u_0$ and $|x|^{1/2}B_0$ belong to $L^2(R^3)$, then the suitable weak solution is regular in the region $\{(x,t) \mid |x|^2 t > N_1\}$ with absolute constant N_1 depending only on the initial data.

On the other hand, if the solution is not too singular, the above results imply that

Theorem 2.5. Let u_0 and B_0 belong to $\overset{o}{J}{}^2(R^3)$, and $|x|^{-1/2}u_0$ and $|x|^{-1/2}B_0$ belong to $L^2(R^3)$. Then there exists an absolute constant L_0, if

$$\||x|^{-1/2}u_0\|_2^2 + \||x|^{-1/2}B_0\|_2^2 = L < L_0$$

then the suitable weak solution is regular in the region $\{(x,t) \mid |x|^2 < t(L_0 - L)\}$.

Remarks:

1. For the incompressible Navier-Stokes equations, the same results were obtained by Caffarelli, Kohn and Nirenberg [1] for cauchy problem, and by Maremonti [9] for the exterior problem.

2. Theorem 2.4 and 2.5 are valid for the exterior domain by similar discussion given in [9].

3 Suitable Weak Solutions

In this section, we first define the suitable weak solution to the MHD equations (1.1), then sketch the construction of the suitable weak solutions.

Duvaut and Lions[4] constructed a class of global weak solutions and local strong solutions to the initial boundary value of the three-dimensional incompressible magneto-hydrodynamic equations. General speaking, we call a problem “strong”, if it lies in a space in which the solution of (1.1) and (1.2) is known to be unique. Otherwise, we call a solution “weak”. There are many different choices for the function spaces in which to construct the solution of the initial boundary value problem. The global weak solutions, which are similar to the Leray-Hopf weak solutions to the Navier-Stokes equations, are very important. In fact, the class of weak solutions satisfy: $u, B \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ for any $T>0$,

$$\|u(t)\|_2^2 + \|B(t)\|_2^2 + 2\int_0^t (\|\nabla u(s)\|_2^2 + \|\nabla B(s)\|_2^2)ds \le \|u_0\|_2^2 + \|B_0\|_2^2, \tag{3.1}$$

and (u, B) satisfy the equation (1.1) in the sense of distribution. In order to develop a local theory of partial regularity, we need the localized form of the energy inequality (3.1), which is satisfied by a class of suitable weak solutions.

But it is not clear whether the known weak solutions are suitable weak solutions. So in the following, we show the existence of a class of suitable weak solutions to the magneto-hydrodynamic equations. Since the procedure is similar to one for the incompressible Navier-Stokes system, we only sketch the proof.

Theorem 3.1 Let $u_0, B_0 \in \overset{o}{J}{}^2(\Omega)$ and $u_0 \in W^{4/5,5/3}(\Omega)$. Then there exists a suitable weak solution (u, B, p) to the magneto-hydrodynamic equations in $\Omega \times R^+$, such that

$$u, B \in L^\infty(0,+\infty; \overset{o}{J}{}^2(\Omega)), \qquad \nabla u, \nabla B \in L^2(0,+\infty; L^2(\Omega)), \tag{3.2}$$

$$u, B \in L^{10/3}(0,+\infty; L^{10/3}(\Omega)), \qquad p \in L^{5/3}(0,+\infty; L^{5/3}(\Omega)/R), \tag{3.3}$$

Further,

$$\|u(t)\|_2^2 + \|B(t)\|_2^2 + \int_0^t \Big(\|\nabla(s)\|_2^2 + \|\nabla B(s)\|_2^2\Big)ds \le 4\Big(\|u_0\|_2^2 + \|B_0\|_2^2\Big), \tag{3.4}$$

$$\|u\|_{L^{10/3}(Q_T)}^{10/3} + \|B\|_{L^{10/3}(Q_T)}^{10/3} + \|p\|_{L^{5/3}(Q_T)}^{5/3} \le C\Big(\|u_0\|_{W^{4/5,5/3}(\Omega)}, \|B_0\|_2\Big). \tag{3.5}$$

$u(t)$ and $B(t)$ converge weakly to u_0 and B_0 in $L^2(\Omega)$ respectively, as $t \longrightarrow 0$. Moreover, for $\phi \in C_0^\infty(\Omega \times R^+)$ with $\phi \ge 0$, it holds that, for $0 < t < +\infty$,

$$\begin{aligned} &\int_\Omega (|u(x,t)|^2 + |B(x,t)|^2)\phi(x,t)dx + 2\int_0^t \int_\Omega (|\nabla u(x,s)|^2 + |\nabla B(x,s)|^2)\phi(x,s)dxds \\ \le\ & \int_\Omega (|u_0(x)|^2 + |B_0(x)|^2)\phi(x,0)dx \\ &\quad + \int_0^t \int_\Omega (|u(x,t)|^2 + |B(x,t)|^2)(\phi_t(x,t) + \Delta\phi(x,t))dxdt \\ &\quad + \int_0^t \int_\Omega (u(x,t)\cdot\nabla\phi(x,t))(|u(x,t)|^2 + |B(x,t)|^2 + 2p(x,t))dxdt \\ &\quad -2\int_0^t \int_\Omega (B(x,t)\cdot\nabla\phi(x,t))(u(x,t)\cdot B(x,t))dxdt, \end{aligned} \tag{3.6}$$

and (2.3) is valid for any $\chi \in C_0^\infty(\Omega \times R^+)$ in the sense of distribution.

Proof. Since the proof of Theorem 3.1 is similar to that of Navier-Stokes equations, here we only sketch the construction of the approximate solutions and the deducement of the main estimates. For this purpose, we select u_0^k and B_0^k in $C_{0,\sigma}^\infty(\Omega)$, such that

$$\lim_{k\to\infty} \|u_0^k - u_0\|_2 = \lim_{k\to\infty} \|B_0^k - B_0\|_2 = 0$$

and

$$\|u_0^k\|_{W^{4/5,5/3}(\Omega)} \leq 2\|u_0\|_{W^{4/5,5/3}(\Omega)}, \qquad \|B_0^k\|_2 \leq 2\|B_0\|_2. \tag{3.7}$$

Now we linearize the magneto-hydrodynamic equations (1.1) to constructe the approximate solutions as follows:

$$\begin{cases} \dfrac{\partial u^0}{\partial t} - \Delta u^0 + \nabla p^0 = 0, \\ \dfrac{\partial B^0}{\partial t} - \Delta B^0 = 0, \\ \mathrm{div} u^0 = 0, \qquad \mathrm{div} B^0 = 0, \\ u^0(x,t) = B^0(x,t) = 0, \qquad \text{on } \partial\Omega \\ (u^0(x,0), B^0(x,0)) = (u_0^0(x), B_0^0(x)) \end{cases} \tag{3.8}$$

and for any $k \geq 1$

$$\begin{cases} \dfrac{\partial u^k}{\partial t} - \Delta u^k + (u^{k-1}\cdot\nabla)u^k - (B^{k-1}\cdot\nabla)B^k + \nabla p^k = 0, \\ \dfrac{\partial B^k}{\partial t} - \Delta B^k + (u^{k-1}\cdot\nabla)B^k - (B^{k-1}\cdot\nabla)u^k + \nabla q^k = 0, \\ \mathrm{div} u^k = 0, \qquad \mathrm{div} B^k = 0, \\ u^k(x,t) = B^k(x,t) = 0, \qquad \text{on } \partial\Omega \\ (u^k(x,0), B^k(x,0)) = (u_0^k(x), B_0^k(x)). \end{cases} \tag{3.9}$$

It is obvious that (u^k, B^k, p^k) are well defined for all $k \geq 0$, and (u^k, B^k, p^k) are sufficiently smooth. We multiply the first and the second equation of (3.9) by u^k and B^k respectively, and add the resulting equations to obtain, by the integration by parts, that

$$\begin{aligned} &\|u^k(t)\|_2^2 + \|B^k(t)\|_2^2 + 2\int_0^t \int_\Omega (|\nabla u^k|^2 + |\nabla B^k|^2) dxds \\ &\quad \leq \|u_0^k\|_2^2 + \|B_0^k\|_2^2 \leq 4(\|u_0\|_2^2 + \|B_0\|_2^2). \end{aligned} \tag{3.10}$$

By the Gagliardo-Nirenberg inequality,

$$\|u^k\|_{10/3} \leq C\|u^k\|_2^{2/5}\|\nabla u^k\|_2^{3/5}, \qquad \|B^k\|_{10/3} \leq C\|B^k\|_2^{2/5}\|\nabla B^k\|_2^{3/5}.$$

By (3.10), it follows that

$$\int_0^\infty (\|u^k\|_{10/3}^{10/3} + \|B^k\|_{10/3}^{10/3})ds \le C(\|u_0\|_2^2 + \|B_0\|_2^2)^{5/3}. \tag{3.11}$$

In order to estimate the pressure, we observe that Theorem 3.1 in [6] implies that the pressure p^k can be chosen such that

$$\int_0^\infty \int_\Omega |p^k(x,t)|^{5/3} dxdt \le C(\|u_0\|_{W^{4/5,5/3}(\Omega)}, \|B_0\|_2). \tag{3.12}$$

By the Rellich compactness theorem and the Lions-Aubin Lemma, it is routine to show that

$$\int_0^\infty \int_\Omega \nabla q^k \cdot \phi dxdt \longrightarrow 0, \quad \text{as } \ k \to \infty$$

for any $\phi \in C_0^\infty(\Omega \times R^+)$, and there exists (u, B, p), which is a suitable weak solution to (1.1) in $\Omega \times R^+$. By the lower semicontinuity of weak convergence, (3.4) and (3.5) are valid due to estimates (3.10), (3.11) and (3.12). In order to deduce the generalized energy inequality (3.6), we muitiply the first equation of (3.9) by $u^k\phi$, the second equation of (3.9) by $B^k\phi$ for $\phi \in C_0^\infty(\Omega \times R^+)$ with $\phi \ge 0$, then take the limit as $k \longrightarrow \infty$, after adding the resulting two equations. The rest can be done in exactly way and in [1]. Once we omit the details. □

4 Some Dimensionless Estimates

In this section, we intend to derive some estimates of scaled dimensionless quantities that are needed in the analysis later. By the invariance of (1.1) under translation, we may always shift the center of ball to the point $x_0 = 0$ and $t_0 = 0$. As for the Navier-Stokes equations, the generalized energy inequality (3.6) will be one of the prinsipal tool in our discussion. In order to make use of the generalized energy inequality effectively, we must estimate every terms at the right hand side of (3.6). We start with the terms related to velocity field.

Lemma 4.1 For $r > 0$, there is a constant C independent of r, such that

$$E_p(r) \le CE^{(p-2)/2}(r)\Big(E_2^{(10-3p)/4}(r)E_*^{3(p-2)/4}(r) + E_2(r)\Big) \tag{4.1}$$

with $p \,\epsilon\, [2, 10/3]$.

Proof. By the Sobolev inequality,

$$\begin{aligned}\int_{B_r} |u|^p dx \le \ & C\Big(\int_{B_r} |u|^2 dx\Big)^{(6-p)/4}\Big(\int_{B_r} |\nabla u|^2 dx\Big)^{3(p-2)/4} \\ & + Cr^{-3(p-2)/2}\Big(\int_{B_r} |u|^2 dx\Big)^{p/2}.\end{aligned} \tag{4.2}$$

Integrating in time, we obtain, by the Hölder inequality, that

$$
\begin{aligned}
&\iint_{Q_r} |u|^p dxdt \\
&\le C \max_{-r^2\le t<0} \Big(\int_{B_r} |u|^2 dx\Big)^{(p-2)/2} \Big(\iint_{Q_r} |u|^2 dxdt\Big)^{(10-3p)/4} \Big(\iint_{Q_r} |\nabla u|^2 dxdt\Big)^{3(p-2)/4} \\
&\qquad + Cr^{-3(p-2)/2} \max_{-r^2\le t<0} \Big(\int_{B_r} |u|^2 dx\Big)^{(p-2)/2} \iint_{Q_r} |u|^2 dxdt \\
&\le Cr^{(5-p)} E^{(p-2)/2}(r) \Big(E_2^{(10-3p)/4}(r) E_*^{3(p-2)/4}(r) + E_2(r)\Big),
\end{aligned}
$$

which implies (4.1). □

Lemma 4.2 If any $r>0$, and $5/2 \le p \le 3$, then

$$
\begin{aligned}
E_3(r) \le \; & C E^{(2p-3)/2p}(r) \Big(E_p^{1/p}(r) E_*^{3/2p}(r) + E_p^{(6-p)/p^2}(r) E_*^{(2p-3)/2p}(r) \\
& + E_p^{3(p-1)/p^2}(r) E_*^{(3-p)/p}(r) + E_p^{(p+3)/p^2}(r)\Big)
\end{aligned} \tag{4.3}
$$

for some positive constant C independent of r.

Proof. Applying the interpolation and Sobolev inequalities, we get

$$
\begin{aligned}
\Big(\int_{B_r} |u|^3 dx\Big)^{1/3} \le \; & C\Big(\int_{B_r} |u|^p dx\Big)^{1/(6-p)} \Big(\int_{B_r} |\nabla u|^2 dx\Big)^{(3-p)/(6-p)} \\
& + Cr^{-(3-p)/p} \Big(\int_{B_r} |u|^p dx\Big)^{1/p}.
\end{aligned} \tag{4.4}
$$

It follows from (4.2) and (4.4) that

$$
\begin{aligned}
\|u\|_{L^3(B_r)}^3 \le \; & \|u\|_{L^3(B_r)}^{2(2p-3)/p} \|u\|_{L^3(B_r)}^{(6-p)/p} \\
\le \; & C\Big(\|u\|_{L^2(B_r)}^{(2p-3)/p} \|\nabla u\|_{L^2(B_r)}^{(2p-3)/p} + r^{-(2p-3)/p} \|u\|_{L^2(B_r)}^{2(2p-3)/p}\Big) \\
& \times \Big(\|u\|_{L^p(B_r)} \|\nabla u\|_{L^2(B_r)}^{2(3-p)/p} + r^{-(3-p)(6-p)/p^2} \|u\|_{L^p(B_r)}^{(6-p)/p}\Big).
\end{aligned}
$$

Thus,

$$
\iint_{Q_r} |u|^3 dxdt \le C(I_1 + I_2 + I_3 + I_4). \tag{4.5}
$$

where the terms at the right hand side of (4.5) are defined and can be estimated as follows:

$$
\begin{aligned}
I_1 \; = \; & \int_{-r^2}^0 \|u\|_{L^2(B_r)}^{(2p-3)/p} \|u\|_{L^p(B_r)} \|\nabla u\|_{L^2(B_r)}^{3/p} ds \\
\le \; & r^{2-5/p} \max_{-r^2\le t<0} \|u\|_{L^2(B_r)}^{(2p-3)/p} \Big(\iint_{Q_r} |u|^p dxdt\Big)^{1/p} \Big(\iint_{Q_r} |\nabla u|^2 dxdt\Big)^{3/2p} \\
= \; & r^2 E^{(2p-3)/2p}(r) E_p^{1/p}(r) E_*^{3/2p}(r),
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= r^{-(3-p)(6-p)/p^2}\int_{-r^2}^{0}\|u\|_{L^2(B_r)}^{(2p-3)/p}\|u\|_{L^p(B_r)}^{(6-p)/p}\|\nabla u\|_{L^2(B_r)}^{(2p-3)/p}ds\\
&\le r^{-(30-14p+p^2)/p^2}\max_{-r^2\le t<0}\|u\|_{L^2(B_r)}^{(2p-3)/p}\Big(\iint_{Q_r}|u|^p dxdt\Big)^{(6-p)/p^2}\\
&\qquad\times\Big(\iint_{Q_r}|\nabla u|^2dxdt\Big)^{(2p-3)/2p}\\
&= r^2E^{(2p-3)/2p}(r)E_p^{(6-p)/p^2}(r)E_*^{(2p-3)/2p}(r),\\
I_3 &= r^{-(2p-3)/p}\int_{-r^2}^{0}\|u\|_{L^2(B_r)}^{2(2p-3)/p}\|u\|_{L^p(B_r)}\|\nabla u\|_{L^2(B_r)}^{2(3-p)/p}ds\\
&\le r^{-(2p-3)/p}\max_{-r^2\le t<0}\|u\|_{L^2(B_r)}^{(2p-3)/p}\int_{-r^2}^{0}\|u\|_{L^2(B_r)}^{(2p-3)/p}\|u\|_{L^p(B_r)}\|\nabla u\|_{L^2(B_r)}^{2(3-p)/p}ds\\
&\le Cr^{1-15/2p+9/p^2}\max_{-r^2\le t<0}\|u\|_{L^2(B_r)}^{(2p-3)/p}\int_{-r^2}^{0}\|u\|_{L^p(B_r)}^{3(p-1)/p}\|\nabla u\|_{L^2(B_r)}^{2(3-p)/p}ds\\
&\le r^{5-39/2p+15/p^2}\max_{-r^2\le t<0}\|u\|_{L^2(B_r)}^{(2p-3)/p}\Big(\iint_{Q_r}|u|^p dxdt\Big)^{3(p-1)/p^2}\\
&\qquad\times\Big(\iint_{Q_r}|\nabla u|^2dxdt\Big)^{(3-p)/p}\\
&= Cr^2E^{(2p-3)/2p}(r)E_p^{3(p-1)/p^2}(r)E_*^{(3-p)/p}(r),\\
I_4 &= r^{-3(p^2-4p+6)/p^2}\int_{-r^2}^{0}\|u\|_{L^2(B_r)}^{2(2p-3)/p}\|u\|_{L^p(B_r)}^{(6-p)/p}ds\\
&\le Cr^{3(p-6)/2p^2}\max_{-r^2\le t<0}\|u\|_{L^2(B_r)}^{(2p-3)/p}\int_{-r^2}^{0}\|u\|_{L^p(B_r)}^{(p+3)/p}ds\\
&\le Cr^{-(30+p)/2p^2}\max_{-r^2\le t<0}\|u\|_{L^2(B_r)}^{(2p-3)/p}\Big(\iint_{Q_r}|u|^p dxdt\Big)^{(p+3)/p^2}\\
&= Cr^2E^{(2p-3)/2p}(r)E_p^{(p+3)/p^2}(r).
\end{aligned}
$$

Substituting above estimates into (4.5), we get (4.3). □

Next we turn to the estimates of the terms involving the pressure function. First, we have

Lemma 4.3 For $1 < q \le 5/3$ and $\mu \le \rho/2$, then there exists a positive constant C independent of μ and ρ, such that

$$
P_q(\mu) \le C(\frac{\rho}{\mu})^{5-2q}\Big(E_{2q}(\rho)+F_{2q}(\rho)\Big)+C(\frac{\mu}{\rho})^{2(q-1)}P_q(\rho). \tag{4.6}
$$

Proof. We observe that the pressure satisfies the equation

$$
-\Delta p = \sum_{i,j=1}^{3}\frac{\partial^2}{\partial x_i\partial x_j}\Big(u_iu_j - B_iB_j\Big)
$$

from which one can obtain the following representation for pressure

$$p(x,t) = \int_{B_\rho} D_x^2\Gamma(x-y) : (u\otimes u - B\otimes B)(y)dy + |u(x,t)|^2 - |B(x,t)|^2 + H(x,t) \quad (4.7)$$

for all $(x,t)\in Q_\rho$, where $\Gamma(x)$ is the normalized fundamental solution of Laplace's equations, and H is harmonic on B_ρ for each fixed $t\in(-\rho^2,0)$. And the integral is in the sense of the Cauchy principal value. Let

$$p_0 = \int_{B_\rho} D_x^2\Gamma(x-y) : (u\otimes u - B\otimes B)(y)dy$$

Then, by the Calderón-Zygmund theory on singular integrals, one can get

$$\|p_0\|_{L^q(B_\rho)} \le C(q)(\|u\|^2_{L^{2q}(B_\rho)} + \|B\|^2_{L^{2q}(B_\rho)}). \quad (4.8)$$

Employing the mean value property of harmonic functions, one has, for $\forall x\in B_\mu$, that

$$\begin{aligned} |H(x,t)| &\le \frac{C}{\rho^3}\int_{B_\rho}|H(x,t)|dx \\ &\le \frac{C}{\rho^3}\int_{B_\rho}\Big(|p(x,t)| + |p_0(x,t)| + |u(x,t)|^2 + |B(x,t)|^2\Big)dx. \end{aligned}$$

Thus,

$$\|H\|_{L^q(B_\mu)} \le C(\frac{\mu}{\rho})^{3/q}\Big(\|p\|_{L^q(B_\rho)} + \|u\|^2_{L^{2q}(B_\rho)} + \|B\|^2_{L^{2q}(B_\rho)}\Big). \quad (4.9)$$

It follows from (4.7)-(4.9) that

$$\begin{aligned} \|p\|_{L^q(B_\mu)} &\le \|p_0\|_{L^q(B_\mu)} + \|u\|^2_{L^{2q}(B_\rho)} + \|B\|^2_{L^{2q}(B_\rho)} + \|H\|_{L^q(B_\mu)} \\ &\le C(\frac{\mu}{\rho})^{3/q}\|p\|_{L^q(B_\rho)} + C\Big(\|u\|^2_{L^{2q}(B_\rho)} + \|B\|^2_{L^{2q}(B_\rho)}\Big). \end{aligned}$$

Integrating in time over $(-\mu^2,0)$, we get (4.6). □

Lemma 4.4 Let $\mu\le\rho/2$, then, for any $2\le p\le 10/3$ and $1<q\le 5/3$, one has

$$\begin{aligned} &\mu^{-2}\iint_{Q_\mu}|u||p|dxdt \\ &\le C\Big\{\Big((\frac{\mu}{\rho})^{2/q-1} + (\frac{\rho}{\mu})^{1/2-3(p-2)(q-1)/2q}\Big)E^{1/2-p(q-1)/2q}(\rho)E_p^{(q-1)/q}(\rho)P_q^{1/q}(\rho) \\ &+(\frac{\rho}{\mu})^{5/p-2}\Big(E^{1/2}(\rho)E_*^{1/2}(\rho) + F^{1/2}(\rho)F_*^{1/2}(\rho)\Big)E_p^{1/p}(\rho) \\ &+(\frac{\rho}{\mu})^{7/4}E_p^{1/2(6-p)}(\rho)\Big(E^{(14-3p)/4(6-p)}(\rho)E_*^{(24-5p)/4(6-p)}(\rho)E_2^{(p-2)/4(6-p)}(\rho) \\ &+F^{(14-3p)/4(6-p)}(\rho)F_*^{1/2}(\rho)E_*^{3(4-p)/4(6-p)}(\rho)F_2^{(p-2)/4(6-p)}(\rho)\Big)\Big\} \end{aligned} \quad (4.10)$$

if $1/p+1/q\geq 1$; and

$$\mu^{-2}\iint_{Q_\mu}|u||p|dxdt\leq C\Big\{\Big((\frac{\mu}{\rho})^{1-2/p}+(\frac{\mu}{\rho})^{1+3/q-5/p}\Big)E_p^{1/p}(\rho)P_q^{1/q}(\rho)$$
$$+(\frac{\rho}{\mu})^{5/p-2}\Big(E^{1/2}(\rho)E_*^{1/2}(\rho)+F^{1/2}(\rho)F_*^{1/2}(\rho)\Big)E_p^{1/p}(\rho)$$
$$+(\frac{\rho}{\mu})^{7/4}E_p^{1/2(6-p)}(\rho)\Big(E^{(14-3p)/4(6-p)}(\rho)E_*^{(24-5p)/4(6-p)}(\rho)E_2^{(p-2)/4(6-p)}(\rho)$$
$$+F^{(14-3p)/4(6-p)}(\rho)F_*^{1/2}(\rho)E_*^{3(4-p)/4(6-p)}(\rho)F_2^{(p-2)/4(6-p)}(\rho)\Big)\Big\} \quad (4.11)$$

if $1/p+1/q<1$. Here the positive constant C is independent of μ and ρ.

Proof. Let $\bar{f}_r$ denote the average of f on the ball B_r, i.e., $\bar{f}_r=\frac{1}{|B_r|}\int_{B_r}fdx$. Thus,

$$\iint_{Q_\mu}|u||p|dxdt\leq\iint_{Q_\mu}|u-\bar{u}_\rho||p|dxdt+\iint_{Q_\mu}|\bar{u}_\rho||p|dxdt. \quad (4.12)$$

Now the last term at the right hand side of (4.12) can be estimated as

$$\begin{aligned}
J_1 &= \iint_{Q_\mu}|\bar{u}_\rho||p|dxdt\\
&\leq C\mu^{3-3/q}\int_{-\mu^2}^0|\bar{u}_\rho|\|p\|_{L^q(B_\mu)}ds\\
&\leq C\mu^{3-3/q}\Big(\int_{-\mu^2}^0|\bar{u}_\rho|^{q/(q-1)}ds\Big)^{1-1/q}\|p\|_{L^q(Q_\mu)}\\
&\leq C\mu^{3-3/q}\rho^{-9/2+3/q+3p(q-1)/2q}\max_{-\mu^2\leq t<0}\|u\|_{L^2(B_\rho)}^{1-p(q-1)/q}\|u\|_{L^p(B_\rho)}^{p(q-1)/q}\|p\|_{L^q(Q_\mu)}\\
&= C\mu^2(\frac{\mu}{\rho})^{2/q-1}E^{1/2-p(q-1)/2q}(\rho)E_p^{(q-1)/q}(\rho)P_q^{1/q}(\mu)
\end{aligned}$$

if $1/p+1/q\geq 1$; and

$$\begin{aligned}
J_1 &\leq C\mu^{3-3/q}\Big(\int_{-\mu^2}^0|\bar{u}_\rho|^{q/(q-1)}ds\Big)^{1-1/q}\|p\|_{L^q(Q_\mu)}\\
&\leq C\rho^{-3/p}\mu^{5-5/q-2/p}\|u\|_{L^p(Q_\rho)}\|p\|_{L^q(Q_\mu)}\\
&\leq C\mu^2(\frac{\mu}{\rho})^{1-2/p}E_p^{1/p}(\rho)P_q^{1/q}(\mu)
\end{aligned}$$

if $1/p+1/q<1$.

In order to estimate the first term at the right hand side of (4.12), we use another representation for the pressure

$$p(x,t)=\int_{B_\rho}\nabla_x\Gamma(x-y)\cdot\Big(u\cdot\nabla u-B\cdot\nabla B\Big)(y)dy+H_0(x,t)\equiv p_1(x,t)+H_0(x,t) \quad (4.13)$$

for $x \in B_\rho$ and $t \in (-\mu^2, 0)$. Here H_0 is a harmonic function in $x \in B_\rho$ for each fixed $t \in (-\rho^2, 0)$. Then

$$\begin{aligned}
\iint_{Q_\mu} |u-\bar{u}_\rho||p|dxdt &\le \iint_{Q_\mu} |u-\bar{u}_\rho||p_1|dxdt + \iint_{Q_\mu} |u-\bar{u}_\rho||H_0|dxdt \\
&\le C\Big(\iint_{Q_\mu} |u-\bar{u}_\rho||\bar{p}_\rho|dxdt + \iint_{Q_\mu} |u-\bar{u}_\rho||\overline{|p_1|}_\rho|dxdt \\
&\qquad + \iint_{Q_\mu} |u-\bar{u}_\rho||p_1|dxdt\Big).
\end{aligned}$$

Here we used the mean value property of harmonic functions. In the following, we estimate the right terms of the last inequality.

$$\begin{aligned}
J_2 &= \iint_{Q_\mu} |u-\bar{u}_\rho||\bar{p}_\rho|dxdt \\
&\le C\rho^{-3/q}\int_{-\mu^2}^0 \|p\|_{L^q(B_\rho)}\int_{B_\mu}|u-\bar{u}_\rho|dxdt \\
&\le C\rho^{-3/q}\Big(\iint_{Q_\rho}|p|^q dxdt\Big)^{1/q}\Big(\int_{-\mu^2}^0\Big(\int_{B_\mu}|u-\bar{u}_\rho|dx\Big)^{q/(q-1)}dt\Big)^{1-1/q} \\
&\le C\rho^{-3/q}\Big(\iint_{Q_\rho}|p|^q dxdt\Big)^{1/q}\Big(\int_{-\mu^2}^0\Big(\int_{B_\mu}|u-\bar{u}_\rho|dx\Big)^{p} \\
&\qquad \times\Big(\int_{B_\mu}|u-\bar{u}_\rho|dx\Big)^{q/(q-1)-p}dt\Big)^{1-1/q} \\
&\le C\rho^{-3/q}\mu^{3/2+3(p-2)(q-1)/2q}\max_{-\mu^2\le t<0}\Big(\int_{B_\rho}|u|^2dx\Big)^{1/2-p(q-1)/2q} \\
&\qquad \times\Big(\iint_{Q_\rho}|p|^q dxdt\Big)^{1/q}\Big(\iint_{Q_\rho}|u|^p dxdt\Big)^{(q-1)/q} \\
&= C\mu^2\Big(\frac{\rho}{\mu}\Big)^{1/2-3(p-2)(q-1)/2q}E^{1/2-p(q-1)/2q}(\rho)E_p^{(q-1)/q}(\rho)P_q^{1/q}(\rho)
\end{aligned}$$

if $1/p+1/q \ge 1$; and

$$\begin{aligned}
J_2 &\le C\rho^{-3/q}\Big(\iint_{Q_\rho}|p|^q dxdt\Big)^{1/q}\Big(\int_{-\mu^2}^0\Big(\int_{B_\mu}|u-\bar{u}_\rho|dx\Big)^{q/(q-1)}dt\Big)^{1-1/q} \\
&\le C\rho^{-3/q}\mu^{5-5/p-2/q}\|u\|_{L^p(Q_\rho)}\|p\|_{L^q(Q_\rho)} \\
&\le C\mu^2\Big(\frac{\mu}{\rho}\Big)^{1+3/q-5/p}E_p^{1/p}(\rho)P_q^{1/q}(\rho)
\end{aligned}$$

if $1/p+1/q < 1$.

By the Young inequality and formula (4.13), one has that

$$\begin{aligned}
J_3 &= \iint_{Q_\mu} |u-\bar{u}_\rho||\overline{|p_1|}_\rho|dxdt \\
&\le C\rho^{-2}\int_{-\mu^2}^0\Big(\|u\|_{L^2(B_\rho)}\|\nabla u\|_{L^2(B_\rho)} + \|B\|_{L^2(B_\rho)}\|\nabla B\|_{L^2(B_\rho)}\Big)\int_{B_\mu}|u-\bar{u}_\rho|dxdt
\end{aligned}$$

$$
\begin{aligned}
&\le C\mu^{3-3/p}\rho^{-2}\int_{-\mu^2}^0 \Big(\|u\|_{L^2(B_\rho)}\|\nabla u\|_{L^2(B_\rho)}+\|B\|_{L^2(B_\rho)}\|\nabla B\|_{L^2(B_\rho)}\Big)\|u\|_{L^p(B_\rho)}dt\\
&\le C\mu^{4-5/p}\rho^{-2}\Big(\max_{-\rho^2\le t<0}\|u\|_{L^2(B_\rho)}\Big(\int_{-\rho^2}^0\|\nabla u\|^2_{L^2(B_\rho)}dt\Big)^{1/2}\\
&\qquad +\max_{-\rho^2\le t<0}\|B\|_{L^2(B_\rho)}\Big(\int_{-\rho^2}^0\|\nabla B\|_{L^2(B_\rho)}ds\Big)^{1/2}\Big)\|u\|_{L^p(Q_\rho)}\\
&= C\mu^2(\frac{\rho}{\mu})^{5/p-2}\Big(E^{1/2}(\rho)E_*^{1/2}(\rho)+F^{1/2}(\rho)F_*^{1/2}(\rho)\Big)E_p^{1/p}(\rho),\\
J_4 &= \iint_{Q_\mu}|u-\bar{u}_\rho||p_1|dxdt\\
&\le C\int_{-\mu^2}^0\|u-\bar{u}_\rho\|_{L^4(B_\mu)}\|p_1\|_{L^{4/3}(B_\mu)}dt\\
&\le C\mu^{1/4}\int_{-\mu^2}^0\Big(\|u\|_{L^2(B_\rho)}\|\nabla u\|_{L^2(B_\rho)}\\
&\qquad +\|B\|_{L^2(B_\rho)}\|\nabla B\|_{L^2(B_\rho)}\Big)\|u\|^{p/2(6-p)}_{L^p(B_\rho)}\|\nabla u\|_2^{3(4-p)/2(6-p)}ds\\
&\le C\mu^{1/4}\Big\{\max_{-\rho^2\le t<0}\|u\|^{\frac{(14-3p)}{2(6-p)}}_{L^2(B_\rho)}\Big(\iint_{Q_\rho}|u|^2dxdt\Big)^{\frac{(p-2)}{4(6-p)}}\Big(\iint_{Q_\rho}|\nabla u|^2dxdt\Big)^{\frac{(24-5p)}{4(6-p)}}\\
&\qquad +\max_{-\rho^2\le t<0}\|B\|^{\frac{(14-3p)}{2(6-p)}}_{L^2(B_\rho)}\Big(\iint_{Q_\rho}|b|^2dxdt\Big)^{\frac{(p-2)}{4(6-p)}}\Big(\iint_{Q_\rho}|\nabla u|^2dxdt\Big)^{\frac{3(4-p)}{2(6-p)}}\\
&\qquad\qquad \times\Big(\iint_{Q_\rho}|\nabla B|^2dxdt\Big)^{1/2}\Big\}\Big(\iint_{Q_\rho}|u|^pdxdt\Big)^{1/2(6-p)}\\
&\le (\frac{\rho}{\mu})^{7/4}E_p^{1/2(6-p)}(\rho)\Big(E^{(14-3p)/4(6-p)}(\rho)E_*^{(24-5p)/4(6-p)}(\rho)E_2^{(p-2)/4(6-p)}(\rho)\\
&\qquad +F^{(14-3p)/4(6-p)}(\rho)F_*^{1/2}(\rho)E_*^{3(4-p)/4(6-p)}(\rho)F_2^{(p-2)/4(6-p)}(\rho)\Big)\Big\}
\end{aligned}
$$

Substituting above estimates into (4.12), one derives (4.10) and (4.11). □

5 The Boundedness of Some Scaled Quantities

In this section, we derive the boundedness of some scaled quantities related to the magnetic field and pressure function, which are essential for the deducement of the smallness of some scaled quantities of magnetic field. For this purpose, we will make fully use of the generalized energy inequality (3.6). Let $\phi(x,t)$ be a smooth function with the property that $0\le\phi\le 1$, $\phi\equiv 1$ on Q_r, $\phi\equiv 0$ away from Q_{r_*} with $r_*=2r$, such that

$$
|\nabla\phi|\le\frac{C}{r_*}\quad\text{and}\quad|\frac{\partial\phi}{\partial t}|+|\nabla_x^2\phi|\le\frac{C}{r_*^2}. \tag{5.1}
$$

Using ϕ^2 instead of ϕ in (3.6), it follows that

$$\begin{aligned}
&\int_{B_{r_*}} \Big(|u(x,t)|^2 + |B(x,t)|^2\Big)\phi^2(x,t)dx \\
&\qquad\qquad +2\int_{-r_*^2}^t \int_{B_{r_*}} \Big(|\nabla u(x,s)|^2 + |\nabla B(x,s)|^2\Big)\phi^2(x,s)dxds \\
\leq\ & \int_{-r_*^2}^t \int_{B_{r_*}} \Big(|u(x,t)|^2 + |B(x,t)|^2\Big)((\phi^2)_t(x,t) + \Delta\phi^2(x,t))dxdt \\
&+ \int_{-r_*^2}^t \int_{B_{r_*}} \Big(u(x,t)\cdot\nabla\phi^2(x,t)\Big)\Big(|u(x,t)|^2 + |B(x,t)|^2 + 2p(x,t)\Big)dxdt \\
&\qquad -2\int_{-r_*^2}^t \int_{B_{r_*}} \Big(B(x,t)\cdot\nabla\phi^2(x,t)\Big)\Big(u(x,t)\cdot B(x,t)\Big)dxdt \\
\leq\ & \frac{C}{r^2}\int_{-r_*^2}^t \int_{B_{r_*}} \Big(|u|^2 + |B|^2\Big)dxds + \frac{C}{r}\int_{-r_*^2}^t \int_{B_{r_*}} |u|^3 dxds \\
&+ \frac{C}{r}\int_{-r_*^2}^t \int_{B_{r_*}} |B|^2|u|dxds + \frac{C}{r}\int_{-r_*^2}^t \int_{B_{r_*}} |p||u|dxds. \qquad (5.2)
\end{aligned}$$

Employing the generalized energy inequality (5.2), one can show that

Proposition 5.1 There exist two absolute constants ε and M such that, for some $r_0 > 0$,

i) $E_2(r) \leq \varepsilon$ for all $0 < r \leq r_0$,

ii) $E(r) \leq M$ for all $0 < r \leq r_0$,

iii) $F(r) \leq M$ for all $0 < r \leq r_0$,

then, there is some $r_1 \leq r_0$, such that, for any $0 < r \leq r_1$ and $3 \leq p < 10/3$,

$$A_p(r) \triangleq E(r) + F(r) + E_*(r) + F_*(r) + P_{p/2}(r) \leq M_1 \qquad (5.3)$$

with absolute constant M_1 depending only on C_i and M.

Proof. By (3.4) and (3.5), there are two absolute constants r_2 and M_0 such that

$$A_p(r_2) \leq M_0. \qquad (5.4)$$

Without lost of generality, we assume that $r_0 \leq r_2$. Let $0 < 2r = r_* < \rho \leq r_0$. Then, from (5.2), it follows that

$$\begin{aligned}
A_p(r) \leq\ & C\Big\{E_2(r_*) + F_2(r_*) + E_3(r_*) \\
&+ r_*^{-2}\iint_{Q_{r_*}} |B|^2|u|dxdt + r_*^{-2}\iint_{Q_{r_*}} |p||u|dxdt\Big\} + P_{p/2}(r). \qquad (5.5)
\end{aligned}$$

In the following, we estimate each term at the right hand side of (5.5). First, let $\mu = r$ and $q = p/2$ in (4.6). Then, by the Hölder inequality, one has that

$$P_{p/2}(r) \ \leq\ C(\frac{\rho}{r})^{5-p}\Big(E_p(\rho) + F_p(\rho)\Big) + C(\frac{r}{\rho})^{p-2}P_{p/2}(\rho)$$

$$
\begin{aligned}
&\le C(\frac{r}{\rho})^{p-2}P_{p/2}(\rho)+C(\frac{\rho}{r})^{5-p}\Big(E_3(\rho)+F_p(\rho)\Big)\\
&\le C(\frac{r}{\rho})^{p-2}P_{p/2}(\rho)+C(\frac{\rho}{r})^{5-p}F_p(\rho)\\
&\qquad +C(\frac{\rho}{r})^{5-p}E^{(p-2)/2}(\rho)\Big(E_2^{(10-3p)/4}(\rho)E_*^{3(p-2)/4}(\rho)+E_2(\rho)\Big).
\end{aligned}
$$

Here estimate (4.1) has been used. By Lemma 4.1,

$$
\begin{aligned}
C(\frac{\rho}{r})^{5-p}F_p(\rho) &\le C(\frac{\rho}{r})^{5-p}F^{(p-2)/2}(\rho)\Big(F_2^{(10-3p)/4}(\rho)F_*^{3(p-2)/4}(\rho)+F_2(\rho)\Big)\\
&\le C(\frac{\rho}{r})^{5-p}M^{(p+2)/4}A_p^{3(p-2)/4}(\rho)\\
&\le \frac{1}{8}A_p(\rho)+C(\frac{\rho}{r})^{4(5-p)/(10-3p)}M^{(p+2)/(10-3p)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
P_{p/2}(r)\le\ & C(\frac{r}{\rho})^{p-2}P_{p/2}(\rho)+C(\frac{\rho}{r})^{5-p}\Big(M^{(5p-14)/4}\varepsilon^{(10-3p)/4}+\varepsilon^{1/2}\Big)A_p(\rho)\\
&+\frac{1}{8}A_p(\rho)+C(\frac{\rho}{r})^{4(5-p)/(10-3p)}M^{(p+2)/(10-3p)}. \qquad (5.6)
\end{aligned}
$$

Clearly,

$$
E_2(r_*)+F_2(r_*)\le M+F(r_*)\le 2M. \qquad (5.7)
$$

Next, it follows from (4.1) and the assumptions that

$$
\begin{aligned}
E_3(r_*) &\le CE^{1/2}(r_*)\Big(E_2^{1/2}(r_*)E_*^{3/4}(r_*)+E_2(r_*)\Big)\\
&\le C(\frac{\rho}{r})\Big(M^{1/4}\varepsilon^{1/4}+\varepsilon^{1/2}\Big)A_p(\rho). \qquad (5.8)
\end{aligned}
$$

Let $\mu=r_*$, $p=2$ and $q=p/2$, one has, from (4.10), that

$$
\begin{aligned}
& r_*^{-2}\iint_{Q_{r_*}}|u||p|dxdt\\
&\le C\Big\{\Big((\frac{r}{\rho})^{4/p-1}+(\frac{\rho}{r})^{1/2}\Big)E^{2/p-1/2}(\rho)E_2^{(p-2)/p}(\rho)P_{p/2}^{2/p}(\rho)\\
&\quad +(\frac{\rho}{r})^{1/2}\Big(E^{1/2}(\rho)E_*^{1/2}(\rho)+F^{1/2}(\rho)F_*^{1/2}(\rho)\Big)\\
&\quad +(\frac{\rho}{r})^{7/4}\Big(E^{1/2}(\rho)E_*^{7/8}(\rho)+F^{1/2}(\rho)F_*^{1/2}(\rho)E_*^{3/8}(\rho)\Big)E_2^{1/8}(\rho)\Big\}\\
&\le C\Big\{\Big((\frac{r}{\rho})^{4/p-1}+(\frac{\rho}{r})^{1/2}\Big)M^{4/p-3/2}\varepsilon^{(p-2)/p}\\
&\qquad +(\frac{\rho}{r})^{1/2}\varepsilon^{1/2}+(\frac{\rho}{r})^{7/4}M^{3/8}\varepsilon^{1/8}\Big\}A_p(\rho). \qquad (5.9)
\end{aligned}
$$

Finally, we need to estimate the term $r_*^{-2}\iint_{Q_{r_*}}|B|^2|u|dxdt$. By the interpolation and Sobolev inequalities, one has

$$
\|B\|^2_{L^4(B_{r_*})}\le C\|B\|^{1/2}_{L^2(B_{r_*})}\|\nabla B\|^{3/2}_{L^2(B_{r_*})}+Cr^{-3/2}\|B\|^2_{L^2(B_{r_*})}.
$$

Then,

$$\iint_{Q_{r_*}}|B|^2|u|dxdt \le \int_{-r_*^2}^{0}\|u\|_{L^2(B_{r_*})}\|B\|^2_{L^4(B_{r_*})}dt$$

$$\le \; C\max_{-r_*^2\le t<0}\|B\|^{1/2}_{L^2(B_{r_*})}\max_{-r_*^2\le t<0}\|u\|^{1/2}_{L^2(B_{r_*})}$$

$$\times\Big(\int_{-r_*^2}^{0}\|u\|^2_{L^2(B_{r_*})}dt\Big)^{1/4}\Big(\iint_{Q_{r_*}}|\nabla B|^2dxdt\Big)^{3/4}$$

$$+Cr^{-3/2}\max_{-r_*^2\le t<0}\|B\|_{L^2(B_{r_*})}\Big(\iint_{Q_{r_*}}|u|^2dxdt\Big)^{1/2}\Big(\iint_{Q_{r_*}}|B|^2dxdt\Big)^{1/2}$$

$$\le \; Cr_*^2\Big(F^{1/4}(r_*)F_*^{3/4}(r_*)E^{1/4}(r_*)E_2^{1/4}(r_*)+F^{1/2}(r_*)F_2^{1/2}(r_*)E_2^{1/2}(r_*)\Big).$$

Therefore,

$$r_*^{-2}\iint_{Q_{r_*}}|B|^2|u|dxdt \le C(\frac{\rho}{r})\Big(M^{1/4}\varepsilon^{1/4}+\varepsilon^{1/2}\Big)A_p(\rho). \tag{5.10}$$

Let $r=\lambda\rho$ with $\lambda<1/2$. Substituting estimates (5.6) - (5.10) into (5.5), one gets that

$$A_p(\lambda\rho)\le\frac{1}{8}A_p(\rho)+C_4\lambda^{p-2}A_p(\rho)+g(\lambda,\varepsilon)A_p(\rho)+C(M,\lambda). \tag{5.11}$$

First fix a $\lambda\in(0,1/2)$, such that $C_4\lambda^{p-2}\le 1/8$. Then, let ε small enough such that $g(\lambda,\varepsilon)\le 1/4$. Thus, one has, at last, that

$$A_p(\lambda\rho)\le\frac{1}{2}A_p(\rho)+C_5. \tag{5.12}$$

Iterating the inequality (5.12) k times yields

$$A_p(\lambda^k\rho)\le(\frac{1}{2})^kA_p(\rho)+C_5\Big(1+\frac{1}{2}+\cdots+(\frac{1}{2})^{k-1}\Big).$$

Next, we choose an integer K_0 such that

$$(\frac{1}{2})^{k_0}A_p(r_0)\le(\frac{1}{2})^{k_0}\max\{r_0^{-1},r_0^{-5+2p}\}(C+1+C_2+C_3)\le 2C_6.$$

Define $r_1=\lambda^{k_0}r_0$. For any $0<r\le r_1$, there exists a $k\ge k_0$, such that $\lambda^{k+1}r_0\le r\le\lambda^kr_0$. Thus,

$$A_p(r)\le(\frac{1}{2})^kA_p(r_0)\le(\frac{1}{2})^{k_0}A_p(r_0)\le C(C_5+C_6)\triangleq M_1,$$

which gives the desired. □

Proposition 5.2 There exist absolute constants ε and M, such that, for all $0<r\le r_0$ with some $r_0>0$,

i) $E_2(r)\le\varepsilon$;

ii) $E_*(r)\le M$;

iii) $F_2(r)\leq M$.

Then, there is a $0<r_1\leq r_0$, such that

$$A_{8/3}(r)\triangleq E(r)+F(r)+E_*(r)+F_*(r)+P_{4/3}(r)\leq M_1 \tag{5.13}$$

for every $0<r\leq r_1$ with an absolute constant M_1.

Proof. This proof is similar to that of Proposition 5.1. Here we only point out the differences. As above, we need to estimate the each term at the right hand side of (5.5). But, except the last two terms at the right hand side of (5.5), the estimates of others are same. In order to estimate the last term at the right hand side of (5.5), we use the Lemma 4.4 with $p=3$ and $q=4/3$, and bound the term $E_3(\rho)$ by Lemma 4.2. Then, after some manipulations, one can obtain the desired estimate. Next, we need to estimate the term

$$r_*^{-2}\iint_{Q_{r_*}}|B|^2|u|dxdt.$$

By the Hölder interpolation and Sobolev inequalities, one has that

$$\begin{aligned}
&r_*^{-2}\iint_{Q_{r_*}}|B|^2|u|dxdt\\
&\leq r_*^{-2}\int_{-r_*^2}^{0}\|B\|^2_{L^{8/3}(B_{r_*})}\|u\|_{L^4(B_{r_*})}dt\\
&\leq Cr_*^{-2}\int_{-r_*^2}^{0}\Big(\|B\|^{5/4}_{L^2(B_{r_*})}\|\nabla B\|^{3/4}_{L^2(B_{r_*})}+r_*^{-3/4}\|B\|^2_{L^2(B_{r_*})}\Big)\\
&\qquad\qquad\times\Big(\|u\|^{1/4}_{L^2(B_{r_*})}\|\nabla u\|^{3/4}_{L^2(B_{r_*})}+r_*^{-3/4}\|u\|_{L^2(B_{r_*})}\Big)dt\\
&\leq CF^{1/2}(r_*)\Big(F_2^{1//8}(r_*)F_*^{3/8}(r_*)+F_2^{1/2}(r_*)\Big)\Big(E_2^{1/8}(r_*)E_*^{3/8}(r_*)+E_2^{1/2}(r_*)\Big).
\end{aligned} \tag{5.14}$$

By assumptions,

$$r_*^{-2}\iint_{Q_{r_*}}|B|^2|u|dxdt\leq C\Big((\frac{\rho}{r})^{5/4}+(\frac{\rho}{r})^2\Big)\Big(M^{3/8}\varepsilon^{1/8}+\varepsilon^{1/2}\Big)A_{8/3}(\rho).$$

Due to above estimates, one can deduce an inequality similar to (5.12). Thus, we get desired result by same iterating procedure as that in the proof of Proposition 5.1. □

Proposition 5.3 There exist absolute constants ε and M, such that, for every $0<r\leq r_1$ with some $r_1\leq r_0$,

i) $E_*(r)\leq\varepsilon$;

ii) $F_2(r)\leq M$.

Then, there is an absolute constant M_1, such that

$$A_{8/3}(r)\leq M_1 \tag{5.15}$$

for every $0 < r \le r_1$ with some $r_1 \le r_0$.

Proof. By the Proposition 2.2 in [15], the condition i) implies that, for any $\varepsilon_1 > 0$, there is some $r_0' < r_0$, such that

$$\sup_{0<r\le r_0'} E_2(r) \le \varepsilon_1. \tag{5.16}$$

Thus, Proposition 5.3 follows from (5.16) and Proposition 5.2. □

Proposition 5.4 There exist two absolute constants ε and M, such that, for some $r_0 > 0$ and $3 \le p \le 10/3$,

i) $\sup\limits_{0<r\le r_0} E_p(r) \le \varepsilon$;

ii) $\sup\limits_{0<r\le r_0} F_{2p/(p-1)}(r) \le M$.

Then there is some $r_1 \le r_0$ and an absolute constant M_1, such that

$$A_{2p/(p-1)}(r) \le M_1 \tag{5.17}$$

for every $0 < r \le r_1$.

Proof. By the Hölder inequality, it is obvious that

$$\begin{aligned} r_*^{-2}\iint_{Q_{r_*}} |u||p|dxdt &\le r_*^{-2}\|u\|_{L^p(Q_{r_*}}\|p\|_{L^{p/(p-1)}(Q_{r_*})} \\ &\le E_p^{1/p}(r_*)P_{p/(p-1)}^{(p-1)/p}(r_*) \le C(\frac{\rho}{r})^{(3p-5)/p}E_p^{1/p}(r_*)P_{p/(p-1)}^{(p-1)/p}(\rho). \end{aligned}$$

and

$$r_*^{-2}\iint_{Q_{r_*}} |B|^2|u|dxdt \le E_p^{1/p}(r_*)F_{2p/(p-1)}^{2(p-1)/p}(r_*).$$

We use Lemma 4.3 to treat the term $P_{p/(p-1)}(r)$ as before. It remains to estimate $E_{2p/(p-1)}(\rho)$. If $3 \le p \le 10/3$, then $2p/(p-1) \le p$. By the Hölder inequality, one has

$$E_{2p/(p-1)}(\rho) \le CE_p(\rho).$$

By the iterating proceduce similar to the proof of Proposition 5.1, we can obtain the desired result. Here we omit the details. □

Proposition 5.5 There exist absolute constants ε and M, such that, for some $r_0 > 0$ and $5/2 \le p < 3$,

i) $\sup\limits_{0<r\le r_0} E_p(r) \le \varepsilon$;

ii) $\sup\limits_{0<r\le r_0} F_3(r) \le M$.

Then there is some $r_1 \leq r_0$ and an absolute constant M_1, such that

$$A_3(r) \leq M_1 \tag{5.18}$$

for every $0 < r \leq r_1$.

Proof. By the Hölder inequality, one has

$$r_*^{-2} \iint_{Q_{r_*}} |u||p| dxdt \leq C(\frac{\rho}{r})^2 E_3^{1/3}(\rho) P_{3/2}^{2/3}(\rho)$$

and

$$r_*^{-2} \iint_{Q_{r_*}} |B|^2|u| dxdt \leq C(\frac{\rho}{r})^{2/3} E_3^{1/3}(\rho) F_3^{2/3}(r_*).$$

Term $E_3(\rho)$ can be estimated by Lemma 4.2 as follows:

$$\begin{aligned} E_3(\rho) \quad \leq \quad & CE^{(2p-3)/6p}(\rho)\Big(E_p^{1/3p}(\rho)E_*^{1/2p}(\rho) + E_p^{(6-p)/3p^2}(\rho)E_*^{(2p-3)/6p}(\rho) \\ & + E_p^{(p-1)/p^2}(\rho)E_*^{(3-p)/3p}(\rho) + E_p^{(p+3)/3p^2}(\rho)\Big). \end{aligned}$$

Then, our result follows as before. □

Similarly, one can show that:

Proposition 5.6 There exist absolute constants ε and M, such that, for some $r_0 > 0$ and $5/2 < p \leq 10/3$,

i) $\sup_{0<r\leq r_0} E_p(r) \leq \varepsilon$;

ii) $\sup_{0<r\leq r_0} F(r) \leq M$.

Then there exist some $r_1 \leq r_0$ and an absolute constant M_1, such that

$$A_{2p/(p-1)}(r) \leq M_1 \tag{5.19}$$

for every $0 < r \leq r_1$.

Finally,

Proposition 5.7 If there exists an absolute constant M_1 such that

$$B(r) \stackrel{\Delta}{=} E(r) + F(r) + E_*(r) + F_*(r) \leq M_1 \tag{5.20}$$

for every $0 < r \leq r_1$ with some $r_1 > 0$, then, for $2 \leq p \leq 10/3$,

$$E_p(r) + F_p(r) \leq M_2 \tag{5.21}$$

for every $0 < r \leq r_1$, with an absolute constant M_2. Furthermore, there is $r_2 \leq r_1$, such that, for $2 < q \leq 10/3$,

$$P_{q/2}(r) \leq M_3 \tag{5.22}$$

for any $0 < r \le r_2$ with an absolute constant M_3.

Proof. Note that (5.21) follows from Lemma 4.1 and (5.20). In order to deduce (5.22), we apply Lemma 4.3 with $\rho \le r_1$. By the Hölder inequality,

$$\begin{aligned} P_{q/2}(\mu) &\le C(\frac{\rho}{\mu})^{5-q}\Big(E_q(\rho) + F_q(\rho)\Big) + C(\frac{\mu}{\rho})^{q-2}P_{q/2}(\rho) \\ &\le C(\frac{\mu}{\rho})^{q-2}P_{q/2}(\rho) + C(\frac{\rho}{\mu})^{5-q}M_2. \end{aligned}$$

Let $\mu = \lambda\rho$, then fix λ such that $C\lambda^{q-2} \le 1/2$. One obtains that

$$P_{q/2}(\lambda\rho) \le \frac{1}{2}P_{q/2}(\rho) + CM_2. \tag{5.23}$$

Now (5.22) follows from (5.23) by iteration in the same way as in the proof of proposition 5.1. □

Summing up the above results, we conclude that if the conditions in any one of Proposition 5.1 - 5.6 are satisfied, then

$$A_q(r) \le M_4 \tag{5.24}$$

for $0 < r \le r_2$ and $2 < q \le 10/3$ with an absolute constant M_4.

6 The Smallness of Some Scaled Qualities of Magnetic Field

In this section, we deduce the smallness of some scaled quantities related to the magnetic field by making use of the smallness of certain quantities of the velocity field and the boundedness of some quantities related to the velocity field and the magnetic field, which are obtained in last section. First, one has that

Proposition 6.1 For any $\varepsilon_1 > 0$, there exist absolute constants δ_1, r_1 and M_1 such that $\sup_{0<r\le r_2} (E(r) + E_*(r)) \le M_1$, and one of the following two conditions holds, for some $r_2 > 0$,

i) $\sup_{0<r\le r_2} E_2(r) \le \varepsilon \le \delta_1$;

ii) $\sup_{0<r\le r_2} E_*(r) \le \varepsilon \le \delta_1$.

Then

$$E_3(r) \le \varepsilon_1 \quad \text{for any} \ \ 0 < r \le r_2. \tag{6.1}$$

Proposition 6.1 follows directly from Lemma 4.1.

Proposition 6.2 For any $\varepsilon_2 > 0$, there exists an absolute constant δ_2 such that, for some $r_2 > 0$,

i) $\sup\limits_{0<r\leq r_2} E_3(r) \leq \varepsilon_1 \leq \delta_2$;

ii) $\sup\limits_{0<r\leq r_2} (F_2(r) + F_*(r)) \leq M_1$ with an absolute constant M_1.

Then, there exists a positive constant $r_3 \leq r_2$, such that

$$F_2(r) \leq \varepsilon_2 \quad \text{for all} \;\; 0 < r \leq r_3. \tag{6.2}$$

Proof. Let $\psi(x,t)$ be a smooth cut-off function with properties that $0 \leq \psi(x,t) \leq 1$, $\psi(x,t) \equiv 1$ in $Q_{4\rho/5}$, $\psi(x,t) \equiv 0$ away from Q_ρ^c, such that

$$|\nabla\psi(x,t)| \leq \frac{C}{\rho} \quad \text{and} \;\; |\frac{\partial\psi(x,t)}{\partial t}| + |\nabla_x^2\psi(x,t)| \leq \frac{C}{\rho^2}. \tag{6.3}$$

Set $\chi(x,t) = \psi(x,t)$ in (2.3) to get that

$$\frac{\partial B\psi}{\partial t} - \Delta(B\psi) = B(\frac{\partial\psi}{\partial t} - \Delta\psi) - 2\nabla\psi\cdot\nabla B - \psi(u\cdot\nabla)B + \psi(B\cdot\nabla)u. \tag{6.4}$$

Then, for any $(x,t) \in Q_{4\rho/3}$, the solution $B\psi$ can be represented as

$$\begin{aligned} B\psi(x,t) = \;& \int_{-\rho^2}^t \int_{B_\rho} G(x-y,t-s)\Big(B(\frac{\partial\psi}{\partial t} - \Delta\psi) - 2\nabla\psi\cdot\nabla B\Big)dyds \\ & - \int_{-\rho^2}^t \int_{B_\rho} G(x-y,t-s)\psi u\cdot\nabla B dyds \\ & + \int_{-\rho^2}^t \int_{B_\rho} G(x-y,t-s)\psi B\cdot\nabla u dyds. \end{aligned} \tag{6.5}$$

Here $G(x,t)$ is the normalized fundamentmental solution of the heat equation. By integration by part, one has that

$$\begin{aligned} |\int_{-\rho^2}^t \int_{B_\rho} G(x-y,t-s)\nabla\psi\cdot\nabla B dyds| \leq & \int_{-\rho^2}^t \int_{B_\rho} |\nabla G(x-y,t-s)||\nabla\psi||B|dyds \\ & + \int_{-\rho^2}^t \int_{B_\rho} G(x-y,t-s)|\nabla_y^2\psi||B|dyds \end{aligned}$$

and

$$\begin{aligned} |\int_{-\rho^2}^t \int_{B_\rho} G(x-y,t-s)\psi B\cdot\nabla u dyds| \leq & \int_{-\rho^2}^t \int_{B_\rho} |\nabla G(x-y,t-s)||\psi||B||u|dyds \\ & + \int_{-\rho^2}^t \int_{B_\rho} G(x-y,t-s)|\nabla\psi||B||u|dyds \\ & + \int_{-\rho^2}^t \int_{B_\rho} G(x-y,t-s)|\psi||\nabla B||u|dyds. \end{aligned}$$

Thus, (6.5) implies that, for $(x,t)\in Q_{4\rho/5}$,

$$\begin{aligned}|B(x,t)| \ &\le \int_{-\rho^2}^{t}\int_{B_\rho}\Big\{G(x-y,t-s)\Big(|B|(|\frac{\partial\psi}{\partial t}|+3|\nabla_y^2\psi|)\Big)\\ &\qquad\qquad +2|\nabla G(x-y,t-s)||\nabla\psi||B|\Big\}dyds\\ &\quad +2\int_{-\rho^2}^{t}\int_{B_\rho}G(x-y,t-s)|\psi||u||\nabla B|dyds\\ &\quad +\int_{-\rho^2}^{t}\int_{B_\rho}|\nabla G(x-y,t-s)||\psi||B||u|dyds\\ &\quad +\int_{-\rho^2}^{t}\int_{B_\rho}G(x-y,t-s)|\nabla\psi||B||u|dyds\\ &\triangleq I_5+I_6+I_7+I_8. \end{aligned}\tag{6.6}$$

Note the fact that

$$|\frac{\partial\psi}{\partial t}|+|\nabla\psi|\equiv 0 \quad \text{in} \quad Q_{4\rho/5}\cup Q_\rho^c. \tag{6.7}$$

Then, for any $(x,t)\in Q_\mu$ with $\mu\le 2\rho/5$, one has

$$\begin{aligned} I_5 \ &\le C\rho^{-5}\iint_{Q_\rho}|B(y,s)|dyds\\ &\le C\rho^{-5/2}\Big(\iint_{Q_\rho}|B(y,s)|^2dyds\Big)^{1/2}=C\rho^{-1}F_2^{1/2}(\rho). \end{aligned}$$

Thus

$$\Big(\iint_{Q_\mu}|I_5|^2dxdt\Big)^{1/2}\le C\mu^{5/2}|I_5|\le C\mu^{5/2}\rho^{-1}F_2^{1/2}(\rho). \tag{6.8}$$

Next,

$$\|I_6+I_7\|_{L^2(B_\mu)}\le C\int_{-\rho^2}^{t}(t-s)^{-1/2}\Big(\|B\|_{L^6(B_\rho)}+\|\nabla B\|_{L^2(B_\rho)}\Big)\|u\|_{L^3(B_\rho)}ds.$$

Using the Sobolev inequality

$$\|B\|_{L^6(B_\rho)}\le C\Big(\|\nabla B\|_{L^2(B_\rho)}+\rho^{-1}\|B\|_{L^2(B_\rho)}\Big),$$

one has

$$\|I_6+I_7\|_{L^2(B_\mu)}\le C\int_{-\rho^2}^{t}(t-s)^{-1/2}\Big(\rho^{-1}\|B\|_{L^2(B_\rho)}+\|\nabla B\|_{L^2(B_\rho)}\Big)\|u\|_{L^3(B_\rho)}ds.$$

By the Calderóln -Zygmund theorem on singular integrals and the Hölder inequality, we deduce that

$$\begin{aligned}\|I_6+I_7\|_{L^2(Q_\mu)} \ &\le C\mu^{1/3}\Big(\int_{-\mu^2}^{0}\|I_6+I_7\|_{L^2(B_\mu)}ds\Big)^{1/3}\\ &\le C\mu^{1/3}\|u\|_{L^3(Q_\rho)}\Big(\|\nabla B\|_{L^2(Q_\rho)}+\rho^{-1}\|B\|_{L^2(Q_\rho)}\Big)\\ &\le C\mu^{1/3}\rho^{7/6}E_3^{1/3}(\rho)\Big(F_*^{1/2}(\rho)+F_2^{1/2}(\rho)\Big). \end{aligned}\tag{6.9}$$

Next, taking into account of the fact (6.7), one has that

$$\|I_8\|_{L^2(Q_\mu)} \leq C\mu^{5/2}\rho^{-4}\|B\|_{L^2(Q_\rho)}\|u\|_{L^2(Q_\rho)} \leq C\mu^{5/2}\rho^{-1}E_2^{1/2}(\rho)F_2^{1/2}(\rho). \tag{6.10}$$

Therefore it follows from (6.6) and (6.8) - (6.10) that

$$F_2^{1/2}(\mu) \leq C(\frac{\mu}{\rho})F_2^{1/2}(\rho)\Big(1+E_2^{1/2}(\rho)\Big) + C(\frac{\rho}{\mu})^{7/6}E_3^{1/3}(\rho)\Big(F_2^{1/2}(\rho)+F_*^{1/2}(\rho)\Big),$$

i.e.,

$$F_2(\mu) \leq C_9(\frac{\mu}{\rho})^2F_2(\rho)\Big(1+E_2(\rho)\Big) + C(\frac{\rho}{\mu})^{7/3}E_3^{2/3}(\rho)\Big(F_2(\rho)+F_*(\rho)\Big). \tag{6.11}$$

Let $\mu = \lambda\rho$ and fix λ such that $C_9\lambda^2 \leq 1/4$. Then one has

$$\begin{aligned} F_2(\lambda\rho) &\leq \frac{1}{4}F_2(\rho)(1+\varepsilon_1) + C\lambda^{-7/3}\varepsilon_1^{3/2}M_1 \\ &\leq \frac{1}{2}F_2(\rho) + C\lambda^{-7/3}\varepsilon_1^{2/3}M_1. \end{aligned} \tag{6.12}$$

Here we assume that $\varepsilon_1 \leq 1$. So our result follows from (6.12) by iteration in the same way as in the proof of Proposition 5.1. Also see the proof of Proposition 2.2 in [15]. □

Remark: Similar to the derivation of (6.11), one can show that

$$F_{5/2}(\mu) \leq C(\frac{\mu}{\rho})^{5/2}\Big(F_2^{5/4}(\rho) + F_{10/3}^{3/4}(\rho)E_2^{5/4}(\rho)\Big) + C(\frac{\rho}{\mu})^{5/2}F_{10/3}^{3/43}(\rho)\Big(E_2^{5/4}(\rho)+E_*^{5/4}(\rho)\Big), \tag{6.13}$$

or

$$F_{5/2}(\mu) \leq C(\frac{\mu}{\rho})^{5/2}F_2^{5/4}(\rho)\Big(1+E_{10/3}^{3/4}(\rho)\Big) + C(\frac{\rho}{\mu})^{5/2}E_{10/3}^{4/3}(\rho)\Big(F_2^{5/4}(\rho)+F_*^{5/4}(\rho)\Big). \tag{6.14}$$

Thus, if $E_{10/3}(r) \leq \varepsilon$ and $F_2(r)+F_*(r) \leq M$ for $0 < r \leq r_2$, or $E_*(r) \leq \varepsilon$ and $F_{10/3}(r) \leq M$ for $0 < r \leq r_2$, we can obtain the smallness of $F_{5/2}(r)$ for any $0 < r \leq r_3$ with $r_3 \leq r_2$, by same discussion as above.

Proposition 6.3 For any $\varepsilon_3 > 0$, there exists an absolute constant δ_3 such that if

i) $\sup_{0<r\leq r_2} F_2(r) \leq \varepsilon_2 \leq \delta_3$;

ii) $\sup_{0<r\leq r_2} E_3(r) \leq \varepsilon_1 \leq \delta_3$; and

iii) $\sup_{0<r\leq r_2} A_3(r) \leq M_4$,

then, there is a positive constant $r_4 \leq r_2$ such that

$$A_3(r) \leq \varepsilon_3 \qquad \text{for every} \quad 0 < r \leq r_4. \tag{6.15}$$

Moreover

$$F_3(r) \leq \varepsilon_3 \quad \text{for every} \quad 0 < r \leq r_4. \tag{6.16}$$

Proof. First, we apply inequality (5.5) with $2r = r_* < \rho \le r_2$. By the Hölder inequality,

$$A_3(r) \le P_{3/2}(r) + C\Big(E_2(r_*) + F_2(r_*) + E_3(r_*) + F_3^{2/3}(r_*)E_3^{1/3}(r_*) + E_3^{1/3}(r_*)P_{3/2}^{2/3}(r_*)\Big). \tag{6.17}$$

It follows from Lemma 4.3 that

$$P_{3/2}(r) \le C(\frac{r}{\rho})P_{3/2}(\rho) + C(\frac{\rho}{r})^3\Big(E_3(\rho) + F_3(\rho)\Big). \tag{6.18}$$

Noting that $F_3(\rho)$ can be estimated by Lemma 4.1 with $p = 3$, one may conclude that

$$A_3(r) \le C(\frac{r}{\rho})A_3(\rho) + \varepsilon_1^{1/3}M_4^{2/3} + C(1 + (\frac{\rho}{r}))\varepsilon_1 + C(1 + M_4^{1/4})\varepsilon_2^{1/4}. \tag{6.19}$$

By iteration in the same way as in the proof of Proposition 5.1, (6.15) results from (6.19). And (6.16) follows directly from Lemma 4.1 and (6.15). □

7 The Proofs of the Theorem 2.1 - 2.3

In this section, we will give the spatial gradient estimates on suitable weak solutions of the incompressible magnetohydrodynamic equations provided that some scaled quantities is suitably small for the velocity field and magnetic field. Then we can indicate the proofs of Theorem 2.1 - 2.3. First, one has

Proposition 7.1 There exists an absolute constant $\varepsilon_4 > 0$, such that, for any suitable weak solution $(u, B, p)(x, t)$ to the magnetohydrodynamic equations satisfying (3.4) and (3.5), if there is a $r_4 > 0$, such that one of the following three conditions holds

i) Either $\sup_{0<r\le r_4}(E(r) + F(r)) < +\infty$ or $\sup_{0<r\le r_4}(E_*(r) + F_*(r)) < +\infty$ and

$$E_2(r) + F_2(r) \le \varepsilon_4 \qquad \text{for all} \quad 0 < r \le r_4,$$

ii) $\sup_{0<r\le r_4}(E_*(r) + F_*(r)) \le \varepsilon_4$,

iii) $\sup_{0<r\le r_4}(E_p(r) + F_p(r)) \le \varepsilon_4$ for some $5/2 \le p \le 10/3$,

then, there is some $r_5 \le r_4$, such that

$$\sup_{Q_{r/2}}\Big(|\nabla u| + |\nabla B|\Big) \le Cr^{-2} \quad \text{ for } 0 < r \le r_5$$

with an absolute constant C.

This Proposition is only direct extension to the magnetohydrodynamic equations of the corresponding results of incompressible Navier-Stokes equations in [15]. So the proof is exactly same as that of Theorem 3.1 in [15]. Thus we omit the details.

It should be noted that the results in Theorem 2.1 - 2.3 follow directly from Proposition 5.1 - 5.7, Proposition 6.1 - 6.3 and Proposition 7.1. Thus we complete the proof of Theorem 2.1 - 2.3.

8 The Proofs of Theorem 2.4 and 2.5

In this section, we will establish the further regularity for solutions to the magneto-hydrodynamic equations, by making use of the results obtained in conjunction with extra hypotheses on the given initial data, as for the incompressible Navier-Stokes equations. In this section, we study the case $\Omega = R^3$. The ideas and techniques in this section are borrowed from [1]. Firstly, we show that (u, B) is regular for $t > C$ with an absolute constant C. More precisely, we have

Proposition 8.1. Let (u, B, p) be a suitable weak solution to the magneto-hydrodynamic equations. There exists an absolute constant $\varepsilon_5 > 0$, such that, if

$$\int_0^t \int_{|x-y|\le t} |u|^3(y,s)dyds \le \varepsilon_5 t, \tag{8.1}$$

and

$$\int_0^t \int_{|x-y|\le t} |B|^3(y,s)dyds \le C_{10} t \tag{8.2}$$

for some constant C_{10}, then $|\nabla u| + |\nabla B|$ is uniformly bounded in some neighborhood of (x,t). In particular, $|\nabla u|+|\nabla B|$ is locally uniformlly bounded as $t > N_0(\|u_0\|_2^2+\|B_0\|_2^2)^{3/2}$ with an absolute constant N_0.

Proof. Let $r = \sqrt{t}$. Then, (8.1) and (8.2) imply that

$$r^{-2} \iint_{Q_r(x,t)} |u|^3 dyds \le \varepsilon_5,$$

$$r^{-2} \iint_{Q_r(x,t)} |B|^3 dyds \le C_{10}.$$

Thus, the first results follows from Theorem 2.2.

In the following, we prove the second result. By the interpolation and Sobolev inequalities and (3.4), one has that

$$\int_0^t \int_{R^3} (|u|^3 + |B|^3)dxdt \le Ct^{1/4}(\|u_0\|_2^2 + \|B_0\|_2^2)^{3/2}.$$

Therefore

$$\int_0^t \int_{R^3} (|u|^3 + |B|^3) dxdt \le \varepsilon_5 t$$

provided that $t > N_0(\|u_0\|_2^2 + \|B_0\|_2^2)^{3/2}$ with some absolute constant N_0. So the desired result follows from Theorem 2.2 again. □

Next, we turn to the proof of Theorem 2.4. First, we need,

Lemma 8.2. Let $u_0, B_0 \in \overset{o}{J}{}^2(R^3)$ and $|x|^{1/2}u_0, |x|^{1/2}B_0 \in L^2(R^3)$. Then the weak solution satisfies that

$$\frac{1}{2}\int_{R^3} \Big(|u(x,t)|^2 + |B(x,t)|^2\Big)|x|dxdt + \int_0^t \int_{R^3} \Big(|\nabla u|^2 + |\nabla B|^2\Big)|x|dxdt \le F(t) \qquad (8.3)$$

for almost every $t > 0$ with $F(t) = \||x|^{1/2}u_0\|_2^2 + \||x|^{1/2}B_0\|_2^2 + C(\|u_0\|_2^2 + \|B_0\|_2^2)t^{1/2} + C(\|u_0\|_2^2 + \|B_0\|_2^2)^{3/2}t^{1/4}$.

Proof. Let $\chi(r)$ be the smooth cut-off function on $r \ge 0$, such that $0 \le \chi \le 1$, $\chi \equiv 1$ for $r \le 1$ and $\chi \equiv 0$ for $r \ge 2$. For constants $0 < \varepsilon \ll \lambda < 1$, we set

$$\phi(x) = \frac{1}{2}(\lambda^2 + |x|^2)^{1/2}\chi((\varepsilon/\lambda)|x|)$$

in (2.2). Then (8.3) follows by the discussion similar to the proof of Lemma 8.2 in [1]. □

Proof of Theorem 2.4. We follow the discussion in [1]. By the first order interpolation inequality with weight and (8.3), one has that

$$\||x|^{1/2}u\|_{10/3}^{10/3} \le C\||x|^{1/2}u\|_2^{4/3}\||x|^{1/2}\nabla u\|_2^2$$

$$\||x|^{1/2}u\|_{10/3}^{10/3} \le C\||x|^{1/2}u\|_2^{4/3}\||x|^{1/2}\nabla u\|_2^2.$$

Therefore, one obtains that

$$\int_0^t \||x|^{1/2}u\|_{10/3}^{10/3} ds \le CF(t)^{5/3}$$

$$\int_0^t \||x|^{1/2}u\|_{10/3}^{10/3} ds \le CF(t)^{5/3}. \qquad (8.4)$$

Let $r = \sqrt{t}$. Then the Proposition 8.1 implies that, if (x,t) is a singular point, then

$$\iint_{Q_r(x,t)} |u|^3 dxdt \ge \varepsilon_5 t \quad \text{or} \quad \iint_{Q_r(x,t)} |B|^3 dxdt \ge C_{10} t \qquad (8.5)$$

and

$$t \le N_0(\|u_0\|_2^2 + \|B_0\|_2^2)^{3/2}. \qquad (8.6)$$

Now let $R = |x|$. If $R \ge 2r$, by the Hölder inequality, (8.4) and (8.5),

$$\varepsilon_5 t \le Cr^{1/2}\|u\|_{10/3}^3 \le Cr^{1/2}R^{-3/2}F(t)^{5/3} \le Ct^{1/4}R^{-3/2},$$

or

$$C_{10}t \le Cr^{1/2}\|B\|_{10/3}^3 \le Cr^{1/2}R^{-3/2}F(t)^{5/3} \le Ct^{1/4}R^{-3/2},$$

in which the fact (8.6) has been used. Thus,

$$|x|^2 t = R^2 t \le C_{11}.$$

If $R \le 2r$, then

$$|x|^2 t \le 4t^2.$$

Therefore,

$$|x|^2 t \le N_1 = \max\{C_{11}, 4N_0^2(\|u_0\|_2^2 + \|B_0\|_2^2)^3\},$$

which shows that Theorem 2.4 is valid. $\square$

To prove Theorem 2.5, we need an a priori estimate,

Lemma 8.3. Let $u_0, B_0 \in \overset{o}{J}{}^2(R^3)$ and $|x|^{-1/2}u_0, |x|^{-1/2}B_0 \in L^2(R^3)$. Then there is an absolute constant $L_0 > 0$, such that, if $\||x|^{-1/2}u_0\|_2^2 + \||x|^{-1/2}B_0\|_2^2 = L < L_0$, then the inequality

$$\int_{R^3}\Big\{\frac{|u|^2+|B|^2}{|x-t\xi|} + (L_0 - L - |\xi|^2 t)\exp\{\frac{1}{L_0}\int_0^t\int_{R^3}\frac{|\nabla u|^2+|\nabla B|^2}{|x-t\xi|}dxdt\}\Big\}dx \le L_0 \quad (8.7)$$

holds for every $\xi \in R^3$ and $t > 0$ with $|\xi|^2 t < L_0 - L$.

The proof is similar to that of Lemma 8.3 in [1]. Here we omit the details.

The Proof of Theorem 2.5. We follow the discussion in [1]. We want to show that $|\nabla u| + |\nabla B|$ is is bounded at (x_0, t_0) whenever

$$|x_0|^2 < t_0(L_0 - L). \quad (8.8)$$

To this end, setting $\xi = t_0^{-1}x_0$ in (8.7), one gets that

$$\int_0^t\int_{R^3}\frac{|\nabla u|^2}{|x-t\xi|}dxdt \le +\infty \quad (8.9)$$

only if

$$t \le t_0^2\frac{L_0 - L}{|x_0|^2}.$$

And (8.8) shows that (8.9) is valid for some $t > t_0$. For any $(x,t) \in Q_r(x_0,t_0)$, then

$$|x - t\xi| \le |x - t_0\xi| + |t - t_0||\xi| \le r + r^2|\xi|.$$

If $r|\xi| \le 1$, then

$$\max_{t_0 - r^2 \le t \le t_0} r^{-1}\int_{B_r(x_0,t_0)}|B|^2 dx \le 2\max_{t_0 - r^2 \le t \le t_0}\int_{R^3}\frac{|B|^2}{|x-t\xi|}dx \le 2L_0, \quad (8.10)$$

and

$$r^{-1}\iint_{Q_r(x_0,t_0)}|\nabla u|^2 dxdt \le 2\iint_{Q_r(x_0,t_0)}\frac{|\nabla u|^2}{|x-t\xi|}dxdt.$$

From the property of the absolute continuity of the integral and (8.9), one can conclude that

$$\lim_{r\to 0^+}\frac{1}{r}\iint_{Q_r(x_0,t_0)}|\nabla u|^2 dxdt = 0. \tag{8.11}$$

Therefore, from (8.10) and (8.11), Theorem 2.3 implies the result.

Corollary. Let $u_0, B_0 \in \overset{o}{J}{}^2(R^3)$. If, for some $R>0$,

$$\int_{|x|>R}\left(|\nabla u_0|^2+|\nabla B_0|^2\right)dx < +\infty,$$

then, $|\nabla u|+|\nabla B|$ is bounded in the region $\{|x|>R'\}$ for some $R'\ge R$.

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