Arnold diffusion in Hamiltonian Systems 1: *a priori* Unstable Case

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Abstract. By using variational method and under generic conditions we show that Arnold diffusion exists in *a priori* unstable and time-periodic Hamiltonian systems with multiple degrees of freedom.

1, Introduction

In this paper we consider *a priori* unstable and time-periodic Hamiltonian systems with arbitrary $n+1$ degrees of freedom. The Hamiltonian has the form

$$H(u,v,t)=h_1(p)+h_2(x,y)+P(u,v,t) \tag{1.1}$$

where $u=(q,x)$, $v=(p,y)$, $(p,q)\in\mathbb{R}\times\mathbb{T}$, $(x,y)\in\mathbb{T}^n\times\mathbb{R}^n$, P is a time-1-periodic small perturbation. $H\in C^r$ ($r\geq 3$) is assumed to satisfy the following hypothesis:

H1, h_1+h_2 is a convex function in v, i.e., Hessian matrix $\partial^2_{vv}(h_1+h_2)$ is positive definite. It is finite everywhere and has superlinear growth in v, i.e., $(h_1+h_2)/\|v\|\to\infty$ as $\|v\|\to\infty$.

H2, it is *a priori* unstable in the sense that the Hamiltonian flow $\Phi^t_{h_2}$ determined by h_2 has a non-degenerate hyperbolic fixed point $(x,y)=(0,0)$, the function $h_2(x,0):\mathbb{T}^n\to\mathbb{R}$ attains its strict maximum at $x=0$ mod 2π. We set $h_2(0,0)=0$.

Here, we do not assume the condition that the hyperbolic fixed point $(x,y)=(0,0)$ is connected to itself by its stable manifold and unstable manifold, i.e., $W^s(0,0)\equiv W^u(0,0)$. Such condition appears not natural when $n>1$.

Let $\mathcal{B}_{\epsilon,K}$ denote a ball in the function space $C^r(\{(u,v,t)\in\mathbb{T}^{n+1}\times\mathbb{R}^{n+1}\times\mathbb{T}:\|v\|\leq K\}\to\mathbb{R})$, centered at the origin with radius of ϵ. Now we can state the our main result of this paper, which is a higher dimensional version of the theorem formulated by Arnold in [Ar1] where it was assumed that $n=1$.

Theorem 1.1. *Let $A<B$ be two arbitrarily given numbers and assume H satisfies the above two conditions. There exist a small number $\epsilon>0$, a large number $K>0$ and a residual set in $\mathcal{S}_{\epsilon,K}\subset\mathcal{B}_{\epsilon,K}$ such that for each $P\in\mathcal{S}_{\epsilon,K}$ there exists an orbit of the Hamiltonian flow which connects the region with $p<A$ to the region with $p>B$.*

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In his celebrated paper [Ar1], Arnold constructed an example of nearly integrable Hamiltonian system with two and half degrees of freedom, in which there are some unstable orbits in the sense that the action along these orbits undergoes substantial variation. Such orbits are usually called diffusion orbits. Although this example does not have generic property, Arnold still asked whether there is such a phenomenon for a "typical" small perturbation (cf. [Ar2,Ar3]).

Variational method has its advantage in the study of Arnold diffusion problem, it needs less geometrical structure information of the system. In our previous paper [CY], by using variational arguments, we have shown that the diffusion orbits exist in generic *a priori* unstable Hamiltonian systems with two and half degrees of freedom. Mather has announced ([Ma5]) that, under so-called cusp residual condition, Arnold diffusion exists in *a priori* stable systems with two degrees of freedom in time-periodic case, or with three degrees of freedom in autonomous case. Some announcement was also made in [Xi] earlier. Using geometrical method, some demonstration was provided in [DLS] as well as in [Tr] to show that diffusion orbits exist in some types of *a priori* unstable and time-periodic Hamiltonian systems with two degrees of freedom.

In this paper we still use variational arguments to construct diffusion orbits. In order to use variational method, we put the problem of consideration into Lagrangian formalism. Using Legendre transformation $\mathcal{L}^*: H \to L$ we obtain the Lagrangian

$$L(u, \dot{u}, t) = \max_{v} \{\langle v, \dot{u}\rangle - H(u, v, t)\}. \tag{1.2}$$

Here $\dot{u} = \dot{u}(u, v, t)$ is implicitly determined by $\dot{u} = \frac{\partial H}{\partial v}$. We denote by $\mathcal{L}: (u, v, t) \to (u, \dot{u}, t)$ the coordinate transformation determined by the Hamiltonian H.

Roughly speaking, we construct diffusion orbits by connecting different Mañé sets, along which the Lagrange action attains its local minimum. To construct local connecting orbits between different Mañé sets, we introduce so-called pseudo connecting orbit sets. These sets contain the minimal configurations of some modified Lagrangian which do not necessarily generate orbits determined by the Lagrangian L. Based on the upper semi-continuity of the set functions, from Lagrangian to Mañé set and to pseudo connecting orbit set, and on the understanding of these sets with respect to the configuration manifold and its finite covering, we show that each configuration in the pseudo connecting orbit set generates a real orbit of the Lagrangian L which connects some Mañé set to another Mañé set nearby if this Mañé set has some kind of topological triviality. Such construction does not need the manifold structure of the Mather sets, and is applicable to systems with arbitrary degrees of freedom. Thus, some global connecting orbits can be constructed if some so-called generalized transition chain is established. Such a chain does exist in the system we study in this paper.

In the Lagrangian formalism, the Hamiltonian equation (1.1) is equivalent to the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) - \frac{\partial L}{\partial u} = 0. \tag{1.3}$$

This equation corresponds to the critical point of the functional

$$A_c(\gamma) = \int (L - \eta_c)(\gamma, \dot{\gamma}, t)dt,$$

where η_c is a closed 1-form whose de-Rham cohomology $[\eta_c] = c \in H^1(M, \mathbb{R})$. L and η_c can be thought as the function defined on $TM \times \mathbb{T}$.

To apply the Mather theory on positive definite Lagrangian systems we introduce a modified Lagrangian

$$\tilde{L} = L_0(\dot{u})\rho(\dot{u}) + (1 - \rho(\dot{u}))L(u, \dot{u}, t),$$

in which $L_0(\dot{u})$ is strictly convex in $\dot{u}$ and has super-linear growth in $\|\dot{u}\|$; $\rho(\dot{u}) = 1$ when $\|\dot{u}\| \geq 2K$, $\rho(\dot{u}) = 0$ when $\|\dot{u}\| \leq K$. We choose sufficiently large K so that the diffusion orbits we search for remain in the region $\{\|\dot{u}\| \leq K\}$. Clearly, we can choose some $\rho(\dot{u})$ so that $\tilde{L}$ is convex in $\dot{u}$ also. This system is integrable near infinity, so each solution is defined for all $t \in \mathbb{R}$. Therefore we can assume that the Lagrangian L satisfies the conditions suggested by Mather [Ma3]:

Positive definiteness. For each $(u, t) \in M \times \mathbb{T}$, the Lagrangian function is strictly convex in velocity: the Hessian $L_{\dot{u}\dot{u}}$ is positive definite;

Super-linear growth. We suppose that L has fiber-wise super-linear growth: for each $(u, t) \in M \times \mathbb{T}$, we have $L/\|\dot{u}\| \to \infty$ as $\|\dot{u}\| \to \infty$.

Completeness. All solutions of the Lagrangian equations are well defined for all $t \in \mathbb{R}$.

Let $I = [a, b]$ be a compact interval of time. A curve $\gamma \in C^1(I, M)$ is called a c-minimizer or a c-minimal curve if it minimizes the action among all curves $\xi \in C^1(I, M)$ which satisfy the same boundary conditions:

$$A_c(\gamma) = \min_{\substack{\xi(a)=\gamma(a) \\ \xi(b)=\gamma(b)}} \int_a^b (L - \eta_c)(d\xi(t), t)dt.$$

If J is a non compact interval, the curve $\gamma \in C^1(J, M)$ is said a c-minimizer if $\gamma|_I$ is c-minimal for any compact interval $I \subset J$. An orbit $X(t)$ of Φ^t is called c-minimizing if the curve $\pi \circ X$ is c-minimizing, where the operator π is the standard projection from tangent bundle to the underling manifold along the fibers, a point $(z, s) \in TM \times \mathbb{R}$ is c-minimizing if its orbit $\Phi^t(z, s)$ is c-minimizing. We use $\tilde{\mathcal{G}}_L(c) \subset TM \times \mathbb{R}$ to denote the set of minimal orbits of $L - \eta_c$ (the c-minimal orbits of L). We shall drop the subscript L when it is clear which Lagrangian is under consideration. It is not necessary to assume the periodicity of L in t for the definition of $\tilde{\mathcal{G}}$. When it is periodic in t, $\tilde{\mathcal{G}}(c) \subset TM \times \mathbb{R}$ is a nonempty compact subset of $TM \times \mathbb{T}$, invariant for the Euler-Lagrange flow ϕ_L^t.

The definition of action along a C^1-curve can be extended to the action on a probability measure. Let $\mathfrak{M}$ be the set of Borel probability measures on $TM \times \mathbb{T}$. For each $\nu \in \mathfrak{M}$, the action $A_c(\nu)$ is defined as the following:

$$A_c(\nu) = \int (L - \eta_c) d\nu.$$

Mather has proved [Ma3] that for each first de Rham cohomology class c there is a probability measure μ which minimizes the actions over $\mathfrak{M}$

$$A_c(\mu) = \inf_{\nu \in \mathfrak{M}} \int (L - \eta_c) d\nu.$$

This μ is invariant to the Euler-Lagrange flow. We use $\tilde{\mathcal{M}}(c)$ to denote the support of the measure and call it Mather set. We use $-\alpha(c) = A_c(\mu)$ to denote the minimum c-action, it defines a function α: $H^1(M, \mathbb{R}) \to \mathbb{R}$, usually called α-function. Its Legendre transformation β: $H_1(M, \mathbb{R}) \to \mathbb{R}$ is usually called β-function. Both functions are convex, finite everywhere and have super-linear growth [Ma3].

To define Aubry set and Mañé set we let

$$\begin{aligned}
h_c((m,t),(m',t')) = \min_{\substack{\gamma \in C^1([t,t'],M) \\ \gamma(t)=m, \gamma(t')=m'}} \int_t^{t'} (L - \eta_c)(d\gamma(s), s) ds + (t'-t)\alpha(c), \\
F_c((m,s),(m',s')) = \inf_{\substack{t = s \bmod 1 \\ t' = s' \bmod 1 \\ t'-t \geq 1}} h_c((m,t),(m',t')) \\
h_c^\infty((m,s),(m',s')) = \liminf_{\substack{s = t \bmod 2\pi \\ s' = t' \bmod 2\pi \\ t'-t \to \infty}} h_c((m,t),(m',t')), \\
h_c^k(m,m') = h_c((m,0),(m',k)), \\
h_c^\infty(m,m') = h_c^\infty((m,0),(m',0)), \\
F_c(m,m') = F_c((m,s),(m',s')) \\
d_c(m,m') = h_c^\infty(m,m') + h_c^\infty(m',m).
\end{aligned}$$

It was showed in [Ma4] that d_c is a pseudo-metric on the set $\{x \in M : h_c^\infty(x,x) = 0\}$. A curve $\gamma \in C^1(\mathbb{R}, M)$ is called c-semi-static if

$$A_c(\gamma|_{[a,b]}) + \alpha(c)(b-a) = F_c(\gamma(a), \gamma(b), a \bmod 1, b \bmod 1)$$

for each $[a,b] \subset \mathbb{R}$. A curve $\gamma \in C^1(\mathbb{R}, M)$ is called c-static if, in addition

$$A_c(\gamma|_{[a,b]}) + \alpha(c)(b-a) = -F_c(\gamma(b), \gamma(a), b \bmod 1, a \bmod 1)$$

for each $[a,b] \subset \mathbb{R}$. An orbit $X(t) = (d\gamma(t), t \bmod 1)$ is called c-static (semi-static) if γ is c-static (semi-static). We call the Mañé set $\tilde{\mathcal{N}}(c)$ the union of global c-semi-static orbits, the set $\tilde{\mathcal{A}}(c)$ is defined as the union of global c-static orbits, we call it Aubry set. We can also define corresponding Aubry sets and Mañé sets for some covering manifold $\tilde{M}$ respectively. Obviously, the c-static (semi-static) orbits for $\tilde{M}$ is not necessarily c-static (semi-static) for M.

We use $\mathcal{M}(c)$, $\mathcal{A}(c)$, $\mathcal{N}(c)$ and $\mathcal{G}(c)$ to denote the standard projection of $\tilde{\mathcal{M}}(c)$, $\tilde{\mathcal{A}}(c)$, $\tilde{\mathcal{N}}(c)$ and $\tilde{\mathcal{G}}(c)$ from $TM\times\mathbb{T}$ to $M\times\mathbb{T}$ respectively. We have the following inclusions ([Be])

$$\tilde{\mathcal{M}}(c)\subseteq\tilde{\mathcal{A}}(c)\subseteq\tilde{\mathcal{N}}(c)\subseteq\tilde{\mathcal{G}}(c).$$

It was showed in [Ma4] that the inverse of the projection is Lipschitz when it is restricted to $\mathcal{A}(c)$ and $\mathcal{M}(c)$.

In the following we use the symbol $\tilde{\mathcal{N}}_s(c)=\tilde{\mathcal{N}}(c)|_{t=s}$ to denote the time section of a Mañé set, and so on. We use Φ^t_H, Φ_H to denote the Hamiltonian flow generated by H and its time-1-map, and ϕ^t_L, ϕ_L to denote the Lagrangian flow generated by L and its time-1-map respectively.

This paper is organized as follows. In the section 2 we introduce so-called pseudo connecting orbit set and establish the upper semi-continuity of these sets. Such property shall be used to show the existence of local minimal orbits connecting some Mañé set to another Mañé set nearby. In the section 3, we investigate the topological structure of the Mañé sets and the pseudo connecting orbit sets, they correspond to those cohomology classes through which the diffusion orbits shall be constructed. The Mañé set becomes larger if we consider a finite covering of the manifold. In the section 4, by making use of the upper semi-continuity of Mañé sets, the existence of local connecting orbits is established if the Mañé set has some kind of triviality. The section 5 is devoted to the construction of diffusion orbits if there is a so-called generalized transition chain along the corresponding path in the first de-Rham cohomology space. To show the generic condition we establish some Hölder continuity of the barrier functions in the section 6, with which the generic property is proved in the last section.

In this paper, the only assumption on h_2 is the existence of a non-degenerate minimal fixed point with respect to the Lagrangian flow $\mathcal{L}(\Phi^t_{h_2})$. Thus, the main result and the method developed in this paper can be applied to study the diffusion in *a priori* stable Hamiltonian systems with arbitrary n degrees of freedom. We shall present it in our next paper.

2, Upper semi-continuity

The construction of diffusion orbits depends on the upper semi-continuity of some set functions.

Lemma 2.1. *We assume $L\in C^r(TM\times\mathbb{R},\mathbb{R})$ $(r\geq 2)$ satisfies the positive definite, superlinear-growth and completeness conditions, where M is a compact, connected Riemanian manifold. Considered as the function of t, L is assumed periodic for $t\in(-\infty,0]$ and for $t\in[1,\infty)$. Then the map $L\to\tilde{\mathcal{G}}_L\subset TM\times\mathbb{R}$ is upper semi-continuous. As an immediate consequence, $\tilde{\mathcal{G}}(c)$ is a non-empty compact set in $TM\times\mathbb{T}$ and the map $c\to\tilde{\mathcal{G}}(c)$ is upper semi-continuous if L is periodic in t.*

The proof of this lemma was provided in [Be2,CY]. We can consider t is defined on $(\mathbb{T}\vee[0,1]\vee\mathbb{T})/\sim$, where $\sim$ is defined by identifying $\{0\}\in[0,1]$ with some point on one circle and identifying $\{1\} \in [0,1]$ with some point on another circle. Let $U_k = \{(\zeta, q, t) : (q,t) \in M \times (\mathbb{T} \vee [0,1] \vee \mathbb{T})/\sim, \ \|\zeta\| \le k, \}$, $\cup_{k=1}^{\infty} U_k = TM \times \mathbb{R}$. Let $L_i \in C^r(TM \times \mathbb{T}, \mathbb{R})$. We say L_i converges to L if for each $\epsilon > 0$ and each U_k there exists i_0 such that $\|L - L_i\|_{U_k} \le \epsilon$ if $i \ge i_0$.

In the application, the set $\tilde{\mathcal{G}}(c)$ seems too big to be used for the construction of connecting orbits in interesting problems. Mañé sets seem good candidates. In the time-periodic case, Mañé set can be a proper subset of $\tilde{\mathcal{G}}(c)$, $\tilde{\mathcal{N}}(c) \subsetneq \tilde{\mathcal{G}}(c)$. It is closely related to the problem whether the Lax-Oleinik semi-group converges or not, some example can be found in [FM]. To establish some connection between two Mañé sets we consider a modified Lagrangian

$$L_{\eta,\mu,\psi} = L - \eta - \mu - \psi$$

where η is a closed 1-form on M such that $[\eta] = c$, μ is a 1-form depending on t in the way that the restriction of μ on $\{t \le 0\}$ is 0, the restriction on $\{t \ge 1\}$ is a closed 1-form $\bar{\mu}$ on M with $[\bar{\mu}] = c' - c$. ψ is a function on $M \times \mathbb{R}$ and $\psi = 0$ in $(-\infty, 0] \cup [1, \infty)$. Let $m, m' \in M$, we define

$$h_{\eta,\mu,\psi}^{T_0,T_1}(m,m') = \inf_{\substack{\gamma(-T_0)=m \\ \gamma(T_1)=m'}} \int_{-T_0}^{T_1} L_{\eta,\mu,\psi}(d\gamma(t),t)dt + T_0\alpha(c) + T_1\alpha(c').$$

Clearly $\exists\ m^* \in M$ and some constants $C_{\eta,\mu}$, $C_{\eta,\mu,\psi}$, independent of T_0, T_1, such that

$$\begin{aligned} h_{\eta,\mu,\psi}^{T_0,T_1}(m,m') &\le h_c^{T_0}(m,m^*) + h_{c'}^{T_2}(m^*,m') + C_{\eta,\mu} \\ &\le C_{\eta,\mu,\psi}. \end{aligned}$$

Thus its limit infimum is bounded

$$h_{\eta,\mu,\psi}^{\infty}(m,m') = \liminf_{T_0,T_1\to\infty} h_{\eta,\mu,\psi}^{T_0,T_1}(m,m') \le C_{\eta,\mu,\psi}.$$

Let $\{T_0^i\}_{i\in\mathbb{Z}_+}$ and $\{T_1^i\}_{i\in\mathbb{Z}_+}$ be the sequence of positive integers such that $T_j^i \to \infty$ $(j = 0,1)$ as $i \to \infty$ and the following limit exists

$$\lim_{i\to\infty} h_{\eta,\mu,\psi}^{T_0^i,T_1^i}(m,m') = h_{\eta,\mu,\psi}^{\infty}(m,m').$$

Let $\gamma_i(t,m,m')$: $[-T_0^i, T_1^i] \to M$ be a minimizer connecting m and m'

$$h_{\eta,\mu,\psi}^{T_0^i,T_1^i}(m,m') = \int_{-T_0^i}^{T_1^i} L_{\eta,\mu,\psi}(d\gamma_i(t),t)dt + T_0^i\alpha(c) + T_1^i\alpha(c').$$

It is not difficult to see that for any compact interval $[a,b]$ there is some $I \in \mathbb{Z}_+$ such that the set $\{\gamma_i\}_{i\ge I}$ is pre-compact in $C^1([a,b],M)$.

Lemma 2.2. *Let γ: $\mathbb{R} \to M$ be an accumulation point of $\{\gamma_i\}$. If $s \geq 1$ then*

$$A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau]) = \inf_{\substack{\tau_1-\tau\in\mathbb{Z},\tau_1>s \\ \gamma^*(s)=\gamma(s) \\ \gamma^*(\tau_1)=\gamma(\tau)}} \int_s^{\tau_1} L_{\eta,\mu,\psi}(d\gamma^*(t),t)dt + (\tau_1-\tau)\alpha(c'); \tag{2.1a}$$

if $\tau \leq 0$ then

$$A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau]) = \inf_{\substack{s_1-s\in\mathbb{Z},s_1<\tau \\ \gamma^*(s_1)=\gamma(s) \\ \gamma^*(\tau)=\gamma(\tau)}} \int_{s_1}^{\tau} L_{\eta,\mu,\psi}(d\gamma^*(t),t)dt - (s_1-s)\alpha(c); \tag{2.1b}$$

if $s \leq 0$ and $\tau \geq 1$ then

$$\begin{aligned} A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau]) = &\inf_{\substack{s_1-s\in\mathbb{Z},\tau_1-\tau\in\mathbb{Z} \\ s_1\leq 0,\tau_1\geq 1 \\ \gamma^*(s_1)=\gamma(s) \\ \gamma^*(\tau_1)=\gamma(\tau)}} \int_{s_1}^{\tau_1} L_{\eta,\mu,\psi}(d\gamma^*(t),t)dt \\ &- (s_1-s)\alpha(c) - (\tau_1-\tau)\alpha(c'). \end{aligned} \tag{2.1c}$$

Proof: To show that let us suppose the contrary, for instance, (2.15b) does not hold. Thus there would exist $\Delta > 0$, $s < \tau \leq 0$, $s_1 < \tau \leq 0$, $s_1 - s \in \mathbb{Z}$ and a curve γ^*: $[s_1,\tau] \to M$ with $\gamma^*(s_1) = \gamma(s)$, $\gamma^*(\tau) = \gamma(\tau)$ such that

$$A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau]) \geq \int_{s_1}^{\tau} L_{\eta,\mu,\psi}(d\gamma^*(t),t)dt - (s_1-s)\alpha(c) + \Delta.$$

Let $\epsilon = \frac{1}{4}\Delta$. By the definition of limit infimum there exist $T_0^{i_0} > 0$ and $T_1^{i_0} > 0$ such that

$$h_{\eta,\mu,\psi}^{T_0,T_1}(m_0,m_1) \geq h_{\eta,\mu,\psi}^{\infty}(m_0,m_1) - \epsilon, \qquad \forall\ T_0 \geq T_0^{i_0},\ T_1 \geq T_1^{i_0}, \tag{2.2}$$

there exist subsequences $T_j^{i_k}$ $(j = 0,1,\ k = 0,1,2,\cdots)$ such that for all $k > 0$

$$T_0^{i_k} - T_0^{i_0} \geq s - s_1, \tag{2.3}$$

$$|h_{\eta,\mu,\psi}^{T_0^{i_k},T_1^{i_k}}(m_0,m_1) - h_{\eta,\mu,\psi}^{\infty}(m_0,m_1)| < \epsilon. \tag{2.4}$$

By taking a further subsequence we can assume $\gamma_{i_k} \to \gamma$. In this case, we can choose sufficiently large k such that $\gamma_{i_k}(s)$ and $\gamma_{i_k}(\tau)$ are so close to $\gamma(s)$ and $\gamma(\tau)$ respectively that we can construct a curve $\gamma_{i_k}^*$: $[s_1,\tau] \to M$ which has the same endpoints as γ_{i_k}: $\gamma_i^*(s_1) = \gamma_i(s)$, $\gamma_i^*(\tau) = \gamma_i(\tau)$ and satisfies the following

$$A_{L_{\eta,\mu,\psi}}(\gamma_{i_k}|[s,\tau]) \geq \int_{s_1}^{\tau} L_{\eta,\mu,\psi}(d\gamma_{i_k}^*(t),t)dt - (s_1-s)\alpha(c) + \frac{3}{4}\Delta. \tag{2.5}$$

Let $T_0' = T_0^{i_k} + (s - s_1)$, if we extend $\gamma^*_{i_k}$ to $\mathbb{R} \to M$ such that

$$\gamma^*_{i_k} = \begin{cases} \gamma_{i_k}(t - s_1 + s), & t \le s_1, \\ \gamma^*_{i_k}(t), & s_1 \le t \le \tau, \\ \gamma_{i_k}(t), & t \ge \tau, \end{cases}$$

then we obtain from (2.4) and (2.5) that

$$\begin{aligned} h^{T_0', T_1^{i_k}}_{\eta,\mu,\psi}(m_0, m_1) \le & A_{L_{\eta,\mu,\psi}}(\gamma^*_{i_k}|[-T_0', T_1^{i_k}]) - T_1^{i_k}\alpha(c') - T_0'\alpha(c) \\ \le & A_{L_{\eta,\mu,\psi}}(\gamma_{i_k}|[-T_0^{i_k}, T_1^{i_k}]) - T_1^{i_k}\alpha(c') - T_0^{i_k}\alpha(c) - \frac{3}{4}\Delta \\ \le & h^{\infty}_{\eta,\mu,\psi}(m_0, m_1) - 2\epsilon. \end{aligned}$$

but this contradicts (2.2) since $T_0' \ge T_0^{i_0}$ and $T_1^{i_k} \ge T_1^{i_0}$, guaranteed by (2.3). (2.1a) and (2.1c) can be proved in the same way. □

With this lemma it is natural to define

$$\tilde{\mathcal{C}}_{\eta,\mu,\psi} = \{d\gamma \in \tilde{\mathcal{G}}_{L_{\eta,\mu,\psi}} : \ (2.1a)\ (2.1b) \text{ and } (2.1c) \text{ hold } \}.$$

Although the elements in this set are not necessarily the orbits of the Lagrangian flow determined by L, the α-limit set of each element in this set is contained in $\tilde{\mathcal{N}}(c)$, the ω-limit set is contained in $\tilde{\mathcal{N}}(c')$. Due to this reason, we call it pseudo connecting orbit set. Obviously $\tilde{\mathcal{C}}_{\eta,0,0} = \tilde{\mathcal{N}}(c)$. For convenience we may drop the subscript ψ in the symbol when it is equal to zero, i.e. $\tilde{\mathcal{C}}_{\eta,\mu} := \tilde{\mathcal{C}}_{\eta,\mu,0}$.

Lemma 2.3. *The map $(\eta,\mu,\psi) \to \tilde{\mathcal{C}}_{\eta,\mu,\psi}$ is upper semi-continuous. $\tilde{\mathcal{C}}_{\eta,0,0} = \tilde{\mathcal{N}}(c)$ if $[\eta] = c$. Consequently, the map $c \to \tilde{\mathcal{N}}(c)$ is upper semi-continuous.*

Proof: Let $\eta_i \to \eta$, $\mu_i \to \mu$ and $\psi_i \to \psi$ let $\gamma_i \in \tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}$ and let γ be an accumulation point of the set $\{\gamma_i \in \tilde{\mathcal{C}}_{\eta_i,\mu_i,,\psi_i}\}_{i \in \mathbb{Z}^+}$. Clearly, $\gamma \in \tilde{\mathcal{C}}_{\eta,\mu,\psi}$. If $\gamma \notin \tilde{\mathcal{C}}_{\eta,\mu,\psi}$ there would be two point $\gamma(s)$,$\gamma(\tau) \in M$ such that one of the following three possible cases takes place. Either $\gamma(s)$ and $\gamma(\tau) \in M$ can be connected by another curve γ^*: $[s+n,\tau] \to M$ with smaller action

$$A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau]) < A_{L_{\eta,\mu,\psi}}(\gamma^*|[s+n,\tau]) - n\alpha(c)$$

in the case $\tau < 0$; or there would a curve γ^*: $[s,\tau+n] \to M$ such that

$$A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau]) < A_{L_{\eta,\mu,\psi}}(\gamma^*|[s,\tau+n]) - n\alpha(c')$$

in the case $s \ge 1$, or when $s \le 0$ and $\tau \ge 1$ there would be a curve γ^*: $[s+n_1, \tau+n_2] \to M$ such that

$$A_{L_{\eta,\mu,\psi}}(\gamma|[s,\tau]) < A_{L_{\eta,\mu,\psi}}(\gamma^*|[s+n_1,\tau+n_2]) - n_1\alpha(c) - n_2\alpha(c')$$

where $s + n_1 \le 0$, $\tau + n_2 \ge 1$. Since γ is an accumulation point of γ_i, for any small $\epsilon > 0$, there would be sufficiently large i such that $\|\gamma - \gamma_i\|_{C^1[s,t]} < \epsilon$, it follows that $\gamma_i \notin \tilde{\mathcal{C}}_{\eta_i,\mu_i,\psi_i}$ but that is absurd.

Let us consider the case that $\mu=0$ and $\psi=0$. In this case, $L-\eta$ is periodic in t. If some orbit $\gamma\in\tilde{\mathcal{C}}_{\eta,0,0}$: $\mathbb{R}\to M$ is not semi-static, then there exist $s<\tau\in\mathbb{R}$, $n\in\mathbb{Z}$, $\Delta>0$ and a curve γ^*: $[s,\tau+n]\to M$ such that $\gamma^*(s)=\gamma(s)$, $\gamma^*(\tau+n)=\gamma(\tau)$ and

$$A_{L_{\eta,0,0}}(\gamma|[s,\tau])\geq A_{L_{\eta,0,0}}(\gamma^*|[s,\tau+n])-n\alpha(c)+\Delta.$$

We can extend γ^* to $[s_1,\tau_1+n]\to M$ such that $s_1\leq\min\{s,0\}$, $\min\{\tau_1,\tau_1+n\}\geq 1$, $\tau_1\geq\tau$ and

$$\gamma^*=\begin{cases}\gamma(t), & s_1\leq t\leq s,\\ \gamma^*(t), & s\leq t\leq \tau+n,\\ \gamma(t-n), & \tau+n\leq t\leq\tau_1+n.\end{cases}$$

Since $L-\eta$ is periodic in t, we would have

$$A_{L_{\eta,0,0}}(\gamma|[s_1,\tau_1])\geq A_{L_{\eta,0,0}}(\tau^*\gamma|[s_1,\tau_1+n])-n\alpha(c)+\Delta.$$

but this contradicts to (2.1c). $\square$

The upper semi-continuity of the map $(\eta,\mu,\psi)\to\tilde{\mathcal{C}}_{\eta,\mu,\psi}$ will be fully exploited to construct connecting orbits between two different Mañé sets if they are closed to each other.

3, Structure of some $\tilde{\mathcal{N}}(c)$ and $\tilde{\mathcal{C}}_{\eta,\mu,\psi}$

It is natural to study the topological structure of the relevant Mañé sets if we want to construct the connecting orbits between them.

Let L be the Lagrangian obtained from H in (1.1) by the Legendre transformation, it has the form as follows:

$$L(u,\dot u,t)=\ell_1(\dot q)+\ell_2(x,\dot x)+L_1(u,\dot u,t), \tag{3.1}$$

here $\ell_i=\mathcal{L}^*(h_i)$ for $i=1,2$, L_1 is a small perturbation. The perturbation term of the Lagrangian L_1 and the perturbation term of the Hamiltonian P is related by an operator $\Delta\mathcal{L}^*$ induced by the Legendre transformation $L_1=\Delta\mathcal{L}^*(P)=\mathcal{L}^*(h_1+h_2+P)-\mathcal{L}^*(h_1+h_2)$. We denote by $\dot{\mathcal{B}}_{\epsilon,K}$ the ball in $C^r(\{(u,\dot u,t)\in\mathbb{T}^{k+n}\times\mathbb{R}^{k+n}\times\mathbb{T}:\|\dot u\|\leq K\}\to\mathbb{R})$, centered at the origin with radius of ϵ. Obviously, when $\epsilon\leq 1$, there exists $\varrho>0$ such that

$$\Delta\mathcal{L}^*(\mathcal{B}_{\epsilon,2K})\subset\dot{\mathcal{B}}_{\varrho\epsilon,K}.$$

Let $c=(c_q,c_x)$ denote a cohomology class in $H^1(\mathbb{T}^{k+n},\mathbb{R})$ where $c_q\in\mathbb{R}^k$ and $c_x\in\mathbb{R}^n$. To obtain the result of this paper we choose $k=1$, but the demonstration in the following 3, 4 and 5 sections applies for arbitrary k.

Lemma 3.1. *Given some large number $K>0$ and a small number $\delta>0$. There exists a small number $\epsilon=\epsilon(\delta)>0$, if $c_q\in\{\max_{1\leq i\leq k}|c_{q_i}|\leq K\}$ and if $P\in\mathcal{B}_{\epsilon,2K}$, then there exists an n-dimensional convex set $\mathcal{D}(c_q)$ which contains $\{c_q=$* constant, $\|c_x\|<C_x\}$ *with small $C_x>0$ such that*

1, for each $c \in \{\|c_x\| < C_x\}$*, the Mañé set* $\tilde{\mathcal{N}}(c) \subset \{\|x\| < \delta\}$*;*

2, for each $c \in \text{int}(\mathcal{D}(c_q))$*, the Mather set* $\tilde{\mathcal{M}}(c) \subset \{\|x\| < \delta\}$*, for each* $c \in \{c_q = \text{constant}\}\backslash\overline{\mathcal{D}(c_q)}$ *and each c-minimal measure* μ*,* $\rho_x(\mu) \neq 0$*;*

3, if $\mathcal{M}(c)$ *is uniquely ergodic for each* $c \in \text{int}(\mathcal{D}(c_q))$*, then* $\mathcal{N}(c) \subset \{\|x\| < \delta\}$ *for each* $c \in \text{int}(\mathcal{D}(c_q))$*.*

The interior of $\mathcal{D}(c_q)$ is in the sense that we think $\mathcal{D}(c_q)$ as a set in $\mathbb{R}^n$. We denote the rotation vector of μ by $\rho(\mu) = (\rho_q(\mu), \rho_x(\mu))$.

Proof: Note $\min_{\dot{u}}(L - \langle c, \dot{u}\rangle) = -h_1(c_q) - h_2(x, c_x) - P(u, c, t)$. Given $0 < \lambda < 1$, by the assumption (**H2**) we find that there exist $E_1' > 0$ and $C_x' > 0$ such that if $\|c_q\| \leq K$ $\|c_x\| \leq C_x'$, if $\epsilon' \leq E_1'\lambda^2\delta^2$ and if $P \in \mathcal{B}_{\epsilon', 2K}$, then

$$L - \langle c, \dot{u}\rangle + h_1(c_q) \geq 3\epsilon', \qquad \forall\ (u, \dot{u}, t) \in \{\|x\| > \lambda\delta\},$$

$$L - \langle c, \dot{u}\rangle + h_1(c_q) \leq 2\epsilon', \qquad \forall\ (u, \dot{u}, t) \in \Big\{\|x\| < \frac{1}{2}\lambda\delta,\ \dot{x} = 0, \|c\| \leq K\Big\}.$$

This implies that for each c-minimal measure μ we have

$$\int_{TM\times\mathbb{T}\backslash\{\|x\|\leq\lambda\delta\}} d\mu = 0$$

Thus, for each c-semi static curve $\gamma : \mathbb{R} \to M$ there exist two sequences $t_j^{\pm} \to \infty$ as $j \to \infty$ so that $\gamma(-t_j^-), \gamma(t_j^+) \in \{\|x\| \leq \lambda\delta\}$.

To continue the proof, let us consider the Lagrangian ℓ_2 first. To each absolutely continuous curve $\gamma_x : [t_0, t_1] \to \mathbb{T}^n$ with $\gamma_x(t_0), \gamma_x(t_1) \in \{\|x\| = \lambda\delta\}$ we can associate an element $[\gamma_x] \in H_1(\mathbb{T}^n, \{\|x\| \leq \lambda\delta\}, \mathbb{Z}) = \mathbb{Z}^n$. Let $|[\gamma_x]| = \sum_{i=1}^n |[\gamma_x]_i|$. If $\gamma_x(t) \notin \{\|x\| < \lambda\delta\}$ for all $t \in (t_0, t_1)$ and there is some $t^* \in (t_0, t_1)$ such that $\gamma_x(t^*) \notin \{\|x\| < \delta\}$, then there exist $E_1, E_2 > 0$ such that

$$\int_{t_1}^{t_2} \ell_2(d\gamma_x(t))dt \geq E_1\lambda^2\delta^2(t_1 - t_0) + E_2(\delta^2 + |[\gamma_x]|). \tag{3.2}$$

Here, we have made use of the super-linear growth in $\dot{x}$ and the the hypothesis (**H2**). Let $\xi_x : [t_0, t_1] \to \mathbb{T}^n$ $(t_1 - t_0 \geq 1)$ be a minimal curve of ℓ_2 joining two points in $\{\|x\| \leq \lambda\delta\}$. Since $\{x = \dot{x} = 0\}$ is hyperbolic, we see that $[\xi_x] = 0$ and there exists $E_3 > 0$ such that

$$\int_{t_1}^{t_2} \ell_2(d\xi_x(t))dt \leq E_3\lambda^2\delta^2. \tag{3.3}$$

We choose

$$\lambda \leq \sqrt{\frac{E_2}{2E_3}}, \qquad C_x \leq \min\left\{C_x', \frac{E_2}{2\pi}\right\}, \qquad \epsilon \leq \min\left\{\frac{E_1'}{\varrho}, \frac{E_1}{4}\right\}\lambda^2\delta^2. \tag{3.4}$$

If δ is suitably small, if $P\in\mathcal{B}_{\epsilon,2K}$ and if $\gamma=(\gamma_q,\gamma_x):\mathbb{R}\to\mathbb{T}^k\times\mathbb{T}^n$ is a c-semi static curve for L such that $\gamma(t_0),\gamma(t_1)\in\{\|x\|\le\delta\}$, $\gamma(t)\notin\{\|x\|\le\delta\}$ and $[\gamma_x]\neq 0$ we find $t_1-t_0\ge 1$. In this case, we construct a curve $\xi=(\xi_q,\xi_x):\mathbb{R}\to\mathbb{T}^k\times\mathbb{T}^n$ such that $\xi_q(t)=\gamma_q(t)$,

$$\xi_x(t)=\begin{cases}\gamma_x(t) & t\le t_0,\\ \xi_x(t) & t_0\le t\le t_1,\\ \gamma_x(t) & t\ge t_1.\end{cases}$$

In this case we obtain from (3.2), (3.3) and (3.4) that

$$\begin{aligned}&\int_{t_0}^{t_1}\Big(L(d\gamma(t),t)-L(d\xi(t),t)-\langle c,\dot\gamma(t)-\dot\xi(t)\rangle\Big)dt\\ \ge&\int_{t_0}^{t_1}\Big(\ell_2(d\gamma_x(t))-\ell_2(d\xi_x(t))\Big)dt-2\pi C_x|[\gamma_x]|-2\epsilon(t_1-t_0)\\ \ge&\frac{1}{2}(E_1\lambda^2\delta^2(t_1-t_0)+E_2\delta^2)>0,\end{aligned}$$

but this contradicts the fact that γ is c-semi static. This proves the first part of the lemma.

To continue the proof, we define

$$\begin{aligned}\mathcal{D}(c_q)=\Big\{&c\in H^1(\mathbb{T}^k\times\mathbb{T}^n,\mathbb{R}):c_q=\text{constant},\\ &\exists\, c\text{-minimal measure }\mu\text{ such that }\rho_x(\mu)=0\Big\}.\end{aligned}$$

Obviously, it is an n-dimensional convex disk and contains $\{c_q=\text{constant},\ \|c_x\|\le C_x\}$. In fact, if μ is a c-minimal measure for some $c\in\text{int}(\mathcal{D}(c_q))$ then it is also a $(c_q,0)$-minimal measure. To see it, let us note a fact:

Proposition 3.2. *Let $c',c^*\in H^1(M,\mathbb{R})$, μ' and μ^* be the corresponding minimal measures respectively. If $\langle c'-c^*,\rho(\mu')\rangle=\langle c'-c^*,\rho(\mu^*)\rangle=0$, then $\alpha(c')=\alpha(c^*)$.*

Proof: By the definition of the α-function we find that

$$\begin{aligned}-\alpha(c')=&\int(L-\eta_{c'})d\mu'\\ =&\int(L-\eta_{c^*})d\mu^*+\langle c^*-c',\rho(\mu)\rangle\\ \ge&-\alpha(c^*).\end{aligned}$$

In the same way, we have $-\alpha(c^*)\ge-\alpha(c')$. $\square$

It follows from this proposition that $\alpha(c) =$ constant for all $c \in \mathcal{D}(c_q)$. For each $c \in \text{int}(\mathcal{D}(c_q))$ if there was a c-minimal measure μ_1 such that $\rho_x(\mu_1) \neq 0$, then $\exists\, c' = (c_q, c'_x) \in \text{int}(\mathcal{D}(c_q))$ such that $\langle c_x - c'_x, \rho_x(\mu_1)\rangle < 0$. Thus

$$\begin{aligned}-\alpha(c^*) =& A_{c^*}(\mu_1)\\ =& A_{c'}(\mu_1) + \langle c_x - c'_x, \rho(\mu_1)\rangle\\ >& -\alpha(c').\end{aligned}$$

On the other hand, from the definition of $\mathcal{D}(c_q)$ and from the proposition 3.2 we obtain that $\alpha(c') = \alpha(c^*)$. The contradiction implies that $\rho_x(\mu) = 0$ for each c-minimal measure when $c \in \text{int}(\mathcal{D}(c_q))$. Consequently, for each c-minimal measure μ

$$\int (L - \eta_c) d\mu = \int (L - \eta_{c_q}) d\mu,$$

here, $\eta_c = (\eta_{c_q}, \eta_{c_x})$ is any closed 1-form such that $[\eta_c] = (c_q, c_x) \in H^1(\mathbb{T}^{k+n}, \mathbb{R})$. Therefore, $\text{supp}(\mu) \subset \{\|x\| \le \delta\}$. This proves the second part of the lemma.

Finally, let us consider the case that the c-minimal measure μ_c is always uniquely ergodic for each $c \in \text{int}(\mathcal{D}(c_q))$. It is easy to see that $\exists\ \mu$ such that $\mu = \mu_c$ for all $c \in \text{int}(\mathcal{D}(c_q))$. Let $d\gamma \in \tilde{\mathcal{N}}(c)$. Note $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{A}}(c)$ in this case. For each $\xi \in \mathcal{M}_0(c)$, if $k_{ij} \to \infty$ $(i = 1, 2)$ as $j \to \infty$ are the two sequences such that $d\gamma(-k_{1j}), d\gamma(k_{2j}) \to \pi^{-1}(\xi)$, then we claim that

$$\lim_{j\to\infty} \int_{-k_{1j}}^{k_{2j}} \dot\gamma_{x_i}(t) dt = 0, \qquad \forall\ 1 \le i \le n. \tag{3.5}$$

In fact, for any $\xi \in \mathcal{M}_0(c)$ there exist two sequences $k_{ij} \to \infty$ as $j \to \infty$ $(i = 1, 2)$ such that $d\gamma(-k_{1j}) \to \pi^{-1}(\xi)$ and $d\gamma(k_{2j}) \to \pi^{-1}(\xi)$ as $j \to \infty$. Since γ is c-static, it follows that

$$h_c^{k_{1j}}(\gamma(-k_{1j}), \gamma(0)) + h_c^{k_{2j}}(\gamma(0), \gamma(k_{2j})) \to 0.$$

If (3.5) does not hold, by choosing a subsequence again (we use the same symbol) there would be some $1 \le i \le n$ such that

$$\left| \lim_{j\to\infty} \int_{-k_1 j}^{k_2 j} \dot\gamma_{x_i}(t) dt \right| \ge 2\pi > 0.$$

In this case, let us consider the barrier function $B^*_{c'}$ where all other components of $c' \in \mathbb{R}^{k+n}$ are the same as those of c except for the component for x_j. Since $c - c' = (0, \cdots, 0, c_{x_i} - c'_{x_i}, 0, \cdots, 0)$, we obtain from the proposition 3.2 that $\alpha(c') = \alpha(c)$, so

$$\begin{aligned}B_{c'}(\gamma(0)) \le& \liminf_{j\to\infty} \int_{-k_{1j}}^{k_{2j}} (L(d\gamma(t), t) - \langle c', \dot\gamma(t)\rangle + \alpha(c'))\, dt\\ \le& \liminf_{j\to\infty} \int_{-k_{1j}}^{k_{2j}} (L(d\gamma(t), t) - \langle c, \dot\gamma(t)\rangle + \alpha(c))\, dt\\ &+ (c_{x_i} - c'_{x_i}) \lim_{j\to\infty} \int_{-k_{1j}}^{k_{2j}} \dot\gamma_{x_i}(t) dt\\ \le& -2|c_{x_i} - c'_{x_i}|\pi < 0\end{aligned}$$

as we can choose $c_{x_i} > c'_{x_i}$ or $c_{x_i} < c'_{x_i}$ accordingly. But this is absurd since barrier function is non-negative.

Let us derive from (3.5) that there is no c-semi-static orbit that is not contained in $\{\|x\| \leq \delta\}$. In fact, we find that $d\gamma \in \tilde{\mathcal{N}}((c_q, 0))$. To see that, we obtain from (3.5) that the term $\langle c_x, \dot{\gamma}_x \rangle$ has no contribution to the action along the curve $\gamma|_{[-k_{1j}, k_{2j}]}$:

$$\int_{-k_{1j}}^{k_{2j}} (L - \langle c_q, \dot{\gamma}_q \rangle - \langle c_x, \dot{\gamma}_x \rangle) dt \to \int_{-k_{1j}}^{k_{2j}} (L - \langle c_q, \dot{\gamma}_q \rangle) dt, \tag{3.6}$$

as $j \to \infty$. Note $k_{ij} \to \infty$ as $j \to \infty$ $(i = 1, 2)$. If $d\gamma \notin \tilde{\mathcal{N}}((c_1, 0))$, there would exist $j' \in \mathbb{Z}^+$, $k' \in \mathbb{Z}$, $E > 0$ and a curve ζ: $[-k_{1j}, k_{2j} + k'] \to M$ such that $\zeta(-k_{1j'}) = \gamma(-k_{1j'})$, $\zeta(k_{2j'} + k') = \gamma(k_{2j'})$

$$\begin{aligned} &\int_{-k_{1j'}}^{k_{2j'}} (L(d\gamma(t), t) - \langle c_q, \dot{\gamma}_q \rangle + \alpha(c)) dt \\ \geq &\int_{-k_{1j'}}^{k_{2j'} + k'} (L(d\zeta(t), t) - \langle c_q, \dot{\zeta}_q \rangle + \alpha(c)) dt + E \\ \geq &F_{(c_q, 0)}(\gamma(-k_{1j'}), \gamma(k_{2j'})) + E \end{aligned} \tag{3.7}$$

and

$$\left| \int_{-k_{1j'}}^{k_{2j'} + k'} \dot{\zeta}_{x_i} dt \right| \to 0, \qquad \forall\, 1 \leq i \leq n. \tag{3.8}$$

The second condition (3.8) follows from the facts that $\tilde{\mathcal{N}}((c_q, 0)) \subset \tilde{\Sigma}$ and $\gamma(-k_{ij}) \to \xi \in \mathcal{M}_0((c_q, 0)) = \mathcal{M}_0(c)$. Let $j - j'$ be sufficiently large, we construct a curve ζ': $[-k_{1j}, k_{2j} + k'] \to M$ such that

$$\zeta'(t) = \begin{cases} \gamma(t), & t \in [-k_{1j}, -k_{1j'}]; \\ \zeta(t), & t \in [-k_{1j'}, k_{2j'} + k']; \\ \gamma(t - k'), & t \in [k_{2j'} + k', k_{2j} + k']. \end{cases}$$

It follows from (3.5∼8) that

$$\begin{aligned} \int_{-k_{1j}}^{k_{2j} + k'} (L(d\zeta'(t), t) - \langle c, \dot{\zeta}' \rangle) dt &< \int_{-k_{1j}}^{k_{2j}} (L(d\gamma(t) - \langle c_q, \dot{\gamma}_q \rangle) dt - E \\ &\leq \int_{-k_{1j}}^{k_{2j}} (L(d\gamma(t), t) - \langle c, \dot{\gamma} \rangle) dt - \frac{E}{2}, \end{aligned}$$

but this contradicts to the property that $d\gamma \in \tilde{\mathcal{N}}(c)$. □

Remark: The first part of the lemma can be proved by using the upper-semi continuity of Mañé set on Lagrangian functions. But the dependence on ϵ is not so clear as here (cf (3.4)) if we prove it in that way.

From the proof of the first part of the lemma 3.1 we can see

Lemma 3.3. *Let $c \in \{\|c_x\| < C_x\}$ and $b - a \geq 1$. For small number $d > 0$ there exits $\epsilon > 0$ and $\delta > 0$, such that if $|L_1| \leq \epsilon$ and if γ: $[a, b] \to \mathbb{T}^{k+n}$ is a c-minimizer connecting points $\gamma(a), \gamma(b) \in \{\|x\| \leq \delta\}$, then $\|\gamma_x(t)\| < d$ for all $t \in [a, b]$.*

The structure of Mañé set and pseudo connecting orbit set depends on what configuration manifold we choose for our consideration. In the following, when necessary, we use $\tilde{\mathcal{N}}(c, M)$, $\tilde{\mathcal{C}}_{\eta,\mu,\psi}(M)$ to specify the manifold on which these sets are defined. We shall omit M in that symbol when it is clearly defined. We do not intend to consider the general case. Instead, let us consider some special case which is sufficient for the purpose of this paper. According to the lemma 3.1, for a cohomology class $c = (c_q, 0)$, the support of the c-minimal measure is contained in $N_\delta = \{\|x\| \leq \delta\}$, a δ-neighborhood of the lower dimensional torus. To each curve γ: $(a, b) \to M$ such that $\gamma(a) \in N_\delta$ and $\gamma(b) \in N_\delta$ we can associate an element $[\gamma] = ([\gamma]_1, [\gamma]_2, \cdots, [\gamma]_n) \in H_1(M, N_\delta, \mathbb{Z})$. Here, a can be a finite number or $-\infty$, b can be a finite number or ∞. From the proof in [Be1] we can see that there exists a homoclinic orbit $d\gamma$ such that the first component of its relative homology is not zero: $[\gamma]_1 \neq 0$.

Lemma 3.4. *Let $c = (c_q, 0)$, $\tilde{M} = \mathbb{T}^k \times (2\mathbb{T}) \times \mathbb{T}^{n-1}$. If γ: $\mathbb{R} \to M$ is a minimal homoclinic curve such that*

$$\begin{aligned}
&\liminf_{\substack{T_0\to\infty\\ T_1\to\infty}} \Big\{ \int_{-T_0}^{T_1} (L - \eta_c)(d\gamma(t), t)dt + (T_0 + T_1)\alpha(c) \Big\} \\
=&\liminf_{\substack{T_0\to\infty\\ T_1\to\infty}} \min_{\substack{\xi(-T_0)\in N_\delta\\ \xi(T_1)\in N_\delta\\ [\xi]_1\neq 0}} \Big\{ \int_{-T_0}^{T_1} (L - \eta_c)(d\xi(t), t)dt + (T_0 + T_1)\alpha(c) \Big\},
\end{aligned}$$

then $\{d\gamma(t), t\} \subset \tilde{\mathcal{N}}(c, \tilde{M})$.

Proof: If we think $\tilde{M}$ as the configuration manifold, the minimal measure has at least two ergodic components. The lift of a homoclinic orbit to $T\tilde{M}$ is an orbit joining the different lift of the support of the minimal measure. Recall the definition of the barrier function introduced by Mather in [Ma4]

$$B_c^*(m) = \min\{h_c^\infty(\xi, m) + h_c^\infty(m, \zeta) - h_c^\infty(\xi, \zeta) : \forall\ \xi, \zeta \in \mathcal{M}_0(c)\},$$

we obtain the result immediately. $\square$

Mañé announced in [Me] that there should be a transition chain of semi-static orbits among different ergodic components of the minimal measure, it has been partially proved in [CP].

We can also define the Mañé set $\tilde{\mathcal{N}}(c, \tilde{M})$ from another point of view. Let $c = (c_q, 0)$, $\xi \in N_\delta$, $\zeta \in N_\delta$, we define

$$h^k_{c,e_1}(\xi, \zeta) = \inf_{\substack{\gamma(0)=\xi\\ \gamma(k)=\zeta\\ [\gamma]_1\neq 0}} \int_0^k (L - \eta_c)(d\gamma(t), t)dt + k\alpha(c),$$

$$h^{k_1,k_2}_{c,e_1}(\xi,m,\zeta)=\inf_{\substack{\gamma(-k_1)=\xi\\ \gamma(0)=m\\ \gamma(k_2)=\zeta\\ [\gamma]_1\neq 0}}\int_{-k_1}^{k_2}(L-\eta_c)(d\gamma(t),t)dt+(k_1+k_2)\alpha(c),$$

$$h^{\infty}_{c,e_1}(\xi,\zeta)=\liminf_{k\to\infty}h^{k}_{c,e_1}(\xi,\zeta),$$

$$h^{\infty}_{c,e_1}(\xi,m,\zeta)=\liminf_{k\to\infty}h^{k}_{c,e_1}(\xi,m,\zeta),$$

$$B^*_{c,e_1}(m)=\inf\{h^{\infty}_{c,e_1}(\xi,m,\zeta)-h^{\infty}_{c,e_1}(\xi,\zeta):\ \xi,\zeta\in\mathcal{M}_0(c)\}.$$

Let π_1: $\tilde{M}\to M$ be the standard projection. Clearly, we have

Lemma 3.5. *Assume L has the form of (3.1), $c=(c_q,0)$, then*

$$\pi_1\mathcal{N}_0(c,\tilde{M})=\{B^*_{c,e_1}=0\}\cup\{B^*_c=0\},$$

$$\pi_1\mathcal{N}_0(c,\tilde{M})\backslash\mathcal{N}_0(c,M)\neq\varnothing.$$

Recall we have introduced a modified Lagrangian $L_{\eta,\mu,\psi}=L-\eta-\mu-\psi$. Let $T_0\in\mathbb{Z}_+$, $T_1\in\mathbb{Z}_+$, we define

$$h^{T_0,T_1}_{\eta,\mu,\psi,e_1}(m_0,m_1)=\inf_{\substack{\xi(-T_0)=m_0\\ \xi(T_1)=m_1\\ [\xi]_1\neq 0}}\int_{-T_0}^{T_1}L_{\eta,\mu,\psi}(d\xi(t),t)+T_0\alpha(c)+T_1\alpha(c').$$

Let us study $\tilde{\mathcal{C}}_{\eta,\mu,\psi}(\tilde{M})$:

Lemma 3.6. *Let $c=(c_q,0)$, $c'=(c'_q,0)$, $[\eta]=c$ and $[\bar{\mu}]=c'-c$. If ψ is suitably small, then*

$$\pi_1\mathcal{C}_{\eta,\mu,\psi}(\tilde{M})\backslash\mathcal{C}_{\eta,\mu,\psi}(M)\neq\varnothing.$$

Proof: For m_0, $m_1\in N_\delta$, positive integers T_0^i, $T_1^i\in\mathbb{Z}_+$, we let $\gamma_i(t,m_0,m_1,e_1)$: $[-T_0^i,T_1^i]\to M$ be a minimal curve joining m_0 and m_1 such that $[\gamma_i]_1\neq 0$ and

$$h^{T_0^i,T_1^i}_{\eta,\mu,\psi,e_1}(m_0,m_1)=\int_{-T_0^i}^{T_1^i}L_{\eta,\mu,\psi}(d\gamma_i(t),t)dt+T_0^i\alpha(c)+T_1^i\alpha(c').$$

Let $\{T_0^i\}_{i\in\mathbb{Z}_+}$ and $\{T_1^i\}_{i\in\mathbb{Z}_+}$ be the sequence of positive integers such that $T_j^i\to\infty$ $(j=0,1)$ as $i\to\infty$ and the following limit exists

$$\lim_{i\to\infty}h^{T_0^i,T_1^i}_{\eta,\mu,\psi,e_1}(m_0,m_1)=\liminf_{T_0,T_1\to\infty}h^{T_0,T_1}_{\eta,\mu,\psi,e_1}(m_0,m_1)=h^{\infty}_{\eta,\mu,\psi,e_1}(m_0,m_1).$$

Let $\tilde{\gamma}_i$ be the lift of γ_i in the covering space $\tilde{M}$, it is a $\tilde{M}$-minimal curve. Clearly, the set of accumulation points of the set $\{\gamma_i\}$ contains a curve γ: $\mathbb{R}\to M$ with $[\gamma]_1\neq 0$.

On the other hand, if $|\psi|$ is suitably small and m_0, $m_1 \in N_\delta$, the hyperbolic structure of ℓ_2 guarantees that

$$h^{\infty}_{\eta,\mu,\psi}(m_0,m_1) < h^{\infty}_{\eta,\mu,\psi,e_1}(m_0,m_1).$$

In other words, these $\tilde{M}$-minimal curves $\{\gamma_i\}$ are not M-minimal curve. Consequently, γ is not a M-minimal curve. This completes the proof. □

4, Existence of local connecting orbits

To begin with, let us consider the construction of diffusion orbits in Arnold's example from the variational point of view. There, each Mañé set under consideration properly contains the corresponding Mather set if we study the problem in a covering manifold $\tilde{M} = \mathbb{T} \times 2\mathbb{T}$. Under the small perturbation the stable manifold of each invariant circle transversally intersects the unstable manifold of the same invariant circle. It implies that the set $\pi_1\mathcal{N}_0(c,\tilde{M})\backslash(\mathcal{M}_0(c,M)+\delta)$ is non-empty but topologically trivial for each c under consideration. The main goal of this section is to show that if a system has such a property for some c, then for all c' sufficiently close to c, $\tilde{\mathcal{N}}_{c'}$ can be connected with $\tilde{\mathcal{N}}_c$ by some local minimal orbits.

Definition 4.1. *Let $c=(c_q,0)$, $c'=(c'_q,0)$, $N_\delta=\{\|x\|\le\delta\}$. We assume that $\tilde{\mathcal{N}}(c)\subset \mathrm{int}(N_\delta)$ and $\tilde{\mathcal{N}}(c')\subset \mathrm{int}(N_\delta)$. Let γ: $\mathbb{R}\to M$ be an absolutely continuous curve such that $\gamma(t)\in N_\delta$ when $|t|\ge T$, a sufficiently large number, and such that $[\gamma]_1\neq 0$. We say $d\gamma$ is a local minimal orbit of L that connects $\tilde{\mathcal{N}}(c)$ and $\tilde{\mathcal{N}}(c')$ if*

1, $d\gamma(t)$ is the solution of the Euler-Lagrange equation (1.3), the α- and ω-limit sets of $d\gamma$ are in $\tilde{\mathcal{N}}(c)$ and $\tilde{\mathcal{N}}(c')$ respectively.

2, There exist two open and connected sets V_0 and V_1 such that $\bar{V}_0\subset N_\delta\backslash\mathcal{M}_0(c)$, $\bar{V}_1\subset N_\delta\backslash\mathcal{M}_0(c')$, $\gamma(-T_0)$ and $\gamma(T_1)$ are in the interior of V_0 and V_1 respectively, where $T_0\in\mathbb{Z}_+$ and $T_1\in\mathbb{Z}_+$. For any $(m_0,m_1)\in\partial(V_0\times V_1)$, we have

$$\begin{aligned}\min\Big\{&h^{T_0,T_1}_{\eta,\mu,\psi}(m_0,m_1,e_1)+h^{\infty}_c(\xi,m_0)+h^{\infty}_{c'}(m_1,\zeta):\\ &\xi\in\mathcal{M}_0(c)\cap\pi(\alpha(d\gamma)|_{t=0}),\zeta\in\mathcal{M}_0(c')\cap\pi(\omega(d\gamma)|_{t=0})\Big\}\\ -\liminf_{\substack{T'_0\to\infty\\ T'_1\to\infty}}&\int_{-T'_0}^{T'_1}(L-\eta-\mu-\psi)(d\gamma(t),t)dt-T'_0\alpha(c)-T'_1\alpha(c')\\ &>0,\end{aligned}\tag{4.1}$$

where ψ is a non-negative function $M\times\mathbb{R}\to\mathbb{R}^+$ such that $\psi=0$ when $t\le 0$ or when $t\ge 1$; μ is a 1-form on $M\times\mathbb{R}$ such that $\mu=0$ when $t\le 0$, $\mu=\bar{\mu}$ is closed when $t\ge 1$, $[\bar{\mu}]=c'-c$, η is a closed 1-form on M with $[\eta]=c$.

Since $\pi(\omega(d\gamma))\subset\mathcal{N}(c')\subset N_\delta$ and $\pi(\alpha(d\gamma))\subset\mathcal{N}(c)\subset N_\delta$, $[\gamma|_{T_1\le t<\infty}]$ and $[\gamma|_{-\infty<t\le -T_0}]$ are well defined. Indeed, recall the lemma 3.3, we can see $[\gamma|_{T_1\le t<\infty}]=0$ and $[\gamma|_{-\infty<t\le -T_0}]=0$. That is why we use $h^{\infty}_c(\xi,m_0)$ and $h^{\infty}_{c'}(m_1,\zeta)$ in this definition. We do not intend to discuss local minimal curves in the most general case, the above definition is introduced for the special purpose of this paper.

Obviously, (4.1) is equivalent to that

$$\begin{aligned} & h^{T_0,T_1}_{\eta,\mu,\psi}(m_0,m_1,e_1)+h^\infty_c(\xi,m_0)+h^\infty_{c'}(m_1,\zeta) \\ & -\int_{-T_0}^{T_1}(L-\eta-\mu-\psi)(d\gamma(t),t)dt-T_0'\alpha(c)-T_1'\alpha(c') \\ & -h^\infty_c(\xi,\gamma(-T_0))-h^\infty_{c'}(\gamma(T_1),\zeta) \\ & \quad >0, \end{aligned} \tag{4.1'}$$

holds for each $\xi\in\mathcal{M}_0(c)\cap\pi(\alpha(d\gamma)|_{t=0})$ and each $\zeta\in\mathcal{M}_0(c')\cap\pi(\omega(d\gamma)|_{t=0})$.

Lemma 4.2. *We assume that $|L_1|\le\epsilon$ so that Lemma 3.3 holds. Given a cohomology class $c=(c_q,0)$, we assume that $\pi_1\mathcal{N}_0(c,\tilde{M})\backslash N_\delta$ is totally disconnected, then there exists $\epsilon_1>0$ such that for each $c'=(c_q',0)$ with $\|c_q'-c_q\|\le\epsilon_1$, there exists an orbit $d\gamma$ of the Euler-Lagrange equation determined by L which connecting $\tilde{\mathcal{N}}_c$ with $\tilde{\mathcal{N}}_{c'}$. γ: $\mathbb{R}\to M$ is a local minimal curve for L in the sense of the definition 4.1.*

Proof: Since $\pi_1\mathcal{N}_0(c,\tilde{M})\backslash N_\delta$ is totally disconnected, there is an open, connected and homotopically trivial set O and a small positive number $t_0>0$ such that

$$O\cap\pi_1\mathcal{N}(c,\tilde{M})|_{0\le t\le t_0}\backslash N_\delta\neq\varnothing,\qquad \partial O\cap\pi_1\mathcal{N}(c,\tilde{M})|_{0\le t\le t_0}=\varnothing.$$

Clearly, we can define a non-negative function $f\in C^r(M,\mathbb{R})$ such that

$$f(q,x)\begin{cases} =0 & (q,x)\in N_\delta\cup(\pi_1\mathcal{N}(c,\tilde{M})|_{0\le t\le\tau}\backslash O), \\ =1 & (q,x)\in O, \\ <1 & \text{elsewhere.}\end{cases}$$

We choose a C^r-function $\rho:\mathbb{R}\to[0,1]$ such that $\rho=0$ on $t\in(-\infty,0]\cup[t_0,\infty)$, $0<\rho\le1$ on $t\in(0,t_0)$. Let $\lambda\ge0$ be a positive number,

$$\psi(q,x,t)=\lambda\rho(t)f(q,x),$$

By the upper semi-continuity of the set-valued function $(\eta,\mu,\psi)\to\mathcal{C}_{\eta,\mu,\psi}(\tilde{M})$ we see that $\mathcal{C}_{\eta,0,\psi}(\tilde{M})|_{0\le t\le t_0}\cap\partial O=\varnothing$ if $\lambda>0$ is suitably small. By the choice of ψ, we have $\tilde{\mathcal{C}}_{\eta,0,\psi}(M)=\tilde{\mathcal{N}}(c,M)$. Consequently, we obtain from the lemma 3.6 that

$$\varnothing\neq\Big\{\pi_1\mathcal{C}_{\eta,0,\psi}(\tilde{M})\backslash\mathcal{C}_{\eta,0,0}(M)\Big\}_{0\le t\le t_0}\subset O.$$

Since O is homotopically trivial, there exists a closed 1-form $\bar{\mu}$ such that $[\bar{\mu}]=c'-c$ and $\mathrm{supp}(\bar{\mu})\cap O=\varnothing$. Let $\rho_1\in C^r(\mathbb{R},[0,1])$ such that $\rho_1=0$ on $(-\infty,0]$, $0<\rho_1<1$ on $(0,t_0)$ and $\rho_1=1$ on $[t_0,\infty)$, let $\mu=\lambda_1\rho_1(t)\bar{\mu}$ and set

$$L_{\eta,\mu,\psi}=L-\eta-\mu-\psi.$$

By using the upper semi-continuity and the lemma 3.6 again we find that

$$\varnothing\neq\Big\{\pi_1\mathcal{C}_{\eta,\mu,\psi}(\tilde{M})\backslash\mathcal{C}_{\eta,\mu,\psi}(M)\Big\}_{0\le t\le t_0}\subset O$$

if $\lambda_1>0$ is suitably small. Let $\gamma\in\pi_1\mathcal{C}_{\eta,\mu,\psi}(\tilde{M})\backslash\mathcal{C}_{\eta,\mu,\psi}(M)$. Note that $f\equiv1$ in O, $\mathrm{supp}(\bar{\mu})\cap O=\varnothing$, $d\gamma$: $TM\to\mathbb{R}$ is obviously a solution of the Euler-Lagrange equation, $\alpha(d\gamma)\subset\tilde{\mathcal{N}}(c)$ and $\omega(d\gamma)\subset\tilde{\mathcal{N}}(c')$.

Since we have assumed that $\pi_1\mathcal{C}_0(c,\tilde{M})\backslash N_\delta$ is totally disconnected, by the upper semi-continuity, there obviously are two open and connected sets V_0 and V_1 such that $\bar{V}_0\subset N_\delta\backslash\mathcal{M}_0(c)$, $\bar{V}_1\subset N_\delta\backslash\mathcal{M}_0(c')$ and (4.1) holds. $\square$

Let us compare $\pi_1\mathcal{C}_{\eta,0,\psi}(\tilde{M})\backslash\mathcal{C}_{\eta,0,\psi}(M)$ with $\pi_1\mathcal{N}(c,\tilde{M})\backslash\mathcal{N}(c,M)$. If $\gamma(t)$ is a minimal curve in $\pi_1\mathcal{N}(c,\tilde{M})\backslash\mathcal{N}(c,M)$, then its time k translation $\gamma(t+k)$ is also a minimal curve for each $k\in\mathbb{Z}$. By the choice of the open set O and the function ψ, we see that each orbit $d\gamma$ in $\pi_1\tilde{\mathcal{N}}(c,\tilde{M})\backslash\tilde{\mathcal{N}}(c,M)$ may still an orbit of the Euler-Lagrange equation determined by $L-\psi$, but only those curves remain to be minimal if they pass through O when $t\in[0,t_0]$.

5, Construction of global connecting orbits

The goal of this section is to construct some orbits which connect $\tilde{\mathcal{N}}(c)$ with $\tilde{\mathcal{N}}(c')$ if c and c' are connected by a generalized transition chain in $H^1(\mathbb{T}^k\times\mathbb{T}^n,\mathbb{R})$.

Definition 5.1. *Let $\tilde{M}$ be a finite covering of a compact manifold M and let c, c' be two cohomolgy classes in $H^1(M,\mathbb{R})$. We say that c is joined with c' by a generalized transition chain if there is a continuous curve Γ: $[0,1]\to H^1(M,\mathbb{R})$ such that $\Gamma(0)=c$, $\Gamma(1)=c'$ and for each $\tau\in[0,1]$ at least one of the following cases takes place:*

(I), *$\mathcal{N}_0(\Gamma(\tau),M)$ is topologically trivial; and*

(II), *$\pi_1\mathcal{N}_0(\Gamma(\tau),\tilde{M})\backslash(\mathcal{N}_0(\Gamma(\tau),M)+\delta)$ is totally disconnected.*

In this paper, we do not intend to establish a theorem of the existence of connecting orbits between two cohomology classes in the most general case when they are joined by a generalized transition chain. Instead, we restrict ourselves to the special case:

Theorem 5.2. *Let $M=\mathbb{T}^k\times\mathbb{T}^n$, $\tilde{M}=\mathbb{T}^k\times 2\mathbb{T}\times\mathbb{T}^{n-1}$, the Lagrangian L be given by (3.1). We assume that two first cohomology classes $c=(c_q,0)$ and $c'=(c'_q,0)$ are joined by a generalized transition chain Γ: $[0,1]\to H^1(M,\mathbb{R})\cap\{c_x=0\}$. Moreover, $\pi_1\mathcal{N}_0(\Gamma(\tau),\tilde{M})\backslash(\mathcal{N}_0(\Gamma(\tau),M)+\delta)$ is totally disconnected, we assume $\mathcal{M}(\Gamma(\tau),M)$ is uniquely ergodic. There exist a large positive numbers $K>0$ and small positive number $\epsilon>0$ such that if $L_1\in\dot{\mathcal{B}}_{\epsilon,K}$, then there exists an orbit of the Euler-Lagrange equation (1.3) $d\gamma$: $\mathbb{R}\to TM$ that has the property: $\alpha(d\gamma)\subset\tilde{\mathcal{N}}(c)$ and $\omega(d\gamma)\subset\tilde{\mathcal{N}}(c')$.*

Remark: We feel that the uniquely ergodic condition here is not necessary, but this condition is automatically satisfied if $k=1$ as we shall see later.

Proof: Since the map $c\to\mathcal{N}(c,M)$ has upper semi-continuity, there are finite open intervals $\{I_i\}_{0\le i\le k}$ such that

1, $\cup I_i\supset[0,1]$, $I_i\cap I_{i+1}\neq\varnothing$ and $I_i\cap I_{i\pm2}=\varnothing$;

2, each I_i is defined in this way: if for all $t\in I_i$ the case (I) happens, then for all $t\in I_{i-1}\cup I_{i+1}$ the case (II) happens.

Without of losing generality and for the simplification of notation, we study the case that $I_1\cup I_2\cup I_3\supset[0,1]$, the case (II) takes place for each $t\in I_1\cup I_3$ while the case (I) takes place for each $t\in I_2$. The general case can be treated in the same way.

By the assumptions, there exists a finite sequence $\{s_i\}_{0\le i\le i_3}$ such that $s_i\in I_1$ for each integer $i\in[0,i_1]$, $s_i\in I_2$ for each integer $i\in[i_1,i_2]$ and $s_i\in I_3$ for each integer $i\in[i_2,i_3]$. For each integer $i\in[0,i_1]\cup[i_2,i_3]$:

1, there exists an orbit $d\gamma_i$: $\mathbb{R}\to TM$ of the Euler-Lagrange equation determined by L such that $\alpha(d\gamma)\subset\tilde{\mathcal{N}}(c_i)$ and $\omega(d\gamma)\subset\tilde{\mathcal{N}}(c_{i+1})$, where $c_i=\Gamma(s_i)$;

2, there is a non-negative function $\psi_i(q,x,t)\le\lambda_i$, a small number, such that $\psi_i=0$ when $t\le 0$ or when $t\ge t_0$. For each fixed t, $\psi=$ constant when it is restricted in an open, connected and homotopically trivial set O_i which has the properties:

$$O_i\cap\mathcal{N}(c_i,\tilde M)|_{0\le t\le t_0}\backslash N_\delta\neq\varnothing,\qquad \partial O_i\cap\mathcal{N}(c_i,\tilde M)|_{0\le t\le t_0}=\varnothing,$$

$$\gamma_i(t)|_{0\le t\le t_0}\in\mathrm{int}(O_i);$$

3, there exist a closed 1-forms η_i with $[\eta_i]=c_i$ and a 1-form μ_i depending on t in the way that the restriction of μ_i on $\{t\le 0\}$ is 0, the restriction on $\{t\ge\tau\}$ is a closed 1-form $\bar\mu_i$ on M such that $[\bar\mu_i]=c_{i+1}-c_i$. The support of μ_i is disjoint with $O_i\times[0,\tau]$. γ_i: $\mathbb{R}\to M$ is a local minimal curve of $L_{\eta,\mu,\psi}=L-\psi-\eta-\mu$ in the sense of the definition 4.1. Consequently, there exist two open $(k+n)$-dimensional topological disks V_i^+ and V_{i+1}^- with $\bar V_i^+\subset N_\delta\backslash\mathcal{M}_0(c_i)$, $\bar V_{i+1}^-\subset N_\delta\backslash\mathcal{M}_0(c_{i+1})$, two positive integers $\tilde T_i^0$, $\tilde T_i^1$ and a positive small number $\epsilon_i^*>0$ such that

$$\begin{aligned}\min\Big\{&h_{c_i}^\infty(\xi,m_0)+h_{\eta_i,\mu_i,\psi_i,e_1}^{\tilde T_i^0,\tilde T_i^1}(m_0,m_1)+h_{c_{i+1}}^\infty(m_1,\zeta):\\ &(m_0,m_1)\in\partial(V_i^+\times V_{i+1}^-)\Big\}\\ \ge\min\Big\{&h_{c_i}^\infty(\xi,m_0)+h_{\eta_i,\mu_i,\psi_i,e_1}^{\tilde T_i^0,\tilde T_i^1}(m_0,m_1)+h_{c_{i+1}}^\infty(m_1,\zeta):\\ &(m_0,m_1)\in V_i^+\times V_{i+1}^-\Big\}+5\epsilon_i^*,\end{aligned}\tag{5.1}$$

where $\xi\in\mathcal{M}_0(c_i),\zeta\in\mathcal{M}_0(c_{i+1})$. Note (5.1) is independent of the choice of ξ and ζ since ergodicity of $M(c_i)$ is assumed for each i.

For each integer $i\in[i_1,i_2]$, there exist two closed 1-forms η_i, $\bar\mu$ defined on M, a 1-form μ_i defined on $(u,t)\in M\times\mathbb{R}$ and an open set $U_i\subset M$ such that $[\eta_i]=c_i$, $[\bar\mu_i]=c_{i+1}-c_i$, $\mu_i=0$ when $t\le 0$, $\mu_i=\bar\mu_i$ when $t\ge t_0>0$, μ_i is closed on $U_i\times[0,t_0]$ and

$$\mathcal{C}_{\eta_i,\mu_i}(t)+\delta_i\subset U_i,\qquad \text{when } t\in[0,t_0]\tag{5.2}$$

where $\delta_i>0$ is a small number. The possibility of choosing μ_i and U_i in this way is the consequence of the following argument. By the assumption on I_2, for each $\tau\in I_2$, there is an open neighborhood U_τ of $\mathcal{N}_0(\Gamma(\tau))$ in the configuration manifold M such that such that the inclusion map $H_1(U_\tau,\mathbb{Z})\to H_1(M,\mathbb{Z})$ is the zero map, thus for each $c' \in H^1(M,\mathbb{R})$ there exists a closed 1-form $\bar\mu$ such that $[\bar\mu] = c' - \Gamma(\tau)$ and $\mathrm{supp}(\bar\mu) \cap U_\tau = \varnothing$. Let $\rho \in C^r(\mathbb{R},[0,1])$ such that $\rho = 0$ on $(-\infty, 0]$, $0 < \rho < 1$ on $(0, t_0)$ and $\rho = 1$ on $[t_0, \infty)$, let $\mu = \lambda\rho(t)\bar\mu$. The upper semi-continuity of the map $(\eta,\mu) \to \tilde{\mathcal{C}}_{\eta,\mu}$ guarantees that $\mathcal{C}_{\eta,\mu}(t) \subset U_\tau$ if λ is sufficiently small.

By the compactness of the manifold M, for a small $\epsilon_i^* > 0$ there exists $(\breve T_i^0, \breve T_i^1) = (\breve T_i^0, \breve T_i^1)(\epsilon_i^*) \in (\mathbb{Z}^+, \mathbb{Z}^+)$ such that

$$h^{T_0,T_1}_{\eta_i,\mu_i}(m_0,m_1) \ge h^\infty_{\eta_i,\mu_i}(m_0,m_1) - \epsilon_i^*, \tag{5.3}$$

holds for all $T_0 \ge T_i^0$, $T_1 \ge T_i^1$ and for all $(m_0,m_1) \in M \times M$. Obviously, given (m_0,m_1) there are infinitely many $T_0 \ge T_i^0$ and $T_1 \ge T_i^1$ such that

$$|h^{T_0,T_1}_{\eta_i,\mu_i}(m_0,m_1) - h^\infty_{\eta_i,\mu_i}(m_0,m_1)| \le \epsilon_i^*. \tag{5.4}$$

Let $\gamma_i(t,m_0,m_1,T_0,T_1) : [-T_0,T_1] \to M$ be the minimizer of $h^{T_0,T_1}_{\eta_i,\mu_i}(m_0,m_1)$, it follows from the lemma 2.2 that if $\epsilon_i^* > 0$ is sufficiently small, $T_0 > \breve T_i^0$ and $T_1 \breve T_i^1$ are chosen sufficiently large so that (5.4) holds, then

$$d\gamma_i(t,m_0,m_1,T_0,T_1) \in \tilde{\mathcal{C}}_{\eta_i,\mu_i}(t) + \delta_i, \qquad \forall\ 0 \le t \le 1. \tag{5.5}$$

From the Lipschitze property of $h^{T_0,T_1}_{\eta_i,\mu_i}(m_0,m_1)$ in (m_0,m_1) there exist $\hat T_j^0(\epsilon_i^*) > \breve T_i^0(\epsilon_i^*)$ and $\hat T_j^1(\epsilon_i^*) > \breve T_i^1(\epsilon_i^*)$ so that for each (m_0,m_1) there are $T_j = T_j(m_0,m_1)$ with $\breve T_i^j(\epsilon_i^*) \le T_j \le \hat T_i^j(\epsilon_i^*)$ $(j = 0,1)$ such that (5.4) and (5.5) hold. Note that for different (m_0,m_1) we may need different $T_j \ge \breve T_i^j$ $(j = 0,1)$.

Before we go back to consider those i with $0 \le i < i_1$ or $i_2 \le i < i_3$, let us observe some facts. We can define the set of forward and backward semi-static curves:

$$\tilde{\mathcal{N}}^+(c) = \{(z,s) \in TM \times \mathbb{T} : \pi \circ \phi_L^t(z,s)|_{[0,+\infty)} \text{ is } c\text{-semi-static}\},$$

$$\tilde{\mathcal{N}}^-(c) = \{(z,s) \in TM \times \mathbb{T} : \pi \circ \phi_L^t(z,s)|_{(-\infty,0]} \text{ is } c\text{-semi-static}\}.$$

Proposition 5.3. *If $\mathcal{M}(c)$ is uniquely ergodic, $u \in \mathcal{A}_0(c)$, then there exists a unique $v \in T_uM$ such that $(u,v) \in \tilde{\mathcal{N}}_0^+(c)$* (or $\tilde{\mathcal{N}}_0^-(c)$)*. Moreover, $(u,v) \in \tilde{\mathcal{A}}_0(c)$.*

Proof: Let us suppose the contrary. Then there would exist $(u,v) \in \tilde{\mathcal{A}}_0(c)$ and a forward c-semi-static curve $\gamma_+(t)$ with $\gamma_+(0) = u$ and $\dot\gamma_+(0) \ne v$. In this case, for any $u_1 \in \mathcal{M}_0(c)$ there exist two sequences $k_i, k_i' \to \infty$ such that $\pi \circ \phi_L^{k_i}(u,v) \to u_1, \gamma_+(k_i') \to u_1$ and

$$\begin{aligned} h_c^\infty(u,u_1) &= \lim_{k_i\to\infty} \int_0^{k_i} (L-\eta_c)(\phi_L^t(u,v),t)dt + k_i\alpha(c) \\ &= \lim_{k_i'\to\infty} \int_0^{k_i'} (L-\eta_c)(d\gamma_+(t),t)dt + k_i\alpha(c). \end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
&h_c^\infty(\pi\circ\phi_L^{-1}(u,v),u_1)\\
=&F_c(\pi\circ\phi_L^{-1}(u,v),u)+h_c^\infty(u,u_1)\\
=&F_c(\pi\circ\phi_L^{-1}(u,v),u)+\lim_{k_i'\to\infty}\int_0^{k_i'}(L-\eta_c)(d\gamma_+(t),t)dt\\
>&h_c^\infty(\pi\circ\phi_L^{-1}(u,v),u_1)
\end{aligned}$$

where the last inequality follows from the facts that $\dot\gamma_+(0)\neq v$ and the minimizer must be a C^1-curve. But this is absurd. □

Proposition 5.4. *Assume $\mathcal{M}(c)$ is uniquely ergodic, then for all $\zeta\in\mathcal{M}(c)$ and all $m_0,m_1\in M$, we have*

$$h_c^\infty(m_0,\zeta)+h_c^\infty(\zeta,m_1)=h_c^\infty(m_0,m_1).$$

Proof: By definition,

$$h_c^\infty(m_0,\zeta)+h_c^\infty(\zeta,m_1)\geq h_c^\infty(m_0,m_1).$$

for all $m_0,m_1,\zeta\in M$. Let γ_T: $[0,T]\to M$ be a c-minimal curve connecting m_0 with m_1. As $\mathcal{M}(c)$ is uniquely ergodic, for any $\epsilon>0$, there exists positive integer $T(\epsilon)$ such that for each integer $T\geq T(\epsilon)$ there is $T_1<T$ with the property $\gamma_T(T_1)\in\mathcal{M}(c)+\epsilon$. Let $T_2=T-T_1$. In this case, we have

$$h_c^T(m_0,m_1)=h_c^{T_1}(m_0,\gamma_T(T_1))+h_c^{T_2}(\gamma_T(T_1),m_1)$$

We claim that $T_1\to\infty$ and $T-T_1\to\infty$ as $\epsilon\to 0$. Indeed, if T_1 is bounded by some finite number, then there would be a point $u\in\mathcal{M}_0(c)$ and a vector $v\in T_uM$ such that $\phi^t(u,v)$ is a forward c-semi-static orbit as $t\to\infty$ with $(u,v)\notin\tilde{\mathcal{A}}(c)$. But this contradicts to the proposition 5.3. Clearly, there exist $\zeta\in\mathcal{M}(c)$ and a subsequence $\{T_i\}$ such that $\gamma_{T_i}(T_1)\to\zeta$ as $T_i\to\infty$. It implies that

$$h_c^\infty(m_0,\zeta)+h_c^\infty(\zeta,m_1)\leq h_c^\infty(m_0,m_1).$$

As $\mathcal{M}(c)$ is uniquely ergodic, for any $\xi\in\mathcal{A}(c)|_{t=0}$

$$\begin{aligned}
&h_c^\infty(m_0,\xi)+h_c^\infty(\xi,m_1)\\
=&h_c^\infty(m_0,\zeta)+h_c^\infty(\zeta,\xi)+h_c^\infty(\xi,\zeta)+h_c^\infty(\zeta,m_1)\\
=&h_c^\infty(m_0,\zeta)+h_c^\infty(\zeta,m_1).
\end{aligned}$$

This completes the proof. □

Let $m_0, m_1 \in M$, let γ_T: $[0,T] \to M$ be a c-minimizer connecting m_0 with m_1. For each integer $i \in [0, i_1] \cup [i_2, i_3]$, by the similar reason to obtain (5.3) and (5.4), there exists $\breve{T}_i(\epsilon_i^*) > 0$, independent of m_0 and m_1, such that

$$h_{c_i}^T(m_0, m_1) \geq h_{c_i}^\infty(m_0, \zeta) + h_{c_i}^\infty(\zeta, m_1) - \epsilon_i^*, \quad \forall\, T \geq \breve{T}_i(\epsilon_i^*),\ \zeta \in \mathcal{M}(c) \tag{5.6}$$

there exists $\hat{T}_i(\epsilon_i^*) > \breve{T}_i(\epsilon_i^*)$ such that for each (m_0, m_1) the following holds for some integer T such that $\breve{T}_i(\epsilon_i^*) \leq T \leq \hat{T}_i(\epsilon_i^*)$ and

$$|h_{c_i}^T(m_0, m_1) - h_{c_i}^\infty(m_0, \zeta) - h_{c_i}^\infty(\zeta, m_1)| \leq \epsilon_i^*, \quad \forall\, \zeta \in \mathcal{M}(c). \tag{5.7}$$

We define τ_i inductively for $0 \leq i \leq i_3$. We let $\tau_0 = 0$, for $0 < i < i_1$ and for $i_2 < i < i_3$ we choose τ_i such that

$$\breve{T}_i + \tilde{T}_{i-1}^1 + \tilde{T}_i^0 \leq \tau_i - \tau_{i-1} \leq \hat{T}_i + \tilde{T}_{i-1}^1 + \tilde{T}_i^0; \tag{5.8}$$

for $i_1 < i \leq i_2$ we choose those τ_i such that

$$\max\{\breve{T}_i^0, \breve{T}_{i-1}^1 + 1\} \leq \tau_i - \tau_{i-1} \leq \max\{\hat{T}_i^0, \hat{T}_{i-1}^1 + 1\}. \tag{5.9}$$

To consider the case that $i = i_1$ we note that both $\hat{T}_{i_1}$ and $\hat{T}_{i_1}^0$ can be taken large enough such that for any $m_0, m_1 \in M$ there exist $T(m_0, m_1)$, $T_0(m_0, m_1)$ with $\max\{\breve{T}_{i_1}, \breve{T}_{i_1}^0\} \leq T(m_0, m_1), T_0(m_0, m_1) \leq \max\{\hat{T}_{i_1}, \hat{T}_{i_1}^0\}$ such that (5.4) holds if we set $T_0 = T_0(m_0, m_1)$ there; (5.7) holds if we set $T = T(m_0, m_1)$ there; (5.3) holds for each $T_0 \geq T_0(m_0, m_1)$ and (5.6) holds for each $T \geq T(m_0, m_1)$. Thus, we choose

$$\tilde{T}_{i_1-1}^1 + \max\{\breve{T}_{i_1}, \breve{T}_{i_1}^0\} \leq \tau_{i_1} - \tau_{i_1-1} \leq \tilde{T}_{i_1-1}^1 + \max\{\hat{T}_{i_1}, \hat{T}_{i_1}^0\}. \tag{5.10}$$

For the same reason, we cam choose large enough $\hat{T}_{i_2}$ and $\hat{T}_{i_2}^1$ and set the range for τ_{i_2}:

$$\max\{\breve{T}_{i_2-1}^1, \breve{T}_{i_2}\} + \tilde{T}_{i_2}^0 \leq \tau_{i_2} - \tau_{i_2-1} \leq \max\{\hat{T}_{i_2-1}^1, \hat{T}_{i_2}\} + \tilde{T}_{i_2}^0 \tag{5.11}$$

We define an index set for $\vec{\tau} = (\tau_1, \tau_2, \cdots, \tau_{i_3-2}, \tau_{i_3-1})$:

$$\Lambda = \Big\{\vec{\tau} \in \mathbb{Z}^{i_3-1} :\ (5.8 \sim 11) \text{ hold}\Big\}.$$

Consider τ as the time translation $(q, x, t) \to (q, x, t + \tau)$ on $M \times \mathbb{R}$, let $\psi_i \equiv 0$ for $i_1 \leq i < i_2$, we define a modified Lagrangian

$$\tilde{L} = L - \eta_0 - \sum_{i=0}^{i_3-1} (-\tau_i)^*(\mu_i + \psi_i).$$

For $(m,m')\in M\times M$, $Z=(z_0^+,z_1^-,z_1^+,\cdots,z_{i_1-1}^+,z_{i_1}^-,z_{i_2}^+,z_{i_2+1}^-,\cdots,z_{i_3-1}^+,z_{i_3}^-)\in V_0^+\times V_1^-\times\cdots\times V_{i_1}^-\times V_{i_2}^+\times\cdots\times V_{i_3-1}^+\times V_{i_3}^-=\mathbb{V}$ we let

$$h_{\tilde{L}}^{K,K'}(m,m',Z,\vec{\tau})=\inf_{\substack{\gamma(-K)=m\\ \gamma(K')=m'\\ \gamma(\tau_i-\tilde{T}_i^0)=z_i^+\\ \gamma(\tau_i+\tilde{T}_i^1)=z_{i+1}^-\\ [\gamma|_{t\in J_i}]_1\neq 0\\ i\in\mathbb{I}}}\int_{-K}^{K'+\tau_{i_3-1}+\tilde{T}_{i_3-1}^1+\hat{T}_{i_3}}\tilde{L}(d\gamma(t),t)dt$$
$$+\sum_{i=1}^{i_3-1}(\tau_i-\tau_{i-1})\alpha(c_i)+K\alpha(c_0)+K'\alpha(c_{i_3})$$

where $J_i=[\tau_i-\tilde{T}_i^0,\tau_i+\tilde{T}_i^1]$ and $\mathbb{I}=\{1,2,\cdots,i_1-1,i_2+1,\cdots i_3-1\}$.

Let $h_{\tilde{L}}^{K,K'}(m,m')$ be the minimizer of $h_{\tilde{L}}^{K,K'}(m,m',Z,\vec{\tau})$ over $\mathbb{V}$ and Λ in z and $\vec{\tau}$ respectively:

$$h_{\tilde{L}}^{K,K'}(m,m')=\min_{\vec{\tau}\in\Lambda,z\in\mathbb{V}}h_{\tilde{L}}^{K,K'}(m,m',Z,\vec{\tau}),$$

let $K_j,K_j'\to\infty$ be the subsequence such that

$$\lim_{K_j,K_j'\to\infty}h_{\tilde{L}}^{K_j,K_j'}(m,m')=\liminf_{\substack{K\to\infty\\ K'\to\infty}}h_{\tilde{L}}^{K,K'}(m,m'),$$

and let $\gamma(t;K_j,K_j',m,m')$ be the minimal curve, we claim that $d\gamma(t;K_j,K_j',m,m')$ is a solution of the Euler-Lagrange equation determined by L if K_j and K_j' are sufficiently large. Indeed,

1, for each $i_1\le i<i_2$, we have

$$(-\tau_i)^*\gamma(t;K_j,K_j')\in\mathcal{C}_{\eta_i,\mu_i}(t)+\delta_i\subset U_i,\qquad\text{when } \tau_i\le t\le\tau_i+1.\tag{5.12}$$

In fact, let us choose $m_i=\gamma(\tau_{i-1}+1)$, $m_i'=\gamma(\tau_{i+1})$. Since $\gamma(t;K_j,K_j',m,m')$ is the minimizer of $h_{\tilde{L}}^{K,K'}(m,m',Z,\vec{\tau})$ over Λ, thus

$$A_{\tilde{L}}((-\tau_i)^*\gamma|_{\tau_{i-1}+1}^{\tau_{i+1}})+(\tau_i-\tau_{i-1}+1)\alpha(c_i)+(\tau_{i+1}-\tau_i)\alpha(c_{i+1})$$
$$=\inf_{\substack{\xi(-T_0)=m_i\\ \xi(T_1)=m_i'\\ \check{T}_i^0\le T_0\le\hat{T}_i^0\\ \check{T}_i^1\le T_1\le\hat{T}_i^1}}\int_{-T_0}^{T_1}(L-\eta_i-\mu_i)(d\xi(t),t)dt+T_0\alpha(c_i)+T_1\alpha(c_{i+1}),$$

from which and (5.2), (5.5) as well as (5.9) it follows that (5.12) holds. Consequently, $\gamma(t;K_j,K_j')|_{\tau_i\le t\le\tau_i+1}$ falls into the region where $(-\tau_i)^*\mu_i$ is closed. Thus, $d\gamma(t;K_j,K_j')$ is the solution of the Euler-Lagrange equation determined by L when $\tau_i\le t\le\tau_i+1$;

2, for $0 \le i < i_1$ and $i_2 \le i < i_3$, there is a local minimal curve connecting $m_i \in V_i^+$ and $m_{i+1} \in V_{i+1}^-$ with the time $\tilde{T}_i^0 + \tilde{T}_i^1$, so, it is easy to see that

$$(-\tau_i)^*\gamma(t)|_{0\le t\le \tau} \in \mathrm{int}(O_i)$$

if both V_i^+ and V_{i+1}^- are chosen suitably small. That also implies that $d\gamma(t; K_j, K_j')$ is the solution of the Euler-Lagrange equation determined by L when $\tau_i \le t \le \tau_i + 1$ for $0 \le i < i_1$ and $i_2 \le i < i_3$;

3, We claim that the curve γ does not touch the boundary of V_i^+ at the time $t = \tau_i - \tilde{T}_i^0$ and does not touch the boundary of V_{i+1}^- at the time $t = \tau_i + \tilde{T}_i^1$ for each $0 \le i < i_1$ and for each $i_2 \le i < i_3$. If $(\gamma(\tau_i - \tilde{T}_i^0), \gamma(\tau_i + \tilde{T}_i^1)) = (m_i, m_i') \in \partial(V_i^+ \times V_{i+1}^-)$ for some i, let $m_{i-1}' = \gamma(\tau_{i-1} + \tilde{T}_{i-1}^1)$ and $m_{i+1} = \gamma(\tau_{i+1} - \tilde{T}_{i+1}^1)$, from (5.1) we can see that there exist $(\bar{m}_i, \bar{m}_i') \in V_i^+ \times V_{i+1}^-$ such that for $\xi \in \mathcal{M}_0(c_i)$, $\zeta \in \mathcal{M}_0(c_{i+1})$:

$$\begin{aligned}
&h_{c_i}^{T_i}(m_{i-1}', m_i) + h_{\eta_i,\mu_i,\psi_i}^{\tilde{T}_i^0,\tilde{T}_i^1}(m_i, m_i', e_1) + h_{c_{i+1}}^{T_{i+1}}(m_i', m_{i+1}) \\
\ge &h_{c_i}^{\infty}(\xi, m_i) + h_{\eta_i,\mu_i,\psi_i}^{\tilde{T}_i^0,\tilde{T}_i^1}(m_i, m_i', e_1) + h_{c_{i+1}}^{\infty}(m_i', \zeta) \\
&+ h_{c_i}^{\infty}(m_{i-1}', \xi) + h_{c_{i+1}}^{\infty}(\zeta, m_{i+1}) - 2\epsilon_i^* \\
\ge &h_{c_i}^{\infty}(\xi, \bar{m}_i) + h_{\eta_i,\mu_i,\psi_i}^{\tilde{T}_i^0,\tilde{T}_i^1}(\bar{m}_i, \bar{m}_i', e_1) + h_{c_{i+1}}^{\infty}(\bar{m}_i', \zeta) \\
&+ h_{c_i}^{\infty}(m_{i-1}', \xi) + h_{c_{i+1}}^{\infty}(\zeta, m_{i+1}) + 3\epsilon_i^* \\
\ge &h_{c_i}^{T_i'}(m_{i-1}', \bar{m}_i) + h_{\eta_i,\mu_i,\psi_i}^{\tilde{T}_i^0,\tilde{T}_i^1}(\bar{m}_i, \bar{m}_i', e_1) + h_{c_{i+1}}^{T_{i+1}'}(\bar{m}_i', m_{i+1}) + \epsilon_i^*
\end{aligned}$$

but this contradicts to the fact that γ is a minimal curve of $\tilde{L}$ on $\mathbb{V}$ and Λ. In above argument, (5.6) and (5.7) are used to obtain the first and the third inequality, (5.1) is used to obtain the second inequality.

Let $K_j, K_j' \to \infty$, let γ_∞: $\mathbb{R} \to M$ be an accumulation point of $\{\gamma(t, K_j, K_j')\}$. Obviously, $\alpha(d\gamma_\infty) \subset \tilde{\mathcal{N}}(c_i)$ and $\omega(d\gamma_\infty) \subset \tilde{\mathcal{N}}(c_{i+1})$.

6, Hölder continuity

The task in this section is to build up some Hölder continuous dependence of h_c^∞ on some parameters if we set $k = 1$. These properties will be used to show that there is a generic set for perturbation where the conditions for the theorem are satisfied.

Let Φ_H^t be the Hamiltonian flow determined by H, it is a small perturbation of $\Phi_{h_1+h_2}^t$. Let Φ_H and $\Phi_{h_1+h_2}$ be their time-1-maps. As the cylinder $\mathbb{T} \times \mathbb{R} \times \{(x, y) = (0, 0)\} = \Sigma_0$ is the normally hyperbolic invariant manifold for $\Phi_{h_1+h_2}$ and the *a priori* unstable condition is assumed, it follows from the fundamental theorem of normally hyperbolic invariant manifold (cf. [HPS]) that there is $\epsilon = \epsilon(A, B) > 0$ such that if $\|P\|_{C^r} \le \epsilon$ on the region $\{\|(p, y)\| \le \max(|A|, |B|) + 1\}$ the map Φ_H^{s+k} $(k \in \mathbb{Z})$ also has a C^{r-1} invariant manifold $\Sigma(s) \subset \mathbb{R}^{n+1} \times \mathbb{T}^{n+1}$, provided that $r \ge 2$. This manifold is a small deformation of the manifold $\Sigma_0|_{\{|p|\le \max(|A|,|B|)+1\}}$, and is also normally hyperbolic and time-1-periodic. Let $\Sigma=\Sigma(0)$, it can be considered as the image of a map ψ: $\Sigma_0\to\mathbb{R}^n\times\mathbb{T}^n$, $\Sigma=\{p,q,x(p,q),y(p,q)\}$. This map induces a 2-form $\Psi^*\omega$ on Σ_0

$$\Psi^*\omega=\left(1+\sum_{i=1}^n\frac{\partial(x_i,y_i)}{\partial(p,q)}\right)dp\wedge dq.$$

Since the second de Rham co-homology group of Σ_0 is trivial, by using Moser's argument on the isotopy of symplectic forms [Mo], we find that there exists a diffeomorphism Ψ_1 on $\Sigma_0|_{\{|p|\le \max(|A|,|B|)+1\}}$ such that

$$(\Psi\circ\Psi_1)^*\omega=dp\wedge dq.$$

Since Σ is invariant for Φ_H and $\Phi_H^*\omega=\omega$, we have

$$\Big((\Psi\circ\Psi_1)^{-1}\circ\Phi_H\circ(\Psi\circ\Psi_1)\Big)^*dp\wedge dq=dp\wedge dq$$

i.e. $(\Psi\circ\Psi_1)^{-1}\circ\Phi_H\circ(\Psi\circ\Psi_1)$ preserves the standard area. Clearly, it is exact and twist since it is a small perturbation of Φ_{h_1}. In this sense, we say that the restriction of Φ_H on Σ is obviously area-preserving and twist. If $r>4$ there are many invariant homotopically non-trivial curves, including many KAM curves. All these curves are Lipschitz. Given $\rho\in\mathbb{R}$ there is an Aubry-Mather set with rotation number ρ, which is either an invariant circle, or a Denjoy set if $\rho\in\mathbb{R}\backslash\mathbb{Q}$, or periodic orbits if $\rho\in\mathbb{Q}$. Under the generic condition we can assume there is no homotopically non-trivial invariant curves with rational rotation number for $\Phi_H|_\Sigma$, there is only one minimal periodic orbit on Σ for each rational rotation number.

Let us consider the Legendre transformation $\mathcal{L}$. By abuse of terminology we continue to denote $\Sigma(s)$ and its image under the Legendre transformation by the same symbol. Let

$$\tilde\Sigma=\bigcup_{s\in\mathbb{T}}(\Sigma(s),s),$$

which has the normal hyperbolicity as well. Under the Legendre transformation those Aubry-Mather sets (invariant curves, Denjoy sets or minimal periodic orbits) on Σ correspond to the support of some c-minimal measures. Recall $H^1(M,\mathbb{R})=\mathbb{R}^{n+1}$. So we have

Lemma 3.1 (k=1). *Given some large number $K>0$ and a small number $\delta>0$ there exists a small number $\epsilon=\epsilon(\delta)$, if $L_1\in\mathcal{B}_{\epsilon,K}$ and if $|c_q|\le K$ then there exists an n-dimensional convex set $\mathcal{D}(c_q)$ which contains a neighborhood of $(c_q,0)\cap\mathbb{R}^n$ such that for each $c\in\mathrm{int}(\mathcal{D}(c_q))$ the Mañé set $\tilde{\mathcal{N}}(c)\subset\tilde\Sigma$, the Mather set $\mathcal{M}_0(c)$ is the Aubry-Mather set for the twist map. If the rotation number is irrational, then $\mathcal{M}(c)$ is uniquely ergodic.*

Proof: We have shown that $\mathcal{M}(c)\subset N_\delta$. The normal hyperbolicity guarantees that the invariant set in N_δ must be in the invariant cylinder. The time 1 map restricted on the cylinder is then an area-preserving twist map. $\square$

Consider a cohomology class $c = (c_q, 0) \in H^1(M, \mathbb{R})$ such that it corresponds to an invariant circle Γ in Σ with irrational rotation number. In the Hamiltonian formalism, $\Gamma = \{(p, q, x, y) \in \mathbb{R}^{n+1} \times \mathbb{T}^{n+1} : (p, x, y) = (p, x, y)(q), q \in \mathbb{T}\}$. Based on each point on this circle, there is a C^{r-1}-stable fiber as well as a C^{r-1}-unstable fiber. These stable (unstable) fibers C^{r-2}-depends on the base point and make up the local stable (unstable) manifold of that circle which are the graph of a Lipschitz function in a small neighborhood of the circle, i.e.

$$W^{u,s}_{loc}(\Gamma) = \Big\{(q, x, (p, y)^{u,s}(q, x)) : (q, x) \in N_\delta \subset T^{n+1}\Big\}$$

where $(p, y)(q, x)$ is a Lipschitz function of (q, x).

Lemma 6.1. *There exists a $C^{1,1}$ function $S^{s,u}$: $\{\|x\|\} \to \mathbb{R}$ and a constant vector $c \in \mathbb{R}^{n+1}$ such that $W^{s,u}_{loc}(\Gamma) = \{(q, x), dS^{s,u}(q, x) + c : (q, x) \in N_\delta\}$.*

We use $C^{k,\alpha}$ to denote those functions whose k-th derivative is of α-Hölder.

Proof: Let us consider the stable manifold. By the condition there is a Lipschitz function (p, y): $N_\delta \to \mathbb{R}$ such that $W^s_{loc}(\Gamma) = \{(q, x, (p, y)^s(q, x)) : (q, x) \in N_\delta\}$. Let σ be an 2-dimensional disk in W^s_{loc}. Since σ is in the stable manifold, $\Phi^k_H(\partial\sigma)$ approaches uniformly to Γ, i.e. $\Phi^k_H(\partial\sigma)$ is such a closed curve going from a point to another point and returning back along almost the same path when k is sufficiently large. As Φ_H is a symplectic diffeomorphism we have

$$\begin{aligned}\iint_\sigma (dp \wedge dq + \sum_{i+1}^{n} dy_i \wedge dx_i) &= \oint_{\partial\sigma} (pdq + \sum_{i+1}^{n} y_i dx_i) \\ &= \oint_{\Phi^k_H(\partial\sigma)} (pdq + \sum_{i+1}^{n} y_i dx_i) \\ &= 0.\end{aligned}$$

Note the function $(p, y)^s(q, x)$ is Lipschitz, it is differentiable almost everywhere in N_δ. As σ is arbitrarily chosen, for almost $(q, x) \in N_\delta$ the following holds:

$$\frac{\partial p}{\partial x_i} = \frac{\partial y_i}{\partial q}, \qquad \frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i}, \qquad \forall\ 1 \le i, j \le n. \tag{6.1}$$

Consequently, there exists a $C^{1,1}$-function S^s_c and $c = (c_q, 0) \in \mathbb{R}^{n+1}$ such that $(p, y)^s = dS^s_c + c$. In the same way, we obtain a $C^{1,1}$-function S^u_c and $c' = (c'_q, 0) \in \mathbb{R}^{n+1}$ such that $(p, y)^u = dS^u_c + c'$. Since W^s_{loc} intersects W^u_{loc} on the whole Γ, $c' = c$. □

Indeed, for almost all initial points $(q, x, (p, y)^s(q, x)) \in W^s$, $(p, y)^s$ is differentiable at all $\Phi^k_H(q, x, (p, y)^s(q, x))$ for all $k \in \mathbb{Z}^+$. To see that, let $O \subset N_\delta$ be an open set, for each k there is a full Lebesgue measure set $O_k \subset \pi(\Phi^k_H\{O, (p, y)^s(O)\})$ where $(p, y)^s$ is differentiable. Since Φ_H is a diffeomorphism, the set

$$O^* = \bigcap_{k=0}^{\infty} \pi\Big(\Phi^{-k}_H\{O_k, (p, y)^s(O_k)\}\Big)$$

is a full Lebesgue measure subset of O. For any point $(q,x)\in O^*$, $(p,y)^s$ is differentiable at the points $\pi(\Phi_H^k\{(q,x),(p,y)^s(q,x)\}$ for all $k\in\mathbb{Z}^+$.

Let us consider the Hamiltonian flow. The local stable (unstable) manifold is a graph of some function

$$\tilde{W}^{s,u}_{loc}=\{(q,x,t),(p,y)^{s,u}(q,x,t):(q,x,t)\in N_\delta\times\mathbb{T}\}.$$

Obviously, we have $((p,y)^{s,u},t)^*\Omega=0$ in $M\times\mathbb{T}$, where $\Omega=\sum dx_i\wedge dy_i+dq\wedge dp-dH\wedge dt$. Thus, in the covering space $\mathbb{R}^{n+2}$ there exists a $C^{1,1}$-function $\bar{S}^{s,u}_c(q,x,t)$ such that $d\bar{S}^{s,u}_c=(p,y)^{s,u}(q,x,t)-H((p,y)^{s,u}(q,x,t),q,x,t)dt$. Consequently, there exists a function $S^{s,u}_c(q,x,t)\in C^{1,1}(N_\delta\times\mathbb{T},\mathbb{R})$ and $c=(c_q,0)$ such that

$$L^{s,u}=L-c_q(\dot q+\partial_qS^{s,u}_c)-\langle\partial_xS^{s,u}_c,\dot x\rangle-\partial_tS^{s,u}_c$$

attains its minimum at $\mathcal{L}W^{s,u}$ as the function of $(\dot q,\dot x)$. Note $L^{s,u}_{(\dot q,\dot x)}-\partial_{(q,x)}S^{s,u}_c$ is Lipschitz, $dL^{s,u}_{(\dot q,\dot x)}/dt$ and $L^{s,u}_{(q,x)}$ exist almost everywhere. Since $\mathcal{L}W^{s,u}$ is a manifold made up by the trajectories of the Euler-Lagrange flow, it follows from Euler-Lagrange equation $dL_{\dot q,\dot x}/dt=L_{q,x}$ and (6.1) that $L_{q,x}|_{\mathcal{L}W^{s,u}_{loc}}=0$ almost everywhere. The absolute continuity of L implies that $L^{s,u}|_{\mathcal{L}W^{s,u}_{loc}}$ is a function of t alone. So, by adding some function of t to $S^{s,u}_c$, $L^{s,u}|_{\mathcal{L}W^{s,u}_{loc}}=-\alpha(c)$.

Lemma 6.2. *Let $c=(c_q,0)$. If Γ is an invariant circle in the cylinder, the Aubry-Mather set is uniquely ergodic, then for each $\xi\in\pi(\Gamma)$ and each $m\in N_\delta$, we have*

$$h^\infty_c(\xi,m)=S^u_c(m)-S^u_c(\xi),\qquad h^\infty_c(m,\xi)=S^s_c(\xi)-S^s_c(m).\tag{6.2}$$

Proof: Since there are the local stable manifold $W^s(\Gamma)$ and the unstable manifold $W^u(\Gamma)$ to the invariant circle Γ, for each point $m\in N_\delta$ there is a unique c-minimal orbit $\gamma^{s,u}(t)$ such that $\gamma^{s,u}(0)=m$ and $\gamma^{s,u}(k)\to\pi(\Gamma)$ as $\mathbb{Z}\ni k\to\pm\infty$. Let $\xi\in\mathcal{M}_0(c)$, there is an integer subsequence $k^{s,u}_i\to\pm\infty$ as $i\to\infty$ such that $\gamma^{s,u}(k^{s,u}_i)\to\xi$ as $i\to\infty$. It means that

$$\lim_{i\to\infty}h^{k^s_i}_c(m,\gamma^s(k^s_i))=h^\infty_c(m,\xi),\qquad \lim_{i\to\infty}h^{-k^u_i}_c(\gamma^u(k^u_i),m)=h^\infty_c(\xi,m).$$

Since $L^{s,u}+\alpha(c)=0$ on $W^{s,u}$, we have

$$\begin{aligned}\int_{k^u_i}^0\Big(L(d\gamma^u_c(k^u_i)-\langle c,\dot\gamma^u_c(k^u_i)\rangle+\alpha(c)\Big)dt=&S^u(\gamma^u_c(0))-S^u(\gamma^u_c(k^u_i)),\\ \int_0^{k^s_i}\Big(L(d\gamma^s_c(k^u_i)-\langle c,\dot\gamma^s_c(k^u_i)\rangle+\alpha(c)\Big)dt=&S^s(\gamma^s_c(k^u_i))-S^s(\gamma^s_c(0)).\end{aligned}$$

That implies that (6.2) holds for each $m\in N_\delta$ and each $\xi\in\mathcal{M}_0(c)$. To see that (6.2) holds for each $\xi\in\pi(\Gamma)$, let us recall that, for a twist map, the sufficient and necessary condition for the existence of an invariant circle is that the Peierl's barrier function is identically equal to zero. Consequently, passing each $\zeta\in\pi(\Gamma)$ there is a regular c-minimal configuration $(\cdots,m_i,\cdots)$ such that $\zeta=m_0$. Since we have assumed the unique ergodicity of the minimal measure, $d_c(\zeta,\xi)=0$ for each $\zeta\in\pi(\Gamma)$ and each $\xi\in\mathcal{M}_0(c)$. Thus, (6.2) holds for any $\xi\in\pi(\Gamma)$. □

We now consider the local stable and unstable manifolds of all invariant circles. Different invariant circle determines different stable and unstable manifolds, i.e. we have a family of these local stable and unstable manifolds. We claim that this family of local stable (unstable) manifolds can be parameterized by some parameter σ so that both S^u_c and S^s_c have $\frac{1}{2}$-Hölder continuity in σ. Indeed we choose one circle and denote it Γ_0 and parameterize another circle Γ_σ by the algebraic area between Γ_σ and Γ_0,

$$\sigma = \int_0^1 (\Gamma_\sigma(q) - \Gamma_0(q))dq.$$

This integration is in the sense that we pull it back to the standard cylinder by $\Psi \circ \Psi_1 \in \mathrm{diff}(\Sigma_0, \Sigma)$. In this way we obtain an one-parameter family curves Γ: $\mathbb{T} \times \mathbb{S} \to \Sigma$ in which $\mathbb{S} \subset [A', B']$ is a closed set. Usually, $\mathbb{S}$ is a Cantor with positive Lebesgue measure. A' and B' correspond to the curves where the action $p \leq A$ and $p \geq B$ respectively. Clearly, for each $\sigma \in \mathbb{S}$, there is only one $c_q = c_q(\sigma)$ such that $\Gamma_\sigma = \tilde{\mathcal{M}}_0(c)$ for $c = (c_q, 0) \in H^1(M, \mathbb{R})$ when the rotation number is irrational. Clearly, c_q is continuous in σ on $\mathbb{S}$. We can think Γ_σ as a map to function space C^0 equipped with supremum norm Γ: $\mathbb{S} \to C^0(\mathbb{T}, \mathbb{R})$,

$$\|\Gamma_{\sigma_1} - \Gamma_{\sigma_2}\| = \max_{q \in \mathbb{T}} |\Gamma(q, \sigma_1) - \Gamma(q, \sigma_2)|.$$

Straight-forward calculation shows

$$|\sigma_1 - \sigma_2| \geq \frac{1}{2C_h} \left(\max_{q \in \mathbb{T}} |\Gamma(q, \sigma_1) - \Gamma(q, \sigma_2)| \right)^2,$$

where C_h is the Lipschitz constant for the twist map, it follows that

$$\|\Gamma_{\sigma_1} - \Gamma_{\sigma_2}\| \leq C_s |\sigma_1 - \sigma_2|^{\frac{1}{2}}$$

where $C_s = \sqrt{2C_h}$. Since the stable (unstable) fibers have C^{r-2}-smoothness on their base points on Σ, $r \geq 3$, $(p, y)^{s,u}_\sigma$ is also $\frac{1}{2}$-Hölder continuous in σ. Let $S^{s,u}_\sigma = S^{s,u}_{c(\sigma)}$, we have

Lemma 6.3. *Restricted in N_δ the functions $S^s_\sigma(m)$, $S^u_\sigma(m)$ are $\frac{1}{2}$-Hölder continuous in $\sigma \in \mathbb{S}$.*

Next, let us consider the dependence of the barrier function on σ and $c_q(\sigma)$ when $\sigma \in \mathbb{S}$. For each $c = (c_q, 0)$, each $m \in M \backslash N_\delta$ and each $\xi \in \mathcal{M}_0(c)$ there exist $m^+, m^- \in N_\delta$ and $k^+, k^- \in \mathbb{Z}^+$ such that

$$h^\infty_c(\xi, m) = h^\infty_c(\xi, m^+) + h^{k^+}_c(m^+, m), \qquad h^\infty_c(m, \xi) = h^{k^-}_c(m, m^-) + h^\infty_c(m^-, \xi).$$

Clearly, there exists a uniform upper bound $K \in \mathbb{Z}$ such that for each $c = (c_q, 0)$, each $m \in M \backslash N_\delta$ and each $\xi \in \mathcal{M}(c)$ we have $k^+ \leq K$, $k^- \leq K$.

Fix $m \in M\backslash N_\delta$, different σ determines different $m^+, m^- \in N_\delta$. For each $\sigma \in \mathbb{S}$, each $m \in M\backslash N_\delta$ can be covered by both $\Phi_H^K(W^u_{loc}(\Gamma_\sigma))$ and $\Phi_H^{-K}(W^s_{loc}(\Gamma_\sigma))$ in the sense that $m \in \pi(\Phi_H^K(W^u_{loc}(\Gamma_\sigma)))$, $m \in \pi(\Phi_H^{-K}(W^s_{loc}(\Gamma_\sigma)))$. Let u be the coordinate of $m \in M$ and let $z = (u, v) \in TM$ be the initial value such that $\phi^t_L(z)$: $[0, \infty) \to TM$ is a forward c-semi-static orbit, assuming $m \in M\backslash N_\delta$. Clearly, $z \in \mathcal{L}(\Phi_H^{-k}(W^s_{loc}(\Gamma_\sigma)))$ Consider another invariant circle $\Gamma_{\sigma'}$ close to Γ_σ. By the $\frac{1}{2}$-Hölder continuity there exists a point $z' = (u', v') \in \mathcal{L}(\Phi_H^{-K}(W^s_{loc}(\Gamma_{\sigma'})))$ with $\|z' - z\| \le C_1\sqrt{|\sigma - \sigma'|}$. We claim that $\exists\ d_1 \ge 0$ such that if $C_1\sqrt{|\sigma - \sigma'|} \le d_1$ then

$$\|\phi^t_L(z') - \phi^t_L(z)\| \le 2\|z' - z\|, \qquad \forall\ t \in [0, k^+]. \tag{6.3}$$

To see that, we write the Lagrange equation into the form of first order ODE:

$$\begin{aligned} \dot v &= \left(\frac{\partial^2 L}{\partial \dot u^2}\right)^{-1}\left(-\frac{\partial^2 L}{\partial \dot u \partial u}v - \frac{\partial^2 L}{\partial \dot u \partial t} + \frac{\partial L}{\partial u}\right), \\ \dot u &= v. \end{aligned} \tag{6.4}$$

Since the convexity of L is assumed this equation is well-posed. If we write (6.4) in the form

$$\dot z = F(z, t)$$

its variational equation along an orbit $d\gamma_\sigma$ as follows

$$\Delta \dot z = DF(d\gamma_\sigma(t), t)\Delta z + f(d\gamma(t), t, \Delta z)$$

where $f(d\gamma(t), t, \Delta z) = O(\|\Delta z\|^2)$ as $\Delta z \to 0$. Let $E(t, \sigma)$ be the fundamental matrix solution of the linearized variational equation. Since the orbits $d\gamma$: $[0, \infty) \to TM$ concerned has the property that $\omega(d\gamma_\sigma) \subset \tilde{\mathcal{N}}(c)$ while we are only interested in those co-homology classes $\{c \in H^1(M, \mathbb{R}) : c = (c_q(\sigma), 0), |c_q| \le \max\{|A|, |B|\} + 1\}$, there exists a large but bounded constant $C_2 = C_2(A, B)$ such that $C_2 \ge \max\{E_{ij}(t, \sigma), E^{-1}_{ij}(t, \sigma) :\ 1 \le i, j \le n,\ 0 \le t \le K, \sigma \in \mathbb{S}\}$ where E_{ij}, E^{-1}_{ij} denote the (i, j)-entry of E and its inverse respectively. Since f is the higher order term of Δz, there exists a small constant $d_1 > 0$ such that

$$\|f(d\gamma(t), t, \Delta z)\| \le \frac{1}{n^2 K C_2^2}\|\Delta z\|, \quad \text{if } \|\Delta z\| \le 2d_1.$$

Since Δz is the solution of the integral equation:

$$\Delta z(t) = \Delta z(0) + E(t)\int_0^t E^{-1}(s) f(d\gamma(s), s, \Delta z(s))ds,$$

we have $\|\Delta z(t)\| \le 2\|\Delta z(0)\|$ for all $t \in [0, K]$ if $\|\Delta z(0)\| \le d_1$.

Let $z\in\mathcal{L}(\Phi_H^{-k}(W^s_{loc}(\Gamma_\sigma)))$ so that $\phi_L^t(z)$: $[0,\infty)\to M$ is a forward c-semi-static orbit. Let $c'=(c_q(\sigma'),0)$. By the $\frac12$-Hölder continuity of $W^s_{loc}(\Gamma_\sigma)$ in σ, there is a point $z'=(u',v')\in\mathcal{L}(\Phi_H^{-k}(W^s_{loc}(\Gamma_{\sigma'})))$ with the property that $u'=u(\sigma')$ $\frac12$-continuously depending on σ', $\phi_L^t(z')$: $[0,\infty)\to M$ is a forward c'-semi static orbit and (6.3) holds. Since the stable manifold may multi-fold, there might be no $z'=(u,v')$ such that $\phi_L^t(z)$ is a c'-semi static orbit and $\|z'-z\|=O(\sqrt{\sigma-\sigma'})$. Let $u_{k^+}=\pi(\phi_L^{k^+}(z))$, $u'_{k^+}=\pi(\phi_L^{k^+}(z'))$, we can write

$$h^\infty_{c(\sigma')}(u,\xi)=E_1(\sigma')+E_2(\sigma')+E_3(\sigma)+c'_qE_4(\sigma')+E_5(c'_q(\sigma))$$

where

$$\begin{aligned}
E_1(\sigma') =&h^\infty_{c(\sigma')}(u,\xi)-h^\infty_{c(\sigma')}(u',\xi),\\
E_2(\sigma') =&h^\infty_{c(\sigma')}(u'_{k^+},\xi),\\
E_3(\sigma') =&\int_0^{k^+}L(d\gamma'(t),t)dt,\\
E_4(\sigma') =&\bar\gamma'_q(k^+)-\bar\gamma'_q(0),\\
E_5(\sigma') =&k^+\alpha(c'),
\end{aligned}$$

$\gamma'(t)=\pi(\phi_L^t(z'))$, $\bar\gamma'$ is the lift of γ' to the universal covering $\mathbb{R}^{n+1}$. By the Lipschitz property of $h^\infty_c(m,m')$ in m,m' and the choice of $u'=u(\sigma)$ we can see that E_i $(i=1,2,3,4)$ has $\frac12$-Hölder continuity in σ'. Recall that the β function for a twist map has no flat piece, and is differentiable at irrational numbers, E_5 is Lipschitz in c. Thus, we can formally define a function $h^\infty_{c,\sigma}=E_1(\sigma)+E_2(\sigma)+E_3(\sigma)+c_qE_4(\sigma)+E_5(c_q)$ regardless of the dependence of c_q on σ. So we obtain

Lemma 6.4. *Assume that $m\in M\backslash N_\delta$, $|c_q|\le\max\{|A|,|B|\}+1$. Let $c=(c_q,0)$ and $\xi\in\mathcal{M}_0(c)$, then we can extend the function $h^\infty_{c(\sigma)}$, defined on $\{A-1\le\sigma\le B+1\}$, to a function $h^\infty_{c,\sigma}(u,\xi)$ defined in a neighborhood of the continuous curve $\{\sigma,c_q(\sigma)\}\subset\mathbb{R}^2$ such that $h^\infty_{c(\sigma)}=h^\infty_{c,\sigma}(u,\xi)|_{c_q=c_q(\sigma)}$ and*

$$\begin{aligned}
|h^\infty_{c,\sigma}(\xi,m)-h^\infty_{c',\sigma'}(\xi,m)|\le C_3(\sqrt{|\sigma-\sigma'|}+|c_q(\sigma)-c_q(\sigma')|),\\
|h^\infty_{c,\sigma}(m,\xi)-h^\infty_{c',\sigma'}(m,\xi)|\le C_3(\sqrt{|\sigma-\sigma'|}+|c_q(\sigma)-c_q(\sigma')|).
\end{aligned}$$

Remark: We do not know whether the function $\sigma\to c_q(\sigma)$ has some Hölder continuity in σ.

7, Generic property

In this section we also assume that $k=1$. The task in this section is to show that there is a residual set in $\mathcal{B}_{\epsilon,K}$ such that if P is in this set then there is a generalized transition chain $\{c\in H^1(M,\mathbb{R}):c_x=0,A\le c_q\le B\}$.

Let us consider this issue from the Hamiltonian dynamics point of view. Since the system is positive definite in action variable $v=(p,y)$, it has a generating function $G(u,u')$ $(u=(q,x))$

$$G(u,u')=\inf_{\substack{\gamma\in C^1([0,1],\tilde M)\\ \gamma(0)=u,\gamma(1)=u'}}\int_0^1 L(\gamma(s),\dot\gamma(s),s)ds,$$

where (u,u') is in the universal covering space $\tilde M\times\tilde M=\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$. Clearly, $G(u+2k\pi,u'+2k\pi)=G(u,u')$ for each $k\in\mathbb{Z}^{n+1}$. The map Φ_H: $(u,v)\to(u',v')$ is given by

$$v'=\partial_{u'}G(u,u'),\qquad v=-\partial_u G(u,u').$$

Let π_2 be the standard projection from $\mathbb{R}^{n+1}\to\mathbb{T}^{n+1}$, let $c\in\mathbb{R}^{n+1}$ and

$$G_c(u,u')=G(u,u')-\langle c,u'-u\rangle$$

then

$$h_c(m,m')=\min_{\substack{\pi_2(u)=m\\ \pi_2(u')=m'}}G_c(u,u')+\alpha(c).$$

We consider the change of the function h_c^∞ when the generating function is subject to a small perturbation $G\to G+G_1$. Let $m\in M\backslash N_\delta$, $\xi\in\mathcal{M}_0(c)$, $c=(c_q,0)$. Let $\{k_i\}$ be a subsequence such that

$$\lim_{i\to\infty}h_c^{k_i}(\xi,m)=\liminf_{k\to\infty}h_c^k(\xi,m).$$

Let $\{u_1,u_2,\cdots,u_{k_i}=m\}$ be the minimal configuration, we claim that there exists $b>0$ such that $u_i\notin\mathcal{B}_b(m)$, the ball centered at m with radius b, for each $1\le i\le k_i-1$. In fact, $h_c^k(m,m)\ge 2A>0$ for each $c=(c_q,0)\in H^1(M,\mathbb{R})$ and for each $k\in\mathbb{Z}^+$. If not, there exists a subsequence k_j such that

$$\lim_{k_j\to\infty}h_c^{k_j}(m,m)=0.$$

It implies that $m\in\mathcal{A}_0(c)$, which contradicts the lemma 3.1. Since $h_c^k(m,m')$ is Lipschitz in m,m', there exists $b>0$ such that if $m'\in\mathcal{B}_b(m)$ then $h_c^k(m,m)\ge A$ for each $k\in\mathbb{Z}^+$. If there is $u_i\in\mathcal{B}_b(m)$ for some $i\in\{k_i\}$, let m' be an accumulation point of $\{u_i\}$ then there exists some $k\in\mathbb{Z}^+$ such that

$$h_c^\infty(\xi,m)=h_c^\infty(\xi,m')+h_c^k(m',m).$$

Consequently,

$$h_c^\infty(\xi,m')\le h_c^\infty(\xi,m)-A.$$

On the other hand, from the Lipschitz property we obtain that

$$h_c^\infty(\xi,m')\ge h_c^\infty(\xi,m)-C_L\|m-m'\|.$$

It leads a contradiction if m' is sufficiently close to m. The contradiction verifies our claim. Consequently, if the generating function subjects to a small perturbation $G(u,u')\to G(u,u')+G_1(u')$, where $\mathrm{supp}(G_1)\subseteq\mathcal{B}_b(m)$, h_c^∞ will also undergo the small perturbation:

$$h_c^\infty(\xi,m')\to h_c^\infty(\xi,m')+G_1(m'),\qquad\forall m'\in\mathcal{B}_b(m),\ \xi\in\mathcal{M}_0(c);$$

while $h_c^\infty(m',\xi)$ remains the unchanged.

Choose $\xi, \zeta \in \mathcal{M}_0(c)$, $m \in M\backslash N_\delta$. The change of $h^\infty_{c,e_1}(\xi, m, \zeta)$ is a little bit complicated when the generating function undergoes the same small perturbation as above. Let $\{k_i\}$ be a subsequence such that

$$\lim_{i\to\infty} h^{k_i}_{c,e_1}(\xi, m, \zeta) = \liminf_{k\to\infty} h^k_{c,e_1}(\xi, m, \zeta).$$

Let $\{u_0 = \xi, u_1, \cdots, u_{l_i} = m, \cdots, u_{k_i} = \zeta\}$ be the minimal configuration that realizes the minimal action $h^{k_i}_{c,e_1}(\xi, m, \zeta)$, denote by γ_i: $[0, k_i] \to M$ the corresponding minimal curve. We claim that there exists $b > 0$ such that there is at most one $u_{j_i} \in \mathcal{B}_b(m)$ for some $1 \le j_i < k_i$, $j_i \neq l_i$ when k_i is sufficiently large. To a curve γ: $[0, k] \to M$ with $\gamma(0), \gamma(k) \in \mathcal{B}_b(m)$ we can associate an element $[\gamma] \in H_1(M, \mathcal{B}_b(m), \mathbb{Z})$. If there were two other points $u_{j_i}, u_{j_i'} \in \mathcal{B}_b(m)$ (without losing of generality we assume $j_i < j_i' < l_i$), then we would have two alternatives:

1, either $[\gamma|_{[j_i,j_i']}]_{x_1} = 0$, or $[\gamma|_{[j_i',l_i]}]_{x_1} = 0$, or both;

2, both $[\gamma|_{[j_i,j_i']}]_{x_1} \neq 0$ and $[\gamma|_{[j_i',l_i]}]_{x_1} \neq 0$.

In the first case, we can cut off a piece $\gamma|_{[j_i,j_i']}$ from the minimal curve and define a curve γ': $[0, k_i - j_i' + j_i] \to M$ such that

$$\gamma'(t) = \begin{cases} \gamma(t) & t \in [0, j_i], \\ \eta(t) & t \in [j_i, j_i + 1], \\ \gamma(t - j_i' + j_i + 1) & t \in [j_i + 1, k_i - j_i' + j_i], \end{cases}$$

where η: $[j_i, j_i + 1] \to M$ is a minimal curve joining $\gamma(j_i)$ with $\gamma(j_i' + 1)$. Clearly, $[\gamma']_{x_1} \neq 0$. Since $\gamma(j_i)$ is close to $\gamma(j_i')$, by the Lipschitz property of $h_c(m, m')$ in m, m', we have

$$\int_0^{k_i - j_i' + j_i} L(d\gamma'(t), t)dt + (k_i - j_i' + j_i)\alpha(c) \le h^{k_i}_{c,e_1}(\xi, m, \zeta) - A. \tag{7.1}$$

To see the absurdity of (7.1), let us observe a simple fact: if some $u_j \in \mathcal{B}_b(m)$, then $k_i - j \to \infty$, $j \to \infty$ as $i \to \infty$. It implies that $k_i - j_i' + j_i \to \infty$. So (7.1) contradicts the definition of h^∞_c if we choose $k_i \to \infty$ being such a subsequence that $\lim_{k_i\to\infty} h^{k_i}_{c,e_1}(\xi, m, \zeta) = h^\infty_{c,e_1}(\xi, m, \zeta)$. For the second alternative, by cutting off one piece $\gamma|_{[j_i,j_i']}$ or both $\gamma|_{[j_i,j_i']}$ and $\gamma|_{[j_i',l_i]}$ we can construct a curve γ' such that $[\gamma']_{x_1} \neq 0$. In the same way we can show this is also impossible. Thus, we have

Lemma 7.1. *Assume that generating function is subject to a small perturbation* $G(u, u') \to G(u, u') + G_1(u')$, *where* $\mathrm{supp}(G_1) \subseteq \mathcal{B}_b(m)$, $m \in M\backslash N_\delta$. *There exists* $b > 0$ *such that for each* $c = (c_q, 0)$ *with* $|c_q| \le \max\{|A|, |B|\} + 1$, *the barrier function undergoes a small perturbation:*

$$B^*_{c,e_1}(u) \to B^*_{c,e_1}(u) + jG_1(u) + \text{a small constant} \qquad j = 1, \text{ or } 2.$$

The next step is to show that the density of the set $\{P \in C^r : \{u \in M\backslash N_\delta : B_c^*(u, e_1) = \min_u B_c^*(u, e_1)\}$ is totally disconnected$\}$. Let $R_d = \{u \in M : |q - q^*| \leq d, |x_i - x_i^*| \leq d, \forall\, 1 \leq i \leq n\} \subset \mathcal{B}_b(u^*)$, $S_{c,\sigma} = B^*_{c(\sigma)} + G_1$, we define

$$Z(\sigma) = \{u \in R_d :\ S_{c,\sigma} = \min_{u \in R_d} S_{c,\sigma}\}.$$

We say a connected set V is non-trivial for R_d if either $\Pi_q(V \cap R_d) = \{q^* - d \leq q \leq q^* + d\}$ or $\Pi_i(V \cap R_d) = \{x_i^* - d \leq x_i \leq x_i^* + d\}$ for some $1 \leq i \leq n$. Here Π_i is the standard projection from $\mathbb{T}^{n+1}$ to its i-th component. Let $M_{d,u^*} = \{u : S(u) = \min_{u \in R_d(u^*)} S\}$, we define a set in the function space $\mathfrak{F}(d, u^*) = C^0(R_d(u^*), \mathbb{R})$,

$$\mathfrak{Z}(d, u^*) = \Big\{S \in \mathfrak{F}(d, u^*) : M_{d,u^*}(S) \text{ contains a set non-trivial for } R_d(u^*)\Big\}.$$

For convenience of notation, we set $x_0 = q$ and define $\mathfrak{Z}_i$ $(i = 0, 1, \cdots, n)$:

$$\begin{aligned} \mathfrak{Z}_0 &= \Big\{S \in \mathfrak{Z}(d, u^*) : \Pi_q(M_{d,u^*}(S)) = \{q^* - d \leq q \leq q^* + d\}\Big\}, \\ \mathfrak{Z}_i &= \Big\{S \in \mathfrak{Z}(d, u^*) : \Pi_i(M_{d,u^*}(S)) = \{x_i^* - d \leq x_i \leq x_i^* + d\}\Big\}. \end{aligned}$$

Clearly:

$$\mathfrak{Z}(d, u^*) = \bigcup_{i=0}^{n} \mathfrak{Z}_i.$$

Our first task is to show for each generating function $G \in C^r(M \times M, \mathbb{R})$ and each $\epsilon > 0$, there is an open and dense set $\mathfrak{H}(d, u^*)$ of $\mathcal{B}_\epsilon(0) \subset C^r(R_d(u^*), \mathbb{R})$, for each $G_1 \in \mathfrak{H}(d, u^*)$, the image of S_σ from $[A', B']$ to $\mathfrak{F}$ has no intersection with the set $\mathfrak{Z}_i$.

Obviously, the set $\mathfrak{Z}_i$ is a closed set and has infinite co-dimension in the following sense, there exists $\mathfrak{N}$, an infinite dimensional subspace of $\mathfrak{F}$, such that $(S + F) \notin \mathfrak{Z}_i$ for all $S \in \mathfrak{Z}_i$ and $F \in \mathfrak{N}\backslash\{0\}$. In fact, for each non constant function $F(x_i) \in C^0([x_i^* - d, x_i^* + d], \mathbb{R})$ with $F(x_i^*) = 0$ and each $S + F \notin \mathfrak{Z}_i$. Thus, we can choose $\mathfrak{N} = C^0([x_i^* - d, x_i^* + d], \mathbb{R})/\mathbb{R}$, which we think as the subspace of $C^0(R_d(u^*), \mathbb{R})$ consisting of those continuous functions independent of other coordinate components x_j $(j \neq i)$.

On the other hand, since $S_{c,\sigma}$: $[A, B] \times [A', B'] \to \mathfrak{F}$ has $\frac{1}{2}$-Hölder continuity, the image of the continuous curve $\{\sigma, c(\sigma)\} \subset [A, B] \times [A', B']$ is compact and its box dimension is not bigger than 4,

$$D_B(\mathfrak{F}_\sigma) \leq 4,$$

where $\mathfrak{F}_\sigma = \{S_{c(\sigma),\sigma} : \sigma \in [A', B']\}$. Clearly, this set is determined by the generating function G.

Lemma 7.2. *There is an open and dense set $\mathfrak{N}^* \subset \mathfrak{N}$ such that for all $F \in \mathfrak{N}^*$*

$$(\mathfrak{F}_\sigma + F) \cap \mathfrak{Z} = \varnothing. \tag{7.2}$$

Proof: The open property is obvious. If there was no density property, for any $k \in \mathbb{Z}$, there would be a k-dimensional ϵ-ball $\mathcal{B}_\epsilon \subset \mathfrak{N}$ for some $\epsilon > 0$, for each $F \in \mathcal{B}_\epsilon$, there would exist $S \in \mathfrak{F}_\sigma$ such that $F + S \in \mathfrak{Z}_i$. For each $S \in \mathfrak{F}_\sigma$ there is only one $F \in \mathcal{B}_\epsilon$ such that $S + F \in \mathfrak{Z}_i$, otherwise, there would be $F' \neq F$ such that $S + F' \in \mathfrak{Z}_i$. Since we can write $F' + S = F' - F + F + S$ where $S + F \in \mathfrak{Z}_i$ and $F' - F \in \mathfrak{N}\backslash\{0\}$, this contradicts to the definition of $\mathfrak{N}$. Given $F \in \mathcal{B}_\epsilon$, there might be more than one element in $\mathfrak{S}_F = \{S \in \mathfrak{F}_\sigma : S + F \in \mathfrak{Z}_i\}$. Given any two $F_1, F_2 \in \mathcal{B}_\epsilon$, for any $S_1 \in \mathfrak{S}_{F_1}$ and any $S_2 \in \mathfrak{S}_{F_2}$ we have

$$\begin{aligned} d(S_1, S_2) &= \max_{u \in R_d(u^*)} |S_1(u) - S_2(u)| \\ &\geq \max_{|x_i - x_i^*| \leq d} \Big| \min_{\substack{|x_j - x_j^*| \leq d \\ j \neq i}} S_1(u) - \min_{\substack{|x_j - x_j^*| \leq d \\ j \neq i}} S_2(u) \Big| \\ &= \max_{|x_i - x_i^*| \leq d} |F_1(x_i) - F_2(x_i)| \\ &= d(F_1, F_2) \end{aligned} \tag{7.3}$$

where $d(\cdot, \cdot)$ denotes the C^0-metric. It follows from (7.3) and the definition of box dimension that

$$D_B(\mathfrak{F}_\sigma) \geq D_B(\mathcal{B}_\epsilon) = k,$$

but this is absurd if we choose $k > 4$. $\square$

We use symbol $2\mathfrak{N} = \{F : \frac{1}{2}F \in \mathfrak{N}\}$. $(2\mathfrak{N}) \cap \mathfrak{N}$ is clearly open and dense. As C^r is dense in C^0, an open and dense set $\mathfrak{H}(d, u^*) \subset C^r(R_d(u^*), \mathbb{R})$ clearly exists such that for each perturbation of generating function $G_1 \in \mathfrak{H}(d, u^*)$, we have

$$\mathfrak{F}_\sigma \cap \mathfrak{Z}(d, u^*) = \varnothing, \qquad \forall\, \sigma \in \mathbb{S},$$

where by abuse of terminology we continue to denote S_σ and its restriction to $R_d(u^*)$ by the same symbol.

Let $U = \overline{M \backslash N_\delta}$, $M_U(S) = \{u : S(u) = \min_{u \in U} S\}$ and

$$\mathfrak{Z} = \Big\{ S \in C^0(U, \mathbb{R}) : \ M_U(S) \text{ is totally disconnected } \Big\}.$$

Given $d_i > 0$, there are finitely many u_{ij} such that $\cup_j R_{d_i}(u_{ij}) \supseteq U$. Thus there exists a sequence $d_i \to 0$ and a countable set $\{u_{ij}\}$ such that

$$\left(\bigcap_{i,j=1}^{\infty} \mathfrak{H}(d_i, u_{ij}) \right) \bigcap \mathfrak{Z} = \varnothing.$$

Recall the lemma 3.5, we have the following

Lemma 7.3. *There exists a residual set $\mathcal{S}_\epsilon \subset \mathcal{B}_\epsilon \subset C^r(U,\mathbb{R})$, for each $G_1 \in \mathcal{S}_\epsilon$*

$$\pi_1 \mathcal{N}_0(c(\sigma),\tilde{M})\backslash(\mathcal{N}_0(c(\sigma),M)+\delta) = \{\text{is totally disconnected}\}.$$

Note we can write $G_1 = \sum_k G_{1k}$ so that each G_{1k} has simply connected support.

The perturbation to the generating function G can be achieved by perturbing the Hamiltonian function $H \to H' = H + \delta H$. To do that, let us introduce a differentiable function κ: $M \to \mathbb{R}$ such that $0 \le \kappa(u-u') \le 1$, $\kappa(u-u') = 1$ if $|u-u'| \le K$ and $\kappa(u-u') = 0$ if $|u-u'| \ge K+1$. We choose sufficiently large K so that $\{\|v\| \le \max(|A|,|B|)+1\}$ is contained in the set where $|u-u'| \le K$. Let Φ' be the map determined by the generating function $G+\kappa G_1$, the symplectic diffeomorphism $\Psi = \Phi' \circ \Phi^{-1}$ is closed to identity. We choose a smooth function $\rho(s)$ with $\rho(0) = 0$ and $\rho(1) = 1$, let Φ'_s be the symplectic map determined by $G+\rho(s)\kappa G_{1k}$, let $\Psi_s = \Phi'_s \circ \Phi^{-1}$. Clearly, Ψ_s defines a symplectic isotopy between identity map and Ψ. Thus, there is a unique family of symplectic vector field X_s: $T^*M \to TT^*M$ such that

$$\frac{d}{ds}\Psi_s = X_s \circ \Psi_s.$$

By the choice of perturbation, there is a simply connected and compact domain D_K such that $\Psi_s|_{T^*M\backslash D_K} = id$. It follows that there is a hamiltonian $H_1(u,v,s)$ such that $dH_1(Y) = dv \wedge du(X_s, Y)$ holds for any vector field Y. Re-parameterizing s by t we can make H_1 smoothly and periodically depend on t. To see that dH_1 is also small, let us mention a theorem of Weinstain [W]. A neighborhood of the identity map in the symplectic diffeomorphism group of a compact symplectic manifold $\mathbf{M}$ can be identified with a neighborhood of the zero in the vector space of closed 1-forms on $\mathbf{M}$. Since Hamiltomorphism is a subgroup of symplectic diffeomorphism, there is a function H', sufficiently close to H, such that $\Phi_{H_1} \circ \Phi_H = \Phi^t_{H'}|_{t=1}$.

The perturbation made to H does not change the dynamics around the cylinder, it means that the set of invariant circles remains unchanged if H is subject to the perturbation constructed this way.

In the case of twist map, each co-homology class corresponds to a unique rotation number. Obviously, for each rotation number $p/q \in \mathbb{Q}$, there is an open and dense set in the space of area-preserving twist maps such that there is only one minimal (p,q)-periodic orbit without homoclinic loop. Take the intersection of countably open dense sets it is a generic property that there is only one minimal (p,q)-periodic orbit without homoclinic loop for all $p,q \in \mathbb{Z}$. Recall that the minimal measure is always uniquely ergodic when the rotation number is irrational, there is a residual set in $\mathcal{B}_{\epsilon,K}$, if L_1 is in this set, then there is a generalized transition chain Γ: $[0,1] \to H^1(M,\mathbb{R}) \cap \{c_x = 0\}$ which connects $\{c_q \le A\}$ with $\{c_q \ge B\}$. For each c in a transition piece, $\mathcal{M}(c)$ is uniquely ergodic, thus the conditions of the theorem 5.2 are satisfied.

Therefore, the proof of the theorem 1.1 is completed.

Remark: Since the time for each orbit drifts from $\{p \le A\}$ to $\{p \ge B\}$ is finite, the smooth dependence of solutions of ODE's on parameters implies that the theorem still holds if the generic condition is replaced by the open and dense condition for perturbation.

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