Zero-Viscosity Limit of the Linearized Compressible Navier-Stokes Equations with Highly Oscillatory Forces in the Half-Plane

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Abstract. We study the asymptotic expansion of solutions to the linearized compressible Navier-Stokes equations with highly oscillatory forces in the half-plane with nonslip boundary conditions for small viscosity. The wave length of oscillations is assumed to be proportional to the square root of the viscosity. By means of asymptotic analysis, we deduce that the zero-viscosity limit of solutions satisfies a linearized Euler system away from the boundary, and oscillations are propagated in a way of linear geometric optics in free space. In a small neighborhood of boundary, a boundary layer appears and satisfies a linearized Prandtl system. There is an interaction between the boundary layer and highly oscillatory waves near the boundary, which is described by an initial-boundary value problem for a Poisson-Prandtl coupled system. Finally, by using the energy method and mode analysis, we obtain the well-posedness of this Poisson-Prandtl coupled problem, and a rigorous theory on the asymptotic analysis of the zero-viscosity limit.

Key words. linearized compressible Navier-Stokes equations, boundary layers, oscillatory waves

AMS subject classification. 35Q30, 76N20, 35B05

1 Introduction

Consider the following initial-boundary value problem for the two-dimensional isentropic compressible Navier-Stokes equations with nonslip boundary conditions in $\{t, x_1 > 0, x_2 \in I\!R\}$:

$$\begin{cases} \partial_t \rho + (v \cdot \nabla)\rho + \rho \nabla \cdot v = f(t,x) \\ \rho(\partial_t v + (v \cdot \nabla)v) + \nabla p = \nabla \cdot (2\mu P + \lambda I_2 \nabla \cdot v) + g(t,x) \\ v|_{x_1=0} = 0 \\ (\rho, v)|_{t=0} = (\rho_0, v_0)(x) \end{cases} \tag{1.1}$$

where f and g represent the source and force terms, $P = \frac{1}{2}\{\partial_{x_j} v_i + \partial_{x_i} v_j\}_{i \times j}$ is a 2×2 matrix with $v = (v_1, v_2)^T$, $p = p(\rho)$ is the equation of state, μ and λ denote the coefficient and the second coefficient of viscosity respectively with $\mu > 0$ and $\mu' = \mu + \lambda \geq 0$. Corresponding to (1.1), the motion of an inviscid compressible fluid is governed by the following Euler equations:

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$$
\begin{cases}
\partial_t \rho + (v\cdot\nabla)\rho + \rho\nabla\cdot v = f(t,x) \\
\rho(\partial_t v + (v\cdot\nabla)v) + \nabla p = g(t,x) \\
v_1|_{x_1=0} = 0 \\
(\rho, v)|_{t=0} = (\rho_0, v_0)(x)
\end{cases}
\tag{1.2}
$$

For simplicity, we assume that μ and μ' are proportional to a parameter, say ϵ^2 with $\epsilon > 0$.

One of the interesting problems is to study the asymptotic convergence of solutions to the Navier-Stokes system (1.1) to the ones of the Euler system (1.2) in the limit of small viscosity. It is expected that uniform convergence can take place only away from the physical boundary $\{x_1 = 0\}$ even for smooth solutions of (1.2) due to the disparity of the boundary conditions in (1.1) and (1.2), and a thin region comes out near the boundary $\{x_1 = 0\}$ (the boundary layer) in which the values of the unknown functions change drastically in the process of this limit.

It is a challenge problem to analyze rigorously this boundary layer phenomena displayed by the actual Navier-Stokes solutions. For the problem of incompressible Navier-Stokes equations, Prandtl carried out a formal analysis in his speech ([6]) in the International Congress of Mathematicians in 1904, and derived a nonlinear degenerate parabolic-elliptic coupled system for the velocity fields in the boundary layer, which is now called the Prandtl system. Under the monotonic assumption on the velocity of the outflow, Oleinik and her collaborators established the local existence of smooth solutions for the boundary value problems of the Prandtl system in 1960's, and their works were surveyed recently in the monography [5]. The existence and uniqueness of global weak solutions to the Prandtl system are recently established by Xin, Zhang [13] and Xin, Zhang and Zhao [14] respectively. In [7, 8], Sammartino and Caflisch obtained the local existence of analytic solutions to the Prandtl system, and a rigorous theory on the boundary layer in incompressible fluids with analytic data in the frame of the abstract Cauchy-Kowaleskaya theory. Grenier ([2, 3]) investigated the stability of boundary layer type solutions to the Euler equations and the instability of solutions to the incompressible Navier-Stokes equations. Till now, there exists no general rigorous theory of viscous boundary layer in the case of nonslip boundary conditions. This is reviewed in [1, 11]. The problem of the viscous boundary layer in the case of slip boundary conditions was studied rigorously by Temam and Wang in [10].

To study the theory of the viscous boundary layer for compressible fluids with nonslip boundary conditions, recently, Xin and Yanagisawa ([12]) obtained a rigorous justification of the Prandtl boundary layer theory for the linearized compressible fluids when the viscosity goes to zero.

The purpose of this paper is to study the asymptotic behaviour of solutions to the linearized compressible Navier-Stokes equations in the half-plane with nonslip boundary conditions perturbed by high frequency oscilllatory force terms, and to investigate the interaction between the linearized boundary layer and rapidly oscillatory waves.

In the case that the oscillation of force terms is propagated along the tangential characteristic field of the boundary with respect to the linearized Euler operator, see (2.6)-(2.9), and the wave length is proportional to the square root of viscosities, we establish a rigorous theory on the boundary layer and its oscillatory behaviour. Roughly speeking, it is shown that the leading profile of solutions to the linearized compressible Navier-Stokes equations can be divided into four terms: the first term is the outflow satisfying the linearized Euler equations, the second term is an oscillatory wave in the whole half-plane, which is propagated along the characteristic field tangential to the boundary associated with the linearized Euler operator, and its amplitude satisfies a linear degenerate parabolic equation with the second order term coming from the viscous term in the linearized Navier-Stokes equations, the third term is the classical linearized Prandtl boundary layer supported in a thin neighborhood of the boundary, and the fourth term has oscillations with the phase being the trace of the oscillatory phase in the force terms, this fourth term together with its vorticity with resepect to the normal variable and the fast variable satisfy an initial-boundary value problem for a Poisson-Prandtl coupled system. This result shows that the zero-viscosity limit of solutions to the linearized compressible Navier-Stokes equations with highly oscillatory forces satisfies the linearized Euler equations away from the boundary, and oscillations are propagated in a way of linear geometric optics in free space. The boundary layer is of the Prandtl type as usual, but the novelties are that oscillations are propagated in the layer, and there is an interaction between the boundary layer and highly oscillatory waves near the boundary. For detail, see Theorem 4.1.

The nonlinear interaction between the boundary layer and high frequency oscillating waves for the artificial viscosity problem of a semilinear hyperbolic system was studied by Gues in [4], for which the leading profiles of solutions have three terms: the first one is the outflow satisfying the hyperbolic problem, the second one is an oscillatory wave in the whole half space, its amplitude satisfies an initial value problema for a degenerate parabolic equation, and the third one describes the boundary layer, which satisfies an initial-boundary value problem for a degenerate parabolic equation. Due to the nonlinearity of the system, problems for these three profiles are coupled each other. Main differences between this paper with Gues' work [4] are that the profile of the boundary layer in the Navier-Stokes system satisfies the Prandtl system even when the force terms without oscillations, and the phase function of oscillations we will study is nonlinear in general, which gives rise to the above fourth profile, describing the oscillations in the boundary layer, while the phase function of the oscillatory waves considered by Gues in [4] is linear and vanishes at the boundary, which implies that the above fourth term does not appear in that case (see Remark 2.1).

Another related work is that of Szepessy in [9], which gave a geometric optics expansion for a linearized viscous shock profile perturbed by a highly oscillatory wave in two space variables.

The remainder of this paper shall be arranged as follows: In §2, we carry out the formal analysis to derive problems for each profile of the asymptotic expansion of the solution to the linearized Navier-Stokes equations with respect to ϵ, proportional to the square root of viscosities, and observe the interesting phenomenon which we mentioned just above. The problem for the Poisson-Prandtl coupled equations is not a classical one. To our knowledge, there is not any literature devoted to this kind problem, so we shall establish the well-posedness of this problem in §3. Finally, in §4, we rigorously justify the formal analysis of §2 for the zero-viscosity limit of the solution to the linearized Navier-Stokes equations.

2 Asymptotic Analysis

Corresponding to the problem (1.1) for the compressible Navier-Stokes equations, let us consider the following linearized problem at a state $V' = (\rho', v_1', v_2')^T$ with high frequency oscillatory force terms in the half-space $\{t, x_1 > 0, x_2 \in I\!R\}$:

$$\begin{cases} A_0(V')\partial_t V^\epsilon + A_1(V')\partial_{x_1} V^\epsilon + A_2(V')\partial_{x_2} V^\epsilon = B(\epsilon^2, D\epsilon^2)V^\epsilon + \Phi(t,x;\frac{\varphi(t,x)}{\epsilon}) \\ M^+ V^\epsilon = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V^\epsilon = 0, \qquad \text{on} \quad x_1 = 0 \\ V^\epsilon|_{t=0} = V_0(x) \end{cases} \tag{2.1}$$

where $V^\epsilon = (\rho^\epsilon, v_1^\epsilon, v_2^\epsilon)^T$, $\Phi(t,x;\theta)$ is periodic in $\theta \in T^1 = I\!R/2\pi Z$,

$$A_0(V') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho' & 0 \\ 0 & 0 & \rho' \end{pmatrix}, \quad A_1(V') = \begin{pmatrix} v_1' & \rho' & 0 \\ c^2 & \rho' v_1' & 0 \\ 0 & 0 & \rho' v_1' \end{pmatrix}, \quad A_2(V') = \begin{pmatrix} v_2' & 0 & \rho' \\ 0 & \rho' v_2' & 0 \\ c^2 & 0 & \rho' v_2' \end{pmatrix}$$

with $c = \sqrt{\frac{dp(\rho')}{d\rho}} > 0$ being the sound speed at V', and

$$B(\epsilon^2, D\epsilon^2)V^\epsilon = \epsilon^2(B_1\partial_{x_1}^2 V^\epsilon + B_2\partial_{x_2}^2 V^\epsilon + B_3\partial_{x_1x_2}^2 V^\epsilon)$$

with $D \geq 0$ being a constant, and

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1+D & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+D \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & D \\ 0 & D & 0 \end{pmatrix}$$

where we assume that $\mu = \epsilon^2$ and $\mu' = D\epsilon^2$.

For convenience we shall assume that the background state V' is smooth. The case of finite regularity can be handled as below, but much more bookkeeping is needed.

Suppose that

$$v_1'|_{x_1=0} = 0. \tag{2.2}$$

For any fixed $(\xi_1, \xi_2) \neq (0,0)$, denote by

$$\tau_1 = -(\xi_1 v_1' + \xi_2 v_2'), \quad \tau_{2,3} = -(\xi_1 v_1' + \xi_2 v_2' \pm c\sqrt{\xi_1^2 + \xi_2^2}) \tag{2.3}$$

the eigenvalues of the symbol $L(\tau, \xi_1, \xi_2)$ associated with the linearized Euler operator at V',

$$L(\partial_t, \partial_x) = A_0(V')\partial_t + A_1(V')\partial_{x_1} + A_2(V')\partial_{x_2} \tag{2.4}$$

which means that τ_k are roots to the following characteristic equation:

$$\det(\tau A_0(V') + \xi_1 A_1(V') + \xi_2 A_2(V')) = 0.$$

Denote by $\{\vec{r}_k\}_{k=1}^3$ and $\{\vec{l}_k\}_{k=1}^3$ the associated right and left eigenvectors respectively,

$$\begin{cases} (\tau_k A_0(V') + \xi_1 A_1(V') + \xi_2 A_2(V'))\vec{r}_k = 0 \\ \vec{l}_k(\tau_k A_0(V') + \xi_1 A_1(V') + \xi_2 A_2(V')) = 0 \end{cases} \tag{2.5}$$

with the normalization

$$\vec{l}_j A_0 \vec{r}_k = \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}.$$

From (2.2), we know that the boundary $\{x_1 = 0\}$ is uniformly characteristic with respect to the eigenvalue $\tau_1 = -(\xi_1 v_1' + \xi_2 v_2')$ associated with the linearized Euler operator (2.4).

As in the classical theory of nonlinear geometric optics, we assume that the oscillation phase $\varphi(t,x)$ in (2.1) satisfies the eikonal equation with respect to τ_1,

$$\partial_t \varphi + v_1' \partial_{x_1}\varphi + v_2'\partial_{x_2}\varphi = 0. \tag{2.6}$$

In this paper, we shall assume

$$\varphi^0(t, x_2) := \varphi(t, 0, x_2) \neq 0. \tag{2.7}$$

Obviously, by using the assumption $v_1'|_{x_1=0}=0$, we get

$$\varphi_t^0 + v_2'(0)\varphi_{x_2}^0 = 0 \tag{2.8}$$

with $v_2'(0)$ denoting $v_2'(t,0,x_2)$.

In this paper, we assume

$$\partial_{x_2}\varphi^0 = \partial_{x_2}\varphi|_{x_1=0} \neq 0 \tag{2.9}$$

at each point of $\{(t,x_2)\in I\!R^+ \times I\!R\}$. If $\varphi_{x_2}^0 \equiv 0$, then from (2.8) we have $\varphi_t^0 = 0$ as well, which implies

$$\varphi^0(t,x_2) \equiv const.$$

yielding no oscillation factor in the boundary layer. The problem in the general case of φ, e.g. $\varphi(t,0,x_2)$ degenerates in a subset of $(t,x_2)\in I\!R^+\times I\!R$ is interesting, and shall be investigated in the future. As we shall see, the case $\varphi(t,0,x_2)\equiv 0$ is easier to handle.

In the case (2.6)—(2.9), we take the following ansatz for the solution of (2.1):

$$V^\epsilon(t,x) = V_{in}^\epsilon(t,x) + V_{bd}^\epsilon(t,x) \tag{2.10}$$

where

$$\begin{cases} V_{in}^\epsilon(t,x) = \sum_{j\geq 0}\epsilon^j\big(a_j(t,x) + c_j(t,x;\frac{\varphi(t,x)}{\epsilon})\big) \\ V_{bd}^\epsilon(t,x) = \sum_{j\geq 0}\epsilon^j\big(b_j(t,x_2;\frac{x_1}{\epsilon}) + d_j(t,x_2;\frac{x_1}{\epsilon},\frac{\varphi^0(t,x_2)}{\epsilon})\big) \end{cases} \tag{2.11}$$

where $c_j(t,x;\theta)$ and $d_j(t,x_2;z,\theta)$ are $2\pi-$periodic in θ with mean value vanishing, and $b_j(t,x_2;z)$ and $d_j(t,x_2;z,\theta)$ are rapidly decreasing in z when $z\to+\infty$.

In the sequel, we shall always denote by $C_p^k(T_\theta^1)$ the set of $k-$th order smooth functions which are $2\pi-$periodic in $\theta\in T^1$, $S(I\!R_z^+)$ the set of smooth functions rapidly decreasing in z when $z\to+\infty$, and $a_j^{(k)}$ $(k=1,2,3)$ the $k-$th component of a_j etc..

Taking the formal expansion as

$$(A_0(V')\partial_t + A_1(V')\partial_{x_1} + A_2(V')\partial_{x_2})V_{in}^\epsilon - B(\epsilon^2, D\epsilon^2)V_{in}^\epsilon - \Phi(t,x;\frac{\varphi(t,x)}{\epsilon}) = \sum_{j\geq -1}\epsilon^j\mathcal{F}_j, \tag{2.12}$$

in ϵ, we have

$$\begin{cases} \mathcal{F}_{-1} = \sum_{k=0}^2 \varphi_{x_k}A_k(V')\partial_\theta c_0 \\ \mathcal{F}_0 = L(\partial_t,\partial_x)(a_0+c_0) - (\varphi_{x_1}^2B_1 + \varphi_{x_2}^2B_2 + \varphi_{x_1}\varphi_{x_2}B_3)\partial_\theta^2 c_0 - \Phi(t,x;\theta) + \sum_{k=0}^2\varphi_{x_k}A_k(V')\partial_\theta c_1 \\ \cdots\cdots \\ \mathcal{F}_j = L(\partial_t,\partial_x)(a_j+c_j) - (\varphi_{x_1}^2B_1 + \varphi_{x_2}^2B_2 + \varphi_{x_1}\varphi_{x_2}B_3)\partial_\theta^2 c_j + \sum_{k=0}^2\varphi_{x_k}A_k(V')\partial_\theta c_{j+1} + f_j \end{cases} \tag{2.13}$$

for each $j\geq 1$, where $\varphi_{x_k} = \partial_{x_k}\varphi$ with $x_0 = t$, and

$$\begin{aligned} f_j = \;& -(\varphi_{x_1x_1}B_1 + \varphi_{x_2x_2}B_2 + \varphi_{x_1x_2}B_3)\partial_\theta c_{j-1} - (2\varphi_{x_1}B_1 + \varphi_{x_2}B_3)\partial_{\theta x_1}^2 c_{j-1} \\ & -(2\varphi_{x_2}B_2 + \varphi_{x_1}B_3)\partial_{\theta x_2}^2 c_{j-1} - (B_1\partial_{x_1}^2 + B_2\partial_{x_2}^2 + B_3\partial_{x_1x_2}^2)(a_{j-2}+c_{j-2}) \end{aligned}$$

with $a_{-1} = c_{-1} = 0$.

Letting $z = \frac{x_1}{\epsilon}$, and

$$(A_0(V')\partial_t + A_1(V')\partial_{x_1} + A_2(V')\partial_{x_2})V_{bd}^\epsilon - B(\epsilon^2, D\epsilon^2)V_{bd}^\epsilon = \sum_{j\geq -1}\epsilon^j\mathcal{G}_j, \tag{2.14}$$

then we have

$$
\begin{cases}
\mathcal{G}_{-1}=(\varphi_t^0A_0(0)+\varphi_{x_2}^0A_2(0))\partial_\theta d_0+A_1(0)\partial_z(b_0+d_0)\\
\mathcal{G}_0=L_{bd}(\partial_t,\partial_{x_2})(b_0+d_0)+z(\varphi_t^0A_0'(0)+\varphi_{x_2}^0A_2'(0))\partial_\theta d_0+zA_1'(0)\partial_z(b_0+d_0)\\
\qquad -B_1\partial_z^2b_0-(B_1\partial_z^2+(\varphi_{x_2}^0)^2B_2\partial_\theta^2+\varphi_{x_2}^0B_3\partial_{z\theta}^2)d_0\\
\qquad +(\varphi_t^0A_0(0)+\varphi_{x_2}^0A_2(0))\partial_\theta d_1+A_1(0)\partial_z(b_1+d_1)\\
\cdots\cdots\\
\mathcal{G}_j=L_{bd}(\partial_t,\partial_{x_2})(b_j+d_j)+z(\varphi_t^0A_0'(0)+\varphi_{x_2}^0A_2'(0))\partial_\theta d_j+zA_1'(0)\partial_z(b_j+d_j)\\
\qquad -(B_1\partial_z^2+(\varphi_{x_2}^0)^2B_2\partial_\theta^2+\varphi_{x_2}^0B_3\partial_{z\theta}^2)d_j-B_1\partial_z^2b_j\\
\qquad +(\varphi_t^0A_0(0)+\varphi_{x_2}^0A_2(0))\partial_\theta d_{j+1}+A_1(0)\partial_z(b_{j+1}+d_{j+1})+g_j
\end{cases}
\tag{2.15}
$$

for any $j\geq 1$, where g_j depends smoothly on $\{b_k,d_k\}_{k\leq j-1}$ and their derivatives up to order two, $A_k(0)=A_k(V')|_{x_1=0}$, $A_k'(0)=\partial_{x_1}(A_k(V'))|_{x_1=0}$, and

$$L_{bd}(\partial_t,\partial_{x_2})=A_0(0)\partial_t+A_2(0)\partial_{x_2}.$$

From the equations in (2.1) and the assumption that each term (b_j,d_j) in V_{bd}^ϵ is rapidly decreasing in z when $z\to+\infty$, it is natural to set

$$\mathcal{F}_j\equiv 0 \qquad \text{and} \qquad \mathcal{G}_j\equiv 0 \tag{2.16}$$

in (2.12) and (2.14) respectively for all $j\geq -1$.

The next step is to derive the governing problems for various order of profiles from (2.16) and initial and boundary conditions given in (2.1).

Let $\{\vec{r}_k(\nabla\varphi),\vec{l}_k(\nabla\varphi)\}_{k=1}^3$ be the right and left eigenvectors given in (2.5) associated with $(\xi_1,\xi_2)=(\varphi_{x_1},\varphi_{x_2})$.

It follows from $\mathcal{F}_{-1}=0$ that

$$c_0(t,x;\theta)=v_0(t,x;\theta)\vec{r}_1(\nabla\varphi) \tag{2.17}$$

with $v_0(t,x;\theta)$ being a scalar function.

Acting the mean value operator

$$\mathbf{m}_\theta(u)=\frac{1}{2\pi}\int_0^{2\pi}u(\theta)d\theta$$

on the equation $\mathcal{F}_0=0$, we deduce

$$L(\partial_t,\partial_x)a_0=\mathbf{m}_\theta(\Phi) \tag{2.18}$$

and the difference between (2.18) and $\mathcal{F}_0=0$ gives

$$L(\partial_t,\partial_x)c_0-(\varphi_{x_1}^2B_1+\varphi_{x_2}^2B_2+\varphi_{x_1}\varphi_{x_2}B_3)\partial_\theta^2c_0-\Phi+\mathbf{m}_\theta(\Phi)=-\sum_{k=0}^{2}\varphi_{x_k}A_k(V')\partial_\theta c_1. \tag{2.19}$$

Multiplying $\vec{l}_1(\nabla\varphi)$ from the left of (2.19), and using (2.17), it follows that $v_0(t,x;\theta)$ satisfies the following problem:

$$
\begin{cases}
[(\vec{l}_1A_0\vec{r}_1)\partial_t+(\vec{l}_1A_1\vec{r}_1)\partial_{x_1}+(\vec{l}_1A_2\vec{r}_1)\partial_{x_2}]v_0+\vec{l}_1(A_0\partial_t\vec{r}_1+A_1\partial_{x_1}\vec{r}_1+A_2\partial_{x_2}\vec{r}_1)v_0\\
\qquad -\vec{l}_1(\varphi_{x_1}^2B_1+\varphi_{x_2}^2B_2+\varphi_{x_1}\varphi_{x_2}B_3)\vec{r}_1\partial_\theta^2v_0=\vec{l}_1(\Phi-\mathbf{m}_\theta(\Phi))\\
v_0|_{t=0}=0
\end{cases}.
\tag{2.20}
$$

Noting that the vector field

$$(\vec{l}_1A_0\vec{r}_1)\partial_t+(\vec{l}_1A_1\vec{r}_1)\partial_{x_1}+(\vec{l}_1A_2\vec{r}_1)\partial_{x_2}$$

is tangential to the boundary $\{x_1=0\}$, and

$$\vec{l}_1(\varphi_{x_1}^2 B_1+\varphi_{x_2}^2 B_2+\varphi_{x_1}\varphi_{x_2}B_3)\vec{r}_1=\frac{1}{\rho'}(\varphi_{x_1}^2+\varphi_{x_2}^2)>0$$

the problem (2.20) is the one for a linear degenerate parabolic equation, which can be easily solved.

To solve a_0 from (2.18), we need to impose a boundary data for $a_0^{(2)}$ on $\{x_1=0\}$.

It follows from the ansatz (2.10) and (2.11) that for any $j\geq 0$, the $0(\epsilon^j)-$term of the boundary condition $M^+V^\epsilon|_{x_1=0}=0$ in (2.1) gives

$$a_j^{(k)}(t,x)+c_j^{(k)}(t,x;\theta)+b_j^{(k)}(t,x_2;z)+d_j^{(k)}(t,x_2;z,\theta^0)=0 \tag{2.21}$$

on $\{x_1=0,z=0,\theta=\theta^0\}$ for $k\in\{2,3\}$. Since $c_j^{(k)}$ and $d_j^{(k)}$ are $2\pi-$periodic in θ and θ^0, with mean values vanishing respectively, the condition (2.21) is equivalent to

$$\begin{cases} a_j^{(k)}(t,x)+b_j^{(k)}(t,x_2;z)=0 \quad \text{on } \{x_1=z=0\} \\ c_j^{(k)}(t,x;\theta)+d_j^{(k)}(t,x_2;z,\theta^0)=0 \quad \text{on } \{x_1=z=0,\theta=\theta^0\} \end{cases} \tag{2.22}$$

for $k\in\{2,3\}$.

Thus, we should first study $b_0^{(2)}$ to determine the boundary condition of $a_0^{(2)}$ on $\{x_1=0\}$.

Taking the mean value operator $\mathbf{m}_\theta$ on $\mathcal{G}_{-1}=0$ leads to

$$A_1(0)\partial_z b_0=0. \tag{2.23}$$

So, $\mathcal{G}_{-1}=0$ gives rise to

$$(\varphi_t^0 A_0(0)+\varphi_{x_2}^0 A_2(0))\partial_\theta d_0+A_1(0)\partial_z d_0=0. \tag{2.24}$$

From (2.23), we obtain immediately that

$$\partial_z b_0^{(1)}=\partial_z b_0^{(2)}=0$$

which implies

$$b_0^{(1)}=b_0^{(2)}\equiv 0 \tag{2.25}$$

by using $b_0\in S(I\!R_z^+)$.

Thus, it follows from (2.18) and (2.22) that $a_0(t,x)$ satisfies the following problem for the linearized Euler equations:

$$\begin{cases} (A_0(V')\partial_t+A_1(V')\partial_{x_1}+A_2(V')\partial_{x_2})a_0=\mathbf{m}_\theta(\Phi), \quad t,x_1>0 \\ a_0^{(2)}|_{x_1=0}=0 \\ a_0|_{t=0}=V_0(x) \end{cases}. \tag{2.26}$$

To determine $b_0^{(3)}(t,x_2;z)$, we take the mean value of $\mathcal{G}_0=0$ to deduce

$$L_{bd}(\partial_t,\partial_{x_2})b_0+zA_1'(0)\partial_z b_0+A_1(0)\partial_z b_1=B_1\partial_z^2 b_0 \tag{2.27}$$

and the difference between (2.27) and $\mathcal{G}_0=0$ gives rise to

$$L_{bd}(\partial_t,\partial_{x_2})d_0+z(\varphi_t^0 A_0'(0)+\varphi_{x_2}^0 A_2'(0))\partial_\theta d_0+zA_1'(0)\partial_z d_0-(B_1\partial_z^2+(\varphi_{x_2}^0)^2 B_2\partial_\theta^2+\varphi_{x_2}^0 B_3\partial_{z\theta}^2)d_0$$
$$=-(\varphi_t^0 A_0(0)+\varphi_{x_2}^0 A_2(0))\partial_\theta d_1-A_1(0)\partial_z d_1. \tag{2.28}$$

From the third component of (2.27), we conclude that $b_0^{(3)}(t,x_2;z)$ satisfies the following problem:

$$\begin{cases} (\partial_t + v_2'(0)\partial_{x_2})b_0^{(3)} + z\frac{\partial v_1'(0)}{\partial x_1}\partial_z b_0^{(3)} - \frac{1}{\rho'(0)}\partial_z^2 b_0^{(3)} = 0, \quad t,\ z>0 \\ b_0^{(3)}|_{z=0} = -a_0^{(3)}(t,0,x_2) \\ b_0^{(3)}|_{t=0} = 0 \end{cases} \tag{2.29}$$

where $a_0^{(3)}$ is given by (2.26) .

The problem (2.29) is the one for a linearized Prandtl equation, which has been solved by Xin and Yanagisawa in [12].

Now, let us derive determine $d_0(t,x_2;z,\theta)$ from (2.24) and (2.28).

By using $\varphi_t^0 + v_2'(0)\varphi_{x_2}^0 = 0$, we know that

$$(\varphi_t^0 A_0(0) + \varphi_{x_2}^0 A_2(0))\partial_\theta d + A_1(0)\partial_z d = \begin{pmatrix} \rho'(0)(\varphi_{x_2}^0\partial_\theta d^{(3)} + \partial_z d^{(2)}) \\ c^2(0)\partial_z d^{(1)} \\ c^2(0)\varphi_{x_2}^0\partial_\theta d^{(1)} \end{pmatrix}.$$

Thus, it follows from (2.24) that

$$\varphi_{x_2}^0\partial_\theta d_0^{(3)} + \partial_z d_0^{(2)} = 0 \tag{2.30}$$

and

$$\varphi_{x_2}^0\partial_\theta d_0^{(1)} = 0, \quad \partial_z d_0^{(1)} = 0$$

which implies

$$d_0^{(1)} \equiv 0. \tag{2.31}$$

To solve $(d_0^{(2)}, d_0^{(3)})$, we define $I\!\!E$ by

$$I\!\!E\begin{pmatrix} d^{(1)} \\ d^{(2)} \\ d^{(3)} \end{pmatrix} = \begin{pmatrix} \mathbf{m}_\theta d^{(1)} \\ \varphi_{x_2}^0\partial_\theta d^{(2)} - \partial_z d^{(3)} \end{pmatrix}$$

for any $d = (d^{(1)}, d^{(2)}, d^{(3)})^T \in C^1(I\!\!R_z^+ \times T_\theta^1)$. It is easy to know that for any $d(t,x_2;z,\theta) \in C_p^1(T_\theta^1) \cap S(I\!\!R_z^+)$ with $\mathbf{m}_\theta(d) = 0$, we have

$$I\!\!E((\varphi_t^0 A_0(0) + \varphi_{x_2}^0 A_2(0))\partial_\theta d + A_1(0)\partial_z d) = 0. \tag{2.32}$$

Acting the operator $I\!\!E$ on (2.28) and using (2.32), one gets

$$I\!\!E(\text{left hand side of (2.28)}) = 0. \tag{2.33}$$

Denote by A and B the second and the third components of the left hand side of (2.28) respectively. Then, by using (2.30) and (2.31), we deduce

$$\begin{cases} A = \rho'(0)((\partial_t + v_2'(0)\partial_{x_2})d_0^{(2)} + z\frac{\partial v_1'(0)}{\partial x_1}\partial_z d_0^{(2)} + z\frac{\partial v_2'(0)}{\partial x_1}\varphi_{x_2}^0\partial_\theta d_0^{(2)}) - (\partial_z^2 + (\varphi_{x_2}^0)^2\partial_\theta^2)d_0^{(2)} \\ B = \rho'(0)((\partial_t + v_2'(0)\partial_{x_2})d_0^{(3)} + z\frac{\partial v_1'(0)}{\partial x_1}\partial_z d_0^{(3)} + z\frac{\partial v_2'(0)}{\partial x_1}\varphi_{x_2}^0\partial_\theta d_0^{(3)}) - (\partial_z^2 + (\varphi_{x_2}^0)^2\partial_\theta^2)d_0^{(3)} \end{cases}$$

From (2.33), we obtain

$$\varphi_{x_2}^0\partial_\theta A - \partial_z B = 0$$

which can be explicitely written as

$$(\partial_t + v_2'(0)\partial_{x_2})\omega_0 + z(\frac{\partial v_1'(0)}{\partial x_1}\partial_z + \frac{\partial v_2'(0)}{\partial x_1}\varphi^0_{x_2}\partial_\theta)\omega_0 - \frac{1}{\rho'(0)}(\partial_z^2 + (\varphi^0_{x_2})^2\partial_\theta^2)\omega_0$$
$$-(\frac{\partial v_1'(0)}{\partial x_1}\partial_z + \frac{\partial v_2'(0)}{\partial x_1}\varphi^0_{x_2}\partial_\theta)d_0^{(3)} = 0 \tag{2.34}$$

where $\omega_0(t, x_2; z, \theta) = \varphi^0_{x_2}\partial_\theta d_0^{(2)} - \partial_z d_0^{(3)}$.

Combining (2.30) with (2.34), and using (2.22) one obtains that $(d_0^{(3)}, \omega_0)(t, x_2; z, \theta)$ satisfy the following problem:

$$\begin{cases} (\partial_z^2 + (\varphi^0_{x_2})^2\partial_\theta^2)d_0^{(3)} = -\partial_z\omega_0 \\ (\partial_t + v_2'(0)\partial_{x_2})\omega_0 + z(\frac{\partial v_1'(0)}{\partial x_1}\partial_z + \varphi^0_{x_2}\frac{\partial v_2'(0)}{\partial x_1}\partial_\theta)\omega_0 - \frac{1}{\rho'(0)}(\partial_z^2 + (\varphi^0_{x_2})^2\partial_\theta^2)\omega_0 \\ \qquad -(\frac{\partial v_1'(0)}{\partial x_1}\partial_z + \varphi^0_{x_2}\frac{\partial v_2'(0)}{\partial x_1}\partial_\theta)d_0^{(3)} = 0 \\ d_0^{(3)}|_{z=0} = -c_0^{(3)}(t, 0, x_2; \theta) \\ (\omega_0 + \partial_z d_0^{(3)})|_{z=0} = -\varphi^0_{x_2}(\partial_\theta c_0^{(2)})(t, 0, x_2; \theta) \\ (d_0^{(3)}, \omega_0) \in S(I\!R_z^+) \\ \omega_0|_{t=0} = 0 \end{cases} \tag{2.35}$$

and $d_0^{(2)}(t, x_2; z, \theta)$ satisfies

$$\begin{cases} (\partial_z^2 + (\varphi^0_{x_2})^2\partial_\theta^2)d_0^{(2)} = \varphi^0_{x_2}\partial_\theta\omega_0 \\ d_0^{(2)}|_{z=0} = -c_0^{(2)}(t, 0, x_2; \theta) \\ d_0^{(2)} \in S(I\!R_z^+) \end{cases} \tag{2.36}$$

where $(c_0^{(2)}, c_0^{(3)})$ are given by (2.17)(2.20).

In summary, by formal analysis, we conclude:

- the leading terms $a_0(t, x)$ and $c_0(t, x; \frac{\varphi(t,x)}{\epsilon}) = v_0(t, x; \frac{\varphi(t,x)}{\epsilon})\vec{r}_1(\nabla\varphi)$ of the inner solution $V_{in}^\epsilon(t, x)$ satisfy the initial-boundary value problems for the linearized Euler equations (2.26) and for the degenerate parabolic equation (2.20) respectively;

- the leading terms $b_0(t, x_2; \frac{x_1}{\epsilon})$ and $d_0(t, x; \frac{x_1}{\epsilon}, \frac{\varphi^0(t,x_2)}{\epsilon})$ of the boundary layer $V_{bd}^\epsilon(t, x)$ satisfy (2.25) and the problems for the Prandtl equation (2.29), the Poisson equation (2.36) and the Poisson-Prandtl coupled equations (2.35) respectively.

The problems for high order terms in expansions of $V_{in}^\epsilon(t, x) + V_{bd}^\epsilon(t, x)$ can be formulated in a similar way. For completeness, let us sketch the idea. Suppose that $\{a_k(t, x), c_k(t, x; \theta), b_k(t, x_2; z), d_k(t, x_2; z, \theta)\}_{k\le j}$ are known already, we want to determine $\{a_{j+1}(t, x), c_{j+1}(t, x; \theta), b_{j+1}(t, x_2; z), d_{j+1}(t, x_2; z, \theta)\}$.

It follows from (2.13) and the fact $\mathbf{m}_\theta(\mathcal{F}_j) = 0$ that

$$L(\partial_t, \partial_x)a_j = (B_1\partial_{x_1}^2 + B_2\partial_{x_2}^2 + B_3\partial_{x_1x_2}^2)a_{j-2} \tag{2.37}$$

and the difference between $\mathcal{F}_j = 0$ and (2.37) gives rise to

$$\sum_{k=0}^{2} \varphi_{x_k} A_k(V')\partial_\theta c_{j+1} = \tilde{f}_j \tag{2.38}$$

where

$$\begin{aligned}\tilde{f}_j = \ & (\varphi_{x_1}^2 B_1 + \varphi_{x_2}^2 B_2 + \varphi_{x_1}\varphi_{x_2} B_3)\partial_\theta^2 c_j - L(\partial_t, \partial_x)c_j + (\varphi_{x_1x_1} B_1 + \varphi_{x_2x_2} B_2 + \varphi_{x_1x_2} B_3)\partial_\theta c_{j-1} \\ & +(2\varphi_{x_1} B_1 + \varphi_{x_2} B_3)\partial_{\theta x_1}^2 c_{j-1} + (2\varphi_{x_2} B_2 + \varphi_{x_1} B_3)\partial_{\theta x_2}^2 c_{j-1} \\ & +(B_1\partial_{x_1}^2 + B_2\partial_{x_2}^2 + B_3\partial_{x_1x_2}^2)c_{j-2}\end{aligned}$$

satisfies $\mathbf{m}_\theta(\tilde{f}_j) = 0$.

If we set

$$c_{j+1}(t, x; \theta) = \sum_{k=1}^{3} v_{j+1}^{(k)}(t, x; \theta)\vec{r}_k(\nabla\varphi), \tag{2.39}$$

then (2.38) yields

$$(\varphi_t - \tau_k(\nabla\varphi))\partial_\theta v_{j+1}^{(k)} = (\vec{l}_k(\nabla\varphi) \cdot \tilde{f}_j)(t, x; \theta), \quad k = 2, 3 \tag{2.40}$$

where $\tau_k(\nabla\varphi)$ are defined in (2.3). Due to the assumption (2.6), we obtain that $(v_{j+1}^{(2)}, v_{j+1}^{(3)})$ can be uniquely determined by (2.40) with $\mathbf{m}_\theta(v_{j+1}^{(2)}, v_{j+1}^{(3)}) = 0$.

To solve $v_{j+1}^{(1)}$, acting $\vec{l}_1(\nabla\varphi)$ from the left on the same equation as (2.38) with j being replaced by $j+1$, and using (2.39), one gets that $v_{j+1}^{(1)}$ satisfies the following problem:

$$\begin{cases} [(\vec{l}_1 A_0 \vec{r}_1)\partial_t + (\vec{l}_1 A_1 \vec{r}_1)\partial_{x_1} + (\vec{l}_1 A_2 \vec{r}_1)\partial_{x_2}]v_{j+1}^{(1)} + \vec{l}_1(A_0\partial_t \vec{r}_1 + A_1\partial_{x_1}\vec{r}_1 + A_2\partial_{x_2}\vec{r}_1)v_{j+1}^{(1)} \\ \qquad -\vec{l}_1(\varphi_{x_1}^2 B_1 + \varphi_{x_2}^2 B_2 + \varphi_{x_1}\varphi_{x_2} B_3)\vec{r}_1\partial_\theta^2 v_{j+1}^{(1)} = h_{j+1} \\ v_{j+1}^{(1)}|_{t=0} = 0 \end{cases} \tag{2.41}$$

which is similar to the problem (2.20), where

$$\begin{aligned} h_{j+1} = \ & \vec{l}_1[(\varphi_{x_1x_1} B_1 + \varphi_{x_2x_2} B_2 + \varphi_{x_1x_2} B_3)\partial_\theta c_j + (2\varphi_{x_1} B_1 + \varphi_{x_2} B_3)\partial_{\theta x_1}^2 c_j \\ & +(2\varphi_{x_2} B_2 + \varphi_{x_1} B_3)\partial_{\theta x_2}^2 c_j + (B_1\partial_{x_1}^2 + B_2\partial_{x_2}^2 + B_3\partial_{x_1x_2}^2)c_{j-1} \\ & -(L(\partial_t, \partial_x) - (\varphi_{x_1}^2 B_1 + \varphi_{x_2}^2 B_2 + \varphi_{x_1}\varphi_{x_2} B_3)\partial_\theta^2)(v_{j+1}^{(2)}\vec{r}_2 + v_{j+1}^{(3)}\vec{r}_3)]. \end{aligned}$$

It follows from (2.22) that in order to determine a_{j+1} from the same equation as (2.37) with j being replaced by $j+1$, one should impose the boundary condition of a_{j+1} as

$$a_{j+1}^{(2)}|_{x_1=0} = -b_{j+1}^{(2)}(t, x_2; 0),$$

thus one needs to study $b_{j+1}^{(2)}$ first.

Acting the averaging operator $\mathbf{m}_\theta$ on $\mathcal{G}_j = 0$ from (2.15), and using the assumption $\mathbf{m}_\theta(d_k) = 0$ for any $k \geq 0$, we get

$$A_1(0)\partial_z b_{j+1} = \tilde{g}_j(t, x_2; z) \tag{2.42}$$

and the difference between $\mathcal{G}_j = 0$ and (2.42) gives rise to

$$(\varphi_t^0 A_0(0) + \varphi_{x_2}^0 A_2(0))\partial_\theta d_{j+1} + A_1(0)\partial_z d_{j+1} = g_j^\star(t, x_2; z, \theta) \tag{2.43}$$

where

$$\begin{cases} \tilde{g}_j(t, x_2; z) = B_1 \partial_z^2 b_j - (A_0(0)\partial_t + A_2(0)\partial_{x_2})b_j - zA_1'(0)\partial_z b_j - \mathbf{m}_\theta(g_j) \\ g_j^\star(t, x_2; z, \theta) = (B_1\partial_z^2 + (\varphi_{x_2}^0)^2 B_2 \partial_\theta^2 + \varphi_{x_2}^0 B_3 \partial_{z\theta}^2)d_j - (A_0(0)\partial_t + A_2(0)\partial_{x_2})d_j \\ \qquad\qquad -z(\varphi_t^0 A_0'(0) + \varphi_{x_2}^0 A_2'(0))\partial_\theta d_j - zA_1'(0)\partial_z d_j - g_j + \mathbf{m}_\theta(g_j) \end{cases}.$$

From (2.42), we deduce immediately that $(b_{j+1}^{(1)}, b_{j+1}^{(2)})$ solve the following problem:

$$\begin{cases} \begin{pmatrix} 0 & \rho'(0) \\ c^2(0) & 0 \end{pmatrix} \begin{pmatrix} \partial_z b_{j+1}^{(1)} \\ \partial_z b_{j+1}^{(2)} \end{pmatrix} = \begin{pmatrix} \tilde{g}_j^{(1)} \\ \tilde{g}_j^{(2)} \end{pmatrix} \\ (b_{j+1}^{(1)}, b_{j+1}^{(2)}) \in S(I\!R_z^+) \end{cases} \tag{2.44}$$

which implies

$$\begin{cases} b_{j+1}^{(1)}(t, x_2; z) = -c^{-2}(0) \int_z^{+\infty} \tilde{g}_j^{(2)}(t, x_2; \xi) d\xi \\ b_{j+1}^{(2)}(t, x_2; z) = -(\rho'(0))^{-1} \int_z^{+\infty} \tilde{g}_j^{(1)}(t, x_2; \xi) d\xi \end{cases} \tag{2.45}$$

Therefore, from the same equation as (2.37) with j being replaced by $j+1$, we know that $a_{j+1}(t, x)$ solves the following problem:

$$\begin{cases} L(\partial_t, \partial_x) a_{j+1} = (B_1 \partial_{x_1}^2 + B_2 \partial_{x_2}^2 + B_3 \partial_{x_1 x_2}^2) a_{j-1} \\ a_{j+1}^{(2)}|_{x_1=0} = (\rho'(0))^{-1} \int_0^{+\infty} \tilde{g}_j^{(1)}(t, x_2; \xi) d\xi \\ a_{j+1}|_{t=0} = 0 \end{cases} \tag{2.46}$$

To determine $b_{j+1}^{(3)}(t, x_2; z)$, we act the averaging operator $\mathbf{m}_\theta$ on $\mathcal{G}_{j+1} = 0$ with $\mathcal{G}_{j+1}$ being given as in (2.15), and obtain

$$L_{bd}(\partial_t, \partial_{x_2}) b_{j+1} + zA_1'(0)\partial_z b_{j+1} - B_1 \partial_z^2 b_{j+1} + A_1(0)\partial_z b_{j+2} + \mathbf{m}_\theta(g_{j+1}) = 0. \tag{2.47}$$

The third component of (2.47) shows that $b_{j+1}^{(3)}$ solves the following initial-boundary value problem for the linearized Prandtl equation:

$$\begin{cases} (\partial_t + v_2'(0)\partial_{x_2}) b_{j+1}^{(3)} + z\frac{\partial v_1'(0)}{\partial x_1} \partial_z b_{j+1}^{(3)} - \frac{1}{\rho'(0)} \partial_z^2 b_{j+1}^{(3)} = -\frac{c^2(0)}{\rho'(0)} \partial_{x_2} b_{j+1}^{(1)} - \mathbf{m}_\theta(g_{j+1}^{(3)}) \\ b_{j+1}^{(3)}|_{z=0} = -a_{j+1}^{(3)}(t, 0, x_2), \quad b_{j+1}^{(3)} \in S(I\!R_z^+) \\ b_{j+1}^{(3)}|_{t=0} = 0 \end{cases} \tag{2.48}$$

where $a_{j+1}^{(3)}$ is the third component of a_{j+1} given in (2.46), and $b_{j+1}^{(1)}$ is given already in (2.45).

It remains to determine $d_{j+1}(t, x_2; z, \theta)$. From (2.43), we get

$$\begin{cases} \varphi_{x_2}^0 \partial_\theta d_{j+1}^{(1)} = \frac{1}{c^2(0)} g_j^{\star(3)}, \quad \partial_z d_{j+1}^{(1)} = \frac{1}{c^2(0)} g_j^{\star(2)} \\ d_{j+1}^{(1)} \in S(I\!R_z^+) \end{cases} \tag{2.49}$$

and

$$\partial_z d_{j+1}^{(2)} + \varphi_{x_2}^0 \partial_\theta d_{j+1}^{(3)} = \frac{1}{\rho'(0)} g_j^{\star(1)}. \tag{2.50}$$

By using the fact (2.32) in (2.43), we know

$$I\!\!E(g_j^\star) = 0$$

which implies especially

$$\partial_z g_j^{\star(3)} - \varphi_{x_2}^0 \partial_\theta g_j^{\star(2)} = 0. \tag{2.51}$$

Obviously, (2.51) is the compatibility condition for solving $d_{j+1}^{(1)}$ from (2.49), and

$$d_{j+1}^{(1)} = -c^{-2}(0) \int_z^{+\infty} g_j^{\star(2)}(t, x_2; \xi, \theta) d\xi. \tag{2.52}$$

Acting the operator $I\!\!E$ on the same equations as in (2.43) with j being replaced by $j+1$, it follows

$$\begin{aligned} &I\!\!E(L_{bd}(\partial_t, \partial_{x_2}) d_{j+1} + z(\varphi_t^0 A_0'(0) + \varphi_{x_2}^0 A_2'(0))\partial_\theta d_{j+1} + zA_1'(0)\partial_z d_{j+1} \\ &-(B_1 \partial_z^2 + (\varphi_{x_2}^0)^2 B_2 \partial_\theta^2 + \varphi_{x_2}^0 B_3 \partial_{z\theta}^2) d_{j+1} + g_{j+1} - \mathbf{m}_\theta(g_{j+1})) = 0. \end{aligned} \tag{2.53}$$

Denote by $\tilde{A}$ and $\tilde{B}$ the second and the third components of the above term on which $I\!\!E$ acts. Due to (2.50), they can be expressed as:

$$\begin{cases} \tilde{A} = & \rho'(0)((\partial_t + v_2'(0)\partial_{x_2}) d_{j+1}^{(2)} + z(\frac{\partial v_1'(0)}{\partial x_1}\partial_z + \frac{\partial v_2'(0)}{\partial x_1}\varphi_{x_2}^0 \partial_\theta) d_{j+1}^{(2)}) - (\partial_z^2 + (\varphi_{x_2}^0)^2 \partial_\theta^2) d_{j+1}^{(2)} \\ & + z\frac{\partial c^2(0)}{\partial x_1}\partial_z d_{j+1}^{(1)} + g_{j+1}^{(2)} - \mathbf{m}_\theta(g_{j+1}^{(2)}) - \frac{D}{\rho'(0)}\partial_z g_j^{\star(1)} \\ \tilde{B} = & \rho'(0)((\partial_t + v_2'(0)\partial_{x_2}) d_{j+1}^{(3)} + z(\frac{\partial v_1'(0)}{\partial x_1}\partial_z + \frac{\partial v_2'(0)}{\partial x_1}\varphi_{x_2}^0 \partial_\theta) d_{j+1}^{(3)}) - (\partial_z^2 + (\varphi_{x_2}^0)^2 \partial_\theta^2) d_{j+1}^{(3)} \\ & + c^2(0)\partial_{x_2} d_{j+1}^{(1)} + z\frac{\partial c^2(0)}{\partial x_1}\varphi_{x_2}^0 \partial_\theta d_{j+1}^{(1)} + g_{j+1}^{(3)} - \mathbf{m}_\theta(g_{j+1}^{(3)}) - \frac{D\varphi_{x_2}^0}{\rho'(0)}\partial_\theta g_j^{\star(1)} \end{cases}$$

We deduce from (2.53) that

$$\omega_{j+1}(t, x_2; z, \theta) = \varphi_{x_2}^0 \partial_\theta d_{j+1}^{(2)} - \partial_z d_{j+1}^{(3)} \tag{2.54}$$

satisfies

$$\begin{aligned} &(\partial_t + v_2'(0)\partial_{x_2})\omega_{j+1} + z(\frac{\partial v_1'(0)}{\partial x_1}\partial_z + \frac{\partial v_2'(0)}{\partial x_1}\varphi_{x_2}^0 \partial_\theta)\omega_{j+1} - \frac{1}{\rho'(0)}(\partial_z^2 + (\varphi_{x_2}^0)^2 \partial_\theta^2)\omega_{j+1} \\ &-(\frac{\partial v_1'(0)}{\partial x_1}\partial_z + \frac{\partial v_2'(0)}{\partial x_1}\varphi_{x_2}^0 \partial_\theta) d_{j+1}^{(3)} = R_{j+1} \end{aligned} \tag{2.55}$$

where

$$R_{j+1} = \frac{1}{\rho'(0)}[\partial_z g_{j+1}^{(3)} - \mathbf{m}_\theta(\partial_z g_{j+1}^{(3)}) - \varphi_{x_2}^0 \partial_\theta g_{j+1}^{(2)} + c^2(0)\partial_{zx_2}^2 d_{j+1}^{(1)} + \frac{\partial c^2(0)}{\partial x_1}\varphi_{x_2}^0 \partial_\theta d_{j+1}^{(1)}] \tag{2.56}$$

with $d_{j+1}^{(1)}$ being given in (2.52).

Combining (2.50), (2.54), (2.55) and (2.22) for the $(j+1)-$case leads to that $(d^{(2)}_{j+1}, d^{(3)}_{j+1}, \omega_{j+1})$ satisfy the following problems:

$$\begin{cases} (\partial_z^2 + (\varphi^0_{x_2})^2\partial_\theta^2)d^{(3)}_{j+1} = \frac{\varphi^0_{x_2}}{\rho'(0)}\partial_\theta g_j^{\star(1)} - \partial_z\omega_{j+1} \\ (\partial_t + v_2'(0)\partial_{x_2})\omega_{j+1} + z(\frac{\partial v_1'(0)}{\partial x_1}\partial_z + \frac{\partial v_2'(0)}{\partial x_1}\varphi^0_{x_2}\partial_\theta)\omega_{j+1} - \frac{1}{\rho'(0)}(\partial_z^2 + (\varphi^0_{x_2})^2\partial_\theta^2)\omega_{j+1} \\ \qquad -(\frac{\partial v_1'(0)}{\partial x_1}\partial_z + \frac{\partial v_2'(0)}{\partial x_1}\varphi^0_{x_2}\partial_\theta)d^{(3)}_{j+1} = R_{j+1} \\ d^{(3)}_{j+1}|_{z=0} = -c^{(3)}_{j+1}(t,0,x_2;\theta) \\ (\omega_{j+1} + \partial_z d^{(3)}_{j+1})|_{z=0} = -\varphi^0_{x_2}(\partial_\theta c^{(2)}_{j+1})(t,0,x_2;\theta) \\ (d^{(3)}_{j+1}, \omega_{j+1}) \in S(I\!R^+_z) \\ \omega_{j+1}|_{t=0} = 0 \end{cases} \tag{2.57}$$

and

$$\begin{cases} (\partial_z^2 + (\varphi^0_{x_2})^2\partial_\theta^2)d^{(2)}_{j+1} = \varphi^0_{x_2}\partial_\theta\omega_{j+1} + \frac{1}{\rho'(0)}\partial_z g_j^{\star(1)} \\ d^{(2)}_{j+1}|_{z=0} = -c^{(2)}_{j+1}(t,0,x_2;\theta) \\ d^{(2)}_{j+1} \in S(I\!R^+_z) \end{cases} \tag{2.58}$$

which are similar to the problems (2.35) and (2.36).

Remark 2.1: When $\varphi|_{x_1=0} = \varphi^0(t,x_2) \equiv 0$, the terms d_j disappear, similar to Gues [4], the boundary conditions (2.21) become as

$$(a^{(k)}_j(t,x) + c^{(k)}_j(t,x;\theta) + b^{(k)}_j(t,x_2;z))|_{x_1=z=\theta=0} = 0$$

for any $j \geq 0, k = 2,3$. In this case, we obtain that $a_0(t,x)$, $c_0(t,x;\theta)$ and $b_0(t,x_2;z))$ satisfy the same problems as (2.26), (2.17)-(2.20) and (2.25)-(2.29), but for $j \geq 1$, $b_j(t,x_2;z)$ satisfies different problems from (2.48) due to the disparity of the boundary conditions.

3 The Study of A Poisson-Prandtl Coupled System

It is clear from problems (2.35), (2.36) and (2.57), (2.58) that in order to determine $(d^{(2)}_j, d^{(3)}_j)$ for any $j \geq 0$, we need to study the following initial-boundary value problem for a Poisson-Prandtl coupled system in $\{t, z > 0, x \in I\!R, \theta \in T^1\}$:

$$\begin{cases} (\partial_z^2 + a^2\partial_\theta^2)u = f(t,x;z,\theta) - \partial_z w \\ (\partial_t + a_1\partial_x)w + z(a_2\partial_z + a_3\partial_\theta)w - a_4^2(\partial_z^2 + a^2\partial_\theta^2)w - (a_2\partial_z + a_3\partial_\theta)u = g(t,x;z,\theta) \\ u|_{z=0} = b_0(t,x;\theta), \quad u \in S(I\!R^+_z) \\ (w + \partial_z u)|_{z=0} = b_1(t,x;\theta), \quad (u,w) \in S(I\!R^+_z) \\ w|_{t=0} = 0 \end{cases} \tag{3.1}$$

for the unknowns (u,w), where (f,g) are rapidly decreasing in z when $z \to +\infty$, and periodic in $\theta \in T^1 = I\!R/2\pi Z$ as well as for $(b_0,b_1)(t,x;\theta)$ with mean values vanishing,

$$\mathbf{m}_\theta(f) = \mathbf{m}_\theta(g) = \mathbf{m}_\theta(b_0) = \mathbf{m}_\theta(b_1) = 0,$$

all coefficients in (3.1) are smooth functions of (t,x), with $a(t,x) \geq a_0$, $a_4(t,x) \geq a_0$ for a positive constant a_0. For simplicity of presentation, we assume that (f,g,b_0,b_1) are smooth, and any order compatibility conditions are satisfied for the problem (3.1).

The goals of this section are to study the solvability of the problem (3.1), and to look for solutions (u,w) which are rapidly decreasing when $z \to +\infty$ and periodic in $\theta \in T^1$ with $\mathbf{m}_\theta(u,w) = 0$, which constitutes the main part of the rigorous justification for the formal analysis given in §2.

To this end, first, we derive a functional representation $u = u(w)$ of u in terms of w from the first and the third equations of (3.1), second, by substituting the relation $u = u(w)$ into the second and the fourth equations of (3.1), we can solve the unknown $w = w(t,x;z,\theta)$.

To derive the representation $u = u(w)$, we first consider the following boundary value problem:

$$\begin{cases} (\partial_z^2 + a^2\partial_\theta^2)u = F(t,x;z,\theta) \\ u|_{z=0} = b_0(t,x;\theta), \quad u \in S(I\!R_z^+) \end{cases} \tag{3.2}$$

where F is rapidly decreasing when $z \to +\infty$, and (b_0, F) are periodic in $\theta \in T^1$ with mean values vanishing.

Obviously, to solve the problem (3.2) is equivalent to study the following problem:

$$\begin{cases} (\partial_z^2 + a^2\partial_\theta^2)u = F(t,x;z,\theta) \\ u|_{z=0} = b_0(t,x;\theta), \quad u_z|_{z=0} = u_0(t,x;\theta) \end{cases} \tag{3.3}$$

where u_0, periodic in $\theta \in T^1$ with $\mathbf{m}_\theta(u_0) = 0$, will be determined by $(b_0(t,x;\theta), F(t,x;z,\theta))$ such that the problem (3.3) admits a unique solution $u(t,x;z,\theta) \in C_p^2(T_\theta^1) \cap S(I\!R_z^+)$ with $\mathbf{m}_\theta(u) = 0$.

Denote by

$$\begin{cases} F(t,x;z,\theta) = \sum_{k\neq 0} F^{(k)}(t,x;z)e^{ik\theta} \\ b_0(t,x;\theta) = \sum_{k\neq 0} b_0^{(k)}(t,x)e^{ik\theta} \end{cases} \tag{3.4}$$

the Fourier expansions of (F, b_0) with respect to $\theta \in T^1$.

Lemma 3.1: *The necessary and sufficient condition for the solution $u(t,x;z,\theta)$ of (3.3) to be rapidly decreasing when $z \to +\infty$ is*

$$\begin{aligned} u_0(t,x;\theta) = \quad & -\sum_{k=1}^{\infty}(kab_0^{(k)} + \int_0^\infty e^{-ka\xi}F^{(k)}(t,x;\xi)d\xi)e^{ik\theta} \\ & + \sum_{k=-1}^{-\infty}(kab_0^{(k)} - \int_0^\infty e^{ka\xi}F^{(k)}(t,x;\xi)d\xi)e^{ik\theta}. \end{aligned} \tag{3.5}$$

Proof: (1) First, we solve the following problem:

$$\begin{cases} (\partial_z^2 + a^2\partial_\theta^2)w = F(t,x;z,\theta) \\ w|_{z=0} = 0, \quad w_z|_{z=0} = w_0(t,x;\theta). \end{cases} \tag{3.6}$$

We will find $w_0(t,x;\theta)$, periodic in $\theta \in T^1$ with $\mathbf{m}_\theta(w_0) = 0$, such that the solution $w(t,x;z,\theta)$ to (3.6) is rapidly decreasing when $z \to +\infty$.

If we set

$$\begin{cases} w(t,x;z,\theta) = \sum_{k\neq 0} w^{(k)}(t,x;z)e^{ik\theta} \\ w_0(t,x;\theta) = \sum_{k\neq 0} w_0^{(k)}(t,x)e^{ik\theta} \end{cases} \tag{3.7}$$

then the problem (3.6) is equivalent to the following one for $w^{(k)}(t,x;z)$:

$$\begin{cases} (\partial_z^2 - k^2a^2)w^{(k)} = F^{(k)}(t,x;z) \\ w^{(k)}|_{z=0} = 0, \quad w_z^{(k)}|_{z=0} = w_0^{(k)}(t,x;\theta) \end{cases} \tag{3.8}$$

for any $k \neq 0$.

Obviously, the solution to (3.8) can be expressed as

$$\begin{aligned} w^{(k)}(t,x;z) = \ & \tfrac{1}{2ka}(w_0^{(k)}(t,x) + \textstyle\int_0^z e^{-ka\xi}F^{(k)}(t,x;\xi)d\xi)e^{kaz} \\ & -\tfrac{1}{2ka}(w_0^{(k)}(t,x) + \textstyle\int_0^z e^{ka\xi}F^{(k)}(t,x;\xi)d\xi)e^{-kaz}. \end{aligned} \tag{3.9}$$

When $k > 0$, the necessary condition for $w^{(k)} \in S(I\!R_z^+)$ is

$$\lim_{z\to+\infty}(w_0^{(k)}(t,x) + \int_0^z e^{-ka\xi}F^{(k)}(t,x;\xi)d\xi) = 0$$

which implies

$$w_0^{(k)}(t,x) = -\int_0^\infty e^{-ka\xi}F^{(k)}(t,x;\xi)d\xi. \tag{3.10}$$

Substituting (3.10) into (3.9) yields

$$\begin{aligned} w^{(k)}(t,x;z) = \ & -\tfrac{1}{2ka}\textstyle\int_z^\infty e^{ka(z-\xi)}F^{(k)}(t,x;\xi)d\xi + \tfrac{1}{2ka}\int_0^\infty e^{-ka(z+\xi)}F^{(k)}(t,x;\xi)d\xi \\ & -\tfrac{1}{2ka}\textstyle\int_0^z e^{ka(\xi-z)}F^{(k)}(t,x;\xi)d\xi. \end{aligned} \tag{3.11}$$

Since $F^{(k)} \in S(I\!R_z^+)$, we deduce

$$\int_0^\infty e^{-ka(z+\xi)}F^{(k)}(t,x;\xi)d\xi \in S(I\!R_z^+)$$

and

$$\int_z^\infty e^{ka(z-\xi)}F^{(k)}(t,x;\xi)d\xi \in S(I\!R_z^+).$$

On the other hand, we have

$$\left|z^l\int_0^z e^{ka(\xi-z)}F^{(k)}(t,x;\xi)d\xi\right| \leq \sum_{0\leq j\leq l}\begin{pmatrix} l \\ j \end{pmatrix}\left|\int_0^z (z-\xi)^{l-j}\xi^j e^{-ka(z-\xi)}F^{(k)}(t,x;\xi)d\xi\right|$$

which is bounded for any $l \geq 0$ by using $F^{(k)} \in S(I\!R_z^+)$. Thus, we also have

$$\int_0^z e^{ka(\xi-z)}F^{(k)}(t,x;\xi)d\xi \in S(I\!R_z^+).$$

Therefore, the function $w^{(k)}(t,x;z)$ given in (3.11) is rapidly decreasing when $z \to +\infty$.

Similarly, we deduce that when $k < 0$, the necessary and sufficient condition for $w^{(k)}$ given in (3.9) belonging to $S(I\!R_z^+)$ is:

$$w_0^{(k)}(t,x) = -\int_0^\infty e^{ka\xi}F^{(k)}(t,x;\xi)d\xi \tag{3.12}$$

and in this case, the solution to (3.8) can be expressed as:

$$\begin{aligned} w^{(k)}(t,x;z) = \ & \tfrac{1}{2ka}\textstyle\int_z^\infty e^{ka(\xi-z)}F^{(k)}(t,x;\xi)d\xi + \tfrac{1}{2ka}\int_0^z e^{ka(z-\xi)}F^{(k)}(t,x;\xi)d\xi \\ & -\tfrac{1}{2ka}\textstyle\int_0^\infty e^{ka(\xi+z)}F^{(k)}(t,x;\xi)d\xi. \end{aligned} \tag{3.13}$$

Combining (3.10), (3.11), (3.12) with (3.13) shows that

$$\begin{aligned} w(t,x;z,\theta) = & \sum_{k=1}^{\infty} \tfrac{1}{2ka}[\int_0^{\infty} e^{-ka(z+\xi)}F^{(k)}(t,x;\xi)d\xi - \int_0^{z} e^{-ka(z-\xi)}F^{(k)}(t,x;\xi)d\xi \\ & - \int_z^{\infty} e^{ka(z-\xi)}F^{(k)}(t,x;\xi)d\xi]e^{ik\theta} + \sum_{k=-1}^{-\infty} \tfrac{1}{2ka}[\int_0^{z} e^{ka(z-\xi)}F^{(k)}(t,x;\xi)d\xi \\ & - \int_0^{\infty} e^{ka(z+\xi)}F^{(k)}(t,x;\xi)d\xi + \int_z^{\infty} e^{-ka(z-\xi)}F^{(k)}(t,x;\xi)d\xi]e^{ik\theta} \\ & \in S(I\!R_z^+) \end{aligned} \tag{3.14}$$

is the unique solution to

$$\begin{cases} (\partial_z^2 + a^2\partial_\theta^2)w = F(t,x;z,\theta) \\ w|_{z=0} = 0 \\ w_z|_{z=0} = -\sum_{k=1}^{\infty}\int_0^{\infty} e^{k(i\theta - a\xi)}F^{(k)}(t,x;\xi)d\xi - \sum_{k=-1}^{-\infty}\int_0^{\infty} e^{k(i\theta + a\xi)}F^{(k)}(t,x;\xi)d\xi \end{cases} \tag{3.15}$$

(2) Let $v = u - w$ with u being the solution to (3.3). Then v solves the following problem:

$$\begin{cases} (\partial_z^2 + a^2\partial_\theta^2)v = 0 \\ v|_{z=0} = b_0(t,x;\theta) \\ v_z|_{z=0} = u_0(t,x;\theta) + \sum_{k=1}^{\infty}\int_0^{\infty} e^{k(i\theta - a\xi)}F^{(k)}(t,x;\xi)d\xi + \sum_{k=-1}^{-\infty}\int_0^{\infty} e^{k(i\theta + a\xi)}F^{(k)}(t,x;\xi)d\xi \end{cases} \tag{3.16}$$

Denote by

$$\begin{cases} v(t,x;z,\theta) = \sum_{k\neq 0} v^{(k)}(t,x;z)e^{ik\theta} \\ u_0(t,x;\theta) = \sum_{k\neq 0} u_0^{(k)}(t,x)e^{ik\theta} \end{cases}$$

the Fourier expansions of (v, u_0). Then, (3.16) yields

$$\begin{cases} (\partial_z^2 - k^2a^2)v^{(k)} = 0 \\ v^{(k)}|_{z=0} = b_0^{(k)}(t,x) \\ v_z^{(k)}|_{z=0} = \begin{cases} u_0^{(k)}(t,x) + \int_0^{\infty} e^{-ka\xi}F^{(k)}(t,x;\xi)d\xi, & k \geq 1 \\ u_0^{(k)}(t,x) + \int_0^{\infty} e^{ka\xi}F^{(k)}(t,x;\xi)d\xi, & k \leq -1 \end{cases} \end{cases}. \tag{3.17}$$

It follows that

$$\begin{aligned} v^{(k)}(t,x;z) = & [\tfrac{1}{2}b_0^{(k)} + \tfrac{1}{2ka}(u_0^{(k)} + \int_0^{+\infty} e^{-ka\xi}F^{(k)}(t,x;\xi)d\xi)]e^{kaz} \\ & +[\tfrac{1}{2}b_0^{(k)} - \tfrac{1}{2ka}(u_0^{(k)} + \int_0^{+\infty} e^{-ka\xi}F^{(k)}(t,x;\xi)d\xi)]e^{-kaz} \end{aligned} \tag{3.18}$$

when $k > 0$, and

$$\begin{aligned} v^{(k)}(t,x;z) = & [\tfrac{1}{2}b_0^{(k)} + \tfrac{1}{2ka}(u_0^{(k)} + \int_0^{+\infty} e^{ka\xi}F^{(k)}(t,x;\xi)d\xi)]e^{kaz} \\ & +[\tfrac{1}{2}b_0^{(k)} - \tfrac{1}{2ka}(u_0^{(k)} + \int_0^{+\infty} e^{ka\xi}F^{(k)}(t,x;\xi)d\xi)]e^{-kaz} \end{aligned} \tag{3.19}$$

when $k < 0$.

From (3.18) and (3.19), we conclude that one should have the condition (3.5) to guarantee $v^{(k)} \in S(I\!R_z^+)$, and in this case we have

$$v(t,x;z,\theta) = \sum_{k=1}^{\infty} b_0^{(k)}(t,x)e^{-k(az - i\theta)} + \sum_{k=-1}^{-\infty} b_0^{(k)}(t,x)e^{k(az+i\theta)}. \tag{3.20}$$

Therefore, we have shown that the necessary and sufficient condition for the problem (3.3) to have a unique solution $u \in S(I\!R_z^+)$ is (3.5), and the solution is given by

$$\begin{aligned} u(t,x;z,\theta) = \quad & \sum_{k=1}^{\infty} b_0^{(k)}(t,x)e^{-k(az-i\theta)} + \sum_{k=-1}^{-\infty} b_0^{(k)}(t,x)e^{k(az+i\theta)} \\ & + \sum_{k=1}^{\infty} \frac{1}{2ka}[\int_0^{\infty} e^{-ka(z+\xi)}F^{(k)}(t,x;\xi)d\xi - \int_0^z e^{-ka(z-\xi)}F^{(k)}(t,x;\xi)d\xi \\ & - \int_z^{\infty} e^{ka(z-\xi)}F^{(k)}(t,x;\xi)d\xi]e^{ik\theta} + \sum_{k=-1}^{-\infty} \frac{1}{2ka}[\int_0^z e^{ka(z-\xi)}F^{(k)}(t,x;\xi)d\xi \\ & - \int_0^{\infty} e^{ka(z+\xi)}F^{(k)}(t,x;\xi)d\xi + \int_z^{\infty} e^{-ka(z-\xi)}F^{(k)}(t,x;\xi)d\xi]e^{ik\theta} \end{aligned} \tag{3.21}$$

which is also the unique solution to the problem (3.2). ¶

For the problem (3.1), let the Fourier expansion of w be

$$w(t,x;z,\theta) = \sum_{k\neq 0} w^{(k)}(t,x;z)e^{ik\theta}.$$

Using Lemma 3.1 and (3.21), we conclude

Proposition 3.2: *For the problem (3.1), the solution u has the following representation in term of w:*

$$\begin{aligned} u(t,x;z,\theta) = \quad & \sum_{k=1}^{\infty} b_0^{(k)}(t,x)e^{k(i\theta-az)} + \sum_{k=-1}^{-\infty} b_0^{(k)}(t,x)e^{k(i\theta+az)} \\ & + \sum_{k=1}^{\infty} \frac{1}{2}\{\int_0^{\infty} e^{-ka(z+\xi)}(\frac{f^{(k)}(t,x;\xi)}{ka} - w^{(k)}(t,x;\xi))d\xi \\ & - \int_0^z e^{-ka(z-\xi)}(\frac{f^{(k)}(t,x;\xi)}{ka} + w^{(k)}(t,x;\xi))d\xi \\ & - \int_z^{\infty} e^{ka(z-\xi)}(\frac{f^{(k)}(t,x;\xi)}{ka} - w^{(k)}(t,x;\xi))d\xi\}e^{ik\theta} \\ & - \sum_{k=-1}^{-\infty} \frac{1}{2}\{\int_0^{\infty} e^{ka(z+\xi)}(\frac{f^{(k)}(t,x;\xi)}{ka} + w^{(k)}(t,x;\xi))d\xi \\ & - \int_0^z e^{ka(z-\xi)}(\frac{f^{(k)}(t,x;\xi)}{ka} + w^{(k)}(t,x;\xi))d\xi \\ & - \int_z^{\infty} e^{-ka(z-\xi)}(\frac{f^{(k)}(t,x;\xi)}{ka} + w^{(k)}(t,x;\xi))d\xi\}e^{ik\theta} \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \partial_z u|_{z=0} = \quad & \sum_{k=-1}^{-\infty} kab_0^{(k)}(t,x)e^{ki\theta} - \sum_{k=1}^{\infty} kab_0^{(k)}(t,x)e^{ki\theta} \\ & - \sum_{k=1}^{\infty} \int_0^{\infty} e^{k(i\theta-a\xi)}(f^{(k)}(t,x;\xi) - \partial_\xi w^{(k)}(t,x;\xi))d\xi \\ & - \sum_{k=-1}^{-\infty} \int_0^{\infty} e^{k(i\theta+a\xi)}(f^{(k)}(t,x;\xi) - \partial_\xi w^{(k)}(t,x;\xi))d\xi. \end{aligned} \tag{3.23}$$

As mentioned at the beginning of this section, to solve the problem (3.1), one needs the compatibility conditions satisfied. Now, we can state the compatibility conditions precisely as follows:

(1) The zero-th order compatibility condition for the problem (3.1).

From the initial data $w|_{t=0} = 0$, we have $\partial_z w|_{t=0} = 0$. Thus, from the first and third equations of (3.1), the datum $u_0(x,z,\theta) = u|_{t=0}$ should satisfy the problem:

$$\begin{cases} (\partial_z^2 + a_0^2\partial_\theta^2)u_0 = f(0,x;z,\theta) \\ u_0|_{z=0} = b_0(0,x,\theta), \quad u_0 \in S(I\!R_z^+) \end{cases} \tag{3.24}$$

where $a_0(x) = a(0,x)$.

If we denote by

$$\begin{cases} f(0,x;z,\theta) = \sum_{k\neq 0} f_0^{(k)}(x,z)e^{ik\theta} \\ b_0(0,x;\theta) = \sum_{k\neq 0} b_{00}^{(k)}(x)e^{ik\theta} \end{cases}$$

the Fourier expansions, then by using (3.21) we obtain

$$\begin{aligned} u_0(x,z,\theta) = \; & \sum_{k=1}^{\infty} b_{00}^{(k)}(x)e^{-k(az-i\theta)} + \sum_{k=-1}^{-\infty} b_{00}^{(k)}(t,x)e^{k(az+i\theta)} \\ & + \sum_{k=1}^{\infty} \tfrac{1}{2ka}[\int_0^{\infty} e^{-ka(z+\xi)} f_0^{(k)}(x,\xi)d\xi - \int_0^z e^{-ka(z-\xi)} f_0^{(k)}(x,\xi)d\xi \\ & - \int_z^{\infty} e^{ka(z-\xi)} f_0^{(k)}(x,\xi)d\xi]e^{ik\theta} + \sum_{k=-1}^{-\infty} \tfrac{1}{2ka}[\int_0^z e^{ka(z-\xi)} f_0^{(k)}(x,\xi)d\xi \\ & - \int_0^{\infty} e^{ka(z+\xi)} f_0^{(k)}(x,\xi)d\xi + \int_z^{\infty} e^{-ka(z-\xi)} f_0^{(k)}(x,\xi)d\xi]e^{ik\theta} \end{aligned} . \tag{3.25}$$

Therefore, from the fourth equation in (3.1), we conclude the following zero-th order compatibility condition for (3.1):

$$b_1(0,x;\theta) = \partial_z u_0|_{z=0} \tag{3.26}$$

where $u_0(x,z,\theta)$ is given by (3.25).

(2) The $k-$th order compatibility condition for the problem (3.1) for any fixed integer $k \geq 1$.

As above, in the discussion of compatibility conditions of (3.1) up to order $k-1$, one should have the data $u_l(x,z,\theta) = \partial_t^l u|_{t=0}$ and $w_l(x,z,\theta) = \partial_t^l w|_{t=0}$ for any integer $l \leq k-1$ in terms of (f,g,b_0,b_1). From the second equation in (3.1), we immediately obtain the data $w_k(x,z,\theta) = \partial_t^k w|_{t=0}$ in terms of $\{u_l, w_l\}_{l\leq k-1}$. By differentiating the first equation of (3.1) $k-$times with respect to t, and applying Lemma 3.1 to solve the problem:

$$\begin{cases} (\partial_z^2 + a_0^2\partial_\theta^2)u_k = F_k(x,z,\theta) \\ u_k|_{z=0} = (\partial_t^k b_0)(0,x,\theta), \quad u_k \in S(I\!R_z^+) \end{cases}$$

with $F_k(x,z,\theta) = (\partial_t^k f - \partial_t^k(a^2\partial_\theta^2 u) + a^2\partial_\theta^2\partial_t^k u - \partial_z\partial_t^k w)|_{t=0}$ being given in terms of $\{u_l\}_{l\leq k-1}$ and $\{w_l\}_{l\leq k}$, we determine the data $u_k(x,z,\theta) = \partial_z^k u|_{t=0}$.

In this way, we get formulae of $\{u_k, w_k\}$ in terms of (f,g,b_0,b_1). From the boundary condition of (3.1), it follows that the $k-$th order compatibility condition should be

$$(w_k + \partial_z u_k)|_{z=0} = (\partial_t^k b_1)(0,x;\theta) \tag{3.27}$$

which can be explicitly formulated in terms of (f,g,b_0,b_1).

It follows from Proposition 3.2 that to solve the problem (3.1), it suffices to use (3.22) and (3.23) to study the following problem for w:

$$\begin{cases} (\partial_t + a_1\partial_x)w + z(a_2\partial_z + a_3\partial_\theta)w - a_4^2(\partial_z^2 + a^2\partial_\theta^2)w - (a_2\partial_z + a_3\partial_\theta)u = g \\ w|_{z=0} = b_1(t,x;\theta) - \partial_z u(t,x;0,\theta), \quad w \in S(I\!R_z^+) \\ w|_{t=0} = 0. \end{cases} \tag{3.28}$$

The compatibility conditions for the problem (3.28) follow immediately from those for the problem (3.1) given as above.

Denote by

$$\begin{cases} w(t,x;z,\theta)=\sum_{k\neq 0} w^{(k)}(t,x;z)e^{ik\theta} \\ g(t,x;z,\theta)=\sum_{k\neq 0} g^{(k)}(t,x;z)e^{ik\theta} \\ b_1(t,x;\theta)=\sum_{k\neq 0} b_1^{(k)}(t,x)e^{ik\theta} \end{cases} \tag{3.29}$$

the Fourier expansions with respect to $\theta\in T^1$.

It follows from (3.28) that $w^{(k)}(t,x;z)$ satisfies the following problem:

$$\begin{cases} (\partial_t+a_1\partial_x)w^{(k)}+z(a_2\partial_z+ika_3)w^{(k)}-a_4^2(\partial_z^2-k^2a^2)w^{(k)}+a_2w^{(k)} \\ \qquad +\frac{k}{2}(ia_3-aa_2)[\int_0^{+\infty} e^{-ka(z+\xi)}(w^{(k)}(t,x;\xi)-\frac{f^{(k)}(t,x;\xi)}{ka})d\xi \\ \qquad +\int_0^z e^{-ka(z-\xi)}(w^{(k)}(t,x;\xi)+\frac{f^{(k)}(t,x;\xi)}{ka})d\xi] \\ \qquad -\frac{k}{2}(aa_2+ia_3)[\int_z^{+\infty} e^{ka(z-\xi)}(w^{(k)}(t,x;\xi)-\frac{f^{(k)}(t,x;\xi)}{ka})d\xi \\ \qquad =g^{(k)}(t,x;z)-k(aa_2+ia_3)b_0^{(k)}e^{-kaz} \\ w^{(k)}|_{z=0}=b_1^{(k)}+kab_0^{(k)}+\int_0^{+\infty} e^{-ka\xi}(f^{(k)}(t,x;\xi)-\partial_\xi w^{(k)}(t,x;\xi))d\xi \\ w^{(k)}|_{t=0}=0, \quad w^{(k)}\in S(I\!R_z^+) \end{cases} \tag{3.30}$$

for any $k\geq 1$, and

$$\begin{cases} (\partial_t+a_1\partial_x)w^{(k)}+z(a_2\partial_z+ika_3)w^{(k)}-a_4^2(\partial_z^2-k^2a^2)w^{(k)}+a_2w^{(k)} \\ \qquad +\frac{k}{2}(aa_2+ia_3)[\int_0^{+\infty} e^{ka(z+\xi)}(w^{(k)}(t,x;\xi)+\frac{f^{(k)}(t,x;\xi)}{ka})d\xi \\ \qquad +\int_0^z e^{ka(z-\xi)}(w^{(k)}(t,x;\xi)-\frac{f^{(k)}(t,x;\xi)}{ka})d\xi] \\ \qquad +\frac{k}{2}(aa_2-ia_3)[\int_z^{+\infty} e^{-ka(z-\xi)}(w^{(k)}(t,x;\xi)+\frac{f^{(k)}(t,x;\xi)}{ka})d\xi \\ \qquad =g^{(k)}(t,x;z)+k(aa_2+ia_3)b_0^{(k)}e^{kaz} \\ w^{(k)}|_{z=0}=b_1^{(k)}-kab_0^{(k)}+\int_0^{+\infty} e^{ka\xi}(f^{(k)}(t,x;\xi)-\partial_\xi w^{(k)}(t,x;\xi))d\xi \\ w^{(k)}|_{t=0}=0, \quad w^{(k)}\in S(I\!R_z^+) \end{cases} \tag{3.31}$$

for any $k\leq -1$.

The boundary conditions of $w^{(k)}(t,x;z)$ at $\{z=0\}$ given in (3.30) and (3.31) can be expressed as:

$$\begin{cases} \int_0^{+\infty} e^{-ka\xi}(f^{(k)}-kaw^{(k)})(t,x;\xi)d\xi+b_1^{(k)}(t,x)+kab_0^{(k)}(t,x)=0, \quad k\geq 1 \\ \int_0^{+\infty} e^{ka\xi}(f^{(k)}+kaw^{(k)})(t,x;\xi)d\xi+b_1^{(k)}(t,x)-kab_0^{(k)}(t,x)=0, \quad k\leq -1 \end{cases}. \tag{3.32}$$

In terms of the transformation:

$$Y^{(k)}(t,x;z)=\begin{cases} \int_z^{+\infty} e^{ka(z-\xi)}w^{(k)}(t,x;\xi)d\xi, \quad k\geq 1 \\ \int_z^{+\infty} e^{ka(\xi-z)}w^{(k)}(t,x;\xi)d\xi, \quad k\leq -1, \end{cases} \tag{3.33}$$

problems (3.30), (3.31) and (3.32) can be reformulated as

$$\left\{\begin{array}{l} (\partial_t + a_1\partial_x)Y^{(k)} - a_4^2(\partial_z^2 - k^2a^2)Y^{(k)} + z(a_2\partial_z + ika_3)Y^{(k)} \\ \qquad\qquad +ka_5\int_z^{+\infty} e^{ka(z-\xi)}Y^{(k)}(t,x;\xi)d\xi + ka_6\int_0^z e^{ka(\xi-z)}Y^{(k)}(t,x;\xi)d\xi \\ \qquad\qquad = G^{(k)}(t,x;z) \\ Y^{(k)}|_{z=0} = W_0^{(k)}(t,x), \quad Y^{(k)} \in S(I\!R_z^+) \\ Y^{(k)}|_{t=0} = 0 \end{array}\right. \tag{3.34}$$

for any $k \geq 1$, and

$$\left\{\begin{array}{l} (\partial_t + a_1\partial_x)Y^{(k)} - a_4^2(\partial_z^2 - k^2a^2)Y^{(k)} + z(a_2\partial_z + ika_3)Y^{(k)} \\ \qquad\qquad +ka_5\int_z^{+\infty} e^{ka(\xi-z)}Y^{(k)}(t,x;\xi)d\xi + ka_6\int_0^z e^{ka(z-\xi)}Y^{(k)}(t,x;\xi)d\xi \\ \qquad\qquad = G^{(k)}(t,x;z) \\ Y^{(k)}|_{z=0} = W_0^{(k)}(t,x), \quad Y^{(k)} \in S(I\!R_z^+) \\ Y^{(k)}|_{t=0} = 0 \end{array}\right. \tag{3.35}$$

for any $k \leq -1$, where

$$\left\{\begin{array}{l} G^{(k)} = \int_z^{+\infty} e^{ka(z-\xi)}g^{(k)}(t,x;\xi)d\xi + \frac{aa_2-ia_3}{4ka^2}(\int_0^{+\infty} e^{-ka(z+\xi)}f^{(k)}(t,x;\xi)d\xi \\ \qquad\qquad + \int_z^{+\infty} e^{ka(z-\xi)}f^{(k)}(t,x;\xi)d\xi + \int_0^z e^{ka(\xi-z)}f^{(k)}(t,x;\xi)d\xi) \\ \qquad\qquad - \frac{aa_2+ia_3}{2a}\int_z^{\infty}(\xi - z)e^{ka(z-\xi)}f^{(k)}(t,x;\xi)d\xi + \frac{aa_2-ia_3}{2ka^2}e^{-kaz}b_1^{(k)}(t,x) \\ W_0^{(k)}(t,x) = b_0^{(k)} + \frac{1}{ka}b_1^{(k)} + \frac{1}{ka}\int_0^{+\infty} e^{-ka\xi}f^{(k)}(t,x;\xi)d\xi \end{array}\right.$$

for any $k \geq 1$,

$$\left\{\begin{array}{l} G^{(k)} = \int_z^{+\infty} e^{ka(\xi-z)}g^{(k)}(t,x;\xi)d\xi - \frac{aa_2+ia_3}{4ka^2}(\int_0^{+\infty} e^{ka(z+\xi)}f^{(k)}(t,x;\xi)d\xi \\ \qquad\qquad + \int_z^{\infty} e^{ka(\xi-z)}f^{(k)}(t,x;\xi)d\xi + \int_0^z e^{ka(z-\xi)}f^{(k)}(t,x;\xi)d\xi) \\ \qquad\qquad - \frac{aa_2-ia_3}{2a}\int_z^{\infty}(\xi - z)e^{ka(\xi-z)}f^{(k)}(t,x;\xi)d\xi - \frac{aa_2+ia_3}{2ka^2}e^{kaz}b_1^{(k)}(t,x) \\ W_0^{(k)}(t,x) = b_0^{(k)} - \frac{1}{ka}b_1^{(k)} - \frac{1}{ka}\int_0^{+\infty} e^{ka\xi}f^{(k)}(t,x;\xi)d\xi \end{array}\right.$$

for any $k \leq -1$, and

$$\left\{\begin{array}{l} a_5 = a_t + a_1a_x + \frac{aa_2+ia_3}{2}, \quad a_6 = -\frac{1}{2}(aa_2 + ia_3), \quad k \geq 1 \\ a_5 = -(a_t + a_1a_x + \frac{aa_2-ia_3}{2}), \quad a_6 = \frac{1}{2}(aa_2 + ia_3), \quad k \leq -1 \end{array}\right. .$$

The compatibility conditions for problems (3.34) and (3.35) can be easily formulated in a classical way. For example, the zero-th order compatibility condition for (3.34) is

$$W_0^{(k)}(0,x) = 0$$

and the first order one is

$$G^{(k)}(0,x;0) = (\partial_t W_0^{(k)})(0,x).$$

It is not difficult to verify that the compatibility conditions for problems (3.34) and (3.35) are implied directly by those for the problem (3.1).

Now, we study the problem (3.34) under the assumption that any order compatibility condition of (3.34) is satisfied, and the problem (3.35) can be studied in the same way. The problem (3.34) shall be solved in the following steps:

STEP 1: Let $\chi(z) \in C_0^\infty(I\!R)$ be an arbitrary smooth function with compact support and $\chi(0) = 1$. Then, the function

$$Y_0^{(k)}(t, x; z) = \chi(z) W_0^{(k)}(t, x)$$

satisfies the initial and boundary conditions given in (3.34) due to the compatibility conditions.

Use the transformation $\tilde{Y}^{(k)} = Y^{(k)} - Y_0^{(k)}$ if necessary. It suffices to study the problem (3.34) in the special case $Y^{(k)}|_{z=0} \equiv 0$, which will be assumed in the sequel.

STEP 2: Construct an approximate solution sequence $\{Y_n^{(k)}\}_{n\geq 1}$ of (3.34) by solving the following problem for each $n \geq 1$:

$$\begin{cases} (\partial_t + a_1\partial_x)Y_n^{(k)} - a_4^2(\partial_z^2 - k^2a^2)Y_n^{(k)} + z(a_2\partial_z + ika_3)Y_n^{(k)} - \frac{1}{n}\partial_x^2 Y_n^{(k)} \\ \qquad +ka_5\int_z^{+\infty} e^{ka(z-\xi)}Y_{n-1}^{(k)}(t, x; \xi)d\xi + ka_6\int_0^z e^{ka(\xi-z)}Y_{n-1}^{(k)}(t, x; \xi)d\xi = G^{(k)}(t, x; z) \\ Y_n^{(k)}|_{z=0} = 0, \quad Y_n^{(k)} \in S(I\!R_z^+) \\ Y_n^{(k)}|_{t=0} = 0 \end{cases} \tag{3.36}$$

where $Y_0^{(k)}(t, x, z) \equiv 0$.

It remains to study the properties of the sequence $\{Y_n^{(k)}\}_{n\geq 1}$. Most of this part will follow the idea of Xin and Yanagisawa in §4 of [12] for studying the linearized Prandtl system.

In the sequel, for any $j \in I\!N$, we shall denote by C_j a constant depending only upon the bounds of derivatives of coefficients appeared in the equation in (3.36) up to order j.

(1) The boundedness in L^2- norm.

Denote by $< z >= (1+z^2)^{\frac{1}{2}}$, and $\Omega = I\!R_+^2 = \{(x, z) \in I\!R^2 |\ z > 0\}$. For any fixed integer $l \in I\!N$, multiplying the equation of (3.36) by $< z >^{2l}\ \overline{Y}_n^{(k)}$, and integrating the resulting equation over Ω, one gets

$$\begin{aligned} &\frac{d}{dt}\int_\Omega < z >^{2l} |Y_n^{(k)}|^2 dxdz - \int_\Omega \partial_x a_1 < z >^{2l} |Y_n^{(k)}|^2 dxdz \\ &\qquad +2k^2\int_\Omega a^2 a_4^2 < z >^{2l} |Y_n^{(k)}|^2 dxdz + 4l\mathcal{R}\int_\Omega a_4^2 z < z >^{2(l-1)} Y_n^{(k)}\partial_z\overline{Y}_n^{(k)} dxdz \\ &\qquad +2\int_\Omega a_4^2 < z >^{2l} |\partial_z Y_n^{(k)}|^2 dxdz + \frac{2}{n}\int_\Omega < z >^{2l} |\partial_x Y_n^{(k)}|^2 dxdz + 2kA_0 \\ &\qquad = 2\mathcal{R}\int_\Omega < z >^{2l} G^{(k)}\overline{Y}_n^{(k)} dxdz \end{aligned} \tag{3.37}$$

where $\mathcal{R}(\cdot)$ denotes the real part of the related functions, and

$$\begin{aligned} A_0 = \quad &\mathcal{R}\int_\Omega < z >^{2l} \overline{Y}_n^{(k)}(a_5\int_z^{+\infty} e^{ka(z-\xi)}Y_{n-1}^{(k)}(t, x; \xi)d\xi \\ &+a_6\int_0^z e^{ka(\xi-z)}Y_{n-1}^{(k)}(t, x; \xi)d\xi)dxdz. \end{aligned} \tag{3.38}$$

A simple computation leads to

$$\begin{aligned}
&\int_\Omega <z>^{2l}\ |\overline{Y}_n^{(k)}\int_0^z e^{ka(\xi-z)}Y_{n-1}^{(k)}(t,x;\xi)d\xi|dxdz\\
&\qquad\le \tfrac{1}{2}\int_\Omega\int_0^z <z>^{2l}\ e^{ka(\xi-z)}(|Y_n^{(k)}(t,x;z)|^2+|Y_{n-1}^{(k)}(t,x;\xi)|^2)d\xi dxdz\\
&\qquad\le \tfrac{c(l,a_0)}{k}\int_\Omega <z>^{2l}\ (|Y_n^{(k)}|^2+|Y_{n-1}^{(k)}|^2)dxdz
\end{aligned}\tag{3.39}$$

where $c(l,a_0)$ is a constant depending only upon $l\in I\!N$ and a_0 satisfying $0<a_0\le a(t,x)$.

Similarly, we have

$$\begin{aligned}
&\int_\Omega <z>^{2l}\ |\overline{Y}_n^{(k)}\int_z^{+\infty} e^{ka(z-\xi)}Y_{n-1}^{(k)}(t,x;\xi)d\xi|dxdz\\
&\qquad\le \tfrac{c(l,a_0)}{k}\int_\Omega <z>^{2l}\ (|Y_n^{(k)}|^2+|Y_{n-1}^{(k)}|^2)dxdz.
\end{aligned}\tag{3.40}$$

Substituting (3.39) and (3.40) into (3.38) shows that

$$|A_0|\le\frac{C_0}{k}\int_\Omega <z>^{2l}\ (|Y_n^{(k)}|^2+|Y_{n-1}^{(k)}|^2)dxdz.\tag{3.41}$$

Combining (3.41) and (3.37), we get

$$\begin{aligned}
&\tfrac{d}{dt}\int_\Omega <z>^{2l}\ |Y_n^{(k)}|^2dxdz+\int_\Omega <z>^{2l}\ |\partial_z Y_n^{(k)}|^2dxdz+k^2\int_\Omega <z>^{2l}\ |Y_n^{(k)}|^2dxdz\\
&\quad\le C_0\int_\Omega <z>^{2l}\ (|Y_n^{(k)}|^2+|Y_{n-1}^{(k)}|^2)dxdz+\int_\Omega <z>^{2l}\ |G^{(k)}|^2dxdz.
\end{aligned}\tag{3.42}$$

which implies that

$$\begin{aligned}
&\max_{0\le t\le T}\int_\Omega <z>^{2l}\ |Y_n^{(k)}|^2dxdz+\int_0^T\int_\Omega <z>^{2l}\ (|\partial_z Y_n^{(k)}|^2+k^2|Y_n^{(k)}|^2)dxdzdt\\
&\qquad\le\int_0^T\int_\Omega e^{2C_0(T-t)}<z>^{2l}\ |G^{(k)}|^2dxdzdt
\end{aligned}\tag{3.43}$$

holds for any $T\ge 0$ and $n\in I\!N$ by using the following result.

Lemma 3.3: *Given nonnegative functions $f\in C^0[0,\infty)$, $b_n\in C^0[0,\infty)$, $a_n\in C^1[0,\infty)$ satisfying $a_n(0)\le a$ for a constant a for any $n\in I\!N$, if we have*

$$a_n'(t)+b_n(t)\le C_0(a_n(t)+a_{n-1}(t))+f(t),\quad \forall n\ge 1$$

for a constant $C_0\ge 0$ independent of n, then the estimate

$$a_n(t)+\int_0^t e^{C_0(t-s)}b_n(s)ds\le ae^{2C_0t}+\int_0^t e^{2C_0(t-s)}f(s)ds$$

holds for any $n\in I\!N$.

This Gronwall type estimate can be obtained by induction on n.

(2) Estimates of spatial tangential derivatives $Y_{n,\alpha}^{(k)}=\partial_x^\alpha Y_n^{(k)}$.

For any $\alpha \in I\!N$, set $Y_{n,\alpha}^{(k)} = \partial_x^\alpha Y_n^{(k)}$, and act ∂_x^α on the problem of (3.36). Then $Y_{n,\alpha}^{(k)}$ solves the following problem:

$$\begin{cases} (\partial_t + a_1\partial_x)Y_{n,\alpha}^{(k)} - a_4^2(\partial_z^2 - k^2a^2)Y_{n,\alpha}^{(k)} + z(a_2\partial_z + ika_3)Y_{n,\alpha}^{(k)} - \frac{1}{n}\partial_x^2 Y_{n,\alpha}^{(k)} \\ \qquad +ka_5\int_z^{+\infty} e^{ka(z-\xi)}Y_{n-1,\alpha}^{(k)}(\cdot,\xi)d\xi + ka_6\int_0^z e^{ka(\xi-z)}Y_{n-1,\alpha}^{(k)}(\cdot,\xi)d\xi + R_\alpha = \partial_x^\alpha G^{(k)} \\ Y_{n,\alpha}^{(k)}|_{z=0} = 0, \quad Y_{n,\alpha}^{(k)} \in S(I\!R_z^+) \\ Y_{n,\alpha}^{(k)}|_{t=0} = 0 \end{cases} \tag{3.44}$$

where

$$\begin{aligned} R_\alpha = \ & [\partial_x^\alpha, a_1\partial_x - a_4^2(\partial_z^2 - k^2a^2) + z(a_2\partial_z + ika_3)]Y_n^{(k)} \\ & +k\sum_{0<j\leq\alpha}\begin{pmatrix}\alpha \\ j\end{pmatrix}(\partial_x^j a_5\int_z^{+\infty}\partial_x^{\alpha-j}(e^{ka(z-\xi)}Y_{n-1}^{(k)}(\cdot,\xi))d\xi \\ & +\partial_x^j a_6\int_0^z\partial_x^{\alpha-j}(e^{ka(\xi-z)}Y_{n-1}^{(k)}(\cdot,\xi))d\xi) \\ & +ka_5\int_z^{+\infty}[\partial_x^\alpha, e^{ka(z-\xi)}]Y_{n-1}^{(k)}(t,x;\xi)d\xi + ka_6\int_0^z[\partial_x^\alpha, e^{ka(\xi-z)}]Y_{n-1}^{(k)}(t,x;\xi)d\xi \end{aligned}.$$

Similar to (3.42), by multiplying the equation in (3.44) by $< z >^{2l}\ \overline{Y}_{n,\alpha}^{(k)}$ for any fixed $l \in I\!N$, and integrating the resulting equation over Ω, we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega < z >^{2l}\ |Y_{n,\alpha}^{(k)}|^2 dxdz + \int_\Omega < z >^{2l}\ (|\partial_z Y_{n,\alpha}^{(k)}|^2 + k^2|Y_{n,\alpha}^{(k)}|^2)dxdz \\ \leq C_0\int_\Omega < z >^{2l}\ (|Y_{n,\alpha}^{(k)}|^2 + |Y_{n-1,\alpha}^{(k)}|^2)dxdz + \int_\Omega < z >^{2l}\ |\partial_x^\alpha G^{(k)}|^2 dxdz \\ -2\mathcal{R}\int_\Omega < z >^{2l}\ R_\alpha \overline{Y}_{n,\alpha}^{(k)} dxdz. \end{aligned} \tag{3.45}$$

On the other hand, we have

$$\begin{aligned} & |\int_\Omega\ < z >^{2l}\ R_\alpha\overline{Y}_{n,\alpha}^{(k)}dxdz| \\ & \quad \leq C_0\int_\Omega < z >^{2l}\ |Y_{n,\alpha}^{(k)}|^2 dxdz + \epsilon\int_\Omega < z >^{2l}\ (|\partial_z Y_{n,\alpha}^{(k)}|^2 + k^2|Y_{n,\alpha}^{(k)}|^2)dxdz \\ & \qquad + \sum_{0<j\leq\alpha}\frac{C_j}{\epsilon}\int_\Omega < z >^{2l}\ (|\partial_z Y_{n,\alpha-j}^{(k)}|^2 + k^2|Y_{n,\alpha-j}^{(k)}|^2 + < z >^2\ |Y_{n,\alpha-j}^{(k)}|^2)dxdz \\ & \qquad + \sum_{0<j\leq\alpha} C_j\int_\Omega < z >^{2l}\ (< z >^2\ |\partial_z Y_{n,\alpha-j}^{(k)}|^2 + |Y_{n-1,\alpha-j}^{(k)}|^2)dxdz \end{aligned} \tag{3.46}$$

for any $\epsilon > 0$.

Substituting (3.46) into (3.45), and letting ϵ be small, we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega < z >^{2l}\ |Y_{n,\alpha}^{(k)}|^2 dxdz + \int_\Omega < z >^{2l}\ (|\partial_z Y_{n,\alpha}^{(k)}|^2 + k^2|Y_{n,\alpha}^{(k)}|^2)dxdz \\ \leq C_0\int_\Omega < z >^{2l}\ (|Y_{n,\alpha}^{(k)}|^2 + |Y_{n-1,\alpha}^{(k)}|^2)dxdz + \int_\Omega < z >^{2l}\ |\partial_x^\alpha G^{(k)}|^2 dxdz \\ + \sum_{0<j\leq\alpha} C_j(\int_\Omega < z >^{2(l+1)}\ (|\partial_z Y_{n,\alpha-j}^{(k)}|^2 + |Y_{n,\alpha-j}^{(k)}|^2)dxdz \\ + \sum_{0<j\leq\alpha} C_j\int_\Omega < z >^{2l}\ (|Y_{n-1,\alpha-j}^{(k)}|^2 + k^2|Y_{n,\alpha-j}^{(k)}|^2)dxdz) \end{aligned}$$

which implies

$$\max_{0\le t\le T}\int_\Omega <z>^{2l}\ |Y_{n,\alpha}^{(k)}|^2 dxdz+\int_0^T\int_\Omega <z>^{2l}\ (|\partial_z Y_{n,\alpha}^{(k)}|^2+k^2|Y_{n,\alpha}^{(k)}|^2)dxdzdt \\ \le C(T)\sum_{j=0}^{\alpha}\int_0^T\int_\Omega <z>^{2(l+\alpha-j)}\ |\partial_x^j G^{(k)}|^2 dxdzdt \tag{3.47}$$

by using Lemma 3.3 and induction on $\alpha\in I\!N$.

(3) Estimates of derivatives $\partial_{t,x}^{\alpha}Y_n^{(k)}$ for any $\alpha\in I\!N^2$.

For any fixed integer $j\ge 0$, set $V_{n,j}^{(k)}=\partial_t^j Y_n^{(k)}$.

It follows from (3.36) that $V_{n,1}^{(k)}$ solves the following problem:

$$\begin{cases} (\partial_t+a_1\partial_x)V_{n,1}^{(k)}-a_4^2(\partial_z^2-k^2a^2)V_{n,1}^{(k)}+z(a_2\partial_z+ika_3)V_{n,1}^{(k)}-\frac{1}{n}\partial_x^2 V_{n,1}^{(k)} \\ \qquad +ka_5\int_z^{+\infty}e^{ka(z-\xi)}V_{n-1,1}^{(k)}(\cdot,\xi)d\xi+ka_6\int_0^z e^{ka(\xi-z)}V_{n-1,1}^{(k)}(\cdot,\xi)d\xi+Q_1=\partial_t G^{(k)} \\ V_{n,1}^{(k)}|_{z=0}=0,\quad V_{n,1}^{(k)}\in S(I\!R_z^+) \\ V_{n,1}^{(k)}|_{t=0}=V_{n,1,0}^{(k)}(x,z) \end{cases} \tag{3.48}$$

where $V_{n,1,0}^{(k)}=G^{(k)}(0,x;z)$, and

$$\begin{aligned} Q_1=\ & [\partial_t, a_1\partial_x-a_4^2(\partial_z^2-k^2a^2)+z(a_2\partial_z+ika_3)]Y_n^{(k)} \\ & +k\partial_t a_5\int_z^{+\infty}e^{ka(z-\xi)}Y_{n-1}^{(k)}(t,x;\xi)d\xi+k\partial_t a_6\int_0^z e^{ka(\xi-z)}Y_{n-1}^{(k)}(t,x;\xi)d\xi \\ & +ka_5\int_z^{+\infty}[\partial_t, e^{ka(z-\xi)}]Y_{n-1}^{(k)}(t,x;\xi)d\xi+ka_6\int_0^z[\partial_t, e^{ka(\xi-z)}]Y_{n-1}^{(k)}(t,x;\xi)d\xi. \end{aligned}$$

Multiplying the equation in (3.48) by $<z>^{2l}\ \overline{V}_{n,1}^{(k)}$ for any fixed $l\in I\!N$, and integrating the resulting equation over Ω, we deduce

$$\begin{aligned} & \frac{d}{dt}\int_\Omega <z>^{2l}\ |V_{n,1}^{(k)}|^2dxdz+\int_\Omega <z>^{2l}\ (|\partial_z V_{n,1}^{(k)}|^2+k^2|V_{n,1}^{(k)}|^2)dxdz \\ & \qquad \le C_0\int_\Omega <z>^{2l}\ (|V_{n,1}^{(k)}|^2+|V_{n-1,1}^{(k)}|^2)dxdz+\int_\Omega <z>^{2l}\ |\partial_t G^{(k)}|^2 dxdz \\ & \qquad\quad -2\mathcal{R}\int_\Omega <z>^{2l}\ Q_1\overline{V}_{n,1}^{(k)}dxdz. \end{aligned} \tag{3.49}$$

It is not difficult to have

$$\begin{aligned} & |\int_\Omega <z>^{2l}\ Q_1\overline{V}_{n,1}^{(k)}dxdz| \\ & \qquad \le C_0\int_\Omega <z>^{2l}\ |V_{n,1}^{(k)}|^2dxdz+\epsilon\int_\Omega <z>^{2l}\ (|\partial_z V_{n,1}^{(k)}|^2+k^2|V_{n,1}^{(k)}|^2)dxdz \\ & \qquad\quad +\frac{C_1}{\epsilon}\int_\Omega <z>^{2l}\ (|\partial_z Y_n^{(k)}|^2+k^2|Y_n^{(k)}|^2)dxdz \\ & \qquad\quad +\int_\Omega <z>^{2(l+1)}\ (|\partial_z Y_n^{(k)}|^2+|Y_n^{(k)}|^2)dxdz \\ & \qquad\quad +C_1\int_\Omega <z>^{2l}\ (|\partial_x Y_n^{(k)}|^2+k^2|Y_n^{(k)}|^2+|Y_{n-1}^{(k)}|^2)dxdz \end{aligned}$$

for any $\epsilon>0$.

Thus, from (3.49) we obtain

$$\frac{d}{dt}\int_\Omega <z>^{2l}\ |V_{n,1}^{(k)}|^2dxdz+\int_\Omega <z>^{2l}\ (|\partial_zV_{n,1}^{(k)}|^2+k^2|V_{n,1}^{(k)}|^2)dxdz$$
$$\le C_0\int_\Omega <z>^{2l}\ (|V_{n,1}^{(k)}|^2+|V_{n-1,1}^{(k)}|^2)dxdz+\int_\Omega <z>^{2l}\ |\partial_tG^{(k)}|^2dxdz$$
$$+C_1(\int_\Omega <z>^{2l}\ (|\partial_xY_n^{(k)}|^2+k^2|Y_n^{(k)}|^2+|Y_{n-1}^{(k)}|^2)dxdz$$
$$+\int_\Omega <z>^{2(l+1)}\ (|\partial_zY_n^{(k)}|^2+|Y_n^{(k)}|^2)dxdz) \tag{3.50}$$

which implies

$$\max_{0\le t\le T}\ \int_\Omega <z>^{2l}\ |V_{n,1}^{(k)}|^2dxdz+\int_0^T\int_\Omega <z>^{2l}\ (|\partial_zV_{n,1}^{(k)}|^2+k^2|V_{n,1}^{(k)}|^2)dxdzdt$$
$$\le C(T)\int_0^T\int_\Omega <z>^{2l}\ (|\partial_xG^{(k)}|^2+|\partial_tG^{(k)}|^2+|G^{(k)}|^2)dxdzdt \tag{3.51}$$

due to Lemma 3.3 and (3.47).

For any $j\in I\!N$, $V_{n,j}^{(k)}$ satisfies the following problem:

$$\begin{cases}(\partial_t+a_1\partial_x)V_{n,j}^{(k)}-a_4^2(\partial_z^2-k^2a^2)V_{n,j}^{(k)}+z(a_2\partial_z+ika_3)V_{n,j}^{(k)}-\frac{1}{n}\partial_x^2V_{n,j}^{(k)}\\ \qquad +ka_5\int_z^{+\infty}e^{ka(z-\xi)}V_{n-1,j}^{(k)}(\cdot,\xi)d\xi+ka_6\int_0^z e^{ka(\xi-z)}V_{n-1,j}^{(k)}(\cdot,\xi)d\xi+Q_j=\partial_t^jG^{(k)}\\ V_{n,j}^{(k)}|_{z=0}=0,\quad V_{n,j}^{(k)}\in S(I\!R_z^+)\\ V_{n,j}^{(k)}|_{t=0}=V_{n,j,0}^{(k)}(x,z)\end{cases} \tag{3.52}$$

where

$$\begin{aligned}Q_j=\ &[\partial_t^j,a_1\partial_x-a_4^2(\partial_z^2-k^2a^2)+z(a_2\partial_z+ika_3)]Y_n^{(k)}\\ &+k\sum_{0<m\le j}\begin{pmatrix}j\\ m\end{pmatrix}(\partial_t^ma_5\int_z^{+\infty}\partial_t^{j-m}(e^{ka(z-\xi)}Y_{n-1}^{(k)}(t,x;\xi))d\xi\\ &+\partial_t^ma_6\int_0^z\partial_t^{j-m}(e^{ka(\xi-z)}Y_{n-1}^{(k)}(t,x;\xi))d\xi)\\ &+ka_5\int_z^{+\infty}[\partial_t^j,e^{ka(z-\xi)}]Y_{n-1}^{(k)}(t,x;\xi)d\xi+ka_6\int_0^z[\partial_t^j,e^{ka(\xi-z)}]Y_{n-1}^{(k)}(t,x;\xi)d\xi\end{aligned}$$

and

$$\begin{aligned}V_{n,j,0}^{(k)}=\ &\partial_t^{j-1}G^{(k)}-(a_1\partial_x-a_4^2(\partial_z^2-k^2a^2)+z(a_2\partial_z+ika_3)-\tfrac{1}{n}\partial_x^2)V_{n,j-1,0}^{(k)}\\ &-ka_5\int_z^{+\infty}e^{ka(z-\xi)}V_{n-1,j-1,0}^{(k)}d\xi-ka_6\int_0^z e^{ka(\xi-z)}V_{n-1,j-1,0}^{(k)}d\xi-Q_{j-1}|_{t=0}\end{aligned} \tag{3.53}$$

is defined by induction on j with $V_{n,1,0}^{(k)}=G^{(k)}(0,x,z)$.

Multiplying the equation in (3.52) by $<z>^{2l}\ \overline{V}_{n,j}^{(k)}$, and integrating the resulting equation over Ω, one gets

$$\frac{d}{dt}\int_\Omega <z>^{2l}\ |V_{n,j}^{(k)}|^2dxdz+\int_\Omega <z>^{2l}\ (|\partial_zV_{n,j}^{(k)}|^2+k^2|V_{n,j}^{(k)}|^2)dxdz$$
$$\le C_0\int_\Omega <z>^{2l}\ (|V_{n,j}^{(k)}|^2+|V_{n-1,j}^{(k)}|^2)dxdz+\int_\Omega <z>^{2l}\ |\partial_t^jG^{(k)}|^2dxdz$$
$$-2\mathcal{R}\int_\Omega <z>^{2l}\ Q_j\overline{V}_{n,j}^{(k)}dxdz. \tag{3.54}$$

On the other hand, we have

$$
\begin{aligned}
&|\int_\Omega <z>^{2l}\ Q_j \overline{V}^{(k)}_{n,j} dxdz| \\
&\qquad \le C_0 \int_\Omega <z>^{2l}\ |V^{(k)}_{n,j}|^2 dxdz + \epsilon \int_\Omega <z>^{2l}\ (|\partial_z V^{(k)}_{n,j}|^2 + k^2 |V^{(k)}_{n,j}|^2) dxdz \\
&\qquad\quad + \sum_{0<m\le j} \frac{C_m}{\epsilon} \int_\Omega <z>^{2l}\ (|\partial_z V^{(k)}_{n,j-m}|^2 + k^2 |V^{(k)}_{n,j-m}|^2 \\
&\qquad\qquad + <z>^2\ |V^{(k)}_{n,j-m}|^2 + |V^{(k)}_{n-1,j-m}|^2) dxdz \\
&\qquad\quad + \sum_{0<m\le j} C_m (\int_\Omega <z>^{2l}\ (<z>^2\ |\partial_z V^{(k)}_{n,j-m}|^2 + |\partial_x V^{(k)}_{n,j-m}|^2) dxdz
\end{aligned}
\tag{3.55}
$$

for any $\epsilon > 0$.

Substituting (3.55) into (3.54), and letting ϵ be small, we obtain

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega <z>^{2l}\ |V^{(k)}_{n,j}|^2 dxdz + \int_\Omega <z>^{2l}\ (|\partial_z V^{(k)}_{n,j}|^2 + k^2 |V^{(k)}_{n,j}|^2) dxdz \\
&\qquad \le C_0 \int_\Omega <z>^{2l}\ (|V^{(k)}_{n,j}|^2 + |V^{(k)}_{n-1,j}|^2) dxdz + \int_\Omega <z>^{2l}\ |\partial_t^j G^{(k)}|^2 dxdz \\
&\qquad\quad + \sum_{0<m\le j} C_m (\int_\Omega <z>^{2(l+1)}\ (|\partial_z V^{(k)}_{n,j-m}|^2 + |V^{(k)}_{n,j-m}|^2) dxdz \\
&\qquad\quad + \int_\Omega <z>^{2l}\ (|V^{(k)}_{n-1,j-m}|^2 + k^2 |V^{(k)}_{n,j-m}|^2 + |\partial_x V^{(k)}_{n,j-m}|^2) dxdz).
\end{aligned}
\tag{3.56}
$$

Thus, to complete the estimate on $V^{(k)}_{n,2}$, we should study $\partial_x V^{(k)}_{n,1}$ first.

It follows from (3.52) that $\partial_x^p V^{(k)}_{n,j} = \partial_x^p \partial_t^j Y^{(k)}_n$ satisfies the following problem:

$$
\begin{cases}
(\partial_t + a_1\partial_x - a_4^2(\partial_z^2 - k^2 a^2) + z(a_2\partial_z + ika_3) - \frac{1}{n}\partial_x^2)\partial_x^p V^{(k)}_{n,j} \\
\qquad + ka_5 \int_z^{+\infty} e^{ka(z-\xi)} \partial_x^p V^{(k)}_{n-1,j} d\xi + ka_6 \int_0^z e^{ka(\xi-z)} \partial_x^p V^{(k)}_{n-1,j} d\xi + Q_{j,p} = \partial_x^p \partial_t^j G^{(k)} \\
\partial_x^p V^{(k)}_{n,j}|_{z=0} = 0, \quad \partial_x^p V^{(k)}_{n,j} \in S(I\!R_z^+) \\
\partial_x^p V^{(k)}_{n,j}|_{t=0} = \partial_x^p V^{(k)}_{n,j,0}(x,z)
\end{cases}
\tag{3.57}
$$

where $V^{(k)}_{n,j,0}(x,z)$ is given in (3.53), and

$$
\begin{aligned}
Q_{j,p} = \quad & \partial_x^p Q_j + [\partial_x^p, a_1\partial_x - a_4^2(\partial_z^2 - k^2 a^2) + z(a_2\partial_z + ika_3)] V^{(k)}_{n,j} \\
& + k \sum_{0<m\le p} \begin{pmatrix} p \\ m \end{pmatrix} (\partial_x^m a_5 \int_z^{+\infty} \partial_x^{p-m} (e^{ka(z-\xi)} V^{(k)}_{n-1,j}(t,x;\xi)) d\xi \\
& + \partial_x^m a_6 \int_0^z \partial_x^{p-m} (e^{ka(\xi-z)} V^{(k)}_{n-1,j}(t,x;\xi)) d\xi) \\
& + ka_5 \int_z^{+\infty} [\partial_x^p, e^{ka(z-\xi)}] V^{(k)}_{n-1,j}(t,x;\xi) d\xi + ka_6 \int_0^z [\partial_x^p, e^{ka(\xi-z)}] V^{(k)}_{n-1,j}(t,x;\xi) d\xi
\end{aligned}
$$

with Q_j being given in (3.52).

Multiplying the equation in (3.57) by $<z>^{2l}\ \partial_x^p \overline{V}^{(k)}_{n,j}$, and integrating the resulting equation over Ω, one gets

$$\frac{d}{dt}\int_\Omega <z>^{2l}\ |\partial_x^p V_{n,j}^{(k)}|^2 dxdz+\int_\Omega <z>^{2l}\ (|\partial_z\partial_x^p V_{n,j}^{(k)}|^2+k^2|\partial_x^p V_{n,j}^{(k)}|^2)dxdz$$
$$\le C_0\int_\Omega <z>^{2l}\ (|\partial_x^p V_{n,j}^{(k)}|^2+|\partial_x^p V_{n-1,j}^{(k)}|^2)dxdz+\int_\Omega <z>^{2l}\ |\partial_x^p\partial_t^j G^{(k)}|^2 dxdz$$
$$-2\mathcal{R}\int_\Omega <z>^{2l}\ Q_{j,p}\partial_x^p\overline{V}_{n,j}^{(k)}dxdz. \tag{3.58}$$

A direct computation shows

$$|\ \int_\Omega <z>^{2l}\ Q_{j,p}\partial_x^p\overline{V}_{n,j}^{(k)}dxdz|$$
$$\le C_0\int_\Omega <z>^{2l}\ |\partial_x^p V_{n,j}^{(k)}|^2dxdz+\epsilon\int_\Omega <z>^{2l}\ (|\partial_z\partial_x^p V_{n,j}^{(k)}|^2+k^2|\partial_x^p V_{n,j}^{(k)}|^2)dxdz$$
$$+\sum_{0<m\le j,q\le p}\frac{C_{q+m}}{\epsilon}\int_\Omega <z>^{2(l+1)}\ (|\partial_z\partial_x^q V_{n,j-m}^{(k)}|^2+|\partial_x^q V_{n,j-m}^{(k)}|^2)dxdz$$
$$+k^2\sum_{0<m\le j,q\le p}\frac{C_{q+m}}{\epsilon}\int_\Omega <z>^{2l}\ |\partial_x^q V_{n,j-m}^{(k)}|^2dxdz$$
$$+\sum_{q<p}\frac{C_q}{\epsilon}\int_\Omega <z>^{2l}\ (|\partial_z\partial_x^q V_{n,j}^{(k)}|^2+k^2|\partial_x^q V_{n,j}^{(k)}|^2)dxdz$$
$$+\sum_{q<p}C_q\int_\Omega <z>^{2l}\ (<z>^2\ |\partial_z\partial_x^q V_{n,j}^{(k)}|^2+|\partial_x^q V_{n-1,j}^{(k)}|^2)dxdz$$
$$+\sum_{1\le q\le p}C_q\int_\Omega <z>^{2l}\ |\partial_x^q V_{n,j}^{(k)}|^2dxdz+\sum_{0<m\le j}C_m\int_\Omega <z>^{2l}\ |\partial_x^{q+1}V_{n,j-m}^{(k)}|^2dxdz$$
$$+\sum_{q\le p,0<m\le j}C_{q+m}\int_\Omega <z>^{2l}\ |\partial_x^q V_{n-1,j-m}^{(k)}|^2dxdz.$$

Thus, (3.58) yields that

$$\frac{d}{dt}\ \int_\Omega <z>^{2l}\ |\partial_x^p V_{n,j}^{(k)}|^2dxdz+\int_\Omega <z>^{2l}\ (|\partial_z\partial_x^p V_{n,j}^{(k)}|^2+k^2|\partial_x^p V_{n,j}^{(k)}|^2)dxdz$$
$$\le C_0\int_\Omega <z>^{2l}\ (|\partial_x^p V_{n,j}^{(k)}|^2+|\partial_x^p V_{n-1,j}^{(k)}|^2)dxdz+\int_\Omega <z>^{2l}\ |\partial_x^p\partial_t^j G^{(k)}|^2dxdz$$
$$+C_{j+p}\{\sum_{0<m\le j,q\le p}(\int_\Omega <z>^{2(l+1)}\ (|\partial_z\partial_x^q V_{n,j-m}^{(k)}|^2+|\partial_x^q V_{n,j-m}^{(k)}|^2)dxdz$$
$$+\int_\Omega <z>^{2l}\ (k^2|\partial_x^q V_{n,j-m}^{(k)}|^2+|\partial_x^q V_{n-1,j-m}^{(k)}|^2)dxdz$$
$$+\sum_{q<p}\int_\Omega <z>^{2l}\ (<z>^2\ |\partial_z\partial_x^q V_{n,j}^{(k)}|^2+k^2|\partial_x^q V_{n,j}^{(k)}|^2+|\partial_x^q V_{n-1,j}^{(k)}|^2)dxdz$$
$$+\sum_{0<m\le j}\int_\Omega <z>^{2l}\ |\partial_x^{q+1}V_{n,j-m}^{(k)}|^2dxdz. \tag{3.59}$$

By using (3.47) and (3.51) in (3.59) for the case $j=1$ and $p=1$, and using Lemma 3.3, we get that $\partial_x V_{n,1}^{(k)}=\partial_t\partial_x Y_n^{(k)}$ satisfies:

$$\max_{0\le t\le T}\int_\Omega <z>^{2l}\ \ |\partial_x V_{n,1}^{(k)}|^2dxdz+\int_0^T\int_\Omega <z>^{2l}\ (|\partial_z\partial_x V_{n,1}^{(k)}|^2+k^2|\partial_x V_{n,1}^{(k)}|^2)dxdzdt$$
$$\le C(T)(\sum_{j=0}^2\int_0^T\int_\Omega <z>^{2(l+2-j)}\ |\partial_x^j G^{(k)}|^2dxdzdt$$
$$+\sum_{j=0}^1\int_0^T\int_\Omega <z>^{2(l+1-j)}\ |\partial_x^j\partial_t G^{(k)}|^2dxdzdt). \tag{3.60}$$

It follows from (3.60), (3.47) and (3.51) in (3.56) for the case $j=2$, and Lemma 3.3 that

$V_{n,2}^{(k)} = \partial_t^2 Y_n^{(k)}$ satisfies:

$$\begin{aligned}&\max_{0\leq t\leq T}\int_\Omega <z>^{2l}|V_{n,2}^{(k)}|^2dxdz+\int_0^T\!\!\int_\Omega <z>^{2l}(|\partial_z V_{n,2}^{(k)}|^2+k^2|V_{n,2}^{(k)}|^2)dxdzdt\\&\leq C(T)(\int_\Omega <z>^{2l}|V_{n,2,0}^{(k)}|^2dxdz+\sum_{|\alpha|\leq 2}\int_0^T\!\!\int_\Omega <z>^{2(l+2-|\alpha|)}|\partial_{t,x}^\alpha G^{(k)}|^2dxdzdt).\end{aligned}\tag{3.61}$$

Similarly, to estimate $V_{n,3}^{(k)}$, one needs to study $\partial_x V_{n,2}^{(k)}$ first, which can be bounded if $\partial_x^2 V_{n,1}^{(k)}$ can be stimated due to (3.59). However, we can deduce from (3.59) that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega <z>^{2l}|\partial_x^2 V_{n,1}^{(k)}|^2dxdz+\int_\Omega <z>^{2l}(|\partial_z\partial_x^2 V_{n,1}^{(k)}|^2+k^2|\partial_x^2 V_{n,1}^{(k)}|^2)dxdz\\&\quad\leq C_0\int_\Omega <z>^{2l}(|\partial_x^2 V_{n,1}^{(k)}|^2+|\partial_x^2 V_{n-1,1}^{(k)}|^2)dxdz+\int_\Omega <z>^{2l}|\partial_x^2\partial_t G^{(k)}|^2dxdz\\&\qquad+C\{\sum_{q\leq 2}(\int_\Omega <z>^{2(l+1)}(|\partial_z\partial_x^q Y_n^{(k)}|^2+|\partial_x^q Y_n^{(k)}|^2)dxdz\\&\qquad\quad+\int_\Omega <z>^{2l}(k^2|\partial_x^q Y_n^{(k)}|^2+|\partial_x^q Y_{n-1}^{(k)}|^2)dxdz)\\&\qquad+\int_\Omega <z>^{2l}(k^2|\partial_x V_{n,1}^{(k)}|^2+k^2|V_{n,1}^{(k)}|^2+|\partial_x V_{n-1,1}^{(k)}|^2+|V_{n-1,1}^{(k)}|^2)dxdz\\&\qquad+\int_\Omega <z>^{2l}|\partial_x^3 Y_n^{(k)}|^2dxdz+\int_\Omega <z>^{2(l+1)}|\partial_z\partial_x V_{n,1}^{(k)}|^2dxdz+|\partial_z V_{n,1}^{(k)}|^2)dxdz\}\end{aligned}$$

which implies

$$\begin{aligned}&\max_{0\leq t\leq T}\int_\Omega <z>^{2l}|\partial_x^2 V_{n,1}^{(k)}|^2dxdz+\int_0^T\!\!\int_\Omega <z>^{2l}(|\partial_z\partial_x^2 V_{n,1}^{(k)}|^2+k^2|\partial_x^2 V_{n,1}^{(k)}|^2)dxdzdt\\&\qquad\leq C(T)(\sum_{j=0}^3\int_0^T\!\!\int_\Omega <z>^{2(l+3-j)}|\partial_x^j G^{(k)}|^2dxdzdt\\&\qquad\quad+\sum_{j=0}^2\int_0^T\!\!\int_\Omega <z>^{2(l+2-j)}|\partial_x^j\partial_t G^{(k)}|^2dxdzdt)\end{aligned}\tag{3.62}$$

by using Lemma 3.3, (3.47) and (3.60).

It follows from (3.62) in (3.59) for the case $j=2$ and $p=1$ that $\partial_x V_{n,2}^{(k)} = \partial_t^2\partial_x Y_n^{(k)}$ satisfies:

$$\begin{aligned}&\frac{d}{dt}\int_\Omega <z>^{2l}|\partial_x V_{n,2}^{(k)}|^2dxdz+\int_\Omega <z>^{2l}(|\partial_z\partial_x V_{n,2}^{(k)}|^2+k^2|\partial_x V_{n,2}^{(k)}|^2)dxdz\\&\quad\leq C_0\int_\Omega <z>^{2l}(|\partial_x V_{n,2}^{(k)}|^2+|\partial_x V_{n-1,2}^{(k)}|^2)dxdz+\int_\Omega <z>^{2l}|\partial_x\partial_t^2 G^{(k)}|^2dxdz\\&\qquad+C_2\{\sum_{q=0}^1\int_\Omega <z>^{2(l+1)}(|\partial_z\partial_x^q V_{n,1}^{(k)}|^2+|\partial_x^q V_{n,1}^{(k)}|^2+|\partial_z\partial_x^q Y_n^{(k)}|^2+|\partial_x^q Y_n^{(k)}|^2)dxdz\\&\qquad+\sum_{q=0}^1\int_\Omega <z>^{2l}(k^2|\partial_x^q V_{n,1}^{(k)}|^2+k^2|\partial_x^q Y_n^{(k)}|^2+|\partial_x^q V_{n-1,1}^{(k)}|^2+|\partial_x^q Y_{n-1}^{(k)}|^2)dxdz\\&\qquad+\int_\Omega <z>^{2l}(|\partial_x^2 V_{n,1}^{(k)}|^2+k^2|V_{n,2}^{(k)}|^2+|\partial_x^2 Y_n^{(k)}|^2+|V_{n-1,2}^{(k)}|^2)dxdz\\&\qquad+\int_\Omega <z>^{2(l+1)}|\partial_z V_{n,2}^{(k)}|^2dxdz\}\end{aligned}$$

which implies

$$\begin{aligned}&\max_{0\leq t\leq T}\int_\Omega <z>^{2l}|\partial_x V_{n,2}^{(k)}|^2dxdz+\int_0^T\!\!\int_\Omega <z>^{2l}(|\partial_z\partial_x V_{n,2}^{(k)}|^2+k^2|\partial_x V_{n,2}^{(k)}|^2)dxdzdt\\&\qquad\leq C(T)(\sum_{j=0}^1\int_\Omega <z>^{2(l+1-j)}|\partial_x^j V_{n,2,0}^{(k)}|^2dxdz\\&\qquad\quad+\sum_{p+q\leq 3,p\leq 2}\int_0^T\!\!\int_\Omega <z>^{2(l+3-p-q)}|\partial_x^q\partial_t^p G^{(k)}|^2dxdzdt).\end{aligned}\tag{3.63}$$

by using Lemma 3.3, (3.60), (3.51), (3.61) and (3.62).

Now, (3.56) for the case $j=3$ shows

$$\begin{aligned}
&\tfrac{d}{dt}\int_\Omega <z>^{2l}\ |V_{n,3}^{(k)}|^2 dxdz + \int_\Omega <z>^{2l}\ (|\partial_z V_{n,3}^{(k)}|^2 + k^2|V_{n,3}^{(k)}|^2)dxdz \\
&\quad \le C_0\int_\Omega <z>^{2l}\ (|V_{n,3}^{(k)}|^2 + |V_{n-1,3}^{(k)}|^2)dxdz + \int_\Omega <z>^{2l}\ |\partial_t^3 G^{(k)}|^2 dxdz \\
&\quad\quad + C_3\{\textstyle\sum_{0<j\le 3}(\int_\Omega <z>^{2(l+1)}\ (|\partial_z V_{n,3-j}^{(k)}|^2 + |V_{n,3-j}^{(k)}|^2)dxdz \\
&\quad\quad + \int_\Omega <z>^{2l}\ (|V_{n-1,3-j}^{(k)}|^2 + k^2|V_{n,3-j}^{(k)}|^2)dxdz \\
&\quad\quad + \int_\Omega <z>^{2l}\ (|\partial_x V_{n,2}^{(k)}|^2 + |\partial_x V_{n,1}^{(k)}|^2 + |\partial_x Y_n^{(k)}|^2)dxdz\}
\end{aligned}$$

which implies

$$\begin{aligned}
&\max_{0\le t\le T}\ \int_\Omega <z>^{2l}\ |V_{n,3}^{(k)}|^2 dxdz + \int_0^T\!\!\int_\Omega <z>^{2l}\ (|\partial_z V_{n,3}^{(k)}|^2 + k^2|V_{n,3}^{(k)}|^2)dxdzdt \\
&\quad \le C(T)(\int_\Omega <z>^{2l}\ |V_{n,3,0}^{(k)}|^2 dxdz + \textstyle\sum_{j=0}^1 \int_\Omega <z>^{2(l+1-j)}\ |\partial_x^j V_{n,2,0}^{(k)}|^2 dxdz \\
&\quad\quad + \textstyle\sum_{0\le|\alpha|\le 3}\int_0^T\!\!\int_\Omega <z>^{2(l+3-|\alpha|)}\ |\partial_{t,x}^\alpha G^{(k)}|^2 dxdzdt).
\end{aligned} \tag{3.64}$$

By induction on $|\alpha|\in I\!N$, we deduce the following bound on $\partial_{t,x}^\alpha Y_n^{(k)}$:

$$\begin{aligned}
&\max_{0\le t\le T}\ \int_\Omega <z>^{2l}\ |\partial_{t,x}^\alpha Y_n^{(k)}|^2 dxdz + \int_0^T\!\!\int_\Omega <z>^{2l}\ (|\partial_z\partial_{t,x}^\alpha Y_n^{(k)}|^2 + k^2|\partial_{t,x}^\alpha Y_n^{(k)}|^2)dxdzdt \\
&\quad \le C(T)(\textstyle\sum_{j=2}^{|\alpha|}\sum_{m=0}^{|\alpha|-j}\int_\Omega <z>^{2(l+|\alpha|-j-m)}\ |\partial_x^m\partial_t^j Y_n^{(k)}(t=0)|^2 dxdz \\
&\quad\quad + \textstyle\sum_{|\beta|\le|\alpha|}\int_0^T\!\!\int_\Omega <z>^{2(l+|\alpha|-|\beta|)}\ |\partial_{t,x}^\beta G^{(k)}|^2 dxdzdt)
\end{aligned} \tag{3.65}$$

for any $\alpha\in I\!N^2$.

(4) Estimates of normal derivatives $W_{n,\alpha,j}^{(k)} = \partial_z^j\partial_{t,x}^\alpha Y_n^{(k)}$ for any $j\in I\!N$ and $\alpha = (\alpha_1,\alpha_2)\in I\!N^2$.

Note that

$$\partial_z^j\partial_t^{\alpha_1}\partial_x^{\alpha_2}Y_n^{(k)}(t,x,z) = \partial_z^j\partial_t^{\alpha_1}\partial_x^{\alpha_2}Y_n^{(k)}(0,x,z) + \int_0^t \partial_z^j\partial_t^{\alpha_1+1}\partial_x^{\alpha_2}Y_n^{(k)}(s,x,z)ds$$

which implies the following estimate:

$$\begin{aligned}
&\max_{0\le t\le T}\ \int_\Omega <z>^{2l}\ |\partial_z\partial_t^{\alpha_1}\partial_x^{\alpha_2}Y_n^{(k)}|^2 dxdz \le \int_\Omega <z>^{2l}\ |\partial_z\partial_t^{\alpha_1}\partial_x^{\alpha_2}Y_n^{(k)}(t=0)|^2 dxdz \\
&\quad + C(T)(\textstyle\sum_{j=2}^{|\alpha|+1}\sum_{m=0}^{|\alpha|+1-j}\int_\Omega <z>^{2(l+|\alpha|+1-j-m)}\ |\partial_x^m\partial_t^j Y_n^{(k)}(t=0)|^2 dxdz \\
&\quad + \textstyle\sum_{|\beta|\le|\alpha|+1}\int_0^T\!\!\int_\Omega <z>^{2(l+|\alpha|+1-|\beta|)}\ |\partial_{t,x}^\beta G^{(k)}|^2 dxdzdt)
\end{aligned} \tag{3.66}$$

by using (3.65).

If follows from the equation in (3.57) that

$$\int_\Omega <z>^{2l}\ |\partial_z^2\partial_t^{\alpha_1}\partial_x^{\alpha_2}Y_n^{(k)}|^2dxdz$$
$$\le C_0\{\int_\Omega <z>^{2(l+1)}\ (|\partial_z\partial_t^{\alpha_1}\partial_x^{\alpha_2}Y_n^{(k)}|^2+k^2|\partial_t^{\alpha_1}\partial_x^{\alpha_2}Y_n^{(k)}|^2)dxdz$$
$$+\int_\Omega <z>^{2l}\ (|\partial_t^{\alpha_1+1}\partial_x^{\alpha_2}Y_n^{(k)}|^2+|\partial_t^{\alpha_1}\partial_x^{\alpha_2+1}Y_n^{(k)}|^2+|\partial_t^{\alpha_1}\partial_x^{\alpha_2+2}Y_n^{(k)}|^2$$
$$+k^4|\partial_t^{\alpha_1}\partial_x^{\alpha_2}Y_n^{(k)}|^2+|\partial_t^{\alpha_1}\partial_x^{\alpha_2}Y_{n-1}^{(k)}|^2+|\partial_t^{\alpha_1}\partial_x^{\alpha_2}G^{(k)}|^2+Q_{\alpha_1,\alpha_2})dxdz\} \tag{3.67}$$

On the other hand, (3.57) yields

$$\int_\Omega\ <z>^{2l}\ |Q_{\alpha_1,\alpha_2}|^2dxdz$$
$$\le C\{\sum_{j\le\alpha_1-1,m\le\alpha_2}\int_\Omega <z>^{2l}\ (|\partial_z^2\partial_t^j\partial_x^mY_n^{(k)}|^2+k^4|\partial_t^j\partial_x^mY_n^{(k)}|^2+|\partial_t^j\partial_x^mY_{n-1}^{(k)}|^2)dxdz$$
$$+\sum_{j\le\alpha_1-1,m\le\alpha_2}\int_\Omega <z>^{2(l+1)}\ (|\partial_z\partial_t^j\partial_x^mY_n^{(k)}|^2+k^2|\partial_t^j\partial_x^mY_n^{(k)}|^2)dxdz$$
$$+\int_\Omega <z>^{2l}\ (|\partial_t^{\alpha_1}\partial_x^{\alpha_2}Y_n^{(k)}|^2+\sum_{j\le\alpha_1-1}|\partial_t^j\partial_x^{\alpha_2+1}Y_n^{(k)}|^2)dxdz$$
$$+\sum_{m\le\alpha_2-1}\int_\Omega <z>^{2l}\ (|\partial_z^2\partial_t^{\alpha_1}\partial_x^mY_n^{(k)}|^2+k^4|\partial_t^{\alpha_1}\partial_x^mY_n^{(k)}|^2+|\partial_t^{\alpha_1}\partial_x^mY_{n-1}^{(k)}|^2)dxdz$$
$$+\sum_{m\le\alpha_2-1}\int_\Omega <z>^{2(l+1)}\ (|\partial_z\partial_t^{\alpha_1}\partial_x^mY_n^{(k)}|^2+k^2|\partial_t^{\alpha_1}\partial_x^mY_n^{(k)}|^2)dxdz\} \tag{3.68}$$

Substituting (3.68) into (3.67) shows

$$\int_\Omega <z>^{2l}\ |\partial_z^2\partial_t^{\alpha_1}\partial_x^{\alpha_2}Y_n^{(k)}|^2dxdz$$
$$\le C\{\sum_{\beta\le\alpha}\int_\Omega <z>^{2(l+1)}\ (|\partial_z\partial_{t,x}^\beta Y_n^{(k)}|^2+k^2|\partial_{t,x}^\beta Y_n^{(k)}|^2)dxdz$$
$$+\int_\Omega <z>^{2l}\ (\sum_{\beta\le\alpha}(k^4|\partial_{t,x}^\beta Y_n^{(k)}|^2+|\partial_{t,x}^\beta Y_{n-1}^{(k)}|^2)+\sum_{\beta<\alpha}|\partial_z^2\partial_{t,x}^\beta Y_n^{(k)}|^2$$
$$+\sum_{j\le\alpha_1}|\partial_t^j\partial_x^{\alpha_2+1}Y_n^{(k)}|^2+|\partial_t^{\alpha_1}\partial_x^{\alpha_2+2}Y_n^{(k)}|^2$$
$$+|\partial_t^{\alpha_1+1}\partial_x^{\alpha_2}Y_n^{(k)}|^2+|\partial_t^{\alpha_1}\partial_x^{\alpha_2}G^{(k)}|^2)dxdz\} \tag{3.69}$$

where the notations $\beta\le\alpha$ and $\beta<\alpha$ for $\alpha,\beta\in I\!N^2$ mean that $\beta_1\le\alpha_1,\beta_2\le\alpha_2$ and $\beta_1\le\alpha_1,\beta_2\le\alpha_2$, $\beta_1+\beta_2<\alpha_1+\alpha_2$ respectively.

By using (3.66) and (3.65) in (3.69), we get

$$\max_{0\le t\le T}\ \int_\Omega <z>^{2l}\ |\partial_z^2\partial_{t,x}^\alpha Y_n^{(k)}|^2dxdz$$
$$\le C\{k^4\sum_{\beta\le\alpha}(\int_\Omega <z>^{2(l+|\alpha|-|\beta|)}\ |\partial_{t,x}^\beta Y_n^{(k)}(t=0)|^2dxdz$$
$$+\int_0^T\int_\Omega <z>^{2(l+|\alpha|-|\beta|)}\ |\partial_{t,x}^\beta G^{(k)}|^2dxdzdt)$$
$$+\sum_{|\beta|\le|\alpha|+2}(\int_\Omega <z>^{2(l+|\alpha|+2-|\beta|)}\ |\partial_{t,x}^\beta Y_n^{(k)}(t=0)|^2dxdz$$
$$+\int_0^T\int_\Omega <z>^{2(l+|\alpha|+2-|\beta|)}\ |\partial_{t,x}^\beta G^{(k)}|^2dxdzdt)$$
$$+\sum_{\beta\le\alpha}\int_\Omega <z>^{2(l+1)}\ |\partial_z\partial_{t,x}^\beta Y_n^{(k)}(t=0)|^2dxdz$$
$$+\sum_{\beta<\alpha}\int_0^T\int_\Omega <z>^{2l}\ |\partial_z^2\partial_{t,x}^\beta Y_n^{(k)}|^2dxdzdt\}$$

and by induction on $\alpha \in I\!N^2$,

$$
\begin{aligned}
\max_{0\le t\le T} \ & \int_\Omega <z>^{2l}\ |\partial_z^2\partial_{t,x}^\alpha Y_n^{(k)}|^2 dxdz \\
& \le C(T)\{k^4 \textstyle\sum_{\beta\le\alpha}(\int_\Omega <z>^{2(l+|\alpha|-|\beta|)}\ |\partial_{t,x}^\beta Y_n^{(k)}(t=0)|^2 dxdz \\
& \quad + \textstyle\int_0^T\int_\Omega <z>^{2(l+|\alpha|-|\beta|)}\ |\partial_{t,x}^\beta G^{(k)}|^2 dxdzdt) \\
& \quad + \textstyle\sum_{|\beta|\le|\alpha|+2}(\int_\Omega <z>^{2(l+|\alpha|+2-|\beta|)}\ |\partial_{t,x}^\beta Y_n^{(k)}(t=0)|^2 dxdz \\
& \quad + \textstyle\int_0^T\int_\Omega <z>^{2(l+|\alpha|+2-|\beta|)}\ |\partial_{t,x}^\beta G^{(k)}|^2 dxdzdt) \\
& \quad + \textstyle\sum_{\beta\le\alpha}\int_\Omega <z>^{2(l+1)}\ |\partial_z\partial_{t,x}^\beta Y_n^{(k)}(t=0)|^2 dxdz\}.
\end{aligned}
\tag{3.70}
$$

Differentiating (3.57) with respect to z and by induction on $j \in I\!N$, one can obtain

$$
\begin{aligned}
& \max_{0\le t\le T}\int_\Omega <z>^{2l}\ |\partial_z^j\partial_{t,x}^\alpha Y_n^{(k)}|^2 dxdz \\
& \le C(T)\{\textstyle\sum_{m=0}^{[j/2]} k^{4m}\sum_{|\beta|\le|\alpha|+j-2m}(\int_\Omega <z>^{2(l+|\alpha|+j-2m-|\beta|)}\ |\partial_{t,x}^\beta Y_n^{(k)}(t=0)|^2 dxdz \\
& \quad + \textstyle\int_0^T\int_\Omega <z>^{2(l+|\alpha|+j-2m-|\beta|)}\ |\partial_{t,x}^\beta G^{(k)}|^2 dxdzdt) \\
& \quad + \textstyle\sum_{|\beta|\le|\alpha|+j-1-2m}\int_\Omega <z>^{2(l+|\alpha|+j-1-2m-|\beta|)}\ |\partial_z\partial_{t,x}^\beta Y_n^{(k)}(t=0)|^2 dxdz \\
& \quad + \max_{0\le t\le T}\textstyle\sum_{m=1}^{j-2}\int_\Omega <z>^{2(l+j-2-m)}\ |\partial_z^m\partial_{t,x}^\alpha G^{(k)}|^2 dxdzdt\}.
\end{aligned}
\tag{3.71}
$$

In summary, we conclude

Proposition 3.4: *Under the assumption that any order compatibility condition of the problem (3.34) is satisfied, the approximate solution sequence* $\{Y_n^{(k)}\}_{n\ge1}$ *constructed by (3.36) is bounded in* $W^{k,\infty}([0,T],H^s(\Omega))$ *for any fixed* $k,s\in I\!N$*; moreover,* $\{Y_n^{(k)}\}_{n\ge1}$ *satisfies the estimates (3.47), (3.65) and (3.71).*

Step 3: The convergence of the approximate solution sequence $\{Y_n^{(k)}\}_{n\ge1}$.

As usual, based on the high order norm boundedness estimate (3.71) of $\{Y_n^{(k)}\}_{n\ge1}$, it suffices to consider the convergence of $\{Y_n^{(k)}\}_{n\ge1}$ in the L^2-norm.

Let $W_n^{(k)} = Y_{n+1}^{(k)} - Y_n^{(k)}$. It follows from (3.36) that $W_n^{(k)}$ solves the following problem

$$
\begin{cases}
(\partial_t + a_1\partial_x)W_n^{(k)} - a_4^2(\partial_z^2 - k^2a^2)W_n^{(k)} + z(a_2\partial_z + ika_3)W_n^{(k)} - \frac{1}{n+1}\partial_x^2 W_n^{(k)} \\
\qquad + ka_5\int_z^{+\infty} e^{ka(z-\xi)}W_{n-1}^{(k)}(\cdot,\xi)d\xi + ka_6\int_0^z e^{ka(\xi-z)}W_{n-1}^{(k)}(\cdot,\xi)d\xi = -\frac{1}{n(n+1)}\partial_x^2 U_n^{(k)} \\
W_n^{(k)}|_{z=0} = 0, \quad W_n^{(k)} \in S(I\!R_z^+) \\
W_n^{(k)}|_{t=0} = 0
\end{cases}
\tag{3.72}
$$

In a way similar to (3.43), we deduce that for all $n \ge 1$,

$$
\begin{aligned}
& \tfrac{d}{dt}\int_\Omega <z>^{2l}\ |W_n^{(k)}|^2 dxdz + \int_\Omega <z>^{2l}\ (|\partial_z W_n^{(k)}|^2 + k^2|W_n^{(k)}|^2)dxdz \\
& \qquad \le C_0\int_\Omega <z>^{2l}\ (|W_n^{(k)}|^2 + |W_{n-1}^{(k)}|^2)dxdz + \tfrac{C_0}{n(n+1)}
\end{aligned}
\tag{3.73}
$$

by using the boundedness of $\{Y_n^{(k)}\}_{n\geq 1}$. Applying Lemma 3.3 in (3.73) yields immediately that

Proposition 3.5: *For any fixed $T>0$ and $l \in I\!N$, it holds that*

$$\max_{0\leq t\leq T}\int_\Omega <z>^{2l} |Y_{n+1}^{(k)} - Y_n^{(k)}|^2 dxdz \longrightarrow 0 \tag{3.74}$$

when n goes to infinite.

Collecting all the results in Step 1 to Step 3, we deduce the existence of a smooth solution $Y^{(k)}$ to (3.34) and (3.35). The uniqueness of this solution is obvious. Combinning this result with the transformation (3.33), Proposition 3.2, we establish the existence and uniqueness of solutions (u,w) to the Poisson-Prandtl coupled problem (3.1).

4 Rigorous Justification of The Zero-Viscosity Limit

In this section, we shall rigorously justify the formal analysis given in §2.

From §3, we know that the problems (2.35)-(2.36) of $(d_0^{(2)}, d_0^{(3)})$, and (2.57)-(2.58) of $\{(d_{(j+1)}^{(2)}, d_{(j+1)}^{(3)})\}_{j\geq 0}$ can be solved under the assumption that certain order compatibility conditions for these problems are satisfied. It thus follows from §2 that each order smooth profile $\{(a_j, c_j, b_j, d_j)\}_{j\geq 0}$ in the formal expansions of solutions

$$V^\epsilon(t,x) \sim \sum_{j\geq 0}\epsilon^j (a_j(t,x) + c_j(t,x;\frac{\varphi(t,x)}{\epsilon}) + b_j(t,x_2;\frac{x_1}{\epsilon}) + d_j(t,x_2;\frac{x_1}{\epsilon},\frac{\varphi^0(t,x_2)}{\epsilon}) \tag{4.1}$$

can be uniquely determined provided that

(H1) all compatibility conditions for the problems (2.26), (2.29), (2.35), (2.46), (2.48) and (2.57) are satisfied.

For any fixed $J \in I\!N$, denote by

$$V_J^\epsilon(t,x) = \sum_{j=0}^{J}\epsilon^j \left(a_j(t,x) + c_j(t,x;\frac{\varphi(t,x)}{\epsilon}) + b_j(t,x_2;\frac{x_1}{\epsilon}) + d_j(t,x_2;\frac{x_1}{\epsilon},\frac{\varphi^0(t,x_2)}{\epsilon})\right)$$

the $J-$th order approximate solution to the problem (2.1), and V^ϵ the exact solution to (2.1) under the assumption

(H2) all compatibility conditions for the problem (2.1) are satisfied.

Then, from the discussion in §2, it is easy to see that $W_J^\epsilon = V^\epsilon - V_J^\epsilon$ solves the following problem

$$\begin{cases} A_0(V')\partial_t W_J^\epsilon + A_1(V')\partial_{x_1} W_J^\epsilon + A_2(V')\partial_{x_2} W_J^\epsilon = B(\epsilon^2, D\epsilon^2)W_J^\epsilon + R_J^\epsilon \\ M^+ W_J^\epsilon = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} W_J^\epsilon = 0, \qquad \text{on} \quad x_1 = 0 \\ W_J^\epsilon|_{t=0} = 0 \end{cases} \tag{4.2}$$

where $R_J^\epsilon(t,x)$ satisfies

$$\|R_J^\epsilon\|_{L^\infty([0,T],L^2(I\!R_+^2))} \leq C\epsilon^{J-1} \tag{4.3}$$

for any $T > 0$ and a constant $C > 0$.

By using the classical theory of the linearized Navier-Stokes equations in the problem (4.2), we immediately conclude

$$\|V^\epsilon - V_J^\epsilon\|_{L^\infty([0,T],L^2(I\!R_+^2))} \le C_1 \epsilon^{J-1} \tag{4.4}$$

which implies

$$\|V^\epsilon - V_J^\epsilon\|_{L^\infty([0,T],L^2(I\!R_+^2))} \le C_2 \epsilon^{J+1} \tag{4.5}$$

for any $J \in I\!N$ with the constant C_2 depending only upon T and J.

In particular, we obtain:

Theorem 4.1: *Under the assumptions (H1) and (H2), the solution $V^\epsilon = (\rho^\epsilon, v_1^\epsilon, v_2^\epsilon)$ of (2.1) has the following asymptotics*

$$V^\epsilon(t,x) = a_0(t,x) + c_0(t,x;\frac{\varphi(t,x)}{\epsilon}) + b_0(t,x_2;\frac{x_1}{\epsilon}) + d_0(t,x_2;\frac{x_1}{\epsilon},\frac{\varphi^0(t,x_2)}{\epsilon}) + O(\epsilon) \tag{4.6}$$

in $L^\infty([0,T],L^2(I\!R_+^2))$ for any $T > 0$, where $a_0(t,x)$ satisfies the problem for the linearized Euler equations (2.26), $c_0 = v_0(t,x;\frac{\varphi(t,x)}{\epsilon})\vec{r}_1(\nabla\varphi)$ with v_0 satisfying the degenerate parabolic equation (2.20), $(b_0^{(1)}, b_0^{(2)}) = 0$ and $b_0^{(3)}(t,x_2;z)$ satisfies the linearized Prandtl equation (2.29), $d_0^{(1)} = 0$, and $(d_0^{(2)}, d_0^{(3)})(t,x_2;z,\theta)$ together with its vorticity with respect to $(z,\theta)-$variables satisfy the Poisson-Prandtl coupled system (2.35) and the Poisson equation (2.36) respectively.

Remark 4.2: Both the estimate (4.5) and asymptotic relation (4.6) hold true in high order Sobolev spaces with weighted norms due to the high frequency of oscillations in $\{c_j, d_j\}_{j\ge 0}$ and the multiple scales in boundary layers $\{b_j, d_j\}_{j\ge 0}$, e.g. in $L^\infty([0,T], H_\epsilon^s(I\!R^2))$ with the norm of $H_\epsilon^s(I\!R^2)$ being defined as

$$\|u\|_{s,\epsilon} = (\sum_{|\alpha|\le s} \epsilon^{2|\alpha|}\|\partial_x^\alpha u\|^2_{L^2(I\!R_+^2)})^{\frac{1}{2}}.$$

Finally, for completeness, let us investigate the assumptions (H1) and (H2).

(I) The compatibility condition for the problem of linearized Navier-Stokes equations (2.1) can be formulated in the classical way as follows.

(I1) The zero-th order compatibility condition is:

$$V_0^{(2)} = V_0^{(3)} = 0 \qquad \text{on} \quad \{x_1 = 0\}. \tag{4.7}$$

(I2) The $j-$th order compatibility condition ($j \ge 1$).

Set $\Phi^\epsilon(t,x) = \Phi(t,x;\frac{\varphi(t,x)}{\epsilon})$. For any fixed $j \in I\!N$ with $j \ge 1$, it follows from the equations in (2.1) that

$$\partial_t^j V^\epsilon = (A_0(V'))^{-1}\{B(\epsilon^2, D\epsilon^2)\partial_t^{j-1}V^\epsilon + \partial_t^{j-1}\Phi^\epsilon - [\partial_t^{j-1}A_0(V')\partial_t + A_1(V')\partial_{x_1} + A_2(V')\partial_{x_2}]V^\epsilon)\}$$

by induction on j. By using the initial data $V^\epsilon|_{t=0} = V_0(x)$, we know that $V_j^\epsilon(x) = \partial_t^j V^\epsilon|_{t=0}$ is a linear function of $\{\partial_x^\alpha V_0\}_{|\alpha|\le 2j}$ and $\{\partial_t^k \partial_x^\alpha \Phi^\epsilon(t=0)\}_{k\le j-1, \frac{|\alpha|}{2}+k=j-1}$. Then, the $j-$th order compatibility condition for the problem (2.1) is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_j^\epsilon = 0 \qquad \text{on} \quad \{x_1 = 0\}. \tag{4.8}$$

Next, we study the assumption (H1).

($I\!I$) The compatibility condition for the problem of linearized Euler equations (2.26).

($I\!I$1) The zero-th order compatibility condition is:

$$V_0^{(2)} = 0 \qquad \text{on} \quad \{x_1 = 0\}, \tag{4.9}$$

which is a direct consequence of the zero-th order compatibility condition (4.7) for the problem (2.1).

($I\!I$2) The $j-$th order compatibility condition ($j \geq 1$).

Set $\overline{\Phi}(t,x) = \mathbf{m}_\theta(\Phi)$. As in (I1), for any fixed $j \in I\!N$ with $j \geq 1$, the equation in (2.26) shows that

$$\partial_t^j a_0 = (A_0(V'))^{-1}\{\partial_t^{j-1}\overline{\Phi} \\ -[\partial_t^{j-1} \quad A_0(V')]\partial_t + A_1(V')\partial_{x_1} + A_2(V')\partial_{x_2}]a_0)\}$$

by induction on j. Since $a_0|_{t=0} = V_0(x)$, so that $V_{0,j}(x) = \partial_t^j a_0|_{t=0}$ is a linear function of $\{\partial_x^\alpha V_0\}_{|\alpha|\leq j}$ and $\{\partial_t^k \partial_x^\alpha \overline{\Phi}(t=0)\}_{k\leq j-1, |\alpha|+k=j-1}$. Then, the $j-$th order compatibility condition for the problem (2.26) is

$$V_{0,j}^{(2)} = 0 \qquad \text{on} \quad \{x_1 = 0\}. \tag{4.10}$$

($I\!I\!I$) The compatibility condition for the problem of linearized Prandtl equation (2.29).

($I\!I\!I$1) The zero-th order compatibility condition is:

$$a_0^{(3)} = 0 \qquad \text{on} \quad \{t = x_1 = 0\}, \tag{4.11}$$

which is a simple consequence of the zero-th order compatibility condition (4.7) by noting $a_0|_{t=0} = V_0(x)$ in (2.26).

($I\!I\!I$2) The $j-$th order compatibility condition ($j \geq 1$).

It follows from the equation and the initial data in (2.29) that

$$\partial_t^j b_0^{(3)}|_{t=0} = 0.$$

So, the $j-$th order compatibility condition for the problem (2.29) is

$$\partial_t^j a_0^{(3)} = 0 \qquad \text{on} \quad \{t = x_1 = 0\}, \tag{4.12}$$

where $a_0^{(3)}(t,x)$ is determined by the problem (2.26).

The compatibility conditions for the problems (2.46) and (2.48) can be obtained in the same ways as those for the problems (2.26) and (2.29) given in ($I\!I$) and ($I\!I\!I$) respectively.

Both of problems (2.35) and (2.57) are the special cases of the problem (3.1), so their compatibility conditions can be stated in the same way as that for the problem (3.1) given in §3.

Finally, we should note that in general the compatibility conditions for the problems of profiles $\{a_j, c_j, b_j, d_j\}_{j\geq 0}$ could not be implied by those for the original linearized Navier-Stokes equations (2.1). The simplest case to guarantee all compatibility conditions given as above valid is that

$$\begin{cases} \partial_t^k \partial_x^\alpha \Phi(t,x;\theta) = 0, & \text{on } \{t = x_1 = 0\} \\ \partial_x^\alpha V_0(x) = 0, & \text{on } \{x_1 = 0\} \end{cases}$$

hold for any $k \in I\!N$ and $\alpha \in I\!N^2$.

Acknowledgements. This research was supported by the Zheng Ge Ru Foundation when Ya-Guang Wang were visiting The Institute of Mathematical Sciences in The Chinese

University of Hong Kong. The work of Wang is also partially supported by the NSFC, the Educational Ministry of China and the Shanghai Science and Technology Committee. The work of Xin is supported in part by grants from RGC of HKSAR CUHK-4279/00P and CUHK-4040/02P.

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