Some Uniform Estimates and Blowup Behavior of Global Strong Solutions to the Stokes Approximation Equations for Two-Dimensional Compressible Flows*

Jing Li and Zhouping Xin†
IMS and Department of Mathematics,
The Chinese University of Hong Kong, Shatin, N. T., Hong Kong

Abstract

This paper concerns the global existence and the large time behavior of strong and classical solutions to the two-dimensional Stokes approximation equations for the compressible flows. We consider the unique global strong solution or classical solution to the two-dimensional Stokes approximation equations for the compressible flows together with the space-periodicity boundary condition or the no-stick boundary condition or Cauchy problem for arbitrarily large initial data. First, we prove that the density is bounded from above independent of time in all these cases. Secondly, we show that for the space-periodicity boundary condition or the no-stick boundary condition, if the initial density contains vacuum at least at one point, then the global strong (or classical) solution must blow up as time goes to infinity.

Keywords: Stokes approximation equations; Isentropic compressible fluids; Uniform upper bound; Vacuum; Blowup

1 Introduction

The compressible isentropic Navier-Stokes equations, which are the basic model describing the evolution of a viscous compressible gas, read as follows

$$\begin{cases} \rho_t + div(\rho u) = 0, \\ (\rho u)_t + div(\rho u \otimes u) = \mu \Delta u + \nabla(\xi div u) - \nabla P(\rho), \end{cases} \tag{1.1}$$

where $x \in \Omega \subset R^N, t \in (0,T)$ and $P(\rho) = a\rho^\gamma, a > 0, \gamma > 1$, the viscosity coefficients μ, ξ are assumed to satisfy $\mu > 0$ and $\xi + \mu \geq 0$.

There is huge literature on the studies on the large time existence and behavior of solutions to (1.1). The one-dimensional problem was addressed by Kazhikhov and Shelukhin [11] for sufficiently smooth data, and by Serre [22] [23] and Hoff [7] for discontinuous initial data, where the data are uniformly away from the vacuum. The multidimensional problem (1.1) was investigated by Matsumura and Nishida [17] [18] [19], who proved global existence of smooth solutions for data close to a non-vacuum equilibrium, and later by Hoff for discontinuous initial data [8], and more recently, by

*This research was supported in part by Grants from RGC of HKSAR CUHK4028/04P, CUHK4040/02P and CUHK4279/00P and Zheng Ge Ru Foundation.

†E-mail addresses: jli@ims.cuhk.edu.hk(Jing Li), zpxin@ims.cuhk.edu.hk(Zhouping Xin).

Danchin [3], who obtained existence and uniqueness of global solutions in a functional space invariant by the natural scaling of the associated equations. For the existence of solutions for arbitrary data(which may include vacuum states), the major breakthrough is due to P. L. Lions [14] [15] [16] (see also Feireisl et al [4]), where he obtains global existence of weak solutions - defined as solutions with finite energy - when the exponent γ is suitably large. The only restriction on initial data is that the initial energy is finite, so that the density is allowed to vanish.

Despite the important progress, the regularity and behavior of these weak solutions is completely open. As emphasized in many papers related to compressible fluid dynamics [2], [7], [9]-[11], [21], [22], [24]-[26], the possible appearance of vacuum and uniform upper bound estimate on the density is one of the major difficulties when trying to prove global existence and strong regularity results. In particular, the results of Xin [26] show that there is no global smooth solution (ρ, u) to Cauchy problem for (1.1) with a nontrivial compactly supported initial density, which gives results for finite time blow-up in the presence of vacuum.

The major difficulties in analysis of the compressible Navier-Stokes equations (1.1) are the nonlinearities in both convection and pressure and their interactions. To study the well-posedness of solutions and gain understanding of the key issues, one has been looking into various simplified models of the Navier-Stokes systems. One of the pro-type simplifications of the Navier-Stokes system (1.1) is the Stokes approximation

$$\begin{cases} \rho_t + div(\rho u) = 0, \\ \overline{\rho} u_t - \mu \Delta u - \xi \nabla(divu) + \nabla P = 0, \end{cases} \tag{1.2}$$

where $\overline{\rho} = \text{const.} > 0$ is the mean density, and $P = a\rho^\gamma, a > 0, \gamma > 1$. This is a good approximation for strongly viscous fluids and where the convection is unimportant.

For simplicity, we take $\overline{\rho} = 1, \mu = 1, \xi = 0, a = 1$, and study the system

$$\rho_t + div(\rho u) = 0, \tag{1.3}$$

$$u_t - \Delta u + \nabla P = 0, \tag{1.4}$$

with $P = \rho^\gamma, \gamma > 1$. We are concerned with the initial conditions for the density and the velocity:

$$\rho(0) = \rho_0, u(0) = u_0, \tag{1.5}$$

and three types of boundary conditions:

1) space-periodicity condition, i.e.,

$$\Omega \text{ is a product } \prod_{i=1}^{N}(0, L_i), \text{ and } \rho, u \text{ are } \Omega\text{-periodic;} \tag{1.6}$$

2) Cauchy problem:

$$\Omega = R^N \text{ and (in some weak sense) } \rho, u \text{ vanish at infinity;} \tag{1.7}$$

3) no-stick boundary condition: in this case, Ω is a bounded smooth domain in R^N, and

$$u \cdot n = 0 \text{ and } \begin{cases} curlu = 0 \text{ on } \partial\Omega & \text{if } N = 2, \\ curlu \times n = 0 \text{ on } \partial\Omega & \text{if } N = 3, \end{cases} \tag{1.8}$$

where n is the unit outward normal to $\partial\Omega$. The first condition in (1.8) is the non-penetration boundary condition, while the second one is also known in the form

$$(D(u) \cdot n)_\tau = 0, \tag{1.9}$$

where $D(u)$ is the stress tensor with components

$$D_{ij}(u) = \frac{1}{2}\left(\partial_{x_i} u^j + \partial_{x_j} u^i\right).$$

Condition (1.9) means the tangential component of $D(u) \cdot n$ vanishes on the boundary $\partial\Omega$.

It should be noted that the initial-boundary-value problem (1.3)-(1.5) with the boundary data given either by (1.6) or (1.7) or (1.8) has been throughly studied by many people. In particular, the existence of classical solutions to the 2D initial-boundary-value problem on any finite interval $[0, T](T > 0)$ for arbitrarily large smooth initial data has been proved by Kazhikhov et al [12], Lions [16], Min et al [20], and Chatelon et al [1]. However, it seems to us that the known upper bounds on the density ρ depend on the time T, see [1], [12], [16], [20], so it is impossible to study the large time asymptotic behavior of solutions in the setting in [1], [12], [16], [20]. One of the main purposes of this paper is to derive an uniform time-independent upper bound for the density. As a consequence of the uniform estimate on the bound of density, we show the large time asymptotic behavior of solutions for the strong solutions. Our first result is

Theorem 1.1 *Suppose that $N = 2$ and that*

$$\rho_0 \in W^{l,q}(\Omega) \cap L^1(\Omega), u_0 \in W^{l+1,q}(\Omega) \cap L^2(\Omega) \tag{1.10}$$

for some $q > 2, l \geq 1$. Then problem (1.3)-(1.5) with the boundary condition (1.6) or (1.7) or (1.8) has a unique solution (ρ, u) such that for any $T > 0$,

$$\frac{\partial^k \rho}{\partial t^k} \in L^\infty(0, T; W^{l-k,q}(\Omega)), \frac{\partial^k u}{\partial t^k} \in L^\infty(0, T; W^{l-k+1,q}(\Omega)), \tag{1.11}$$

for any $k, 0 \leq k \leq l$, and moreover,

$$\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^\infty(\Omega)} \leq C, \tag{1.12}$$

and

$$\lim_{t\to\infty} \left(\|R(\cdot, t)\|_{L^\alpha(\Omega)} + \|u(\cdot, t)\|_{L^\beta(\Omega)}\right) = 0, \tag{1.13}$$

with C independent of T and R, α, β such that

$$\begin{cases} R = \rho - \overline{\rho_0}, \ \ any\ \alpha, \beta \in [1, \infty), & if\ \Omega\ is\ bounded, \\ R = \rho, \ \ any\ \alpha \in (1, \infty), \ \ any\ \beta \in (2, \infty), & if\ \Omega = R^2. \end{cases}$$

Remark 1.1 *If $l = 1$, the unique solution is the so-called strong solution; if $l \geq 2$, the unique solution is also a classical one. In this paper, by strong solutions, we mean weak solutions satisfying the equations (1.3) (1.4) almost everywhere in $\Omega \times (0, \infty)$; and by classical solutions, we mean a pair of functions $(\rho, u), \rho \in C^1(\Omega \times (0, \infty)), u \in C^2(\Omega \times (0, \infty))$, such that (1.3) and (1.4) are satisfied everywhere in $\Omega \times (0, \infty)$.*

Remark 1.2 *Under the same conditions, Lions ([16]), Kazhikhov et al ([12]), and Chatelon et al ([1]) proved the same results except (1.12) and (1.13) .*

Remark 1.3 *(1.12) means that the density is bounded from above independent of time; this is the key for the large time dynamical behavior of solutions.*

Theorem 1.1 shows that there exists a unique strong (or classical) solution on $[0, T]$ to the initial-boundary-value problem for the 2D equations (1.2) for any $T > 0$ for smooth data. Furthermore, the large time asymptotic behavior of ρ, u themselves in L^p-norms is given by (1.13). A natural question is what is the large time behavior of the derivatives of this solution. We will give a partial answer to this question. It will be shown that if the initial density contains vacuum at least at a point and the domain Ω is bounded then the global strong solution has to blow up as time goes to infinity, that is

Theorem 1.2 *In addition to the conditions of Theorem 1.1, assume further that there exists some point $x_0 \in \overline{\Omega}$ such that $\rho_0(x_0) = 0$. Then the unique global strong (or classical) solution (ρ, u) to problem (1.3)-(1.5) with the boundary condition (1.6) or (1.8) obtained in Theorem 1.1 has to blow up as $t \to \infty$; that is*

$$\lim_{t\to\infty} \|\nabla\rho(\cdot, t)\|_{L^q(\Omega)} = \infty.$$

Remark 1.4 *It would be interesting to study the existence and large time asymptotic behavior of solutions for the case $q = 2$. This is left for the future.*

Finally, we give a brief outline of the rest of the paper. In Section 2 we collect some elementary facts which are useful for our analysis later. The main results, Theorem 1.1 and Theorem 1.2, are proved in Section 3, 4, 5. It should be noted that although we use some ideas developed in [12] [16], some new elaborate estimates are needed to overcome the difficulties in obtaining the uniform time-independent upper bound estimate for the density. This is achieved by some careful estimates on the deviation of the pressure from its mean value and the difference between the divergence of the velocity field and the deviation of the pressure from its mean value. The case of bounded domains is treated in Section 3. While Section 4 is devoted to the Cauchy problem. Finally, we prove Theorem 1.2 in Section 5.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used and play important roles later.

Consider the following parabolic problem

$$\begin{cases} \varphi_t - \Delta\varphi = f, \\ \varphi(x, 0) = 0, \end{cases} \tag{2.1}$$

supplemented with one of the following three boundary conditions:

$$\varphi(\cdot, t) \text{ is } \Omega\text{-periodic;} \tag{2.2}$$

$$\Omega \text{ bounded, smooth, and } \frac{\partial\varphi}{\partial n} = 0 \text{ on } \partial\Omega; \tag{2.3}$$

$$\Omega = R^N, \text{ and } \varphi \text{ vanishes at infinity.} \tag{2.4}$$

Let Ω be a bounded domain and f is integral on Ω. We denote by $\overline{f}$ the average of f over Ω for bounded domain Ω, i.e.,

$$\overline{f} = \frac{1}{|\Omega|}\int_\Omega f(x)dx.$$

Then the following facts are well-known (see [5] [6]):

Lemma 2.1 *Assume that $p \in (1,\infty), 0 < T \le \infty$. Then for*

$$f \in \begin{cases} \left\{f \in L^p(\Omega\times(0,T)), f \text{ periodic}, \overline{f}=0\right\}, & \text{if (2.2) holds,} \\ \left\{f \in L^p(\Omega\times(0,T)), \overline{f}=0\right\}, & \text{if (2.3) holds,} \\ L^p(\Omega\times(0,T)), & \text{if (2.4) holds,} \end{cases}$$

the problem (2.1) with the boundary condition (2.2) or (2.3) or (2.4) has a unique solution φ such that

$$\varphi_t, D^2\varphi \in L^p(0,T;L^p(\Omega)), \text{ and } \overline{\varphi} = 0 \text{ if } \Omega \text{ is bounded;}$$

moreover, there exists a positive constant A independent of T such that

$$\int_0^T \|\varphi_t(t)\|_{L^p}^p dt + \int_0^T \|\Delta\varphi(t)\|_{L^p}^p dt \le A\int_0^T \|f\|_{L^p}^p dt.$$

Lemma 2.1 yields directly the following derivative estimate.

Lemma 2.2 *Let $r \in (1,\infty), f \in \left(L^r(R^N\times(0,T))\right)^N$. Then solutions of the following parabolic problem:*

$$\begin{cases} \varphi_t - \Delta\varphi = div f, & (x,t) \in R^N\times(0,T) \\ \varphi(x,0)=0, & x \in R^N \\ \text{(2.4) holds,} & \end{cases}$$

satisfy the following estimate

$$\|D\varphi\|_{L^r(R^N\times(0,T))} \le A\|f\|_{L^r(R^N\times(0,T))}$$

where A is a positive constant independent of T.

Lemma 2.1 and the Hodge decomposition lead to the following simple derivative estimate.

Lemma 2.3 *Let $r \in (1,\infty), f \in L^r(\Omega\times(0,T))$ $\left(\overline{f}=0 \text{ for bounded } \Omega\right)$. Then solutions of the following parabolic problem:*

$$\begin{cases} v_t - \Delta v = \nabla f, \\ v(x,0) = 0, \end{cases}$$

supplemented with (1.6) or (1.7) or (1.8), satisfy

$$\|Dv\|_{L^r(\Omega\times(0,T))} \le A\|f\|_{L^r(\Omega\times(0,T))}$$

with A independent of T.

Also, the following estimate will be used later.

Lemma 2.4 ([27]) *Let the function y satisfy*

$$y'(t) \le g(y) + b'(t) \ \ on \ [0,T], y(0) = y^0,$$

with $g \in C(R)$ and $y, b \in W^{1,1}(0,T)$. If $g(\infty) = -\infty$ and $b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1)$ for all $0 \le t_1 < t_2 \le T$ with some $N_0 \ge 0$ and $N_1 \ge 0$, then

$$y(t) \le \max\left\{y^0, \overline{\zeta}\right\} + N_0 < \infty \ \ on \ [0,T],$$

where $\overline{\zeta}$ such that $g(\zeta) \le -N_1$ for $\zeta \ge \overline{\zeta}$.

Finally, we state the well-known Sobolev's inequality.

Lemma 2.5 ([13]) *Assume that $N = 2$ and $\Omega = R^2$ or Ω is a bounded domain in R^2 with piecewise smooth boundary, and that*

$$u \in H_0^1(\Omega) \ \ or \ u \in \left\{u \in H^1(\Omega), \overline{u} = 0\right\} \ \ or \ u \in \left\{\left(H^1(\Omega)\right)^2, u \cdot n|_{\partial\Omega} = 0\right\}.$$

Then there exists a constant C independent of u such that

$$\|u\|_{L^4} \le C\|u\|_{L^2}^{1/2}\|Du\|_{L^2}^{1/2}.$$

3 Proof of Theorem 1.1 on bounded domains

In this section, we will prove Theorem 1.1 in the case of bounded domains, i.e., either boundary condition (1.6) or (1.8) holds. Due to the existence and uniqueness results established in [16] [12], we need only to show that (1.12) and (1.13) hold.

Let $T \in (0, \infty)$ be fixed. In this section and the following one, C denotes a generic positive constant independent of T.

First, we consider the periodic case and deduce (1.12) and (1.13). The boundary condition (1.8) will be treated later.

Case 1. (Periodic case.) Without loss of generality, we assume that

$$\int_\Omega u_0 dx = 0. \tag{3.1}$$

Otherwise, one may change u to $u - \overline{u}_0$, and consider the following system

$$\begin{cases} \rho_t + div(\rho u) + \overline{u}_0 \cdot \nabla\rho = 0, \\ u_t - \Delta u + \nabla P = 0, \\ u(x,0) = u_0 - \overline{u}_0. \end{cases}$$

It is easy to check step by step that the following procedure still holds.

Standard energy estimates for (1.3) – (1.6) yield that

$$\begin{aligned} &\frac{1}{2}\sup_{0\le s\le t}\|u(\cdot,s)\|_{L^2}^2 + \frac{1}{\gamma-1}\sup_{0\le s\le t}\|P(\cdot,s)\|_{L^1} + \int_0^t \|Du\|_{L^2}^2 ds \\ &\quad \le \frac{1}{2}\|u_0\|_{L^2}^2 + \frac{1}{\gamma-1}\|P_0\|_{L^1} \triangleq I_0^2. \end{aligned} \tag{3.2}$$

(3.1), (1.4) and the periodic boundary condition (1.6) lead to

$$\int_\Omega u dx = 0.$$

Thus, we use Lemma 2.5, (3.2) and Poincaré's inequality to derive

$$\int_0^t \left(\|u\|_{L^4}^4 + \|u\|_{L^2}^2 \right) ds \le C. \tag{3.3}$$

Denote by φ and w the unique periodic functions such that

$$u = \nabla\varphi + w, divw = 0, \int_\Omega \varphi dx = 0,$$

and

$$\|\nabla\varphi\|_{L^p} + \|w\|_{L^p} \le C\|u\|_{L^p},$$

for $1 < p < \infty$; and similarly

$$u_0 = \nabla\varphi_0 + w_0, divw_0 = 0, \int_\Omega \varphi_0 dx = 0,$$

and

$$\|\nabla\varphi_0\|_{L^p} + \|w_0\|_{L^p} \le C\|u_0\|_{L^p}.$$

Hence, (1.4) and (1.5) show that w and φ satisfy

$$\begin{cases} w_t - \Delta w = 0, \\ w(x,0) = w_0(x), \end{cases} \tag{3.4}$$

and

$$\begin{cases} \varphi_t - \Delta\varphi + Q = 0, \\ \varphi(x,0) = \varphi_0(x), \end{cases} \tag{3.5}$$

respectively, where $Q \triangleq P - \overline{P}$.

Denote by $S = \varphi_t = \Delta\varphi - Q = divu - Q$. Then, P, Q and S satisfy

$$P_t + div(Pu) + (\gamma - 1)Pdivu = 0, \tag{3.6}$$

$$\begin{cases} Q_t + div(uQ) + (\gamma-1)Qdivu + \gamma\overline{P}divu - (\gamma-1)\overline{Qdivu} = 0, \\ Q(x,0) = Q_0 \triangleq P_0 - \overline{P_0}, \end{cases} \tag{3.7}$$

and

$$\begin{cases} S_t - \Delta S = -Q_t, \\ S(x,0) = \Delta\varphi_0 - Q_0, \end{cases} \tag{3.8}$$

respectively, which follow by direct calculations based on (1.3) (1.4).

Multiplying (3.7) by Q^2 and integrating the result in space, one can obtain after integrating by parts and Hölder's inequality and (3.2) that

$$\begin{aligned}&\|P(t)\|_{L^3}^3 + \int_0^t \|Q(s)\|_{L^4}^4 ds \\ &\qquad \le C + C\|P_0\|_{L^3}^3 + C\int_0^t \|S\|_{L^4}^4 ds + C\int_0^t \|Q\|_{L^2}^2 \|Du\|_{L^2}^2 ds,\end{aligned} \tag{3.9}$$

where one has used the simple fact that $Q^3 \ge P^3/2 - CI_0^6$, due to (3.2).

We first derive some estimates on Q. Rewrite (3.5) as

$$Q = \Delta\varphi - \varphi_t. \tag{3.10}$$

Multiplying (3.10) by Q, then integrating the result over $\Omega \times (0,t)$, one gets by integration by parts and (3.7) that

$$\begin{aligned}&\int_0^t \|Q\|_{L^2}^2 ds \\ &\quad = \int_0^t \int_\Omega Q divu dx ds - (\gamma - 1)\int_0^t \int_\Omega \varphi Q divu dx ds \\ &\qquad - \int_\Omega Q\varphi(x,t)dx + \int_\Omega Q_0\varphi_0 dx \\ &\qquad - \int_0^t \int_\Omega div(Qu)\varphi dx ds - \gamma\overline{P}\int_0^t \int_\Omega divu\varphi dx ds.\end{aligned} \tag{3.11}$$

The terms in (3.11) can be estimated as follows:
It follows from (3.2) that

$$\left|\int_0^t \int_\Omega Q divu dx ds\right| \le \frac{1}{4}\int_0^t \|Q\|_{L^2}^2 ds + C. \tag{3.12}$$

Noticing that

$$\int_\Omega \varphi dx = 0,$$

we use (3.3) and Poincaré's inequality to get

$$\int_0^t \|\varphi\|_{L^4}^4 ds \le C\int_0^t \|u\|_{L^4}^4 ds \le C. \tag{3.13}$$

(3.13) and (3.2) yield that for any $\varepsilon > 0$,

$$\begin{aligned}&\left|\int_0^t \int_\Omega \varphi Q divu dx ds\right| \\ &\quad \le \left(\int_0^t \|Q\|_{L^4}^4 ds\right)^{1/4}\left(\int_0^t \|\varphi\|_{L^4}^4 ds\right)^{1/4}\left(\int_0^t \|divu\|_{L^2}^2 ds\right)^{1/2} \\ &\quad \le \varepsilon\left(\int_0^t \|Q\|_{L^4}^4 ds\right)^{1/2} + C_\varepsilon.\end{aligned} \tag{3.14}$$

Poincaré's inequality and (3.2) lead to

$$\begin{aligned}\left|\int_\Omega Q\varphi(x,t)dx\right| &\le \|Q(\cdot,t)\|_{L^2}\|\varphi(\cdot,t)\|_{L^2} \\ &\le C\|Q(\cdot,t)\|_{L^2}\|u(\cdot,t)\|_{L^2} \\ &\le C_\lambda + \lambda\|Q(\cdot,t)\|_{L^2}^2,\end{aligned} \tag{3.15}$$

for any $\lambda > 0$. Next, it follows from (3.2) and (3.3) that

$$\begin{aligned}\left|\int_0^t\int_\Omega div(Qu)\varphi dxds\right| &= \left|\int_0^t\int_\Omega Qu^i\partial_i\varphi dxds\right| \\ &\le \frac{1}{4}\int_0^t \|Q\|_{L^2}^2 ds + C\int_0^t \|u\|_{L^4}^4 ds \\ &\le \frac{1}{4}\int_0^t \|Q\|_{L^2}^2 ds + C, \end{aligned}\tag{3.16}$$

Finally, (3.2), (3.3) and Poincaré's inequality give

$$\begin{aligned}\left|\gamma\overline{P}\int_0^t\int_\Omega divu\varphi dxds\right| &\le CI_0^2\left(\int_0^t \|Du\|_{L^2}^2 ds\right)^{1/2}\left(\int_0^t \|u\|_{L^2}^2 ds\right)^{1/2} \\ &\le C. \end{aligned}\tag{3.17}$$

Thus, collecting all the estimates (3.12)-(3.17), we deduce from (3.11) that

$$\int_0^t \|Q\|_{L^2}^2 ds \le \varepsilon\left(\int_0^t \|Q\|_{L^4}^4 ds\right)^{1/2} + \lambda\|Q(\cdot,t)\|_{L^2}^2 + C_\lambda + C_\varepsilon. \tag{3.18}$$

To estimate the second term on the right hand side of (3.18), we multiply the equation (3.7) by Q, then integrate the result over Ω to obtain

$$\begin{aligned}\frac{d}{dt}\|Q(\cdot,t)\|_{L^2}^2 &= -(2\gamma-1)\int_\Omega Q^2 divudx - 2\gamma\overline{P}\int_\Omega Qdivudx \\ &\le C\left(\|Q\|_{L^4}^2 + \|Q\|_{L^2}\right)\|Du\|_{L^2}. \end{aligned}\tag{3.19}$$

Thus, we integrate this inequality over $(0,t)$ to derive from (3.18) that

$$\begin{aligned}\|Q(\cdot,t)\|_{L^2}^2 &\le \|Q_0\|_{L^2}^2 + C\left(\int_0^t \|Q\|_{L^4}^4 ds\right)^{1/2} + \int_0^t \|Q\|_{L^2}^2 ds + C \\ &\le \|Q_0\|_{L^2}^2 + C\left(\int_0^t \|Q\|_{L^4}^4 ds\right)^{1/2} + \lambda\|Q(\cdot,t)\|_{L^2}^2 + C_\lambda. \end{aligned}$$

Choosing $\lambda = 1/2$ in this estimate leads to

$$\sup_{0\le s\le t}\|Q(\cdot,s)\|_{L^2}^2 \le C\|Q_0\|_{L^2}^2 + C\left(\int_0^t \|Q\|_{L^4}^4 ds\right)^{1/2} + C. \tag{3.20}$$

The combination of (3.18) with (3.20) gives that

$$\begin{aligned}&\varepsilon\sup_{0\le s\le t}\|Q(\cdot,s)\|_{L^2}^2 + \int_0^t \|Q\|_{L^2}^2 ds \\ &\quad\le C\varepsilon\|Q_0\|_{L^2}^2 + C\varepsilon\left(\int_0^t \|Q\|_{L^4}^4 ds\right)^{1/2} + C_\varepsilon. \end{aligned}\tag{3.21}$$

Next, we turn to the estimate on S. Multiplying the equation (3.4) by $-\Delta w$, then integrating the resulting identity over both space and time, one gets

$$\|Dw(\cdot,t)\|_{L^2}^2 + \int_0^t \|\Delta w\|_{L^2}^2 ds \le \|Dw_0\|_{L^2}^2.$$

Hence,

$$\int_0^t \|w_t\|_{L^2}^2 ds = \int_0^t \|\Delta w\|_{L^2}^2 ds \le C. \tag{3.22}$$

Since

$$\nabla S = \nabla \varphi_t, \|S\|_{L^2} \le C\|DS\|_{L^2},$$

where the second inequality is due to Poincaré's inequality, we can multiply (3.8) by S, and then integrate the resulting identity over both space and time to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|S\|_{L^2}^2 + \|DS\|_{L^2}^2 \\
&\quad = -\int_\Omega Qu\cdot\nabla S dx + (\gamma-1)\int_\Omega Q divu S dx + \gamma\overline{P}\int_\Omega divu S dx \\
&\quad = -\int_\Omega Qu\cdot\nabla\varphi_t dx + (\gamma-1)\int_\Omega (divu)^2 S dx - (\gamma-1)\int_\Omega (divu) S^2 dx \\
&\qquad + \gamma\overline{P}\int_\Omega divu S dx \\
&\quad \le -\frac{1}{2}\int_\Omega Q\frac{\partial |u|^2}{\partial t} dx + C\int_\Omega \left(|Qu||w_t| + |Du|^2|S| + |Du|S^2\right) dx \\
&\qquad + \frac{1}{2}\int_\Omega |DS|^2 dx + C\int_\Omega |Du|^2 dx,
\end{aligned} \tag{3.23}$$

where (3.2) and (3.7) have been used.

The first term on the right hand side in (3.23) can be estimated again by (3.7) that

$$\begin{aligned}
&-\frac{1}{2}\int_\Omega Q\frac{\partial|u|^2}{\partial t} dx \\
&\quad = -\frac{1}{2}\frac{d}{dt}\int_\Omega Q|u|^2 dx + \frac{1}{2}\int_\Omega |u|^2 Q_t dx \\
&\quad \le -\frac{1}{2}\frac{d}{dt}\int_\Omega Q|u|^2 dx + C\int_\Omega \left(|u|^2|Du||Q| + |Du||u|^2\right) dx \\
&\qquad + C\int_\Omega |Q||Du| dx \int_\Omega |u|^2 dx \\
&\quad \le -\frac{1}{2}\frac{d}{dt}\int_\Omega Q|u|^2 dx + C\int_\Omega \left(|u|^2|Du|^2 + |u|^2|Du||S|\right) dx \\
&\qquad + C\|Du\|_{L^2}^2 + C\|u\|_{L^4}^4 + C\|Q\|_{L^2}^2\|Du\|_{L^2}^2 + C\|u\|_{L^2}^4.
\end{aligned} \tag{3.24}$$

We multiply the equation (1.4) by $|u|^2 u$, and integrate the result in space to get

$$\begin{aligned}
\frac{d}{dt}\|u\|_{L^4}^4 &\le C\int_\Omega \left(|u|^2|Du||Q| + |Du|^2|u|^2\right) dx \\
&\le C\int_\Omega \left(|u|^2|Du||S| + |u|^2|Du|^2\right) dx,
\end{aligned} \tag{3.25}$$

It follows from (3.23)-(3.25) that

$$\begin{aligned}
&\int_\Omega \left(P|u|^2 - \overline{P}|u|^2 + \|S\|_{L^2}^2 + \|u\|_{L^4}^4\right) dx + \int_0^t \|DS\|_{L^2}^2 ds \\
&\quad \le C + C\int_0^t \|Qu\|_{L^2}\|w_t\|_{L^2} ds + C\int_0^t \|Q\|_{L^2}^2\|Du\|_{L^2}^2 ds \\
&\qquad + C\int_0^t\int_\Omega \left(|Du|S^2 + |Du|^2|S| + |u|^2|Du||S| + |u|^2|Du|^2\right) dxds.
\end{aligned} \tag{3.26}$$

We can estimate each of the terms on the right hand side of (3.26) as follows: First, (3.22) and (3.3) yield that

$$\begin{aligned}\int_0^t \|Qu\|_{L^2}\|w_t\|_{L^2}ds &\leq C\left(\int_0^t \|Qu\|_{L^2}^2 ds\right)^{1/2}\\ &\leq \varepsilon\left(\int_0^t \|Q\|_{L^4}^4 ds\right)^{1/2}+C_\varepsilon. \end{aligned}\tag{3.27}$$

Next, Lemma 2.5 and $\int_\Omega S dx = 0$ give that

$$\|S\|_{L^4(\Omega)} \leq C\|S\|_{L^2(\Omega)}^{1/2}\|DS\|_{L^2(\Omega)}^{1/2};$$

thus, one has

$$\begin{aligned}&\int_0^t \left\|DuS^2(\cdot,s)\right\|_{L^1} ds\\ &\quad\leq \int_0^t \|Du\|_{L^2}\|S\|_{L^4}^2 ds\\ &\quad\leq C\int_0^t \|Du\|_{L^2}\|S\|_{L^2}\|DS\|_{L^2}ds\\ &\quad\leq \varepsilon\int_0^t \|DS\|_{L^2}^2 ds + C_\varepsilon\int_0^t \|Du\|_{L^2}^2\|S\|_{L^2}^2 ds. \end{aligned}\tag{3.28}$$

We infer from (3.2) that

$$\begin{aligned}&\int_0^t\int_\Omega \left(|Du|^2|S|+|u|^2|Du||S|\right)dxds\\ &\quad\leq \int_0^t \left(\|Du\|_{L^2}\|Du\|_{L^4}\|S\|_{L^4}+\|u\|_{L^4}^2\|Du\|_{L^4}\|S\|_{L^4}\right)ds\\ &\quad\leq C\int_0^t \left(\|Du\|_{L^2}\|Du\|_{L^4}\|S\|_{L^4}+\|u\|_{L^2}\|Du\|_{L^2}\|Du\|_{L^4}\|S\|_{L^4}\right)ds\\ &\quad\leq C\int_0^t \|Du\|_{L^2}\|Du\|_{L^4}\|S\|_{L^4}ds\\ &\quad\leq C\left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/4}\left(\int_0^t \|Du\|_{L^2}^{4/3}\|S\|_{L^2}^{2/3}\|DS\|_{L^2}^{2/3}\right)^{3/4}\\ &\quad\leq C\left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/4}\left(\int_0^t \|DS\|_{L^2}^2 ds\right)^{1/4}\left(\int_0^t \|Du\|_{L^2}^2\|S\|_{L^2}ds\right)^{1/2}\\ &\quad\leq \varepsilon\left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/2}+\varepsilon\int_0^t \|DS\|_{L^2}^2 ds + C_\varepsilon\int_0^t \|Du\|_{L^2}^2\|S\|_{L^2}^2 ds. \end{aligned}\tag{3.29}$$

Lemma 2.5 gives that

$$\begin{aligned}\left\||u|^2\right\|_{L^4} &\leq \left\||u|^2-\overline{|u|^2}\right\|_{L^4}+C\|u\|_{L^2}^2\\ &\leq C\left\||u|^2-\overline{|u|^2}\right\|_{L^2}^{1/2}\left\|D|u|^2\right\|_{L^2}^{1/2}+C\|u\|_{L^2}^2\\ &\leq C\left(\|u\|_{L^4}+\|u\|_{L^2}\right)\|uDu\|_{L^2}^{1/2}+C\|u\|_{L^2}^2. \end{aligned}$$

Hence, we use (3.3) to deduce

$$
\begin{aligned}
&\int_0^t\int_\Omega |u|^2|Du|^2dxds\\
&\quad\le \int_0^t \left\||u|^2\right\|_{L^4}\|Du\|_{L^4}\|Du\|_{L^2}ds\\
&\quad\le \frac12\int_0^t\|uDu\|_{L^2}^2ds + C\int_0^t\left(\|u\|_{L^4}^{4/3}+\|u\|_{L^2}^{4/3}\right)\|Du\|_{L^4}^{4/3}\|Du\|_{L^2}^{4/3}ds + C.
\end{aligned}
$$

Thus, making use of (3.2), we have

$$
\begin{aligned}
&\int_0^t\int_\Omega |u|^2|Du|^2dxds\\
&\quad\le C\int_0^t\left(\|u\|_{L^4}^{4/3}+\|u\|_{L^2}^{4/3}\right)\|Du\|_{L^4}^{4/3}\|Du\|_{L^2}^{4/3}ds + C\\
&\quad\le C\left(\int_0^t\|Du\|_{L^4}^4ds\right)^{1/3}\left(\int_0^t\left(\|u\|_{L^4}^2+\|u\|_{L^2}^2\right)\|Du\|_{L^2}^2ds\right)^{2/3}+C\\
&\quad\le \varepsilon\left(\int_0^t\|Du\|_{L^4}^4ds\right)^{1/2}+C_\varepsilon\left(\int_0^t\|u\|_{L^4}^2\|Du\|_{L^2}^2ds\right)^2+C_\varepsilon\\
&\quad\le \varepsilon\left(\int_0^t\|Du\|_{L^4}^4ds\right)^{1/2}+C_\varepsilon\int_0^t\|u\|_{L^4}^4\|Du\|_{L^2}^2ds+C_\varepsilon.
\end{aligned}\tag{3.30}
$$

Lemma 2.3 yields

$$
\left(\int_0^t\|Du\|_{L^4}^4ds\right)^{1/2}\le C+C\left(\int_0^t\|Q\|_{L^4}^4ds\right)^{1/2}.\tag{3.31}
$$

We use (3.2)(3.9)(3.21)(3.26) $-$ (3.31) to deduce that

$$
\begin{aligned}
&A(t)+\varepsilon\sup_{0\le s\le t}\|Q\|_{L^2}^2+B(t)\\
&\quad\le C\varepsilon\left(\int_0^t\|S\|_{L^4}^4ds\right)^{1/2}+C\varepsilon B(t)+C_\varepsilon\\
&\qquad+C_\varepsilon\int_0^t\|Du\|_{L^2}^2\left(\|S\|_{L^2}^2+\|Q\|_{L^2}^2+\|u\|_{L^4}^4\right)ds\\
&\quad\le C\varepsilon\left(\int_0^t A(t)\|DS(s)\|_{L^2}^2ds\right)^{1/2}+C\varepsilon B(t)+C_\varepsilon\\
&\qquad+C_\varepsilon\int_0^t\|Du\|_{L^2}^2\left(\|S\|_{L^2}^2+\|Q\|_{L^2}^2+\|u\|_{L^4}^4\right)ds\\
&\quad\le C\varepsilon\left(A(t)+B(t)\right)+C_\varepsilon\\
&\qquad+C_\varepsilon\int_0^t\|Du\|_{L^2}^2\left(\|S\|_{L^2}^2+\|Q\|_{L^2}^2+\|u\|_{L^4}^4\right)ds,
\end{aligned}
$$

where

$$
A(t)\triangleq\sup_{0\le s\le t}\int_\Omega\left(|S|^2+|u|^4+P|u|^2\right)(x,s)dx,
$$

and

$$
B(t)\triangleq\int_0^t\left(\|DS\|_{L^2}^2+\|Q\|_{L^2}^2\right)ds.
$$

Choosing ε small enough, we have

$$\begin{aligned}\sup_{0\le s\le t}\left(\|S\|_{L^2}^2+\|Q\|_{L^2}^2+\|u\|_{L^4}^4\right)(s)+\int_0^t\left(\|DS\|_{L^2}^2+\|Q\|_{L^2}^2\right)ds\\ \le C+C\int_0^t\|Du\|_{L^2}^2\left(\|S\|_{L^2}^2+\|Q\|_{L^2}^2+\|u\|_{L^4}^4\right)ds.\end{aligned}$$

Since

$$\int_0^t\|Du\|_{L^2}^2ds\le C,$$

Gronwall's inequality thus gives that

$$\sup_{0\le s\le t}\left(\|S\|_{L^2}^2+\|Q\|_{L^2}^2+\|u\|_{L^4}^4\right)(s)+\int_0^t\left(\|DS\|_{L^2}^2+\|Q\|_{L^2}^2\right)ds\le C. \tag{3.32}$$

The combination of this estimate with (3.9) yields that

$$\sup_{0\le s\le T}\|P(\cdot,s)\|_{L^3}^3+\int_0^T\left(\|Q\|_{L^4}^4+\|Q\|_{L^2}^2\right)ds\le C. \tag{3.33}$$

One deduces from (3.31) and (3.33) that

$$\int_0^t\|Du\|_{L^4}^4ds\le C. \tag{3.34}$$

By the identity

$$\int_\Omega\partial_i\varphi dx=0,$$

(3.32) and the Gagliardo-Nirenberg inequality, one gets

$$\begin{aligned}\|\partial_i\varphi(t)\|_{L^\infty} &\le C\|D\varphi(t)\|_{L^4}^{1/2}\|D^2\varphi(t)\|_{L^4}^{1/2}\\ &\le C\|u(t)\|_{L^4}^{1/2}\|Du(t)\|_{L^4}^{1/2}\\ &\le C\|Du(t)\|_{L^4}^{1/2}.\end{aligned}$$

We deduce from this estimate, (3.34) and Hölder's inequality that

$$\int_0^T\|\partial_j\varphi\|_{L^\infty}^8ds\le C.$$

Since $\overline{u}=0$, It follows from (3.32), (3.33) and the Poincaré-Sobolev inequality that

$$\int_0^T\|u\|_{L^\infty}^8ds\le C.$$

The above two inequalities give that

$$\int_0^T\|u^j\partial_j\varphi\|_{L^\infty}^4ds\le C. \tag{3.35}$$

Set $D_tw=w_t+u\cdot\nabla w$. Using (3.2), we conclude from (1.3) and (3.5) that

$$D_t\left(\log P+\gamma\varphi\right)\le-\gamma P+CI_0^2+\gamma u^j\partial_j\varphi. \tag{3.36}$$

Now, we pass in (3.36) to the Lagrangian coordinates and take $y = \log P$, $g(y) = -\gamma e^y$, and $b(t) = b_1(t) - b_0(t)$ where

$$b_1(t) = \gamma \int_0^t u^j \partial_j \varphi ds + C I_0^2 t \text{ and } b_0(t) = \gamma\varphi.$$

Thus, (3.35) yields that for $0 \le t_1 < t_2 \le T$,

$$\begin{aligned}
|b_1(t_2) - b_1(t_1)| &\le \gamma \int_{t_1}^{t_2} \left\| u^j \partial_j \varphi(\cdot, s) \right\|_{L^\infty} ds + C I_0^2 (t_2 - t_1) \\
&\le C \int_0^T \left\| u^j \partial_j \varphi(\cdot, s) \right\|_{L^\infty}^4 ds + C(t_2 - t_1) \\
&\le C + C(t_2 - t_1).
\end{aligned}$$

(3.32) and Poincaré's inequality give that

$$\begin{aligned}
\sup_{0\le t\le T} |b_0(t)| &\le C \sup_{0\le t\le T} \|\varphi\|_{L^4}^{1/2} \sup_{0\le t\le T} \|D\varphi\|_{L^4}^{1/2} \\
&\le C \sup_{0\le t\le T} \|u\|_{L^4} \\
&\le C.
\end{aligned}$$

Hence, we have

$$|b(t_2) - b(t_1)| \le C + C(t_2 - t_1). \tag{3.37}$$

Since estimate (3.37) holds, the uniform upper bounds for $\log P$ and consequently for ρ follow from Lemma 2.4.

Next, we will prove (1.13).

We claim that we have

$$\lim_{t\to\infty} (\|Q(\cdot, t)\|_{L^2} + \|u(\cdot, t)\|_{L^4}) = 0. \tag{3.38}$$

In fact, set

$$h(t) = \|Q(\cdot, t)\|_{L^2}^2 + \|u(\cdot, t)\|_{L^4}^4.$$

It follows easily from (3.3) and (3.33) that

$$\int_0^\infty h(t) dt \le C.$$

Using (3.3), (3.33) and (3.34), we derive from (3.19) and (3.25) that

$$\begin{aligned}
&\int_0^\infty \left| h'(t) \right| dt \\
&\le C \int_0^\infty \left(\|Du\|_{L^2} \left(\|Q\|_{L^4}^2 + \|Q\|_{L^2} \right) + \left(\|Du\|_{L^4} \|Q\|_{L^4} + \|Du\|_{L^4}^2 \right) \|u\|_{L^4}^2 \right) dt \\
&\le C \int_0^\infty \left(\|Q\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|u\|_{L^4}^4 + \|Du\|_{L^2}^2 + \|Du\|_{L^4}^4 \right) dt \\
&\le C.
\end{aligned}$$

Consequently,

$$\lim_{t\to\infty} h(t) = 0. \tag{3.39}$$

This shows that (3.38) holds true.

It follows from (3.6) that

$$\begin{aligned}
\int_0^\infty \left|\frac{d}{dt}\overline{P}(t)\right| dt &\le (\gamma-1)\int_0^\infty \left|\overline{Qdivu}\right| dt\\
&\le C\int_0^\infty \left(\|Q\|_{L^2}^2+\|Du\|_{L^2}^2\right) dt\\
&\le C.
\end{aligned}$$

This yields that there exists some positive constant ρ_s such that

$$\lim_{t\to\infty}\overline{P}(t)=\rho_s^\gamma, \tag{3.40}$$

since $0<\overline{\rho_0}^\gamma\le\overline{P}\le CI_0^2$. (3.38), (3.40) and (1.12) lead to

$$\lim_{t\to\infty}\|\rho(\cdot,t)-\rho_s\|_{L^\alpha}=0,$$

for any $\alpha\in[1,\infty)$. Hence, we have

$$\rho_s=\overline{\rho_0},$$

due to the fact that $\overline{\rho}(t)\equiv\overline{\rho_0}$. Consequently,

$$\lim_{t\to\infty}\|\rho(\cdot,t)-\overline{\rho_0}\|_{L^\alpha}=0, \tag{3.41}$$

for any $\alpha\in[1,\infty)$.

(1.12) and (3.33) yield that Q satisfies

$$\int_0^\infty \|Q(\cdot,t)\|_{L^p}^p dt\le C \text{ for any } 2\le p<\infty.$$

Hence,

$$\sup_{0\le t<\infty}\|u\|_{L^p}\le C, \tag{3.42}$$

for all $2\le p<\infty$.

It thus follows easily from (3.2), (3.41), (3.42) and (3.38) that (1.13) holds true.

Case 2. (the boundary condition (1.8) holds.) Notice that $u\cdot n|_{\partial\Omega}=0$ yields that Poincaré's inequality still holds, i.e.

$$\|u\|_{L^2}\le C\|Du\|_{L^2},$$

and that for $1<p<\infty$,

$$\|Du\|_{L^p}\le C\left(\|divu\|_{L^p}+\|curlu\|_{L^p}\right). \tag{3.43}$$

Denoting by $\nabla^\perp=(\partial_{x_2},-\partial_{x_1})^T$, we have

$$\Delta u=\nabla divu-\nabla^\perp curlu.$$

Hence, (1.3) – (1.5), (1.8) and Lemma 2.5 yield that (3.2) and (3.3) still hold.

Denote by φ and w the unique functions such that $u = \nabla\varphi + w, divw = 0$, and

$$\begin{cases} \Delta\varphi = divu, \\ \frac{\partial\varphi}{\partial n}\Big|_{\partial\Omega} = 0, \int_\Omega \varphi dx = 0; \end{cases}$$

similarly $u_0 = \nabla\varphi_0 + w_0, divw_0 = 0$, and

$$\begin{cases} \Delta\varphi_0 = divu_0, \\ \frac{\partial\varphi_0}{\partial n}\Big|_{\partial\Omega} = 0, \int_\Omega \varphi_0 dx = 0. \end{cases}$$

Using (1.8), we infer from (1.4) that $curlw$ satisfies

$$\begin{cases} (curlw)_t - \Delta curlw = 0, \\ curlw|_{\partial\Omega} = 0, \\ curlw(x,0) = curlu_0. \end{cases}$$

Hence, we have

$$\|curlw(t)\|_{L^2}^2 + 2\int_0^t \|Dcurlw\|_{L^2}^2 ds \le \|curlu_0\|_{L^2}^2.$$

Since $curlw|_{\partial\Omega} = 0$, this estimate and Lemma 2.5 lead to

$$\int_0^t \|curlu\|_{L^4}^4 ds = \int_0^t \|curlw\|_{L^4}^4 ds \le C. \tag{3.44}$$

Choosing the smooth basis $\nabla\{\psi_i\}$ of $\nabla H^1(\Omega)$, where $\{\psi_i\}$ is the solutions of

$$\begin{cases} -\Delta\psi_i = \lambda_i\psi_i, \\ \frac{\partial\psi_i}{\partial n}\Big|_{\partial\Omega} = 0. \end{cases}$$

We have, for any i,

$$\begin{aligned} 0 &= \int_\Omega (u_t - \Delta u + \nabla P)\cdot\nabla\psi_i dx \\ &= \int_\Omega (-u_t - \nabla divu + \nabla^\perp curlu + \nabla P)\cdot\nabla\psi_i dx \\ &= \int_\Omega (-\varphi_t + \Delta\varphi - P)\Delta\psi_i dx \\ &= \lambda_i \int_\Omega (\varphi_t - \Delta\varphi + P)\psi_i dx. \end{aligned}$$

This yields that φ satisfies

$$\begin{cases} \varphi_t - \Delta\varphi + Q = 0, \\ \frac{\partial\varphi}{\partial n}\Big|_{\partial\Omega} = 0, \int_\Omega \varphi dx = 0, \\ \varphi(x,0) = \varphi_0(x), \end{cases} \tag{3.45}$$

where $Q \triangleq P - \overline{P}$.

Obviously, Q satisfies (3.7).

Denote by $S = \varphi_t$. It follows from (3.45) that S satisfies

$$\begin{cases} S_t - \Delta S = -Q_t, \\ \frac{\partial S}{\partial n}\big|_{\partial\Omega} = 0, \int_\Omega S dx = 0, \\ S(x,0) = \Delta\varphi_0(x) - P_0(x). \end{cases}$$

We use Lemma 2.1 to get

$$\int_0^t \|\Delta\varphi\|_{L^4}^4 ds \le C\left(\|\Delta\varphi_0\|_{L^2}^2 + \int_0^t \|Q\|_{L^4}^4 ds\right).$$

Hence, this estimate, together with (3.43) and (3.44), yields that

$$\int_0^t \|Du\|_{L^4}^4 ds \le C\left(1 + \int_0^t \|Q\|_{L^4}^4 ds\right);$$

that is to say, (3.31) still holds.

We then follow the proof in Case 1 to obtain (1.12) and (1.13) in this case.

4 Proof of Theorem 1.1 for the Cauchy problem

In this section, we treat the Cauchy problem (1.3)-(1.5) and (1.7). Since the main idea is similar to that given in Section 3, we make some slightly modification due to the non-compactness of $\Omega = R^2$ and we just sketch the proof of Theorem 1.1 for this case.

First, standard energy estimates applying to the problem (1.3)-(1.5) and (1.7) show that (3.2) still holds in this case. Using (3.2), we deduce from Lemma 2.5 that

$$\int_0^t \|u\|_{L^4}^4 ds \le C. \tag{4.1}$$

Obviously, P satisfies (3.6). Denote by

$$S = divu - P. \tag{4.2}$$

It is easy to see that S satisfies

$$\begin{cases} S_t - \Delta S = -P_t = div(Pu) + (\gamma - 1)P divu, \\ S(x,0) = S_0(x) = divu_0 - P_0. \end{cases} \tag{4.3}$$

Lemma 2.5 leads to

$$\|S\|_{L^4} \le C\|S\|_{L^2}^{1/2}\|DS\|_{L^2}^{1/2}. \tag{4.4}$$

We multiply (3.6) by P^2 and integrate the resulting identity in both space and time to derive that

$$\|P(t)\|_{L^3}^3 + \int_0^t \|P\|_{L^4}^4 ds \le C\|P_0\|_{L^3}^3 + C\int_0^t \|S\|_{L^4}^4 ds. \tag{4.5}$$

Denote by v and w the unique functions such that $u = v + w, divw = \mathrm{curl}v = 0$, and similarly, $u_0 = v_0 + w_0, divw_0 = \mathrm{curl}v_0 = 0$. One deduces from direct calculations based on (1.3) (1.4) that

$$\begin{cases} v_t = \nabla S, \\ v_t - \Delta v + \nabla P = 0, \\ v(x,0) = v_0(x), \end{cases} \tag{4.6}$$

and

$$\begin{cases} w_t - \Delta w = 0, \\ w(x,0) = w_0(x). \end{cases} \tag{4.7}$$

Hence, similar to (3.22), we have

$$\int_0^t \|w_t\|_{L^2}^2 ds = \int_0^t \|\Delta w\|_{L^2}^2 ds \le C. \tag{4.8}$$

It follows from (4.6) and (4.3) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|S\|_{L^2}^2 + \|DS\|_{L^2}^2 \\ &\quad = -\int_{R^2} Pu \cdot \nabla S dx + (\gamma - 1)\int_{R^2} P divu S dx \\ &\quad \le -\frac{1}{2}\int_{R^2} P\frac{\partial |u|^2}{\partial t} dx + C\int_{R^2} \left(|Pu||w_t| + |Du|^2|S| + |Du|S^2\right) dx. \end{aligned} \tag{4.9}$$

Note that

$$\begin{aligned} &-\frac{1}{2}\int_{R^2} P\frac{\partial |u|^2}{\partial t} dx \\ &\quad = -\frac{1}{2}\frac{d}{dt}\int_{R^2} P|u|^2 dx + \frac{1}{2}\int_{R^2} |u|^2 P_t dx \\ &\quad \le -\frac{1}{2}\frac{d}{dt}\int_{R^2} P|u|^2 dx + C\int_{R^2} |u|^2|Du|P dx \\ &\quad \le -\frac{1}{2}\frac{d}{dt}\int_{R^2} P|u|^2 dx + C\int_{R^2} \left(|u|^2|Du|^2 + |u|^2|Du||S|\right) dx, \end{aligned}$$

and

$$\frac{d}{dt}\|u\|_{L^4}^4 \le C\int_{R^2} \left(|u|^2|Du||S| + |u|^2|Du|^2\right) dx.$$

Using these estimates, we infer from (4.9) that

$$\begin{aligned} &\int_{R^2} \left(P|u|^2 + S^2 + |u|^4\right)(x,t) dx + \int_0^t \|DS\|_{L^2}^2 ds \\ &\quad \le C\int_0^t \int_{R^2} \left(|Du|S^2 + |Du|^2|S| + |u|^2|Du||S| + |u|^2|Du|^2\right) dx ds \\ &\quad\quad + C\int_0^t \|Pu\|_{L^2}\|w_t\|_{L^2} ds + C. \end{aligned} \tag{4.10}$$

We estimate the terms in the right hand side of (4.10) as follows:
First, (4.4) gives that

$$\begin{aligned}
&\int_0^t \left\|DuS^2(\cdot,s)\right\|_{L^1} ds \\
&\quad \le \int_0^t \|Du\|_{L^2}\|S\|_{L^4}^2 ds \\
&\quad \le \varepsilon\int_0^t \|DS\|_{L^2}^2 ds + C_\varepsilon \int_0^t \|Du\|_{L^2}^2\|S\|_{L^2}^2 ds. \qquad (4.11)
\end{aligned}$$

Secondly, similar to (3.29), we deduce from (3.2) and (4.4) that

$$\begin{aligned}
&\int_0^t \int_{R^2} \left(|Du|^2|S| + |u|^2|Du||S|\right) dxds \\
&\quad \le \varepsilon\left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/2} + \varepsilon\int_0^t \|DS\|_{L^2}^2 ds + C_\varepsilon\int_0^t \|Du\|_{L^2}^2\|S\|_{L^2}^2 ds. \qquad (4.12)
\end{aligned}$$

Thirdly, similar to (3.30), we derive from (3.2) and Hölder's inequality that

$$\begin{aligned}
&\int_0^t \int_{R^2} |u|^2|Du|^2 dxds \\
&\quad \le C\int_0^t \|u\|_{L^4}^{4/3}\|Du\|_{L^4}^{4/3}\|Du\|_{L^2}^{4/3} ds \\
&\quad \le \varepsilon\left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/2} + C_\varepsilon\int_0^t \|u\|_{L^4}^4\|Du\|_{L^2}^2 ds. \qquad (4.13)
\end{aligned}$$

Finally, (4.8) and (4.1) yield that

$$\int_0^t \|Pu\|_{L^2}\|w_t\|_{L^2} ds \le \varepsilon\left(\int_0^t \|P\|_{L^4}^4 ds\right)^{1/2} + C_\varepsilon. \qquad (4.14)$$

Lemma 2.3 gives

$$\left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/2} \le C + C\left(\int_0^t \|P\|_{L^4}^4 ds\right)^{1/2}. \qquad (4.15)$$

Using $(3.2)(4.5)(4.11)-(4.14)$, we infer from (4.10) and (4.15) that

$$\begin{aligned}
&A(t) + B(t) \\
&\quad \le C_\varepsilon + C\varepsilon\left(\int_0^t \|S\|_{L^4}^4 ds\right)^{1/2} + C\varepsilon B(t) \\
&\qquad + C_\varepsilon\int_0^t \|Du\|_{L^2}^2\left(\|S\|_{L^2}^2 + \|u\|_{L^4}^4\right) ds \\
&\quad \le C_\varepsilon + C\varepsilon\left(A(t) + B(t)\right) \\
&\qquad + C_\varepsilon\int_0^t \|Du\|_{L^2}^2\left(\|S\|_{L^2}^2 + \|u\|_{L^4}^4\right) ds.
\end{aligned}$$

where

$$A(t) \triangleq \sup_{0\le s\le t}\int_{R^2}\left(|S|^2 + |u|^4 + P|u|^2\right)(x,s)dx,$$

and

$$B(t) \triangleq \int_0^t \|DS\|_{L^2}^2 ds.$$

Choosing ε small enough yields that

$$\begin{aligned}
&\sup_{0\le s\le t} \left(\|S\|_{L^2}^2 + \|u\|_{L^4}^4\right)(s) + \int_0^t \|DS\|_{L^2}^2 ds \\
&\qquad \le C + C\int_0^t \|Du\|_{L^2}^2 \left(\|S\|_{L^2}^2 + \|u\|_{L^4}^4\right) ds.
\end{aligned}$$

Gronwall's inequality thus gives that

$$\sup_{0\le s\le t} \left(\|S\|_{L^2}^2 + \|u\|_{L^4}^4\right)(s) + \int_0^t \|DS\|_{L^2}^2 ds \le C, \tag{4.16}$$

due to (3.2). We use (4.4), (4.5), (4.16) and (4.15) to conclude

$$\sup_{0\le s\le T} \|P(\cdot,s)\|_{L^3}^3 + \int_0^T \left(\|P\|_{L^4}^4 + \|Du\|_{L^4}^4\right) ds \le C. \tag{4.17}$$

The Gagliardo-Nirenberg inequality, together with (3.2) and (4.17), gives that

$$\begin{aligned}
\int_0^T \|u\|_{L^\infty}^3 ds &\le C\int_0^T \|u\|_{L^2}^{1/2} \|Du\|_{L^{5/2}}^{5/2} ds \\
&\le C\int_0^T \|Du\|_{L^{5/2}}^{5/2} ds \\
&\le C.
\end{aligned}$$

We derive from this estimate and (4.17) that

$$\begin{aligned}
\int_0^T \|Pu\|_{L^3}^3 ds &\le \sup_{0\le t\le T} \|P\|_{L^3}^3 \int_0^T \|u\|_{L^\infty}^3 ds \\
&\le C.
\end{aligned} \tag{4.18}$$

Noticing that

$$\int_0^T \|Pdivu\|_{L^p}^p ds \le C,$$

for any $3/2 \le p \le 2$, using (4.18), we deduce from Lemma 2.2 and (4.3) that

$$\int_0^T \|DS\|_{L^3}^3 ds \le C.$$

Hence, this estimate, together with the Gagliardo-Nirenberg inequality and (4.16), leads to

$$\begin{aligned}
\int_0^T \|S\|_{L^\infty}^4 ds &\le C\int_0^T \|S\|_{L^2} \|DS\|_{L^3}^3 ds \\
&\le C.
\end{aligned} \tag{4.19}$$

Set $D_t w = w_t + u\cdot\nabla w$. We conclude from (1.3) and (4.2) that

$$D_t \log P = -\gamma P - \gamma S. \tag{4.20}$$

Now, we pass in (4.20) to the Lagrangian coordinates and take $y = \log P$, $g(y) = -\gamma e^y$, and

$$b(t) = \int_0^t S(x(s), s)ds.$$

Thus, it follows from (4.19) and Hölder's inequality that

$$\begin{aligned} |b(t_2) - b(t_1)| &\leq \int_0^T \|S(\cdot, s)\|_{L^\infty}^4 ds + C(t_2 - t_1) \\ &\leq C + C(t_2 - t_1), \end{aligned}$$

for $0 \leq t_1 < t_2 \leq T$. This estimate and Lemma 2.4 yield that (1.12) holds.

Similarly to Section 3, we can prove that (1.13) holds true for this case. Thus, the proof of Theorem 1.1 is completed.

5 Proof of Theorem 1.2

With the basic estimates (1.12) and (1.13) in Theorem 1.1, we can establish the Theorem 1.2 easily in this section.

Proof of Theorem 1.2. Otherwise, there exist some $C_0 > 0$ and a subsequence $\{t_{n_j}\}_{j=1}^\infty$, $t_{n_j} \to \infty$ such that $\|\nabla\rho(\cdot, t_{n_j})\|_{L^q(\Omega)} \leq C_0$. Hence, for

$$a = \frac{q}{2(q-1)} \in (0, 1),$$

the Poincaré-Sobolev inequality yields that

$$\begin{aligned} &\left\|\rho(x, t_{n_j}) - \overline{\rho_0}\right\|_{C(\overline{\Omega})} \\ &\leq C \left\|\nabla\rho(x, t_{n_j})\right\|_{L^q(\Omega)}^a \left\|\rho(x, t_{n_j}) - \overline{\rho_0}\right\|_{L^2(\Omega)}^{1-a} \\ &\leq C C_0^a \left\|\rho(x, t_{n_j}) - \overline{\rho_0}\right\|_{L^2(\Omega)}^{1-a}, \end{aligned} \tag{5.1}$$

with C independent of t_{n_j}. We deduce from (1.13) that the right hand side of (5.1) goes to 0 as $t_{n_j} \to \infty$. Hence,

$$\left\|\rho(x, t_{n_j}) - \overline{\rho_0}\right\|_{C(\overline{\Omega})} \to 0 \text{ as } t_{n_j} \to \infty. \tag{5.2}$$

On the other hand, for $T > 0$, we introduce the Lagrangian coordinates which are defined as initial data to the Cauchy problem:

$$\begin{cases} \frac{\partial}{\partial s} X(s; t, x) = u(X(s; t, x), s) & 0 \leq s \leq T, \\ X(t; t, x) = x & 0 \leq t \leq T, x \in \overline{\Omega}. \end{cases} \tag{5.3}$$

(1.11) yields that the transformation (5.3) is well-defined. Consequently, on the one hand, we have

$$\rho(x, t) = \rho_0(X(0; t, x)) \exp\left\{-\int_0^t divu(X(s; t, x), s)ds\right\}; \tag{5.4}$$

on the other hand, since, by assumption, there exists some point $x_0 \in \overline{\Omega}$ such that $\rho_0(x_0) = 0$, we get that there exists a $x_0(t) \in \overline{\Omega}$ such that $X(0; t, x_0(t)) = x_0$. Using (5.4), we deduce from (1.11) that

$$\rho(x_0(t), t) \equiv 0 \text{ for all } t \geq 0.$$

So, we conclude from this equality and Hölder's inequality that

$$\begin{aligned} \left\|\rho(x, t_{n_j}) - \overline{\rho_0}\right\|_{C(\overline{\Omega})} &\geq \left|\rho(x_0(t_{n_j}), t_{n_j}) - \overline{\rho_0}\right| \\ &= \overline{\rho_0} > 0, \end{aligned}$$

which contradicts (5.2).

Acknowledgements. This paper is part of the PhD thesis of the first author written under the supervision of the second author at The Institute of Mathematical Sciences of The Chinese University of Hong Kong.

References

[1] F. J. Chatelon; P. Orenga. Some smoothness and uniqueness results for a shallow-water problem. *Adv. Differential Equations,* 3 (1998), no. 1, 155-176.

[2] Y. Cho; H. J. Choe; H. Kim. Unique solvability of the initial boundary value problems for compressible viscous fluids. *J. Math. Pures Appl. (9),* 83 (2004), no. 2, 243-275.

[3] R. Danchin. Global existence in critical spaces for compressible Navier-Stokes equations. *Invent. Math.,* 141 (2000), 579-614.

[4] E. Feireisl; A. Novotny; H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.,* 3 (2001), no. 4, 358–392.

[5] Y. Giga; H. Sohr. Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains. *J. Funct. Anal.,* 102 (1991), no. 1, 72-94.

[6] M. Hieber; J. Prüss. Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations. *Comm. Partial Differential Equations,* 22 (1997), no. 9-10, 1647-1669.

[7] D. Hoff. Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. *Trans. Amer. Math. Soc.,* 303 (1987), no. 1, 169-181.

[8] D. Hoff. Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. *Arch. Rational Mech. Anal.,* 132 (1995), 1-14.

[9] D. Hoff; D. Serre. The failure of continuous dependence on initial data for the Navier-Stokes equations of compressible flow. *SIAM J. Appl. Math.,* 51 (1991), no. 4, 887–898.

[10] D. Hoff; J. Smoller. Non-formation of vacuum states for compressible Navier-Stokes equations. *Comm. Math. Phys.,* 216 (2001), no. 2, 255–276.

[11] A. V. Kazhikhov; V. V. Shelukhin. Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. *J. Appl. Math. Mech.,* 41 (1977), no. 2, 273-282.

[12] A. V. Kazhikhov; V. A. Weigant. Global solutions of equations of potential flows of a compressible viscous fluid for small Reynolds numbers. *Differential Equations,* 30 (1994), no. 6, 935–947.

[13] O. A. Ladyzenskaja; V. A. Solonnikov; N. N. Uraltseva. Linear and quasilinear equations of parabolic type. Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. 1968.

[14] P. L. Lions. Existence globale de solutions pour les equations de Navier-Stokes compressibles isentropiques. *C. R. Acad. Sci. Paris, Sér I Math.,* 316 (1993), 1335-1340.

[15] P. L. Lions. Limites incompressible et acoustique pour des fluides visqueux, compressibles et isentropiques. *C. R. Acad. Sci. Paris Sér. I Math.,* 317 (1993), 1197-1202.

[16] P. L. Lions. Mathematical topics in fluid mechanics. Vol. 2. Compressible models. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.

[17] A. Matsumura; T. Nishida. The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids. *Proc. Japan Acad. Ser. A Math. Sci.,* 55 (1979), no. 9, 337–342.

[18] A. Matsumura; T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.,* 20 (1980), no. 1, 67–104.

[19] A. Matsumura; T. Nishida. The initial boundary value problems for the equations of motion of compressible and heat-conductive fluids. *Comm. Math. Phys.,* 89 (1983), 445-464.

[20] L. Min; A. V. Kazhikhov; S. Ukai. Global solutions to the Cauchy problem of the Stokes approximation equations for two-dimensional compressible flows. *Comm. Partial Differential Equations,* 23 (1998), no. 5-6, 985-1006.

[21] R. Salvi; I. Straškraba. Global existence for viscous compressible fluids and their behavior as $t \to \infty$. *J. Fac. Sci. Univ. Tokyo Sect. IA, Math.,* 40 (1993), 17-51.

[22] D. Serre. Solutions faibles globales des quations de Navier-Stokes pour un fluide compressible. *C. R. Acad. Sci. Paris Sér. I Math.,* 303 (1986), no. 13, 639–642.

[23] D. Serre. On the one-dimensional equation of a viscous, compressible, heat-conducting fluid. *C. R. Acad. Sci. Paris Sér. I Math.,* 303 (1986), no. 14, 703–706.

[24] V. A. Solonnikov. On solvability of an initial boundary value problem for the equations of motion of viscous compressible fluid. *Zap. Nauchn. Sem. LOMI,* 56 (1976), 128–142.

[25] A. Valli; W. M. Zajaczkowski. Navier-Stokes equations for compressible fluids: global existence and qualitative properties of the solutions in the general case. *Comm. Math. Phys.,* 103 (1986), no. 2, 259-296.

[26] Z. Xin. Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density. *Comm. Pure Appl. Math.,* 51 (1998), no. 3, 229–240.

[27] A. A. Zlotnik. Uniform estimates and stabilization of symmetric solutions of a system of quasilinear equations. *Diff. Equations,* 36 (2000), 701-716.