Some Uniform Estimates and Blowup Behavior of Global Strong Solutions to the Stokes Approximation Equations for Two-Dimensional Compressible Flows

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Abstract

This paper concerns the global existence and the large time behavior of strong and classical solutions to the two-dimensional Stokes approximation equations for the compressible flows. We consider the unique global strong solution or classical solution to the two-dimensional Stokes approximation equations for the compressible flows together with the space-periodicity boundary condition or the no-stick boundary condition or Cauchy problem for arbitrarily large initial data. First, we prove that the density is bounded from above independent of time in all these cases. Secondly, we show that for the space-periodicity boundary condition or the no-stick boundary condition, if the initial density contains vacuum at least at one point, then the global strong (or classical) solution must blow up as time goes to infinity.

Keywords: Stokes approximation equations; Isentropic compressible fluids; Uniform upper bound; Vacuum; Blowup

1 Introduction

The compressible isentropic Navier-Stokes equations, which are the basic model describing the evolution of a viscous compressible gas, read as follows

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) = \mu \Delta u + \nabla(\xi \text{div} u) - \nabla P(\rho),
\end{cases}
\]

(1.1)

where \( x \in \Omega \subset R^N, t \in (0, T) \) and \( P(\rho) = a\rho^\gamma, a > 0, \gamma > 1 \), the viscosity coefficients \( \mu, \xi \) are assumed to satisfy \( \mu > 0 \) and \( \xi + \mu \geq 0 \).

There is huge literature on the studies on the large time existence and behavior of solutions to (1.1). The one-dimensional problem was addressed by Kazhikhov and Shelukhin [11] for sufficiently smooth data, and by Serre [22] [23] and Hoff [7] for discontinuous initial data, where the data are uniformly away from the vacuum. The multidimensional problem (1.1) was investigated by Matsumura and Nishida [17] [18] [19], who proved global existence of smooth solutions for data close to a non-vacuum equilibrium, and later by Hoff for discontinuous initial data [8], and more recently, by

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Danchin [3], who obtained existence and uniqueness of global solutions in a functional space invariant by the natural scaling of the associated equations. For the existence of solutions for arbitrary data (which may include vacuum states), the major breakthrough is due to P. L. Lions [14] [15] [16] (see also Feireisl et al [4]), where he obtains global existence of weak solutions - defined as solutions with finite energy - when the exponent $\gamma$ is suitably large. The only restriction on initial data is that the initial energy is finite, so that the density is allowed to vanish.

Despite the important progress, the regularity and behavior of these weak solutions is completely open. As emphasized in many papers related to compressible fluid dynamics [2], [7], [9]-[11], [21], [22], [24]-[26], the possible appearance of vacuum and uniform upper bound estimate on the density is one of the major difficulties when trying to prove global existence and strong regularity results. In particular, the results of Xin [26] show that there is no global smooth solution $(\rho, u)$ to Cauchy problem for (1.1) with a nontrivial compactly supported initial density, which gives results for finite time blow-up in the presence of vacuum.

The major difficulties in analysis of the compressible Navier-Stokes equations (1.1) are the nonlinearities in both convection and pressure and their interactions. To study the well-posedness of solutions and gain understanding of the key issues, one has been looking into various simplified models of the Navier-Stokes systems. One of the pro-type simplifications of the Navier-Stokes system (1.1) is the Stokes approximation

$$
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
\overline{\rho} u_t - \mu \Delta u - \xi \nabla (\text{div} u) + \nabla P &= 0,
\end{aligned}
$$

where $\overline{\rho} = \text{const.} > 0$ is the mean density, and $P = a \rho^\gamma$, $a > 0$, $\gamma > 1$. This is a good approximation for strongly viscous fluids and where the convection is unimportant.

For simplicity, we take $\overline{\rho} = 1$, $\mu = 1$, $\xi = 0$, $a = 1$, and study the system

$$
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
u_t - \Delta u + \nabla P &= 0,
\end{aligned}
$$

with $P = \rho^\gamma$, $\gamma > 1$. We are concerned with the initial conditions for the density and the velocity:

$$
\rho(0) = \rho_0, u(0) = u_0,
$$

and three types of boundary conditions:

1) space-periodicity condition, i.e.,

$$
\Omega \text{ is a product } \prod_{i=1}^{N}(0, L_i), \text{ and } \rho, u \text{ are } \Omega\text{-periodic;}
$$

2) Cauchy problem:

$$
\Omega = \mathbb{R}^N \text{ and (in some weak sense) } \rho, u \text{ vanish at infinity;}
$$

3) no-stick boundary condition: in this case, $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, and

$$
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0 \text{ on } \partial \Omega \\
u_t - \Delta u + \nabla P &= 0 \text{ on } \partial \Omega
\end{aligned}
$$

if $N = 2$,

$$
\begin{aligned}
\text{curl } u = 0 \text{ on } \partial \Omega \\
\text{curl } u \times n = 0 \text{ on } \partial \Omega
\end{aligned}
$$

if $N = 3$,  (1.8)
where \( n \) is the unit outward normal to \( \partial \Omega \). The first condition in (1.8) is the non-penetration boundary condition, while the second one is also known in the form

\[
(D(u) \cdot n)_\tau = 0, \tag{1.9}
\]

where \( D(u) \) is the stress tensor with components

\[
D_{ij}(u) = \frac{1}{2} \left( \partial_{x_i} u^j + \partial_{x_j} u^i \right).
\]

Condition (1.9) means the tangential component of \( D(u) \cdot n \) vanishes on the boundary \( \partial \Omega \).

It should be noted that the initial-boundary-value problem (1.3)-(1.5) with the boundary data given either by (1.6) or (1.7) or (1.8) has been thoroughly studied by many people. In particular, the existence of classical solutions to the 2D initial-boundary-value problem on any finite interval \( [0, T] (T > 0) \) for arbitrarily large smooth initial data has been proved by Kazhikhov et al [12], Lions [16], Min et al [20], and Chatelon et al [1]. However, it seems to us that the known upper bounds on the density \( \rho \) depend on the time \( T \), see [1], [12], [16], [20], so it is impossible to study the large time asymptotic behavior of solutions in the setting in [1], [12], [16], [20]. One of the main purposes of this paper is to derive an uniform time-independent upper bound for the density. As a consequence of the uniform estimate on the bound of density, we show the large time asymptotic behavior of solutions for the strong solutions. Our first result is

**Theorem 1.1** Suppose that \( N = 2 \) and that

\[
\rho_0 \in W^{l,q}(\Omega) \cap L^1(\Omega), \quad u_0 \in W^{l+1,q}(\Omega) \cap L^2(\Omega) \tag{1.10}
\]

for some \( q > 2, l \geq 1 \). Then problem (1.3)-(1.5) with the boundary condition (1.6) or (1.7) or (1.8) has a unique solution \((\rho, u)\) such that for any \( T > 0 \),

\[
\frac{\partial^k \rho}{\partial t^k} \in L^\infty(0, T; W^{l-k,q}(\Omega)), \quad \frac{\partial^k u}{\partial t^k} \in L^\infty(0, T; W^{l-k+1,q}(\Omega)), \tag{1.11}
\]

for any \( k, 0 \leq k \leq l \), and moreover,

\[
\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^\infty(\Omega)} \leq C, \tag{1.12}
\]

and

\[
\lim_{t \to \infty} \left( \|R(\cdot, t)\|_{L^\alpha(\Omega)} + \|u(\cdot, t)\|_{L^\beta(\Omega)} \right) = 0, \tag{1.13}
\]

with \( C \) independent of \( T \) and \( R, \alpha, \beta \) such that

\[
\begin{cases}
R = \rho - \rho_0, \text{ any } \alpha, \beta \in [1, \infty), & \text{if } \Omega \text{ is bounded}, \\
R = \rho, \text{ any } \alpha \in (1, \infty), \text{ any } \beta \in (2, \infty), & \text{if } \Omega = \mathbb{R}^2.
\end{cases}
\]

**Remark 1.1** If \( l = 1 \), the unique solution is the so-called strong solution; if \( l \geq 2 \), the unique solution is also a classical one. In this paper, by strong solutions, we mean weak solutions satisfying the equations (1.3) (1.4) almost everywhere in \( \Omega \times (0, \infty) \); and by classical solutions, we mean a pair of functions \((\rho, u), \rho \in C^1(\Omega \times (0, \infty)), u \in C^2(\Omega \times (0, \infty))\), such that (1.3) and (1.4) are satisfied everywhere in \( \Omega \times (0, \infty) \).
Remark 1.2 Under the same conditions, Lions ([16]), Kazhikhov et al. ([12]), and Chatelon et al. ([1]) proved the same results except (1.12) and (1.13).

Remark 1.3 (1.12) means that the density is bounded from above independent of time; this is the key for the large time dynamical behavior of solutions.

Theorem 1.1 shows that there exists a unique strong (or classical) solution on \([0, T]\) to the initial-boundary-value problem for the 2D equations (1.2) for any \(T > 0\) for smooth data. Furthermore, the large time asymptotic behavior of \(\rho, u\) themselves in \(L^p\)-norms is given by (1.13). A natural question is what is the large time behavior of the derivatives of this solution. We will give a partial answer to this question. It will be shown that if the initial density contains vacuum at least at a point and the domain \(\Omega\) is bounded then the global strong solution has to blow up as time goes to infinity, that is

\[
\lim_{t \to \infty} \|\nabla \rho(\cdot, t)\|_{L^q(\Omega)} = \infty.
\]

Remark 1.4 It would be interesting to study the existence and large time asymptotic behavior of solutions for the case \(q = 2\). This is left for the future.

Finally, we give a brief outline of the rest of the paper. In Section 2 we collect some elementary facts which are useful for our analysis later. The main results, Theorem 1.1 and Theorem 1.2, are proved in Section 3, 4, 5. It should be noted that although we use some ideas developed in [12] [16], some new elaborate estimates are needed to overcome the difficulties in obtaining the uniform time-independent upper bound estimate for the density. This is achieved by some careful estimates on the deviation of the pressure from its mean value and the difference between the divergence of the velocity field and the deviation of the pressure from its mean value. The case of bounded domains is treated in Section 3. While Section 4 is devoted to the Cauchy problem. Finally, we prove Theorem 1.2 in Section 5.

## 2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used and play important roles later.

Consider the following parabolic problem

\[
\begin{align*}
\varphi_t - \Delta \varphi &= f, \\
\varphi(x, 0) &= 0,
\end{align*}
\]

supplemented with one of the following three boundary conditions:

\(\varphi(\cdot, t)\) is \(\Omega\)-periodic; \hspace{1cm} (2.2)

\(\Omega\) bounded, smooth, and \(\frac{\partial \varphi}{\partial n} = 0\) on \(\partial \Omega\); \hspace{1cm} (2.3)

\(\Omega = \mathbb{R}^N\), and \(\varphi\) vanishes at infinity. \hspace{1cm} (2.4)
Let $\Omega$ be a bounded domain and $f$ is integral on $\Omega$. We denote by $\bar{f}$ the average of $f$ over $\Omega$ for bounded domain $\Omega$, i.e.,

$$\bar{f} = \frac{1}{|\Omega|} \int_{\Omega} f(x) \, dx.$$ 

Then the following facts are well-known (see [5] [6]):

**Lemma 2.1** Assume that $p \in (1, \infty)$, $0 < T \leq \infty$. Then for

$$f \in \begin{cases} 
\{ f \in L^p(\Omega \times (0,T)), f \text{ periodic, } \bar{f} = 0 \}, & \text{if (2.2) holds,} \\
\{ f \in L^p(\Omega \times (0,T)), \bar{f} = 0 \}, & \text{if (2.3) holds,} \\
L^p(\Omega \times (0,T)), & \text{if (2.4) holds,}
\end{cases}$$

the problem (2.1) with the boundary condition (2.2) or (2.3) or (2.4) has a unique solution $\varphi$ such that $\varphi_t, D^2 \varphi \in L^p(0,T; L^p(\Omega))$, and $\varphi = 0$ if $\Omega$ is bounded; moreover, there exists a positive constant $A$ independent of $T$ such that

$$\int_0^T \| \varphi_t(t) \|^p_{L^p} dt + \int_0^T \| \Delta \varphi(t) \|^p_{L^p} dt \leq A \int_0^T \| f(t) \|^p_{L^p} dt.$$ 

Lemma 2.1 yields directly the following derivative estimate.

**Lemma 2.2** Let $r \in (1, \infty)$, $f \in (L^r(R^N \times (0,T)))^N$. Then solutions of the following parabolic problem:

$$\begin{cases} 
\varphi_t - \Delta \varphi = \text{div} f, & (x, t) \in R^N \times (0, T) \\
\varphi(x, 0) = 0, & x \in R^N \\
(2.4) \text{ holds},
\end{cases}$$

satisfy the following estimate

$$\| D \varphi \|_{L^r(R^N \times (0,T))} \leq A \| f \|_{L^r(R^N \times (0,T))}$$

where $A$ is a positive constant independent of $T$.

Lemma 2.1 and the Hodge decomposition lead to the following simple derivative estimate.

**Lemma 2.3** Let $r \in (1, \infty)$, $f \in L^r(\Omega \times (0,T))$ ($\bar{f} = 0$ for bounded $\Omega$). Then solutions of the following parabolic problem:

$$\begin{cases} 
v_t - \Delta v = \nabla f, \\
v(x, 0) = 0,
\end{cases}$$

supplemented with (1.6) or (1.7) or (1.8), satisfy

$$\| Dv \|_{L^r(\Omega \times (0,T))} \leq A \| f \|_{L^r(\Omega \times (0,T))}$$

with $A$ independent of $T$. 

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Also, the following estimate will be used later.

**Lemma 2.4 ([27])** Let the function $y$ satisfy

$$y'(t) \leq g(y) + b'(t) \text{ on } [0, T], y(0) = y^0,$$

with $g \in C(R)$ and $y, b \in W^{1,1}(0, T)$. If $g(\infty) = -\infty$ and $b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1)$ for all $0 \leq t_1 < t_2 \leq T$ with some $N_0 \geq 0$ and $N_1 \geq 0$, then

$$y(t) \leq \max\{y^0, \zeta\} + N_0 < \infty \text{ on } [0, T],$$

where $\zeta$ such that $g(\zeta) \leq -N_1$ for $\zeta \geq \zeta$.

Finally, we state the well-known Sobolev’s inequality.

**Lemma 2.5 ([13])** Assume that $N = 2$ and $\Omega = R^2$ or $\Omega$ is a bounded domain in $R^2$ with piecewise smooth boundary, and that

$$u \in H^1_0(\Omega) \text{ or } u \in \left\{ u \in H^1(\Omega), \bar{u} = 0 \right\} \text{ or } u \in \left\{ (H^1(\Omega))^2, u \cdot n|_{\partial \Omega} = 0 \right\}.$$

Then there exists a constant $C$ independent of $u$ such that

$$\|u\|_{L^4} \leq C \|u\|^{1/2}_{L^2} \|Du\|^{1/2}_{L^2}.$$

## 3 Proof of Theorem 1.1 on bounded domains

In this section, we will prove Theorem 1.1 in the case of bounded domains, i.e., either boundary condition (1.6) or (1.8) holds. Due to the existence and uniqueness results established in [16] [12], we need only to show that (1.12) and (1.13) hold.

Let $T \in (0, \infty)$ be fixed. In this section and the following one, $C$ denotes a generic positive constant independent of $T$.

First, we consider the periodic case and deduce (1.12) and (1.13). The boundary condition (1.8) will be treated later.

**Case 1. (Periodic case.)** Without loss of generality, we assume that

$$\int_{\Omega} u_0 dx = 0.$$  \hspace{1cm} (3.1)

Otherwise, one may change $u$ to $u - \bar{u}_0$, and consider the following system

$$\begin{cases}
\rho_t + \text{div}(\rho u) + \bar{u}_0 \cdot \nabla \rho = 0, \\
u_t - \Delta u + \nabla P = 0, \\
u(x, 0) = u_0 - \bar{u}_0.
\end{cases}$$

It is easy to check step by step that the following procedure still holds.

Standard energy estimates for (1.3) – (1.6) yield that

$$\frac{1}{2} \sup_{0 \leq s \leq t} \|u(\cdot, s)\|^2_{L^2} + \frac{1}{\gamma - 1} \sup_{0 \leq s \leq t} \|P(\cdot, s)\|_{L^1} + \int_0^t \|Du\|^2_{L^2} ds \leq \frac{1}{2} \|u_0\|^2_{L^2} + \frac{1}{\gamma - 1} \|P_0\|_{L^1} \triangleq I_0^2.$$  \hspace{1cm} (3.2)
(3.1), (1.4) and the periodic boundary condition (1.6) lead to
\[ \int_{\Omega} u \, dx = 0. \]
Thus, we use Lemma 2.5, (3.2) and Poincaré’s inequality to derive
\[ \int_{0}^{t} \left( \|u\|_{L^4}^4 + \|u\|_{L^2}^2 \right) \, ds \leq C. \]  

(3.3)

Denote by \( \varphi \) and \( w \) the unique periodic functions such that
\[ u = \nabla \varphi + w, \text{div} w = 0, \int_{\Omega} \varphi \, dx = 0, \]
and
\[ \|\nabla \varphi\|_{L^p} + \|w\|_{L^p} \leq C \|u\|_{L^p}, \]
for \( 1 < p < \infty \); and similarly
\[ u_0 = \nabla \varphi_0 + w_0, \text{div} w_0 = 0, \int_{\Omega} \varphi_0 \, dx = 0, \]
and
\[ \|\nabla \varphi_0\|_{L^p} + \|w_0\|_{L^p} \leq C \|u_0\|_{L^p}. \]
Hence, (1.4) and (1.5) show that \( w \) and \( \varphi \) satisfy
\[ \begin{cases} w_t - \Delta w = 0, \\ w(x, 0) = w_0(x), \end{cases} \]  
and
\[ \begin{cases} \varphi_t - \Delta \varphi + Q = 0, \\ \varphi(x, 0) = \varphi_0(x), \end{cases} \]  
respectively, where \( Q \triangleq P - \overline{P}. \)
Denote by \( S = \varphi_t - \Delta \varphi - Q = \text{div} u - Q. \) Then, \( P, Q \) and \( S \) satisfy
\[ P_t + \text{div}(Pu) + (\gamma - 1)P\text{div}u = 0, \]  

(3.6)
\[ \begin{cases} Q_t + \text{div}(uQ) + (\gamma - 1)Q\text{div}u + \gamma P\text{div}u - (\gamma - 1)Q\overline{\text{div}u} = 0, \\ Q(x, 0) = Q_0 \triangleq P_0 - \overline{P_0}, \end{cases} \]  
and
\[ \begin{cases} S_t - \Delta S = -Q_t, \\ S(x, 0) = \Delta \varphi_0 - Q_0, \end{cases} \]  
respectively, which follow by direct calculations based on (1.3) (1.4).
Multiplying (3.7) by $Q^2$ and integrating the result in space, one can obtain after integrating by parts and Hölder’s inequality and (3.2) that
\[
\|P(t)\|_{L^3}^3 + \int_0^t \|Q(s)\|_{L^4}^4 ds
\leq C + C\|P_0\|_{L^3}^3 + C \int_0^t \|S\|_{L^4}^4 ds + C \int_0^t \|Q\|_{L^2}^2 \|Du\|_{L^2}^2 ds,
\tag{3.9}
\]
where one has used the simple fact that $Q^3 \geq P^3/2 - CI_0$, due to (3.2).

We first derive some estimates on $Q$. Rewrite (3.5) as
\[
Q = \Delta \varphi - \varphi_t. \tag{3.10}
\]

Multiplying (3.10) by $Q$, then integrating the result over $\Omega \times (0, t)$, one gets by integration by parts and (3.7) that
\[
\int_0^t \|Q\|_{L^2}^2 ds
= \int_0^t \int_\Omega Q \text{div} u dx ds - (\gamma - 1) \int_0^t \int_\Omega \varphi Q \text{div} u dx ds
- \int_\Omega Q \varphi(x, t) dx + \int_\Omega Q_0 \varphi_0 dx
- \int_0^t \int_\Omega \text{div}(Q_u)\varphi dx ds - \gamma P \int_0^t \int_\Omega \text{div} \varphi dx ds. \tag{3.11}
\]

The terms in (3.11) can be estimated as follows:

It follows from (3.2) that
\[
\left| \int_0^t \int_\Omega Q \text{div} u dx ds \right| \leq \frac{1}{4} \int_0^t \|Q\|_{L^2}^2 ds + C. \tag{3.12}
\]

Noticing that
\[
\int_\Omega \varphi dx = 0,
\]
we use (3.3) and Poincaré’s inequality to get
\[
\int_0^t \|\varphi\|_{L^4}^4 ds \leq C \int_0^t \|u\|_{L^4}^4 ds \leq C. \tag{3.13}
\]

(3.13) and (3.2) yield that for any $\varepsilon > 0$,
\[
\left| \int_0^t \int_\Omega \varphi Q \text{div} u dx ds \right|
\leq \left( \int_0^t \|Q\|_{L^4}^4 ds \right)^{1/4} \left( \int_0^t \|\varphi\|_{L^4}^4 ds \right)^{1/4} \left( \int_0^t \|\text{div} u\|_{L^2}^2 ds \right)^{1/2}
\leq \varepsilon \left( \int_0^t \|Q\|_{L^4}^4 ds \right)^{1/2} + C\varepsilon. \tag{3.14}
\]

Poincaré’s inequality and (3.2) lead to
\[
\left| \int_\Omega Q(\varphi(x, t) dx \right| \leq \|Q(\cdot, t)\|_{L^2} \|\varphi(\cdot, t)\|_{L^2}
\leq C\|Q(\cdot, t)\|_{L^2} \|u(\cdot, t)\|_{L^2}
\leq C\lambda + \lambda \|Q(\cdot, t)\|_{L^2}^2, \tag{3.15}
\]

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for any \( \lambda > 0 \). Next, it follows from (3.2) and (3.3) that
\[
\left| \int_0^t \int_\Omega \text{div}(Qu)\varphi dxds \right| = \left| \int_0^t \int_\Omega Qu^i \partial_i \varphi dxds \right|
\leq \frac{1}{4} \int_0^t \|Q\|^2_{L^2} ds + C \int_0^t \|u\|^4_{L^4} ds
\leq \frac{1}{4} \int_0^t \|Q\|^2_{L^2} ds + C,
\] (3.16)

Finally, (3.2), (3.3) and Poincaré’s inequality give
\[
\left| \gamma \mathcal{P} \int_0^t \int_\Omega divu\varphi dxds \right| \leq C I_0^t \left( \int_0^t \|Du\|^2_{L^2} ds \right)^{1/2} \left( \int_0^t \|u\|^2_{L^2} ds \right)^{1/2} \leq C.
\] (3.17)

Thus, collecting all the estimates (3.12)-(3.17), we deduce from (3.11) that
\[
\int_0^t \|Q\|^2_{L^2} ds \leq \varepsilon \left( \int_0^t \|Q\|^4_{L^4} ds \right)^{1/2} + \lambda \|Q(\cdot,t)\|^2_{L^2} + C\lambda + C \varepsilon.
\] (3.18)

To estimate the second term on the right hand side of (3.18), we multiply the equation (3.7) by \( Q \), then integrate the result over \( \Omega \) to obtain
\[
\frac{d}{dt} \|Q(\cdot,t)\|^2_{L^2} = -(2\gamma - 1) \int_\Omega Q^2 divu dx - 2\gamma \mathcal{P} \int_\Omega Q divu dx \leq C \left( \|Q\|^2_{L^4} + \|Q\|_{L^2} \right) \|Du\|_{L^2}.
\] (3.19)

Thus, we integrate this inequality over \((0,t)\) to derive from (3.18) that
\[
\|Q(\cdot,t)\|^2_{L^2} \leq \|Q_0\|^2_{L^2} + C \left( \int_0^t \|Q\|^4_{L^4} ds \right)^{1/2} + \int_0^t \|Q\|^2_{L^2} ds + C \leq \|Q_0\|^2_{L^2} + C \left( \int_0^t \|Q\|^4_{L^4} ds \right)^{1/2} + \lambda \|Q(\cdot,t)\|^2_{L^2} + C\lambda.
\]

Choosing \( \lambda = 1/2 \) in this estimate leads to
\[
\sup_{0 \leq s \leq t} \|Q(\cdot,s)\|^2_{L^2} \leq C \|Q_0\|^2_{L^2} + C \left( \int_0^t \|Q\|^4_{L^4} ds \right)^{1/2} + C.
\] (3.20)

The combination of (3.18) with (3.20) gives that
\[
\varepsilon \sup_{0 \leq s \leq t} \|Q(\cdot,s)\|^2_{L^2} + \int_0^t \|Q\|^2_{L^2} ds \leq C \varepsilon \|Q_0\|^2_{L^2} + C \varepsilon \left( \int_0^t \|Q\|^4_{L^4} ds \right)^{1/2} + C \varepsilon.
\] (3.21)

Next, we turn to the estimate on \( S \). Multiplying the equation (3.4) by \(-\Delta w\), then integrating the resulting identity over both space and time, one gets
\[
\|Dw(\cdot,t)\|^2_{L^2} + \int_0^t \|\Delta w\|^2_{L^2} ds \leq \|Dw_0\|^2_{L^2}.
\]
where the second inequality is due to Poincaré’s inequality, we can multiply (3.8) by $P$ and then integrate the resulting identity over both space and time to obtain

$$
\int_0^t \|w_t\|_{L^2}^2 ds = \int_0^t \|\Delta w\|_{L^2}^2 ds \leq C.
$$

(3.22)

Hence,

$$\nabla S = \nabla \varphi_t, \|S\|_{L^2} \leq C\|DS\|_{L^2}, \nabla \varphi,\|\varphi\|_{L^2} \leq C\|\varphi\|_{L^2},
$$

where (3.2) and (3.7) have been used.

Since

$$
\int_0^t \|\varphi_t\|_{H^1}^2 dt \leq C,
$$

we can multiply (3.24) by $1/2$ and integrate the result in space to get

$$
\frac{1}{2} \frac{d}{dt} \|S\|_{L^2}^2 + \|DS\|_{L^2}^2
$$

$$
= - \int_\Omega Q u \cdot \nabla S dx + (\gamma - 1) \int_\Omega (divu) S dx - (\gamma - 1) \int_\Omega (divu)^2 S dx
$$

$$
\leq \frac{1}{2} \int_\Omega Q \frac{\partial |u|^2}{\partial t} dx + C \int_\Omega (|Q u| + |Du|^2) S dx + C \int_\Omega |Du|^2 dx
$$

(3.23)

where (3.2) and (3.7) have been used.

The first term on the right hand side in (3.23) can be estimated again by (3.7) that

$$
- \frac{1}{2} \int_\Omega Q \frac{\partial |u|^2}{\partial t} dx
$$

$$
= - \frac{1}{2} \frac{d}{dt} \int_\Omega Q |u|^2 dx + \frac{1}{2} \int_\Omega |u|^2 Q dx
$$

$$
\leq - \frac{1}{2} \frac{d}{dt} \int_\Omega Q |u|^2 dx + \frac{1}{2} \int_\Omega |u|^2 dx
$$

$$
\leq - \frac{1}{2} \frac{d}{dt} \int_\Omega Q |u|^2 dx + C \int_\Omega (|u|^2 |Du| + |u|^2 |Q|) dx
$$

$$
+ C \|Du\|_{L^2}^2 + C \|u\|_{L^4}^4 + C \|Q\|_{L^2}^2 \|Du\|_{L^2}^2 + C \|u\|_{L^2}^4.
$$

(3.24)

We multiply the equation (1.4) by $|u|^2 u$, and integrate the result in space to get

$$
\frac{d}{dt} \|u\|_{L^4}^4 \leq C \int_\Omega (|u|^2 |Du| Q + |Du|^2 |u|^2) dx
$$

$$
\leq C \int_\Omega (|u|^2 |Du| S + |u|^2 |Du|^2) dx,
$$

(3.25)

It follows from (3.23)-(3.25) that

$$
\int_\Omega (P|u|^2 - \overline{P}|u|^2 + \|S\|_{L^2}^2 + \|u\|_{L^4}^4) dx + \int_0^t \|DS\|_{L^2}^2 ds
$$

$$
\leq C + C \int_0^t \|Q u\|_{L^2}^2 w_t dx + C \int_0^t \|Q\|_{L^2}^2 \|Du\|_{L^2}^2 ds
$$

$$
+ C \int_0^t \int_\Omega (|Du|^2 S + |Du|^2 |S| + |u|^2 |Du| S + |u|^2 |Du|^2) dx ds.
$$

(3.26)
We can estimate each of the terms on the right hand side of (3.26) as follows: First, (3.22) and (3.3) yield that
\[
\int_0^t \|Q u\|_{L^2} \|w_t\|_{L^2} ds \leq C \left(\int_0^t \|Q u\|_{L^2}^2 ds\right)^{1/2} \\
\leq \varepsilon \left(\int_0^t \|Q u\|_{L^4}^4 ds\right)^{1/2} + C_\varepsilon. \tag{3.27}
\]
Next, Lemma 2.5 and \(\int_\Omega S dx = 0\) give that
\[
\|S\|_{L^4(\Omega)} \leq C \|S\|_{L^2(\Omega)}^{1/2} \|DS\|_{L^2(\Omega)}^{1/2};
\]
thus, one has
\[
\int_0^t \|Du S^2(\cdot, s)\|_{L^1} ds \\
\leq \int_0^t \|Du\|_{L^2} \|S\|_{L^2}^2 ds \\
\leq C \int_0^t \|Du\|_{L^2} \|S\|_{L^2} \|DS\|_{L^2} ds \\
\leq \varepsilon \int_0^t \|DS\|_{L^2}^2 ds + C\varepsilon \int_0^t \|Du\|_{L^2}^2 \|S\|_{L^2}^2 ds. \tag{3.28}
\]
We infer from (3.2) that
\[
\int_0^t \int_\Omega (|Du|^2 |S| + |u|^2 |Du| |S| \) dx ds \\
\leq \int_0^t \left(\|Du\|_{L^2} \|Du\|_{L^4} \|S\|_{L^4} + \|u\|_{L^4}^2 \|Du\|_{L^4} \|S\|_{L^4}\right) ds \\
\leq C \int_0^t \left(\|Du\|_{L^2} \|Du\|_{L^4} \|S\|_{L^4} + \|u\|_{L^2} \|Du\|_{L^2} \|Du\|_{L^4} \|S\|_{L^4}\right) ds \\
\leq C \int_0^t \|Du\|_{L^2} \|Du\|_{L^4} \|S\|_{L^4} ds \\
\leq C \left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/4} \left(\int_0^t \|Du\|_{L^2}^{4/3} \|S\|_{L^2}^{2/3} \|DS\|_{L^2}^{2/3} ds\right)^{3/4} \\
\leq C \left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/4} \left(\int_0^t \|DS\|_{L^2}^2 ds\right)^{1/4} \left(\int_0^t \|Du\|_{L^2}^2 \|S\|_{L^2} ds\right)^{1/2} \\
\leq \varepsilon \left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/2} + \varepsilon \int_0^t \|DS\|_{L^2}^2 ds + C\varepsilon \int_0^t \|Du\|_{L^2}^2 \|S\|_{L^2}^2 ds. \tag{3.29}
\]
Lemma 2.5 gives that
\[
\|u\|_{L^2}^2 \leq \|u\|_{L^4}^2 - \|u_{\text{mod}}\|_{L^2}^2 + C \|u\|_{L^2}^2 \\
\leq C \|u\|_{L^4}^2 \|u_{\text{mod}}\|_{L^2}^{1/2} \|D u\|_{L^2}^{1/2} + C \|u\|_{L^2}^2 \leq C \left(\|u\|_{L^4} + \|u\|_{L^2}\right) \|Du\|_{L^2}^{1/2} + C \|u\|_{L^2}^2.
\]
Hence, we use (3.3) to deduce
\[
\int_0^t \int_\Omega |u|^2 |Du|^2\,dx\,ds \\
\leq \int_0^t \|u\|_{L^4}^2 \|Du\|_{L^4} \|Du\|_{L^2} \,ds \\
\leq \frac{1}{2} \int_0^t \|u\|_{L^4}^2 \|Du\|_{L^2}^2 \,ds + C \int_0^t \left( \|u\|_{L^4}^{4/3} + \|u\|_{L^2}^{4/3} \right) \|Du\|_{L^4}^{4/3} \|Du\|_{L^2}^{4/3} \,ds + C.
\]
Thus, making use of (3.2), we have
\[
\int_0^t \int_\Omega |u|^2 |Du|^2\,dx\,ds \\
\leq C \int_0^t \left( \|u\|_{L^4}^{4/3} + \|u\|_{L^2}^{4/3} \right) \|Du\|_{L^4}^{4/3} \|Du\|_{L^2}^{4/3} \,ds + C \\
\leq C \left( \int_0^t \|Du\|_{L^4}^2 \,ds \right)^{1/3} \left( \int_0^t \left( \|u\|_{L^4}^2 + \|u\|_{L^2}^2 \right) \|Du\|_{L^2}^2 \,ds \right)^{2/3} + C \\
\leq C \left( \int_0^t \|Du\|_{L^4}^2 \,ds \right)^{1/2} + C \varepsilon \left( \int_0^t \|u\|^4_{L^4} \|Du\|^2_{L^2} \,ds \right)^2 + C \varepsilon \\
\leq C \left( \int_0^t \|Du\|_{L^4}^2 \,ds \right)^{1/2} + C \varepsilon \int_0^t \|u\|^4_{L^4} \|Du\|^2_{L^2} \,ds + C \varepsilon.
\]
Lemma 2.3 yields
\[
\left( \int_0^t \|Du\|_{L^4}^2 \,ds \right)^{1/2} \leq C + C \left( \int_0^t \|Q\|^2_{L^4} \,ds \right)^{1/2}.
\]
We use (3.2)(3.9)(3.21)(3.26) – (3.31) to deduce that
\[
\begin{align*}
A(t) + \varepsilon \sup_{0 \leq s \leq t} \|Q\|^2_{L^2} + B(t) \\
\leq C \varepsilon \left( \int_0^t \|S\|^2_{L^4} \,ds \right)^{1/2} + C \varepsilon B(t) + C \varepsilon \\
+ C \varepsilon \int_0^t \|Du\|^2_{L^2} \left( \|S\|^2_{L^2} + \|Q\|^2_{L^2} + \|u\|^2_{L^4} \right) \,ds \\
\leq C \varepsilon \left( \int_0^t A(t) \|DS\|^2_{L^2} \,ds \right)^{1/2} + C \varepsilon B(t) + C \varepsilon \\
+ C \varepsilon \int_0^t \|Du\|^2_{L^2} \left( \|S\|^2_{L^2} + \|Q\|^2_{L^2} + \|u\|^2_{L^4} \right) \,ds \\
\leq C \varepsilon \left( A(t) + B(t) \right) + C \varepsilon \\
+ C \varepsilon \int_0^t \|Du\|^2_{L^2} \left( \|S\|^2_{L^2} + \|Q\|^2_{L^2} + \|u\|^2_{L^4} \right) \,ds,
\end{align*}
\]
where
\[
A(t) \triangleq \sup_{0 \leq s \leq t} \int_\Omega \left( |S|^2 + |u|^4 + |P|u|^2 \right) (x, s) \,dx,
\]
and
\[
B(t) \triangleq \int_0^t \left( \|DS\|^2_{L^2} + \|Q\|^2_{L^2} \right) \,ds.
\]
Choosing \( \varepsilon \) small enough, we have
\[
\sup_{0 \leq s \leq t} (\|S\|_{L^2}^2 + \|Q\|_{L^2}^2 + \|u\|_{L^4}^4) (s) + \int_0^t (\|DS\|_{L^2}^2 + \|Q\|_{L^2}^2) \, ds \\
\leq C + C \int_0^t \|Du\|_{L^2}^2 (\|S\|_{L^2}^2 + \|Q\|_{L^2}^2 + \|u\|_{L^4}^4) \, ds.
\]

Since
\[
\int_0^t \|Du\|_{L^2}^2 \, ds \leq C,
\]
Gronwall’s inequality thus gives that
\[
\sup_{0 \leq s \leq t} (\|S\|_{L^2}^2 + \|Q\|_{L^2}^2 + \|u\|_{L^4}^4) (s) + \int_0^t (\|DS\|_{L^2}^2 + \|Q\|_{L^2}^2) \, ds \leq C.
\] (3.32)

The combination of this estimate with (3.9) yields that
\[
\sup_{0 \leq s \leq T} \|P(\cdot, s)\|_{L^3}^3 + \int_0^T (\|Q\|_{L^4}^4 + \|Q\|_{L^2}^2) \, ds \leq C.
\] (3.33)

One deduces from (3.31) and (3.33) that
\[
\int_0^t \|Du\|_{L^4}^4 \, ds \leq C.
\] (3.34)

By the identity
\[
\int_{\Omega} \partial_i \varphi \, dx = 0,
\]
(3.32) and the Gagliardo-Nirenberg inequality, one gets
\[
\|\partial_t \varphi(t)\|_{L^\infty} \leq C \|D\varphi(t)\|_{L^4}^{1/2} \|D^2 \varphi(t)\|_{L^4}^{1/2} \\
\leq C \|u(t)\|_{L^4}^{1/2} \|Du(t)\|_{L^4}^{1/2} \\
\leq C \|Du(t)\|_{L^4}^{1/2}.
\]

We deduce from this estimate, (3.34) and Hölder’s inequality that
\[
\int_0^T \|\partial_j \varphi\|_{L^\infty}^8 \, ds \leq C.
\]

Since \( \overline{u} = 0 \), It follows from (3.32), (3.33) and the Poincaré-Sobolev inequality that
\[
\int_0^T \|u\|_{L^\infty}^2 \, ds \leq C.
\]

The above two inequalities give that
\[
\int_0^T \|u^j \partial_j \varphi\|_{L^\infty}^4 \, ds \leq C.
\] (3.35)

Set \( D_tw = w_t + u \cdot \nabla \varphi \). Using (3.2), we conclude from (1.3) and (3.5) that
\[
D_t (\log P + \gamma \varphi) \leq -\gamma P + CT_0^2 + \gamma u^j \partial_j \varphi.
\] (3.36)
Now, we pass in (3.36) to the Lagrangian coordinates and take \( y = \log P \), \( g(y) = -\gamma e^y \), and \( b(t) = b_1(t) - b_0(t) \) where

\[
b_1(t) = \gamma \int_0^t u^j \partial_j \varphi ds + C I_0^2 t \quad \text{and} \quad b_0(t) = \gamma \varphi.
\]

Thus, (3.35) yields that for \( 0 \leq t_1 < t_2 \leq T \),

\[
|b_1(t_2) - b_1(t_1)| \leq \gamma \int_{t_1}^{t_2} \| u^j \partial_j \varphi(\cdot, s) \|_{L^\infty} ds + C I_0^2 (t_2 - t_1)
\]

\[
\leq C \int_0^T \| u^j \partial_j \varphi(\cdot, s) \|_{L^\infty}^4 ds + C (t_2 - t_1)
\]

\[
\leq C + C (t_2 - t_1).
\]

(3.32) and Poincaré’s inequality give that

\[
\sup_{0 \leq t \leq T} \| b_0(t) \| \leq C \sup_{0 \leq t \leq T} \| \varphi \|_{L^4}^{1/2} \sup_{0 \leq t \leq T} \| D \varphi \|_{L^4}^{1/2}
\]

\[
\leq C \sup_{0 \leq t \leq T} \| u \|_{L^4}
\]

\[
\leq C.
\]

Hence, we have

\[
|b(t_2) - b(t_1)| \leq C + C (t_2 - t_1).
\]

(3.37)

Since estimate (3.37) holds, the uniform upper bounds for \( \log P \) and consequently for \( \rho \) follow from Lemma 2.4.

Next, we will prove (1.13).

We claim that we have

\[
\lim_{t \to \infty} (\|Q(\cdot, t)\|_{L^2} + \|u(\cdot, t)\|_{L^4}) = 0.
\]

(3.38)

In fact, set

\[
h(t) = \|Q(\cdot, t)\|_{L^2}^2 + \|u(\cdot, t)\|_{L^4}^4.
\]

It follows easily from (3.3) and (3.33) that

\[
\int_0^\infty h(t)dt \leq C.
\]

Using (3.3), (3.33) and (3.34), we derive from (3.19) and (3.25) that

\[
\int_0^\infty |h'(t)| dt
\]

\[
\leq C \int_0^\infty (\| Du \|_{L^2} (\|Q\|_{L^4}^2 + \|Q\|_{L^2}^2) + (\| Du \|_{L^4} Q \|_{L^4} + \| Du \|_{L^4}^2 \) \| u \|_{L^4}^2 dt
\]

\[
\leq C \int_0^\infty (\|Q\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|u\|_{L^4}^4 + \| Du \|_{L^2}^2 + \| Du \|_{L^4}^4) dt
\]

\[
\leq C.
\]

Consequently,

\[
\lim_{t \to \infty} h(t) = 0.
\]

(3.39)
This shows that (3.38) holds true.

It follows from (3.6) that

\[
\int_0^\infty \left| \frac{d}{dt} \mathbf{P}(t) \right| dt \leq (\gamma - 1) \int_0^\infty |\mathbf{Q} \mathbf{div} u| dt \\
\leq C \int_0^\infty (\|\mathbf{Q}\|_{L^2}^2 + \|\mathbf{D}u\|_{L^2}^2) dt \\
\leq C.
\]

This yields that there exists some positive constant \(\rho_s\) such that

\[
\lim_{t \to \infty} \mathbf{P}(t) = \rho_s^\gamma,
\]

(3.40)

since \(0 < \rho_0^\gamma \leq \mathbf{P} \leq C \rho_0^2\). (3.38), (3.40) and (1.12) lead to

\[
\lim_{t \to \infty} \|\mathbf{\rho}(\cdot, t) - \rho_s\|_{L^\alpha} = 0,
\]

for any \(\alpha \in [1, \infty)\). Hence, we have

\[
\rho_s = \rho_0,
\]

due to the fact that \(\mathbf{P}(t) \equiv \rho_0\). Consequently,

\[
\lim_{t \to \infty} \|\mathbf{\rho}(\cdot, t) - \rho_0\|_{L^\alpha} = 0, \quad (3.41)
\]

for any \(\alpha \in [1, \infty)\).

(1.12) and (3.33) yield that \(\mathbf{Q}\) satisfies

\[
\int_0^\infty \|\mathbf{Q}(\cdot, t)\|_{L^p}^p dt \leq C \quad \text{for any } 2 \leq p < \infty.
\]

Hence,

\[
\sup_{0 \leq t < \infty} \|\mathbf{u}\|_{L^p} \leq C, \quad (3.42)
\]

for all \(2 \leq p < \infty\).

It thus follows easily from (3.2), (3.41), (3.42) and (3.38) that (1.13) holds true.

Case 2. (The boundary condition (1.8) holds.) Notice that \(\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0\) yields that Poincaré’s inequality still holds, i.e.

\[
\|\mathbf{u}\|_{L^2} \leq C\|\mathbf{D}u\|_{L^2},
\]

and that for \(1 < p < \infty\),

\[
\|\mathbf{D}u\|_{L^p} \leq C (\|\mathbf{div} u\|_{L^p} + \|\mathbf{curl} u\|_{L^p}). \quad (3.43)
\]

Denoting by \(\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})^T\), we have

\[
\Delta \mathbf{u} = \nabla \mathbf{div} u - \nabla^\perp \mathbf{curl} u.
\]

Hence, (1.3) – (1.5), (1.8) and Lemma 2.5 yield that (3.2) and (3.3) still hold.
Denote by $\varphi$ and $w$ the unique functions such that $u = \nabla \varphi + w$, $\text{div} w = 0$, and

$$\begin{cases} 
\Delta \varphi = \text{div} u, \\
\frac{\partial \varphi}{\partial n}|_{\partial \Omega} = 0, \int_{\Omega} \varphi dx = 0;
\end{cases}$$

similarly $u_0 = \nabla \varphi_0 + w_0$, $\text{div} w_0 = 0$, and

$$\begin{cases} 
\Delta \varphi_0 = \text{div} u_0, \\
\frac{\partial \varphi_0}{\partial n}|_{\partial \Omega} = 0, \int_{\Omega} \varphi_0 dx = 0.
\end{cases}$$

Using (1.8), we infer from (1.4) that $\text{curl} w$ satisfies

$$\begin{cases} 
(\text{curl} w)_t - \Delta \text{curl} w = 0, \\
\text{curl} w|_{\partial \Omega} = 0, \\
\text{curl} w(x, 0) = \text{curl} u_0.
\end{cases}$$

Hence, we have

$$\|\text{curl} w(t)\|_{L^2}^2 + 2 \int_0^t \|D\text{curl} w\|_{L^2}^2 ds \leq \|\text{curl} u_0\|_{L^2}^2.$$ 

Since $\text{curl} w|_{\partial \Omega} = 0$, this estimate and Lemma 2.5 lead to

$$\int_0^t \|\text{curl} u\|_{L^4}^4 ds = \int_0^t \|\text{curl} w\|_{L^4}^4 ds \leq C. \quad (3.44)$$

Choosing the smooth basis $\nabla \{\psi_i\}$ of $\nabla H^1(\Omega)$, where $\{\psi_i\}$ is the solutions of

$$\begin{cases} 
-\Delta \psi_i = \lambda_i \psi_i, \\
\frac{\partial \psi_i}{\partial n}|_{\partial \Omega} = 0.
\end{cases}$$

We have, for any $i$,

$$0 = \int_{\Omega} (u_t - \Delta u + \nabla P) \cdot \nabla \psi_i dx = \int_{\Omega} (-u_t - \nabla \text{div} u + \nabla \text{curl} u + \nabla P) \cdot \nabla \psi_i dx = \int_{\Omega} (-\varphi_t + \Delta \varphi - P) \Delta \psi_i dx = \lambda_i \int_{\Omega} (\varphi_t - \Delta \varphi + P) \psi_i dx.$$ 

This yields that $\varphi$ satisfies

$$\begin{cases} 
\varphi_t - \Delta \varphi + Q = 0, \\
\frac{\partial \varphi}{\partial n}|_{\partial \Omega} = 0, \int_{\Omega} \varphi dx = 0, \\
\varphi(x, 0) = \varphi_0(x),
\end{cases} \quad (3.45)$$

where $Q \triangleq P - \overline{P}$. 

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Obviously, $Q$ satisfies (3.7).

Denote by $S = \varphi_t$. It follows from (3.45) that $S$ satisfies

$$
\begin{align*}
S_t - \Delta S &= -Q_t, \\
\frac{dS}{dt}|_{\partial \Omega} &= 0, \int_{\Omega} Sdx = 0, \\
S(x,0) &= \Delta\varphi_0(x) - P_0(x).
\end{align*}
$$

We use Lemma 2.1 to get

$$
\int_0^t \|\Delta \varphi\|_{L^4}^4 ds \leq C \left( \|\Delta \varphi_0\|_{L^2}^2 + \int_0^t \|Q\|_{L^4}^4 ds \right).
$$

Hence, this estimate, together with (3.43) and (3.44), yields that

$$
\int_0^t \|Du\|_{L^4}^4 ds \leq C \left( 1 + \int_0^t \|Q\|_{L^4}^4 ds \right);
$$

that is to say, (3.31) still holds.

We then follow the proof in Case 1 to obtain (1.12) and (1.13) in this case.

4 Proof of Theorem 1.1 for the Cauchy problem

In this section, we treat the Cauchy problem (1.3)-(1.5) and (1.7). Since the main idea is similar to that given in Section 3, we make some slightly modification due to the non-compactness of $\Omega = R^2$ and we just sketch the proof of Theorem 1.1 for this case.

First, standard energy estimates applying to the problem (1.3)-(1.5) and (1.7) show that (3.2) still holds in this case. Using (3.2), we deduce from Lemma 2.5 that

$$
\int_0^t \|u\|_{L^4}^4 ds \leq C. \tag{4.1}
$$

Obviously, $P$ satisfies (3.6). Denote by

$$
S = divu - P. \tag{4.2}
$$

It is easy to see that $S$ satisfies

$$
\begin{align*}
S_t - \Delta S &= -P_t = div(Pu) + (\gamma - 1)Pdivu, \\
S(x,0) &= S_0(x) = divu_0 - P_0.
\end{align*} \tag{4.3}
$$

Lemma 2.5 leads to

$$
\|S\|_{L^4} \leq C \|S\|_{L^2}^{1/2} \|DS\|_{L^2}^{1/2}. \tag{4.4}
$$

We multiply (3.6) by $P^2$ and integrate the resulting identity in both space and time to derive that

$$
\|P(t)\|_{L^3}^3 + \int_0^t \|P\|_{L^4}^4 ds \leq C \|P_0\|_{L^3}^3 + C \int_0^t \|S\|_{L^4}^4 ds. \tag{4.5}
$$
Denote by $v$ and $w$ the unique functions such that $u = v + w$, $\text{div} w = \text{curl} v = 0$, and similarly, $u_0 = v_0 + w_0$, $\text{div} w_0 = \text{curl} v_0 = 0$. One deduces from direct calculations based on (1.3) (1.4) that

$$
\begin{align*}
\begin{cases}
    v_t = \nabla S, \\
    v_t - \Delta v + \nabla P = 0, \\
    v(x, 0) = v_0(x),
\end{cases} 
\end{align*}
$$

(4.6)

and

$$
\begin{align*}
\begin{cases}
    w_t - \Delta w = 0, \\
    w(x, 0) = w_0(x).
\end{cases} 
\end{align*}
$$

(4.7)

Hence, similar to (3.22), we have

$$
\int_0^t \|w_t\|_{L^2}^2 ds = \int_0^t \|\Delta w\|_{L^2}^2 ds \leq C.
$$

(4.8)

It follows from (4.6) and (4.3) that

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|S\|_{L^2}^2 + \|DS\|_{L^2}^2 & = - \int_{R^2} Pu \cdot \nabla Sdx + (\gamma - 1) \int_{R^2} P \text{div} u Sdx \\
& \leq -\frac{1}{2} \int_{R^2} P \frac{\partial |u|^2}{\partial t} dx + C \int_{R^2} (|Pu| |w_t| + |Du|^2 |S| + |Du| |S|^2) dx.
\end{align*}
$$

(4.9)

Note that

$$
\begin{align*}
-\frac{1}{2} \int_{R^2} P \frac{\partial |u|^2}{\partial t} dx & = \frac{1}{2} \frac{d}{dt} \int_{R^2} P |u|^2 dx + \frac{1}{2} \int_{R^2} |u|^2 P_t dx \\
& \leq \frac{1}{2} \frac{d}{dt} \int_{R^2} P |u|^2 dx + C \int_{R^2} |u|^2 |Du| P dx \\
& \leq \frac{1}{2} \frac{d}{dt} \int_{R^2} P |u|^2 dx + C \int_{R^2} (|u|^2 |Du|^2 + |u|^2 |Du| |S|) dx,
\end{align*}
$$

and

$$
\frac{d}{dt} \|u\|_{L^4}^4 \leq C \int_{R^2} (|u|^2 |Du| |S| + |u|^2 |Du|^2) dx.
$$

Using these estimates, we infer from (4.9) that

$$
\begin{align*}
\int_{R^2} (Pu |u|^2 + S^2 + |u|^4) (x, t) dx + \int_0^t \|DS\|_{L^2}^2 ds & \leq C \int_0^t \int_{R^2} (|Du| S^2 + |Du|^2 |S| + |u|^2 |Du||S| + |u|^2 |Du|^2) dxds \\
& + C \int_0^t \|Pu\|_{L^2} \|w_t\|_{L^2} ds + C.
\end{align*}
$$

(4.10)
We estimate the terms in the right hand side of (4.10) as follows: First, (4.4) gives that
\[
\int_0^t \|DuS^2(\cdot, s)\|_{L^1} \, ds \\
\leq \int_0^t \|Du\|_{L^2} \|S\|_{L^4}^2 \, ds \\
\leq \varepsilon \int_0^t \|DS\|_{L^2}^2 \, ds + C\varepsilon \int_0^t \|Du\|_{L^2}^2 \|S\|_{L^2}^2 \, ds. \quad (4.11)
\]
Secondly, similar to (3.29), we deduce from (3.2) and (4.4) that
\[
\int_0^t \int_{\mathbb{R}^2} \left( |Du|^2 |S| + |u|^2 |Du| |S| \right) \, dx \, ds \\
\leq \varepsilon \left( \int_0^t \|Du\|_{L^4}^4 \, ds \right)^{1/2} + \varepsilon \int_0^t \|DS\|_{L^2}^2 \, ds + C\varepsilon \int_0^t \|Du\|_{L^2}^2 \|S\|_{L^2}^2 \, ds. \quad (4.12)
\]
Thirdly, similar to (3.30), we derive from (3.2) and Hölder’s inequality that
\[
\int_0^t \int_{\mathbb{R}^2} |u|^2 |Du|^2 \, dx \, ds \\
\leq C \int_0^t \|u\|_{L^4}^{4/3} \|Du\|_{L^4}^{4/3} \|Du\|_{L^2}^{4/3} \, ds \\
\leq \varepsilon \left( \int_0^t \|Du\|_{L^4}^4 \, ds \right)^{1/2} + \varepsilon \int_0^t \|u\|_{L^4}^{4/3} \|Du\|_{L^2}^{2} \, ds. \quad (4.13)
\]
Finally, (4.8) and (4.1) yield that
\[
\int_0^t \|Pu\|_{L^2} \|w_t\|_{L^2} \, ds \leq \varepsilon \left( \int_0^t \|P\|_{L^4}^4 \, ds \right)^{1/2} + C\varepsilon. \quad (4.14)
\]
Lemma 2.3 gives
\[
\left( \int_0^t \|Du\|_{L^4}^4 \, ds \right)^{1/2} \leq C + C \left( \int_0^t \|P\|_{L^4}^4 \, ds \right)^{1/2}. \quad (4.15)
\]
Using (3.2)(4.5)(4.11) – (4.14), we infer from (4.10) and (4.15) that
\[
A(t) + B(t) \\
\leq C\varepsilon + C\varepsilon \left( \int_0^t \|S\|_{L^4}^4 \, ds \right)^{1/2} + C\varepsilon B(t) \\
\quad + C\varepsilon \int_0^t \|Du\|_{L^2}^2 (\|S\|_{L^2}^2 + \|u\|_{L^4}^4) \, ds \\
\leq C\varepsilon + C\varepsilon (A(t) + B(t)) \\
\quad + C\varepsilon \int_0^t \|Du\|_{L^2}^2 (\|S\|_{L^2}^2 + \|u\|_{L^4}^4) \, ds.
\]
where
\[
A(t) \triangleq \sup_{0 \leq s \leq t} \int_{\mathbb{R}^2} \left( |S|^2 + |u|^4 + P|u|^2 \right) (x, s) \, dx,
\]
and

\[ B(t) \triangleq \int_0^t \| DS \|_{L^2}^2 ds. \]

Choosing \( \varepsilon \) small enough yields that

\[
\sup_{0 \leq s \leq t} \left( \| S \|_{L^2}^2 + \| u \|_{L^4}^4 \right) (s) + \int_0^t \| DS \|_{L^2}^2 ds \leq C + C \int_0^t \| Du \|_{L^2}^2 \left( \| S \|_{L^2}^2 + \| u \|_{L^4}^4 \right) ds.
\]

Gronwall’s inequality thus gives that

\[
\sup_{0 \leq s \leq t} \left( \| S \|_{L^2}^2 + \| u \|_{L^4}^4 \right) (s) + \int_0^t \| DS \|_{L^2}^2 ds \leq C,
\]

due to (3.2). We use (4.4), (4.5), (4.16) and (4.15) to conclude

\[
\sup_{0 \leq s \leq T} \left( \| P(\cdot, s) \|_{L^3}^3 + \int_0^T \left( \| P \|_{L^4}^4 + \| Du \|_{L^4}^4 \right) ds \right) \leq C.
\]

The Gagliardo-Nirenberg inequality, together with (3.2) and (4.17), gives that

\[
\int_0^T \| u \|_{L^\infty}^3 ds \leq C \int_0^T \| u \|_{L^2}^{1/2} \| Du \|_{L^{5/2}}^{5/2} ds \leq C \int_0^T \| Du \|_{L^{5/2}}^{5/2} ds \leq C.
\]

We derive from this estimate and (4.17) that

\[
\int_0^T \| Pu \|_{L^3}^3 ds \leq \sup_{0 \leq t \leq T} \| P \|_{L^3}^3 \int_0^T \| u \|_{L^\infty}^3 ds \leq C.
\]

Noticing that

\[
\int_0^T \| Pdiv u \|_{L^p}^p ds \leq C,
\]

for any \( 3/2 \leq p \leq 2 \), using (4.18), we deduce from Lemma 2.2 and (4.3) that

\[
\int_0^T \| DS \|_{L^3}^3 ds \leq C.
\]

Hence, this estimate, together with the Gagliardo-Nirenberg inequality and (4.16), leads to

\[
\int_0^T \| S \|_{L^\infty}^4 ds \leq C \int_0^T \| S \|_{L^2} \| DS \|_{L^3}^3 ds \leq C.
\]

Set \( D_t w = w_t + u \cdot \nabla w \). We conclude from (1.3) and (4.2) that

\[
D_t \log P = -\gamma P - \gamma S.
\]
Now, we pass in (4.20) to the Lagrangian coordinates and take \( y = \log P \), \( g(y) = -\gamma e^y \), and
\[
b(t) = \int_0^t S(x(s), s) \, ds.
\]
Thus, it follows from (4.19) and Hölder’s inequality that
\[
|b(t_2) - b(t_1)| \leq \int_0^T \|S(\cdot, s)\|_{L^\infty}^4 \, ds + C(t_2 - t_1)
\]
for \( 0 \leq t_1 < t_2 \leq T \). This estimate and Lemma 2.4 yield that (1.12) holds.

Similarly to Section 3, we can prove that (1.13) holds true for this case. Thus, the proof of Theorem 1.1 is completed.

5 Proof of Theorem 1.2

With the basic estimates (1.12) and (1.13) in Theorem 1.1, we can establish the Theorem 1.2 easily in this section.

Proof of Theorem 1.2. Otherwise, there exist some \( C_0 > 0 \) and a subsequence \( \{t_{n_j}\}_{j=1}^\infty \), \( t_{n_j} \to \infty \) such that \( \|\nabla \rho(\cdot, t_{n_j})\|_{L^2(\Omega)} \leq C_0 \). Hence, for
\[
a = \frac{q}{2(q - 1)} \in (0, 1),
\]
the Poincaré-Sobolev inequality yields that
\[
\|\rho(x, t_{n_j}) - \rho_0\|_{C(\Omega)} \leq C \|\nabla \rho(x, t_{n_j})\|^a_{L^q(\Omega)} \|\rho(x, t_{n_j}) - \rho_0\|^{1-a}_{L^2(\Omega)}
\]
with \( C \) independent of \( t_{n_j} \). We deduce from (1.13) that the right hand side of (5.1) goes to 0 as \( t_{n_j} \to \infty \). Hence,
\[
\|\rho(x, t_{n_j}) - \rho_0\|_{C(\Omega)} \to 0 \text{ as } t_{n_j} \to \infty.
\]

On the other hand, for \( T > 0 \), we introduce the Lagrangian coordinates which are defined as initial data to the Cauchy problem:
\[
\begin{aligned}
\frac{\partial}{\partial s} X(s; t, x) &= u(X(s; t, x), s) & 0 \leq s \leq T, \\
X(t; t, x) &= x & 0 \leq t \leq T, x \in \Omega.
\end{aligned}
\]
(1.11) yields that the transformation (5.3) is well-defined. Consequently, on the one hand, we have
\[
\rho(x, t) = \rho_0(X(0; t, x)) \exp \left\{ -\int_0^t \text{div}(X(s; t, x), s) \, ds \right\};
\]
(5.4)
on the other hand, since, by assumption, there exists some point \( x_0 \in \overline{\Omega} \) such that \( \rho_0(x_0) = 0 \), we get that there exists a \( x_0(t) \in \overline{\Omega} \) such that \( X(0; t, x_0(t)) = x_0 \). Using (5.4), we deduce from (1.11) that
\[
\rho(x_0(t), t) \equiv 0 \text{ for all } t \geq 0.
\]
So, we conclude from this equality and Hölder’s inequality that
\[
\left\| \rho(x, t_n) - \rho_0 \right\|_{C(\overline{\Omega})} \geq \left| \rho(x_0(t_n), t_n) - \rho_0 \right| = \rho_0 > 0,
\]
which contradicts (5.2).

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References


