Stability of Contact Discontinuities for the 1-D Compressible Navier-Stokes Equations

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Contents

Abstract

In this paper, we study the large time asymptotic behavior of solutions to the one-dimensional compressible Navier-Stokes system toward a contact discontinuity, which is one of the basic wave patterns for the compressible Euler equations. It is proved that such a weak contact discontinuity is a metastable wave pattern, in

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the sense introduced in [23], for the 1-D compressible Navier-Stokes system for polytropic fluid by showing that a viscous contact wave, which approximates the contact discontinuity on any finite time interval for small heat conduction and then runs away from it for large time, is nonlinearly stable with a uniform convergence rate provided that the initial excessive mass is zero. This result is proved by an elaborate combination of elementary energy estimates with a weighted characteristic energy estimate, which makes full use of the underlying structure of the viscous contact wave.

1 Introduction and Main Results

We consider the 1-dimensional compressible Navier-Stokes equations in *Lagrangian* coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu(\frac{u_x}{v})_x, \\ (e + \frac{u^2}{2})_t + (pu)_x = (\kappa\frac{\theta_x}{v} + \mu\frac{uu_x}{v})_x, \end{cases} \tag{1.1}$$

where $x \in R^1$, $t > 0$, $v(x,t) > 0$, $u(x,t)$, $\theta(x,t) > 0$, $e(x,t) > 0$ and $p(x,t)$ are the specific volume, velocity, internal energy, temperature, and pressure respectively, while $\mu > 0$ and $\kappa > 0$ denote the viscosity and heat conduction coefficients respectively. Here we study the perfect fluids so that p and e are given by

$$p = \frac{R\theta}{v}, \quad e = \frac{R}{\gamma - 1}\theta + const. \tag{1.2}$$

where $\gamma > 1$ is the adiabatic exponent and $R > 0$ is the gas constant.

In the ideal fluids, i.e., $\kappa = \mu = 0$, (1.1) becomes the well-known compressible Euler system:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ (e + \frac{u^2}{2})_t + (pu)_x = 0, \end{cases} \tag{1.3}$$

which is one of the most important nonlinear strictly hyperbolic system of conservation laws. The basic waves for the system (1.3) are dilation invariant solutions: shock waves, rarefaction waves (which are nonlinear waves), and contact discontinuities, [20], [2], and the linear combinations of these basic waves, called Riemann solutions, govern both local and large time asymptotic behavior of general solutions to the inviscid Euler system (1.3) [11]. Since the inviscid system (1.3) is an idealization when the dissipative effects are neglected, thus it is of great importance to study the large time asymptotic behavior of solutions to the corresponding viscous systems, such as (1.1), toward the viscous versions of these basic waves. Indeed, there have been great interests and intensive studies in this respect in the development of theory of viscous conservation laws since 1985, started with studies on the nonlinear stability of viscous shock profiles by Goodman [4] and Matsumura-Nishihara [16]. Deeper understanding has been achieved on the asymptotic stability toward nonlinear waves, viscous shock profiles and viscous rarefaction waves, which have been shown to be nonlinearly stable for quite general perturbation for the compressible Navier-Stokes system (1.1) and more general system of viscous strictly hyperbolic conservation laws, and new phenomena have been discovered and new techniques, such as weighted characteristic energy methods and uniform approximate Green's functions, have been developed based on the intrinsic properties of the underlying nonlinear waves, see [9], [10], [13], [14], [21], [12] [22], [18], [19] and the references therein.

However, the problem of stability of contact discontinuities is more subtle and the progress has been less satisfactory, except the studies in [6], [7], [8], [15], [23]. One of the main reasons is the contact discontinuities are associated with linear degenerate fields and are less stable compared with the nonlinear waves for the inviscid system (1.3), [11]. Thus the stabilizing effects around a contact discontinuity should come mainly from the viscosity and heat conductivity in (1.1). A general perturbation of a contact wave may introduces waves in the nonlinear sound wave families, and interactions of these waves with the linear contact waves are some of the major difficulties to overcome, see [23] and [15]. Another technique difficulty is that the viscosity matrix for the compressible Navier-Stokes equations is only semi-positive definite.

The stability toward contact waves for solutions to systems of viscous conservation laws was first studied by Xin in [23], where the metastability of a weak contact discontinuity for the compressible Euler equations with uniform viscosity, was proved by showing that although a contact discontinuity is not an asymptotic attractor for the viscous system, yet a viscous wave, which approximates the contact discontinuity on any finite time interval, is asymptotically nonlinear stable for small generic perturbations and the detail asymptotic behavior can be determined a priorily by initial mass distribution. This was later generalized by Liu-Xin in [15] to show the metastability of contact discontinuities for a class of general systems of nonlinear conservation laws with uniform viscosity, and obtain pointwise asymptotic behavior toward viscous contact wave by approximate fundamental solutions, which also leads to the nonlinear stability of the viscous contact wave in L^p-norms for all $p \geq 1$. However, the theory in [15] and [23] does not apply to the compressible Navier-Stokes system (1.1) since the viscosity matrix in (1.1) is only semi-positive definite.

It was conjectured by Xin in [23] that the metastability of contact discontinuities remains true for the Navier-Stokes system (1.1). Yet, it has remained open since then. For a free boundary value problem for (1.1) with a particle path as free boundary, the nonlinear stability of a viscous contact wave is proved in the super-norm by the elementary energy estimate by Huang-Matsumura-Shi in [7], see also [8]. However, the approach can not be applied here to study the asymptotic behavior toward contact waves for solutions to Cauchy problems of (1.1) since the analysis in [7] depends crucially on the availability of Poincaré type inequality, which can not be true for Cauchy problems.

The main purpose of this paper is to study the large time asymptotic behavior toward contact discontinuities for solutions to initial value problems for the Navier-Stokes system (1.1), and to show that the contact discontinuities are metastable wave patterns for the compressible Navier-Stokes system (1.1) as conjectured in [23]. We will show that for a weak contact discontinuity for the compressible Euler system (1.3), one can construct a viscous contact wave for the Navier-Stokes system (1.1), which is smooth, solves (1.1) asymptotically, and approximates the given contact discontinuity on any finite time interval, and such a viscous contact wave is nonlinearly stable under small initial perturbation with zero mass condition. Here the stability is in super-norm and a rate of convergence can also be obtained. The precise statement of our main results can be found with Theorem 1.1 below.

We now construct the contact wave $(\bar{v}, \bar{u}, \bar{\theta})(x,t)$ for the compressible N-S equations. First, consider the corresponding Euler equations (1.3) with the Riemann initial data

$$\begin{cases} (v,u,\theta)(x,0) = (v_-,0,\theta_-), \ if \ x<0, \\ (v,u,\theta)(x,0) = (v_+,0,\theta_+), \ if \ x>0, \end{cases} \tag{1.4}$$

where $v_\pm$ and $\theta_\pm$ are given positive constants. It is known (see [20]) that the Riemann problem (1.3), (1.4) admits a contact discontinuity

$$(\bar{V}, \bar{U}, \bar{\Theta})(x,t) = \begin{cases} (v_-,0,\theta_-), \ x<0, \\ (v_+,0,\theta_+), \ x>0, \end{cases} \tag{1.5}$$

provided that

$$p_- = \frac{R\theta_-}{v_-} = p_+ = \frac{R\theta_+}{v_+}. \tag{1.6}$$

Motivated by (1.5) and (1.6), we expect

$$\bar{p} = \frac{R\bar{\theta}}{\bar{v}} \approx p_+, \quad |\bar{u}|^2 << 1, \tag{1.7}$$

sufficiently fast. Then the leading order of the energy equation $(1.1)_3$ is

$$\frac{R}{\gamma-1}\theta_t + p_+ u_x = \kappa(\frac{\theta_x}{v})_x. \tag{1.8}$$

Plugging (1.7) into (1.8) and using the mass equation $(1.1)_1$, one has the following nonlinear diffusion equation,

$$\theta_t = a(\frac{\theta_x}{\theta})_x, \quad a = \frac{\kappa p_+(\gamma-1)}{\gamma R^2} > 0. \tag{1.9}$$

From [1] and [3], the nonlinear equation (1.9) admits a unique self similarity solution $\Theta(\xi), \xi = \frac{x}{\sqrt{1+t}}$ with the following boundary condition

$$\Theta(-\infty,t) = \theta_-, \quad \Theta(+\infty,t) = \theta_+.$$

Furthermore, $\Theta(\xi)$ is a monotone function, increasing if $\theta_+ > \theta_-$ and decreasing if $\theta_+ < \theta_-$. Let $\delta = |\theta_+ - \theta_-|$, then Θ satisfies

$$|\Theta_x| = O(\delta)(1+t)^{-\frac{1}{2}} e^{-\frac{\theta_\pm x^2}{4a(1+t)}}, \quad as \ x \to \pm\infty. \tag{1.10}$$

We now define the contact wave $(\bar v,\bar u,\bar\theta)$ for the Naiver-Stokes equation as follows. Suggested by (1.7), we set $\bar v=\frac{R}{p_+}\Theta$. Since there is no dissipation in the equation of mass conservation, $(1.1)_1$, so we require that the mass is conserved for the contact wave. Thus, we set $\bar u=\frac{Ra}{p_+\Theta}\Theta_x$. To conserve the total energy, one sets $\bar\theta=\Theta-\frac{\gamma-1}{2R}\bar u^2$. Thus,

$$\bar v=\frac{R}{p_+}\Theta,\quad \bar u=\frac{Ra}{p_+\Theta}\Theta_x,\quad \bar\theta=\Theta-\frac{\gamma-1}{2R}\bar u^2. \tag{1.11}$$

It is straightforward to check that $(\bar v,\bar u,\bar\theta)$ has the following property

$$\|(\bar v-\bar V,\bar u-\bar U,\bar\theta-\bar\Theta)\|_{L^p}=O(\kappa^{\frac{1}{2p}})(1+t)^{\frac{1}{2p}},\ p\ge 1,$$

which means the contact wave $(\bar v,\bar u,\bar\theta)$ for the Navier-Stokes system (1.1) approximates the contact discontinuity $(\bar V,\bar U,\bar\Theta)$ to the Euler equation (1.3) in L^p norm, $p\ge1$ on any finite time interval as κ tends to zero. More importantly, the contact wave $(\bar v,\bar u,\bar\theta)$ solves the Navier-Stokes system (1.1) time asymptotically, i.e.,

$$\begin{cases}\bar v_t-\bar u_x=0,\\ \bar u_t+\bar p_x=\mu(\frac{\bar u_x}{\bar v})_x+R_{1x},\\ (\bar e+\frac{\bar u^2}{2})_t+(\bar p\bar u)_x=\kappa(\frac{\bar\theta_x}{\bar v})_x+(\frac{\mu}{\bar v}\bar u\bar u_x)_x+R_{2x},\end{cases} \tag{1.12}$$

where

$$R_1=(\frac{\kappa(\gamma-1)}{\gamma R}-\mu)\frac{\bar u_x}{\bar v}+\bar p-p_+=O(\delta)(1+t)^{-1}e^{-\frac{\theta_\pm x^2}{4a(1+t)}},\ as\ |x|\to\infty, \tag{1.13}$$

$$R_2=(\frac{\kappa(\gamma-1)}{R}-\mu)\frac{\bar u\bar u_x}{\bar v}+(\bar p-p_+)\bar u=O(\delta)(1+t)^{-3/2}e^{-\frac{\theta_\pm x^2}{4a(1+t)}},\ as\ |x|\to\infty, \tag{1.14}$$

which can be verified by a direct computation. It should be noted that in (1.12), the larger error term in R_1 is not integrable in time due to its rate of decay as $O(\frac{1}{1+t})$. This is one of the main technical difficulties to be overcome. One of key observations in this paper is that the worst error term R_1 presents only in the equation of conservation of momentum, so its effect can be controlled by the intrinsic dissipation in the nonlinear sound wave families and the viscosity and the heat conductivity in the compressible Navier-Stokes system, which will be sufficient for the super-norm stability for the contact wave $(\bar v,\bar u,\bar\theta)$ as we show later.

We are now ready to state our main results. Let (v,u,θ) be the solutions to the Cauchy problem for the compressible Navier-Stokes equations (1.1) with the initial data

$$(v,u,\theta)(x,t=0)=(v_0,u_0,\theta_0)(x). \tag{1.15}$$

Denote by $(\phi,\psi,\zeta)(x,t)$ the derivation of the solution (v,u,θ) from the contact wave $(\bar v,\bar u,\bar\theta)$, i.e.,

$$(\phi,\psi,\zeta)(x,t)=(v-\bar v,u-\bar u,\theta-\bar\theta)(x,t). \tag{1.16}$$

In order to make using of the structure of the underlying contact wave, exploit the intrinsic dissipation in the compressible Navier-Stokes system, and overcome the strong nonlinearity, we introduce the anti-derivative variables:

$$(\Phi, \Psi, \bar{W})(x,t) = \int_{-\infty}^{x} (\phi, \psi, e + \frac{|u|^2}{2} - \bar{e} - \frac{|\bar{u}|^2}{2})(y,t)dy, \tag{1.17}$$

as is motivated by the theory of viscous shock waves [9]. Due to the conservative structure of both the Navier-Stokes system (1.1) and the approximate system (1.12), it is expected that $(\Phi, \Psi, \bar{W})(\cdot, t) \in L^2(R^1)$ for all $t > 0$ provided that the initial excessive mass is zero, i.e.,

$$(\Phi, \Psi, \bar{W})(+\infty, 0) = \int_{-\infty}^{+\infty} (v_0(x) - \bar{v}(x,0), u_0(x) - \bar{u}(x,0), E_0(x) - \bar{E}(x,0))dx = (0,0,0), \tag{1.18}$$

where $E = e + \frac{|u|^2}{2}$ is the total energy. Then our main results can be stated as follows:

Theorem 1.1. Let $(\bar{v}, \bar{u}, \bar{\theta})(x,t)$ be the contact wave defined in (1.11) with strength $\delta = |\theta_+ - \theta_-| \leq \delta_0$ for some small positive constant δ_0. Then there exists a small positive constant ϵ, such that if the initial data (v_0, u_0, θ_0) satisfies

$$\|(\Phi, \Psi, \bar{W})\|_{L^2} + \|(\phi, \psi, \zeta)\|_{H^1} \leq \epsilon, \tag{1.19}$$

then the system (1.1) admits a unique global solution $(v, u, \theta)(x,t)$ satisfying

$$(\Phi, \Psi, \bar{W}) \in C(0, +\infty; H^2), \tag{1.20}$$

$$\phi \in L^2(0, +\infty; H^1), \tag{1.21}$$

$$(\psi, \zeta) \in L^2(0, +\infty; H^2), \tag{1.22}$$

where the perturbation $(\Phi, \Psi, \bar{W})$ and (ϕ, ψ, ζ) are defined in (1.17) and (1.16). Furthermore, the perturbation $(\Phi, \Psi, \bar{W})$ and (ϕ, ψ, ζ) have the following decay rate,

$$\|(\Phi, \Psi, \bar{W}\|_{L^\infty} \leq C(\epsilon + \delta^{\frac{1}{4}})(1+t)^{-\frac{1}{8} + \bar{C}_0\sqrt{\delta}}, \tag{1.23}$$

$$\|(v - \bar{v}, u - \bar{u}, \theta - \bar{\theta}\|_{L^\infty} \leq C(\epsilon + \delta^{\frac{1}{4}})(1+t)^{-\frac{1}{4}}, \tag{1.24}$$

where $\bar{C}_0$ is a positive constant independent of time.

A few remarks are in order:

Remark 1.2. Theorem 1.1 shows not only that the viscous contact wave $(\bar{v}, \bar{u}, \bar{\theta})$ is nonlinear stable in super-norm with initial perturbations satisfying (1.18), but also a uniform rate of convergence (1.24), is obtained. This is somewhat surprising given that the convergence rate to either the viscous shock wave or viscous rarefaction wave has not been achieved yet for the compressible Navier-Stokes system, see [9], [14], [19]. Moreover, the rate of decay in (1.24) may not be optimal. Motivated by the pointwise behavior toward viscous contact waves for solutions to the Euler system with uniform viscosity (see [23] and [15]), one would conjecture that its decay rate in (1.24) should be improved to $(1+t)^{-\frac{1}{2}}$. However, this will be left for future.

Remark 1.3. The major assumption in our stability theory is the initial zero excessive mass condition, (1.18) (or (1.19)), which excludes the possible presence of diffusion waves in the sound wave families. As it is shown in [23] and [15] for the compressible Euler equations with uniform viscosity, a generic perturbation of a viscous contact wave introduces not only a shift with center of the viscous contact wave, but also nonlinear and linear diffusion waves. Although, it is expected that the same phenomena holds true for the compressible Navier-Stokes system (1.1), yet the fine accurate asymptotic ansatz as in [23] and [15] may not be necessary for the stability theory toward contact waves in the super-norm. Indeed, one of key ideas in this paper is that though it is necessary for point-wise behavior and L^p stability (for $1 \leq p < \infty$) to have more accurate asymptotic ansatz as in [23] and [15], yet the asymptotic behavior toward contact waves in the super-norm can be obtained without the detailed construction of accurate ansatz. This approach seems to work even for generic initial perturbations (i.e., without the zero excessive mass condition (1.18)). However, this will be reported in a forthcoming paper.

Finally, we comment on some of the main difficulties and techniques involved in studying the problem of asymptotic behavior toward the contact waves of solutions to the Cauchy problem for the compressible Navier-Stokes equations, which has been open for quite a while. In contrast to the stability theory of shock profiles [9] and viscous rarefaction waves [14], where the strict monotonicity of the corresponding characteristic speed along the underlying wave play the leading role for the stability analysis, the characteristic speed along the contact wave is constant, and the spatial derivative of the velocity changes sign for the contact waves. These are some of the main difficulties to overcome here. To this end, we will employ the following strategy. First, as it is first observed by Xin in [23] (see also [15]), the characteristic speeds of the sound wave families are strictly monotone across the contact wave which can yield an intrinsic dissipation besides the physical viscosity and heat conductivity in (1.1) by a careful weighted characteristic-energy method. As shown in section 3, see also [23], this intrinsic dissipation yields control on

$$\int\int |\Theta_x|(b_1^2 + b_3^2)dxdt. \tag{1.25}$$

Second, to make use of the intrinsic dissipation and control the nonlinear term, we introduce the anti-derivative of the perturbation (ϕ, ψ, ζ) as dependent variables and work with the integrated error system. Then the standard classical energy method involves the estimates of $R_1\Psi$ and terms with $\bar{u}_x$. Note that the estimate of $R_1\Psi$ is nontrivial since it is equivalent to $\frac{1}{1+t}\Psi$ due to the structure of the contact wave. Fortunately, the term $R_1\Psi$ will be estimated by making use of the control on (1.25) from the intrinsic dissipation in section 3. To overcome the difficulty that $\bar{u}_x$ may change sign, we simply bound it by $C\delta(1+t)^{-1}$ in the estimate, where δ is the strength of the contact wave and C is a uniform constant. With these ideas in mind, we can obtain the energy estimates of the form

$$f_t + \int |\Theta_x|(b_1^2 + b_3^2)dx + g \le C\sqrt{\delta}(1+t)^{-1}(f + \sqrt{\delta}), \tag{1.26}$$

$$h_t \le C\delta(1+t)^{-1}g + C\delta(1+t)^{-\frac{3}{2}}. \tag{1.27}$$

where $f > 0$ contains the square of the L^2 norm of $(\Phi, \Psi, \bar{W})$ and g and h are positive and contain the square of the L^2 norm of $(\Phi_x, \Psi_x, \bar{W}_x)$. One of key observations in this paper is that the square of the L^2 norm of $(\Phi, \Psi, \bar{W})$ may grow in time in the order of $(1+t)^{C\sqrt{\delta}}$, which is slow if the strength of the contact wave, δ, is small, the estimate (1.27) will enable us to obtain the higher order decay in time for the L^2-norms of higher order derivatives. This, in turns, will lead to the desired L^∞-stability estimates in (1.23) and (1.24). It should be noted that the smallness assumption on the wave strength is essential here.

The rest of the paper will be arranged as follows. In Section 2, the compressible Navier-Stokes equations is reformulated to an integrated system. And the Section 3 is devoted to the lower order estimate, while the Section 4 is for the derivative estimate. The stability and convergence rate of the contact discontinuity will be given in Section 5.

2 Reformulated system

To prove the main theorem, we derive the system for the perturbation (ϕ, ψ, ζ) around the contact wave $(\bar{v}, \bar{u}, \bar{\theta})$. Set

$$\phi = v - \bar{v}, \psi = u - \bar{u}, \zeta = \theta - \bar{\theta}, \tag{2.1}$$

and

$$\begin{aligned} \Phi &= \int_{-\infty}^{x} \phi(y,t)dy, \Psi = \int_{-\infty}^{x} \psi(y,t)dy, \\ \bar{W} &= \int_{-\infty}^{x} (e + \frac{|u|^2}{2} - \bar{e} - \frac{|\bar{u}|^2}{2})(y,t)dy. \end{aligned} \tag{2.2}$$

As mentioned in the introduction, we impose $\Phi(\infty, 0) = \Psi(\infty, 0) = \bar{W}(\infty, 0) = 0$ so that the quantities Φ, Ψ and $\bar{W}$ can be defined in some Sobolev space. Naturally, we have $(\phi, \psi) = (\Phi, \Psi)_x$ and $\frac{R}{\gamma-1}\zeta + \frac{1}{2}|\Psi_x|^2 + \bar{u}\Psi_x = \bar{W}_x$.

Subtracting (1.12) from the equation (1.1) and integrating the resulting system yield the following integrated error equations for $(\phi, \psi, \bar{W})$:

$$\begin{cases} \Phi_t - \Psi_x = 0, \\ \Psi_t + p - \bar{p} = \frac{\mu}{v}u_x - \frac{\mu}{\bar{v}}\bar{u}_x - R_1, \\ \bar{W}_t + pu - \bar{p}\bar{u} = \frac{\kappa}{v}\theta_x - \frac{\kappa}{\bar{v}}\bar{\theta}_x + \frac{\mu}{v}uu_x - \frac{\mu}{\bar{v}}\bar{u}\bar{u}_x - R_2. \end{cases} \tag{2.3}$$

Instead of the variable $\bar{W}$, which is the anti-derivative of the total energy, it is more convenient to introduce another variable related to the temperature,

$$W = \frac{\gamma - 1}{R}(\bar{W} - \bar{u}\Psi). \tag{2.4}$$

It follows that

$$\zeta = W_x - Y, \text{ with } Y = \frac{\gamma-1}{R}(\frac{1}{2}\Psi_x^2 - \bar{u}_x\Psi). \tag{2.5}$$

In terms of the new variable W, we can rewrite the system (2.3) as

$$\begin{cases} \Phi_t - \Psi_x = 0, \\ \Psi_t - \frac{p_+}{\bar{v}}\Phi_x + \frac{R}{\bar{v}}W_x = \frac{\mu}{\bar{v}}\Psi_{xx} + (\frac{\mu}{v} - \frac{\mu}{\bar{v}})u_x + J_1 + \frac{R}{\bar{v}}Y - R_1 \doteq \frac{\mu}{\bar{v}}\Psi_{xx} + Q_1, \\ \frac{R}{\gamma-1}W_t + p_+\Psi_x = \frac{\kappa}{\bar{v}}W_{xx} + (\frac{\kappa}{v} - \frac{\kappa}{\bar{v}})\theta_x + \frac{\mu u_x}{v}\Psi_x - R_2 - \bar{u}_t\Psi + J_2 \\ \quad +\bar{u}R_1 - \frac{\kappa}{\bar{v}}Y_x \doteq \frac{\kappa}{\bar{v}}W_{xx} + Q_2, \end{cases} \tag{2.6}$$

where

$$J_1 = \frac{\bar{p}-p_+}{\bar{v}}\Phi_x - [p - \bar{p} + \frac{\bar{p}}{\bar{v}}\Phi_x - \frac{R}{\bar{v}}(\theta - \bar{\theta})] = O(1)(\Phi_x^2 + W_x^2 + Y^2 + |\bar{u}|^4), \tag{2.7}$$

$$J_2 = (p_+ - p)\Psi_x = O(1)(\Phi_x^2 + \Psi_x^2 + W_x^2 + Y^2 + |\bar{u}|^4), \tag{2.8}$$

$$Q_1 = (\frac{\mu}{v} - \frac{\mu}{\bar{v}})u_x + J_1 + \frac{R}{\bar{v}}Y - R_1, \tag{2.9}$$

$$Q_2 = (\frac{\kappa}{v} - \frac{\kappa}{\bar{v}})\theta_x + \frac{\mu u_x}{v}\Psi_x - R_2 - \bar{u}_t\Psi + \bar{u}R_1 + J_2 - \frac{\kappa}{\bar{v}}Y_x. \tag{2.10}$$

In the next section, we will focus on the Cauchy problem for the reformulated system (2.6). Since the local existence is well known, we omit it here for brevity. To prove the global existence, we only need to close the following a priori estimate:

$$N(T) = \sup_{0\le t\le T}\{\|(\Phi,\Psi,W)\|_{L^\infty}^2 + \|(\phi,\psi,\zeta)\|_{L^2}^2 + \|(\phi_x,\psi_x,\zeta_x)\|_{L^2}^2\} \le \varepsilon_0^2, \tag{2.11}$$

where ε_0 is positive small constant depending on the initial data and the strength of the contact wave.

3 Lower order estimate

This section is devoted to the lower order estimates. We start with the elementary energy estimates. Multiplying $(2.6)_1$ by $p_+\Phi$, $(2.6)_2$ by $\bar{v}\Psi$, $(2.6)_3$ by $\frac{R}{p_+}W$ respectively and adding all the resulting equations, we have

$$\begin{aligned} &(\frac{p_+}{2}\Phi^2 + \frac{R^2}{2(\gamma-1)p_+}W^2 + \frac{\bar{v}}{2}\Psi^2)_t + \mu\Psi_x^2 + \frac{R\kappa}{p_+\bar{v}}W_x^2 \\ &= \frac{1}{2}\bar{v}_t\Psi^2 + \bar{v}Q_1\Psi - (\frac{R\kappa}{p_+\bar{v}})_x WW_x + \frac{R}{p_+}WQ_2 + (\cdots)_x, \end{aligned} \tag{3.1}$$

here and in the sequel the notation $(\cdots)_x$ represents the term in the conservative form so that it vanishes after integration. Since it has no effect on the energy estimates, we do not write them out in details for simplicity.

Note that the term $\bar{v}Q_1\Psi$ contains $(1+t)^{-1}\Psi$ which can not be controlled by the dissipation from the viscosity and heat conductivity. As is explained in Section 1, we will control $\bar{v}Q_1\Psi$ by exploiting an intrinsic dissipation associated with the contact wave: the strict monotonicity of the characteristic speeds in the sound wave families. This is achieved by a weighted characteristic energy method as in [23]. To this end, We define

$$m = (\Phi, \Psi, W)^t, \tag{3.2}$$

where $(\cdot,\cdot,\cdot)^t$ means the transpose of the vector $(\cdot,\cdot,\cdot)$, then from (2.6), we have

$$m_t + A_1 m_x = A_2 m_{xx} + A_3, \tag{3.3}$$

where

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{p_+}{\bar{v}} & 0 & \frac{R}{\bar{v}} \\ 0 & \frac{\gamma-1}{R}p_+ & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\mu}{\bar{v}} & 0 \\ 0 & 0 & \frac{\kappa(\gamma-1)}{R\bar{v}} \end{pmatrix}, \tag{3.4}$$

$$A_3 = (0, Q_1, \frac{\gamma-1}{R}Q_2)^t. \tag{3.5}$$

A direct computation shows that the eigenvalues of the matrix A_1 are $\lambda_1, 0, \lambda_3$. Here $\lambda_3 = -\lambda_1 = \sqrt{\frac{\gamma p_+}{\bar{v}}}$. The corresponding normalized left and right eigenvectors can be chosen as

$$l_1 = \sqrt{\frac{1}{2\gamma}}(-1, -\frac{\gamma}{\lambda_3}, \frac{R}{p_+}),\ l_2 = \sqrt{\frac{\gamma-1}{\gamma}}(1, 0, \frac{R}{(\gamma-1)p_+}),\ l_3 = \sqrt{\frac{1}{2\gamma}}(-1, \frac{\gamma}{\lambda_3}, \frac{R}{p_+}), \tag{3.6}$$

$$r_1 = \sqrt{\frac{1}{2\gamma}}(-1, -\lambda_3, \frac{\gamma-1}{R}p_+)^t,\ r_2 = \sqrt{\frac{\gamma-1}{\gamma}}(1, 0, \frac{p_+}{R})^t,\ r_3 = \sqrt{\frac{1}{2\gamma}}(-1, \lambda_3, \frac{\gamma-1}{R}p_+)^t, \tag{3.7}$$

so that,

$$l_i r_j = \delta_{ij},\ i,j = 1,2,3, \quad LA_1R = \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{3.8}$$

where

$$L = (l_1, l_2, l_3)^t,\ R = (r_1, r_2, r_3).$$

Let

$$B = Lm = (b_1, b_2, b_3), \tag{3.9}$$

then multiplying the equations (3.3) by the matrix L yields that

$$B_t + \Lambda B_x = LA_2RB_{xx} + 2LA_2R_xB_x + [(L_t + \Lambda L_x)R + LA_2R_{xx}]B + LA_3. \tag{3.10}$$

We compute

$$LA_2R = A_4 = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{12} \\ b_{13} & b_{12} & b_{11} \end{pmatrix}, \tag{3.11}$$

with

$$\bar{v}b_{11}=\frac{\mu}{2}+\frac{(\gamma-1)^2\kappa}{2\gamma R},\quad \bar{v}b_{12}=\sqrt{\frac{\gamma-1}{2}}\frac{(\gamma-1)\kappa}{\gamma R}, \tag{3.12}$$

$$\bar{v}b_{13}=-\frac{\mu}{2}+\frac{(\gamma-1)^2\kappa}{2\gamma R},\quad \bar{v}b_{22}=\frac{(\gamma-1)\kappa}{\gamma R}. \tag{3.13}$$

A direct but tedious computation shows that the determinant of the matrix $A_4-\lambda I$ is

$$-\lambda^3+(2b_{11}+b_{22})\lambda^2+(2b_{12}^2-2b_{11}b_{22}-b_{11}^2+b_{13}^2)\lambda+(b_{11}^2-b_{13}^2)b_{22}+2b_{12}^2(b_{13}-b_{11}),$$

or more precisely

$$-\lambda(\lambda-\frac{\mu}{\bar{v}})(\lambda-\frac{\kappa(\gamma-1)}{R\bar{v}}).$$

This implies that the symmetric matrix A_4 is nonnegative and its eigenvalues are $0,\frac{\mu}{\bar{v}},\frac{\kappa(\gamma-1)}{R\bar{v}}$. An L^2-estimate on B will be derived from (3.10) by a weighted energy method based on the intrinsic dissipation. For definiteness, we assume that $\Theta_x>0$. The case when $\Theta_x<0$ can be discussed similarly. Let $v_1=\Theta/\theta_+$, then $|v_1-1|\le C\delta$. Multiplying (3.10) by $\bar{B}=(v_1^nb_1,b_2,v_1^{-n}b_3)$ with a large positive integer n which will be chosen later, we have

$$\begin{aligned}&(\frac{v_1^n}{2}b_1^2+\frac{1}{2}b_2^2+\frac{v_1^{-n}}{2}b_3^2)_t-(\frac{v_1^n}{2})_tb_1^2-(\frac{v_1^{-n}}{2})_tb_3^2+\bar{B}_xA_4B_x+\bar{B}A_{4x}B_x\\&-\frac{v_1^{n-1}}{2}(n\lambda_1v_{1x}+v_1\lambda_{1x})b_1^2+\frac{v_1^{-n-1}}{2}(n\lambda_3v_{1x}-v_1\lambda_{3x})b_3^2+(\cdots)_x\\&=2\bar{B}LA_2R_xB_x+\bar{B}[L_tR+LA_2R_{xx}]B+\bar{B}\Lambda L_xRB+\bar{B}LA_3.\end{aligned} \tag{3.14}$$

Let

$$E_1=\int(\frac{p_+}{2}\Phi^2+\frac{R^2}{2(\gamma-1)p_+}W^2+\frac{\bar{v}}{2}\Psi^2)dx+\int(\frac{v_1^n}{2}b_1^2+\frac{1}{2}b_2^2+\frac{v_1^{-n}}{2}b_3^2)dx. \tag{3.15}$$

$$K_1=\int(\mu\Psi_x^2+\frac{R\kappa}{p_+\bar{v}}W_x^2+B_xA_4B_x)dx. \tag{3.16}$$

Note that

$$\begin{aligned}|\int(\bar{B}-B)_xA_4B_xdx|&\le C\delta\int|B_x|^2dx+C\delta\int|B|^2|\Theta_x|^2dx\\&\le C\delta(1+t)^{-1}E_1+C\delta K_1+C\delta\int|\Phi_x|^2dx.\end{aligned} \tag{3.17}$$

Similarly, the terms $\bar{v}_t\Psi^2$, $(\frac{1}{\bar{v}})_xWW_x$, $\bar{B}A_{4x}B_x$, $\bar{B}LA_2R_xB_x$ and $\bar{B}[L_tR+LA_2R_{xx}]B$ satisfy the same estimate. The integrals involving $\bar{B}\Lambda L_xRB$ and $\bar{B}LA_3$ are more difficult and will be studied in more details. By the choice of the characteristic matrix L and R, we have

$$\Lambda L_xR=\frac{1}{2}\lambda_{3x}\begin{pmatrix}1&0&-1\\0&0&0\\1&0&-1\end{pmatrix}, \tag{3.18}$$

$$LA_3 = \begin{pmatrix} \sqrt{\frac{1}{2\gamma}}\frac{\gamma-1}{p_+}Q_2 - \sqrt{\frac{\gamma}{2}}\frac{Q_1}{\lambda_3} \\ \sqrt{\frac{\gamma-1}{\gamma}}\frac{Q_2}{p_+} \\ \sqrt{\frac{1}{2\gamma}}\frac{\gamma-1}{p_+}Q_2 + \sqrt{\frac{\gamma}{2}}\frac{Q_1}{\lambda_3} \end{pmatrix}. \tag{3.19}$$

Thus,

$$\bar{B}\Lambda L_x RB = \frac{1}{2}\lambda_{3x}(v_1^n b_1^2 + v_1^{-n} b_1 b_3 - v_1^n b_1 b_3 - v_1^{-n} b_3^2), \tag{3.20}$$

$$\bar{B}LA_3 = q_1 v_1^n b_1 + q_2 b_2 + q_3 v_1^{-n} b_3, \tag{3.21}$$

where

$$q_1 = \sqrt{\frac{1}{2\gamma}}\frac{\gamma-1}{p_+}Q_2 - \sqrt{\frac{\gamma}{2}}\frac{Q_1}{\lambda_3},\ q_2 = \sqrt{\frac{\gamma-1}{\gamma}}\frac{Q_2}{p_+},\ q_3 = \sqrt{\frac{1}{2\gamma}}\frac{\gamma-1}{p_+}Q_2 + \sqrt{\frac{\gamma}{2}}\frac{Q_1}{\lambda_3}. \tag{3.22}$$

Combining (3.1), (3.14), (3.17), (3.20-3.21) and using the Cauchy inequality, we have by choosing n sufficiently large,

$$\begin{aligned} &E_{1t} + \frac{1}{2}K_1 + 2\int \Theta_x(b_1^2 + b_3^2)dx \le C\delta(1+t)^{-1}E_1 + C\delta K_1 \\ &+C\delta\int \Phi_x^2 dx + I, \end{aligned} \tag{3.23}$$

where

$$I = \int \bar{v}Q_1\Psi dx + \int \frac{R}{p_+}WQ_2 dx + \int (q_1 v_1^n b_1 + q_2 b_2 + q_3 v_1^{-n} b_3)dx. \tag{3.24}$$

Here we have used the fact that, for n large enough,

$$-\frac{1}{2}v_1^{n-1}(n\lambda_1 v_{1x} + v_1\lambda_{1x}) + \frac{1}{2}v_1^{-n-1}(n\lambda_3 v_{1x} - v_1\lambda_{3x}) - \bar{B}\Lambda L_x RB > 2\Theta_x(b_1^2 + b_3^2), \tag{3.25}$$

and v_1^n and v_1^{-n} are closed to 1.

From (3.22), the estimate for I is equivalent to the one for

$$\int |Q_i||b_j|dx, \quad i = 1, 2, \quad j = 1, 3, \tag{3.26}$$

and

$$\int |Q_2||b_2|dx, \tag{3.27}$$

because

$$\Psi = \sqrt{\frac{1}{2\gamma}}\lambda_3(b_3 - b_1). \tag{3.28}$$

Although Q_1 contains the term R_1 with the decay rate $\frac{1}{1+t}$, the terms in (3.26) involving Q_1 can be estimated by the intrinsic dissipation on b_1 and b_3 as shown later. Note that there is no intrinsic dissipation on b_2. Fortunately, there is no Q_1 in (3.27) and Q_2 contains the term R_2 which has a better decay rate $(1+t)^{-3/2}$. For brevity, we only estimate $\int |Q_1||b_1|dx$ and $\int |Q_2||b_2|dx$ as follows for illustration.

Estimation on $\int |Q_1||b_1|dx$:

It follows from (2.9) that

$$\int |Q_1||b_1|dx \le \int |(\frac{\mu}{v}-\frac{\mu}{\bar v})u_x + J_1 + \frac{R}{\bar v}Y||b_1|dx + \int |R_1||b_1|dx =: I_1 + I_2. \tag{3.29}$$

Since

$$\int |J_1||b_1|dx \le C\varepsilon_0(\|\Phi_x\|^2 + K_1) + C\delta(1+t)^{-1}E_1 + C\delta(1+t)^{-\frac{5}{2}}, \tag{3.30}$$

and

$$\begin{aligned}\int |(\frac{\mu}{v}-\frac{\mu}{\bar v})u_x||b_1|dx + \int |Y||b_1|dx \le C\varepsilon_0 K_1 + C(\delta+\varepsilon_0)\|\Phi_x\|^2 \\ +C\varepsilon_0\|\psi_x\|^2 + C\delta(1+t)^{-1}E_1,\end{aligned} \tag{3.31}$$

we obtain

$$I_1 \le C(\delta+\varepsilon_0)(\|\Phi_x\|^2 + K_1) + C\delta(1+t)^{-1}E_1 + C\varepsilon_0\|\psi_x\|^2 + C\delta(1+t)^{-\frac{5}{2}}. \tag{3.32}$$

On the other hand, (1.13) and the Cauchy inequality give

$$I_2 = \int |R_1|b_1|dx \le C\delta \int \Theta_x b_1^2 dx + C\delta(1+t)^{-1}, \tag{3.33}$$

which, together with (3.32), yields

$$\begin{aligned}\int |Q_1||b_1|dx \le\ & C\delta \int \Theta_x b_1^2 dx + C\delta(1+t)^{-1}(E_1+1) \\ & +C(\delta+\varepsilon_0)(\|\Phi_x\|^2 + K_1) + C\varepsilon_0\|\psi_x\|^2.\end{aligned} \tag{3.34}$$

Estimation on $\int |Q_2||b_2|dx$:

Note that

$$\begin{aligned}\int |Q_2||b_2|dx \le\ & \int |(\frac{\kappa}{v}-\frac{\kappa}{\bar v})\theta_x + \frac{\mu u_x}{v}\Psi_x - \bar u_t\Psi + J_2 - \frac{\kappa}{\bar v}Y_x||b_2|dx \\ & + \int |\bar u R_1 - R_2||b_2|dx,\end{aligned} \tag{3.35}$$

due to (2.10). The Cauchy inequality yields

$$\int |Y_x||b_2| + |\bar u R_1 - R_2||b_2|dx \le C(\delta+\varepsilon_0)(K_1 + \|\psi_x\|^2) + C\delta(1+t)^{-1}E_1 + C\delta(1+t)^{-\frac{3}{2}}, \tag{3.36}$$

and

$$\begin{aligned}\int |(\frac{\kappa}{v}-\frac{\kappa}{\bar v})\theta_x + \frac{\mu u_x}{v}\Psi_x||b_2|dx \le\ & C\delta(1+t)^{-1}E_1 + C(\delta+\varepsilon_0)(\|\Phi_x\|^2 + K_1) \\ & +C\varepsilon_0\|(\phi,\psi,\zeta)_x\|^2.\end{aligned} \tag{3.37}$$

Notice that the estimate for $\int |J_2||b_2|dx$ is the same with the one for $\int |J_1||b_1|dx$. Thus, combining (3.30), (3.36) and (3.37) and using the Cauchy inequality, we have

$$\begin{aligned}\int |Q_2||b_2|dx \leq\ & C(\delta+\varepsilon_0)(\|\Phi_x\|^2+K_1+\|(\phi,\psi,\zeta)_x\|^2)\\ &+C\delta(1+t)^{-1}E_1+C\delta(1+t)^{-\frac{3}{2}}.\end{aligned} \tag{3.38}$$

Collecting the estimates (3.23), (3.34) and (3.38), we arrive at our first inequality as follows:

$$E_{1t}+\frac{K_1}{4}+\int \Theta_x(b_1^2+b_3^2)dx \leq C\delta(1+t)^{-1}(E_1+1)+C(\delta+\varepsilon_0)(\|\Phi_x\|^2+\|(\phi,\psi,\zeta)_x\|^2). \tag{3.39}$$

Note that the norm $\|\Phi_x\|$ is not included in K_1. To complete the lower order inequality, we need to estimate Φ_x. From $(2.6)_2$, we have

$$\frac{\mu}{\bar v}\Phi_{xt}-\Psi_t+\frac{p_+}{\bar v}\Phi_x=\frac{R}{\bar v}W_x-Q_1. \tag{3.40}$$

Multiplying (3.40) by Φ_x yields

$$(\frac{\mu}{2\bar v}\Phi_x^2)_t-(\frac{\mu}{2\bar v})_t\Phi_x^2-\Phi_x\Psi_t+\frac{p_+}{\bar v}\Phi_x^2=(\frac{R}{\bar v}W_x-Q_1)\Phi_x. \tag{3.41}$$

Since

$$\Phi_x\Psi_t=(\Phi_x\Psi)_t-(\Phi_t\Psi)_x+\Psi_x^2, \tag{3.42}$$

we obtain

$$(\int \frac{\mu}{2\bar v}\Phi_x^2-\Phi_x\Psi dx)_t+\int \frac{p_+}{2\bar v}\Phi_x^2dx \leq C\int(\Psi_x^2+W_x^2)dx+C\int Q_1^2dx. \tag{3.43}$$

On the other hand, (2.9) and the Cauchy inequality yield that

$$\int Q_1^2dx \leq C\varepsilon_0(K_1+\|\Phi_x\|^2)+C\delta(1+t)^{-\frac{3}{2}}+C\varepsilon_0\|\psi_x\|^2. \tag{3.44}$$

Plugging (3.44) into (3.33) yields

$$(\int \frac{\mu}{2\bar v}\Phi_x^2-\Phi_x\Psi dx)_t+\int \frac{p_+}{4\bar v}\Phi_x^2dx \leq C_1K_1+C_1\delta(1+t)^{-3/2}+C_1\varepsilon_0\|\psi_x\|^2. \tag{3.45}$$

We now choose large constant $\bar C_1>1$ so that

$$\bar C_1E_1+\int \frac{\mu}{2\bar v}\Phi_x^2-\Phi_x\Psi dx \geq \frac{1}{2}\bar C_1E_1+\int \frac{\mu}{4\bar v}\Phi_x^2dx, \quad \frac{\bar C_1}{4}-C_1>\frac{\bar C_1}{8}. \tag{3.46}$$

Hence, by multiplying (3.39) by $\bar C_1$, we have

$$E_{2t}+K_2+\int \Theta_x(b_1^2+b_3^2)dx \leq C\delta(1+t)^{-1}(E_2+1)+C(\delta+\varepsilon_0)\|(\phi,\psi,\zeta)_x\|^2, \tag{3.47}$$

where

$$E_2=\bar C_1E_1+\int \frac{\mu}{2\bar v}\Phi_x^2-\Phi_x\Psi dx, \quad K_2=\frac{\bar C_1}{8}K_1+\int \frac{p_+}{8\bar v}\Phi_x^2dx. \tag{3.48}$$

4 Derivative estimate

To obtain the estimate for the first order derivative of (Φ_x, Ψ_x, W_x). We shall use an energy estimate based on the convex entropy for the Navier-Stokes equations. From (1.1) and (1.12), we have

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + (p-\bar{p})_x = (\frac{\mu}{v}u_x - \frac{\mu}{\bar{v}}\bar{u}_x)_x - R_{1x}, \\ \frac{R}{\gamma-1}\zeta_t + pu_x - \bar{p}\bar{u}_x = (\frac{\kappa}{v}\theta_x - \frac{\kappa}{\bar{v}}\bar{\theta}_x)_x + Q_3, \end{cases} \tag{4.1}$$

where

$$Q_3 = \frac{\mu}{v}u_x^2 - (\frac{\mu\bar{u}\bar{u}_x}{v})_x - R_{2x} + \frac{1}{2}(|\bar{u}|^2)_t + \bar{p}_x\bar{u}_1. \tag{4.2}$$

Multiplying $(4.1)_2$ by ψ, we have

$$(\frac{1}{2}\psi^2)_t - (p-\bar{p})\psi_x + (\frac{\mu}{v}u_x - \frac{\mu}{\bar{v}}\bar{u}_x)\psi_x = -R_{1x}\psi + (\cdots)_x. \tag{4.3}$$

Since $p-\bar{p} = R\bar{\theta}(\frac{1}{v} - \frac{1}{\bar{v}}) + \frac{R\zeta}{v}$, we get

$$(\frac{1}{2}\psi^2)_t - R\bar{\theta}(\frac{1}{v} - \frac{1}{\bar{v}})\phi_t - \frac{R}{v}\zeta\psi_x + \frac{\mu}{v}\psi_x^2 + (\frac{\mu}{v} - \frac{\mu}{\bar{v}})\bar{u}_x\psi_x = -R_{1x}\psi + (\cdots)_x. \tag{4.4}$$

Let

$$\hat{\Phi}(s) = s - 1 - \ln s. \tag{4.5}$$

It is easy to check that $\hat{\Phi}'(1) = 0$ and $\hat{\Phi}(s)$ is strictly convex around $s = 1$. Moreover,

$$\begin{aligned} &\{R\bar{\theta}\hat{\Phi}(\frac{v}{\bar{v}})\}_t = R\bar{\theta}_t\hat{\Phi}(\frac{v}{\bar{v}}) + R\bar{\theta}(-\frac{1}{v} + \frac{1}{\bar{v}})\phi_t \\ &+R\bar{\theta}(-\frac{v}{\bar{v}^2} + \frac{1}{\bar{v}})\bar{v}_t + R\bar{\theta}(-\frac{1}{v} + \frac{1}{\bar{v}})\bar{v}_t \\ &= R\bar{\theta}(-\frac{1}{v} + \frac{1}{\bar{v}})\phi_t - \bar{p}\hat{\Psi}(\frac{v}{\bar{v}})\bar{v}_t + \bar{v}\bar{p}_t\hat{\Phi}(\frac{v}{\bar{v}}), \end{aligned} \tag{4.6}$$

where

$$\hat{\Psi}(s) = s^{-1} - 1 + \ln s. \tag{4.7}$$

Substituting (4.6) into (4.4) yields

$$\begin{aligned} &(\frac{1}{2}\psi^2 + R\bar{\theta}\hat{\Phi}(\frac{v}{\bar{v}}))_t + \bar{p}\hat{\Psi}(\frac{v}{\bar{v}})\bar{v}_t - \frac{R}{v}\zeta\psi_x + \frac{\mu}{v}\psi_x^2 \\ &+(\frac{\mu}{v} - \frac{\mu}{\bar{v}})\bar{u}_x\psi_x = -R_{1x}\psi + \bar{v}\bar{p}_t\hat{\Phi}(\frac{v}{\bar{v}}) + (\cdots)_x. \end{aligned} \tag{4.8}$$

On the other hand, we calculate

$$[\bar{\theta}\hat{\Phi}(\frac{\theta}{\bar{\theta}})]_t = (1 - \frac{\bar{\theta}}{\theta})\zeta_t - \hat{\Psi}(\frac{\theta}{\bar{\theta}})\bar{\theta}_t, \tag{4.9}$$

and

$$\begin{aligned}
&\frac{R}{\gamma-1}(1-\frac{\bar\theta}{\theta})\zeta_t\\
=\;& (1-\frac{\bar\theta}{\theta})\{-pu_x+\bar p\bar u_x+(\frac{\kappa\theta_x}{v}-\frac{\kappa\bar\theta_x}{\bar v})_x+Q_3\}\\
=\;& -\frac{R}{v}\zeta\psi_x+\frac{\zeta}{\theta}(\bar p-p)\bar u_x-(\frac{\zeta}{\theta})_x(\frac{\kappa\theta_x}{v}-\frac{\kappa\bar\theta_x}{\bar v})+\frac{\zeta}{\theta}Q_3+(\cdots)_x\\
=\;& -\frac{R}{v}\zeta\psi_x+\frac{\zeta}{\theta}(\bar p-p)\bar u_x-\frac{\kappa}{v\theta}\zeta_x^2+\frac{\kappa\zeta_x\Phi_x}{v\bar v\theta}\bar\theta_x+\frac{\zeta\theta_x}{\theta^2}(\frac{\kappa\theta_x}{v}-\frac{\kappa\bar\theta_x}{\bar v})+\frac{\zeta}{\theta}Q_3+(\cdots)_x.
\end{aligned} \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.8) gives

$$\begin{aligned}
&(\frac{1}{2}\psi^2+R\bar\theta\hat\Phi(\frac{v}{\bar v})+\frac{R}{\gamma-1}\bar\theta\hat\Phi(\frac{\theta}{\bar\theta}))_t+\frac{\mu}{v}\psi_x^2+\frac{\kappa}{v\theta}\zeta_x^2\\
&=-\bar p\hat\Psi(\frac{v}{\bar v})\bar v_t+\bar v\bar p_t\hat\Phi(\frac{v}{\bar v})-(\frac{\mu}{v}-\frac{\mu}{\bar v})\bar u_x\psi_x-R_{1x}\psi+\frac{\zeta}{\theta}(\bar p-p)\bar u_x\\
&+\frac{\kappa\zeta_x\Phi_x}{v\bar v\theta}\bar\theta_x+\frac{\zeta\theta_x}{\theta^2}(\frac{\kappa\theta_x}{v}-\frac{\kappa\bar\theta_x}{\bar v})+\frac{\zeta}{\theta}Q_3-\frac{R}{\gamma-1}\hat\Psi(\frac{\theta}{\bar\theta})\bar\theta_t+(\cdots)_x.
\end{aligned} \tag{4.11}$$

Let

$$E_3=\int\frac{1}{2}\psi^2+R\bar\theta\hat\Phi(\frac{v}{\bar v})+\frac{R}{\gamma-1}\bar\theta\hat\Phi(\frac{\theta}{\bar\theta})dx \tag{4.12}$$

and

$$K_3=\int\frac{\mu}{v}\psi_x^2+\frac{\kappa}{v\theta}\zeta_x^2dx. \tag{4.13}$$

Note that $\hat\Phi(s)$ is strictly convex around $s=1$ so that there exist positive constants c_1 and c_2,

$$c_1\phi^2\le\hat\Phi(\frac{v}{\bar v})\le c_2\phi^2,\quad c_1\zeta^2\le\hat\Phi(\frac{\theta}{\bar\theta})\le c_2\zeta^2. \tag{4.14}$$

$\hat\Psi(s)$ is also convex around $s=1$ and this leads to

$$\int|\hat\Psi(\frac{v}{\bar v})\bar v_t|dx+\int|\hat\Psi(\frac{\theta}{\bar\theta})\bar\theta_t|dx\le C\delta(1+t)^{-1}K_2+C\delta(1+t)^{-\frac{5}{2}}, \tag{4.15}$$

where we have used $(\phi,\psi)=(\Phi_x,\Psi_x)$, and $\zeta=W_x-Y$.

On the other hand, the Cauchy inequality yields,

$$\int|R_{1x}\psi|dx\le C\delta(1+t)^{-\frac{3}{2}}+C\delta(1+t)^{-1}K_2, \tag{4.16}$$

$$\int|\frac{\zeta}{\theta}(\bar p-p)\bar u_x+\frac{\kappa\zeta_x\Phi_x}{v\bar v\theta}\bar\theta_x|dx\le C\delta(1+t)^{-1}K_2+C\delta\|\zeta_x\|^2+C\delta(1+t)^{-\frac{5}{2}}, \tag{4.17}$$

$$\int|\frac{\zeta\theta_x}{\theta^2}(\frac{\kappa\theta_x}{v}-\frac{\kappa\bar\theta_x}{\bar v})|dx\le C(\delta+\varepsilon_0)\|\zeta_x\|^2+C\delta(1+t)^{-1}K_2+C\delta(1+t)^{-\frac{5}{2}}, \tag{4.18}$$

and

$$\int|\frac{\zeta}{\theta}Q_3|dx\le C\varepsilon_0\|\psi_x\|^2+C\delta(1+t)^{-1}K_2+C\delta(1+t)^{-\frac{5}{2}}. \tag{4.19}$$

Integrating (4.11) with respect to x and using the Cauchy inequality, we get

$$E_{3t}+\frac{1}{2}K_3\le C\delta(1+t)^{-1}K_2+C\delta(1+t)^{-\frac{3}{2}}. \tag{4.20}$$

Since the norm $\|\phi_x\|$ in not included in K_3, to complete the first derivative estimate, we follow the same way to estimate Φ_x in the previous section. We rewrite the equation $(4.1)_2$ as

$$\frac{\mu}{\bar v}\phi_{xt}-\psi_t-(p-\bar p)_x=-(\frac{\mu}{\bar v})_x\psi_x-[(\frac{\mu}{v}-\frac{\mu}{\bar v})u_x]_x+R_{1x}, \tag{4.21}$$

by using the equation of conservation of the mass (4.1). Multiplying (4.21) by ϕ_x, we get

$$\begin{aligned}&(\frac{\mu}{2\bar v}\phi_x^2)_t-(\frac{\mu}{2\bar v})_t\phi_x^2-\psi_t\phi_x-(p-\bar p)_x\phi_x\\&=\{-(\frac{\mu}{\bar v})_x\psi_x+(\frac{\mu\Phi_x}{v\bar v}u_x)_x+R_{1x}\}\phi_x.\end{aligned} \tag{4.22}$$

We compute

$$-(p-\bar p)_x=\frac{\bar p}{\bar v}\phi_x-\frac{R}{\bar v}\zeta_x+(\frac{p}{v}-\frac{\bar p}{\bar v})v_x-(\frac{R}{v}-\frac{R}{\bar v})\theta_x, \tag{4.23}$$

and

$$\phi_x\psi_t=(\phi_x\psi)_t-(\phi_t\psi)_x+\psi_x^2. \tag{4.24}$$

Integrating (4.22) with respect to x and using the Cauchy inequality, we get

$$\begin{aligned}&(\int\frac{\mu}{2\bar v}\phi_x^2-\phi_x\psi dx)_t+\int\frac{\bar p}{2\bar v}\phi_x^2dx\\&\le C_2K_3+C_2\delta(1+t)^{-1}K_2+C_2\delta(1+t)^{-\frac{5}{2}}+C_2\varepsilon_0\int\psi_{xx}^2dx.\end{aligned} \tag{4.25}$$

Here we have used

$$\int|(\frac{p}{v}-\frac{\bar p}{\bar v})v_x\phi_x|dx\le C(\delta+\varepsilon_0)\|\phi_x\|^2+C\delta(1+t)^{-1}K_2+C\delta(1+t)^{-\frac{5}{2}}, \tag{4.26}$$

$$\begin{aligned}&\int|(\frac{\mu\Phi_x}{v\bar v}u_x)_x\phi_x|dx\le C(\delta+\varepsilon_0)\|\phi_x\|^2+C\delta(1+t)^{-1}K_2\\&\quad+C\varepsilon_0\|\psi_{xx}\|^2+C\delta\|\psi_x\|^2+C\int\phi_x^2|\psi_x|dx,\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}&\int\phi_x^2||\psi_x|dx\le C\|\phi_x\|^2\|\psi_x\|^{\frac{1}{2}}\|\psi_{xx}\|^{\frac{1}{2}}\le C\|\phi_x\|^{\frac{1}{2}}\|\psi_x\|^{\frac{1}{2}}(\|\phi_x\|^2+\|\psi_{xx}\|^2)\\&\le C\varepsilon_0(\|\phi_x\|^2+\|\psi_{xx}\|^2).\end{aligned} \tag{4.28}$$

We now derive the higher order estimates. Multiplying $(4.1)_2$ by $-\psi_{xx}$ and $(4.1)_3$ by $-\zeta_{xx}$, we have

$$\begin{aligned}&(\frac{1}{2}\psi_x^2+\frac{R}{2(\gamma-1)}\zeta_x^2)_t+\frac{\mu}{v}\psi_{xx}^2+\frac{\kappa}{v}\zeta_{xx}^2=(p-\bar p)_x\psi_{xx}+\frac{\mu v_x}{v^2}\psi_x\psi_{xx}+(\frac{\mu\Phi_x}{v\bar v}\bar u_x)_x\psi_{xx}\\&\quad+R_{1x}\psi_{xx}+(pu_x-\bar p\bar u_x)\zeta_{xx}+\frac{\kappa v_x}{v^2}\zeta_x\zeta_{xx}+(\frac{\kappa\Phi_x}{v\bar v}\bar\theta_x)_x\zeta_{xx}-Q_3\zeta_{xx}.\end{aligned} \tag{4.29}$$

The Cauchy inequality gives

$$\int|(p-\bar{p})_x\psi_{xx}|dx \le C(K_3+\|\phi_x\|^2)+\int\frac{\mu}{8v}\psi_{xx}^2dx+C\delta(1+t)^{-1}K_2+C\delta(1+t)^{-\frac{5}{2}}, \quad (4.30)$$

$$\int|\frac{\mu v_x}{v^2}\psi_x\psi_{xx}|dx \le C\delta(\|\psi_x\|^2+\|\psi_{xx}\|^2)+C\int|\phi_x||\psi_x||\psi_{xx}|dx, \quad (4.31)$$

$$\int|(pu_x-\bar{p}\bar{u}_x)\zeta_{xx}|dx \le \int\frac{\mu}{8v}\zeta_{xx}^2dx+CK_3+C\delta(1+t)^{-1}K_2+C\delta(1+t)^{-\frac{5}{2}}, \quad (4.32)$$

and

$$\int|Q_3\zeta_{xx}|dx \le C\int\psi_x^2|\zeta_{xx}|dx+C\delta\|\zeta_{xx}\|^2+C\delta(1+t)^{-\frac{7}{2}}. \quad (4.33)$$

On the other hand,

$$\int|\phi_x||\psi_x||\psi_{xx}|dx \le C\|\psi_x\|^{\frac{1}{2}}\|\phi_x\|\|\psi_{xx}\|^{\frac{3}{2}} \le C\varepsilon_0(\|\psi_{xx}\|^2+\|\psi_x\|^2). \quad (4.34)$$

The term $\int\psi_x^2|\zeta_{xx}|dx$ can be estimated similarly. Thus, integrating (4.29) and using (4.30)-(4.34), we have

$$\begin{aligned}&\{\int(\frac{1}{2}\psi_x^2+\frac{R}{2(\gamma-1)}\zeta_x^2)dx\}_t+\int\frac{\mu}{4v}\psi_{xx}^2dx+\int\frac{\kappa}{4v}\zeta_{xx}^2dx\\ &\le C_3(K_3+\|\phi_x\|^2)+C_3\delta(1+t)^{-1}K_2+C_3\delta(1+t)^{-\frac{5}{2}}.\end{aligned} \quad (4.35)$$

We choose large constants $\bar{C}_2>1$, $\bar{C}_3>1$ so that

$$\bar{C}_2E_3+\bar{C}_3\int(\frac{\mu}{2v}\phi_x^2-\phi_x\psi)dx > \frac{1}{2}\bar{C}_2E_3+\frac{\bar{C}_3}{4}\int\frac{\mu}{v}\phi_x^2dx, \quad (4.36)$$

and

$$\frac{1}{2}\bar{C}_2-\bar{C}_3C_2-C_3 > \frac{1}{4}\bar{C}_2, \quad \bar{C}_3\int\frac{\bar{p}}{2\bar{v}}\phi_x^2dx-C_3\|\phi_x\|^2 > \bar{C}_3\int\frac{\bar{p}}{4\bar{v}}\phi_x^2dx. \quad (4.37)$$

Let

$$E_3=\bar{C}_2E_3+\bar{C}_3\int(\frac{\mu}{2v}\phi_x^2-\phi_x\psi)dx+\int(\frac{1}{2}\psi_x^2+\frac{R}{2(\gamma-1)}\zeta_x^2)dx, \quad (4.38)$$

and

$$K_4=\frac{1}{4}\bar{C}_2K_3+\bar{C}_3\int\frac{\bar{p}}{4\bar{v}}\phi_x^2dx+\int\frac{\mu}{4v}\psi_{xx}^2dx+\int\frac{\kappa}{4v}\zeta_{xx}^2dx. \quad (4.39)$$

Then combining (4.20), (4.25) and (4.35) gives

$$E_{4t}+K_4 \le C\delta(1+t)^{-1}K_2+C\delta(1+t)^{-\frac{3}{2}}. \quad (4.40)$$

We observe that the derivative estimate (4.40) is independent of the lower order one (3.48) except the term $C\delta(1+t)^{-1}K_2$. This kind of derivative estimate is crucial for the stability and convergence rate of the contact wave in this paper.

5 Stability and convergence rate

This section is devoted to the stability and convergence rate in the super-norm for the contact wave. Combining (3.48) and (4.40), we have, if δ and ε_0 are small,

$$(E_2 + E_4)_t + K_2 + K_4 \le C_0\sqrt{\delta}(1+t)^{-1}(E_2 + E_4 + \sqrt{\delta}). \tag{5.1}$$

Let

$$E_5 = E_2 + E_4, K_5 = K_2 + K_4, \tag{5.2}$$

then the Granwall's inequality yields

$$E_5 \le C(E_5(0) + \sqrt{\delta})(1+t)^{C_0\sqrt{\delta}}, \quad \int_0^t K_5 dt \le C(E_5(0) + \sqrt{\delta})(1+t)^{C_0\sqrt{\delta}}. \tag{5.3}$$

Since $E_5 \ge c_3\|(\Phi, \Psi, W)\|^2$ for some positive constant c_3, we have

$$\|(\Phi, \Psi, W)\|^2 \le C(E_5(0) + \sqrt{\delta})(1+t)^{C_0\sqrt{\delta}}. \tag{5.4}$$

Note that the estimate for the lower order term in (5.4) is not desirable since it may grow in time. However, the growth rate of the L^2 norm of (Φ, Ψ, W) is of order $C_0\sqrt{\delta}$ relying on the strength of the contact wave. Hence, if the L^2 norm of the derivative for the variable (Φ, Ψ, W) decreases with a decay rate independent of the small parameter δ. Then the Sobolev inequality gives the L^∞ norm decay of the perturbation (Φ, Ψ, W). In fact, multiply (4.40) by $(1+t)$, we have

$$[(1+t)E_4]_t \le C\delta K_2 + E_4 + C\delta(1+t)^{-\frac{1}{2}} \le K_5 + C\delta(1+t)^{-\frac{1}{2}}. \tag{5.5}$$

Integrating (5.5) with respect to t and using (5.3) imply

$$E_4 \le C(E_5(0) + \sqrt{\delta})(1+t)^{-\frac{1}{2}}, \tag{5.6}$$

where we have used the fact that

$$\begin{aligned} E_4 &\le C\|(\phi, \psi, \zeta)\|_{H^1}^2 \le C(\|(\Phi, \Psi, W)_x\|^2 + \|(\phi, \psi, \zeta)_x\|^2) + C\delta(1+t)^{-\frac{3}{2}} \\ &\le CK_5 + C\delta(1+t)^{-\frac{3}{2}}. \end{aligned}$$

Furthermore, since

$$E_4 \ge c_4\|(\phi, \psi, \zeta)\|_{H^1}^2 \ge c_4(\|(\Phi, \Psi, W)_x\|^2 + \|(\phi, \psi, \zeta)_x\|^2) - c_4\delta(1+t)^{-\frac{3}{2}},$$

for some positive constant c_4, from (5.4) and (5.6), we have the decay rate for (Φ, Ψ, W),

$$\begin{aligned} \|(\Phi, \Psi, W)\|_{L^\infty} &\le C\|(\Phi, \Psi, W)\|^{\frac{1}{2}}\|(\Phi_x, \Psi_x, W_x)\|^{\frac{1}{2}} \\ &\le C(E_5(0) + \sqrt{\delta})^{\frac{1}{2}}(1+t)^{-\frac{1}{8}+\frac{1}{4}C_0\sqrt{\delta}}. \end{aligned} \tag{5.7}$$

If the strength of the contact wave δ is small, then the L^∞ norm $\|(\Phi, \Psi, W\|_{L^\infty}$ is uniformly bounded. Since $W = \frac{\gamma-1}{R}(\bar{W} - \bar{u}\Psi)$, we also have

$$\|(\Phi, \Psi, \bar{W})\|_{L^\infty} \le C(E_5(0) + \sqrt{\delta})^{\frac{1}{2}}(1+t)^{-\frac{1}{8}+\frac{1}{4}C_0\sqrt{\delta}} \le C(\epsilon + \delta^{\frac{1}{4}})(1+t)^{-\frac{1}{8}+\bar{C}_0\sqrt{\delta}}, \tag{5.8}$$

where $\bar{C}_0 = \frac{1}{4}C_0$. The decay rate for $\|(\phi, \psi, \zeta)\|_{L^\infty}$ is straightforward by (5.6) as follows:

$$\|(\phi, \psi, \zeta)\|_{L^\infty} \le E_4^{\frac{1}{2}} \le C(\epsilon + \delta^{\frac{1}{4}})(1+t)^{-\frac{1}{4}}. \tag{5.9}$$

Therefore the a priori assumption (2.11) is verified and the proof of Theorem 1.1 is completed.

References

[1] F.V. Atkinson and L.A. Peletier, *Similarity solutions of the nonlinear diffusion equation*, Arch. rational Mech. Anal.,**54**,373-392,1974.

[2] R. Courant, and K. O. Friedrichs, *Supersonic Flows and Shock Waves*, Wiley-Interscience: New York, 1948.

[3] C.T. Duyn and L.A. Peletier, *A class of similarity solution of the nonlinear diffusion equation*, Nonlinear Analysis, T.M.A.,**1**(1977), 223-233.

[4] J. Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Rational Mech. Anal., **95** (4) (1986), 325-344.

[5] L. Hsiao and T.P. Liu, *Nonlinear diffsusive phenomenia of nonliear hyperbolic systems*, Chin. Ann. Math. **14**B:4(1993),465-480.

[6] F.M. Huang and A. Matsumura, *Convergence rate of contact discontinuity for Navier-Stokes equations*, 2004, Preprint.

[7] F.M. Huang, A. Matsumura and X. Shi, *On the stability of contact discontinuity for compressible Navier-Stokes equations with free boundary*, Osaka J. Math. **41** (2004), no. 1, 193–210.

[8] F.M. Huang and H.J. Zhao, *On the global stability of contact discontinuity for compressible Navier-Stokes equations*, Rend. Sem. Mat. Univ. Padova, **109** (2003), 283–305.

[9] S. Kawashima & A. Matsumura, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion*, Comm. Math. Phys., **101** (1) (1985), 97–127.

[10] S. Kawashima, A. Matsumura and K. Nishihara, *Asymptotic behavior of solutions for the equations of a viscous heat-conductive gas*, Proc.Japan Acad., **62**,Ser.A (1986), 249-252.

[11] T.-P. Liu *Linear and nonlinear large time behavior of general systems of hyperbolic conservation laws*, Comm. Pure Appl. Math. 30 (1977), 767-796.

[12] T.-P. Liu, *Nonlinear Stability of Shock Waves for Viscous Conservation Laws*, Mem. Amer. Math. Soc., **56** (329) (1985), 1-108.

[13] T.-P. Liu, *Pointwise convergence to shock waves for viscous conservation laws*, Comm. Pure Appl. Math. **50** (1997), no. 11, 1113–1182.

[14] T.-P. Liu and Z.P. Xin, *Nonlinear stability of rarefaction waves for compressible Navier-Stokes equations*, Comm. Math. Phys. **118** (1988), no. 3, 451-465.

[15] T.-P.Liu and Z.P. Xin, Pointwise decay to contact discontinuities for systems of viscous conservation laws, Asian J. Math. 1 (1997), 34–84.

[16] A. Matsumura & K. Nishihara, *On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math., **2** (1) (1985), 17-25.

[17] A. Matsumura & K. Nishihara, *Asymptotics toward the rarefaction waves of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math., **3** (1) (1985), 3-13.

[18] A. Matsumura and K. Nishihara, *Global stability of the rarefaction wave of a one-dimensional model system for compressible viscous gas*, Commun. Math. Phys., **144**, 1992, 325-335.

[19] K. Nishihara, T. Yang & H.-J. Zhao, *Nonlinear stability of strong rarefaction waves for compressible Navier-Stokes equations*, SIAM J. Math. Anal., Vol. 35, No. 6(2004), 1561-1597.

[20] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, Second Edition, New York, 1994.

[21] A. Szepessy, and Z. P. Xin, *Nonlinear stability of viscous shock waves*, Arch. Rat. Mech. Anal., **122** 1993, 53-103.

[22] A. Szepessy, and K. Zumbrun, *Stability of rarefaction waves in viscous media*, Arch. Rational Mech. Anal. **133** (1996), no. 3, 249–298.

[23] Z.P. Xin, On nonlinear stability of contact discontinuities. Hyperbolic problems: theory, numerics, applications (Stony Brook, NY, 1994), 249–257, World Sci. Publishing, River Edge, NJ, 1996.