# A new regularity criterion for weak solutions to the Navier-Stokes equations

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**Abstract.** In this paper we obtain a new regularity criterion for weak solutions to the 3-D Navier-Stokes equations. We show that if any one component of the velocity field belongs to $L^\alpha([0,T);L^\gamma(\mathbb{R}^3))$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le\frac{1}{2}$, $6<\gamma\le\infty$, then the weak solution actually is regular and unique.

**Titre.** Un nouveau critère de régularité pour les solutions faibles des équations de Navier-Stokes

**Resumé.** Dans cet article, on obtient un nouveau critère de régularité pour les solutions faibles des équations de Navier-Stokes en dimension 3. On démontre que si une conposante quelconque du champ de vitesse appartient à $L^\alpha([0,T];L^\gamma(\mathbb{R}^3))$ avec $\frac{2}{\alpha}+\frac{3}{\gamma}\le\frac{1}{2}$, $6<\gamma\le\infty$, alors la solution faible est régulière et unique.

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# 1 Introduction

We consider the following Cauchy problem for the incompressible Navier-Stokes equations in $\mathbb{R}^3 \times (0, T)$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \Delta u, \\ \mathrm{div} u = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity field, $p(x,t)$ is a scalar pressure, and $u_0(x)$ with $\mathrm{div} u_0 = 0$ in the sense of distribution is the initial velocity field.

The study of the incompressible Navier-Stokes equations in three space dimensions has a long history (see [8, 22]). In the pioneering work [14] and [11], Leray and Hopf proved the existence of its weak solutions $u(x,t) \in L^\infty(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^1(\mathbb{R}^3))$ for given $u_0(x) \in L^2(\mathbb{R}^3)$. But the uniqueness and regularity of the Leray-Hopf weak solutions are still big open problems. In [17], Scheffer began to study the partial regularity theory of the Navier-Stokes equations. Deeper results were obtained by Caffarelli, Kohn and Nirenberg in [5]. Further results can be found in [23] and references there in.

On the other hand, the regularity of a given weak solution $u$ can be shown under additional conditions. In 1962, Serrin [18] proved that if u is a Leray-Hopf weak solution belonging to $L^{\alpha,\gamma} \equiv L^\alpha(0,T; L^\gamma(\mathbb{R}^3))$ with $2/\alpha + 3/\gamma \leq 1, 2 < \alpha < \infty, 3 < \gamma < \infty$, then the solution $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T))$ (recently, Beirão da Veiga [3] add Serrin's condition only on two components of the velocity field). From then on, there are many criterion results added on $u$. In [24] and [9], von Wahl and Giga showed that if $u$ is a weak solution in $C([0,T); L^3(\mathbb{R}^3))$, then $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T))$; Struwe [21] proved the same regularity of $u$ in $L^\infty(0,T; L^3(\mathbb{R}^3)$ provided $\sup_{0<t<T} \|u(x,t)\|_{L^3}$ is sufficiently small and Kozono and Sohr [12] obtained the regularity for the weak solution $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T))$ provided $u(x,t)$ is left continuous with respect to $L^3$-norm for every $t \in (0,T)$. Recently Kozono and Taniuchi [13] showed that if a Leray-Hopf weak solution $u(x,t) \in L^2(0,T; BMO)$, then $u(x,t)$ actually is a strong solution of

(1.1) on $(0,T)$. $L^{\alpha,\gamma}$ is defined by

$$\|u\|_{L^{\alpha,\gamma}} = \begin{cases} \left(\displaystyle\int_0^t \|u(.,\tau)\|_{L^\gamma}^\alpha d\tau\right)^{1/\alpha} & \text{if } 1 \le \alpha < \infty, \\ \operatorname*{ess\ sup}_{0<\tau<t} \|u(.,\tau)\|_{L^\gamma} & \text{if } \alpha = \infty, \end{cases}$$

where

$$\|u(.,\tau)\|_{L^\gamma} = \begin{cases} \left(\displaystyle\int_{\mathbb{R}^3} |u(x,\tau)|^\gamma dx\right)^{1/\gamma} & \text{if } 1 \le \gamma < \infty, \\ \operatorname*{ess\ sup}_{x\in\mathbb{R}^3} |u(x,\tau)| & \text{if } \gamma = \infty. \end{cases}$$

The point is that $\|u_\lambda\|_{L^{\alpha,\gamma}} = \|u\|_{L^{\alpha,\gamma}}$ holds for all $\lambda > 0$ if and only if $2/\alpha + 3/\gamma = 1$, where $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, $p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$ and if $(u,p)$ solves the Navier-Stokes equations, then so does $(u_\lambda, p_\lambda)$ for all $\lambda > 0$. Usually we say that the norm $\|u\|_{L^{\alpha,\gamma}}$ has the scaling dimension zero for $2/\alpha + 3/\gamma = 1$ [5].

Sohr [19] extended Serrin's regularity criterion by introducing Lorentz space in both time and spatial direction, $u \in L^{s,r}(0,T; L^{q,\infty})$ with $2/s + 3/q = 1$, here $L^{p,q}$ is Lorentz space, for weak solutions which satisfy the strong energy inequality. Later on, Sohr [20] extended Serrin's regularity class for weak solutions of the Navier-Stokes equations replacing the $L^q$-space by Sobolev spaces of negative order, $u \in L^s(0,T; H^{-\alpha,q})$ with $2/s + 3/q = 1 - \alpha$, for $0 \le \alpha < 1$.

For the regularity criteria in terms of the gradient of velocity or the pressure, we refer to [1, 2, 4, 6, 7, 26, 27, 28].

Very recently, He [10] added the regularity criterion only on one component of the velocity field. He proved that if any one component of the velocity field of a weak solution belongs to $L^\infty(\mathbb{R}^3 \times (0,T))$, then the weak solution actually is strong.

In the present paper we improve He's [10] result significantly as

**Theorem 1.1** *Suppose $u_0 \in H^1(\mathbb{R}^3)$, and $\mathrm{div} u_0 = 0$ in the sense of distribution. Assume that $u(x,t)$ is a Leray-Hopf weak solution of (1.1) in $(0,T)$. If any component of $u$, say $u_3$ satisfies*

$$u_3 \in L^{\alpha,\gamma} \ \textit{with}\ \frac{2}{\alpha} + \frac{3}{\gamma} \le \frac{1}{2}, \ 4 < \alpha < \infty, \ 6 < \gamma < \infty,$$

*or $u_3 \in L^{4,\infty}$, then $u(x,t)$ is a regular solution in $[0,T)$.*

**Remark 1.1** After this work was finished, the author was informed that J. Neustupa, A. Novotný and P. Penel [16] proved a result analogous to Theorem 1.1 for the suitable weak solution (see [5] for its definition). Moreover, in [16] they asked whether their result is true for a weak solution. Here, our main theorem is an affirmative answer to their question.

**Remark 1.2** The main progress with respect to Serrin's result is of course that only one component of the velocity field is supposed to be "regular", not all of them. This definitely be useful for people who try to construct an example of a nonsmooth solution (they should keep in mind that all three components of $u$ have to be "singular"). The price to pay is that the assumption $u_3 \in L^\alpha([0,T], L^\gamma(\mathbb{R}^3))$ with $\frac{2}{\alpha} + \frac{3}{\gamma} \leq \frac{1}{2}$ is stronger than Serrin's condition and is not invariant under the natural rescaling $u(x,t) \mapsto \lambda u(\lambda x, \lambda^2 t)$. In the author's opinion, it is a real challenging problem to show regularity by adding Serrin's condition only on one component of the velocity field. We hope we can investigate this problem in the near future.

**Remark 1.3** Comparing with the previous regularity criterion [26] $\nabla u_3 \in L^{\alpha,\gamma}$ with $\frac{2}{\alpha} + \frac{3}{\gamma} \leq \frac{3}{2}$, establishing a priori estimates here are much more difficult than those. Actually, it is not difficult to understand roughly, since $\nabla u_3$ involves more information than $u_3$.

Before going to proof, we recall the definition of Leray-Hopf weak solutions (see [8, 22]).

**Definition.** A measurable vector $u$ is called a Leray-Hopf weak solution to the Navier-Stokes equations (1.1), if $u$ satisfies the following properties

(i) $u$ is weakly continuous from $[0,T)$ to $L^2(\mathbb{R}^3)$.

(ii) $u$ verifies (1.1) in the sense of distribution, i.e.,

$$\int_0^T \int_{\mathbb{R}^3} \Big( \frac{\partial \phi}{\partial t} + (u \cdot \nabla)\phi \Big) u dx dt + \int_{\mathbb{R}^3} u_0 \phi(x,0) dx = \int_0^T \int_{\mathbb{R}^3} \nabla u : \nabla \phi dx dt$$

for all $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,T))$ with $\mathrm{div}\phi = 0$, where $A : B = \sum_{i,j}^3 a_{ij} b_{ij}$, $A = (a_{ij})$ and $B = (b_{ij})$ are $3 \times 3$ matrices, and

$$\int_0^T \int_{\mathbb{R}^3} u \cdot \nabla \phi dx dt = 0$$

for every $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,T))$.

(iii) The energy inequality, i.e.,

$$\|u(.,t)\|_{L^2}^2+2\int_0^t\|\nabla u(.,s)\|_{L^2}^2ds\le\|u_0\|_{L^2}^2,\quad 0\le t\le T.$$

By a strong solution we mean a weak solution $u$ such that

$$u\in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$$

It is well-known that strong solutions are regular (say, classical) and unique in the class of weak solutions.

## 2 A priori estimates on the smooth solution

First, we give a very simple interpolation lemma

**Lemma 2.1** *Assume that a measurable function $u(x,t)\in L^{\infty,2}$ and $\nabla u\in L^{2,2}$ on $[0,T)$, then $u\in L^{p,q}$ with $2/p+3/q\ge 3/2$, $p\ge 2, 2\le q\le 6$. Moreover,*

$$\|u\|_{L^{p,q}}\le C_1\|u\|_{L^{\infty,2}}^{\frac{3}{q}-\frac{1}{2}}\|\nabla u\|_{L^{2,2}}^{\frac{3}{2}-\frac{3}{q}}\tag{2.1}$$

*where $C_1=C_1(p,q,T)$. If $\frac{2}{p}+\frac{3}{q}=\frac{3}{2}$, then*

$$\|u\|_{L^{p,q}}\le C_1(q)\|u\|_{L^{\infty,2}}^{1-\frac{2}{p}}\|\nabla u\|_{L^{2,2}}^{\frac{3}{2}-\frac{3}{q}}\tag{2.2}$$

**Proof:**

$$\begin{aligned}
\|u\|_{L^{p,q}} &= \Big(\int_0^t\|u(.,\tau)\|_{L^q}^p d\tau\Big)^{1/p}\\
&\le C_2(q)\Big(\int_0^t\|u(.,\tau)\|_{L^2}^{(1-\theta)p}\|\nabla u(.,\tau)\|_{L^2}^{\theta p}d\tau\Big)^{1/p}\\
&\le C_2(q)\|u\|_{L^{\infty,2}}^{1-\theta}\|\nabla u\|_{L^{2,2}}^{\theta}t^{(\frac{2}{p}+\frac{3}{q}-\frac{3}{2})/2}\\
&\le C_2(q)\|u\|_{L^{\infty,2}}^{1-\theta}\|\nabla u\|_{L^{2,2}}^{\theta}T^{(\frac{2}{p}+\frac{3}{q}-\frac{3}{2})/2}\equiv C_1(p,q,T)\|u\|_{L^{\infty,2}}^{1-\theta}\|\nabla u\|_{L^{2,2}}^{\theta}
\end{aligned}$$

where we use Gagliardo-Nirenberg inequality

$$\frac{1}{q}=\frac{1-\theta}{2}+\theta(\frac{1}{2}-\frac{1}{3}),\quad 2\le q\le 6,\tag{2.3}$$

and Hölder's inequality provided $\theta p\le 2$.
From (2.3), $\theta=\frac{3}{2}-\frac{3}{q}$, we obtain the relation between $p$ and $q$, $\frac{2}{p}+\frac{3}{q}\ge\frac{3}{2}$. If

$\frac{2}{p}+\frac{3}{q}=\frac{3}{2}$, then obviously $C_1(p,q,T)=C_2(q)\equiv C_1(q)$ and $\frac{3}{q}-\frac{1}{2}=1-\frac{2}{p}$. This finishes the proof. $\square$

In order to prove Theorem 1.1, we give a priori estimate on $\omega_3$ first, where $\omega=\mathrm{curl}u=(\omega_1,\omega_2,\omega_3)$.

**Lemma 2.2** *Suppose $u_0\in H^1(\mathbb{R}^3)$ with $\mathrm{div}u_0=0$. Assume that $(u,p)$ is a smooth solution in $\mathbb{R}^3\times(0,T)$, which satisfies the energy inequality, with $\nabla u\in L^{\infty,2}$ and $\Delta u\in L^{2,2}$. If $u_3\in L^{\alpha,\gamma}(\mathbb{R}^3\times(0,T))$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le\frac{1}{2}$, $6<\gamma<\infty$, or $u_3\in L^{4,\infty}$, then for $0\le t<T$*

$$\|\omega_3(.,t)\|_{L^2}^2+\int_0^t\|\nabla\omega_3(.,\tau)\|_{L^2}^2d\tau$$
$$\le\left\{\begin{array}{ll}\|\omega_3^0\|_{L^2}^2+C_3\|u_3\|_{L^{\alpha,\gamma}}^2\|\nabla u\|_{L^{\infty,2}}^{4/\alpha}\|\Delta u\|_{L^{2,2}}^{6/\gamma} & \text{if } 6<\gamma<\infty,\\ \|\omega_3^0\|_{L^2}^2+C(\|u_0\|_{L^2})\|u_3\|_{L^{4,\infty}}^2\|\nabla u\|_{L^{\infty,2}} & \text{if } (\alpha,\gamma)=(4,\infty),\end{array}\right. \tag{2.4}$$

*where $C_3=C_3(\alpha,\gamma,T,\|u_0\|_{L^2})$ and $\omega^0(x)$ is the initial datum for $\omega$.*

**Proof:** Vorticity $\omega=\mathrm{curl}u$ satisfies

$$\left\{\begin{array}{l}\dfrac{\partial\omega}{\partial t}+(u\cdot\nabla)\omega=(\omega\cdot\nabla)u+\Delta\omega,\\ \mathrm{div}u=0,\\ \mathrm{curl}u=\omega,\\ \omega(x,0)=\omega^0(x).\end{array}\right. \tag{2.5}$$

Multiplying the first equation of (2.5) by $\omega_3$, and integrating on $\mathbb{R}^3\times(0,t)$, after suitable integration by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\|\omega_3(.,t)\|_{L^2}^2+\int_0^t\|\nabla\omega_3(.,\tau)\|_{L^2}^2d\tau\\ &\quad\le\int_0^t\int_{\mathbb{R}^3}|(\omega\cdot\nabla\omega_3)u_3|dxd\tau+\frac{1}{2}\|\omega_3^0\|_{L^2}^2\\ &\quad\le\frac{1}{2}\int_0^t\|\nabla\omega_3(.,\tau)\|_{L^2}^2d\tau+\frac{1}{2}\int_0^t\int_{\mathbb{R}^3}\omega^2u_3^2dxd\tau+\frac{1}{2}\|\omega_3^0\|_{L^2}^2\\ &\quad\le\frac{1}{2}\int_0^t\|\nabla\omega_3(.,\tau)\|_{L^2}^2d\tau+\int_0^t\int_{\mathbb{R}^3}|u_3|^2|\nabla u|^2dxd\tau+\frac{1}{2}\|\omega_3^0\|_{L^2}^2\end{aligned}$$

where we use the inequality $|\omega|^2 \leq 2|\nabla u|^2$. Now we give an estimate of the second term on the right hand side of the above inequality

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^3} |u_3|^2 |\nabla u|^2 dx d\tau &\leq \int_0^t \|u_3\|_{L^\gamma}^2 \|\nabla u\|_{L^q}^{2\theta} \|\nabla u\|_{L^2}^{2(1-\theta)} d\tau \\ &\leq \|u_3\|_{L^{\alpha,\gamma}}^2 \|\nabla u\|_{L^{p,q}}^{2\theta} \|\nabla u\|_{L^{2,2}}^{2(1-\theta)}\end{aligned}$$

where $p$, $q$ and $\theta$ satisfy

$$\begin{cases} \frac{2}{\alpha} + \frac{2\theta}{p} + \frac{2(1-\theta)}{2} = 1, \\ \frac{2}{\gamma} + \frac{2\theta}{q} + \frac{2(1-\theta)}{2} = 1. \end{cases} \tag{2.6}$$

Additional condition added on $p$ and $q$, due to Lemma 2.1, is

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2}. \tag{2.7}$$

(2.6) and (2.7) can be solved easily with

$$\begin{cases} \theta = \frac{2}{\alpha} + \frac{3}{\gamma}, \ \text{if } 6 < \gamma < \infty; \ \theta = \frac{1}{2}, \ \text{if } \gamma = \infty; \\ p = \frac{2(2\gamma+3\alpha)}{3\alpha}, \ \text{if } 6 < \gamma < \infty; \ p = \infty, \ \text{if } \gamma = \infty; \\ q = \frac{2(2\gamma+3\alpha)}{2\gamma+\alpha}, \ \text{if } 6 < \gamma < \infty; \ q = 2, \ \text{if } \gamma = \infty. \end{cases} \tag{2.8}$$

Then (2.4) follows from Lemma 2.1 and energy inequality for the Leray-Hopf weak solution. □

The main result of this section is the following a priori estimate on the velocity field.

**Theorem 2.3** *Under the same assumption of Lemma 2.2, we have*

$$\sup_{0\leq t<T} \|\nabla u(.,t)\|_{L^2}^2 + \int_0^T \|\Delta u(.,\tau)\|_{L^2}^2 d\tau \leq C_4 \tag{2.9}$$

*where $C_4$ depends on $T$, $\alpha$, $\gamma$, $\|\nabla u_0\|_{L^2}$, $\|u_0\|_{L^2}$ and $\|u_3\|_{L^{\alpha,\gamma}}$.*

**Remark 2.1** Not only we use Theorem 2.3 to prove the main theorem, but Theorem 2.3 itself is also very interesting and useful.

**Proof:** We can rewrite the first equation of the Navier-Stokes equations (1.1) as

$$\frac{\partial u}{\partial t} + \omega \times u + \frac{1}{2}\nabla |u|^2 + \nabla p = \Delta u. \tag{2.10}$$

Multiply the equation (2.10) by $\Delta u$ and integrate on $\mathbb{R}^3\times(0,t)$, after suitable integration by parts, one obtains

$$\frac{1}{2}\|\nabla u(.,t)\|_{L^2}^2+\int_0^t\|\Delta u(.,\tau)\|_{L^2}^2d\tau=\int_0^t\int_{\mathbb{R}^3}(\omega\times u)\cdot\Delta u dxd\tau+\frac{1}{2}\|\nabla u_0\|_{L^2}^2 \tag{2.11}$$

let

$$\begin{aligned}
I &= \int_0^t\int_{\mathbb{R}^3}(\omega\times u)\cdot\Delta u dxd\tau\\
&\le \int_0^t\int_{\mathbb{R}^3}|\omega_2u_3\Delta u_1|dxd\tau+\int_0^t\int_{\mathbb{R}^3}|\omega_3u_2\Delta u_1|dxd\tau+\int_0^t\int_{\mathbb{R}^3}|\omega_3u_1\Delta u_2|dxd\tau\\
&\quad+\int_0^t\int_{\mathbb{R}^3}|\omega_1u_3\Delta u_2|dxd\tau+\Big|\int_0^t\int_{\mathbb{R}^3}\omega_1u_2\Delta u_3dxd\tau-\int_0^t\int_{\mathbb{R}^3}\omega_2u_1\Delta u_3dxd\tau\Big|\\
&\equiv I_1+I_2+I_3+I_4+|I_5+I_6|.
\end{aligned}$$

We will estimate the terms one by one.

**Case 1.** $u_3\in L^{\alpha,\gamma}$, with $\frac{2}{\alpha}+\frac{3}{\gamma}\le\frac{1}{2}$, for $6<\gamma<\infty$.

$$\begin{aligned}
I_1 &= \int_0^t\int_{\mathbb{R}^3}|\omega_2u_3\Delta u_1|dxd\tau\\
&\le \int_0^t\|u_3\|_{L^\gamma}\|\omega_2\|_{L^{\frac{2\gamma}{\gamma-2}}}\|\Delta u_1\|_{L^2}d\tau \quad \Big(\text{Hölder's inequality}\Big)\\
&\le C_5''\int_0^t\|u_3\|_{L^\gamma}\|\omega_2\|_{L^2}^{\frac{\gamma-3}{\gamma}}\|\Delta u\|_{L^2}^{\frac{\gamma+3}{\gamma}}d\tau\\
&\quad\Big(\text{Gagliardo-Nirenberg inequality and Calderón-Zygmund inequality}\Big)\\
&\le \frac{1}{24}\|\Delta u\|_{L^{2,2}}^2+C_5'\int_0^t\|\nabla u\|_{L^2}^2\|u_3\|_{L^\gamma}^{\frac{2\gamma}{\gamma-3}}d\tau \quad \Big(\text{Young inequality}\Big)\\
&\le \frac{1}{24}\|\Delta u\|_{L^{2,2}}^2+C_5'\sup_{0\le\tau<t}\|\nabla u(.,\tau)\|_{L^2}^2\|u_3\|_{L^{\alpha,\gamma}}^{\frac{2\gamma}{\gamma-3}}t^{\frac{1-(2/\alpha+3/\gamma)}{1-3/\gamma}}\\
&\quad\Big(\text{Hölder's inequality for } \tfrac{2\gamma}{\gamma-3}\le\alpha\Big)
\end{aligned}$$

Hence

$$I_1\le\frac{1}{24}\|\Delta u\|_{L^{2,2}}^2+C_5\|\nabla u\|_{L^{\infty,2}}^2\|u_3\|_{L^{\alpha,\gamma}}^{\frac{2\gamma}{\gamma-3}}, \tag{2.12}$$

where $C_5 = C_5(\alpha, \gamma, T)$.

$$\begin{aligned} I_2 &\le \frac{1}{24}\|\Delta u\|^2_{L^{2,2}} + 6\int_0^t \|u_2\|^2_{L^a}\|\omega_3\|^2_{L^b} d\tau \\ &\quad \Big(\text{Hölder's and Young inequality } \tfrac{1}{a} + \tfrac{1}{b} = \tfrac{1}{2}\Big) \\ &\le \frac{1}{24}\|\Delta u\|^2_{L^{2,2}} + 6\|u_2\|^2_{L^{p,a}}\|\omega_3\|^2_{L^{q,b}} \quad \Big(\text{Hölder's inequality } \tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{2}\Big) \end{aligned}$$

Now we want to apply Lemma 2.1 on $\|w_3\|_{L^{q,b}}$, so $a$, $b$, $p$ and $q$ satisfies

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad \frac{2}{q} + \frac{3}{b} = \frac{3}{2}. \tag{2.13}$$

(2.13) can be solved as

$$\begin{cases} p = \infty, \ \ a = 3; \\ q = 2, \ \ b = 6. \end{cases} \tag{2.14}$$

Then Lemma 2.2 tells us

$$\|\omega_3\|^2_{L^{2,6}} \le C_6\|u_3\|^2_{L^{\alpha,\gamma}}\|\nabla u\|^{\frac{4}{\alpha}}_{L^{\infty,2}}\|\Delta u\|^{\frac{6}{\gamma}}_{L^{2,2}} + C_7, \tag{2.15}$$

where $C_6$ depends on $\alpha$,$\gamma$, $T$ and $\|u_0\|_{L^2}$, while $C_7$ depends on $\|\omega_3^0\|_{L^2}$ only.

On the other hand,

$$\begin{aligned} \|u_2\|^2_{L^{\infty,3}} &\le \|u\|^2_{L^{\infty,3}} \le \|u\|_{L^{\infty,2}}\|u\|_{L^{\infty,6}} \\ &\le C_8\|\nabla u\|_{L^{\infty,2}} \quad \big(\text{Energy inequality and Sobolev inequality}\big) \end{aligned}$$

Therefore $I_2$ can be estimated as

$$I_2 \le \frac{1}{24}\|\Delta u\|^2_{L^{2,2}} + C_9\|\nabla u\|^{1+4/\alpha}_{L^{\infty,2}}\|\Delta u\|^{6/\gamma}_{L^{2,2}}\|u_3\|^2_{L^{\alpha,\gamma}} + C_{10}\|\nabla u\|_{L^{\infty,2}}, \tag{2.16}$$

where $C_9$ depends on $\alpha$, $\gamma$, $T$ and $\|u_0\|_{L^2}$, while $C_{10}$ depends on $\|u_0\|_{L^2}$ and $\|\omega_3^0\|_{L^2}$.

$I_3$ is similar to $I_2$,

$$I_3 \le \frac{1}{24}\|\Delta u\|^2_{L^{2,2}} + C_9\|\nabla u\|^{1+4/\alpha}_{L^{\infty,2}}\|\Delta u\|^{6/\gamma}_{L^{2,2}}\|u_3\|^2_{L^{\alpha,\gamma}} + C_{10}\|\nabla u\|_{L^{\infty,2}} \tag{2.17}$$

and $I_4$ is similar to $I_1$,

$$I_4 \le \frac{1}{24}\|\Delta u\|^2_{L^{2,2}} + C_5 \sup_{0\le\tau<t}\|\nabla u(.,\tau)\|^2_{L^2}\|u_3\|^{\frac{2\gamma}{\gamma-3}}_{L^{\alpha,\gamma}}. \tag{2.18}$$

$$
\begin{aligned}
I_5 &= \int_0^t \int_{\mathbb{R}^3} (\omega_1 u_2)\Delta u_3 dx d\tau \\
&= \int_0^t \int_{\mathbb{R}^3} (\partial_2 u_3) u_2 \Delta u_3 dx d\tau - \int_0^t \int_{\mathbb{R}^3} (\partial_3 u_2) u_2 \Delta u_3 dx d\tau \equiv I_5^1 + I_5^2
\end{aligned}
$$

$$
|I_5^1| \leq 3\int_0^t \int_{\mathbb{R}^3} u_2^2 (\partial_2 u_3)^2 dx d\tau + \frac{1}{12}\int_0^t \int_{\mathbb{R}^3} (\Delta u_3)^2 dx d\tau \tag{2.19}
$$

$$
\begin{aligned}
\int_0^t \int_{\mathbb{R}^3} u_2^2 (\partial_2 u_3)^2 dx d\tau &= -\Big( \int_0^t \int_{\mathbb{R}^3} (\partial_2^2 u_3) u_3 u_2^2 + u_3 (\partial_2 u_3) \partial_2 (u_2^2) dx d\tau \Big) \\
&\leq \int_0^t \int_{\mathbb{R}^3} |(\partial_2^2 u_3) u_3 u_2^2| dx d\tau + \int_0^t \int_{\mathbb{R}^3} |u_3 (\partial_2 u_3) \partial_2 (u_2^2)| dx d\tau \\
&\equiv I_5^{1,1} + I_5^{1,2}
\end{aligned}
$$

$$
I_5^{1,1} = \int_0^t \int_{\mathbb{R}^3} |(\partial_2^2 u_3) u_3 u_2^2| dx d\tau \leq \|\Delta u\|_{L^{2,2}} \|u_3\|_{L^{\alpha,\gamma}} \|u_2\|^2_{L^{a',b'}}, \tag{2.20}
$$

where

$$
\frac{1}{2} + \frac{1}{\alpha} + \frac{2}{a'} = 1 \quad \text{and} \quad \frac{1}{2} + \frac{1}{\gamma} + \frac{2}{b'} = 1.
$$

Actually $a'$ and $b'$ are constants determined by $\alpha$ and $\gamma$ respectively with

$$
a' = \frac{4\alpha}{\alpha - 2}, \quad b' = \frac{4\gamma}{\gamma - 2}.
$$

And $\|u_2\|_{L^{a',b'}}$ can be controlled as

$$
\begin{aligned}
\|u_2\|^2_{L^{a',b'}} &\leq \|u\|^2_{L^{\frac{4\alpha}{\alpha-2}, \frac{4\gamma}{\gamma-2}}} \leq \|u\|_{L^{\frac{2\alpha}{\alpha-2}, \frac{3\gamma}{\gamma-3}}} \|u\|_{L^{\infty,6}} \\
&\leq C_{11} \|\nabla u\|_{L^{\infty,2}},
\end{aligned} \tag{2.21}
$$

where we have used Lemma 2.1 on $\|_{L^{\frac{2\alpha}{\alpha-2}, \frac{3\gamma}{\gamma-3}}}$, since

$$
\frac{2}{\frac{2\alpha}{\alpha-2}} + \frac{3}{\frac{3\gamma}{\gamma-3}} = 2 - \left(\frac{2}{\alpha} + \frac{3}{\gamma}\right) \geq \frac{3}{2},
$$

and $C_{11}$ is a constants which depends on $\alpha$, $\gamma$, $T$ and $\|u_0\|_{L^2}$ only.

Return to (2.20) and use Young inequality, then we obtain

$$
I_5^{1,1} \leq \frac{1}{144}\|\Delta u\|^2_{L^{2,2}} + C_{12} \|u_3\|^2_{L^{\alpha,\gamma}} \|\nabla u\|^2_{L^{\infty,2}}. \tag{2.22}
$$

$$I_5^{1,2} = \int_0^t \int_{\mathbb{R}^3} |u_3(\partial_2 u_3)\partial_2(u_2^2)|dxd\tau \le 2\|u_3\|_{L^{\alpha,\gamma}}\|\nabla u\|^2_{L^{p_1,q_1}}\|u_2\|_{L^{a_1,b_1}} \tag{2.23}$$

where

$$\begin{cases} \frac{1}{\alpha} + \frac{2}{p_1} + \frac{1}{a_1} = 1, \\ \frac{1}{\gamma} + \frac{2}{q_1} + \frac{1}{b_1} = 1. \end{cases} \tag{2.24}$$

$a_1$ and $b_1$ are required satisfies

$$\frac{2}{a_1} + \frac{3}{b_1} \ge \frac{3}{2} \tag{2.25}$$

(2.24) and (2.25) can be solved as

$$p_1 = 4, \quad q_1 = 3, \quad a_1 = \frac{2\alpha}{\alpha - 2}, \quad b_1 = \frac{3\gamma}{\gamma - 3}. \tag{2.26}$$

It follows from (2.23) and (2.26) that

$$\begin{aligned} I_5^{1,2} &\le C_{13}\|u_3\|_{L^{\alpha,\gamma}}\|\nabla u\|_{L^{\infty,2}}\|\Delta u\|_{L^{2,2}} \\ &\le \frac{1}{144}\|\Delta u\|^2_{L^{2,2}} + C_{14}\|u_3\|^2_{L^{\alpha,\gamma}}\|\nabla u\|^2_{L^{\infty,2}} \end{aligned} \tag{2.27}$$

where $C_{14}$ depends on $\alpha$, $\gamma$, $T$ and $\|u_0\|_{L^2}$ only.

Combining (2.22) and (2.27) together and substituting into (2.19), then

$$|I_5^1| \le \frac{1}{8}\|\Delta u\|^2_{L^{2,2}} + C_{15}\|u_3\|^2_{L^{\alpha,\gamma}}\|\nabla u\|^2_{L^{\infty,2}}, \tag{2.28}$$

where $C_{15}$ depends on $\alpha$, $\gamma$, $T$ and $\|u_0\|_{L^2}$ only.

One can see that $I_5^2$ is a difficult term, so we want to deal with it later. Now we pay our attention to $I_6$,

$$\begin{aligned} I_6 &= -\int_0^t \int_{\mathbb{R}^3} \omega_2 u_1 \Delta u_3 dxd\tau \\ &= \int_0^t \int_{\mathbb{R}^3} (\partial_1 u_3)u_1\Delta u_3 dxd\tau - \int_0^t \int_{\mathbb{R}^3} (\partial_3 u_1)u_1 \Delta u_3 dxd\tau \equiv I_6^1 + I_6^2 \end{aligned}$$

$I_6^1$ can be treated similarly as $I_5^1$,

$$|I_6^1| \le \frac{1}{8}\|\Delta u\|^2_{L^{2,2}} + C_{15}\|u_3\|^2_{L^{\alpha,\gamma}}\|\nabla u\|^2_{L^{\infty,2}}. \tag{2.29}$$

The remaining term which has to be treated is

$$I_5^2 + I_6^2 = -\int_0^t \int_{\mathbb{R}^3} \big((\partial_3 u_2)u_2 + (\partial_3 u_1)u_1\big)\Delta u_3 dxd\tau. \tag{2.30}$$

Since we have no additional conditions on the components $u_1$ and $u_2$, $I_5^2 + I_6^2$ is more difficult to handle. Fortunately, we can circumvent the difficult by the following identity.

$$\begin{aligned}
&\frac{1}{2}\int_0^t\int_{\mathbb{R}^3}\partial_1(u_1^2+u_2^2)\Delta u_1 dxd\tau + \frac{1}{2}\int_0^t\int_{\mathbb{R}^3}\partial_2(u_1^2+u_2^2)\Delta u_2 dxd\tau \\
&+\frac{1}{2}\int_0^t\int_{\mathbb{R}^3}\partial_3(u_1^2+u_2^2)\Delta u_3 dxd\tau = \frac{1}{2}\int_0^t\int_{\mathbb{R}^3}\nabla(u_1^2+u_2^2)\cdot\Delta u dxd\tau = 0
\end{aligned}$$

Therefore from (2.30),

$$\begin{aligned}
|I_5^2 + I_6^2| &= \Big|\frac{1}{2}\int_0^t\int_{\mathbb{R}^3}\partial_1(u_1^2+u_2^2)\Delta u_1 dxd\tau + \frac{1}{2}\int_0^t\int_{\mathbb{R}^3}\partial_2(u_1^2+u_2^2)\Delta u_2 dxd\tau\Big| \\
&\leq \frac{1}{12}\int_0^t\int_{\mathbb{R}^3}(\Delta u)^2 dxd\tau + \frac{3}{4}\int_0^t\int_{\mathbb{R}^3}\big(\partial_1(u_1^2+u_2^2)\big)^2 + \big(\partial_2(u_1^2+u_2^2)\big)^2 dxd\tau \\
&\equiv \frac{1}{12}\int_0^t\int_{\mathbb{R}^3}(\Delta u)^2 dxd\tau + \frac{1}{4}R \qquad (2.31)
\end{aligned}$$

By integration by parts,

$$\begin{aligned}
R &\leq 6\int_0^t\int_{\mathbb{R}^3}u_1^2\big((\partial_2 u_1)^2 + (\partial_1 u_1)^2\big)dxd\tau + 6\int_0^t\int_{\mathbb{R}^3}u_2^2\big((\partial_1 u_2)^2 + (\partial_2 u_2)^2\big)dxd\tau \\
&= 2\int_0^t\int_{\mathbb{R}^3}u_1^3(-\partial_1^2 u_1 - \partial_2^2 u_1)dxd\tau + 2\int_0^t\int_{\mathbb{R}^3}u_2^3(-\partial_1^2 u_2 - \partial_2^2 u_2)dxd\tau
\end{aligned}$$

Note that $\omega_3 = \partial_1 u_2 - \partial_2 u_1$ and $\text{div} u = 0$, the following identity is obtained by direct computation.

$$\partial_2\omega_3 = \partial_1\partial_2 u_2 - \partial_2^2 u_1 = -\partial_1^2 u_1 - \partial_2^2 u_1 - \partial_1\partial_3 u_3 \qquad (2.32)$$

$$\partial_1\omega_3 = \partial_1^2 u_2 - \partial_2\partial_1 u_1 = \partial_1^2 u_2 + \partial_2^2 u_2 + \partial_2\partial_3 u_3 \qquad (2.33)$$

Using (2.32) and (2.33), we obtain

$$\begin{aligned}
R &\leq 2\int_0^t\int_{\mathbb{R}^3}u_1^3(\partial_2\omega_3 + \partial_1\partial_3 u_3)dxd\tau + 2\int_0^t\int_{\mathbb{R}^3}u_2^3(-\partial_1\omega_3 + \partial_2\partial_3 u_3)dxd\tau \\
&\equiv R_1 + R_2 \qquad (2.34)
\end{aligned}$$

Using integration by parts and Young inequality, one has

$$\begin{aligned}
R_1 &= 2\int_0^t\int_{\mathbb{R}^3} u_1^3(\partial_2\omega_3+\partial_1\partial_3 u_3)dxd\tau \\
&= 6\int_0^t\int_{\mathbb{R}^3} u_1^2\big((\partial_2 u_1)^2+(\partial_1 u_1)^2\big)dxd\tau = -6\int_0^t\int_{\mathbb{R}^3} u_1^2\omega_3\partial_2 u_1 dxd\tau \\
&\quad +12\int_0^t\int_{\mathbb{R}^3} u_1u_3\partial_3 u_1\partial_1 u_1 dxd\tau + 6\int_0^t\int_{\mathbb{R}^3} u_1^2u_3\partial_1\partial_3 u_1 dxd\tau \\
&\le 12\int_0^t\int_{\mathbb{R}^3} |u_1u_3\partial_3 u_1\partial_1 u_1| dxd\tau + 6\int_0^t\int_{\mathbb{R}^3} |u_1^2u_3\partial_1\partial_3 u_1| dxd\tau \\
&\quad +3\int_0^t\int_{\mathbb{R}^3} u_1^2(\partial_2 u_1)^2 dxd\tau + 3\int_0^t\int_{\mathbb{R}^3} u_1^2\omega_3^2 dxd\tau \\
&\le 12\int_0^t\int_{\mathbb{R}^3} |u_1u_3\partial_3 u_1\partial_1 u_1| dxd\tau + 6\int_0^t\int_{\mathbb{R}^3} |u_1^2u_3\partial_1\partial_3 u_1| dxd\tau \\
&\quad +3\int_0^t\int_{\mathbb{R}^3} u_1^2\omega_3^2 dxd\tau + \frac{1}{2}R_1
\end{aligned}$$

Then

$$R_1 \le \int_0^t\int_{\mathbb{R}^3} 6u_1^2\omega_3^2 + 24|u_1u_3\partial_3 u_1\partial_1 u_1| + 12|u_1^2u_3\partial_1\partial_3 u_1| dxd\tau \tag{2.35}$$

The terms in (2.35) are similar to the terms which have been treated in $I_2$, $I_5^{1,2}$ and $I_5^{1,1}$ respectively. We would like to write down the estimates directly instead of the detailed computation.

$$\begin{aligned}
R_1 &\le C_9\|\nabla u\|_{L^{\infty,2}}^{1+4/\alpha}\|\Delta u\|_{L^{2,2}}^{6/\gamma}\|u_3\|_{L^{\alpha,\gamma}}^2 \\
&\quad +\frac{1}{6}\|\Delta u\|_{L^{2,2}}^2 + 12C_{15}\|u_3\|_{L^{\alpha,\gamma}}^2\|\nabla u\|_{L^{\infty,2}}^2 + C_{10}\|\nabla u\|_{L^{\infty,2}}
\end{aligned} \tag{2.36}$$

$R_2$ can be treated similarly, so we get the estimate of $|I_5^2+I_6^2|$ with

$$\begin{aligned}
|I_5^2+I_6^2| &\le \frac{1}{6}\|\Delta u\|_{L^{2,2}}^2 + 4C_{15}\|u_3\|_{L^{\alpha,\gamma}}^2\|\nabla u\|_{L^{\infty,2}}^2 \\
&\quad +\frac{1}{2}C_9\|\nabla u\|_{L^{\infty,2}}^{1+4/\alpha}\|\Delta u\|_{L^{2,2}}^{6/\gamma}\|u_3\|_{L^{\alpha,\gamma}}^2 + \frac{1}{2}C_{10}\|\nabla u\|_{L^{\infty,2}}
\end{aligned} \tag{2.37}$$

Combine (2.12), (2.16), (2.17), (2.18), (2.28), (2.29) and (2.37) together and substitute into (2.11), then we obtain

$$\begin{aligned}
&\frac{1}{2}\|\nabla u(.,t)\|_{L^2}^2 + \|\Delta u\|_{L^{2,2}}^2 \\
&\quad\le \frac{7}{12}\|\Delta u\|_{L^{2,2}}^2 + 2C_5\|\nabla u\|_{L^{\infty,2}}^2\|u_3\|_{L^{\alpha,\gamma}}^{\frac{2\gamma}{\gamma-3}} \\
&\qquad + \frac{5}{2}C_9\|\nabla u\|_{L^{\infty,2}}^{1+4/\alpha}\|\Delta u\|_{L^{2,2}}^{6/\gamma}\|u_3\|_{L^{\alpha,\gamma}}^2 \\
&\qquad + 8C_{15}\|u_3\|_{L^{\alpha,\gamma}}^2\|\nabla u\|_{L^{\infty,2}}^2 + \frac{5}{2}C_{10}\|\nabla u\|_{L^{\infty,2}} + \frac{1}{2}\|\nabla u_0\|_{L^2}^2 \qquad (2.38)
\end{aligned}$$

We will consider the case that $2/\alpha + 3/\gamma = 1/2$ first. Using Young inequality on the right hand side of (2.38), we obtain

$$\begin{aligned}
\frac{1}{2}\|\nabla u(.,t)\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^{2,2}}^2 \le\ & \left(C_{16}\|u_3\|_{L^{\alpha,\gamma}}^\alpha + 8C_{15}\|u_3\|_{L^{\alpha,\gamma}}^2 + \frac{1}{8}\right)\|\nabla u\|_{L^{\infty,2}}^2 \\
& + \frac{25}{2}C_{10}^2 + \frac{1}{2}\|\nabla u_0\|_{L^2}^2, \qquad (2.39)
\end{aligned}$$

where $C_{16}$ depends on $\alpha$, $\gamma$, $T$ and $\|u_0\|_{L^2}$ only.

Now we choose $0 < t_0 \le T$, such that

$$\|u_3\|_{L^{\alpha,\gamma}} = \left(\int_0^{t_0}\|u_3(.,\tau)\|_{L^\gamma}^\alpha d\tau\right)^{1/\alpha}$$

satisfies

$$C_{16}\|u_3\|_{L^{\alpha,\gamma}}^\alpha + 8C_{15}\|u_3\|_{L^{\alpha,\gamma}}^2 \le \frac{1}{8} \ \text{ on } (0,t_0). \qquad (2.40)$$

Putting (2.40) into (2.39), we obtain that

$$\sup_{0\le t\le t_0}\|\nabla u(.,t)\|_{L^2}^2 + \int_0^{t_0}\|\Delta u(.,\tau)\|_{L^2}^2 d\tau \le 50C_{10}^2 + 2\|\nabla u_0\|_{L^2}^2 \qquad (2.41)$$

Then we can repeat the above process from $t_0$ with $u(t_0)$ as its initial data for the problem (1.1) and get for $t_0 < t < T$

$$\begin{aligned}
&\frac{1}{2}\|\nabla u(.,t)\|_{L^2}^2 + \frac{1}{4}\int_{t_0}^t\|\Delta u(.,\tau)\|_{L^2}^2 d\tau \\
&\quad\le \left(C_{16}\|u_3\|_{L^{\alpha,\gamma}}^\alpha + 8C_{15}\|u_3\|_{L^{\alpha,\gamma}}^2 + \frac{1}{8}\right)\|\nabla u\|_{L^{\infty,2}}^2 \\
&\qquad + C_{17}^2 + \frac{1}{2}\|\nabla u(.,t_0)\|_{L^2}^2,
\end{aligned}$$

where $C_{17}$ depends on $\|\omega_3(.,t_0)\|_{L^2}$ which is bounded by $\|\nabla u(.,t_0)\|_{L^2}$, while the norm $\|u_3\|_{L^{\alpha,\gamma}}$ is given by

$$\|u_3\|_{L^{\alpha,\gamma}} = \left(\int_{t_0}^{t} \|u_3(.,\tau)\|_{L^\gamma}^{\alpha} d\tau\right)^{1/\alpha}.$$

Then for $t_1 - t_0$ sufficiently small, $t_0 < t_1 < T$, the following inequality holds

$$C_{16}\|u_3\|_{L^{\alpha,\gamma}}^{\alpha} + 8C_{15}\|u_3\|_{L^{\alpha,\gamma}}^{2} \le \frac{1}{8}, \quad \text{on } (t_0, t_1)$$

and consequently

$$\begin{aligned}\sup_{t_0\le\tau\le t_1} \|\nabla u(.,\tau)\|_{L^2} + \int_{t_0}^{t_1} \|\Delta u(.,\tau)\|_{L^2}^2 d\tau &< 4C_{17}^2 + 2\|\nabla u(.,t_0)\|_{L^2}^2 \\ &\le C(\alpha,\gamma,T,\|u_0\|_{L^2},\|\nabla u_0\|_{L^2}).\end{aligned}$$

Note that $u_3 \in L^{\alpha,\gamma}$ on $[0,T)$, and the coefficients involving $\|u_3\|_{L^{\alpha,\gamma}}$ in (2.39), $C_{15}$, $C_{16}$, depend only on $T$, $\alpha$, $\gamma$, $\|u_0\|_{L^2}$, therefore the above process only can be done for finite times. More precisely, we can get

$$\sup_{0\le t<T} \|\nabla u(.,t)\|_{L^2}^2 + \int_0^T \|\Delta u(.,\tau)\|_{L^2}^2 d\tau \le C_4 \tag{2.42}$$

where $C_4$ depends on $T$, $\alpha$, $\gamma$, $\|\nabla u_0\|_{L^2}$, $\|u_0\|_{L^2}$ and $\|u_3\|_{L^{\alpha,\gamma}}$.

Actually, the above process is a standard bootstrap argument. If one sets

$$f(t) = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{4}\int_0^t \|\Delta u(s)\|_{L^2}^2 ds,$$

what (2.39) really shows is that there exist $h > 0$, $\kappa < 1$ and $C > 0$ such that

$$f(t+\tau) \le f(t) + \kappa \sup_{0\le s\le\tau} f(t+s) + C,$$

whenever $0 \le t \le t+\tau \le T$ and $\tau \le h$. It follows that

$$\sup_{0\le\tau\le h} f(t+\tau) \le \frac{1}{1-\kappa}\left(f(t) + C\right),$$

hence by induction

$$f(t) + \frac{C}{\kappa} \le \left(\frac{1}{1-\kappa}\right)^{1+\frac{t}{h}} \left(f(0) + \frac{C}{\kappa}\right), \quad 0 \le t \le T.$$

The expression in the right-hand side depends explicitly on $h$, which is taken so that

$$\sup_{0\le t\le T-h}\int_t^{t+h}\|u_3(s)\|_{L^\gamma}^\alpha ds$$

is sufficiently small, which can be achieved by the integrability of $u_3$ in the space $L^\alpha([0,T],L^\gamma(\mathbb{R}^3))$.

The case with $2/\alpha+3/\gamma<1/2$ can be treated similarly, since the sum of the power index on the norm $\|\nabla u\|_{L^{\infty,2}}$ and $\|\Delta u\|_{L^{2,2}}$ is less than or equiv to 2, the bounds of the left hand side of (2.9) can be obtained.

**Case 2.** $u_3\in L^{4,\infty}$.

Actually, this case can be treated as a limit case for $\alpha=4$ and $\gamma=\infty$. Letting $\alpha=4$ and taking limit as $\gamma\to\infty$ in (2.38), one has the following estimate

$$\begin{aligned}
&\frac{1}{2}\|\nabla u(.,t)\|_{L^2}^2+\frac{1}{8}\|\Delta u\|_{L^{2,2}}^2\\
\le\ & C_{18}\|u_3\|_{L^{4,\infty}}^2\|\nabla u\|_{L^{\infty,2}}^2+C_{19}\|\nabla u\|_{L^{\infty,2}}+\frac{1}{2}\|\nabla u_0\|_{L^2}^2\\
\le\ & \left(C_{18}\|u_3\|_{L^{4,\infty}}^2+\frac{1}{4}\right)\|\nabla u\|_{L^{\infty,2}}^2+C_{19}^2+\frac{1}{2}\|\nabla u_0\|_{L^2}^2
\end{aligned}\tag{2.43}$$

where $C_{18}$ is an absolute constant, while $C_{19}$ depends on $\|u_0\|_{L^2}$ and $\|\nabla u_0\|_{L^2}$ only.

Then just as the argument of case 1, by the integrability of $\|u_3\|_{L^\infty}$ with respect to time variable, (2.9) can be obtained, and where $C_4$ depends only on $\|u_0\|_{L^2}$, $\|\nabla u_0\|_{L^2}$ and $\|u_3\|_{L^{4,\infty}}$.

The proof is complete. $\square$

# 3 Proof of Theorem 1.1

After we establish the key estimate in section 2, the proof of Theorem 1.1 is straightforward.

It is well known [25] that there is a unique strong solution $\tilde{u}\in L^\infty(0,T_0;H^1(\mathbb{R}^3))\cap u\in L^2(0,T_0;H^2(\mathbb{R}^3))$ to (1.1),for some $0<T_0$, for any given $u_0\in H^1(\mathbb{R}^3)$ with $\mathrm{div}u_0=0$. Since $u$ is a Leray-Hopf weak solution which satisfies the energy inequality, we have according to the uniqueness result, $u\equiv\tilde{u}$ on $[0,T_0)$. By the

a priori estimate (2.9) in Theorem 2.3 and standard continuation argument, the local strong solution $u$ can be extended to time $T$. So we have proved $u$ actually is a strong solution on $[0,T)$. This completes the proof for Theorem 1.1.

The following corollary follows from Theorem 1.1 directly.

**Corollary 3.1** *Suppose* $u_0 \in H^1(\mathbb{R}^3)$, *and* $\operatorname{div} u_0 = 0$ *in the sense of distribution. Assume that* $u(x,t)$ *is a Leray-Hopf weak solution of (1.1) in* $(0,T)$. *If* $\nabla u_3 \in L^{p,q}$ *with* $2/p + 3/q \le 3/2$, *for* $2 < q < 3$, *then* $u(x,t)$ *is a strong solution on* $[0,T)$.

**Proof:** By Gagliardo-Nirenberg inequality

$$\|u_3\|_{L^{\alpha,\gamma}} \le C_{27}\|u_3\|_{L^{a,b}}^{1-\theta}\|\nabla u_3\|_{L^{p,q}}^{\theta} \tag{3.1}$$

where

$$\frac{1}{\gamma} = \frac{1-\theta}{b} + \theta\big(\frac{1}{q} - \frac{1}{3}\big) \quad \text{and} \quad \frac{1}{\alpha} = \frac{1-\theta}{a} + \frac{\theta}{p}. \tag{3.2}$$

From (3.2), one obtains

$$\frac{2}{\alpha} + \frac{3}{\gamma} = (1-\theta)\big(\frac{2}{a} + \frac{3}{b}\big) + \theta\big(\frac{2}{p} + \frac{3}{q} - 1\big). \tag{3.3}$$

Since $2/\alpha + 3/\gamma \le 1/2$ and $2/a + 3/b \ge 3/2$, it follows from (3.3) that

$$\frac{\frac{5}{2}\theta - 1}{\theta} \ge \frac{2}{p} + \frac{3}{q}. \tag{3.4}$$

When $\theta = 1$, the function $\frac{\frac{5}{2}\theta-1}{\theta}$ obtains its maximal value $\frac{3}{2}$. But when $\theta = 1$, we have a restriction on $q$ with $q < 3$. In this case, (3.1) reduced to

$$\|u_3\|_{L^{p,\frac{3q}{3-q}}} \le C_{28}\|\nabla u_3\|_{L^{p,q}}, \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} \le \frac{3}{2}, \quad \text{for} \quad 2 < q < 3. \tag{3.5}$$

Thanks to (3.5), Corollary 2.4 follows from Theorem 1.1 directly. The proof is complete. $\square$

**Remark 3.1** In [26], the author proves the regularity criterion for $\nabla u_3 \in L^{p,q}$ with $2/p + 3/q = \frac{3}{2}$, for all $q \ge 3$.

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