Asymptotic behavior of the solutions to the 2D dissipative quasi-geostrophic flows

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Abstract: In this paper we derive a decay rate of the L^2-norm of the solution to the 2-D dissipative quasi-geostrophic flows comparing with the corresponding linear equation. We use a new, concise and direct method to avoid using the Fourier splitting technique completely and make the paper be self-contained without using any previous decay result.

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1 Introduction

In this paper, we consider the two-dimensional quasi-geostrophic equation in $\mathbb{R}^2$,

$$\begin{cases} \dfrac{\partial\theta}{\partial t} + u\cdot\nabla\theta + (-\Delta)^{\alpha}\theta = f, \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $\alpha \in (0,1]$, $\theta(x,t)$ is the potential temperature, f is the force and it is assumed to be zero in what follows just for simplicity, the fluid velocity $u =$

$(u_1, u_2) \in \mathbb{R}^2$ is determined from θ by a stream function Ψ,

$$(u_1, u_2) = \left(-\frac{\partial \Psi}{\partial x_2}, \frac{\partial \Psi}{\partial x_1}\right),$$

while the function Ψ satisfies

$$(-\Delta)^{1/2}\Psi = -\theta.$$

The operator $(-\Delta)^\gamma$ ($\gamma > 0$) is defined by [9]

$$\widehat{(-\Delta)^\gamma} f(\xi) = |\xi|^{2\gamma}\hat{f},$$

where $\hat{f}$ denotes the Fourier transform of f. As usual, we write $(-\Delta)^{1/2}$ as Λ.

By reduction to the special case of solutions with constant potential vorticity in the interior and constant buoyancy frequency, the inviscid 2D quasi-geostrophic equations can be derived from the general quasi-geostrophic equations. And (1.1) is obtained if the dissipative mechanisms are incorporated into the inviscid 2D quasi-geostrophic equations. From the mathematical view point, this model (1.1) is striking similar to the 3D hydrodynamics equations, say the Navier-Stokes equations, although (1.1) is considerably simpler than the 3D Navier-Stokes equations. Moreover (1.1) with $\alpha = 1/2$ is analogous to the 3D Navier-Stokes equations dimensionally. It is proved that the weak solutions to (1.1) globally exist, but the regularity and uniqueness are still big open problems, just as the situation for 3D Navier-Stokes equations [3]. In [4], the strong solution is unique and exists locally, and it is unique among the weak solutions for $\alpha \in (1/2, 1)$. In other words, the weak solutions must coincide with the strong solution occupied with the same initial datum, as long as the strong solution exists.

This paper is concerned with the decay rate of the solutions to (1.1) in the L^2-norm. We consider the linear equation corresponding to (1.1)

$$\begin{cases} \dfrac{\partial \theta}{\partial t} + \Lambda^{2\alpha}\theta = 0, \\ \theta(x, 0) = \theta_0(x). \end{cases} \tag{1.2}$$

The solution of (1.2) can be represent by the fundamental solution as

$$\Theta(t) = e^{t\Lambda^{2\alpha}}\theta_0 = G_\alpha(t) * \theta_0,$$

where G_α is given from the Fourier transform as

$$\hat{G_\alpha}(\xi,t) = e^{-|\xi|^{2\alpha}t}.$$

In this paper, we give a decay rate of the L^2-norm of the solution in term of the decay rate of the linear equation (1.2) and the fractional power index α in (1.1). More precisely, it reads

Theorem 1.1 *Let $\alpha \in (1/2, 1)$ and $\theta_0 \in L^2(\mathbb{R}^2)$. If the solution $e^{t\Lambda^{2\alpha}}\theta_0(t)$ to the linear equation (1.2) satisfies*

$$\left\| e^{t\Lambda^{2\alpha}}\theta_0 \right\|_{L^2} \leq C(1+t)^{-\beta}, \quad t \geq 0, \tag{1.3}$$

for some $\beta > 0$. Then there exists a weak solution $\theta(t)$ to (1.1) such that

$$\|\theta(t)\|_{L^2} \leq C(1+t)^{-\gamma}, \quad \text{with } \gamma = \min\left\{\beta, \frac{1}{\alpha}\right\}. \tag{1.4}$$

Moreover, the solution $\theta(t)$ to (1.1) is asymptotic equivalent to the solution $e^{t\Lambda^{2\alpha}}\theta_0$ of (1.2) in the sense that

$$\left\| \theta(t) - e^{t\Lambda^{2\alpha}}\theta_0 \right\|_{L^2} \leq C(1+t)^{-\frac{1}{\alpha}}. \tag{1.5}$$

Remark 1.1 Theorem 1.1 is motivated mainly by an analogue result for the 3D Navier-Stokes equations was proved by Wiegner in [6, 11]. The usual method to prove the asymptotic behavior is the so called Fourier splitting method, which was used first by Schonbek [5] on the decay of solutions for parabolic conservation laws. Later on she used it to do several results for the Navier-Stokes equations, c.f. [6, 7]. However, we will show Theorem 1.1 by a new, direct and much simpler method, which completely avoids using the Fourier splitting technique. It is testified that this strategy (see section 4) can be used widely. For example, we [12] proved the result of Wiegner by this method very recently.

Remark 1.2 In [4], Constantin and Wu (page 940, Theorem 3.1) proved that there exists a weak solution $\theta(x,t)$ such that

$$\|\theta(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2\alpha}}, \tag{1.6}$$

and in general, the decay rate $-\frac{1}{2\alpha}$ is optimal in the sense that there exists some initial datum such that the corresponding solution to (1.1) satisfies

$$\|\theta(t)\|_{L^2} \geq C(1+t)^{-\frac{1}{2\alpha}}.$$

However, it is easy to find that there are solutions to (1.2) have exponentially decay. For example $\hat{\theta}_0(\xi) = 0$, for $|\xi| \leq r$ with $r > 0$, then the solution satifies

$$\left\| e^{t\Lambda^{2\alpha}} \theta_0 \right\|_{L^2}^2 = \int_{\mathbb{R}^2} e^{-2|\xi|^{2\alpha} t} |\hat{\theta}_0|^2(\xi) d\xi \leq e^{-2r^{2\alpha} t} \|\theta_0\|_{L^2}^2.$$

So in this sense, (1.4) is a significant achievement for the decay rate of solutions to (1.1) comparing with that of the linear equation (1.2).

Also in [4] (page 944, Theorem 4.3), it was proved that the difference $\theta(t) - \Theta(t)$ between a weak solution $\theta(t)$ of the quasi-geostrophic equation (1.1) and solution $\Theta(t)$ of the linear quasi-geostrophic equation (1.2) with the same data $\theta_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ satisfies

$$\|\theta(t) - \Theta(t)\|_{L^2} \leq C(1+t)^{\frac{1}{2} - \frac{1}{\alpha}}.$$

Comparing with their result, first Theorem 1.1 holds for any $\theta_0 \in L^2(\mathbb{R}^2)$. Secondly the decay rate $-\frac{1}{\alpha}$ is much better than that of theirs, $\frac{1}{2} - \frac{1}{\alpha}$.

Remark 1.3 In section 3, under the restriction of $\alpha \in (2/3, 1)$, a rough estimate for $\|\nabla\theta(t)\|_{L^2}$ is shown by a direct and simple method (energy method) instead of using the so called Fourier splitting method. Another advantage of this method is making this paper be self-contained and without using any other previous decay results. On the other hand, in the appendix, we show that the solution satisfies

$$\|\nabla\theta(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2\alpha}}, \tag{1.7}$$

by using the known decay result (1.6) and inequality (5.1).

2 Preliminaries

We denote the Riesz transform in $\mathbb{R}^2$ by $\mathcal{R}_j$, $j = 1, 2$ as

$$\widehat{\mathcal{R}_j f} = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi).$$

The operator $\mathcal{R}^\perp$ is defined by

$$\mathcal{R}^\perp f = \left(-\partial_{x_2} \Lambda^{-1} f, -\partial_{x_1} \Lambda^{-1} f\right) = \left(-\mathcal{R}_2 f, \mathcal{R}_1 f\right),$$

so the relation between u and θ is given by $u = \mathcal{R}^\perp \theta$. Moreover, we have

Lemma 2.1 *There exists a constant $C(p)$ depending only on p such that*

$$\left\|\Lambda^{\delta} u\right\|_{L^p} \leq C(p)\left\|\Lambda^{\delta} \theta\right\|_{L^p}, \tag{2.1}$$

for all $\delta \geq 0$, $1<p<\infty$. If $p=2$, the inequality (2.1) actually is an identity.

The proof can be finished by following from the boundedness of the Riesz transforms in L^p, c.f. [9].

The next lemma is concerned with an embedding for the fractional Sobolev spaces.

Lemma 2.2 *Let $2<p<\infty$ and $\delta=1-\frac{2}{p}$, then there exists a constant $C(p)$ such that*

$$\|f\|_{L^p} \leq C(p)\left\|\Lambda^{\delta} f\right\|_{L^2}, \tag{2.2}$$

for all $f \in \mathcal{S}'$.

Since $\hat{f}=|\xi|^{-\delta}|\xi|^{\delta} \hat{f}$, from the inverse Fourier transform,

$$f=I^{\delta}\left(\Lambda^{\delta} f\right),$$

where I^{δ} is the Riesz potential of order δ. Hence (2.2) follows from the boundedness of I^{δ} form $L^2(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$, with $\frac{2}{p}=1-\delta$, c.f. [9].

By Parseval's equality, it follows from Lemma 2.2 that

$$\begin{aligned}
\|f\|_{L^p} & \leq C\left\|\Lambda^{\delta} f\right\|_{L^2}=C\left(\int_{\mathbb{R}^2}|\xi|^{2 \delta} \hat{f}^2(\xi) d \xi\right)^{1/2} \\
& = C\left(\int_{\mathbb{R}^2} \hat{f}^{2a}(\xi)|\xi|^{2 \delta} \hat{f}^{2(1-a)}(\xi) d \xi\right)^{1/2} \\
& \leq C\|f\|_{L^2}^{a}\left\|\Lambda^{\frac{\delta}{1-a}} f\right\|_{L^2}^{1-a},
\end{aligned}$$

where we used Hölder's inequality. So we have the following fractional type Gagliardo-Nirenberg inequality

$$\|f\|_{L^p} \leq C\|f\|_{L^2}^{a}\left\|\Lambda^{\sigma} f\right\|_{L^2}^{1-a}, \tag{2.3}$$

with

$$\frac{1}{p}=a \frac{1}{2}+(1-a)\left(\frac{1}{2}-\frac{\sigma}{2}\right), \quad 0 \leq a \leq 1.$$

3 A rough decay estimate for $\|\nabla\theta(t)\|_{L^2}$

Direct computation yields

$$\|\Lambda\theta\|_{L^2}^2 = \int_{\mathbb{R}^2} |\xi|^2 \hat{\theta}^2(\xi) d\xi = \int_{\mathbb{R}^2} \left|\xi\hat{\theta}(\xi)\right|^2 d\xi = \|\nabla\theta\|_{L^2}^2,$$

which implies that $\|\nabla\theta(t)\|_{L^2}$ is equivalent to $\|\Lambda\theta\|_{L^2}$. The goal of this section is to give a rough decay estimate for $\|\nabla\theta(t)\|_{L^2}$ based on delicate $L^p - H^s$ estimates.

Assume that the solution is smooth in what follows in this section, i.e., we do the formal computation first. Multiplying $\Lambda^2\theta$ on both sides , we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\Lambda\theta|^2 dx + \int_{\mathbb{R}^2} \left|\Lambda^{\alpha+1}\theta\right|^2 dx \le \left|\int_{\mathbb{R}^2} (u\cdot\nabla\theta)\Lambda^2\theta dx\right|. \tag{3.1}$$

Due to the divergence free of the velocity field u, we have

$$\begin{aligned} \left|\int_{\mathbb{R}^2} (u\cdot\nabla\theta)\Lambda^2\theta dx\right| &= \left|\int_{\mathbb{R}^2} \mathrm{div}(u\theta)\Lambda^2\theta dx\right| \\ &\le \int_{\mathbb{R}^2} |\xi|^2 \left|\xi_1\widehat{\theta u_1}(\xi) + \xi_2\widehat{\theta u_2}(\xi)\right| \left|\hat{\theta}(\xi)\right| d\xi \\ &\le \frac{1}{2}\left\|\Lambda^{\alpha+1}\theta\right\|_{L^2}^2 + \frac{1}{2}\left\|\Lambda^{2-\alpha}(\theta u)\right\|_{L^2}^2 \end{aligned} \tag{3.2}$$

Using the product estimate [10], we have

$$\left\|\Lambda^{2-\alpha}(\theta u)\right\|_{L^2} \le C\left(\|u\|_{L^p}\left\|\Lambda^{2-\alpha}\theta\right\|_{L^q} + \|\theta\|_{L^p}\left\|\Lambda^{2-\alpha}u\right\|_{L^q}\right),$$

with $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$. Submitting this inequality to (3.2) and using Lemma 2.1, (3.1) is reduced to

$$\frac{d}{dt}\|\Lambda\theta\|_{L^2}^2 + \left\|\Lambda^{\alpha+1}\theta\right\|_{L^2}^2 \le C\|\theta\|_{L^p}^2\left\|\Lambda^{2-\alpha}\theta\right\|_{L^q}^2. \tag{3.3}$$

Taking $p = \frac{2}{2\alpha-1}$ and $q = \frac{1}{1-\alpha}$, then by the embedding lemma (Lemma 2.2) and the fractional type Gagliardo-Nirenberg inequality (2.3), we get

$$\|\theta\|_{L^{\frac{2}{2\alpha-1}}}^2\left\|\Lambda^{2-\alpha}\theta\right\|_{L^{\frac{1}{1-\alpha}}}^2 \le C\|\theta\|_{L^2}^{\frac{6\alpha-4}{\alpha}}\|\Lambda^\alpha\theta\|_{L^2}^{\frac{4-4\alpha}{\alpha}}\left\|\Lambda^{\alpha+1}\theta\right\|_{L^2}^2, \tag{3.4}$$

provided that $\frac{2}{3} < \alpha < 1$. Putting (3.4) into (3.3), we obtain

$$\frac{d}{dt}\|\Lambda\theta\|_{L^2}^2 \le \left(C\|\theta\|_{L^2}^{\frac{6\alpha-4}{\alpha}}\|\Lambda^\alpha\theta\|_{L^2}^{\frac{4-4\alpha}{\alpha}} - 1\right)\left\|\Lambda^{\alpha+1}\theta\right\|_{L^2}^2. \tag{3.5}$$

Similarly, we can get a differential inequality for $\|\Lambda^\alpha\theta\|_{L^2}$,

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\alpha\theta\|_{L^2}^2+\left\|\Lambda^{2\alpha}\theta\right\|_{L^2}^2\leq\frac{1}{2}\left\|\Lambda^{2\alpha}\theta\right\|_{L^2}^2+C\|\theta\|_{L^{\frac{2}{2\alpha-1}}}^2\|\Lambda\theta\|_{L^{\frac{1}{1-\alpha}}}^2. \tag{3.6}$$

Hence, using the Gagliardo-Nirenberg inequality (2.3) and rewriting (3.6), we have

$$\frac{d}{dt}\|\Lambda^\alpha\theta\|_{L^2}^2\leq\left(C\|\theta\|_{L^2}^{\frac{6\alpha-4}{\alpha}}\|\Lambda^\alpha\theta\|_{L^2}^{\frac{4-4\alpha}{\alpha}}-1\right)\left\|\Lambda^{2\alpha}\theta\right\|_{L^2}^2. \tag{3.7}$$

On the other hand, multiplying the equation (1.1) by θ, and integrating for both space and time, then

$$\|\theta(.,t)\|_{L^2}^2+2\int_0^t\|\Lambda^\alpha\theta(.,s)\|_{L^2}^2\,ds=\|\theta_0\|_{L^2}^2,\quad\text{for all } t\geq 0.$$

Therefore, there exists a time t_0 such that

$$\|\theta(t_0)\|_{L^2}^{\frac{6\alpha-4}{\alpha}}\|\Lambda^\alpha\theta(t_0)\|_{L^2}^{\frac{4-4\alpha}{\alpha}}\leq\|\theta_0\|_{L^2}^{\frac{6\alpha-4}{\alpha}}\|\Lambda^\alpha\theta(t_0)\|_{L^2}^{\frac{4-4\alpha}{\alpha}}\leq\frac{1}{C},$$

where C is the bigger constant of these in (3.5) and (3.7).

From this equation (3.7) and the choice of t_0, we have

$$\frac{d}{dt}\|\Lambda^\alpha\theta\|_{L^2}\leq 0,\quad\text{for all } t\geq t_0. \tag{3.8}$$

Combining (3.5) and (3.8), one has

$$\frac{d}{dt}\|\Lambda\theta\|_{L^2}\leq 0,\quad\text{for all } t\geq t_0. \tag{3.9}$$

Then we want to obtain the integrability for $\|\Lambda\theta\|_{L^2}$. Multiplying (1.1) by $\Lambda^{2-2\alpha}\theta$, similar computation yields

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^{1-\alpha}\theta\right\|_{L^2}^2+\|\Lambda\theta\|_{L^2}^2\leq\frac{1}{2}\|\Lambda\theta\|_{L^2}^2+C\|\theta\|_{L^2}^{\frac{6\alpha-4}{\alpha}}\|\Lambda^\alpha\theta\|_{L^2}^{\frac{4-4\alpha}{\alpha}}\|\Lambda\theta\|_{L^2}^2. \tag{3.10}$$

Now we assume that

$$\|\theta(t_1)\|_{L^2}^{\frac{6\alpha-4}{\alpha}}\|\Lambda^\alpha\theta(t_1)\|_{L^2}^{\frac{4-4\alpha}{\alpha}}\leq\|\theta_0\|_{L^2}^{\frac{6\alpha-4}{\alpha}}\|\Lambda^\alpha\theta(t_1)\|_{L^2}^{\frac{4-4\alpha}{\alpha}}\leq\frac{1}{2C},\quad\text{for some } t_1>t_0.$$

Then integrating (3.10) with respect to time on $[t_1,t]$, $t>t_1$,

$$\begin{aligned}&\left\|\Lambda^{1-\alpha}\theta(t)\right\|_{L^2}^2+\int_{t_1}^t\|\Lambda\theta(s)\|_{L^2}^2\,ds\\ &\leq\ \left\|\Lambda^{1-\alpha}\theta(t_1)\right\|_{L^2}^2+C\int_0^t\|\theta(s)\|_{L^2}^{\frac{6\alpha-4}{\alpha}}\|\Lambda^\alpha\theta(s)\|_{L^2}^{\frac{4-4\alpha}{\alpha}}\|\Lambda\theta(s)\|_{L^2}^2\,ds.\end{aligned} \tag{3.11}$$

From (3.9) and (3.11), by the choice of t_1, we obtain

$$(t-t_1)\|\Lambda\theta(t)\|_{L^2}^2 \le \int_{t_1}^t \|\Lambda\theta(s)\|_{L^2}^2 ds \le 2\left\|\Lambda^{1-\alpha}\theta(t_1)\right\|_{L^2}^2.$$

So we get a rough decay estimate as

$$\|\Lambda\theta(t)\|_{L^2} = \|\nabla\theta(t)\|_{L^2} \le C(1+t)^{-1/2}, \quad \text{for all } t \ge 0. \tag{3.12}$$

Remark 3.1 In [8], a faster decay rate

$$\|\nabla\theta(t)\|_{L^2} \le C(1+t)^{-1/\alpha}$$

was proved by using Fourier splitting method and the known result (1.6). But here (3.12) is enough for our purpose.

4 Proof of the main theorem

The first part of the proof is formal, that is, we assume the solution is smooth. Actually, it is somehow enough, since we can give a rigorous proof by applying the first part to a sequence of 'retarded mollification', just as what done in [1, 4, 8].

We present the solution by the fundamental solution of (1.2) as

$$\theta(t) = e^{t\Lambda^{2\alpha}}\theta_0 - \int_0^t e^{(t-s)\Lambda^{2\alpha}}(u\cdot\nabla\theta)(s)ds. \tag{4.1}$$

Note that

$$u\cdot\nabla\theta = \mathrm{div}(\theta u) - \theta\mathrm{div}u = \mathrm{div}(u\theta),$$

so the solution has another form

$$\theta(t) = e^{t\Lambda^{2\alpha}}\theta_0 - \int_0^t e^{(t-s)\Lambda^{2\alpha}}\mathrm{div}(u\theta)(s)ds. \tag{4.2}$$

So directly, form (4.1) and (4.2), we have

$$\|\theta(t)\|_{L^2} \le \left\|e^{t\Lambda^{2\alpha}}\theta_0\right\|_{L^2} + \int_0^t \left\|e^{(t-s)\Lambda^{2\alpha}}(u\cdot\nabla\theta)\right\|_{L^2}(s)ds. \tag{4.3}$$

and

$$\|\theta(t)\|_{L^2} \le \left\|e^{t\Lambda^{2\alpha}}\theta_0\right\|_{L^2} + \int_0^t \left\|e^{(t-s)\Lambda^{2\alpha}}\mathrm{div}(u\theta)\right\|_{L^2}(s)ds. \tag{4.4}$$

By Parseval's equality, it follows that

$$\begin{aligned}
\left\|e^{t\Lambda^{2\alpha}}(u\cdot\nabla\theta)\right\|_{L^2}^2 &= \int_{\mathbb{R}^2} e^{-2|\xi|^{2\alpha}t}\left|\widehat{u\cdot\nabla\theta}\right|^2(\xi)d\xi \\
&\leq \left\|\widehat{u\cdot\nabla\theta}\right\|_{L^\infty}^2 \int_{\mathbb{R}^2} e^{-2|\xi|^{2\alpha}t}d\xi \leq C\left\|u\cdot\nabla\theta\right\|_{L^1}^2 \int_0^\infty e^{-2r^{2\alpha}t}rdr \\
&\leq C\|u\|_{L^2}^2\|\nabla\theta\|_{L^2}^2 t^{-1/\alpha}\int_0^\infty e^{-2r^{2\alpha}}rdr \\
&\leq Ct^{-1/\alpha}\|\theta\|_{L^2}^2\|\nabla\theta\|_{L^2}^2. \qquad (4.5)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left\|e^{t\Lambda^{2\alpha}}\mathrm{div}(u\theta)\right\|_{L^2}^2 &\leq \int_{\mathbb{R}^2} e^{-2|\xi|^{2\alpha}t}|\xi|^2\left|\widehat{u\theta}\right|^2(\xi)d\xi \\
&\leq \left\|\widehat{u\theta}\right\|_{L^\infty}^2 \int_{\mathbb{R}^2} e^{-2|\xi|^{2\alpha}t}|\xi|^2 d\xi \leq C\left\|u\theta\right\|_{L^1}^2 \int_0^\infty e^{-2r^{2\alpha}t}r^3dr \\
&\leq C\|u\|_{L^2}^2\|\theta\|_{L^2}^2 t^{-2/\alpha}\int_0^\infty e^{-2r^{2\alpha}}r^3dr \\
&\leq Ct^{-2/\alpha}\|\theta\|_{L^2}^4. \qquad (4.6)
\end{aligned}$$

Combining (3.12), (4.3) and (4.5), it follows that

$$\|\theta(t)\|_{L^2} \leq C(1+t)^{-\beta} + C\int_0^t (t-s)^{-\frac{1}{2\alpha}}(1+s)^{-1/2}\|\theta(s)\|_{L^2}ds. \qquad (4.7)$$

Then, from (4.7), we have

$$\begin{aligned}
(1+t)^\beta\|\theta(t)\|_{L^2} &\leq C + C(1+t)^\beta Q(t)\int_0^t (t-s)^{-\frac{1}{2\alpha}}(1+s)^{-1/2-\beta}ds \\
&= C + C(1+t)^\beta Q(t)\int_0^{t/2} (t-s)^{-\frac{1}{2\alpha}}(1+s)^{-1/2-\beta}ds \\
&\quad + C(1+t)^\beta Q(t)\int_{t/2}^t (t-s)^{-\frac{1}{2\alpha}}(1+s)^{-1/2-\beta}ds \qquad (4.8)
\end{aligned}$$

with

$$Q(t) = \max_{0\leq s\leq t}\left\{(1+s)^\beta\|\theta(s)\|_{L^2}\right\}.$$

By direct computation,

$$\int_0^{t/2}(t-s)^{-\frac{1}{2\alpha}}(1+s)^{-1/2-\beta}ds \leq Ct^{-\frac{1}{2\alpha}}\begin{cases} 1 & \text{if } 1/2+\beta>1, \\ \ln(e+t) & \text{if } 1/2+\beta=1, \\ (1+t)^{1-1/2-\beta} & \text{if } 1/2+\beta<1, \end{cases}$$

and

$$\int_{t/2}^{t}(t-s)^{-\frac{1}{2\alpha}}(1+s)^{-1/2-\beta}ds \le C(1+t)^{-\beta-1/2}t^{\frac{2\alpha-1}{2\alpha}}.$$

Hence, if $\beta < \frac{1}{2\alpha}$, then

$$C(1+t)^{\beta}\int_{0}^{t}(t-s)^{-\frac{1}{2\alpha}}(1+s)^{-1/2-\beta}ds \to 0, \quad \text{as} \quad t\to\infty.$$

So there exists a t_0 sufficiently large such that

$$C(1+t)^{\beta}\int_{0}^{t}(t-s)^{-\frac{1}{2\alpha}}(1+s)^{-1/2-\beta}ds \le \frac{1}{2}, \quad \text{for any } t\ge t_0.$$

Then from (4.8), we have

$$(1+t)^{\beta}\|\theta(t)\|_{L^2} \le C + \frac{1}{2}Q(t), \quad \text{for } t\ge t_0. \tag{4.9}$$

Let

$$\hat{Q}(t) = \max_{t_0\le s\le t}\left\{(1+s)^{\beta}\|\theta(s)\|_{L^2}\right\},$$

then (4.9) can be reduced to

$$(1+t)^{\beta}\|\theta(t)\|_{L^2} \le C + \frac{1}{2}Q(t_0) + \frac{1}{2}\hat{Q}(t), \quad \text{for } t\ge t_0. \tag{4.10}$$

Now taking maximum for $t\in[t_0,T]$ on both sides of (4.10), we obtain

$$\hat{Q}(T) \le C + \frac{1}{2}Q(t_0) + \frac{1}{2}\hat{Q}(T), \quad \text{for } T\ge t_0.$$

Therefore,

$$(1+t)^{\beta}\|\theta(t)\|_{L^2} \le 2C + \max_{0\le s\le t_0}\{(1+s)^{\beta}\|\theta(s)\|_{L^2}\} < \infty,$$

due to the energy inequality.

Now, we assume $\beta \ge \frac{1}{2\alpha}$. Since

$$(1+t)^{-\beta} < (1+t)^{-\frac{\alpha+1}{4\alpha}}, \quad \text{and} \quad \frac{\alpha+1}{4\alpha} < \frac{1}{2\alpha},$$

it follows from the above step that

$$\|\theta(t)\|_{L^2} \le C(1+t)^{-\frac{\alpha+1}{4\alpha}}. \tag{4.11}$$

Thanks to (4.5) and (4.6), we obtain that

$$\begin{aligned}\|\theta(t)\|_{L^2} &\le C(1+t)^{-\beta}+C\int_0^{t/2}(t-s)^{-\frac{1}{\alpha}}\|\theta(s)\|_{L^2}^2ds\\ &\quad +C\int_{t/2}^{t}(t-s)^{-\frac{1}{2\alpha}}(1+s)^{-1/2}\|\theta(s)\|_{L^2}ds \qquad (4.12)\end{aligned}$$

Then by the rough estimate (4.11), it follows that

$$\int_0^{t/2}(t-s)^{-\frac{1}{\alpha}}\|u(s)\|_{L^2}^2ds\le C\int_0^{t/2}(t-s)^{-\frac{1}{\alpha}}(1+s)^{-\frac{\alpha+1}{2\alpha}}ds\le Ct^{-\frac{1}{\alpha}}.$$

So (4.12) reduces to

$$\|\theta(t)\|_{L^2}\le C(1+t)^{-\gamma}+C\int_{t/2}^{t}(t-s)^{-\frac{1}{2\alpha}}(1+s)^{-1/2}\|\theta(s)\|_{L^2}ds, \qquad (4.13)$$

where $\gamma=\min\left\{\beta,\frac{1}{\alpha}\right\}$.

Multiplying $(1+t)^\gamma$ on the both sides of (4.13), we have

$$\begin{aligned}(1+t)^\gamma\|\theta(t)\|_{L^2} &\le C+C(1+t)^\gamma Q(t)\int_{t/2}^{t}(t-s)^{-\frac{1}{2\alpha}}(1+s)^{-1/2-\gamma}ds\\ &\le C+CQ(t)(1+t)^{-1/2-\gamma}t^{\frac{2\alpha-1}{2\alpha}},\end{aligned}$$

with $Q(t)=\max_{0\le s\le t}\{(1+s)^\gamma\|\theta(s)\|_{L^2}\}$.

Therefore

$$(1+t)^\gamma\|\theta(t)\|_{L^2}\le\max\left\{2C,\max_{0\le s\le t_0}\{(1+s)^\gamma\|\theta(s)\|_{L^2}\}\right\}<\infty,$$

where t_0 satisfies $C(1+t_0)^{-1/2-\gamma}t_0^{1/4}\le 1/2$.

Let $\theta(t)$ be the solution to (1.1) and $w=\theta(t)-e^{\Lambda^{2\alpha}t}\theta_0$, then $w(t)$ satisfies

$$\begin{cases}\dfrac{\partial w}{\partial t}+u\cdot\nabla\theta+\Lambda^{2\alpha}w=0,\\ w(x,0)=0.\end{cases} \qquad (4.14)$$

So the solution $w(t)$ to (4.16) can be write as

$$w(t)=\int_0^t e^{(t-s)\Lambda^{2\alpha}}(u\cdot\nabla\theta)(s)ds. \qquad (4.15)$$

Comparing with (4.1), there is no linear term $e^{\Lambda^{2\alpha}t}\theta_0$ in the presentation of $w(t)$. By the above argument, it is easy to see that (1.5) follows from (4.15) directly.

The formal proof is complete.

To make the proof rigorous, we apply the formal proof on the approximate sequence, which are smooth solutions to

$$\frac{\partial \theta_n}{\partial t} + u_n \cdot \nabla \theta_n + (-\Delta)^\alpha \theta_n = 0. \tag{4.16}$$

In (4.16), $u_n = \Psi_{\delta_n}(\theta_n)$ is obtained from θ_n by

$$\Psi_{\delta_n}(\theta_n) = \int_0^\infty \psi(\tau) \mathcal{R}^\perp \theta_n(t - \delta_n \tau) d\tau,$$

where the function ψ is smooth and has support in the interval $[1, 2]$, and $\int_0^\infty \psi(s)\, ds = 1$, (see the similar construction for the 3D Navier-Stokes equations in [1]). For each n, it is easy to find that the values of u_n depend only on the values of θ_n in $[t - 2\delta_n, t - \delta]$, so the equation is (4.16) is linear. As stated in [4], the θ_n converges to a weak solution θ strongly in L^2 for almost every t. Hence

$$\|\theta(t)\|_{L^2} \le \|\theta_n(t) - \theta(t)\|_{L^2} + \|\theta_n(t)\|_{L^2} \le C(1+t)^{-\gamma},$$

with $\gamma = \min\{\beta, \frac{1}{\alpha}\}$.

5 Appendix

Let us recall an maximum principle inequality for (1.1)

$$\|\theta(t)\|_{L^q} \le \|\theta_0\|_{L^q}, \quad \text{for any } q \ge 2. \tag{5.1}$$

For the proof we refer the reader to [2].

Multiplying the equation (1.1) by $\Lambda^2 \theta$ and integrating in $\mathbb{R}^2$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Lambda\theta\|_{L^2}^2 + \left\|\Lambda^{1+\alpha}\theta\right\|_{L^2}^2 &\le \frac{1}{4}\left\|\Lambda^{1+\alpha}\theta\right\|_{L^2}^2 + \frac{1}{2}\|\theta\|_{L^{\frac{4}{2\alpha-1}}} \left\|\Lambda^{2-\alpha}\theta\right\|_{L^{\frac{4}{3-2\alpha}}} \\ &\le \frac{1}{4}\left\|\Lambda^{1+\alpha}\theta\right\|_{L^2}^2 + C\left\|\Lambda^{3/2}\theta\right\|_{L^2}^2, \end{aligned} \tag{5.2}$$

where we used (2.2) and (5.1).

Now we use Fourier splitting method. Let $B_R = \{\xi : |\xi| \le R\}$. Then

$$\begin{aligned} \left\|\Lambda^{3/2}\theta\right\|_{L^2}^2 &= \int_{B_R} |\xi|^3 |\hat{\theta}|^2(\xi) d\xi + \int_{\mathbb{R}^2 \setminus B_R} |\xi|^3 |\hat{\theta}|^2(\xi) d\xi \\ &\le R^3 \|\theta\|_{L^2}^2 + CR^{1-2\alpha}\left\|\Lambda^{1+\alpha}\theta\right\|_{L^2}^2 \end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
\|\Lambda^{1+\alpha}\theta\|_{L^2}^2 &\geq \int_{\mathbb{R}^2\setminus B_R} |\xi|^{2+2\alpha}|\hat{\theta}|^2(\xi)d\xi \geq R^{2\alpha}\int_{\mathbb{R}^2\setminus B_R} |\xi|^2|\hat{\theta}|^2(\xi)d\xi \\
&= R^{2\alpha}\left(\int_{\mathbb{R}^2} |\xi|^2|\hat{\theta}|^2(\xi)d\xi + \int_{B_R} |\xi|^2|\hat{\theta}|^2(\xi)d\xi\right) \\
&\geq R^{2\alpha}\|\Lambda\theta\|_{L^2}^2 - R^{2+2\alpha}\|\theta\|_{L^2}^2 \qquad (5.4)
\end{aligned}$$

Putting (5.3) and (5.4) into (5.2) and letting R large enough such that $CR^{1-2\alpha} \leq \frac{1}{4}$, we have

$$\frac{d}{dt}\|\Lambda\theta\|_{L^2}^2 + R^{2\alpha}\|\Lambda\theta\|_{L^2}^2 \leq CR^{2+2\alpha}(1+t)^{-\frac{1}{\alpha}}, \qquad (5.5)$$

due to the decay rate (1.6).

Then (5.5) implies (1.7). Indeed, multiplying $e^{R^{2\alpha}t}$ on (5.5) and integrating with respect to t, we obtained

$$\begin{aligned}
\|\Lambda\theta\|_{L^2}^2 &\leq e^{-R^{2\alpha}t}\|\Lambda\theta_0\|_{L^2}^2 + CR^{2+2\alpha}\int_0^t e^{-R^{2\alpha}(t-s)}(1+s)^{-\frac{1}{\alpha}}ds \\
&\leq e^{-R^{2\alpha}t}\|\Lambda\theta_0\|_{L^2}^2 + CR^{2+2\alpha}(1+t)^{-\frac{1}{\alpha}}.
\end{aligned}$$

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