Unique Continuation for the Symmetric Regularized Long Wave Equation

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Abstract The unique continuation property has been intensively studied for a long time due to the important role that plays in the applications. The validity of the unique continuation property for symmetric regularized long wave equation is showed in this paper. The result is established by using an appropriate Carleman-type estimate for a partial differential operators closely related to our problem.

Key Words: Unique continuation, Symmetric regularized long wave equation, Carleman type estimates, Treves's inequality

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1 Introduction

A symmetric version of regularized long wave equation

$$u_{xxt} - u_t = \rho_x + uu_x, \tag{1}$$

$$\rho_t + u_x = 0. \tag{2}$$

has been proposed as a model for propagation of weakly nonlinear ion acoustic and space-charge waves [1]. The symmetric regularized long wave equation (1), (2) results first from a weakly nonlinear analysis of the cold-electron fluid equations, the cold-electron equations are as follows:

$$\rho_t + (\rho u)_x = 0, \qquad u_t + (\frac{1}{2}u^2)_x = \phi_x, \quad \phi_{xx} + \phi_{yy} = \rho. \tag{3}$$

where ρ, u, and ϕ are the dimensionless electron charge density, fluid velocity, and electrostatic potential, respectively. Obviously, eliminating ρ from (1) and (2), we get a symmetric regularized long wave equation

$$u_{tt} - u_{xx} + (\frac{1}{2}u^2)_{xt} - u_{xxtt} = 0, \tag{4}$$

The SRLW equation (4) is explicitly symmetry in the x and t derivatives. The SRLW equation (4) arises also in many other areas of mathematical physics [2-5]. The existence and uniqueness of global solutions for the SRLW equation are obtained by Guo Boling in

[6]. Recently, Chen Lin [7] studied the orbital stability and instability of solitary wave solutions of the generalized SRLW equations

$$u_{tt} - u_{xx} + f(u)_{xt} - u_{xxtt} = 0. \tag{5}$$

In this paper, we will show that the unique continuation property holds for the symmetric regularized long wave equation(SRLWE)

$$u_{tt} - u_{xx} + (\frac{1}{p+1}u^{p+1})_{xt} - u_{xxtt} = 0, \tag{6}$$

where $p \geq 1$ is an integer. We consider (6) in $-\infty < x < +\infty, t \geq 0$. We will show that, for the SRLW equation, if we consider a solution u of (6) in a suitable function space,for example $u \in C(R_t; H^4_{loc}(R_x))$ with $u_t \in C(R_t; H^3_{loc}(R_x)), u_{tt} \in C(R_t; H^2_{loc}(R_x))$ and u vanishes in an open subset Ω of $R_x \times R_t$ then, $u \equiv 0$ in the horizontal component of Ω. We recall that the horizontal component of an open subset $\Omega \subseteq R_x \times R_t$ is defined as the union of all segments $t =$ constant in $R_x \times R_t$ which contain a point of Ω.

The unique continuation property (UCP) has immediate applications to inverse problems and to optimal control theory(see Isakov [8] and Lions [9]) and has been intensively studied for a long time. An important work on the subject was done by J C Saut and B Scheurer [10]. They showed the validity of the UCP for a class of equations which includes

$$u_t + \frac{\partial^{2k+1}u}{\partial x^{2k+1}} + \sum_{j=0}^{2k} a_j(x,t)\frac{\partial^j u}{\partial x^j} = 0. \tag{7}$$

Such class includes the well known Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0. \tag{8}$$

(take $k = 1, a_1(x,t) = u$ and $a_j \equiv 0$ for all $j \neq 1$ in (7)). An alternative proof was given by Zhang B [11] using inverse scattering results. D Tataru [12] deduced some Carleman's type inequalities to study the UCP for Schrodinger's type equation. Zhang Bingyu [13] proved the validity of the UCP for nonlinear Schrodinger equation by using inverse scattering approach. In 1997, Bourgain [14] introduced a different approach and prove that, if a solution u to a dispersive equation has compact support in a nontrivial time interval $I = [t_1, t_2]$ then u vanishes identically. His argument is based on the analyticity of the nonlinear term and the dispersion relation of the linear part of the equation. It also applies to higher order dispersive nonlinear models, and to higher spatial dimensions. More recently, Kenig et al.[15] proposed a new method and proved that, if a sufficiently smooth solution u to a generalized KdV equation is supported in a half line at two different instants of time then u vanishes identically. In [16] Kenig et al. studied unique continuation property of solutions of nonlinear Schrödinger equations. In [17] Davila and Menzala proved that the solutions of the BBM equation and Boussinesq's equation enjoy the so-called unique continuation property.

We base our analysis in finding appropriate Carleman-type estimates for the linearized equations associated with (6). In order to do this we use a well known inequality due to F.Treves combined with existing result on the Cauchy problem associated with (6).

The plan of this paper is as follows. Some notations and spaces are given in Section 2, along with the statement of Treves's inequality and a corollary in the appropriate form we did use in this paper. In section 3, some results concerning the well posedness as well as some technical theorems we shall use in the next section are presented. Section 4 contains our main result and its proof. We first prove a Carleman-type inequality for an operator closely related to our problem. As a consequence we get a local UCP for the solutions of (6) and the final result follows by connectiveness.

2 Notation and Treves' inequality

We begin with a brief synopsis of our notational conventions and function-space designations. The time derivative will be denoted by a subscript t, thus, $u_t = \frac{\partial u}{\partial t}, u_{tt} = \frac{\partial^2 u}{\partial t^2}$, etc. Spatial derivative will be denoted by a subscript x, thus, $u_x = \frac{\partial u}{\partial x}, u_{xx} = \frac{\partial^2 u}{\partial x^2}$, etc. The space of C^∞ functions defined on $\Omega \subset R$ with real values and compact support will be denoted by $C_0^\infty(\Omega)$. By $L^p(\Omega)$ we shall denote the space (classes of) functions in Ω with p^{th} power is integrable, with the norm $\|f\|^p_{L^p(\Omega)} = \int_\Omega |f(x)|^p dx, \quad 1 \leq p < +\infty$. By $L^\infty(\Omega)$ we denote the space of measurable essentially bounded functions in Ω with the norm $\|f\|_{L^\infty(\Omega)} = ess \cdot sup_{x\in\Omega}|f(x)|$. If $f : R \to R$ belongs locally to $L^p(R)$ we write $f \in L^p_{loc}(R)$. For each $s \in R$, the Sobolev space of order s is the completion of the Schwartz space $S(R)$ with respect to the norm

$$\|f\|^2_{H^s} = \int_{-\infty}^{+\infty} (1 + |y|^2)^s |\hat{f}(y)|^2 dy$$

where $\hat{f}$ denotes the Fourier transform of f. If X is a Banach space we denote by $C(0, T; X)$ the space of continuous functions $f : [0, T] \to X$. Sometimes we will also consider the space $C(R; X)$ or $C(-T, T; X)$. By $L^2(-T, T; X)$ we denote the space of functions $u : (-T, T)| \to X$ such that $\int_{-T}^T \|u(s)\|^2_X ds < +\infty$. In order to point out the variation of the time or spatial variables sometimes we write R_t or R_x instead of R.

Now we state Treves' inequality and a corollary in the appropriate form we did use it in this paper.

By D_j we denote partial derivative $D_j = \frac{\partial}{\partial x_j}$ with respect to the variable $x_j (1 \leq j \leq n), D = (D_1, D_2, \cdots, D_n$. If $X = (X_1, X_2, \cdots, X_n$, let $C[X]$ be the algebra of polynomials in n variables. If $P \in C[X]$ and p has constant coefficients and degree m then we consider the differential operator $P(D) = \sum\limits_{|\alpha|\leq m} a_\alpha D^\alpha$ of order m where $D^\alpha = D_1^{\alpha_1}, \cdots, D_n^{\alpha_n}$ and $|\alpha| = \sum\limits_{j=1}^n \alpha_j$. By definition $P^{(\beta)}(X) = \frac{\partial^{|\beta|} P(X)}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$ where β is given by $\beta = (\beta_1, \cdots, \beta_n) \in R^n$.

Theorem 1 ([19]). Let $P = P(D)$ a differential operator of order m with constant coefficients. Then, for any multi-index $\alpha, \xi \in R^n$ and $\phi \in C_0^\infty(R^n)$ we have the inequality

$$\begin{aligned} &\frac{2^{|\alpha|}}{\alpha!} \xi^{2\alpha} \int_{R^n} |P^{(\alpha)}(D)\phi|^2 exp(\psi(y, \xi)) dy \\ &\leq C(m, \alpha) \int_{R^n} |P(D)\phi|^2 exp(\psi(y, \xi)) dy, \end{aligned} \quad (9)$$

where

$$\psi = \psi(y,\xi) = \sum_{j=1}^{n} y_j^2 \xi_j^2, \quad \xi^{2\alpha} = \xi_1^{2\alpha_1} \cdots \xi_n^{2\alpha_n}.$$

if $\xi = (\xi_1, \cdots, \xi_n), \alpha! = \alpha_1! \cdots \alpha_n!$ if $\alpha = (\alpha_1, \cdots, \alpha_n) \in N^n$. The constant $C(m,\alpha)$ is given by

$$C(m,\alpha) = \begin{cases} \sup_{|r+\alpha| \le m} \binom{r+\alpha}{\alpha}, & if \quad |\alpha| \le m, \\ 0, & if \quad |\alpha| > m \end{cases}$$

Corollary 1. Let $P = P(D) = P(\frac{\partial}{\partial x}, \frac{\partial}{\partial t})$ be a differential operator with constant coefficients and order m. Let $\delta > 0$ and $\varphi(x,t) = (x-\delta)^2 + \delta^2 t^2$, then the inequality

$$\begin{aligned} &\frac{1}{\alpha!} 2^{2|\alpha|} \tau^{|\alpha|} \delta^{2\alpha_2} \int_{R^2} |P^{(\alpha)}(D)\phi|^2 exp(2\tau\varphi) dxdt \\ &\le C(m,\alpha) \int_{R^2} |P(D)\phi|^2 exp(2\tau\varphi) dxdt \end{aligned} \tag{10}$$

holds $\forall \phi \in C_0^\infty(R^2), \forall \tau > 0$ and $\alpha = (\alpha_1, \alpha_2) \in N^2$.

Proof. We use the above Theorem with the differential operator

$$Q(D) = P(D+a) = P(\frac{\partial}{\partial x} + 2\tau\delta, \frac{\partial}{\partial t}),$$

that is $a = (2\tau\delta, 0), \tau > 0, y = (x,t), \xi = (\xi_1, \xi_2) = (\sqrt{2\tau}, \sqrt{2\tau})$. Thus, $\psi(y,\xi) = 2\tau(x^2 + \delta^2 t^2)$ and $2^{|\alpha|}\xi^{2\alpha} = 2^{2|\alpha|}\tau^{|\alpha|}\delta^{2\alpha_2}$. Inequality (9) reads

$$\begin{aligned} &\frac{1}{\alpha!} 2^{2|\alpha|} \tau^{|\alpha|} \delta^{2\alpha_2} \int_{R^2} |P^{(\alpha)}(D+a)\phi|^2 exp(2\tau(x^2 + \delta^2 t^2)) dxdt \\ &\le C(m,\alpha) \int_{R^2} |P(D+a)\phi|^2 exp(2\tau(x^2 + \delta^2 t^2)) dxdt \end{aligned} \tag{11}$$

$\forall \phi \in C_0^\infty(R^2)$ and any $\tau > 0$. Multiply both sides of (11) by $exp(2\tau\delta^2)$ to obtain

$$\begin{aligned} &\frac{1}{\alpha!} 2^{2|\alpha|} \tau^{|\alpha|} \delta^{2\alpha_2} \int_{R^2} |exp(2\tau\delta x) P^{(\alpha)}(D+a)\phi|^2 exp(2\tau(x^2 + \delta^2 t^2)) dxdt \\ &\le C(m,\alpha) \int_{R^2} |exp(2\tau\delta x) P(D+a)\phi|^2 exp(2\tau(x^2 + \delta^2 t^2)) dxdt \end{aligned} \tag{12}$$

$\forall \phi \in C_0^\infty(R^2), \forall \tau > 0$ and $\alpha = (\alpha_1, \alpha_2) \in N^2$. In particular, we can choose $\phi = \tilde{\phi} exp(-2\tau\delta x)$ where $\tilde{\phi} \in C_0^\infty(R^2)$. Observing that

$$exp(2\tau\delta x) P(D+a)[\tilde{\phi} exp(-2\tau\delta x)] = P(D)\tilde{\phi}$$

and

$$exp(2\tau\delta x) P^{(\alpha)}(D+a)[\tilde{\phi} exp(-2\tau\delta x)] = P^{(\alpha)}(D)\tilde{\phi}$$

because $a = (2\tau\delta, 0)$ then the proof of (10) is complete.

3 Some Preliminary results

By standard method, we can easily deduce the global well posedness theorem as follows.

Theorem 2. Consider the initial value problem

$$u_{tt} - u_{xx} + (\frac{1}{p+1}u^{p+1})_{xt} - u_{xxtt} = 0,$$
$$u(x,0) = g(x), u_t(x,0) = h(x), \tag{13}$$

in $-\infty < x < \infty, t > 0$ where p is an integer ≥ 1 . Assume that $g(x) \in H^s(R), h(x) \in H^s(R), s \geq 1$. Then, there exists a unique solution $u \in C(R^+;, H^s(R_x))$, with $u_t \in C(R^+; H^s(R_x)), u_{tt} \in C(R^+;, H^s(R_x))$.

Proof. The result can be established analogous to those for the initial value problem

$$u_t + u_x + u^p u_x - u_{xxt} = 0, \quad x \in R, t > 0,$$
$$u(x,0) = u_0(x), \quad x \in R. \tag{14}$$

by Bisognin et al in [20].

Theorem 3. Let us consider the differential operator

$$L = \frac{\partial^2}{\partial t^2} - \frac{\partial^4}{\partial x^2 \partial t^2} + k_1 \frac{\partial^4}{\partial x^4} + k_2 \frac{\partial^4}{\partial x^3 \partial t} + $$
$$+ f_1(x,t)\frac{\partial^2}{\partial x^2} + f_2(x,t)\frac{\partial^2}{\partial x \partial t} + f_3(x,t)\frac{\partial}{\partial x} \tag{15}$$

where k_1, k_2 are two real constants and $f_1, f_2, f_3 \in L^{\infty}_{loc}(R^2)$. Let $\delta > 0$ and $B_\delta = \{(x,t) \in R^2$ such that $x^2 + t^2 < \delta^2\}$. Then , the following inequality

$$\begin{aligned}
&\tau^2\delta^2 \int_{B_\delta} |3k_2\phi_{xx} - 4\phi_{xt}|^2 exp(2\tau\varphi)dxdt \\
&+ 128\tau^3\delta^4 \int_{B_\delta} |\phi_x|^2 exp(2\tau\varphi)dxdt \\
&+ 256\tau^4\delta^4 \int_{B_\delta} |\phi|^2 exp(2\tau\varphi)dxdt \\
&+ 4\tau^2\delta^4 \int_{B_\delta} |\phi - \phi_{xx}|^2 exp(2\tau\varphi)dxdt \\
&\leq 16 \int_{B_\delta} |L\phi|^2 exp(2\tau\varphi)dxdt
\end{aligned} \tag{16}$$

holds for any $\phi \in C_0^\infty(B_\delta)$ and $\tau > 0$ such that

$$\tau \geq Max\{\frac{4}{\delta^2}\|f_1\|_{L^\infty(B_\delta)}, \frac{\sqrt{2}}{\delta}\|f_2\|_{L^\infty(B_\delta)}, \frac{\sqrt{3|k_2|\|f_2\|_{L^\infty(B_\delta)}}}{2\sqrt[4]{2}\delta}, \frac{\|f_3\|^{\frac{2}{3}}_{L^\infty(B_\delta)}}{2\delta^{\frac{4}{3}}}, \frac{1}{8}\}$$

In (16) $\varphi = (x-\delta)^2 + \delta^2 t^2$.

Proof. We use Treves' inequality with the operator

$$P = \frac{\partial^2}{\partial t^2} - \frac{\partial^4}{\partial x^2 \partial t^2} + k_1 \frac{\partial^4}{\partial x^4} + k_2 \frac{\partial^4}{\partial x^3 \partial t}. \tag{17}$$

with the notation given in Section 2 we have that

$$P(\xi_1, \xi_2) = \xi_2^2 - \xi_1^2 \xi_2^2 + k_1 \xi_1^4 + k_2 \xi_1^3 \xi_2. \tag{18}$$

In this case we just need four terms of inequality (10): a) $\alpha = (1,1)$, b) $\alpha = (1,2)$ c) $\alpha = (2,2)$, and d)$\alpha = (0,2)$. Straightforward calculations show that if $\alpha = (1,1)$ then $C(m,\alpha) = C(4,(1,1)) = 4$, and $P^{(1,1)}(\frac{\partial}{\partial x}, \frac{\partial}{\partial t})\phi = 3k_2\phi_{xx} - 4\phi_{xt}$ for any $\phi \in C_0^\infty(B_\delta)$. Therefore, in case $\alpha = (1,1)$, it follows from (10) that

$$\begin{aligned} &4\tau^2\delta^2 \int_{B_\delta} |3k_2\phi_{xx} - 4\phi_{xt}|^2 exp(2\tau\varphi)dxdt \\ &\leq \int_{B_\delta} |\phi_{tt} - \phi_{xxtt} + k_1\phi_{xxxx} + k_2\phi_{xxxt}|^2 exp(2\tau\varphi)dxdt \end{aligned} \tag{19}$$

for any $\phi \in C_0^\infty(B_\delta)$ and $\tau > 0$. If $\alpha = (1,2)$ then $C(m,\alpha) = C(4,(1,2)) = 3$ and $P^{(1,2)}(\frac{\partial}{\partial x}, \frac{\partial}{\partial t})\phi = -4\phi_x$ for any $\phi \in C_0^\infty(B_\delta)$. Thus, in case $\alpha = (1,2)$, inequality (10) reads

$$\begin{aligned} &512\tau^3\delta^4 \int_{B_\delta} |\phi_x|^2 exp(2\tau\varphi)dxdt \\ &\leq 3\int_{B_\delta} |\phi_{tt} - \phi_{xxtt} + k_1\phi_{xxxx} + k_2\phi_{xxxt}|^2 exp(2\tau\varphi)dxdt \end{aligned} \tag{20}$$

for any $\phi \in C_0^\infty(B_\delta)$ and $\tau > 0$. If $\alpha = (2,2)$ then $C(m,\alpha) = C(4,(2,2)) = 1$ and $P^{(2,2)}(\frac{\partial}{\partial x}, \frac{\partial}{\partial t})\phi = -4\phi$ for any $\phi \in C_0^\infty(B_\delta)$. Therefore, (10) implies that

$$\begin{aligned} &1024\tau^4\delta^4 \int_{B_\delta} |\phi|^2 exp(2\tau\varphi)dxdt \\ &\leq \int_{B_\delta} |\phi_{tt} - \phi_{xxtt} + k_1\phi_{xxxx} + k_2\phi_{xxxt}|^2 exp(2\tau\varphi)dxdt \end{aligned} \tag{21}$$

for any $\phi \in C_0^\infty(B_\delta)$ and $\tau > 0$. Finally, if $\alpha = (0,2)$ then $C(m,\alpha) = C(4,(0,2)) = 6$ and $P^{(0,2)}(\frac{\partial}{\partial x}, \frac{\partial}{\partial x})\phi = 2\phi - 2\phi_{xx}$ for any $\phi \in C_0^\infty(B_\delta)$. In this case, (10) becomes

$$\begin{aligned} &16\tau^2\delta^4 \int_{B_\delta} |\phi - \phi_{xx}|^2 exp(2\tau\varphi)dxdt \\ &\leq 3\int_{B_\delta} |\phi_{tt} - \phi_{xxtt} + k_1\phi_{xxxx} + k_2\phi_{xxxt}|^2 exp(2\tau\varphi)dxdt \end{aligned} \tag{22}$$

Adding (19), (20), (21) and (22) yields

$$\tau^2\delta^2\int_{B_\delta}|3k_2\phi_{xx}-4\phi_{xt}|^2exp(2\tau\varphi)dxdt$$
$$+128\tau^3\delta^4\int_{B_\delta}|\phi_x|^2exp(2\tau\varphi)dxdt$$
$$+256\tau^4\delta^4\int_{B_\delta}|\phi|^2exp(2\tau\varphi)dxdt$$
$$+4\tau^2\delta^4\int_{B_\delta}|\phi-\phi_{xx}|^2exp(2\tau\varphi)dxdt$$
$$\leq 2\int_{B_\delta}|\phi_{tt}-\phi_{xxtt}+k_1\phi_{xxxx}+k_2\phi_{xxxt}|^2exp(2\tau\varphi)dxdt \tag{23}$$

Now, if

$$\tau\geq Max\{\frac{4}{\delta^2}\|f_1\|_{L^\infty(B_\delta)},\frac{\sqrt{2}}{\delta}\|f_2\|_{L^\infty(B_\delta)},\frac{\sqrt{3|k_2|\|f_2\|_{L^\infty(B_\delta)}}}{2\sqrt[4]{2}\delta},\frac{\|f_3\|^{\frac{2}{3}}_{L^\infty(B_\delta)}}{2\delta^{\frac{4}{3}}},\frac{1}{8}\}$$

then

$$\int_{B_\delta}|f_1(x,t)\phi_{xx}|^2exp(2\tau\varphi)dxdt$$
$$+\int_{B_\delta}|f_2(x,t)\phi_{xt}|^2exp(2\tau\varphi)dxdt+\int_{B_\delta}|f_3(x,t)\phi_x|^2exp(2\tau\varphi)dxdt$$
$$\leq\|f_1\|^2_{L^\infty(B_\delta)}\int_{B_\delta}|\phi_{xx}|^2exp(2\tau\varphi)dxdt+\|f_2\|^2_{L^\infty(B_\delta)}\int_{B_\delta}|\phi_{xt}|^2exp(2\tau\varphi)dxdt$$
$$+\|f_3\|^2_{L^\infty(B_\delta)}\int_{B_\delta}|\phi_x|^2exp(2\tau\varphi)dxdt$$

$$\leq\frac{\tau^2\delta^4}{16}\int_{B_\delta}|\phi-\phi_{xx}|^2exp(2\tau\varphi)dxdt+16\tau^4\delta^4\int_{B_\delta}|\phi|^2exp(2\tau\varphi)dxdt$$
$$+8\tau^3\delta^4\int_{B_\delta}|\phi_x|^2exp(2\tau\varphi)dxdt+\frac{\tau^2\delta^2}{16}\int_{B_\delta}|3k_2\phi_{xx}-4\phi_{xt}|^2exp(2\tau\varphi)dxdt$$
$$\leq\frac{1}{8}\int_{B_\delta}|\phi_{tt}-\phi_{xxtt}+k_1\phi_{xxxx}+k_2\phi_{xxxt}|^2exp(2\tau\varphi)dxdt \tag{24}$$

Since

$$|\phi_{tt}-\phi_{xxtt}+k_1\phi_{xxxx}+k_2\phi_{xxxt}|^2$$
$$=|L\phi-f_1\phi_{xx}-f_2\phi_{xx}-f_3\phi_x|^2$$
$$\leq 4|L\phi|^2+4|f_1\phi_{xx}|^2+4|f_2\phi_{xt}|^2+4|f_3\phi_x|^2,$$

Thus the right hand side of (24) is bounded by

$$\leq\frac{1}{2}\int_{B_\delta}|L\phi|^2exp(2\tau\varphi)dxdt+\frac{1}{2}\int_{B_\delta}|f_1\phi_{xx}|^2exp(2\tau\varphi)dxdt$$
$$+\frac{1}{2}\int_{B_\delta}|f_2\phi_{xt}|^2exp(2\tau\varphi)dxdt+\frac{1}{2}\int_{B_\delta}|f_3\phi_x|^2exp(2\tau\varphi)dxdt \tag{25}$$

Now, from (24) and (25) we deduce that

$$\begin{aligned}
&\int_{B_\delta} |f_1\phi_{xx}|^2 exp(2\tau\varphi)dxdt + \int_{B_\delta} |f_2\phi_{xt}|^2 exp(2\tau\varphi)dxdt \\
&+ \int_{B_\delta} |f_3\phi_x|^2 exp(2\tau\varphi)dxdt \\
&\leq \int_{B_\delta} |L\phi|^2 exp(2\tau\varphi)dxdt. \qquad (26)
\end{aligned}$$

Returning to inequality (24) we conclude that

$$\int_{B_\delta} |\phi_{tt} - \phi_{xxtt} + k_1\phi_{xxxx} + k_2\phi_{xxxt}|^2 exp(2\tau\varphi)dxdt \leq 8\int_{B_\delta} |L\phi|^2 exp(2\tau\varphi)dxdt.$$

which together with inequality (23) proves Theorem 3.

Corollary 2. Let $T > 0$. Under the assumptions of Theorem 3 then inequality (16) holds if we replace $\phi(x,t)$ by a function $v(x,t)$ such that $v \in L^2(-T,T;H^4_{loc}(R_x))$ with $v_t \in L^2(-T,T;H^3_{loc}(R_x))$ and $v_{tt} \in L^2(-T,T;H^2_{loc}(R_x))$ and the support of v is compact set contained in $B_\delta = \{(x,t) : x^2 + t^2 < \delta^2\}$.

Proof. Let $\{\rho_\varepsilon\}$ be a regularizing sequence (in two variables) and consider $v_\varepsilon = \rho_\varepsilon * v$ where $*$ denotes the usual convolution. It follows that $v_\varepsilon \in C_0^\infty(B_\delta)$ therefore inequality (16) holds for v_ε, that is

$$\begin{aligned}
&\tau^2\delta^2 \int_{B_\delta} |3k_2\frac{\partial^2 v_\varepsilon}{\partial x^2} - 4\frac{\partial^2 v_\varepsilon}{\partial x\partial t}|^2 exp(2\tau\varphi)dxdt \\
&+ 128\tau^3\delta^4 \int_{B_\delta} |\frac{\partial v_\varepsilon}{\partial x}|^2 exp(2\tau\varphi)dxdt \\
&+ 256\tau^4\delta^4 \int_{B_\delta} |v_\varepsilon|^2 exp(2\tau\varphi)dxdt \\
&+ 4\tau^2\delta^4 \int_{B_\delta} |v_\varepsilon - \frac{\partial^2 v_\varepsilon}{\partial x^2}|^2 exp(2\tau\varphi)dxdt \\
&\leq 16\int_{B_\delta} |Lv_\varepsilon|^2 exp(2\tau\varphi)dxdt \qquad (27)
\end{aligned}$$

where L is given by (15) and $\tau \geq Max\{\frac{4}{\delta^2}\|f_1\|_{L^\infty(B_\delta)}, \frac{\sqrt{2}}{\delta}\|f_2\|_{L^\infty(B_\delta)}, \frac{\sqrt{3|k_2|\|f_2\|_{L^\infty(B_\delta)}}}{2\sqrt[4]{2}\delta}, \frac{\|f_3\|^{\frac{2}{3}}_{L^\infty(B_\delta)}}{2\delta^{\frac{4}{3}}}, \frac{1}{8}\}$. Since $D^\alpha v_\varepsilon = \rho_\varepsilon * D^\alpha v$ where D^α denotes either one of the operators: $I = identity$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial t}$, $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial x\partial t}$, $\frac{\partial^4}{\partial x^2\partial t^2}$, $\frac{\partial^4}{\partial x^4}$, $\frac{\partial^4}{\partial x^3\partial t}$ then, we have that $D^\alpha v_\alpha \to D^\alpha v$ in $L^2(B_\delta)$ as $\varepsilon \to 0^+$. Consequently we have that

$$\|(D^\alpha v_\alpha)exp(\tau\varphi) - (D^\alpha v)exp(\tau\varphi)\|_{L^2(B_\delta)} \leq C\|(D^\alpha v_\alpha) - (D^\alpha v)\|_{L^2(B_\delta)} \to 0$$

as $\varepsilon \to 0$. Here C is a positive constant depending only on τ and δ. Similarly we can show that

$$(f_1(x,t)\frac{\partial^2 v_\varepsilon}{\partial x^2})exp(\tau\varphi) \to (f_1(x,t)\frac{\partial^2 v}{\partial x^2})exp(\tau\varphi);$$

$$(f_2(x,t)\frac{\partial^2 v_\varepsilon}{\partial x \partial t})exp(\tau\varphi) \to (f_2(x,t)\frac{\partial^2 v}{\partial x \partial t})exp(\tau\varphi)$$
$$(f_3(x,t)\frac{\partial v_\varepsilon}{\partial x})exp(\tau\varphi) \to (f_3(x,t)\frac{\partial v}{\partial x})exp(\tau\varphi)$$

in $L^2(B_\delta)$ as $\varepsilon \to 0^+$. We have shown that $L(v_\varepsilon)exp(\tau\varphi) \to (Lv)exp(\tau\varphi)$ in $L^2(B_\delta)$ as $\varepsilon \to 0^+$ which allows us to pass to the limit in (25) to conclude the proof of Corollary 2.

4 Main Result and Its Proof

Theorem 4. Let $T > 0$ and $f_1, f_2, f_3 \in L^\infty_{loc}(R_x \times (-T,T)), k \in R$. Consider the differential operator

$$L = \frac{\partial^2}{\partial t^2} - \frac{\partial^4}{\partial x^2 \partial t^2} + k_1\frac{\partial^4}{\partial x^4} + k_2\frac{\partial^4}{\partial x^3 \partial t} + f_1\frac{\partial^2}{\partial x^2} + f_2\frac{\partial^2}{\partial x \partial t} + f_3\frac{\partial}{\partial x}.$$

Let $u = u(x,t)$ be a solution of $Lu = 0$ in $R_x \times (-T,T)$ such that $u \in L^2(-T,T; H^4_{loc}(R_x)), u_t \in L^2(-T,T; H^3_{loc}(R_x))$ and $u_{tt} \in L^2(-T,T; H^2_{loc}(R_x))$. Let

$$\tilde{u} = \begin{cases} u, & if \quad t \geq 0, \\ 0, & if \quad t < 0. \end{cases}$$

Let $\alpha > 0$ and suppose that $\tilde{u} \equiv 0$ in the region $\{(x,t) : x < \alpha t\}$ intercepted with a neighborhood of $(0,0)$. Then there exists a neighborhood O_2 (in the plane xt) such that $\tilde{u} \equiv 0$ in O_2.

Proof. Let $0 < \delta < min\{1, \alpha\}$ and $B_\delta = \{(x,t) : x^2 + \delta^2 < \delta^2\}$. Choose $h \in C^\infty_0(B_\delta)$ such that $h \equiv 1$ in a neighborhood O_1 and define $v = v(x,t)$ as $v = h\tilde{u}$. It follows that $v \in L^2(-T,T; H^4_{loc}(R_x)), v_t \in L^2(-T,T; H^3_{loc}(R_x))$ and $v_{tt} \in L^2(-T,T; H^2_{loc}(R_x))$ and it has compact support in B_δ. Using Corollary 2 we obtain that

$$16\tau^4\delta^4 \int_{B_\delta} |v|^2 exp(2\tau\varphi)dxdt \leq \int_{B_\delta} |Lv|^2 exp(2\tau\varphi)dxdt, \tag{28}$$

where $\tau \geq Max\{\frac{4}{\delta^2}\|f_1\|_{L^\infty(B_\delta)}, \frac{\sqrt{2}}{\delta}\|f_2\|_{L^\infty(B_\delta)}, \frac{\sqrt{3|k_2|\|f_2\|_{L^\infty(B_\delta)}}}{2\sqrt[4]{2}\delta}, \frac{\|f_3\|^{\frac{2}{3}}_{L^\infty(B_\delta)}}{2\delta^{\frac{4}{3}}}, \frac{1}{8}\}$. and $\varphi = (x-\delta)^2 + \delta^2 t^2$. Since $Lv = 0$ in O_1 then integration on the right hand side of (28) is only over $B_\delta - O_1$. If (x,t) belongs to the support of v then $0 \leq \alpha t \leq x < \delta < 1$. In that region and for $(x,t) \neq (0,0)$ we have the inequalities

$$\begin{aligned} &\varphi = (x-\delta)^2 + \delta^2 t^2 \leq (\alpha t - \delta)^2 + \delta^2 t^2 \\ &= \delta^2 + (\alpha^2 + \delta^2)t^2 - 2\alpha\delta t < \delta^2. \end{aligned}$$

Since $\varphi(0,0) = \delta^2$ then, it follows that if (x,t) belongs to the support of Lv then there exists $\varepsilon > 0, (\varepsilon < \delta^2)$ such that $\varphi(x,t) \leq \delta^2 - \varepsilon$. Now, we choose a neighborhood O_2 of

$(0,0)$ such that $\varphi(x,t) > \delta^2 - \varepsilon$ in O_2. From (28) and the above construction we obtain the inequalities

$$
\begin{aligned}
&16\tau^4\delta^4 exp(2\tau(\delta^2-\varepsilon))\int_{O_2}|v|^2dxdt \\
&\le 16\tau^4\delta^4 exp(2\tau(\delta^2-\varepsilon))\int_{B_\delta}|v|^2dxdt \\
&\le 16\tau^4\delta^4\int_{B_\delta}|v|^2 exp(2\tau(\delta^2-\varepsilon))dxdt \le \int_{B_\delta}|Lv|^2 exp(2\tau\varphi)dxdt \\
&\le \int_{B_\delta - O_1}|Lv|^2 exp(2\tau\varphi)dxdt \\
&\le exp(2\tau(\delta^2-\varepsilon))\int_{B_\delta - O_1}|Lv|^2dxdt
\end{aligned}
$$

which gives us

$$\int_{O_2}|v|^2dxdt \le \frac{1}{16\tau^4\delta^4}\int_{B_\delta - O_1}|Lv|^2dxdt \tag{29}$$

Letting $\tau \to +\infty$ in (29), it follows that $v \equiv 0$ in O_2. Since $\tilde{u} \equiv v$ in $O_1 \supset O_2$ then the proof of Theorem 4 is complete.

Similarly, we can also show

Theorem 5. Let $T > 0$ and $f_1, f_2, f_3 \in L^\infty_{loc}(R_x \times (-T,T)), k \in R$. Consider the differential operator

$$L = \frac{\partial^2}{\partial t^2} - \frac{\partial^4}{\partial x^2\partial t^2} + k_1\frac{\partial^4}{\partial x^4} + k_2\frac{\partial^4}{\partial x^3\partial t} + f_1\frac{\partial^2}{\partial x^2} + f_2\frac{\partial^2}{\partial x\partial t} + f_3\frac{\partial}{\partial x}.$$

Let $u = u(x,t)$ be a solution of $Lu = 0$ in $R_x \times (-T,T)$ such that $u \in L^2(-T,T; H^4_{loc}(R_x)), u_t \in L^2(-T,T; H^3_{loc}(R_x))$ and $u_{tt} \in L^2(-T,T; H^2_{loc}(R_x))$. Let

$$\tilde{u} = \begin{cases} 0, & if \quad t > 0, \\ u, & if \quad t \le 0. \end{cases}$$

Let $\alpha < 0$ and suppose that $\tilde{u} \equiv 0$ in the region $\{(x,t) : x < \alpha t\}$ intercepted with a neighborhood of $(0,0)$. Then there exists a neighborhood O_3 (in the plane xt) such that $\tilde{u} \equiv 0$ in O_3.

Theorem 6. Let $T > 0$ and $u(x,t)$ be a solution of

$$u_{tt} - u_{xx} - u_{xxtt} + a(x,t)u_{xt} + b(x,t)u_x = 0 \tag{30}$$

in $-\infty < x, t < +\infty$ such that $u \in L^2(-T,T; H^4_{loc}(R_x)), u_t \in L^2(-T,T; H^3_{loc}(R_x))$ and $u_{tt} \in L^2(-T,T; H^2_{loc}(R_x))$. In (29) it is assume that the coefficient $a(x,t), b(x,t) \in L^\infty_{loc}(R_x \times (-T,T))$. Let γ be a circumference passing through the origin $(0,0)$. Suppose that $u \equiv 0$ in the interior of the circle (with boundary γ) in a neighborhood of $(0,0)$. Then, there exists a neighborhood of $(0,0)$ where $u \equiv 0$.

Proof. In order to simplify the calculations, let us assume that the circumference (a piece of it) γ is given by $x = g(t)$ with $g''(t) < 0$ in a neighborhood of $(0,0)$. Thus, $u \equiv 0$

in the region $\{(x,t), x < g(t)\}$. It follows that there exists $w \neq 0, (w \neq 1)$ such that $u \equiv 0$ in a neighborhood of $(0,0)$ in the region $\{(x,t), x < h(t)\}$ where

$$h(t) = \begin{cases} wt, & if \quad t \geq 0, \\ -\frac{1}{w}t, & if \quad t < 0. \end{cases}$$

Now, we consider the following change of variables

$$(x,t) \mapsto (x - h(t) + |t|, t) = (y,s).$$

In the new variables the function $u = u(y,s)$ satisfies

$$\begin{cases} L_1 u = 0, & if \quad s \geq 0, \\ L_2 u = 0, & if \quad s < 0. \end{cases}$$

where

$$L_1 = \frac{\partial^2}{\partial s^2} - \frac{\partial^4}{\partial y^2 \partial s^2} - (1-w)^2 \frac{\partial^4}{\partial y^4} - 2(1-w)\frac{\partial^4}{\partial y^3 \partial s}$$
$$+ [a(y,s)(1-w) + w^2 - 2w]\frac{\partial^2}{\partial y^2} + [a(y,s) + 2(1-w)]\frac{\partial^2}{\partial y \partial s} + c(y,s)\frac{\partial}{\partial y}$$

and

$$L_2 = \frac{\partial^2}{\partial s^2} - \frac{\partial^4}{\partial y^2 \partial s^2} - (\frac{1}{w} - 1)^2 \frac{\partial^4}{\partial y^4} - 2(\frac{1}{w} - 1)\frac{\partial^4}{\partial y^3 \partial s}$$
$$+ [a(y,s)(\frac{1}{w} - 1) + \frac{1}{w^2} - 2\frac{1}{w}]\frac{\partial^2}{\partial y^2} + [a(y,s) + 2(\frac{1}{w} - 1)]\frac{\partial^2}{\partial y \partial s} + c(y,s)\frac{\partial}{\partial y}$$

Observe that $u \equiv 0$ in the region $\{(y,s), y < |s|\}$ in a neighborhood of $(0,0)$. We use Theorem 4 and Theorem 5 to conclude that there exists a neighborhood of $(0,0)$ in the plane ys where $u \equiv 0$. Returning to the original variables xt we conclude the proof of Theorem 6.

Theorem 7. (Unique continuation). Let $T > 0$ and $u = u(x,t)$ be a solution of the generalized symmetric regularized long wave equation

$$u_{tt} - u_{xx} - u_{xxtt} + u^p u_{xt} + p u^{p-1} u_t u_x = 0$$

in $-\infty < x, t < +\infty$ where $p \geq 1$. Assume that $u \in L^\infty(-T,T; H^4_{loc}(R_x)), u_t \in L^\infty(-T,T; H^3_{loc}(R_x))$ and $u_{tt} \in L^\infty(-T,T; H^2_{loc}(R_x))$. If $u \equiv 0$ in an open subset Ω of $R_x \times (-T,T)$, then $u \equiv 0$ in the horizonal component of Ω.

Proof. Since $a(x,t) = u^p \in L^\infty_{loc}(R_x \times (-T,T)), b(x,t) = pu^{p-1}u_t \in L^\infty_{loc}(R_x \times (-T,T))$ then it is sufficient to prove the above result for the equation

$$u_{tt} - u_{xx} - u_{xxtt} + a(x,t)u_{xt} + b(x,t)u_x = 0$$

Let

$$\Lambda = \{(x,t) \in R_x \times (-T,T) \;\; such \;\; that \;\; u \equiv 0 \;\; in \;\; a \;\; neighbourhood \;\; of \;\; (x,t)\}.$$

We want to show that Λ coincides with the horizontal component of Ω which we denote by Ω_1. Suppose that they are not equal. Let $P \in \Lambda$ and $Q \in \Omega_1$. Let Γ be a continuous curve (contained in Ω_1 joining p to Q. Let us parameterize Γ by a function $f(s), 0 \leq s \leq 1$ with $f(0) = P, f(1) = Q$. Let $r_0 > 0, r_0 < dist(\Gamma, \partial\Omega_1)$, (where $\partial\Omega_1$ denotes the boundary of Ω_1) and such that the ball $B_{r_0}(P)$ centered at P with radius r_0 is contained in Λ. Let $r_1 < \frac{r_0}{4}$ and consider the set

$$\Lambda_1 = \{(x,t) \in \Lambda \quad such \quad that \quad u \equiv 0 \quad in \quad B_{r_1}(x,t) \cap \Omega_1\}$$

where $B_{r_1}(x,t)$ denotes the ball of radius r_1 centered in (x,t). Let $s_0 = \sup\{0 \leq s \leq 1 \quad such \quad that \quad f(\tau) \in \Lambda_1 \quad whenever \quad 0 \leq \tau \leq s\}$. We claim that $s_0 = 1$. Clearly, this claim proves Theorem 7 because $Q = f(1) \in \Lambda_1$ therefore $u \equiv 0$ in $B_{r_1}(Q) \cap \Omega_1$. Consequently $u \equiv 0$ in Ω_1 because Ω was arbitrary chosen. Remains to prove the claim. First, let us show that $B_{\frac{r_1}{2}}(f(s_0)) \subseteq \Lambda$. In fact, given $\varepsilon > 0, \varepsilon < \frac{r_1}{2}$ there exists $\delta > 0, \delta < s_0$ such that

$$|f(s_0) - f(s_0 - \delta)| < \varepsilon < \frac{r_1}{2}$$

therefore if $w \in B_{\frac{r_1}{2}}(f(s_0))$ then

$$|w - f(s_0 - \delta)| \leq |w - f(s_0)| + |f(s_0) - f(s_0 - \delta)| < r_1.$$

since $f(s_0 - \delta) \in \Lambda_1$ then $f(s_0 - \delta) \in \Lambda$ and $u \equiv 0$ in

$$B_{r_1}(f(s_0 - \delta)) \quad (because \quad B_{r_1}(f(s_0 - \delta)) \subseteq \Omega_1).$$

Thus, $w \in \Lambda_1 \subseteq \Lambda$. Finally, let us suppose that $s_0 < 1$. We use Theorem 6 to deduce that for each element of

$$F = \{y \in \Omega_1 \quad such \quad that \quad |y - f(s_0)| = \frac{r_1}{2}\}$$

there exists a neighborhood where $u \equiv 0$. using the compactness of F we conclude that there exists $\varepsilon_1 > 0$ such that $u \equiv 0$ in $B_{\frac{r_1}{2}+\varepsilon_1}(f(s_0))$, this implies that we could find $\delta_1 > 0$ and ξ_0 such that for $0 \leq \xi_0 \leq s_0 + \delta_1$ we have $f(\xi_0) \in \Lambda_1$ which contradicts the definition of s_0. This proves Theorem 7.

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