Contact Discontinuity with General Perturbations for Gas Motions

Feimin Huang*
Institute of Applied Mathematics
AMSS, Academia Sinica, China
Zhouping Xin†
The Institute of Mathematical Sciences,
The Chinese University of Hong Kong, Hong Kong.
Tong Yang‡
Lie Bie Ju Centre of Mathematics
City University of Hong Kong, Hong Kong

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*Email address: fhuang@mail.amt.ac.cn, Huang's research is supported in part by the National Natural Science Foundation of China under contract 10471138.

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‡Email address: matyang@math.cityu.edu.hk, Yang's research is supported in part by the RGC Competitive Earmarked Research Grant, CityU 103004.

Abstract

The contact discontinuity is one of the basic wave patterns in gas motions. The stability of contact discontinuities with general perturbations is a long standing open problem. One of the reasons is that contact discontinuities are linearly degenerate waves in the nonlinear settings, like the Navier-Stokes equations and the Boltzmann equation. The nonlinear diffusion waves generated by the perturbations in sound-wave families couple and interact with the contact discontinuity and then cause analytic difficulties. Another reason is that in contrast to the basic nonlinear waves, shock waves and rarefaction waves, for which the corresponding characteristic speeds are strictly monotone, the characteristic speed is constant across a contact discontinuity, and the derivative of contact wave decays slower than the one for rarefaction wave. In this paper, we succeed in obtaining the time asymptotic stability of a damped contact wave pattern with an convergence rate for the Navier-Stokes equations and the Boltzmann equation in a uniform way. One of the key observations is that even though the energy estimate involving the lower order may grow at the rate $(1+t)^{\frac{1}{2}}$, it can be compensated by the decay in the energy estimate for derivatives which is of the order of $(1+t)^{-\frac{1}{2}}$. Thus, these reciprocal order of decay rates for the time evolution of the perturbation are essential to close the priori estimate containing the uniform bounds of the L^∞ norm on the lower order estimate and then it gives the decay of the solution to the contact wave pattern.

1 Introduction

The study of fluid motion has a very long history and the pioneering work on nonlinear wave phenomena dates back to Riemann in 1860s on gas dynamics. Now it is well known that the hyperbolic conservation laws in the form of

$$U_t + F(U)_x = 0, \tag{1.1}$$

have three basic wave patterns in one dimensional space. And as a typical example of (1.1), Euler equations consist of conservation of mass, momentum and energy. Among these basic wave patterns, two are nonlinear waves, shock and rarefaction wave, and the other one is linearly degenerate wave, contact discontinuity. These dilation invariant solutions [52], [15], and their linear superposition in the increasing oeder of characteristic speed, called Riemann solutions, govern both the local and large time asymptotic behavior of general solutions to the inviscid Euler system [36]. Since the inviscid system (1.1) is an idealization when the dissipative effects are neglected, thus it is of great importance to study the large time asymptotic behavior of solutions to the corresponding viscous systems in the form of

$$U_t + F(U)_x = (B(U)U_x)_x, \tag{1.2}$$

toward the viscous versions of these basic waves. As a basic system for viscous fluid, the Navier-Stokes equations which include the effects of viscosity and heat conductivity, have the above wave phenomena which are smoothed out by the dissipative effect. Furthermore, coming from statistics physics for rarefied gas, the Boltzmann equation which describes the macroscopic and microscopic aspects in the non-equilibrium gas motion, has similar wave phenomena as we will show later.

In this paper, we are going to study the stability of the linearly degenerate wave, i.e., damped contact discontinuity, with general perturbations for both the Navier-Stokes equations and the Boltzmann equation in a uniform way. It is somehow surprising that the energy method can be applied to capture the coupling of the contact discontinuity with the diffusion waves created by the perturbations in the sound wave families so that a priori estimate can be closed with a convergence rate on the solution to the wave profile time asymptotically.

In the first part of the paper, we will consider the Navier-Stokes equations. Indeed, there have been great interests and intensive studies in the respect of wave phenomena in the development of the mathematical theory for viscous systems of conservation laws since 1985, started with studies on the nonlinear stability of viscous shock profiles by Goodman [21] and Matsumura-Nishihara [44]. Deeper understanding has been achieved on the asymptotic stability toward nonlinear waves, viscous shock profiles and viscous rarefaction waves, which have been shown to be nonlinearly stable with quite general perturbations for the compressible Navier-Stokes system and more general system of viscous strictly hyperbolic conservation laws (1.2). Moreover, some new phenomena have been discovered and new techniques, such as weighted characteristic energy methods and uniform approximate Green's functions, have been developed based on the intrinsic properties of the underlying nonlinear waves, see [30], [32], [38], [54], [37] [55], [46], [48] and the references therein.

However, the problem of stability of contact discontinuities is more subtle and the progress has been less satisfactory, except the studies in [25], [27], [29], [40], [59]. One of the main reasons is the contact discontinuities are associated with linear degenerate fields and are less stable compared with the nonlinear waves for the inviscid system (1.1), [36]. Thus the stabilizing effects around a damped contact wave pattern for Navier-Stokes equations should come mainly from the viscosity and heat conductivity. A general perturbation of a contact wave may introduce waves in the nonlinear sound wave families, and interactions of these waves with the linear contact wave are some of the major difficulties to overcome, see [59], [40] and [27]. Another technical difficulty is that the viscosity matrix for the compressible Navier-Stokes equations is only semi-positive definite.

The mathematical aspect on the stability toward contact waves for solutions to systems of viscous conservation laws was first studied by Xin in [59], where the metastability of a weak contact discontinuity for the compressible Euler equations with uniform viscosity, was proved by showing that although a contact discontinuity is not an asymptotic attractor for the viscous system, yet a viscous contact wave, which approximates the contact discontinuity on any finite time interval, is asymptotically nonlinear stable for small generic perturbations and the detailed asymptotic behavior can be determined a priorly by initial mass distribution. This was later generalized by Liu-Xin in [40] to show the metastability of contact discontinuities for a class of general systems of nonlinear conservation laws with uniform viscosity, and obtain pointwise asymptotic behavior toward viscous contact wave by approximate fundamental solutions, which also leads to the nonlinear stability of the viscous contact wave in L^p-norms for all $p \geq 1$. However, the theory in [40] and [59] does not apply to the compressible Navier-Stokes system since its viscosity matrix $B(U)$ in (1.2) is only semi-positive definite.

For a free boundary value problem to the Navier-Stokes equations with a particle path as free boundary, the nonlinear stability of a viscous contact wave is proved in the super-norm through the energy method by Huang-Matsumura-Shi in [25]. However, the approach can not be applied here to study the asymptotic behavior toward contact waves for solutions to Cauchy problems of the Navier-Stokes equations since the analysis in [25] depends crucially on the availability of Poincaré type inequality, which does not hold in the whole space. Recently, a more satisfactory answer was obtained in [27] which shows that for a weak contact discontinuity for the compressible Euler system, one can construct a smooth viscous contact wave for the Navier-Stokes system solves Euler equations asymptotically, and approximates the given contact discontinuity on any finite time interval, and such a viscous contact wave is nonlinearly stable under small initial perturbation with zero mass condition. There the stability is in sup-norm and a rate of convergence is also obtained. Notice that the convergence rate to either the viscous shock wave or viscous rarefaction wave has not been achieved yet for the compressible Navier-Stokes system, see [30], [38], [48]. However, the rate of decay obtained in the form of $(1+t)^{-\frac{1}{4}}$ may not be optimal. Motivated by the pointwise behavior toward viscous contact waves for solutions to the Euler system with uniform viscosity (see [59] and [40]), one would conjecture that its decay rate could be improved to $(1+t)^{-\frac{1}{2}}$.

In [27], the major assumption in the stability theory is the initial zero excessive mass condition which excludes the possible presence of diffusion waves in the sound wave families. As it is shown in [59] and [40] for the compressible Euler equations with uniform viscosity, a generic perturbation of a viscous contact wave introduces not only a shift with center of the viscous contact wave, but also nonlinear and linear diffusion waves. Although, it is expected that the same phenomena remain true for the compressible Navier-Stokes system, yet a rigorous mathematical proof has remained to be given. Note that the fine accurate asymptotic ansatz as in [59] and [40] may not be necessary for the stability theory toward contact waves in the super-norm. The main purpose of this paper is to overcome the difficulty for the excessive mass and obtain the stability and convergence rate for the viscous contact waves. Therefore, it gives a satisfactory answer to the problem on the stability of contact discontinuity in the gas motion.

In the second part of the paper, we consider the stability of contact wave profile for the Boltzmann equation. The Boltzmann equation is a fundamental equation, which gives a statistical description of the time evolution of particles in rarefied gas. It takes the form of

$$f_t + \xi \cdot \nabla f = Q(f, f), \ (f, x, t, \xi) \in \mathbf{R} \times \mathbf{R^3} \times \mathbf{R}^+ \times \mathbf{R}^3, \tag{1.3}$$

where f is the distribution function of the particles and $Q(f, f)$ is the collision operator which gives the gain and loss rate of the particle distribution function through collision. The detailed definition of each terms in (1.3) will be given in the next section.

Since its derivation by Boltzmann in 1872, the mathematical problems on (1.3) have been extensively studied with fruitful results. Among them, we mention a few as: the renormalized solution, fluid dynamic limits, global existence around a global Maxwellian, regularity of the solutions, cf. [5], [6], [16], [19], [34] and references therein. Since they are not directly related to our problem considered here, we will not discuss them in details. Notice that the energy method making uses of the spectrum properties of the linearized operator which was from Grad to Ukai gives a good description of the perturbation of a global Maxwellian, cf. [22], [51], [56], [57]. Recently, the energy method based on the decomposition has been developed and used for the study of existence, stability and large time behavior of the solutions. One way is to decompose the solution and the equation around the local Maxwellian so that the techniques used for fluid dynamics can be applied and solution around non-trivial time asymptotic solution profile can be studied clearly, cf. [41] and [43]. Another way is to decompose the solution around a global Maxwellian as in [23] for the problems on space periodic solutions.

One of the most important properties of the Boltzmann equation is its asymptotic equivalence to the macroscopic fluid dynamics equations. In fact, the first order of the Hilbert expansion for the Boltzmann equation is the system of Euler equation and the second order of the Chapman-Enskog expansion gives the system of the Navier-Stokes equations. Hence, one can expect the wave phenomena for the macroscopic fluid dynamics also exist in the solutions to the Boltzmann equation. In fact, there is a series of work on the wave phenomena for the Boltzmann equation starting from the existence of shock profile proved by [9]. Recently, the nonlinear stability of shock profiles, rarefaction wave profiles and contact wave for the Boltzmann equation are also studied through energy method and a decomposition of the solution and the equation into fluid and non-fluid components [43], [42], [28]. As a continuation in this direction, we consider the stability of the contact discontinuity with generic perturbation in this paper.

One of the fundamental properties of the Boltzmann equation is the celebrated H-theorem which implies that the solution is time irreversible so that the mathematical entropy is decreasing in time for non-equilibrium gas. There are two ways to view this dissipative effect. One is from the linearized version of the collision operator which dissipates on the sub-space(non-fluid components) orthogonal to the null space(fluid components) of this linearized operator. This in some sense implies that the gas approaches to equilibrium as time tends to infinity. Another consideration comes from the dissipation through the fluid entropy in the nonlinear setting. In this case, the dissipative effect indeed corresponds to those from the viscosity and heat conductivity as for the Navier-Stokes equations.

Now we come back to the stability of a wave pattern. For a non-trivial solution profile connecting two different global Maxwellians at $x = \pm\infty$, it is reasonable and better to decompose the Boltzmann equation and its solution with respect to the local Maxwellian. This kind of decomposition was introduced in [41], [43] by rewritting the Boltzmann equation into a fluid-type dynamics system with the non-fluid component appearing in the source terms, coupled with an equation for the time evolution of the non-fluid component. In fact, set, cf. [41], [47],

$$f(x,t,\xi) = M(x,t,\xi) + G(x,t,\xi),$$

where the local Maxwellian M and G represent the fluid and non-fluid components in the solution respectively. Here, the local Maxwellian M is defined by the five conserved quantities, that is, the mass density $\rho(x,t)$, momentum $m(x,t) = \rho(x,t)u(x,t)$, and energy density $(E(x,t)+\frac{1}{2}|u(x,t)|^2)$. As presented in the next section, the governing system for the fluid components is of fluid-type so that the techniques for Navier-Stokes equations can be applied with some extra terms coming from the non-fluid component. Moreover, the dissipative effect of the linearized operator on the non-fluid component helps to close the energy estimate for the Boltzmann equation.

Similar to the Navier-Stokes equations, the dissipative effect in the Boltzmann equation also spreads out the contact discontinuity so that it behaves like a nonlinear diffusion wave. In the ansatz given in the next section, we can see that the contact wave profile for the Boltzmann equation is exactly the local Maxwellian defined by the contact wave profile for the corresponding Navier-Stokes equations. By using the fluid dynamic structure of the system for conserved quantities and the dissipation on the non-fluid component, we succeed in obtaining similar growth and decay rates for different order of energy estimates. As in the case for Navier-Stokes equations, the cancellation between the growth and decay rates in lower and higher order estimates leads to an uniform L^∞ estimate on the solution to the Boltzmann equation thus the convergence to the local Maxwellian defined by the contact wave pattern time asymptotically.

The rest of the paper will be arranged as follows. In the next section, we will give the ansatz to each problem and state the main results in this paper. The proofs of the theorems for the Navier-Stokes equations and the Boltzmann equation will be given in Sections 3 and 4 respectively.

2 Ansatz and main theorems

2.1 Compressible Navier-Stokes equations

Consider the one dimensional compressible Navier-Stokes equations in Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu(\frac{u_x}{v})_x, \\ (e+\frac{u^2}{2})_t + (pu)_x = (\kappa\frac{\theta_x}{v} + \mu\frac{uu_x}{v})_x, \end{cases} \tag{2.a1}$$

where $v(x,t)>0$ denotes the specific volume, $u(x,t)$ the velocity, $\theta(x,t)>0$ the absolute temperature, $\mu>0$ the viscosity and $\kappa>0$ the coefficient of heat conduction. Here we study the perfect gas so that the pressure p and the internal energy e are given respectively by

$$p = \frac{R\theta}{v}, \quad e = \frac{R}{\gamma-1}\theta + const. \tag{2.a2}$$

where $\gamma>1$ is the adiabatic exponent and $R>0$ is the gas constant. The initial data $(v_0,u_0,\theta_0)(x)$ satisfies

$$(v_0,u_0,\theta_0)(x) \to (v_\pm, 0, \theta_\pm) \quad \text{as} \quad x\to\pm\infty \tag{2.a3}$$

and

$$p_- := \frac{R\theta_-}{v_-} = \frac{R\theta_+}{v_+} := p_+. \tag{2.a4}$$

We are interested in the asymptotic behavior toward to contact discontinuity for solutions to the compressible Navier-Stokes system with general initial perturbation. First, we recall the contact wave $(\bar{v},\bar{u},\bar{\theta})(x,t)$ for the compressible N-S equations defined in [27]. For the corresponding Euler equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ (e + \dfrac{u^2}{2})_t + (pu)_x = 0, \end{cases} \tag{2.a5}$$

a contact discontinuity takes the form

$$(\bar{V},\bar{U},\bar{\Theta})(x,t) = \begin{cases} (v_-, 0, \theta_-), \ x < 0, \\ (v_+, 0, \theta_+), \ x > 0. \end{cases} \tag{2.a6}$$

if the positive constants $v_\pm$ and $\theta_\pm$ satisfy (2.a4). In the setting of compressible Navier-Stokes equations, the contact wave profile $(\bar{v},\bar{u},\bar{\theta})(x,t)$ becomes smooth and behaves as a diffusion wave due to the dissipation effect. From [27], the pressure of the profile $(\bar{v},\bar{u},\bar{\theta})(x,t)$ is almost constant, i.e.

$$\bar{p} = \frac{R\bar{\theta}}{\bar{v}} \approx p_+, \tag{2.a7}$$

which indicates the leading part of the energy equation $(2.a1)_3$ is

$$\frac{R}{\gamma - 1}\theta_t + p_+ u_x = \kappa(\frac{\theta_x}{v})_x. \tag{2.a8}$$

By (2.a8) and the mass equation $(2.a1)_1$, we obtain a nonlinear diffusion equation,

$$\theta_t = a(\frac{\theta_x}{\theta})_x, \quad a = \frac{\kappa p_+(\gamma-1)}{\gamma R^2} > 0. \tag{2.a9}$$

From [2] and [17], (2.a9) has a unique self similarity solution $\Theta(\xi), \xi = \frac{x}{\sqrt{1+t}}$ with the boundary conditions

$$\Theta(-\infty,t) = \theta_-, \quad \Theta(+\infty,t) = \theta_+.$$

Furthermore, by letting $\delta = |\theta_+ - \theta_-|$, Θ satisfies

$$\begin{cases} |\Theta_x| = O(\delta)(1+t)^{-\frac{1}{2}} e^{-\frac{\theta_\pm x^2}{4a(1+t)}}, \quad as\ x \to \pm\infty, \\ |\Theta - \theta_-| \le C\delta e^{-\frac{\theta_\pm x^2}{8a(1+t)}}, \ x < 0, \quad |\Theta - \theta_+| \le C\delta e^{-\frac{\theta_\pm x^2}{8a(1+t)}}, \ x > 0. \end{cases} \tag{2.a10}$$

Once Θ is determined, the contact wave profile $(\bar{v},\bar{u},\bar{\theta})$ is then defined as follows:

$$\bar{v} = \frac{R}{p_+}\Theta, \quad \bar{u} = \frac{Ra}{p_+\Theta}\Theta_x, \quad \bar{\theta} = \Theta - \frac{\gamma-1}{2R}\bar{u}^2. \tag{2.a11}$$

It is straightforward to check that $(\bar{v},\bar{u},\bar{\theta})$ satisfies

$$\|(\bar{v} - \bar{V}, \bar{u} - \bar{U}, \bar{\theta} - \bar{\Theta})\|_{L^p} = O(\kappa^{\frac{1}{2p}})(1+t)^{\frac{1}{2p}}, \ p \ge 1,$$

which means the nonlinear diffusion wave $(\bar{v}, \bar{u}, \bar{\theta})$ approximates the contact discontinuity $(\bar{V}, \bar{U}, \bar{\Theta})$ to the Euler equation (2.a3) in L^p norm, $p \geq 1$ on any finite time interval as the heat conductivity coefficient κ tends to zero.

On the other hand, substituting (2.a11) into (2.a1), we get

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ \bar{u}_t + \bar{p}_x = \mu(\frac{\bar{u}_x}{\bar{v}})_x + R_{1x}, \\ (\bar{e} + \frac{\bar{u}^2}{2})_t + (\bar{p}\bar{u})_x = \kappa(\frac{\bar{\theta}_x}{\bar{v}})_x + (\frac{\mu}{\bar{v}}\bar{u}\bar{u}_x)_x + R_{2x}, \end{cases} \tag{2.a12}$$

where

$$R_1 = (\frac{\kappa(\gamma-1)}{\gamma R} - \mu)\frac{\bar{u}_x}{\bar{v}} + \bar{p} - p_+ = O(\delta)(1+t)^{-1}e^{-\frac{\theta_\pm x^2}{4a(1+t)}}, \ as\ |x| \to \infty, \tag{2.a13}$$

$$R_2 = (\frac{\kappa(\gamma-1)}{R} - \mu)\frac{\bar{u}\bar{u}_x}{\bar{v}} + (\bar{p} - p_+)\bar{u} = O(\delta)(1+t)^{-\frac{3}{2}}e^{-\frac{\theta_\pm x^2}{4a(1+t)}}, \ as\ |x| \to \infty. \tag{2.a14}$$

Denote the conserved quantities by

$$m(x,t) = (v, u, \theta + \frac{\gamma-1}{2R}u^2)^t, \quad \bar{m}(x,t) = (\bar{v}, \bar{u}, \bar{\theta} + \frac{\gamma-1}{2R}\bar{u}^2)^t, \tag{2.a15}$$

where $()^t$ means the transpose of the vector (). At the far fields $x = \pm\infty$, the vectors m and $\bar{m}$ are the same, that is $m_\pm = (v_\pm, 0, \theta_\pm)^t$. Since we consider the general initial perturbation here, the integral $\int_{-\infty}^{\infty}(m(x,0) - \bar{m}(x,0))dx$ may not be zero in general. Hence, the mass distributes in all characteristic fields when time evolves, which introduces diffusion waves in the two nonlinear sound wave families as in [59] and [40]. Thus we need to construct two diffusion waves θ_1 and θ_3 to carry the exceed mass in the first and third characteristic fields respectively. In fact, let

$$A(v, u, \theta) = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{p}{v} & 0 & \frac{R}{v} \\ -\frac{(\gamma-1)pu}{Rv} & \frac{\gamma-1}{R}p & \frac{(\gamma-1)u}{v} \end{pmatrix}, \tag{2.a16}$$

be the Jacobi matrix of the flux $(-u, p, \frac{\gamma-1}{R}pu)^t$. Then, it is straightforward to check that the first eigenvalue of $A(v_-, 0, \theta_-)$ is $\lambda_1^- = -\sqrt{\frac{\gamma p_-}{v_-}}$ with right eigenvector

$$r_1^- = (-1, \lambda_1^-, \frac{\gamma-1}{R}p_-)^t. \tag{2.a17}$$

Similarly, the third eigenvalue and right eigenvector of $A(v_+, 0, \theta_+)$ are respectively $\lambda_3^+ = \sqrt{\frac{\gamma p_+}{v_+}}$ and

$$r_3^+ = (-1, \lambda_3^+, \frac{\gamma-1}{R}p_+)^t. \tag{2.a18}$$

By strict hyperbolicity, the vectors r_1^-, $m_+ - m_- = (v_+ - v_-, 0, \theta_+ - \theta_-)^t$ and r_3^+ are linearly independent in R^3. Thus, the integral $\int_{-\infty}^{\infty}(m(x,0) - \bar{m}(x,0))dx$ can be distributed as follows

$$\int_{-\infty}^{\infty}(m(x,0) - \bar{m}(x,0))dx = \bar{\theta}_1 r_1^- + \bar{\theta}_2(m_+ - m_-) + \bar{\theta}_3 r_3^+, \tag{2.a19}$$

with some constants $\bar{\theta}_i, i = 1, 2, 3$. Now we can define the ansatz $\tilde{m}(x, t)$ by

$$\tilde{m}(x,t) = \bar{m}(x + \bar{\theta}_2, t) + \bar{\theta}_1\theta_1 r_1^- + \bar{\theta}_3\theta_3 r_3^+, \tag{2.a20}$$

where

$$\theta_1(x,t) = \frac{1}{\sqrt{4\pi(1+t)}}e^{-\frac{(x-\lambda_1^-(1+t))^2}{4(1+t)}}, \quad \theta_3(x,t) = \frac{1}{\sqrt{4\pi(1+t)}}e^{-\frac{(x-\lambda_3^+(1+t))^2}{4(1+t)}}, \tag{2.a21}$$

satisfying

$$\theta_{1t} + \lambda_1^-\theta_{1x} = \theta_{1xx}, \quad \theta_{3t} + \lambda_3^+\theta_{3x} = \theta_{3xx}, \tag{2.a22}$$

and $\int_{-\infty}^{\infty} \theta_i(x,t)dx = 1$ for $i = 1, 3$, and all $t \geq 0$ respectively. More precisely, the ansatz $\tilde{m}$ has the following expression

$$\tilde{m}(x,t) = (\tilde{v}, \tilde{u}, \tilde{\theta} + \frac{\gamma-1}{2R}\tilde{u}^2)^t(x,t), \tag{2.a23}$$

with

$$\begin{aligned}
&\tilde{v}(x,t) = \bar{v}(x+\bar{\theta}_2,t) - \bar{\theta}_1\theta_1 - \bar{\theta}_3\theta_3,\\
&\tilde{u}(x,t) = \bar{u}(x+\bar{\theta}_2,t) + \lambda_1^-\bar{\theta}_1\theta_1 + \lambda_3^+\bar{\theta}_3\theta_3,\\
&\tilde{\theta}(x,t) = \bar{\theta}(x+\bar{\theta}_2,t) + \frac{\gamma-1}{2R}\bar{u}^2(x+\bar{\theta}_2,t) + \frac{\gamma-1}{R}p_+(\bar{\theta}_1\theta_1 + \bar{\theta}_3\theta_3) - \frac{\gamma-1}{2R}\tilde{u}^2.
\end{aligned} \tag{2.a24}$$

Furthermore, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}(m(x,0) - \tilde{m}(x,0))dx\\
&= \int_{-\infty}^{\infty}(m(x,0) - \bar{m}(x,0))dx + \int_{-\infty}^{\infty}(\bar{m}(x,0) - \tilde{m}(x,0))dx\\
&= \bar{\theta}_2(m_+ - m_-) + \int_{-\infty}^{\infty}(\bar{m}(x,0) - \bar{m}(x+\bar{\theta}_2,0))dx = 0.
\end{aligned} \tag{2.a25}$$

Without loss of generality, we can assume that $\bar{\theta}_2 = 0$ from now on. By straightforward computation, the ansatz $\tilde{m}$ satisfies

$$\begin{cases}
\tilde{v}_t - \tilde{u}_x = \tilde{R}_{1x},\\
\tilde{u}_t + \tilde{p}_x = \mu(\frac{\tilde{u}_x}{\tilde{v}})_x + \tilde{R}_{2x},\\
(\tilde{e} + \frac{\tilde{u}^2}{2})_t + (\tilde{p}\tilde{u})_x = \kappa(\frac{\tilde{\theta}_x}{\tilde{v}})_x + (\frac{\mu}{\tilde{v}}\tilde{u}\tilde{u}_x)_x + \tilde{R}_{3x},
\end{cases} \tag{2.a26}$$

where

$$\tilde{R}_1 = -\bar{\theta}_1\theta_{1x} - \bar{\theta}_3\theta_{3x}, \tag{2.a27}$$

$$\tilde{R}_2 = R_1 + \mu(\tfrac{\bar{u}_x}{\bar{v}} - \tfrac{\tilde{u}_x}{\tilde{v}}) + (\lambda_1^-\bar{\theta}_1\theta_{1x} + \lambda_3^+\bar{\theta}_3\theta_{3x}) + (\tilde{p} - \bar{p} - {\lambda_1^-}^2\bar{\theta}_1\theta_1 - {\lambda_3^+}^2\bar{\theta}_3\theta_3), \tag{2.a28}$$

and

$$\begin{aligned}
\tilde{R}_3 = \ & R_2 + \kappa(\frac{\bar{\theta}_x}{\bar{v}} - \frac{\tilde{\theta}_x}{\tilde{v}}) + \mu(\frac{\bar{u}\bar{u}_x}{\bar{v}} - \frac{\tilde{u}\tilde{u}_x}{\tilde{v}}) + p_+(\bar{\theta}_1\theta_{1x} + \bar{\theta}_3\theta_{3x})\\
&+(\tilde{p}\tilde{u} - \bar{p}\bar{u} - p_+\lambda_1^-\bar{\theta}_1\theta_1 - p_+\lambda_3^+\bar{\theta}_3\theta_3).
\end{aligned} \tag{2.a29}$$

Notice that from (2.a10), (2.a21) and (2.a24),

$$\begin{aligned}\tilde p-\bar p &= -\frac{\bar p}{\bar v}(\tilde v-\bar v)+\frac{R}{\bar v}(\tilde\theta-\bar\theta)+O(1)[(\tilde v-\bar v)^2+(\tilde\theta-\bar\theta)^2]\\ &= \frac{\gamma p_-}{v_-}\bar\theta_1\theta_1+\frac{\gamma p_+}{v_+}\bar\theta_3\theta_3\\ &\quad +O(\delta+\bar\theta_1^2+\bar\theta_3^2)\frac{1}{1+t}(e^{-\frac{cx^2}{1+t}}+e^{-\frac{c(x-\lambda_1^-(1+t))^2}{1+t}}+e^{-\frac{c(x-\lambda_3^+(1+t))^2}{1+t}}),\end{aligned} \tag{2.a30}$$

for some positive constant $c>0$. Also we have the similar estimate for $\tilde p\tilde u-\bar p\bar u$. Thus we have the following properties for $\tilde R_i, i=1,2,3$:

$$\tilde R_i=O(\delta+\bar\theta_1^2+\bar\theta_3^2)\frac{1}{1+t}(e^{-\frac{cx^2}{1+t}}+e^{-\frac{c(x-\lambda_1^-(1+t))^2}{1+t}}+e^{-\frac{c(x-\lambda_3^+(1+t))^2}{1+t}}). \tag{2.a31}$$

We can now state our main result for the compressible Navier-Stokes (2.a1)-(2.a2). First, we denote the perturbation around the ansatz $(\tilde v,\tilde u,\tilde\theta)$ by

$$\phi(x,t)=v-\tilde v,\quad \psi(x,t)=u-\tilde u,\quad \zeta(x,t)=\theta-\tilde\theta. \tag{2.a32}$$

Then set

$$\begin{aligned}&\Phi(x,t)=\int_{-\infty}^x\phi(y,t)dy,\qquad \Psi(x,t)=\int_{-\infty}^x\psi(y,t)dy,\\ &\bar W(x,t)=\int_{-\infty}^x(e+\frac{|u|^2}{2}-\tilde e-\frac{|\tilde u|^2}{2})(y,t)dy.\end{aligned} \tag{2.a33}$$

Notice that the quantities $(\Phi,\Psi,\bar W)$ can be well defined in some Sobolev space since the compressible Navier-Stokes equations (2.a1) and the system (2.a26) are in the conservative forms and $(\Phi,\Psi,\bar W)(\pm\infty,0)=0$ due to (2.a25).

The precise statement of our first result is as follows.

Theorem 1. Let $(\tilde v,\tilde u,\tilde\theta)(x,t)$ be defined in (2.a24) and $\delta=|\theta_+-\theta_-|$. Then there exist positive constants δ_0 and ϵ, such that if $\delta\le\delta_0$ and the initial data (v_0,u_0,θ_0) satisfies

$$\|(\Phi,\Psi,\bar W)\|_{L^2}+\|m-\bar m\|_{H^1}\le\epsilon, \tag{2.a34}$$

then the system (2.a1) admits a unique global solution $(v,u,\theta)(x,t)$ satisfying

$$(\Phi,\Psi,\bar W)\in C(0,+\infty;H^2), \tag{2.a35}$$

$$\phi\in L^2(0,+\infty;H^1),\quad (\psi,\zeta)\in L^2(0,+\infty;H^2). \tag{2.a36}$$

Furthermore, the solution satisfies

$$\|(v-\tilde v,u-\tilde u,\theta-\tilde\theta\|_{L^\infty}\le C(\epsilon+\delta_0^{\frac12})(1+t)^{-\frac14}. \tag{2.a37}$$

Remark 2. It should be noted that the constraint (2.a34) on initial data can be satisfied easily due to (2.a25) if $(\phi,\psi,\zeta)(x,0)$ decay fast enough at $x=\pm\infty$.

2.2 Boltzmann equation

Since the profile studied is in one space dimension, we consider the Boltzmann equation with "slab symmetry"

$$f_t + \xi_1 f_x = Q(f,f),\ (f,x,t,\xi) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}^+\times\mathbf{R}^3, \tag{2.b1}$$

where $f(x,t,\xi)$ represents the distributional density of particles at space-time (x,t) with velocity ξ. For monatomic gas, the rotational invariance of the molecules leads to the collision operator $Q(f,f)$ as a bilinear collision operator in the form of, cf. [7]:

$$Q(f,g)(\xi) \equiv \frac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{S}^2_+}\Big(f(\xi')g(\xi'_*)+f(\xi'_*)g(\xi')-f(\xi)g(\xi_*)-f(\xi_*)g(\xi)\Big)B(|\xi-\xi_*|,\theta)\ d\xi_* d\Omega,$$

with θ being the angle between the relative velocity and the unit vector Ω. Here $\mathbf{S}^2_+ = \{\Omega \in \mathbf{S}^2 :\ (\xi-\xi_*)\cdot\Omega \geq 0\}$. The conservation of momentum and energy gives the following relation between velocities before and after collision:

$$\begin{cases} \xi' = \xi - [(\xi-\xi_*)\cdot\Omega]\,\Omega, \\ \xi'_* = \xi_* + [(\xi-\xi_*)\cdot\Omega]\,\Omega. \end{cases}$$

Here we consider the Boltzmann equation for the two basic models, i.e., the hard sphere model and the hard potential with angular cut-off. In these two cases, the collision kernel $B(|\xi-\xi_*|,\theta)$ takes the forms

$$B(|\xi-\xi_*|,\theta) = |(\xi-\xi_*,\Omega)|,$$

and

$$B(|\xi-\xi_*|,\theta) = |\xi-\xi_*|^{\frac{n-5}{n-1}}b(\theta), \quad b(\theta)\in L^1([0,\pi]),\ n>5,$$

respectively. Here, n is the index in the inverse power potentials proportional to r^{1-n} with r being the distance between two particles. The following analysis can be generalized to other kernels with similar property. But we will not discuss them here.

For a non-trivial solution profile connecting two different global Maxwellians at $x = \pm\infty$, we decompose the Boltzmann equation and its solution with respect to the local Maxwellian. This kind of decomposition was introduced in [41], [43] by rewriting the Boltzmann equation into a fluid-type dynamics system with the non-fluid component appearing in the source terms, coupled with an equation for the time evolution of the non-fluid component. In fact, set, cf. [41], [47],

$$f(x,t,\xi) = M(x,t,\xi) + G(x,t,\xi),$$

where the local Maxwellian M and G represent the fluid and non-fluid components in the solution respectively. Here, the local Maxwellian M is defined by the five conserved quantities, that is, the mass density $\rho(x,t)$, momentum $m(x,t) = \rho(x,t)u(x,t)$, and energy density $(E(x,t) + \frac{1}{2}|u(x,t)|^2)$:

$$\begin{cases} \rho(x,t) \equiv \displaystyle\int_{\mathbf{R}^3} f(x,t,\xi)d\xi, \\ m^i(x,t) \equiv \displaystyle\int_{\mathbf{R}^3} \psi_i(\xi)f(x,t,\xi)d\xi \text{ for } i=1,2,3, \\ \left[\rho\left(E+\frac{1}{2}|u|^2\right)\right](x,t) \equiv \displaystyle\int_{\mathbf{R}^3}\psi_4(\xi)f(x,t,\xi)d\xi, \end{cases} \tag{2.b2}$$

as

$$M \equiv M_{[\rho,u,\theta]}(x,t,\xi) \equiv \frac{\rho(x,t)}{\sqrt{(2\pi R\theta(x,t))^3}} \exp\left(-\frac{|\xi-u(x,t)|^2}{2R\theta(x,t)}\right). \tag{2.b3}$$

Here $\theta(x,t)$ is the temperature which is related to the internal energy E by $E = \frac{3}{2}R\theta$ with R being the gas constant, and $u(x,t)$ is the fluid velocity. It is well known that the collision invariants $\psi_\alpha(\xi)$ are given by, cf. [7]:

$$\begin{cases} \psi_0(\xi) \equiv 1, \\ \psi_i(\xi) \equiv \xi_i \quad \text{for} \quad i=1,2,3, \\ \psi_4(\xi) \equiv \frac{1}{2}|\xi|^2, \end{cases}$$

satisfying

$$\int_{\mathbf{R}^3} \psi_j(\xi) Q(h,g) d\xi = 0, \quad \text{for} \quad j=0,1,2,3,4.$$

In the sequel, the inner product of h and g in $L^2_\xi(\mathbf{R}^3)$ with respect to a given Maxwellian $\tilde{M}$ is defined by:

$$\langle h,g\rangle_{\tilde{M}} \equiv \int_{\mathbf{R}^3} \frac{1}{\tilde{M}} h(\xi)g(\xi)d\xi,$$

when the integral is well defined. If $\tilde{M}$ is the local Maxwellian M, with respect to the corresponding inner product, the macroscopic space is spanned by the following five pairwise orthogonal functions

$$\begin{cases} \chi_0(\xi) \equiv \frac{1}{\sqrt{\rho}}M, \\ \chi_i(\xi) \equiv \frac{\xi_i - u_i}{\sqrt{R\theta\rho}}M \quad \text{for} \quad i=1,2,3, \\ \chi_4(\xi) \equiv \frac{1}{\sqrt{6\rho}}(\frac{|\xi-u|^2}{R\theta} - 3)M, \\ < \chi_i, \chi_j >= \delta_{ij}, \;\; i,j=0,1,2,3,4. \end{cases}$$

Using these five basic functions, we define the macroscopic projection P_0 and microscopic projection P_1 as follows:

$$\begin{cases} P_0 h \equiv \sum_{j=0}^{4} < h, \chi_j > \chi_j, \\ P_1 h \equiv h - P_0 h. \end{cases}$$

The projections P_0 and P_1 are orthogonal and satisfy

$$P_0P_0 = P_0, P_1P_1 = P_1, P_0P_1 = P_1P_0 = 0.$$

A function $h(\xi)$ is called microscopic or non-fluid if

$$\int h(\xi)\psi_j(\xi)d\xi = 0, \;\; j=0,1,2,3,4.$$

Under this decomposition, the solution $f(x,t,\xi)$ of the Boltzmann equation satisfies

$$P_0 f = M, \ \ P_1 f = G,$$

and the Boltzmann equation becomes

$$(M+G)_t + \xi_1(M+G)_x = 2Q(M,G) + Q(G,G),$$

which is equivalent to the following fluid-type system for the fluid components (see [41], [42], [43] for details):

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = -\int \xi_1^2 G_x d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = -\int \xi_1 \xi_i G_x d\xi, \ i = 2,3 \\ (\rho(e + \frac{|u|^2}{2}))_t + (\rho u_1 (e + \frac{|u|^2}{2}) + p u_1)_x = -\int \frac{1}{2}\xi_1 |\xi|^2 G_x d\xi, \end{cases} \tag{2.b4}$$

or more precisely,

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = \frac{4}{3}(\mu(\theta) u_{1x})_x - \int \xi_1^2 \Theta_x d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = (\mu(\theta) u_{ix})_x - \int \xi_1 \xi_i \Theta_x d\xi, \ i = 2,3 \\ (\rho(e + \frac{|u|^2}{2}))_t + (\rho u_1 (e + \frac{|u|^2}{2}) + p u_1)_x = (\lambda(\theta)\theta_x)_x \\ \quad + \frac{4}{3}(\mu(\theta) u_1 u_{1x})_x + \sum_{i=2}^{3} (\mu(\theta) u_i u_{ix})_x - \int \frac{1}{2}\xi_1 |\xi|^2 \Theta_x d\xi, \end{cases} \tag{2.b5}$$

together with the equation for the non-fluid component G:

$$G_t + P_1(\xi_1 M_x) + P_1(\xi_1 G_x) = L_M G + Q(G,G). \tag{2.b6}$$

It follows from (2.b6) that

$$G = L_M^{-1}(P_1(\xi_1 M_x)) + \Theta$$

with

$$\Theta = L_M^{-1}(G_t + P_1(\xi_1 G_x) - Q(G,G)). \tag{2.b7}$$

Here L_M is the linearized operator of the collision operator with respect to the local Maxwellian M:

$$L_M h = Q(M,h) + Q(h,M),$$

and the null space N of L_M is spanned by the macroscopic variables:

$$\chi_j, \ \ j = 0,1,2,3,4.$$

Furthermore, there exists a positive constant $\sigma_0(\rho,u,\theta) > 0$ such that for any function $h(\xi) \in N^\perp$, see [22],

$$< h, L_M h > \leq -\sigma_0 < \nu(|\xi|) h, h >,$$

where $\nu(|\xi|)$ is the collision frequency. For the hard sphere and the hard potential with angular cut-off, the collision frequency $\nu(|\xi|)$ has the following property

$$0 < \nu_0 < \nu(|\xi|) \leq c(1+|\xi|)^{\beta},$$

for some positive constants ν_0, c and $0 < \beta \leq 1$. In the above presentation, we have normalized the gas constant R to be $\frac{2}{3}$ for simplicity so that $e = \theta$ and $p = \frac{2}{3}\rho\theta$. Notice also that the viscosity coefficient $\mu(\theta) > 0$ and the heat conductivity coefficient $\lambda(\theta) > 0$ are smooth functions of the temperature θ.

Since our problem is in one dimensional space $x \in \mathbf{R}$, in the macroscopic level, it is more convenient to rewrite the system and the equation by using the *Lagrangian* coordinates as in the study of conservation laws. That is, consider the coordinate transformation:

$$x \Rightarrow \int_0^x \rho(y,t)dy, \qquad t \Rightarrow t.$$

We will still denote the *Lagrangian* coordinates by (x,t) for simplicity of notation. The system (2.b1) and (2.b4) in the Lagrangian coordinates become, respectively,

$$f_t - \frac{u_1}{v} f_x + \frac{\xi_1}{v} f_x = Q(f,f), \tag{2.b8}$$

and

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = -\int \xi_1^2 G_x d\xi, \\ u_{it} = -\int \xi_1 \xi_i G_x d\xi, \ i = 2,3 \\ (e + \frac{|u|^2}{2})_t + (pu_1)_x = -\int \frac{1}{2}\xi_1 |\xi|^2 G_x d\xi. \end{cases} \tag{2.b9}$$

Moreover, (2.b5) and (2.b6) take the form

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = \frac{4}{3}(\frac{\mu(\theta)}{v} u_{1x})_x - \int \xi_1^2 \Theta_{1x} d\xi, \\ u_{it} = (\frac{\mu(\theta)}{v} u_{ix})_x - \int \xi_1 \xi_i \Theta_{1x} d\xi, \ i = 2,3 \\ (e + \frac{|u|^2}{2})_t + (pu_1)_x = (\frac{\lambda(\theta)}{v}\theta_x)_x + \frac{4}{3}(\frac{\mu(\theta)}{v} u_1 u_{1x})_x \\ + \sum_{i=2}^3 (\frac{\mu(\theta)}{v} u_i u_{ix})_x - \int \frac{1}{2}\xi_1 |\xi|^2 \Theta_{1x} d\xi, \end{cases} \tag{2.b10}$$

and

$$G_t - \frac{u_1}{v} G_x + \frac{1}{v} P_1(\xi_1 M_x) + \frac{1}{v} P_1(\xi_1 G_x) = L_M G + Q(G,G), \tag{2.b11}$$

with

$$G = L_M^{-1}(\frac{1}{v} P_1(\xi_1 M_x)) + \Theta_1,$$

and

$$\Theta_1 = L_M^{-1}(G_t - \frac{u_1}{v}G_x + \frac{1}{v}P_1(\xi_1 G_x) - Q(G,G)). \qquad (2.b12)$$

Notice that system (2.b10) can be regarded as the compressible Navier-Stokes equations (2.a1) with some source terms coming from non-fluid components. Analogous to (2.a1), we construct the ansatz for the wave profile of the Boltzmann equation as follows. First, let $\Theta(\frac{x}{\sqrt{1+t}})$ be the unique self-similarity solution of the following nonlinear diffusion equation

$$\Theta_t = (a(\Theta)\Theta_x)_x, \quad \Theta(-\infty,t) = \theta_-, \quad \Theta(+\infty,t) = \theta_+, \qquad (2.b13)$$

where the function $a(s) = \frac{9p_+\lambda(s)}{10\theta} > 0$. Notice that (2.b13) is exactly the same as the diffusion equation (2.a9) for the compressible Naiver-Stokes equations when $\gamma = \frac{5}{3}$, $R = \frac{2}{3}$ and $\kappa = \lambda(\theta)$. We then define

$$\bar{v} = \frac{2}{3p_+}\Theta, \quad \bar{u}_1 = \frac{2a(\Theta)}{3p_+}\Theta_x, \quad \bar{u}_i = 0,\ i = 2,3, \quad \bar{\theta} = \Theta - \frac{1}{2}|\bar{u}|^2. \qquad (2.b14)$$

Let $\delta = |\theta_+ - \theta_-|$. It can be verified by a straightforward computation that $(\bar{v},\bar{u},\bar{\theta})$ satisfies

$$\begin{cases} \bar{v}_t - \bar{u}_{1x} = 0, \\ \bar{u}_{1t} + \bar{p}_x = \frac{4}{3}(\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{1x})_x + R_{1x}, \\ \bar{u}_{it} = (\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_{ix})_x, i = 2,3, \\ (\bar{\theta} + \frac{|\bar{u}|^2}{2})_t + (\bar{p}\bar{u}_1)_x = (\frac{\lambda(\bar{\theta})}{\bar{v}}\bar{\theta}_x)_x + \frac{4}{3}(\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_1\bar{u}_{1x})_x \\ \quad +(\sum_{i=2}^{3}\frac{\mu(\bar{\theta})}{\bar{v}}\bar{u}_i\bar{u}_{ix})_x + R_{2x}, \end{cases} \qquad (2.b15)$$

where

$$R_1 = \frac{2}{3p_+}a(\Theta)\Theta_t + \bar{p} - p_+ - \frac{4\mu(\bar{\theta})}{3\bar{v}}\bar{u}_{1x} = O(\delta)(1+t)^{-1}e^{-\frac{cx^2}{1+t}}, \qquad (2.b16)$$

$$R_2 = \frac{1}{\bar{v}}(\lambda(\Theta)\Theta_x - \lambda(\bar{\theta})\bar{\theta}_x) + (\bar{p} - p_+)\bar{u}_1 - \frac{4\mu(\bar{\theta})}{3\bar{v}}\bar{u}_1\bar{u}_{1x} = O(\delta)(1+t)^{-3/2}e^{-\frac{cx^2}{1+t}} \qquad (2.b17)$$

with some positive constant $c > 0$.

Let $m = (v, u_1, \theta + \frac{1}{2}|u|^2)$ and $\bar{m} = (\bar{v}, \bar{u}_1, \bar{\theta} + \frac{1}{2}|\bar{u}|^2)$. Since $\int_{-\infty}^{\infty}(m(x,0) - \bar{m}(x,0))dx$ is usually not zero, we have to introduce two diffusion waves in the sound wave families as shown in the previous subsection. Let

$$A_- = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{p_-}{v_-} & 0 & \frac{2}{3v_-} \\ 0 & p_- & 0 \end{pmatrix}, \quad A_+ = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{p_+}{v_+} & 0 & \frac{2}{3v_+} \\ 0 & p_+ & 0 \end{pmatrix}, \qquad (2.b18)$$

be the Jacobi matrices of the flux $(-u, p, pu)^t$ at $(v_-, 0, \theta_-)$ and $(v_+, 0, \theta_+)$ respectively. It is easy to check that $\lambda_1^- = -\sqrt{\frac{5p_-}{3v_-}}$ is the first eigenvalue of A_- with $r_1^- = (-1, \lambda_1^-, p_-)^t$

being the corresponding eigenvector. And $\lambda_3^+ = \sqrt{\frac{5p_+}{3v_+}}$ and $r_3^+ = (-1, \lambda_3^+, p_+)^t$ are those of the third family of A_+. Since r_1^-, $(v_+ - v_-, 0, \theta_+ - \theta_-)^t$ and r_3^+ are linearly independent in R^3 by strict hyperbolicity, we have

$$\int_{-\infty}^{\infty} (m(x,0) - \bar{m}(x,0))dx = \bar{\theta}_1 r_1^- + \bar{\theta}_2 (v_+ - v_-, 0, \theta_+ - \theta_-)^t + \bar{\theta}_3 r_3^+ \tag{2.b19}$$

with unique constants $\bar{\theta}_i, i = 1,2,3$. The ansatz $\tilde{m}(x,t)$ for m is defined as

$$\tilde{m}(x,t) = \bar{m}(x + \bar{\theta}_2, t) + \bar{\theta}_1 \theta_1 r_1^- + \bar{\theta}_3 \theta_3 r_3^+, \tag{2.b20}$$

where

$$\theta_1(x,t) = \frac{1}{\sqrt{4\pi(1+t)}} e^{-\frac{(x-\lambda_1^-(1+t))^2}{4(1+t)}}, \quad \theta_3(x,t) = \frac{1}{\sqrt{4\pi(1+t)}} e^{-\frac{(x-\lambda_3^+(1+t))^2}{4(1+t)}}, \tag{2.b21}$$

satisfying $\theta_{1t} + \lambda_1^- \theta_{1x} = \theta_{1xx}$, $\theta_{3t} + \lambda_3^+ \theta_{3x} = \theta_{3xx}$ and $\int_{-\infty}^{\infty} \theta_i(x,t)dx = 1$ for $i = 1,3$ and all $t \geq 0$. Similar to (2.a25), we now have $\int_{-\infty}^{\infty} (m(x,0) - \tilde{m}(x,0))dx = 0$. Notice that $\int_{-\infty}^{\infty} u_i(x,0)dx$ may not be zero either for $i = 2,3$. For this, we define

$$\bar{u}_i(x,t) = \bar{\theta}_{i+2} \frac{1}{\sqrt{4\pi(1+t)}} e^{-\frac{x^2}{4(1+t)}}, \ i = 2,3, \tag{2.b22}$$

where $\bar{\theta}_{i+2} = \int_{-\infty}^{\infty} u_i(x,0)dx$. It is obvious that $\int_{-\infty}^{\infty} (u_i(x,0) - \tilde{u}_i(x,0))dx = 0, i = 2,3$.

Finally, our ansatz is defined as

$$\begin{aligned}
&\tilde{v}(x,t) = \bar{v}(x+\bar{\theta}_2, t) - \bar{\theta}_1\theta_1 - \bar{\theta}_3\theta_3, \\
&\tilde{u}_1(x,t) = \bar{u}_1(x+\bar{\theta}_2, t) + \lambda_1^- \bar{\theta}_1\theta_1 + \lambda_3^+ \bar{\theta}_3\theta_3, \\
&\tilde{u}_i = \frac{\bar{\theta}_{i+2}}{\sqrt{4\pi(1+t)}} e^{-\frac{x^2}{4(1+t)}}, \ i = 2,3, \\
&\tilde{\theta}(x,t) = \bar{\theta}(x+\bar{\theta}_2, t) + \frac{1}{2}|\bar{u}|^2(x+\bar{\theta}_2, t) + p_+(\bar{\theta}_1\theta_1 + \bar{\theta}_3\theta_3) - \frac{1}{2}|\tilde{u}|^2.
\end{aligned} \tag{2.b23}$$

Here $(\tilde{v}, \tilde{u}, \tilde{\theta})$ satisfies

$$\tilde{m}(x,t) = (\tilde{v}, \tilde{u}_1, \tilde{\theta} + \frac{1}{2}|\tilde{u}|^2)^t(x,t). \tag{2.b24}$$

Without loss of generality, we also assume that $\bar{\theta}_2 = 0$. It is straightforward to show that

$$\begin{cases}
\tilde{v}_t - \tilde{u}_x = \tilde{R}_{1x}, \\
\tilde{u}_{1t} + \tilde{p}_x = \frac{4}{3}(\mu(\tilde{\theta})\frac{\tilde{u}_{1x}}{\tilde{v}})_x + \tilde{R}_{2x}, \\
\tilde{u}_{it} = (\frac{\mu(\tilde{\theta})\tilde{u}_{ix}}{\tilde{v}})_x + (\tilde{R}_{i+1})_x, i = 2,3, \\
(\tilde{e} + \frac{|\tilde{u}|^2}{2})_t + (\tilde{p}\tilde{u}_1)_x = (\lambda(\tilde{\theta})\frac{\tilde{\theta}_x}{\tilde{v}})_x + \frac{4}{3}(\frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_1\tilde{u}_{1x})_x \\
\quad +(\sum_{i=2}^{3}\frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_i\tilde{u}_{ix})_x + \tilde{R}_{5x},
\end{cases} \tag{2.b25}$$

where

$$\tilde{R}_1 = -\bar{\theta}_1\theta_{1x} - \bar{\theta}_3\theta_{3x}, \tag{2.b26}$$

$$\begin{aligned}\tilde{R}_2 = R_1 + \frac{4}{3}(\frac{\mu(\bar{\theta})\bar{u}_{1x}}{\bar{v}} - \frac{\mu(\tilde{\theta})\tilde{u}_{1x}}{\tilde{v}}) + (\lambda_1^-\bar{\theta}_1\theta_{1x} + \lambda_3^+\bar{\theta}_3\theta_{3x}) \\ +(\tilde{p} - \bar{p} - \lambda_1^{-2}\bar{\theta}_1\theta_1 - \lambda_3^{+2}\bar{\theta}_3\theta_3),\end{aligned} \tag{2.b27}$$

$$R_{i+1} = \tilde{u}_{ix} - \frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_{ix}, i = 2, 3, \tag{2.b28}$$

and

$$\begin{aligned}\tilde{R}_5 = \quad & R_2 + (\frac{\lambda(\bar{\theta})\bar{\theta}_x}{\bar{v}} - \frac{\lambda(\tilde{\theta})\tilde{\theta}_x}{\tilde{v}}) + \frac{4}{3}(\frac{\mu(\bar{\theta})\bar{u}_1\bar{u}_{1x}}{\bar{v}} - \frac{\mu(\tilde{\theta})\tilde{u}_1\tilde{u}_{1x}}{\tilde{v}}) - \sum_{i=2}^{3}\frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_i\tilde{u}_{ix} \\ & +p_+(\bar{\theta}_1\theta_{1x} + \bar{\theta}_3\theta_{3x}) + (\tilde{p}\tilde{u}_1 - \bar{p}\bar{u}_1 - p_+\lambda_1^-\bar{\theta}_1\theta_1 - p_+\lambda_3^+\bar{\theta}_3\theta_3).\end{aligned} \tag{2.b29}$$

By the same argument for (2.a1), it can be shown that, for $i = 1, 2, 3, 4, 5$:

$$\tilde{R}_i = O(\delta + \sum_{i=1}^{5}|\bar{\theta}_i|)\frac{1}{1+t}(e^{-\frac{cx^2}{1+t}} + e^{-\frac{c(x-\lambda_1^-(1+t))^2}{1+t}} + e^{-\frac{c(x-\lambda_3^+(1+t))^2}{1+t}}) \tag{2.b30}$$

holds with some positive constant $c > 0$.

With above preparation, we are ready to state the result on the stability of the contact wave pattern for the Boltzmann equation (2.b1). Denote the perturbation around the ansatz $(\tilde{v}, \tilde{u}, \tilde{\theta})$ by

$$\phi(x,t) = v - \tilde{v}, \quad \psi(x,t) = u - \tilde{u}, \quad \zeta(x,t) = \theta - \tilde{\theta}. \tag{2.b31}$$

Then set

$$\begin{aligned}\Phi(x,t) = \int_{-\infty}^{x}\phi(y,t)dy, \qquad \Psi(x,t) = \int_{-\infty}^{x}\psi(y,t)dy, \\ \bar{W}(x,t) = \int_{-\infty}^{x}(e + \frac{|u|^2}{2} - \tilde{e} - \frac{|\tilde{u}|^2}{2})(y,t)dy,\end{aligned} \tag{2.b32}$$

so that the quantities Φ, Ψ ad $\bar{W}$ can be well defined in some Sobolev space. The second main theorem is as follows.

Theorem 3. Let $(\tilde{v}, \tilde{u}, \tilde{\theta})(x,t)$ be the ansatz defined in (2.b23) with $\delta = |\theta_+ - \theta_-|$. Then there exist small positive constants δ_0, ϵ and global Maxwellian $M_* = M_{[\rho_*, u_*, \theta_*]}$, such that if $\delta \le \delta_0$ and the initial data satisfies

$$\begin{aligned}\{\|(\Phi, \Psi, \bar{W})\|_{L_x^2} + \|(\phi, \psi, \zeta)\|_{H_x^1} + \sum_{|\alpha|=2}\|\partial^\alpha f\|_{L_x^2(L_\xi^2(\frac{1}{\sqrt{M_*}}))} \\ + \sum_{0\le|\alpha|\le 1}\|\partial^\alpha G\|_{L_x^2(L_\xi^2(\frac{1}{\sqrt{M_*}}))}\}|_{t=0} \le \epsilon,\end{aligned} \tag{2.b33}$$

then the Cauchy problem (2.b8) admits a unique global solution $f(x,t,\xi)$ satisfying

$$\|f(x,t,\xi) - M_{[\tilde{v},\tilde{u},\tilde{\theta}]}\|(t)_{L_x^\infty(L_\xi^2(\frac{1}{\sqrt{M_*}}))} \le C(\epsilon + \delta_0^{\frac{1}{2}})(1+t)^{-\frac{1}{4}}. \tag{2.b34}$$

Here $f(\xi) \in L^2_\xi(\frac{1}{\sqrt{M_*}})$ means that $\frac{f(\xi)}{\sqrt{M_*}} \in L^2_\xi(\mathbf{R}^3)$.

Remark 4. The estimate for the higher derivatives on the solution can be obtained similarly, provided that the initial data has the same order regularity.

3 Compressible Navier-Stokes equations

This section is devoted to the stability analysis for the compressible Navier-Stokes system (2.a1). Our proof will be given in the following four subsections. In subsection 3.1, we reformulate the stability problem for the compressible Navier-Stokes equations in terms of the integrated variables in (2.a33). And the subsection 3.2 is devoted to the basic lower order estimates, while the subsection 3.3 is for the derivative estimate. The stability and convergence rate of the contact wave for the compressible Navier-Stokes system (2.a1) will be given in subsection 3.4.

3.1 Reformulated system

To prove Theorem 1, we first reformulate the system (2.a1) in terms of the perturbation (ϕ, ψ, ζ) around the ansatz $(\tilde{v}, \tilde{u}, \tilde{\theta})$ defined in (2.a24). By (2.a32) and (2.a33), it is easy to check that $(\phi, \psi) = (\Phi, \Psi)_x$ and $\frac{R}{\gamma-1}\zeta + \frac{1}{2}|\Psi_x|^2 + \tilde{u}\Psi_x = \bar{W}_x$.

Subtracting (2.a25) from the equation (2.a1) and integrating the resulting system yield

$$\begin{cases} \Phi_t - \Psi_x = -\tilde{R}_1, \\ \Psi_t + p - \tilde{p} = \dfrac{\mu}{v}u_x - \dfrac{\mu}{\tilde{v}}\tilde{u}_x - \tilde{R}_2, \\ \bar{W}_t + pu - \tilde{p}\tilde{u} = \dfrac{\kappa}{v}\theta_x - \dfrac{\kappa}{\tilde{v}}\tilde{\theta}_x + \dfrac{\mu}{v}uu_x - \dfrac{\mu}{\tilde{v}}\tilde{u}\tilde{u}_x - \tilde{R}_3. \end{cases} \tag{3.1}$$

Since the variable $\bar{W}$ is the anti-derivative of the total energy, not the temperature, it is more convenient to introduce the variable

$$W = \frac{\gamma-1}{R}(\bar{W} - \tilde{u}\Psi). \tag{3.2}$$

It follows that

$$\zeta = W_x - Y, \text{ with } Y = \frac{\gamma-1}{R}(\frac{1}{2}\Psi_x^2 - \tilde{u}_x\Psi). \tag{3.3}$$

Using the new variable W and linearizing the left hand side of the system (3.1), we have

$$\begin{cases} \Phi_t - \Psi_x = -\tilde{R}_1, \\ \Psi_t - \dfrac{p_+}{\tilde{v}}\Phi_x + \dfrac{R}{\tilde{v}}W_x = \dfrac{\mu}{\tilde{v}}\Psi_{xx} + (\dfrac{\mu}{v} - \dfrac{\mu}{\tilde{v}})u_x + J_1 + \dfrac{R}{\tilde{v}}Y - \tilde{R}_2 \doteq \dfrac{\mu}{\tilde{v}}\Psi_{xx} + Q_1, \\ \dfrac{R}{\gamma-1}W_t + p_+\Psi_x = \dfrac{\kappa}{\tilde{v}}W_{xx} + (\dfrac{\kappa}{v} - \dfrac{\kappa}{\tilde{v}})\theta_x + \dfrac{\mu u_x}{v}\Psi_x - \tilde{R}_3 - \tilde{u}_t\Psi + J_2 \\ \quad +\tilde{u}\tilde{R}_2 - \dfrac{\kappa}{\tilde{v}}Y_x \doteq \dfrac{\kappa}{\tilde{v}}W_{xx} + Q_2, \end{cases} \tag{3.4}$$

where

$$J_1 = \frac{\tilde{p} - p_+}{\tilde{v}}\Phi_x - [p - \tilde{p} + \frac{\tilde{p}}{\tilde{v}}\Phi_x - \frac{R}{\tilde{v}}(\theta - \tilde{\theta})] = O(1)(\Phi_x^2 + W_x^2 + Y^2 + |\tilde{u}|^4), \tag{3.5}$$

$$J_2 = (p_+ - p)\Psi_x = O(1)(\Phi_x^2 + \Psi_x^2 + W_x^2 + Y^2 + |\tilde{u}|^4), \tag{3.6}$$

$$Q_1 = (\frac{\mu}{v} - \frac{\mu}{\tilde{v}})u_x + J_1 + \frac{R}{\tilde{v}}Y - \tilde{R}_2, \tag{3.7}$$

$$Q_2 = (\frac{\kappa}{v} - \frac{\kappa}{\tilde{v}})\theta_x + \frac{\mu u_x}{v}\Psi_x - \tilde{R}_3 - \tilde{u}_t\Psi + \tilde{u}\tilde{R}_2 + J_2 - \frac{\kappa}{\tilde{v}}Y_x. \tag{3.8}$$

Since the local existence of (3.4) is well known, we omit it here for brevity. To prove Theorem 1, we now only need to close the following a priori estimate:

$$N(T) = \sup_{0\le t\le T}\{\|(\Phi,\Psi,W)\|_{L^\infty}^2 + \|(\phi,\psi,\zeta)\|_{H_x^1}^2\} \le \varepsilon_0^2, \tag{3.9}$$

where ε_0 is a positive small constant depending on the initial data and the strength of the contact wave. By (2.a19), it is obvious that $|\bar{\theta}_1| + |\bar{\theta}_3| \le C\varepsilon_0$ for some constant $C > 0$.

3.2 Lower order estimate

WE now derive the basic energy estimates on (Φ, Ψ, W). Multiplying $(3.4)_1$ by $p_+\Phi$, $(3.4)_2$ by $\tilde{v}\Psi$, $(3.4)_3$ by $\frac{R}{p_+}W$ respectively and adding all the resulting equations, we have

$$\begin{aligned}&(\frac{p_+}{2}\Phi^2 + \frac{R^2}{2(\gamma-1)p_+}W^2 + \frac{\tilde{v}}{2}\Psi^2)_t + \mu\Psi_x^2 + \frac{R\kappa}{p_+\tilde{v}}W_x^2\\ &= \frac{1}{2}\tilde{v}_t\Psi^2 + \tilde{v}Q_1\Psi - (\frac{R\kappa}{p_+\tilde{v}})_x WW_x + \frac{R}{p_+}WQ_2 - \tilde{R}_1 p_+\Phi + (\cdots)_x,\end{aligned} \tag{3.10}$$

here and in the sequel the notation $(\cdots)_x$ represents the term in the conservative form so that it vanishes after integration. Since it has no effect on the energy estimates, we do not write them out in details for clear presentation. Let $\bar{\delta} = \delta + |\bar{\theta}_1| + |\bar{\theta}_3|$ and

$$E_1 = \int(\frac{p_+}{2}\Phi^2 + \frac{R^2}{2(\gamma-1)p_+}W^2 + \frac{\tilde{v}}{2}\Psi^2)dx,\ K_1 = \int(\mu\Psi_x^2 + \frac{R\kappa}{p_+\tilde{v}}W_x^2)dx. \tag{3.11}$$

Then (2.a24), (2.a31) and the Cauchy inequality give

$$|\int \tilde{v}_t\Psi^2 dx| + |\int(\frac{R\kappa}{p_+\tilde{v}})_x WW_x dx| \le C\bar{\delta}(1+t)^{-1}E_1 + C\bar{\delta}\|W_x\|^2, \tag{3.12}$$

and

$$|\int \tilde{R}_1\Phi dx| \le C\bar{\delta}(1+t)^{-1}E_1 + C\bar{\delta}(1+t)^{-\frac{1}{2}}. \tag{3.13}$$

On the other hand,

$$\int |Q_1||\Psi|dx \le \int |(\frac{\mu}{v} - \frac{\mu}{\tilde{v}})u_x + J_1 + \frac{R}{\tilde{v}}Y||\Psi|dx + \int |\tilde{R}_2||\Psi|dx =: I_1 + I_2. \tag{3.14}$$

Since

$$\int |J_1||\Psi|dx \le C\varepsilon_0(\|\Phi_x\|^2 + K_1) + C\bar{\delta}(1+t)^{-1}E_1 + C\bar{\delta}(1+t)^{-\frac{5}{2}}, \tag{3.15}$$

and

$$\begin{aligned} &\int |(\frac{\mu}{v} - \frac{\mu}{\tilde{v}})u_x||\Psi|dx + \int |Y||\Psi|dx \le C\varepsilon_0 K_1 + C(\bar{\delta} + \varepsilon_0)\|\Phi_x\|^2 \\ &+C\varepsilon_0\|\psi_x\|^2 + C\bar{\delta}(1+t)^{-1}E_1, \end{aligned} \tag{3.16}$$

we obtain

$$I_1 \le C(\bar{\delta} + \varepsilon_0)(\|\Phi_x\|^2 + K_1) + C\bar{\delta}(1+t)^{-1}E_1 + C\varepsilon_0\|\psi_x\|^2 + C\bar{\delta}(1+t)^{-\frac{5}{2}}. \tag{3.17}$$

Note that

$$I_2 = \int |\tilde{R}_2\Psi|dx \le C\bar{\delta}(1+t)^{-1}E_1 + C\bar{\delta}(1+t)^{-\frac{1}{2}}. \tag{3.18}$$

Thus, combining (3.17) and (3.18) yields

$$\begin{aligned} \int |Q_1||\Psi|dx \le \ & C\bar{\delta}(1+t)^{-1}E_1 + C\bar{\delta}(1+t)^{-\frac{1}{2}} \\ & +C(\bar{\delta} + \varepsilon_0)(\|\Phi_x\|^2 + K_1) + C\varepsilon_0\|\psi_x\|^2. \end{aligned} \tag{3.19}$$

Similarly, we have

$$\begin{aligned} \int |Q_2||W|dx \le \ & C(\bar{\delta} + \varepsilon_0)(\|\Phi_x\|^2 + K_1 + \|(\phi,\psi,\zeta)_x\|^2) \\ & +C\bar{\delta}(1+t)^{-1}E_1 + C\bar{\delta}(1+t)^{-\frac{3}{2}}. \end{aligned} \tag{3.20}$$

From (3.12), (3.13), (3.19) and (3.20), we obtain our first estimate on lower order terms

$$E_{1t} + \frac{K_1}{2} \le C\bar{\delta}(1+t)^{-1}E_1 + C\bar{\delta}(1+t)^{-\frac{1}{2}} + C(\bar{\delta} + \varepsilon_0)(\|\Phi_x\|^2 + \|(\phi,\psi,\zeta)_x\|^2). \tag{3.21}$$

Notice that K_1 does not contain the term $\|\Phi_x\|^2$. To complete the lower order estimate, we need to estimate Φ_x. From $(3.4)_2$, we have

$$\frac{\mu}{\tilde{v}}\Phi_{xt} - \Psi_t + \frac{p_+}{\tilde{v}}\Phi_x = \frac{R}{\tilde{v}}W_x - Q_1 - \frac{\mu}{\tilde{v}}\tilde{R}_{1x}. \tag{3.22}$$

Multiplying (3.22) by Φ_x yields

$$(\frac{\mu}{2\tilde{v}}\Phi_x^2)_t - (\frac{\mu}{2\tilde{v}})_t\Phi_x^2 - \Phi_x\Psi_t + \frac{p_+}{\tilde{v}}\Phi_x^2 = (\frac{R}{\tilde{v}}W_x - Q_1 - \frac{\mu}{\tilde{v}}\tilde{R}_{1x})\Phi_x. \tag{3.23}$$

Since

$$\Phi_x\Psi_t = (\Phi_x\Psi)_t - (\Phi_t\Psi)_x + \Psi_x^2 - \tilde{R}_1\Psi_x, \tag{3.24}$$

we obtain

$$(\int \frac{\mu}{2\tilde{v}}\Phi_x^2 - \Phi_x\Psi dx)_t + \int \frac{p_+}{2\tilde{v}}\Phi_x^2 dx \le C\int(\Psi_x^2 + W_x^2)dx + C\int Q_1^2 dx + C\bar{\delta}(1+t)^{-\frac{3}{2}}. \tag{3.25}$$

On the other hand, it follows from (3.7) and the Cauchy inequality that

$$\int Q_1^2 dx \le C\varepsilon_0(K_1 + \|\Phi_x\|^2) + C\bar{\delta}(1+t)^{-\frac{3}{2}} + C\varepsilon_0\|\psi_x\|^2. \tag{3.26}$$

Plugging (3.26) into (3.25) yields

$$(\int \frac{\mu}{2\tilde{v}}\Phi_x^2 - \Phi_x\Psi dx)_t + \int \frac{p_+}{4\tilde{v}}\Phi_x^2 dx \le C_1K_1 + C_1\bar{\delta}(1+t)^{-3/2} + C_1\varepsilon_0\|\psi_x\|^2. \tag{3.27}$$

We now choose large constant $\tilde{C}_1 > 1$ so that

$$\tilde{C}_1E_1 + \int \frac{\mu}{2\tilde{v}}\Phi_x^2 - \Phi_x\Psi dx \ge \frac{1}{2}\tilde{C}_1E_1 + \int \frac{\mu}{4\tilde{v}}\Phi_x^2 dx, \quad \frac{\tilde{C}_1}{2} - C_1 > \frac{\tilde{C}_1}{4}. \tag{3.28}$$

Hence, by multiplying (3.21) by $\tilde{C}_1$ together with (3.27), we have obtained the basic energy estimate:

$$E_{2t} + K_2 \le C\bar{\delta}(1+t)^{-1}E_2 + C\bar{\delta}(1+t)^{-\frac{1}{2}} + C(\bar{\delta}+\varepsilon_0)\|(\phi,\psi,\zeta)_x\|^2, \tag{3.29}$$

where

$$E_2 = \tilde{C}_1E_1 + \int \frac{\mu}{2\tilde{v}}\Phi_x^2 - \Phi_x\Psi dx, \quad K_2 = \frac{\tilde{C}_1}{4}K_1 + \int \frac{p_+}{8\tilde{v}}\Phi_x^2 dx. \tag{3.30}$$

3.3 Derivative estimate

In this subsection, we shall estimate the derivatives of (Φ, Ψ, W). From (2.a1) and (2.a25), we have

$$\begin{cases} \phi_t - \psi_x = -\tilde{R}_{1x}, \\ \psi_t + (p-\tilde{p})_x = (\frac{\mu}{v}u_x - \frac{\mu}{\tilde{v}}\tilde{u}_x)_x - \tilde{R}_{2x}, \\ \frac{R}{\gamma-1}\zeta_t + pu_x - \tilde{p}\tilde{u}_x = (\frac{\kappa}{v}\theta_x - \frac{\kappa}{\tilde{v}}\tilde{\theta}_x)_x + Q_3, \end{cases} \tag{3.31}$$

where

$$Q_3 = \frac{\mu}{v}u_x^2 - (\frac{\mu\tilde{u}\tilde{u}_x}{\tilde{v}})_x - \tilde{R}_{3x} + \frac{1}{2}(\tilde{u}^2)_t + \tilde{p}_x\tilde{u}. \tag{3.32}$$

Multiplying $(3.31)_2$ by ψ yields

$$(\frac{1}{2}\psi^2)_t - (p-\tilde{p})\psi_x + (\frac{\mu}{v}u_x - \frac{\mu}{\tilde{v}}\tilde{u}_x)\psi_x = -\tilde{R}_{2x}\psi + (\cdots)_x. \tag{3.33}$$

Since $p - \tilde{p} = R\tilde{\theta}(\frac{1}{v} - \frac{1}{\tilde{v}}) + \frac{R\zeta}{v}$, we get

$$\begin{aligned} &(\frac{1}{2}\psi^2)_t - R\tilde{\theta}(\frac{1}{v} - \frac{1}{\tilde{v}})\phi_t - \frac{R}{v}\zeta\psi_x + \frac{\mu}{v}\psi_x^2 + (\frac{\mu}{v} - \frac{\mu}{\tilde{v}})\tilde{u}_x\psi_x \\ &= -\tilde{R}_{2x}\psi + R\tilde{\theta}(\frac{1}{v} - \frac{1}{\tilde{v}})\tilde{R}_{1x} + (\cdots)_x. \end{aligned} \tag{3.34}$$

Set

$$\hat{\Phi}(s) = s - 1 - \ln s. \tag{3.35}$$

It is easy to check that $\hat{\Phi}'(1) = 0$ and $\hat{\Phi}(s)$ is strictly convex around $s = 1$. Moreover,

$$\begin{aligned} &\{R\tilde{\theta}\hat{\Phi}(\frac{v}{\tilde{v}})\}_t = R\tilde{\theta}_t\hat{\Phi}(\frac{v}{\tilde{v}}) + R\tilde{\theta}(-\frac{1}{v} + \frac{1}{\tilde{v}})\phi_t \\ &+R\tilde{\theta}(-\frac{v}{\tilde{v}^2} + \frac{1}{\tilde{v}})\tilde{v}_t + R\tilde{\theta}(-\frac{1}{v} + \frac{1}{\tilde{v}})\tilde{v}_t \\ &= R\tilde{\theta}(-\frac{1}{v} + \frac{1}{\tilde{v}})\phi_t - \tilde{p}\hat{\Psi}(\frac{v}{\tilde{v}})\tilde{v}_t + \tilde{v}\tilde{p}_t\hat{\Phi}(\frac{v}{\tilde{v}}), \end{aligned} \tag{3.36}$$

where

$$\hat{\Psi}(s) = s^{-1} - 1 + \ln s. \tag{3.37}$$

Substituting (3.36) into (3.34) yields

$$\begin{aligned}&(\frac{1}{2}\psi^2 + R\tilde{\theta}\hat{\Phi}(\frac{v}{\tilde{v}}))_t + \tilde{p}\hat{\Psi}(\frac{v}{\tilde{v}})\tilde{v}_t - \frac{R}{v}\zeta\psi_x + \frac{\mu}{v}\psi_x^2\\&+(\frac{\mu}{v} - \frac{\mu}{\tilde{v}})\tilde{u}_x\psi_x = -\tilde{R}_{2x}\psi + R\tilde{\theta}(\frac{1}{v} - \frac{1}{\tilde{v}})\tilde{R}_{1x} + \tilde{v}\tilde{p}_t\hat{\Phi}(\frac{v}{\tilde{v}}) + (\cdots)_x.\end{aligned} \tag{3.38}$$

On the other hand, we calculate

$$[\tilde{\theta}\hat{\Phi}(\frac{\theta}{\tilde{\theta}})]_t = (1 - \frac{\tilde{\theta}}{\theta})\zeta_t - \hat{\Psi}(\frac{\theta}{\tilde{\theta}})\tilde{\theta}_t, \tag{3.39}$$

and

$$\begin{aligned}&\frac{R}{\gamma-1}(1 - \frac{\tilde{\theta}}{\theta})\zeta_t = (1 - \frac{\tilde{\theta}}{\theta})\{-pu_x + \tilde{p}\tilde{u}_x + (\frac{\kappa\theta_x}{v} - \frac{\kappa\tilde{\theta}_x}{\tilde{v}})_x + Q_3\}\\&= -\frac{R}{v}\zeta\psi_x + \frac{\zeta}{\theta}(\tilde{p} - p)\tilde{u}_x - (\frac{\zeta}{\theta})_x(\frac{\kappa\theta_x}{v} - \frac{\kappa\tilde{\theta}_x}{\tilde{v}}) + \frac{\zeta}{\theta}Q_3 + (\cdots)_x\\&= -\frac{R}{v}\zeta\psi_x + \frac{\zeta}{\theta}(\tilde{p} - p)\tilde{u}_x - \frac{\kappa}{v\theta}\zeta_x^2 + \frac{\kappa\zeta_x\Phi_x}{v\tilde{v}\theta}\tilde{\theta}_x + \frac{\zeta\theta_x}{\theta^2}(\frac{\kappa\theta_x}{v} - \frac{\kappa\tilde{\theta}_x}{\tilde{v}}) + \frac{\zeta}{\theta}Q_3 + (\cdots)_x.\end{aligned} \tag{3.40}$$

Substituting (3.39) and (3.40) into (3.38) gives

$$\begin{aligned}&(\frac{1}{2}\psi^2 + R\tilde{\theta}\hat{\Phi}(\frac{v}{\tilde{v}}) + \frac{R}{\gamma-1}\tilde{\theta}\hat{\Phi}(\frac{\theta}{\tilde{\theta}}))_t + \frac{\mu}{v}\psi_x^2 + \frac{\kappa}{v\theta}\zeta_x^2\\&= -\tilde{p}\hat{\Psi}(\frac{v}{\tilde{v}})\tilde{v}_t + \tilde{v}\tilde{p}_t\hat{\Phi}(\frac{v}{\tilde{v}}) - (\frac{\mu}{v} - \frac{\mu}{\tilde{v}})\tilde{u}_x\psi_x - \tilde{R}_{2x}\psi + R\tilde{\theta}(\frac{1}{v} - \frac{1}{\tilde{v}})\tilde{R}_{1x} + \frac{\zeta}{\theta}(\tilde{p} - p)\tilde{u}_x\\&+\frac{\kappa\zeta_x\Phi_x}{v\tilde{v}\theta}\tilde{\theta}_x + \frac{\zeta\theta_x}{\theta^2}(\frac{\kappa\theta_x}{v} - \frac{\kappa\tilde{\theta}_x}{\tilde{v}}) + \frac{\zeta}{\theta}Q_3 - \frac{R}{\gamma-1}\hat{\Psi}(\frac{\theta}{\tilde{\theta}})\tilde{\theta}_t + (\cdots)_x.\end{aligned} \tag{3.41}$$

Denote

$$E_3 = \int \frac{1}{2}\psi^2 + R\tilde{\theta}\hat{\Phi}(\frac{v}{\tilde{v}}) + \frac{R}{\gamma-1}\tilde{\theta}\hat{\Phi}(\frac{\theta}{\tilde{\theta}})dx,\ K_3 = \int \frac{\mu}{v}\psi_x^2 + \frac{\kappa}{v\theta}\zeta_x^2 dx. \tag{3.42}$$

Notice that $\hat{\Phi}(s)$ is strictly convex around $s = 1$. Thus there exist positive constants c_1 and c_2 such that,

$$c_1\phi^2 \le \hat{\Phi}(\frac{v}{\tilde{v}}) \le c_2\phi^2, \quad c_1\zeta^2 \le \hat{\Phi}(\frac{\theta}{\tilde{\theta}}) \le c_2\zeta^2. \tag{3.43}$$

Notice that $\hat{\Psi}(s)$ is also convex around $s = 1$. This leads to

$$\int |\hat{\Psi}(\frac{v}{\tilde{v}})\tilde{v}_t|dx + \int |\hat{\Psi}(\frac{\theta}{\tilde{\theta}})\tilde{\theta}_t|dx \le C\bar{\delta}(1+t)^{-1}K_2 + C\bar{\delta}(1+t)^{-\frac{5}{2}}, \tag{3.44}$$

where we have used $(\phi, \psi) = (\Phi_x, \Psi_x)$, and $\zeta = W_x - Y$. On the other hand, the Cauchy inequality yields,

$$\int |R_{1x}\psi|dx \le C\bar{\delta}(1+t)^{-\frac{3}{2}} + C\bar{\delta}(1+t)^{-1}K_2, \tag{3.45}$$

$$\int |\frac{\zeta}{\theta}(\tilde{p}-p)\tilde{u}_x + \frac{\kappa\zeta_x\Phi_x}{v\tilde{v}\theta}\tilde{\theta}_x|dx \le C\bar{\delta}(1+t)^{-1}K_2 + C\bar{\delta}\|\zeta_x\|^2 + C\bar{\delta}(1+t)^{-\frac{5}{2}}, \tag{3.46}$$

$$\int |\frac{\zeta\theta_x}{\theta^2}(\frac{\kappa\theta_x}{v} - \frac{\kappa\tilde{\theta}_x}{\tilde{v}})|dx \le C(\bar{\delta}+\varepsilon_0)\|\zeta_x\|^2 + C\bar{\delta}(1+t)^{-1}K_2 + C\bar{\delta}(1+t)^{-\frac{5}{2}}, \tag{3.47}$$

and

$$\int |\frac{\zeta}{\theta}Q_3|dx \le C\varepsilon_0\|\psi_x\|^2 + C\bar{\delta}(1+t)^{-1}K_2 + C\bar{\delta}(1+t)^{-\frac{5}{2}}. \tag{3.48}$$

Integrating (3.41) with respect to x, we have

$$E_{3t} + \frac{1}{2}K_3 \le C\bar{\delta}(1+t)^{-1}K_2 + C\bar{\delta}(1+t)^{-\frac{3}{2}}. \tag{3.49}$$

As before, we need to estimate $\|\phi_x\|^2$ separately. We follow the same argument in the previous subsection for $\|\Phi_x\|^2$ by rewriting the equation $(3.31)_2$ as

$$\frac{\mu}{\tilde{v}}\phi_{xt} - \psi_t - (p-\tilde{p})_x = -(\frac{\mu}{\tilde{v}})_x\psi_x - [(\frac{\mu}{v} - \frac{\mu}{\tilde{v}})u_x]_x + \tilde{R}_{2x} - \frac{\mu}{\tilde{v}}\tilde{R}_{1xx}. \tag{3.50}$$

Multiplying (3.50) by ϕ_x, we get

$$\begin{aligned} &(\frac{\mu}{2\tilde{v}}\phi_x^2)_t - (\frac{\mu}{2\tilde{v}})_t\phi_x^2 - \psi_t\phi_x - (p-\tilde{p})_x\phi_x \\ &= \{-(\frac{\mu}{\tilde{v}})_x\psi_x + (\frac{\mu\Phi_x}{v\tilde{v}}u_x)_x + \tilde{R}_{2x} - \frac{\mu}{\tilde{v}}\tilde{R}_{1xx}\}\phi_x. \end{aligned} \tag{3.51}$$

Note that

$$-(p-\tilde{p})_x = \frac{\tilde{p}}{\tilde{v}}\phi_x - \frac{R}{\tilde{v}}\zeta_x + (\frac{p}{v} - \frac{\tilde{p}}{\tilde{v}})v_x - (\frac{R}{v} - \frac{R}{\tilde{v}})\theta_x, \tag{3.52}$$

and

$$\phi_x\psi_t = (\phi_x\psi)_t - (\phi_t\psi)_x + \psi_x^2 - \tilde{R}_{1x}\psi_x. \tag{3.53}$$

Integrating (3.51) with respect to x, we have

$$\begin{aligned} &(\int \frac{\mu}{2\tilde{v}}\phi_x^2 - \phi_x\psi dx)_t + \int \frac{\tilde{p}}{2\tilde{v}}\phi_x^2 dx \\ &\le C_2K_3 + C_2\bar{\delta}(1+t)^{-1}K_2 + C_2\bar{\delta}(1+t)^{-\frac{5}{2}} + C_2\varepsilon_0\int \psi_{xx}^2 dx. \end{aligned} \tag{3.54}$$

Here we have used

$$\int |(\frac{p}{v} - \frac{\tilde{p}}{\tilde{v}})v_x\phi_x|dx \le C(\bar{\delta}+\varepsilon_0)\|\phi_x\|^2 + C\bar{\delta}(1+t)^{-1}K_2 + C\bar{\delta}(1+t)^{-\frac{5}{2}}, \tag{3.55}$$

$$\begin{aligned} &\int |(\frac{\mu\Phi_x}{v\tilde{v}}u_x)_x\phi_x|dx \le C(\bar{\delta}+\varepsilon_0)\|\phi_x\|^2 + C\bar{\delta}(1+t)^{-1}K_2 \\ &+C\varepsilon_0\|\psi_{xx}\|^2 + C\bar{\delta}\|\psi_x\|^2 + C\int \phi_x^2|\psi_x|dx, \end{aligned} \tag{3.56}$$

and

$$\begin{aligned} &\int \phi_x^2||\psi_x|dx \le C\|\phi_x\|^2\|\psi_x\|^{\frac{1}{2}}\|\psi_{xx}\|^{\frac{1}{2}} \le C\|\phi_x\|^{\frac{1}{2}}\|\psi_x\|^{\frac{1}{2}}(\|\phi_x\|^2 + \|\psi_{xx}\|^2) \\ &\le C\varepsilon_0(\|\phi_x\|^2 + \|\psi_{xx}\|^2). \end{aligned} \tag{3.57}$$

Finally, to estimate the higher order derivatives of (ψ,ζ), we multiply $(3.31)_2$ by $-\psi_{xx}$ and $(3.31)_3$ by $-\zeta_{xx}$ to get

$$\begin{aligned}&(\frac{1}{2}\psi_x^2+\frac{R}{2(\gamma-1)}\zeta_x^2)_t+\frac{\mu}{v}\psi_{xx}^2+\frac{\kappa}{v}\zeta_{xx}^2=(p-\tilde{p})_x\psi_{xx}+\frac{\mu v_x}{v^2}\psi_x\psi_{xx}+(\frac{\mu\Phi_x}{v\tilde{v}}\tilde{u}_x)_x\psi_{xx}\\&\quad+\tilde{R}_{2x}\psi_{xx}+(pu_x-\tilde{p}\tilde{u}_x)\zeta_{xx}+\frac{\kappa v_x}{v^2}\zeta_x\zeta_{xx}+(\frac{\kappa\Phi_x}{v\tilde{v}}\tilde{\theta}_x)_x\zeta_{xx}-Q_3\zeta_{xx}.\end{aligned}\tag{3.58}$$

The Cauchy inequality gives

$$\int|(p-\tilde{p})_x\psi_{xx}|dx\le C(K_3+\|\phi_x\|^2)+\int\frac{\mu}{8v}\psi_{xx}^2dx+C\bar{\delta}(1+t)^{-1}K_2+C\bar{\delta}(1+t)^{-\frac{5}{2}},\tag{3.59}$$

$$\int|\frac{\mu v_x}{v^2}\psi_x\psi_{xx}|dx\le C\bar{\delta}(\|\psi_x\|^2+\|\psi_{xx}\|^2)+C\int|\phi_x||\psi_x||\psi_{xx}|dx,\tag{3.60}$$

$$\int|(pu_x-\tilde{p}\tilde{u}_x)\zeta_{xx}|dx\le\int\frac{\mu}{8v}\zeta_{xx}^2dx+CK_3+C\bar{\delta}(1+t)^{-1}K_2+C\bar{\delta}(1+t)^{-\frac{5}{2}},\tag{3.61}$$

and

$$\int|Q_3\zeta_{xx}|dx\le C\int\psi_x^2|\zeta_{xx}|dx+C\bar{\delta}\|\zeta_{xx}\|^2+C\bar{\delta}(1+t)^{-\frac{7}{2}}.\tag{3.62}$$

On the other hand,

$$\int|\phi_x||\psi_x||\psi_{xx}|dx\le C\|\psi_x\|^{\frac{1}{2}}\|\phi_x\|\|\psi_{xx}\|^{\frac{3}{2}}\le C\varepsilon_0(\|\psi_{xx}\|^2+\|\psi_x\|^2).\tag{3.63}$$

The term $\int\psi_x^2|\zeta_{xx}|dx$ can be estimated similarly. Thus, integrating (3.58) and using (3.59)-(3.63), we have

$$\begin{aligned}&\{\int(\frac{1}{2}\psi_x^2+\frac{R}{2(\gamma-1)}\zeta_x^2)dx\}_t+\int\frac{\mu}{4v}\psi_{xx}^2dx+\int\frac{\kappa}{4v}\zeta_{xx}^2dx\\&\le C_3(K_3+\|\phi_x\|^2)+C_3\bar{\delta}(1+t)^{-1}K_2+C_3\bar{\delta}(1+t)^{-\frac{5}{2}}.\end{aligned}\tag{3.64}$$

We now choose constants $\tilde{C}_2>1$, $\tilde{C}_3>1$ large enough so that

$$\tilde{C}_2E_3+\tilde{C}_3\int(\frac{\mu}{2v}\phi_x^2-\phi_x\psi)dx>\frac{1}{2}\tilde{C}_2E_3+\frac{\tilde{C}_3}{4}\int\frac{\mu}{v}\phi_x^2dx,\tag{3.65}$$

and

$$\frac{1}{2}\tilde{C}_2-\tilde{C}_3C_2-C_3>\frac{1}{4}\tilde{C}_2,\quad\tilde{C}_3\int\frac{\tilde{p}}{2\tilde{v}}\phi_x^2dx-C_3\|\phi_x\|^2>\tilde{C}_3\int\frac{\tilde{p}}{4\tilde{v}}\phi_x^2dx.\tag{3.66}$$

Let

$$E_4=\tilde{C}_2E_3+\tilde{C}_3\int(\frac{\mu}{2v}\phi_x^2-\phi_x\psi)dx+\int(\frac{1}{2}\psi_x^2+\frac{R}{2(\gamma-1)}\zeta_x^2)dx,\tag{3.67}$$

and

$$K_4=\frac{1}{4}\tilde{C}_2K_3+\tilde{C}_3\int\frac{\tilde{p}}{4\tilde{v}}\phi_x^2dx+\int\frac{\mu}{4v}\psi_{xx}^2dx+\int\frac{\kappa}{4v}\zeta_{xx}^2dx.\tag{3.68}$$

Then combining (3.49), (3.54) and (3.64) gives

$$E_{4t} + K_4 \le C\bar{\delta}(1+t)^{-1}K_2 + C\bar{\delta}(1+t)^{-\frac{3}{2}}. \tag{3.69}$$

It should be noted that under the a priori assumption (3.9), the estimate (3.69) on the derivatives of $(\Phi, \Psi, \bar{W})$ does not involve the lower order estimate (3.29) except the term $C\bar{\delta}(1+t)^{-1}K_2$ which has a time-decay factor. This is essential in the proof for the decay rate in the next subsection.

3.4 Decay rate

It follows from (3.29) and (3.69) that

$$E_{5t} + K_5 \le C_0\bar{\delta}(1+t)^{-1}E_5 + C_0\bar{\delta}(1+t)^{-\frac{1}{2}}, \tag{3.70}$$

where

$$E_5 = E_2 + E_4, K_5 = K_2 + K_4. \tag{3.71}$$

Multiplying (3.70) by $(1+t)^{-C_0\bar{\delta}}$ and using the Granwall's inequality yield

$$E_5 \le C(E_5(0) + \bar{\delta})(1+t)^{\frac{1}{2}}, \qquad \int_0^t K_5 dt \le C(E_5(0) + \bar{\delta})(1+t)^{\frac{1}{2}}, \tag{3.72}$$

if $C_0\bar{\delta} < \frac{1}{2}$. Since $E_5 \ge c_3\|(\Phi, \Psi, W)\|^2$ for some positive constant c_3, we have

$$\|(\Phi, \Psi, W)\|^2 \le C(E_5(0) + \bar{\delta})(1+t)^{\frac{1}{2}}. \tag{3.73}$$

Notice that the upper bound for the L^2 norm of (Φ, Ψ, W) grows with the rate $(1+t)^{\frac{1}{2}}$. However, as we will show later that the L^2 norm of (Φ_x, Ψ_x, W_x) decays with the rate $(1+t)^{-\frac{1}{2}}$. Hence the Sobolev inequality implies that the L^∞ norm of (Φ, Ψ, W) is uniformly bounded if $\bar{\delta}$ and the initial data are small. In fact, multiply (3.69) by $(1+t)$, we have

$$[(1+t)E_4]_t \le C\bar{\delta}K_2 + E_4 + C\bar{\delta}(1+t)^{-\frac{1}{2}} \le K_5 + C\bar{\delta}(1+t)^{-\frac{1}{2}}. \tag{3.74}$$

Integrating (3.74) with respect to t and using (3.72) imply

$$E_4 \le C(E_5(0) + \bar{\delta})(1+t)^{-\frac{1}{2}}, \tag{3.75}$$

where we have used the fact that

$$\begin{aligned} E_4 &\le C\|(\phi, \psi, \zeta)\|^2_{H^1} \le C(\|(\Phi, \Psi, W)_x\|^2 + \|(\phi, \psi, \zeta)_x\|^2) + C\bar{\delta}(1+t)^{-\frac{3}{2}} \\ &\le CK_5 + C\bar{\delta}(1+t)^{-\frac{3}{2}}. \end{aligned}$$

Furthermore, since

$$E_4 \ge c_4\|(\phi, \psi, \zeta)\|^2_{H^1} \ge c_4(\|(\Phi, \Psi, W)_x\|^2 + \|(\phi, \psi, \zeta)_x\|^2) - c_4\bar{\delta}(1+t)^{-\frac{3}{2}},$$

for some positive constant c_4, due to (3.73) and (3.75), we have,

$$\|(\Phi, \Psi, W)\|_{L^\infty} \;\le\; C\|(\Phi, \Psi, W)\|^{\frac{1}{2}}\|(\Phi_x, \Psi_x, W_x)\|^{\frac{1}{2}} \le C(E_5(0) + \bar{\delta})^{\frac{1}{2}}. \tag{3.76}$$

Since $W=\frac{\gamma-1}{R}(\tilde{W}-\tilde{u}\Psi)$, we also have

$$\|(\Phi,\Psi,\tilde{W})\|_{L^\infty}\leq C(E_5(0)+\bar{\delta})^{\frac{1}{2}}. \tag{3.77}$$

The decay rate for $\|(\phi,\psi,\zeta)\|_{L^\infty}$ follows directly from (3.75) as follows:

$$\|(\phi,\psi,\zeta)\|_{L^\infty}\leq CE_4^{\frac{1}{2}}\leq C(\epsilon^2+\bar{\delta})^{\frac{1}{2}}(1+t)^{-\frac{1}{4}}. \tag{3.78}$$

Therefore the a priori assumption (3.9) is verified and Theorem 1 is proved.

4 Boltzmann equation

In this section, we investigate the stability of the contact wave pattern constructed in (2.b14) for the Boltzmann equation (2.b8). The arrangement of this section is as follows: in subsection 4.1, the fluid type system (2.b10) is reformulated in terms of the integrated variable (Φ,Ψ,W); the subsection 4.2 is devoted to the lower order estimate, while the subsection 4.3 is for the derivative estimate; the stability and decay rate of the contact wave for the Boltzmann equation (2.b8) is given in subsection 4.4.

4.1 Reformulated system

This subsection is devoted to the Boltzmann equation (2.b8). First, we denote the perturbation by

$$\phi=v-\tilde{v},\psi=u-\tilde{u},\zeta=\theta-\tilde{\theta}, \tag{4.1}$$

and define

$$\begin{aligned}\Phi&=\int_{-\infty}^x\phi(y,t)dy,\Psi=\int_{-\infty}^x\psi(y,t)dy,\\ \bar{W}&=\int_{-\infty}^x(e+\frac{|u|^2}{2}-\tilde{e}-\frac{|\tilde{u}|^2}{2})(y,t)dy,\end{aligned} \tag{4.2}$$

which satisfy $(\phi,\psi)=(\Phi,\Psi)_x$ and $\zeta+\frac{1}{2}|\Psi_x|^2+\sum_{i=1}^3\tilde{u}_i\Psi_{ix}=\bar{W}_x$. Then subtracting (2.b25) from the equation (2.b10) and integrating the resulting system, we have

$$\left\{\begin{aligned}&\Phi_t-\Psi_{1x}=-\tilde{R}_1,\\ &\Psi_{1t}+p-\tilde{p}=\frac{4}{3}\frac{\mu(\theta)}{v}u_{1x}-\frac{4}{3}\frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_{1x}-\int\xi_1^2\Theta_1d\xi-\tilde{R}_2,\\ &\Psi_{it}=\frac{\mu(\theta)}{v}u_{ix}-\frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_{ix}-\int\xi_1\xi_i\Theta_1d\xi-\tilde{R}_{i+1},\ i=2,3,\\ &\bar{W}_t+pu_1-\tilde{p}\tilde{u}_1=\frac{\lambda(\theta)}{v}\theta_x-\frac{\lambda(\tilde{\theta})}{\tilde{v}}\tilde{\theta}_x+\frac{4}{3}\frac{\mu(\theta)}{v}u_1u_{1x}-\frac{4}{3}\frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_1\tilde{u}_{1x}\\ &\quad+\sum_{i=2}^3\frac{\mu(\theta)}{v}u_iu_{ix}-\sum_{i=2}^3\frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_i\tilde{u}_{ix}-\int\frac{1}{2}\xi_1|\xi|^2\Theta_1d\xi-\tilde{R}_5.\end{aligned}\right. \tag{4.3}$$

Let

$$W=\bar{W}-\tilde{u}_1\Psi_1. \tag{4.4}$$

It follows that

$$\zeta = W_x - Y, \text{ with } Y = \frac{1}{2}|\Psi_x|^2 - \tilde{u}_{1x}\Psi_1 + \tilde{u}_2\Psi_{2x} + \tilde{u}_3\Psi_{3x}. \tag{4.5}$$

Using the new variable W and linearizing the system (4.3), we have

$$\begin{cases} \Phi_t - \Psi_{1x} = -\tilde{R}_1, \\ \Psi_{1t} - \frac{p_+}{\tilde{v}}\Phi_x + \frac{2}{3\tilde{v}}W_x = \frac{4}{3}\frac{\mu(\tilde{\theta})}{\tilde{v}}\Psi_{1xx} - \int \xi_1^2\Theta_1 d\xi + \frac{4}{3}(\frac{\mu(\theta)}{v} - \frac{\mu(\tilde{\theta})}{\tilde{v}})u_{1x} \\ \quad + J_1 + \frac{2}{3\tilde{v}}Y - \tilde{R}_2 \doteq \frac{4}{3}\frac{\mu(\tilde{\theta})}{\tilde{v}}\Psi_{1xx} - \int \xi_1^2\Theta_1 d\xi + Q_1, \\ \Psi_{it} = \frac{\mu(\tilde{\theta})}{\tilde{v}}\Psi_{ixx} - \int \xi_1\xi_i\Theta_1 d\xi + (\frac{\mu(\theta)}{v} - \frac{\mu(\tilde{\theta})}{\tilde{v}})u_{ix} - \tilde{R}_{i+1} \\ \quad \doteq \frac{\mu(\tilde{\theta})}{\tilde{v}}\Psi_{ixx} - \int \xi_1\xi_i\Theta_1 d\xi + Q_i, \ i = 2, 3, \\ W_t + p_+\Psi_{1x} = \frac{\lambda(\tilde{\theta})}{\tilde{v}}W_{xx} - \int \frac{1}{2}\xi_1|\xi|^2\Theta_1 d\xi + \tilde{u}_1\int \xi_1^2\Theta_1 d\xi + (\frac{\lambda(\theta)}{v} - \frac{\lambda(\tilde{\theta})}{\tilde{v}})\theta_x \\ \quad + \frac{4u_{1x}}{3}\frac{\mu(\theta)}{v}\Psi_{1x} + \sum_{i=2}^{3}[\frac{\mu(\theta)}{v}u_iu_{ix} - \frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_i\tilde{u}_{ix}] - \tilde{R}_5 - \tilde{u}_{1t}\Psi_1 + J_2 \\ \quad + \tilde{u}_1\tilde{R}_2 - \frac{\lambda(\tilde{\theta})}{\tilde{v}}Y_x \doteq \frac{\lambda(\tilde{\theta})}{\tilde{v}}W_{xx} - \int \frac{1}{2}\xi_1|\xi|^2\Theta_1 d\xi + \tilde{u}_1\int \xi_1^2\Theta_1 d\xi + Q_4, \end{cases} \tag{4.6}$$

where

$$J_1 = \frac{\tilde{p} - p_+}{\tilde{v}}\Phi_x - [p - \tilde{p} + \frac{\tilde{p}}{\tilde{v}}\Phi_x - \frac{2}{3\tilde{v}}(\theta - \tilde{\theta})] = O(1)(\Phi_x^2 + (\theta - \tilde{\theta})^2 + |\tilde{u}|^4), \tag{4.7}$$

$$J_2 = (p_+ - p)\Psi_{1x} = O(1)(\Phi_x^2 + \Psi_{1x}^2 + (\theta - \tilde{\theta})^2 + |\tilde{u}|^4), \tag{4.8}$$

$$Q_1 = \frac{4}{3}(\frac{\mu(\theta)}{v} - \frac{\mu(\tilde{\theta})}{\tilde{v}})u_{1x} + J_1 + \frac{2}{3\tilde{v}}Y - \tilde{R}_2, \tag{4.9}$$

$$Q_i = (\frac{\mu(\theta)}{v} - \frac{\mu(\tilde{\theta})}{\tilde{v}})u_{ix} - \tilde{R}_{i+1}, \ i = 2, 3, \tag{4.10}$$

$$\begin{aligned} Q_4 = (\frac{\lambda(\theta)}{v} - \frac{\lambda(\tilde{\theta})}{\tilde{v}})\theta_x + \frac{4u_{1x}}{3}\frac{\mu(\theta)}{v}\Psi_{1x} - \tilde{R}_5 - \tilde{u}_{1t}\Psi_1 + \tilde{u}_1\tilde{R}_2 \\ + \sum_{i=2}^{3}[\frac{\mu(\theta)}{v}u_iu_{ix} - \frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_i\tilde{u}_{ix}] + J_2 - \frac{\lambda(\tilde{\theta})}{\tilde{v}}Y_x. \end{aligned} \tag{4.11}$$

Since in the right hand side of (4.6), there is some non-fluid component denoted by Θ_1, the system (4.6) is closed together with the equation (2.b6) for the microscopic component. We rewrite (2.b6) as follows.

$$\begin{aligned} \tilde{G}_t - L_M\tilde{G} = & -\frac{1}{Rv\theta}P_1[\xi_1(\frac{|\xi - u|^2}{2\theta}(\theta - \bar{\theta})_x + \xi \cdot (u - \bar{u})_x)M] \\ & + \frac{u_1}{v}G_x - \frac{1}{v}P_1(\xi_1 G_x) + Q(G, G) - \bar{G}_t, \end{aligned} \tag{4.12}$$

where

$$\bar{G} = \frac{1}{Rv\theta} L_M^{-1}\{P_1[\xi_1(\frac{|\xi-u|^2}{2\theta}\tilde{\theta}_x + \xi\cdot\tilde{u}_x)M]\}, \quad \tilde{G} = G - \bar{G}. \tag{4.13}$$

Notice that in (4.12) and (4.13), we have subtracted $\bar{G}$ from G because $\|\tilde{\theta}_x\|^2_{L^2_x}$ is not integrable with respect to time t.

We shall work on the reformulated system (4.6). Since the local existence is now standard as in the discussions in [23], [56], to prove the global existence, we only need to close the following a priori estimate:

$$\begin{aligned} N(T) = & \sup_{0\le t\le T}\{\|(\Phi,\Psi,W)\|^2_{L^\infty} + \|(\phi,\psi,\zeta)\|^2_{H^1} \\ & + \int_R\int_{R^3}(\frac{\tilde{G}^2}{M_*} + \sum_{|\alpha|=1}\frac{(\partial^\alpha G)^2}{M_*} + \sum_{|\alpha|=2}\frac{(\partial^\alpha f)^2}{M_*}d\xi dx)\} \le \varepsilon_0^2, \end{aligned} \tag{4.14}$$

where ε_0 is positive small constant depending on the initial data and M_* is a global Maxwellian chosen later for any $T>0$. Here, it is worthy to pointing out that (4.14) also gives the a priori assumptions on $\|(\phi_t,\psi_t,\zeta_t)\|$, $\|\partial^\alpha(\phi,\psi,\zeta)\|$ and $\int\int\frac{|\partial^\alpha G|^2}{M_*}d\xi dx$ ($|\alpha|=2$). In fact, from (2.b9) and (4.14), we have

$$\|(\phi_t,\psi_t,\zeta_t)\|^2 \le C(\|(\phi_x,\psi_x,\zeta_x)\|^2 + \int\int\frac{G_x^2}{M_*}d\xi dx + \bar{\delta}^2(1+t)^{-\frac{1}{2}}) \le C(\varepsilon_0+\bar{\delta})^2, \tag{4.15}$$

where we have used

$$(\int\xi_1^2 G_x d\xi)^2 \le C\int\frac{G_x^2}{M_*}d\xi, \tag{4.16}$$

and

$$\begin{aligned} &\|(\phi_t,\psi_t,\zeta_t)\|^2 \le C\|(v_t,u_t,\theta_t)\|^2 + C\bar{\delta}^2(1+t)^{-\frac{3}{2}}, \\ &\|(v_x,u_x,\theta_x)\|^2 \le C\|(\phi_x,\psi_x,\zeta_x)\|^2 + C\bar{\delta}^2(1+t)^{-\frac{1}{2}}. \end{aligned} \tag{4.17}$$

To derive the a priori assumption on $\|\partial^\alpha(\phi,\psi,\zeta)\|$, ($|\alpha|=2$), we use the definition of ρ, $m=\rho u$ and $\rho(\theta+\frac{1}{2}|u|^2)$. Let $|\alpha|=2$. By (2.b2), we obtain

$$\|\partial^\alpha(\rho,m,\rho(\theta+\frac{1}{2}|u|^2))\|^2 \le C\int\int\frac{|\partial^\alpha f|^2}{M_*}d\xi dx \le C\varepsilon_0^2. \tag{4.18}$$

This yields that

$$\|\partial^\alpha(\phi,\psi,\zeta)\|^2 \le C\varepsilon_0 + C\bar{\delta}^2(1+t)^{-\frac{3}{2}} \le C(\varepsilon_0+\bar{\delta})^2, \quad |\alpha|=2. \tag{4.19}$$

Finally, we have, for $|\alpha|=2$,

$$\int\int\frac{|\partial^\alpha G|^2}{M_*}d\xi dx \le 2(\int\int\frac{|\partial^\alpha f|^2}{M_*}d\xi dx + \int\int\frac{|\partial^\alpha M|^2}{M_*}d\xi dx) \le C(\varepsilon_0+\bar{\delta})^2. \tag{4.20}$$

4.2 Lower order estimates

Before proving the a priori estimate (4.14), we list some basic lemmas based on the celebrated H-theorem for later use. The first lemma is from [20].

Lemma 4.1. There exists a positive constant C such that

$$\int_{R^3}\frac{\nu(|\xi|)^{-1}Q(f,g)^2}{M}d\xi\le C\{\int_{R^3}\frac{\nu(|\xi|)f^2}{M}d\xi\cdot\int_{R^3}\frac{g^2}{M}+\int_{R^3}\frac{f^2}{M}d\xi\cdot\int_{R^3}\frac{\nu(|\xi|)g^2}{M}\},$$

where M can be any Maxwellian so that the above integrals are well defined.

Based on Lemma 4.1, the following three lemmas are proved in [42]. The proofs are straightforward by using the Cauchy inequality.

Lemma 4.2. If $\theta/2<\theta_*<\theta$, then there exist two positive constants $\tilde\sigma=\tilde\sigma(\rho,u,\theta;\rho_*,u_*,\theta_*)$ and $\eta_0=\eta_0(\rho,u,\theta;\rho_*,u_*,\theta_*)$ such that if $|\rho-\rho_*|+|u-u_*|+|\theta-\theta_*|<\eta_0$, we have for $h(\xi)\in N^\perp$,

$$-\int_{R^3}\frac{hL_Mh}{M_*}d\xi\ge\tilde\sigma\int_{R^3}\frac{\nu(|\xi|)h^2}{M_*}d\xi, \tag{4.21}$$

where $M_*=M_{[\rho_*,u_*,\theta_*]}$ and the definition of $M_{[\rho,u,\theta]}$ can be found in (2.b3).

Lemma 4.3. Under the assumptions in Lemma 4.2, we have

$$\begin{cases}\displaystyle\int_{R^3}\frac{\nu(|\xi|)}{M}|L_M^{-1}h|^2d\xi\le\tilde\sigma^{-2}\int_{R^3}\frac{\nu(|\xi|)^{-1}h^2}{M}d\xi,\\ \displaystyle\int_{R^3}\frac{\nu(|\xi|)}{M_*}|L_M^{-1}h|^2d\xi\le\tilde\sigma^{-2}\int_{R^3}\frac{\nu(|\xi|)^{-1}h^2}{M_*}d\xi\end{cases} \tag{4.22}$$

for each $h(\xi)\in N^\perp$.

Lemma 4.4. Under the conditions in Lemma 4.2, there exists a constant $C>0$ such that for positive constants k and λ, it holds tha

$$|\int_{R^3}\frac{g_1P_1(|\xi|^kg_2)}{M_*}d\xi-\int_{R^3}\frac{g_1|\xi|^kg_2}{M_*}d\xi|\le C\int_{R^3}\frac{\lambda|g_1|^2+\lambda^{-1}|g_2|^2}{M_*}d\xi.$$

We now derive the lower order estimates. Multiplying $(4.6)_1$ by $p_+\Phi$, $(4.6)_2$ by $\tilde v\Psi_1$, $(4.6)_3$ by Ψ_i, $(4.6)_4$ by $\frac{2}{3p_+}W$ respectively and adding all the resulting equations, we have

$$\begin{aligned}&(\frac{p_+}{2}\Phi^2+\frac{W^2}{3p_+}+\frac{\tilde v}{2}\Psi_1^2+\frac12\sum_{i=2}^3\Psi_i^2)_t+\frac{4\mu(\tilde\theta)}{3}\Psi_{1x}^2+\sum_{i=2}^3\frac{\mu(\tilde\theta)}{\tilde v}\Psi_{ix}^2+\frac{2\lambda(\tilde\theta)}{3p_+\tilde v}W_x^2\\ =\;&-p_+\tilde R_1\Phi+\frac12\tilde v_t\Psi_1^2+\tilde vQ_1\Psi_1-(\frac{4\mu(\tilde\theta)}{3})_x\Psi_1\Psi_{1x}-\sum_{i=2}^3(\frac{\mu(\tilde\theta)}{\tilde v})_x\Psi_i\Psi_{ix}\\ &-(\frac{2\lambda(\tilde\theta)}{3p_+\tilde v})_xWW_x+\sum_{i=2}^3Q_i\Psi_i+\frac{2W}{3p_+}Q_4+NF_1+(\cdots)_x,\end{aligned} \tag{4.23}$$

where

$$NF_1 = -\tilde{v}\Psi_1 \int \xi_1^2\Theta_1 d\xi - \sum_{i=2}^{3} \Psi_i \int \xi_1\xi_i\Theta_1 d\xi + \frac{2W}{3p_+}(\tilde{R}_1 \int \xi_1^2\Theta_1 d\xi - \int \frac{1}{2}\xi_1|\xi|^2\Theta_1 d\xi). \tag{4.24}$$

Notice that (4.23) is almost the same as (3.10) for the Navier-Stokes equation except $\sum_{i=2}^{3}(\frac{\mu(\tilde{\theta})}{\tilde{v}})_x\Psi_i\Psi_{ix}$, $\sum_{i=2}^{3} Q_i\Psi_i$ and NF_1. Therefore we can follow the same argument in section 3 for compressible Navier-Stokes equations to estimate all terms in (4.23) except NF_1. That is, analogous to (3.21), we have

$$\begin{aligned} E_{1t} + \frac{1}{2}K_1 &\le C\bar{\delta}(1+t)^{-1}E_1 + C\bar{\delta}(1+t)^{-\frac{1}{2}} \\ &+C(\bar{\delta}+\varepsilon_0)(\|\Phi_x^2\| + \|(\phi,\psi,\zeta)_x\|^2) + \int NF_1 dx, \end{aligned} \tag{4.25}$$

where

$$E_1 = \int \frac{p_+}{2}\Phi^2 + \frac{W^2}{3p_+} + \frac{\tilde{v}}{2}\Psi_1^2 + \frac{1}{2}\sum_{i=2}^{3}\Psi_i^2 dx,\ K_1 = \int \sum_{i=1}^{3}\frac{\mu(\tilde{\theta})}{\tilde{v}}\Psi_{ix}^2 + \frac{2\lambda(\tilde{\theta})}{3p_+\tilde{v}}W_x^2 dx. \tag{4.26}$$

By (4.24), to estimate $\int NF_1 dx$, we only need to estimate $I_1 =: -\int \tilde{v}\Psi_1 \int \xi_1^2\Theta_1 d\xi dx$ since other terms in $\int NF_1 dx$ can be estimated similarly. Let M_* be a global Maxwellian with the state (ρ_*, u_*, θ_*) satisfying $\frac{1}{2}\theta < \theta_* < \theta$ and $|\rho-\rho_*| + |u-u_*| + |\theta-\theta_*| \le \eta_0$ so that Lemma 4.2 holds. By the definition of Θ_1, (see (2.b12)), we have

$$\begin{aligned} I_1 &= -\int \tilde{v}\Psi_1 \int \xi_1^2 L_M^{-1}(G_t) d\xi dx + \int \tilde{v}\Psi_1 \int \xi_1^2 \frac{u_1}{v} L_M^{-1}(G_x) d\xi dx \\ &- \int \tilde{v}\Psi_1 \frac{1}{v} \int \xi_1^2 L_M^{-1}[P_1(\xi_1 G_x)] d\xi dx + \int \tilde{v}\Psi_1 \int \xi_1^2 L_M^{-1}[Q(G,G)] d\xi dx =: \sum_{i=1}^{4} I_1^i. \end{aligned} \tag{4.27}$$

(4.27) can be estimated term by term. For the integral I_1^1, we have

$$I_1^1 = -\int \tilde{v}\Psi_1 \int \xi_1^2 L_M^{-1}(\tilde{G}_t) d\xi dx - \int \tilde{v}\Psi_1 \int \xi_1^2 L_M^{-1}(\bar{G}_t) d\xi dx =: I_1^{11} + I_1^{12}. \tag{4.28}$$

Note that the linearized operator L_M^{-1} satisfies, for any $h \in N^\perp$,

$$\begin{aligned} (L_M^{-1}h)_t &= L_M^{-1}(h_t) - 2L_M^{-1}\{Q(L_M^{-1}h, M_t)\}, \\ (L_M^{-1}h)_x &= L_M^{-1}(h_x) - 2L_M^{-1}\{Q(L_M^{-1}h, M_x)\}. \end{aligned} \tag{4.29}$$

Then we have

$$\begin{aligned} I_1^{11} &= -\int \tilde{v}\Psi_1 \int \xi_1^2 (L_M^{-1}\tilde{G})_t d\xi dx - 2\int \tilde{v}\Psi_1 \int \xi_1^2 L_M^{-1}\{Q(L_M^{-1}\tilde{G}, M_t)\} d\xi dx \\ &= -(\int \tilde{v}\Psi_1 \int \xi_1^2 L_M^{-1}\tilde{G} d\xi dx)_t + \int (\tilde{v}\Psi_1)_t \int \xi_1^2 L_M^{-1}\tilde{G} d\xi dx \\ &\quad -2\int \tilde{v}\Psi_1 \int \xi_1^2 L_M^{-1}\{Q(L_M^{-1}\tilde{G}, M_t)\} d\xi dx. \end{aligned} \tag{4.30}$$

The Hölder inequality and Lemma 4.3 yield

$$|\int \xi_1^2 L_M^{-1}\tilde{G}d\xi|^2 \le C\int \xi_1^4\nu(|\xi|)^{-1}M_*d\xi\cdot\int\frac{\nu(|\xi|)}{M_*}|L_M^{-1}\tilde{G}|^2d\xi \le C\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2d\xi, \quad (4.31)$$

Moreover, from Lemmas 4.1-4.3, we have

$$\begin{aligned}|\int \xi_1^2 L_M^{-1}\{Q(L_M^{-1}\tilde{G},M_t)\}d\xi|^2 &\le C\int\frac{\nu(|\xi|)}{M_*}|L_M^{-1}\{Q(L_M^{-1}\tilde{G},M_t)\}|^2d\xi\\ &\le C\int\frac{\nu(|\xi|)^{-1}}{M_*}|Q(L_M^{-1}\tilde{G},M_t)|^2d\xi \le C\int\frac{\nu(|\xi|)}{M_*}|L_M^{-1}\tilde{G}|^2d\xi\cdot\int\frac{\nu(|\xi|)}{M_*}|M_t|^2d\xi \quad (4.32)\\ &\le C(v_t^2+u_t^2+\theta_t^2)\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2d\xi.\end{aligned}$$

Combining (4.30-4.32) gives

$$\begin{aligned}I_1^{11} \le& -(\int\tilde{v}\Psi_1\int\xi_1^2L_M^{-1}\tilde{G}d\xi dx)_t + C\bar{\delta}(1+t)^{-\frac{3}{2}} + C\varepsilon_1\int|\Psi_{1t}|^2dx\\ &+C_{\varepsilon_1}\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2d\xi dx + C\varepsilon_0\|(\phi_t,\psi_t,\zeta_t)\|^2,\end{aligned} \quad (4.33)$$

where ε_1 is small positive constant to be chosen later. On the other hand, by (4.13), we have

$$\begin{aligned}|I_1^{12}| &= |\int\tilde{v}\Psi_1\int\xi_1^2L_M^{-1}(\bar{G}_t)d\xi dx|\\ &\le C\int|\Psi_1|(|(\tilde{u}_{xt},\tilde{\theta}_{xt})|+|(\tilde{u}_x,\tilde{\theta}_x)||(v_t,u_t,\theta_t)|)dx\\ &\le C\bar{\delta}(1+t)^{-1}E_1 + C\bar{\delta}(1+t)^{-\frac{3}{2}} + C\bar{\delta}\|(\phi_t,\psi_t,\zeta_t)\|^2,\end{aligned} \quad (4.34)$$

which, together with (4.33), implies

$$\begin{aligned}I_1^{1} \le& -(\int\tilde{v}\Psi_1\int\xi_1^2L_M^{-1}\tilde{G}d\xi dx)_t + C\bar{\delta}(1+t)^{-1}E_1 + C\varepsilon_1\int|\Psi_{1t}|^2dx\\ &+C_{\varepsilon_1}\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2d\xi dx + C\bar{\delta}(1+t)^{-\frac{3}{2}} + C(\bar{\delta}+\varepsilon_0)\|(\phi_t,\psi_t,\zeta_t)\|^2.\end{aligned} \quad (4.35)$$

The estimation on I_1^i, $i=2,4$ is relatively easy by using the Cauchy inequality and Lemmas 4.1-4.3 given as follows. In fact, direct computation yields

$$\begin{aligned}|I_1^2| &\le C\int\int\frac{\nu(|\xi|)}{M_*}|G_x|^2d\xi dx + C\int\Psi_1^2u_1^2dx\\ &\le C\bar{\delta}(1+t)^{-1}E_1 + C\varepsilon_0K_1 + C\int\int\frac{\nu(|\xi|)}{M_*}|G_x|^2d\xi dx.\end{aligned} \quad (4.36)$$

On the other hand, since

$$\begin{aligned}|\int\xi_1^2L_M^{-1}\{Q(G,G)\}d\xi| &\le C(\int\frac{\nu(|\xi|)}{M_*}|L_M^{-1}\{Q(G,G)\}|^2d\xi)^{\frac{1}{2}}\\ &\le C(\int\frac{\nu(|\xi|)^{-1}}{M_*}|Q(G,G)|^2d\xi)^{\frac{1}{2}} \le C\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2d\xi + C\bar{\delta}(1+t)^{-\frac{1}{2}},\end{aligned} \quad (4.37)$$

we have

$$|I_1^4| \leq C(\bar{\delta}+\varepsilon_0)\int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dx + C\bar{\delta}(1+t)^{-1}E_1 + C\bar{\delta}(1+t)^{-\frac{1}{2}}. \tag{4.38}$$

The estimation on I_1^3 is similar to the one for I_1^1. First, notice that

$$P_1(\xi_1 G_x) = \{P_1(\xi_1 G)\}_x + \sum_{j=0}^{4} < \xi_1 G, \chi_j > P_1(\chi_{jx}), \tag{4.39}$$

which will be useful for the following analysis. Then, it follows from (4.29), (4.39) and Lemmas 4.1-4.4 that

$$\begin{aligned} I_1^3 = & \int (\frac{\tilde{v}}{v}\Psi_1)_x \int \xi_1^2 L_M^{-1}[P_1(\xi_1 G)] d\xi dx \\ & - \int \frac{\tilde{v}}{v}\Psi_1 \int \xi_1^2 L_M^{-1}[\sum_{j=0}^{4} < \xi_1 G, \chi_j > P_1(\chi_{jx})] d\xi dx \\ & -2\int \frac{\tilde{v}}{v}\Psi_1 \int \xi_1^2 L_M^{-1}\{Q(L_M^{-1}[P_1(\xi_1 G)], M_x)\} d\xi dx \\ & \leq C_{\varepsilon_1}\int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dx + C\bar{\delta}(1+t)^{-1}E_1 + C(\varepsilon_0+\varepsilon_1)(K_1 + \|\Phi_x\|^2) \\ & + C\bar{\delta}(1+t)^{-\frac{1}{2}} + C\varepsilon_0\|(\phi_x,\psi_x,\zeta_x)\|^2, \end{aligned} \tag{4.40}$$

where we have used the fact that

$$| < \xi_1 G, \chi_j > |^2 \leq C\int \frac{\nu(|\xi|)\tilde{G}^2}{M_*} d\xi + C(\tilde{\theta}_x^2 + \tilde{u}_x^2).$$

By (4.27), (4.35-4.36), (4.38) and (4.40), we have

$$\begin{aligned} I_1 \leq & -(\int \tilde{v}\Psi_1 \int \xi_1^2 L_M^{-1}\tilde{G} d\xi dx)_t + C\bar{\delta}(1+t)^{-1}E_1 + C\varepsilon_1 \int |\Psi_{1t}|^2 dx \\ & + C(\varepsilon_0+\varepsilon_1)(K_1 + \|\Phi_x\|^2) + C_{\varepsilon_1}\int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dx \\ & + C\int\int \frac{\nu(|\xi|)}{M_*}|G_x|^2 d\xi dx + C(\bar{\delta}+\varepsilon_0)\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C\bar{\delta}(1+t)^{-\frac{1}{2}}. \end{aligned} \tag{4.41}$$

The estimates on the other terms of $\int NF_1 dx$ are similar. Therefore, collecting (4.25) and (4.41) gives

$$\begin{aligned} & E_{1t} + (\int\int \hat{A}(\xi,\Phi,\Psi,W) L_M^{-1}\tilde{G} d\xi dx)_t + \frac{1}{4}K_1 \\ & \leq C_1\bar{\delta}(1+t)^{-1}E_1 + C_1(\bar{\delta}+\varepsilon_0+\varepsilon_1)(\|(\Phi_t,\Psi_t,W_t)\|^2 + \|\Phi_x\|^2) \\ & + C_{\varepsilon_1}\int\int \frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2 d\xi dx + C_1\int\int \frac{\nu(|\xi|)}{M_*}(|G_x|^2 + |G_t|^2) d\xi dx \\ & + C_1(\bar{\delta}+\varepsilon_0)\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 + C_1\bar{\delta}(1+t)^{-\frac{1}{2}}, \end{aligned} \tag{4.42}$$

where we have used the smallness of $\bar{\delta}$ and ε_0. Here $\hat{A}$ is a linear function of (Φ, Ψ, W) and a polynomial of ξ.

Note that K_1 does not contain the term $\|\Phi_x\|^2$. To complete the lower order inequality, we have to estimate Φ_x. From $(4.6)_2$, we have

$$\frac{4\mu(\tilde{\theta})}{3\tilde{v}}\Phi_{xt} - \Psi_{1t} + \frac{p_+}{\tilde{v}}\Phi_x = \frac{2}{3\tilde{v}}W_x - \frac{4\mu(\tilde{\theta})}{3\tilde{v}}\tilde{R}_{1x} - Q_1 + \int \xi_1^2\Theta_1 d\xi. \tag{4.43}$$

Multiplying (4.43) by Φ_x yields

$$\begin{aligned}&(\frac{2\mu(\tilde{\theta})}{3\tilde{v}}\Phi_x^2)_t - (\frac{2\mu(\tilde{\theta})}{3\tilde{v}})_t\Phi_x^2 - \Phi_x\Psi_{1t} + \frac{p_+}{\tilde{v}}\Phi_x^2 \\ &= (\frac{2}{3\tilde{v}}W_x - \frac{4\mu(\tilde{\theta})}{3\tilde{v}}\tilde{R}_{1x} - Q_1 + \int \xi_1^2\Theta_1 d\xi)\Phi_x.\end{aligned} \tag{4.44}$$

Since

$$\Phi_x\Psi_{1t} = (\Phi_x\Psi_1)_t - (\Phi_t\Psi_1)_x + \Psi_{1x}^2 - \tilde{R}_1\Psi_{1x}, \tag{4.45}$$

we obtain

$$\begin{aligned}&(\int \frac{2\mu(\tilde{\theta})}{3\tilde{v}}\Phi_x^2 - \Phi_x\Psi_1 dx)_t + \int \frac{p_+}{2\tilde{v}}\Phi_x^2 dx \\ &\le C\int (\Psi_{1x}^2 + W_x^2)dx + C\bar{\delta}(1+t)^{-3/2} + \int Q_1^2 dx + \int |\int \xi_1^2\Theta_1 d\xi|^2 dx.\end{aligned} \tag{4.46}$$

By (4.9) and the Cauchy inequality, one has

$$\int Q_1^2 dx \le C\varepsilon_0(K_1 + \|\Phi_x\|^2) + C\bar{\delta}(1+t)^{-\frac{3}{2}} + C\varepsilon_0\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2. \tag{4.47}$$

On the other hand, Lemmas 4.1-4.3 imply

$$\begin{aligned}&\int |\int \xi_1^2\Theta_1 d\xi|^2 dx \le C\int\int \frac{\nu(|\xi|)}{M_*}(|G_x|^2 + |G_t|^2)d\xi dx + C\int |\tilde{\theta}_x|^4 dx \\ &+C(\bar{\delta}+\varepsilon_0)\int\int \frac{\nu(|\xi|)|\tilde{G}|^2}{M_*}d\xi dx \le C(\bar{\delta}+\varepsilon_0)\int\int \frac{\nu(|\xi|)|\tilde{G}|^2}{M_*}d\xi dx \\ &+C\int\int \frac{\nu(|\xi|)}{M_*}(|G_x|^2 + |G_t|^2)d\xi dx + C\bar{\delta}(1+t)^{-\frac{3}{2}}.\end{aligned} \tag{4.48}$$

Plugging (4.48) and (4.47) into (4.46) yields

$$\begin{aligned}&(\int \frac{2\mu(\tilde{\theta})}{3\tilde{v}}\Phi_x^2 - \Phi_x\Psi_1 dx)_t + \int \frac{p_+}{4\tilde{v}}\Phi_x^2 dx \\ &\le C_2K_1 + C_2\bar{\delta}(1+t)^{-3/2} + C_2\int\int \frac{\nu(|\xi|)}{M_*}(|G_x|^2 + |G_t|^2)d\xi dx \\ &+C_2(\bar{\delta}+\varepsilon_0)\int\int \frac{\nu(|\xi|)|\tilde{G}|^2}{M_*} + C_2(\bar{\delta}+\varepsilon_0)\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2.\end{aligned} \tag{4.49}$$

The microscopic component $\tilde{G}$ can be estimated through the equation (4.12). Multiplying (4.12) by $\frac{\tilde{G}}{M_*}$, we get

$$\begin{aligned}(\tfrac{\tilde{G}^2}{2M_*})_t - \tfrac{\tilde{G}}{M_*}L_M\tilde{G} = \quad & \{-\frac{1}{Rv\theta}P_1[\xi_1(\frac{|\xi-u|^2}{2\theta}(\theta-\tilde{\theta})_x + \xi\cdot(u-\tilde{u})_x)M] \\ & +\frac{u_1}{v}G_x - \frac{1}{v}P_1(\xi_1 G_x) + Q(G,G) - \tilde{G}_t\}\cdot\frac{\tilde{G}}{M_*},\end{aligned} \tag{4.50}$$

Integrating (4.50) with respect to ξ and x and using the Cauchy inequality and Lemmas 4.1-4.4, we have

$$\begin{aligned}&(\int\int\frac{\tilde{G}^2}{2M_*}d\xi dx)_t + \frac{\tilde{\sigma}}{2}\int\int\frac{\nu(|\xi|)\tilde{G}^2}{M_*}d\xi dx \\ &\leq C_3\bar{\delta}(1+t)^{-3/2} + C_3(\|\psi_x\|^2 + \|\zeta_x\|^2) + C_3\int\int\frac{\nu(|\xi|)G_x^2}{M_*}d\xi dx.\end{aligned} \tag{4.51}$$

On the other hand, since $(\Phi,\Psi,W)_t$ can be represented by $(\Phi,\Psi,W)_x$ and $(\Phi,\Psi,W)_{xx}$ from the equation (4.6), we can get an estimate for (Φ_t,Ψ_t,W_t) as follows.

$$\begin{aligned}&\int|(\Phi,\Psi,W)_t|^2dx \leq C_4K_1 + C_4\int|\Phi_x|^2dx + C_4\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2 \\ &+C_4\bar{\delta}(1+t)^{-\frac{3}{2}} + C_4\int\int\frac{\nu(|\xi|)(|\tilde{G}|^2+|G_x|^2+|G_t|^2)}{M_*}d\xi dx.\end{aligned} \tag{4.52}$$

We can now complete the lower order estimate. Since $\hat{A}$ is a linear function of the vector (Φ,Ψ,W) and a polynomial of ξ, we get

$$|\int\int\hat{A}L_M^{-1}\tilde{G}d\xi dx| \leq \frac{1}{4}E_1 + C\int\int\frac{\tilde{G}^2}{M_*}d\xi dx. \tag{4.53}$$

We choose large constants $\tilde{C}_1>1$, $\tilde{C}_2>1$, $\tilde{C}_3>1$ and small constant ε_1 so that

$$\begin{aligned}&\tilde{C}_1E_1 + \tilde{C}_1\int\int\hat{A}L_M^{-1}\tilde{G}d\xi dx + \tilde{C}_2\int\frac{2\mu(\tilde{\theta})}{3\tilde{v}}\Phi_x^2 - \Phi_x\Psi_1dx \\ &+\tilde{C}_3\int\int\frac{\tilde{G}^2}{2M_*}d\xi dx \geq \frac{1}{2}\tilde{C}_1E_1 + \tilde{C}_2\int\frac{\mu(\tilde{\theta})}{3\tilde{v}}\Phi_x^2dx + \frac{\tilde{C}_3}{4}\int\int\frac{\tilde{G}^2}{M_*}d\xi dx,\end{aligned} \tag{4.54}$$

$$\begin{aligned}&(\frac{\tilde{C}_1}{4} - C_2\tilde{C}_2 - \tilde{C}_1C_1\varepsilon_1C_4)K_1 + \int(\tilde{C}_2\frac{p_+}{4\tilde{v}} - \tilde{C}_1C_1\varepsilon_1(1+C_4))\Phi_x^2dx \\ &> \frac{\tilde{C}_1}{8}K_1 + \tilde{C}_2\int\frac{p_+}{8\tilde{v}}\Phi_x^2dx,\end{aligned} \tag{4.55}$$

and

$$\frac{\tilde{\sigma}}{2}\tilde{C}_3 - \tilde{C}_1C_1\varepsilon_1C_4 - C_{\varepsilon_1}\tilde{C}_1 > \frac{\tilde{\sigma}}{4}\tilde{C}_3. \tag{4.56}$$

Hence, by multiplying (4.42) by $\tilde{C}_1$, (4.49) by $\tilde{C}_2$, (4.51) by $\tilde{C}_3$, (4.52) by $C_1(\bar{\delta}+\varepsilon_0+\varepsilon_1)\tilde{C}_1$ and adding all these inequalities together, we have

$$\begin{aligned}&E_{2t}+K_2\leq C_5\bar{\delta}(1+t)^{-1}E_2+C_5\int\int\frac{\nu(|\xi|)}{M_*}(|G_x|^2+|G_t|^2)d\xi dx\\&+C_5\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2+C_5\bar{\delta}(1+t)^{-\frac{1}{2}},\end{aligned}\tag{4.57}$$

where

$$E_2=\tilde{C}_1E_1+\tilde{C}_1\int\int\hat{A}L_M^{-1}\tilde{G}d\xi dx+\tilde{C}_2\int\frac{2\mu(\tilde{\theta})}{3\tilde{v}}\Phi_x^2-\Phi_x\Psi_1dx+\tilde{C}_3\int\int\frac{\tilde{G}^2}{2M_*}d\xi dx,\tag{4.58}$$

$$K_2=\frac{\tilde{C}_1}{8}K_1+\tilde{C}_2\int\frac{p_+}{8\tilde{v}}\Phi_x^2dx+\|(\Phi,\Psi,W)_t\|^2+\frac{\tilde{\sigma}}{4}\tilde{C}_3\int\int\frac{\tilde{G}^2}{M_*}d\xi dx.\tag{4.59}$$

4.3 Derivative estimate

To obtain the estimate for the first order derivative of (Φ_x,Ψ_x,W_x). We shall follow the approach of section 3 for the Navier-Stokes system. From (2.b10) and (2.b25), we have

$$\begin{cases}\phi_t-\psi_{1x}=-\tilde{R}_{1x},\\ \psi_{1t}+(p-\tilde{p})_x=\frac{4}{3}(\frac{\mu(\theta)}{v}u_{1x}-\frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_{1x})_x-\tilde{R}_{2x}-\int\xi_1^2\Theta_{1x}d\xi,\\ \psi_{it}=(\frac{\mu(\theta)}{v}u_{ix}-\frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{u}_{ix})_x-(\tilde{R}_{i+1})_x-\int\xi_1\xi_i\Theta_{1x}d\xi, i=2,3,\\ \zeta_t+pu_{1x}-\tilde{p}\tilde{u}_{1x}=(\frac{\lambda(\theta)}{v}\theta_x-\frac{\lambda(\tilde{\theta})}{\tilde{v}}\tilde{\theta}_x)_x+Q_5\\ \quad+\sum_{i=1}^3u_i\int\xi_1\xi_i\Theta_{1x}d\xi-\frac{1}{2}\int\xi_1|\xi|^2\Theta_{1x}d\xi,\end{cases}\tag{4.60}$$

where

$$Q_5=\frac{4}{3}\frac{\mu(\theta)}{v}u_{1x}^2+\sum_{i=2}^3\frac{\mu(\theta)}{v}u_{ix}^2-\tilde{R}_{5x}+\frac{1}{2}(|\tilde{u}|^2)_t+\tilde{p}_x\tilde{u}_1.\tag{4.61}$$

Following the same argument for the Navier-Stokes system, see (3.41), one has

$$\begin{aligned}&(\frac{1}{2}\sum_{i=1}^3\psi_i^2+R\tilde{\theta}\hat{\Phi}(\frac{v}{\tilde{v}})+\tilde{\theta}\hat{\Phi}(\frac{\theta}{\tilde{\theta}}))_t+\frac{4}{3}\frac{\mu(\theta)}{v}\psi_{1x}^2+\sum_{i=2}^3\frac{\mu(\theta)}{v}\psi_{ix}^2+\frac{\lambda(\theta)}{v\theta}\zeta_x^2\\&=-\tilde{p}\hat{\Psi}(\frac{v}{\tilde{v}})\tilde{v}_t+\tilde{v}\tilde{p}_t\hat{\Phi}(\frac{v}{\tilde{v}})-\frac{4}{3}(\frac{\mu(\theta)}{v}-\frac{\mu(\tilde{\theta})}{\tilde{v}})\tilde{u}_{1x}\psi_{1x}-\sum_{i=2}^3(\frac{\mu(\theta)}{v}-\frac{\mu(\tilde{\theta})}{\tilde{v}})\tilde{u}_{ix}\psi_{ix}\\&-\sum_{i=1}^3(\tilde{R}_{i+1})_x\psi_i+R\tilde{\theta}(\frac{1}{v}-\frac{1}{\tilde{v}})\tilde{R}_{1x}+\frac{\zeta}{\theta}(\tilde{p}-p)\tilde{u}_{1x}-\frac{\zeta_x}{\theta}(\frac{\lambda(\theta)}{v}-\frac{\lambda(\tilde{\theta})}{\tilde{v}})\tilde{\theta}_x\\&+\frac{\zeta\theta_x}{\theta^2}(\frac{\lambda(\theta)\theta_x}{v}-\frac{\lambda(\tilde{\theta})\tilde{\theta}_x}{\tilde{v}})+\frac{\zeta}{\theta}Q_5-\hat{\Psi}(\frac{\theta}{\tilde{\theta}})\tilde{\theta}_t+NF_2+(\cdots)_x,\end{aligned}\tag{4.62}$$

where

$$NF_2 = -\sum_{i=1}^{3}\int \xi_1\xi_i\Theta_{1x}d\xi\psi_i + \frac{\zeta}{\theta}(\sum_{i=1}^{3}u_i\int \xi_1\xi_i\Theta_{1x}d\xi - \frac{1}{2}\int \xi_1|\xi|^2\Theta_{1x}d\xi). \tag{4.63}$$

Let

$$E_3 = \int \frac{1}{2}\sum_{i=1}^{3}\psi_i^2 + R\tilde{\theta}\hat{\Phi}(\frac{v}{\tilde{v}}) + \tilde{\theta}\hat{\Phi}(\frac{\theta}{\tilde{\theta}})dx \tag{4.64}$$

and

$$K_3 = \int \frac{4}{3}\frac{\mu(\theta)}{v}\psi_{1x}^2 + \sum_{i=2}^{3}\frac{\mu(\theta)}{v}\psi_{ix}^2 + \frac{\lambda(\theta)}{v\theta}\zeta_x^2dx. \tag{4.65}$$

Similar to (3.49), one has

$$E_{3t} + \frac{1}{2}K_3 \le C\bar{\delta}(1+t)^{-1}K_2 + C\bar{\delta}(1+t)^{-\frac{3}{2}} + \int NF_2dx. \tag{4.66}$$

Here, we consider the term $\int\int \xi_1^2\Theta_{1x}d\xi\psi_1dx$ since other terms in $\int NF_2dx$ can be estimated similarly. By (4.48), one has

$$\begin{aligned}|\int\int \xi_1^2\Theta_{1x}d\xi\psi_1dx| = |\int\int \xi_1^2\Theta_1d\xi\psi_{1x}dx| \le \frac{1}{8}K_3 + C\bar{\delta}(1+t)^{-\frac{3}{2}}\\ +C\sum_{|\alpha|=1}\int\int \frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2d\xi dx + C(\bar{\delta}+\varepsilon_0)\int\int \frac{\nu(|\xi|)\tilde{G}^2}{M_*}d\xi dx.\end{aligned} \tag{4.67}$$

Collecting (4.66) and (4.67) yields

$$\begin{aligned}E_{3t} + \frac{1}{4}K_3 \le C_6(1+t)^{-1}K_2 + C_6\bar{\delta}(1+t)^{-\frac{3}{2}}\\ +C_6\sum_{|\alpha|=1}\int\int \frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2d\xi dx + C_6(\bar{\delta}+\varepsilon_0)\int\int \frac{\nu(|\xi|)\tilde{G}^2}{M_*}d\xi dx.\end{aligned} \tag{4.68}$$

Note that the term $\|\phi_x\|^2$ in not included in K_3. To complete the first derivative estimate, we follow the same way in estimating Φ_x in the previous subsection. We rewrite the equation $(4.60)_2$ as

$$\begin{aligned}&\frac{4}{3}\frac{\mu(\tilde{\theta})}{\tilde{v}}\phi_{xt} - \psi_{1t} - (p-\tilde{p})_x\\ &= -\frac{4}{3}(\frac{\mu(\tilde{\theta})}{\tilde{v}})_x\psi_{1x} - \frac{4}{3}\frac{\mu(\tilde{\theta})}{\tilde{v}}\tilde{R}_{1xx} - \frac{4}{3}[(\frac{\mu(\theta)}{v} - \frac{\mu(\tilde{\theta})}{\tilde{v}})u_{1x}]_x + \tilde{R}_{2x} + \int \xi_1^2\Theta_{1x}d\xi,\end{aligned} \tag{4.69}$$

by using the equation of conservation of the mass (4.60). Multiplying (4.69) by ϕ_x, we get

$$\begin{aligned}\frac{2}{3}(\frac{\mu(\tilde{\theta})}{\tilde{v}}\phi_x^2)_t - \frac{2}{3}(\frac{\mu(\tilde{\theta})}{\tilde{v}})_t\phi_x^2 - \psi_{1t}\phi_x - (p-\tilde{p})_x\phi_x = \{-\frac{4}{3}(\frac{\mu(\tilde{\theta})}{\tilde{v}})_x\psi_{1x}\\ -\frac{4\mu(\tilde{\theta})}{3\tilde{v}}\tilde{R}_{1xx} - \frac{4}{3}[(\frac{\mu(\theta)}{v} - \frac{\mu(\tilde{\theta})}{\tilde{v}})u_{1x}]_x + \tilde{R}_{2x} + \int \xi_1^2\Theta_{1x}d\xi\}\phi_x.\end{aligned} \tag{4.70}$$

Since

$$-(p-\tilde{p})_x = \frac{\tilde{p}}{\tilde{v}}\phi_x - \frac{R}{\tilde{v}}\zeta_x + (\frac{p}{v}-\frac{\tilde{p}}{\tilde{v}})v_x - (\frac{R}{v}-\frac{R}{\tilde{v}})\theta_x, \tag{4.71}$$

and

$$\phi_x\psi_{1t} = (\phi_x\psi_1)_t - (\phi_t\psi_1)_x + \psi_{1x}^2 - \tilde{R}_{1x}\psi_{1x}, \tag{4.72}$$

integrating (4.70) with respect to x and using the Cauchy inequality yield

$$\begin{aligned}
&(\int \frac{2\mu(\tilde{\theta})}{3\tilde{v}}\phi_x^2 - \phi_x\psi_1 dx)_t + \int \frac{\tilde{p}}{2\tilde{v}}\phi_x^2 \\
&\leq C_7K_3 + C_7\bar{\delta}(1+t)^{-1}K_2 + C_7\bar{\delta}(1+t)^{-\frac{5}{2}} + C_7\varepsilon_0\int \psi_{1xx}^2 dx \\
&\quad + C_7\varepsilon_0\sum_{0\leq|\alpha|\leq 1}\int\int \frac{\nu(|\xi|)}{M_*}|\partial^\alpha\tilde{G}|^2 d\xi dx + C_7\sum_{|\alpha|=2}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2 d\xi dx.
\end{aligned} \tag{4.73}$$

Here we have used

$$\begin{aligned}
&\int |\int \xi_1^2\Theta_{1x}d\xi|^2 dx \leq C\bar{\delta}(1+t)^{-\frac{5}{2}} + C(\bar{\delta}+\varepsilon_0)\sum_{0\leq|\alpha|\leq 1}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha\tilde{G}|^2 d\xi dx \\
&\quad + C\sum_{|\alpha|=2}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2 d\xi dx,
\end{aligned} \tag{4.74}$$

due to Lemmas 4.1-4.4. To estimate $(\phi,\psi,\zeta)_t$, we use the original equation (2.b9). For example, multiplying $(2.b9)_2$ by ψ_{1t}, we have

$$\psi_{1t}^2 + \tilde{u}_{1t}\psi_{1t} + (p-\tilde{p})_x\psi_{1t} + \tilde{p}_x\psi_{1t} = -\int \xi_1^2 G_x d\xi\psi_{1t}. \tag{4.75}$$

Integrating (4.75) with respect to x and using (4.71) give

$$\begin{aligned}
&\int \psi_{1t}^2 dx \leq C_7(K_3 + \|\phi_x\|^2) + C_7\bar{\delta}(1+t)^{-1}K_2 + C_7\bar{\delta}(1+t)^{-\frac{5}{2}} \\
&\quad + C_7\sum_{|\alpha|=1}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2 d\xi dx.
\end{aligned} \tag{4.76}$$

Similar estimates hold for ϕ_t, ψ_{2t}, ψ_{3t} and ζ_t. Thus we choose large constants $\tilde{C}_4$ and $\tilde{C}_5$ so that

$$\tilde{C}_4E_3 + \tilde{C}_5\int\frac{2\mu(\tilde{\theta})}{3\tilde{v}}\phi_x^2 - \phi_x\psi_1 dx \geq \frac{\tilde{C}_4}{2}E_3 + \tilde{C}_5\int\frac{\mu(\tilde{\theta})}{3\tilde{v}}\phi_x^2 dx, \tag{4.77}$$

and

$$\frac{1}{4}\tilde{C}_4 - (\tilde{C}_5+5)C_7 \geq \frac{1}{8}\tilde{C}_4, \quad \tilde{C}_5\int\frac{\tilde{p}}{2\tilde{v}}\phi_x^2 dx - 5C_7\|\phi_x\|^2 \geq \frac{\tilde{C}_5}{4}\int\frac{\tilde{p}}{\tilde{v}}\phi_x^2 dx. \tag{4.78}$$

Let

$$E_4 = \tilde{C}_4E_3 + \tilde{C}_5\int\frac{2\mu(\tilde{\theta})}{3\tilde{v}}\phi_x^2 - \phi_x\psi_1 dx + \int\int\frac{|\tilde{G}|^2}{2M_*}d\xi dx, \tag{4.79}$$

$$K_4=\frac{1}{8}\tilde{C}_4K_3+\frac{\tilde{C}_5}{4}\int\frac{\tilde{p}}{\tilde{v}}\phi_x^2dx+\int(\phi_t^2+|\psi_t|^2+\zeta_t^2)dx+\frac{\tilde{\sigma}}{4}\int\int\frac{\nu(|\xi|)}{M_*}|\tilde{G}|^2dx. \tag{4.80}$$

Then we have estimate on the (ϕ,ψ,ζ), from (4.51), (4.68), (4.73) and (4.76-4.78),

$$\begin{aligned}&E_{4t}+K_4\le C_8\bar{\delta}(1+t)^{-\frac{3}{2}}+C_8\bar{\delta}(1+t)^{-1}K_2+C_8\varepsilon_0\int\psi_{1xx}^2dx\\&+C_8\sum_{1\le|\alpha|\le2}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2d\xi dx.\end{aligned}\tag{4.81}$$

Next we derive the higher order derivative estimate. Applying ∂_x to (4.60) yields

$$\begin{cases}\phi_{xt}-\psi_{1xx}=-\tilde{R}_{1xx},\\ \psi_{1xt}-\dfrac{p}{v}\phi_{xx}+\dfrac{R}{v}\zeta_{xx}=\dfrac{4}{3}(\dfrac{\mu(\theta)}{v}\psi_{1xx})_x+Q_6-\displaystyle\int\xi_1^2\Theta_{1xx}d\xi,\\ \psi_{ixt}=(\dfrac{\mu(\theta)}{v}\psi_{ixx})_x+Q_{i+5}-\displaystyle\int\xi_1\xi_i\Theta_{1xx}d\xi,\ i=2,3,\\ \zeta_{xt}+p\psi_{1xx}=(\dfrac{\lambda(\theta)}{v}\zeta_{xx})_x+Q_9+(\displaystyle\sum_{i=1}^3u_i\int\xi_1\xi_i\Theta_{1x}d\xi)_x-\dfrac{1}{2}\int\xi_1|\xi|^2\Theta_{1xx}d\xi,\end{cases}\tag{4.82}$$

where

$$\begin{aligned}Q_6=&\frac{p-\tilde{p}}{v}\tilde{v}_{xx}+\frac{\phi}{v}\tilde{p}_{xx}+\frac{2v_x}{v}(p-\tilde{p})_x+\frac{2\tilde{p}_x}{v}\phi_x\\&+\frac{4}{3}((\frac{\mu(\theta)}{v})_x\psi_{1x})_x+\frac{4}{3}[(\frac{\mu(\theta)}{v}-\frac{\mu(\tilde{\theta})}{\tilde{v}})\tilde{u}_{1x}]_{xx}-\tilde{R}_{2xx},\end{aligned}\tag{4.83}$$

$$Q_{i+5}=((\frac{\mu(\theta)}{v})_x\psi_{ix})_x+[(\frac{\mu(\theta)}{v}-\frac{\mu(\tilde{\theta})}{\tilde{v}})\tilde{u}_{ix}]_{xx}-(\tilde{R}_{i+1})_{xx},\tag{4.84}$$

$$Q_9=-p_xu_{1x}-p\tilde{u}_{1xx}+(\tilde{p}\tilde{u}_{1x})_x+Q_{5x}+((\frac{\lambda(\theta)}{v})_x\zeta_x)_x+[(\frac{\lambda(\theta)}{v}-\frac{\lambda(\tilde{\theta})}{\tilde{v}})\tilde{\theta}_x]_{xx}.\tag{4.85}$$

Then multiplying $(4.82)_1$ by $p\phi_x$, $(4.82)_2$ by $v\psi_{1x}$, $(4.82)_3$ by ψ_{ix}, $(4.82)_4$ by $\frac{R}{p}\zeta_x$, we have

$$\begin{aligned}&(\frac{p}{2}\phi_x^2+\frac{v}{2}\psi_{1x}^2+\sum_{i=2}^3\psi_{ix}^2+\frac{R}{2p}\zeta_x^2)_t-\frac{p_t}{2}\phi_x^2-\frac{v_t}{2}\psi_{1x}^2-(\frac{R}{2p})_t\zeta_x^2+p_x\psi_{1x}\phi_x\\&+\frac{4\mu(\theta)}{3}\psi_{1xx}^2+\sum_{i=2}^3\frac{\mu(\theta)}{v}\psi_{ixx}^2+\frac{R\lambda(\theta)}{vp}\zeta_{xx}^2=-p\tilde{R}_{1xx}\phi_x-\frac{4\mu(\theta)}{3v}\psi_{1xx}v_x\psi_{1x}\\&-\frac{\lambda(\theta)}{v}\zeta_{xx}(\frac{R}{p})_x\zeta_x+vQ_6\psi_{1x}+\sum_{i=2}^3Q_{i+5}\psi_{ix}+\frac{R}{p}Q_9\zeta_x+NF_3+(\cdots)_x,\end{aligned}\tag{4.86}$$

where

$$\begin{aligned}NF_3=&-v\psi_{1x}\int\xi_1^2\Theta_{1xx}d\xi-\sum_{i=2}^3\psi_{ix}\int\xi_1\xi_i\Theta_{1xx}d\xi\\&+\frac{R}{p}\zeta_x[(\sum_{i=1}^3u_i\int\xi_1\xi_i\Theta_{1x}d\xi)_x-\frac{1}{2}\int\xi_1|\xi|^2\Theta_{1xx}d\xi].\end{aligned}\tag{4.87}$$

Let

$$E_5=\int \frac{p}{2}\phi_x^2+\frac{v}{2}\psi_{1x}^2+\sum_{i=2}^{3}\psi_{ix}^2+\frac{R}{2p}\zeta_x^2dx, \tag{4.88}$$

$$K_5=\int \frac{4\mu(\theta)}{3}\psi_{1xx}^2+\sum_{i=2}^{3}\frac{\mu(\theta)}{v}\psi_{ixx}^2+\frac{R\lambda(\theta)}{vp}\zeta_{xx}^2dx. \tag{4.89}$$

The estimates for $\int vQ_6\psi_{1x}dx$, $\int Q_{i+5}\psi_{ix}dx$, $\int \frac{R}{p}Q_9\zeta_x dx$ are straightforward. For instance, by a direct computation, we have

$$|\int vQ_6\psi_{1x}dx|\le \frac{1}{8}K_5+C\bar{\delta}(1+t)^{-\frac{5}{2}}+C(\bar{\delta}+\varepsilon_0)K_4+C\bar{\delta}(1+t)^{-2}K_2. \tag{4.90}$$

Hence we only need to estimate $\int NF_3dx$. As before, it is sufficient to consider the term $\int v\psi_{1x}\int \xi_1^2\Theta_{1xx}d\xi dx$. By (4.74),

$$\begin{aligned}
&|\int v\psi_{1x}\int \xi_1^2\Theta_{1xx}d\xi dx|\le |\int v_x\int \xi_1^2\Theta_{1x}d\xi\psi_{1x}dx|+|\int v\int \xi_1^2\Theta_{1x}d\xi\psi_{1xx}dx|\\
&\le C(\bar{\delta}+\varepsilon_0)K_4+\frac{1}{16}K_5+C\bar{\delta}(1+t)^{-\frac{5}{2}}+C\sum_{|\alpha|=2}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2d\xi dx\\
&\quad +C(\bar{\delta}+\varepsilon_0)\sum_{|\alpha|=1}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2d\xi dx.
\end{aligned} \tag{4.91}$$

Integrating (4.86) with respect to x and using (4.90) and (4.91), we have

$$\begin{aligned}
&E_{5t}+\frac{1}{4}K_5\le C_9(\bar{\delta}+\varepsilon_0)K_4+C_9\bar{\delta}(1+t)^{-\frac{5}{2}}+C_9\sum_{|\alpha|=2}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2d\xi dx\\
&\quad +C_9(\bar{\delta}+\varepsilon_0)\sum_{|\alpha|=1}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2d\xi dx+C_9\bar{\delta}(1+t)^{-2}K_2.
\end{aligned} \tag{4.92}$$

To get the estimation for ϕ_{xx}, we use the first momentum equation of (2.b9). Applying ∂_x on $(2.b9)_2$, we have

$$\psi_{1xt}+\tilde{u}_{1xt}+(p-\tilde{p})_{xx}+\tilde{p}_{xx}=-\int \xi_1^2G_{xx}d\xi. \tag{4.93}$$

Note that

$$(p-\tilde{p})_{xx}=-\frac{p}{v}\phi_{xx}+\frac{R}{v}\zeta_{xx}-\frac{1}{v}(p-\tilde{p})\tilde{v}_{xx}-\frac{\phi}{v}\tilde{p}_{xx}-\frac{2v_x}{v}(p-\tilde{p})_x-\frac{2\tilde{p}_x}{v}\phi_x.$$

Multiplying (4.93) by $-\phi_{xx}$ and using (4.71) imply

$$\begin{aligned}
&-(\psi_{1x}\phi_{xx})_t+\int\frac{p}{2v}\phi_{xx}^2dx\le C_{10}K_5+C_{10}\bar{\delta}(1+t)^{-\frac{7}{2}}+C_{10}(\bar{\delta}+\varepsilon_0)K_4\\
&\quad +C_{10}\bar{\delta}(1+t)^{-2}K_2+C_{10}\sum_{|\alpha|=2}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2d\xi dx.
\end{aligned} \tag{4.94}$$

To estimate $(\phi,\psi,\zeta)_{xt}$ and $(\phi,\psi,\zeta)_{tt}$, we use the original fluid-type equation (2.b9) again. Here we only consider the case $\int \psi_{1xt}^2 dx$ since the other terms can be estimated similarly. From $(2.b9)_2$ and (4.93), we have

$$\psi_{1xt} = -(p-\tilde{p})_{xx} - \tilde{p}_{xx} - \int \xi_1^2 G_{xx} d\xi - \tilde{u}_{1xt}. \tag{4.95}$$

Similarly,

$$\begin{aligned}\int \psi_{1xt}^2 dx \leq & C_{11}\bar{\delta}(1+t)^{-\frac{3}{2}} + C_{11}K_5 + C_{11}\varepsilon_0 K_4 + C_{11}\bar{\delta}(1+t)^{-2}K_2 \\ & + C_{11}\sum_{|\alpha|=2}\int\int \frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2 d\xi dx.\end{aligned} \tag{4.96}$$

Choosing $\tilde{C}_6$ to be a large constant, we have

$$\begin{aligned}\tilde{C}_6(E_5 - \int \psi_{1x}\phi_{xx}dx)_t + \sum_{|\alpha|=2}\int |\partial^\alpha(\phi,\psi,\zeta)|^2 dx \leq C_{12}\bar{\delta}(1+t)^{-\frac{5}{2}} \\ + C_{12}(\bar{\delta}+\varepsilon_0)K_4 + C_{12}(\bar{\delta}+\varepsilon_0)\sum_{|\alpha|=1}\int\int \frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2 d\xi dx \\ + C_{12}\sum_{|\alpha|=2}\int\int \frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2 d\xi dx + C_{12}\bar{\delta}(1+t)^{-2}K_2.\end{aligned} \tag{4.97}$$

To close the above estimate, we also need to estimate the derivatives on the non-fluid component G, i.e., $\partial^\alpha G, 1 \leq |\alpha| \leq 2$. Applying ∂_x on (2.b11), we have

$$\begin{aligned}& G_{xt} - (\frac{u_1}{v}G_x)_x + \{\frac{1}{v}P_1(\xi_1 M_x)\}_x + \{\frac{1}{v}P_1(\xi_1 G_x)\}_x \\ & = L_M G_x + 2Q(M_x, G) + 2Q(G_x, G).\end{aligned} \tag{4.98}$$

Since

$$P_1(\xi_1 M_x) = \frac{1}{Rv\theta}P_1[\xi_1(\frac{|\xi-u|^2}{2\theta}\theta_x + \xi\cdot u_x)M],$$

we have

$$|\{\frac{1}{v}P_1(\xi_1 M_x)\}_x| \leq C(v_x^2 + u_x^2 + \theta_x^2 + |\theta_{xx}| + |u_{xx}|)|\hat{B}(\xi)|M,$$

where $\hat{B}(\xi)$ is a polynomial of ξ. This yields that

$$\int\int |\{\frac{1}{v}P_1(\xi_1 M_x)\}_x \frac{G_x}{M_*}| d\xi dx \leq \frac{\tilde{\sigma}}{8}\int\int \frac{\nu(|\xi|)}{M_*}G_x^2 d\xi dx + CK_5 + C(\bar{\delta}+\varepsilon_0)K_4 + C\bar{\delta}(1+t)^{-\frac{3}{2}}.$$

Also, we have

$$\begin{aligned}|\int\int Q(M_x, G)\frac{G_x}{M_*}d\xi dx| \leq & \frac{\tilde{\sigma}}{8}\int\int \frac{\nu(|\xi|)}{M_*}G_x^2 d\xi dx \\ & + C\int(\int \frac{\nu(|\xi|)}{M_*}M_x^2 d\xi \times \int\int \frac{\nu(|\xi|)}{M_*}G^2 d\xi) dx \\ \leq & \frac{\tilde{\sigma}}{8}\int\int \frac{\nu(|\xi|)}{M_*}G_x^2 d\xi dx + C(\bar{\delta}+\varepsilon_0)K_4 + C\bar{\delta}(1+t)^{-\frac{3}{2}}.\end{aligned}$$

Thus, multiplying (4.98) by $\frac{G_x}{M_*}$ and using the Cauchy inequality and Lemmas 4.1-4.4, we get

$$\begin{aligned}&(\int\int \frac{G_x^2}{2M_*}d\xi dx)_t+\frac{\tilde\sigma}{2}\int\int\frac{\nu(|\xi|)}{M_*}G_x^2d\xi dx\le C\bar\delta(1+t)^{-\frac{3}{2}}+C(\bar\delta+\varepsilon_0)K_4\\&+C\int\int\frac{\nu(|\xi|)}{M_*}G_{xx}^2d\xi dx+CK_5.\end{aligned}\tag{4.99}$$

Similarly,

$$\begin{aligned}&(\int\int \frac{G_t^2}{2M_*}d\xi dx)_t+\frac{\tilde\sigma}{2}\int\int\frac{\nu(|\xi|)}{M_*}G_t^2d\xi dx\le C\bar\delta(1+t)^{-\frac{3}{2}}+C(\bar\delta+\varepsilon_0)K_4\\&+C(\bar\delta+\varepsilon_0)\int\int\frac{\nu(|\xi|)}{M_*}G_x^2d\xi dx+C\int\int\frac{\nu(|\xi|)}{M_*}G_{xt}^2d\xi dx+C\int\psi_{xt}^2+\zeta_{xt}^2dx.\end{aligned}\tag{4.100}$$

Finally, we give the highest order estimate needed to control $\int\psi_{1x}\phi_{xx}dx$ and $\int\int\frac{\nu(|\xi|)|\partial^\alpha G|^2}{M_*}d\xi dx$, $|\alpha|=2$ in (4.97). To estimate $\int\psi_{1x}\phi_{xx}dx$, it is sufficient to study the a priori estimate for $\int\int\frac{1}{M_*}|\partial^\alpha f|^2d\xi dx$, ($|\alpha|=2$) due to (4.18) and (4.19). To this end, apply ∂^α ($|\alpha|=2$) on (2.b8), we have

$$(\partial^\alpha f)_t-\partial^\alpha(\frac{u_1-\xi_1}{v}f_x)=\partial^\alpha Q(f,f)=\partial^\alpha[L_MG+Q(G,G)].\tag{4.101}$$

Multiply (4.101) by $\frac{\partial^\alpha f}{M_*}=\frac{\partial^\alpha M}{M_*}+\frac{\partial^\alpha G}{M_*}$, we obtain

$$\begin{aligned}&(\frac{|\partial^\alpha f|^2}{2M_*})_t+\sum_{|\beta|=1}C(\alpha,\beta)\partial^{\alpha-\beta}(\frac{u_1-\xi_1}{v})\partial^\beta f_x\frac{\partial^\alpha f}{M_*}-L_M\partial^\alpha G\cdot\frac{\partial^\alpha G}{M_*}\\&=\partial^\alpha(\frac{u_1-\xi_1}{v})f_x\frac{\partial^\alpha f}{M_*}-(\frac{u_1-\xi_1}{2v})_x\frac{|\partial^\alpha f|^2}{M_*}+L_M\partial^\alpha G\cdot\frac{\partial^\alpha M}{M_*}\\&+(\sum_{|\beta|=1}2Q(\partial^{\alpha-\beta}M,\partial^\alpha G)+2Q(\partial^\alpha M,G))\frac{\partial^\alpha f}{M_*}+\partial^\alpha Q(G,G)\frac{\partial^\alpha f}{M_*}+(\cdots)_x,\end{aligned}\tag{4.102}$$

where we have used

$$\partial^\alpha L_MG=L_M\partial^\alpha G+\sum_{|\beta|=1}2Q(\partial^{\alpha-\beta}M,\partial^\beta G)+2Q(\partial^\alpha M,G),\ |\alpha|=2.$$

Since $M_x,M_t\in N$, $P_1(\partial^\alpha M)$ does not contain $\partial^\alpha(v,u,\theta)$. Thus, we have

$$\begin{aligned}&|\int\int\frac{L_M\partial^\alpha G\cdot\partial^\alpha M}{M}d\xi dx|=|\int\int\frac{L_M\partial^\alpha G\cdot P_1(\partial^\alpha M)}{M}d\xi dx|\\&\le C(\bar\delta+\varepsilon_0)K_4+C\bar\delta(1+t)^{-\frac{3}{2}}+\frac{\tilde\sigma}{8}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2d\xi dx,\end{aligned}\tag{4.103}$$

and

$$\begin{aligned}&|\int\int L_M\partial^\alpha G\cdot\partial^\alpha M(\frac{1}{M_*}-\frac{1}{M})d\xi dx|\\&\le\frac{\tilde\sigma}{8}\int\int\frac{\nu(|\xi|)}{M_*}|\partial^\alpha G|^2d\xi dx+C\eta_0\|\partial^\alpha(\phi,\psi,\zeta)\|^2+C(\bar\delta+\varepsilon_0)K_4+C\bar\delta(1+t)^{-\frac{3}{2}},\end{aligned}\tag{4.104}$$

where the small constant η_0 is defined in Lemma 4.2. Thus integrating (4.102) and using $f = M + G$ and Lemma 4.2, we have

$$
\begin{aligned}
&(\sum_{|\alpha|=2} \int\int \frac{|\partial^\alpha f|^2}{2M_*} d\xi dx)_t + \frac{1}{2}\tilde{\sigma} \sum_{|\alpha|=2} \int\int \frac{\nu(|\xi|)}{M_*} |\partial^\alpha G|^2 d\xi dx \\
&\leq C\bar{\delta}(1+t)^{-\frac{3}{2}} + C(\bar{\delta}+\varepsilon_0) \sum_{|\alpha|=1} \int\int \frac{\nu(|\xi|)}{M_*} |\partial^\alpha G|^2 d\xi dx + C(\bar{\delta}+\varepsilon_0) K_4 \\
&+ C(\eta_0 + \bar{\delta} + \varepsilon_0) \sum_{|\alpha|=2} \|\partial^\alpha(\phi,\psi,\zeta)\|^2.
\end{aligned}
\tag{4.105}
$$

By (4.18) and (4.19), we choose suitable constants $\hat{C}_i > 1$, $i = 1, 2, 3, 4$, so that

$$
\begin{aligned}
E_6 &= \hat{C}_1 E_4 + \hat{C}_2 \tilde{C}_6 (E_5 - \int \psi_{1x}\phi_{xx} dx) + \hat{C}_3 \sum_{|\alpha|=1} \int\int \frac{1}{M_*} |\partial^\alpha G|^2 d\xi dx \\
&+ \hat{C}_4 \sum_{|\alpha|=2} \int\int \frac{1}{M_*} |\partial^\alpha f|^2 d\xi dx \geq \|(\phi,\psi,\zeta)\|_{H^1}^2 + \int\int \frac{1}{M_*} |\tilde{G}|^2 d\xi dx \\
&+ \sum_{|\alpha|=1} \int\int \frac{1}{M_*} |\partial^\alpha G|^2 d\xi dx + \sum_{|\alpha|=2} \int\int \frac{1}{M_*} |\partial^\alpha f|^2 d\xi dx - C\bar{\delta}^2 (1+t)^{-\frac{3}{2}}.
\end{aligned}
\tag{4.106}
$$

Let

$$
K_6 = \int\int \frac{\nu(|\xi|)}{M_*} |\tilde{G}|^2 d\xi dx + \sum_{1\leq|\alpha|\leq 2} \int\int \frac{\nu(|\xi|)}{M_*} |\partial^\alpha G|^2 d\xi dx + \sum_{1\leq|\alpha|\leq 2} |\partial^\alpha(\phi,\psi,\zeta)|^2. \tag{4.107}
$$

Then using (4.81), (4.97), (4.99-4.100) and (4.105), we obtain the final energy estimate

$$
E_{6t} + K_6 \leq C\bar{\delta}(1+t)^{-\frac{3}{2}} + C\bar{\delta}(1+t)^{-1} K_2. \tag{4.108}
$$

4.4 Decay rate

By combining (4.57) and (4.108) and choosing a large constant $\hat{C}_5$, we have

$$
E_{7t} + K_7 \leq C_0 \bar{\delta}(1+t)^{-1} E_7 + C_0 \bar{\delta}(1+t)^{-\frac{1}{2}}, \tag{4.109}
$$

where

$$
E_7 = E_2 + \hat{C}_5 E_6, \quad K_7 = K_2 + \hat{C}_5 K_6. \tag{4.110}
$$

Notice that (4.108) and (4.109) have the same forms as (3.69) and (3.70) for the compressible Navier-Stokes equations. Following the same argument in subsection 3.4, we have

$$
E_7 \leq C(E_7(0) + \bar{\delta})(1+t)^{\frac{1}{2}}, \qquad \int_0^t K_7 dt \leq C(E_7(0) + \bar{\delta})(1+t)^{\frac{1}{2}}, \tag{4.111}
$$

and

$$
E_6 \leq C(E_7(0) + \bar{\delta})(1+t)^{-\frac{1}{2}}. \tag{4.112}
$$

Since $(\phi, \psi) = (\Phi, \Psi)_x$ and $\zeta = W_x - Y$, $E_7 \geq c\|(\Phi, \Psi, W)\|^2$ and $E_6 \geq c\|(\phi, \psi, \zeta)\|^2$ for some positive constant c, we have

$$\|(\Phi, \Psi, W)\|_{L^\infty}^2 \leq C\|(\Phi, \Psi, W)\|\|(\Phi_x, \Psi_x, W_x)\| \leq C(E_7(0) + \bar{\delta}). \tag{4.113}$$

Now it remains to justify the decay rate of (2.b34). By (4.112), we have

$$\begin{aligned}
&\int\int \frac{|f(x,t,\xi) - M_{[\tilde{\rho},\tilde{u},\tilde{\theta}]}|^2}{M_*} d\xi dx \leq \int\int \frac{|M - M_{[\tilde{\rho},\tilde{u},\tilde{\theta}]}|^2}{M_*} d\xi dx + \int\int \frac{G^2}{M_*} d\xi dx \\
&\leq C\|(\phi, \psi, \zeta)\|^2 + C\int\int \frac{G^2}{M_*} d\xi dx \leq C(E_7(0) + \bar{\delta})(1+t)^{-\frac{1}{2}},
\end{aligned} \tag{4.114}$$

and

$$\begin{aligned}
&\int\int \frac{|f_x(x,t,\xi) - (M_{[\tilde{\rho},\tilde{u},\tilde{\theta}]})_x|^2}{M_*} d\xi dx \\
&\leq \int\int \frac{|M_x - (M_{[\tilde{\rho},\tilde{u},\tilde{\theta}]})_x|^2}{M_*} d\xi dx + \int\int \frac{G_x^2}{M_*} d\xi dx \\
&\leq C\bar{\delta}^2(1+t)^{-1}\|(\phi, \psi, \zeta)\|^2 + C\|(\phi_x, \psi_x, \zeta_x)\|^2 + C\int\int \frac{G^2}{M_*} d\xi dx \\
&\leq C(E_7(0) + \bar{\delta})(1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{4.115}$$

(2.b34) follows directly from (4.114) and (4.115) and the Sobolev inequality. And this completes the proof of Theorem 3.

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