Existence of a Stationary Solution for the Modified King-Ward Tumor Growth Model

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Abstract

This paper studies existence of a stationary solution to a tumor growth model proposed by King and Ward in 1997 and 1998, with biologically reasonable modifications. Mathematical formulation of this problem is a two-point free boundary problem of a system of ordinary differential equations, one of which is singular at the boundary points. By using the Schauder fixed point theorem we prove existence of a solution for this problem.

Key words: Free boundary problem, tumor growth, stationary solution, singular differential equation.

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1 Introduction

Recent development of mathematical modelling of tumor growth has introduced many new interesting free boundary problems. A big portion of such free boundary problems contain two parts of partial differential equations: One or several reaction diffusion equations describing the diffusion of nutrients and inhibitors within the tumor, and a number of first-order hyperbolic conservation laws involving source terms mimicking the movement of various tumor cells (proliferating cells, quiescent cells, and necrotic cells) [1–3, 7–12]. This kind of tumor growth models are initiated by Ward and King [11, 12]. They are currently the main flow in the river of tracking the mechanism of tumor growth using mathematical tools. Rigorous mathematical analysis of such free boundary problems has made some progress (see the references cited in [7]), and is shown to be a hard but very interesting topic of research.

In this paper we study a tumor growth model developed from that of Ward and King [11, 12]. This model assumes that the tumor is ball-shaped and all known and unknown functions are spherically symmetric in the space variable. It contains five unknown functions: (1) $R = R(t)$, the radius of the tumor, where t denotes the time variable, (2) $c = c(r, t)$, the concentration of nutrients (regarded as a single species), where r denotes the radial space variable, (3) $l = l(r, t)$, the density of live tumor cells, (4) $d = d(r, t)$, the density of dead tumor cells, and (5) $\vec{v} = \vec{v}(r, t)$, the velocity of cell movement. Within the tumor the unknown functions c, l, d and $\vec{v}$ satisfy the following system of equations:

$$\frac{\partial c}{\partial t} = D\triangle c - f(c)l, \tag{1.1}$$

$$\frac{\partial l}{\partial t} + \nabla \cdot (\vec{v} l) = K_B(c)l - K_D(c)l, \tag{1.2}$$

$$\frac{\partial d}{\partial t} + \nabla \cdot (\vec{v} d) = K_D(c)l - \mu d, \tag{1.3}$$

$$l + d = N. \tag{1.4}$$

Here D is the diffusion coefficient which is supposed to be constant, $f(c)$ is the consumption rate of nutrients by live tumor cells in unit volume when the nutrient supply is at the level c, $K_B(c)$ is the birth rate of tumor cells when nutrient supply is at the level c, $K_D(c)$ is the death rate of tumor cells when nutrient supply is at the level c, μ is a positive constant representing the dissolution rate of dead tumor cells, and N is another positive constant, denoting the constant density of the mixture of live and dead tumor cells which forms the tumor tissue. It is assumed that when a dead cell is dissolved, it is immediately removed from the tumor so that it no longer occupies any volume of the tumor.

The tumor is sustained with a constant nutrient supply from its surface, so that

$$c(R(t), t) = a \quad \text{for} \quad t \geq 0, \tag{1.5}$$

where a is a positive constant. Since all known and unknown functions are spherically symmetric, the movement of tumor cells must also be spherically symmetric. Thus there is a scaler function $v = v(r, t)$ such that $\vec{v} = v\vec{r^o}$, where $\vec{r^o}$ denotes the unit vector in the radial direction. The equation governing the movement of tumor surface is given by

$$\frac{dR(t)}{dt} = v(R(t), t) \quad \text{for} \quad t > 0. \tag{1.6}$$

Equations (1.1)–(1.6) complemented with suitable initial value conditions form a mathematical model for the growth of a tumor in the early stage containing both live cells and dead cells. This model is essentially due to Ward and King [11, 12], with two points of modifications: (1) In the original model of Ward and King, instead of the equation (1.3) the following equation is used to describe the evolution and movement of dead cells:

$$\frac{\partial d}{\partial t} + \nabla \cdot (\vec{v} d) = K_D(c)l. \tag{1.7}$$

By this equation, dead cells do not undergo dissolution in the process of tumor growth so that the amount of dead cells will accumulate unboundedly as time goes to infinity. This will lead to unbounded growth of the tumor [11, 12], which is not realistic because experimental observation shows that an evolutionary tumor sustained with constant nutrient supply will evolve to a dormant state as time goes to infinity [1–10]. Thus we replace the equation (1.7) with (1.3). (2) In the original model of Ward and King there is a convection term $\nabla \cdot (\vec{v} c)$ on the left-hand side of the equation (1.1), but here we omit such a term. This means that we do not consider convection of nutrient material caused by tumor cell movement, and regard nutrient diffusion and tumor cell movement as two independent processes.

Pettet *et al* [10] established a different tumor growth model that contains only live cells distinguished in two different states: proliferating state and quiescent state. Their model has some similarity in mathematical spirit with the above model, but the equation for nutrient diffusion is different: Since all cells are alive, the equation (1.1) is replaced by

$$\frac{\partial c}{\partial t} = D \triangle c - f(c). \tag{1.8}$$

A more general tumor growth model distinguishing all three species of tumor cells (proliferating cells, quiescent cells, and necrotic cells) was given by Friedman [7].

Explicit expressions of the functions $f(c)$, $K_B(c)$ and $K_D(c)$ can be found from Ward and King [11, 12]. Here we do not reminisce those expressions, but instead summarize main properties of these functions as follows:

(A_1) $f \in C^1[0,\infty)$, $f'(c) > 0$ for $c \geq 0$, and $f(0) = 0$.

(A_2) $K_B \in C^1[0,\infty)$, $K_B'(c) > 0$ for $c \geq 0$, and $K_B(0) = 0$.

(A_3) $K_D \in C^1[0,\infty)$, $K_D'(c) < 0$, $K_D(c) \geq 0$ for $c \geq 0$, and $K_D(0) > 0$.

Later on we shall assume that $f(c)$, $K_B(c)$ and $K_D(c)$ are any functions satisfying these properties. We shall also make the following additional assumptions:

(A_4) $K_B(a) > K_D(a)$.

(A_5) $\mu > K_D(0)$.

Global well-posedness of the initial value problem of the system (1.1)–(1.6) has been established by Cui and Friedman [6] in more general settings. In this paper we consider stationary solution of the system (1.1)–(1.6). We shall prove that the system (1.1)–(1.6) has at least one stationary solution.

We re-scale the space variable such that $D = 1$, normalize unknown functions c, l, d and v such that $a = 1$ and $N = 1$, and re-define known functions in accordance with such re-scaling and normalization. We denote by $(c(r), l(r), d(r), v(r), R)$ the stationary solution of the re-scaled and normalized system of equations. Since by (1.4) and normality of l, d we have

$$d(r) = 1 - l(r) \quad (0 \leq r \leq R), \tag{1.9}$$

we only need to consider $(c(r), l(r), v(r), R)$. It is easy to verify that these unknowns satisfy the following system of equations:

$$c''(r) + \frac{2}{r}c'(r) = f(c(r))l(r) \quad (0 < r < R), \tag{1.10}$$

$$c'(0) = 0, \quad c(R) = 1, \tag{1.11}$$

$$v(r)l'(r) = [K_M(c(r)) - K_D(c(r))]l(r) - K_M(c(r))l^2(r) \quad (0 < r < R), \tag{1.12}$$

$$v'(r) + \frac{2}{r}v(r) = K_M(c(r))l(r) - \mu \quad (0 < r < R), \tag{1.13}$$

$$v(0) = 0, \tag{1.14}$$

$$v(R) = 0, \tag{1.15}$$

where

$$K_M(c) = K_B(c) + \mu. \tag{1.16}$$

Note that $c(r)$ and $l(r)$ should also satisfy the following conditions:

$$c(r) \geq 0, \quad 0 \leq l(r) \leq 1 \quad (0 \leq r \leq R). \tag{1.17}$$

Hence, we need to consider existence of solutions for the system of equations (1.10)–(1.15) subject to conditions (1.17). Note that (1.10)–(1.15) is a two-point free boundary problem of a 3-system of differential equations.

Due to the boundary condition (1.14) and the free-boundary condition (1.15), the equation (1.12) is singular (or degenerate) at the two boundary points $r = 0$ and $r = R$, which causes the main difficulty of this problem. It has some similar features as the problem studied by Cui and Friedman in [5], where unique existence of a stationary solution for the tumor model of Pettet *et al* [10] is established by using the shooting method with two shooting parameters. The problem studied in [5] has also four unknowns $c(r)$, $p(r)$, $v(r)$ and R, where $c(r)$, $v(r)$ and R have the same meaning as the corresponding unknowns in the present problem, while $p(r)$ denotes the normalized stationary density of proliferating cells (and $q(r) = 1 - p(r)$ is the normalized stationary density of quiescent cells). Equations for $p(r)$, $v(r)$ and R are similar (with some minor difference) to the corresponding equations for $l(r)$, $v(r)$ and R in the present problem, but the equation for $c(r)$ is different from (1.9), having the following form:

$$c''(r) + \frac{2}{r}c'(r) = f(c(r)) \quad (0 < r < R). \tag{1.18}$$

Since this equation does not contain other unknown functions, $c(r)$ can be solved as a function of R by using the boundary conditions (1.11). It follows that the number of differential equations in the problem studied in [5] can be reduced from 3 to 2. For the present problem, however, no similar reduction is available. This difference is crucial, because it determines that the shooting method used in [5] does not apply to the present problem. To see more clearly about this, we make a brief introduction to the shooting method used in [5]: Solving the equation (1.18) under the initial conditions $c(0) = \lambda$ and $c'(0) = 0$, we get a function $c(r) = c_\lambda(r)$ defined for $r \geq 0$, depending on a parameter λ. For every $\lambda \in (0, 1)$ there exists a unique $R_\lambda > 0$ such that $0 < c_\lambda(r) < 1$ for $0 \leq r < R_\lambda$ and $c_\lambda(R_\lambda) = 1$. Thus the mapping $\lambda \to R_\lambda$ transforms the problem of finding $R > 0$ such that the free boundary condition $v(R) = 0$ is satisfied into an equivalent problem of finding $\lambda \in (0, 1)$ such that $v(R_\lambda) = 0$. Replacing $c(r)$ with $c_\lambda(r)$ and temporarily removing the free boundary condition, we get an initial value problem for a 2-system of differential equations involving a parameter λ. Due to the initial condition $v(0) = 0$, one of the two equations is singular at the starting point $r = 0$. It turns out, as shown in [5, §5], that there exists a critical value $0 < \lambda_\infty < 1$ for λ, such that this initial value problem has a unique solution for each $\lambda_\infty < \lambda < 1$, whereas it has an 1-parameter family of solutions $(p_{\lambda\nu}, v_{\lambda\nu})$ ($\nu \in R$) for every $0 < \lambda < \lambda_\infty$. Analysis of *a priori* properties of the solution of the free boundary problem shows that the value of λ corresponding to the solution of the free boundary problem belongs to the interval $(0, \lambda_\infty)$. Thus the shooting method is performed in two steps: First find a special $\nu = \nu^*$ for every $\lambda \in (0, \lambda_\infty)$ such that $(p_\lambda, v_\lambda) = (p_{\lambda\nu^*}, v_{\lambda\nu^*})$ has the "best" approximation properties with the solution of the free boundary problem, and next find the value λ^* of λ such that $(p_{\lambda^*}, v_{\lambda^*})$ is the solution of the free boundary problem. The arguments of finding ν^* and λ^* are based a weak comparison result between different solutions of the initial value problem (see [5, Lemma 9.1]). For the problem (1.10)–(1.15), however, we cannot get similar comparison between different solutions due to complex relations among the three unknown functions $c(r)$, $l(r)$ and $v(r)$. This leads to failure of the two-parameter shooting method of [5] when it is applied to the problem (1.10)–(1.15). In this paper, we shall use a different approach to solve this problem. This new approach is based on the Schauder fixed point theorem. We explain the idea of this approach in the following paragraph.

By making the change of variables

$$\bar{r} = \frac{r}{R}, \quad \bar{c}(\bar{r}) = c(r), \quad \bar{l}(\bar{r}) = l(r), \quad \bar{v}(\bar{r}) = \frac{v(r)}{R}, \tag{1.19}$$

the two-point free boundary problem (1.10)–(1.15) is transformed into the following ordinary two-point boundary value problem (we omit all bars to simplify the notation):

$$c''(r) + \frac{2}{r}c'(r) = R^2 f(c(r))l(r) \quad (0 < r < 1), \tag{1.20}$$

$$c'(0) = 0, \quad c(1) = 1, \tag{1.21}$$

$$v(r)l'(r) = [K_M(c(r)) - K_D(c(r))]l(r) - K_M(c(r))l^2(r) \quad (0 < r < 1), \tag{1.22}$$

$$v'(r) + \frac{2}{r}v(r) = K_M(c(r))l(r) - \mu \quad (0 < r < 1), \tag{1.23}$$

$$v(0) = 0, \tag{1.24}$$

$$v(1) = 0. \tag{1.25}$$

To solve this problem, we introduce a mapping S: $(c, l) \to (\tilde{c}, \tilde{l})$ as follows: Given $(c, l) \in C[0,1] \times C[0,1]$ satisfying

$$\begin{gathered} 0 \le c(r) \le 1, \quad 0 \le l(r) \le 1 \quad (0 \le r \le 1), \quad c(1) = 1, \quad l(1) = 1, \quad \text{and} \\ c,\ l \ \text{ monotone nondecreasing,} \end{gathered} \tag{1.26}$$

we denote by $\tilde{c}(r, R)$ the unique solution of the problem

$$\begin{cases} \tilde{c}'' + \frac{2}{r}\tilde{c}' = R^2 l(r) f(\tilde{c}) \quad (0 < r < 1), \\ \tilde{c}'|_{r=0} = 0, \quad \tilde{c}|_{r=1} = 1 \end{cases} \tag{1.27}$$

(for any given $R > 0$), and by $v(r)$ the unique solution of the problem

$$\begin{cases} v' + \frac{2}{r}v = K_M(c(r))l(r) - 3\int_0^1 K_M(c(\rho))l(\rho)\rho^2 d\rho \quad (0 < r < 1), \\ v(0) = 0, \end{cases} \tag{1.28}$$

i.e.,

$$v(r) = \frac{1}{r^2}\int_0^r K_M(c(\rho))l(\rho)\rho^2 d\rho - r\int_0^1 K_M(c(\rho))l(\rho)\rho^2 d\rho \ \ (0 < r \le 1), \quad v(0) = 0. \tag{1.29}$$

It is clear that $v(1) = 0$. Moreover, the monotonicity of $c(r)$ and $l(r)$ ensures that $v(r) < 0$ for all $0 < r < 1$. It follows that the differential equation

$$v(r)\tilde{l}' = [K_M(\tilde{c}(r, R)) - K_D(\tilde{c}(r, R))]\tilde{l} - K_M(\tilde{c}(r, R))\tilde{l}^2 \quad (0 < r < 1) \tag{1.30}$$

has a unique solution satisfying similar conditions as $l(r)$, and we denote by $\tilde{l}(r, R)$ this solution. It can be proved that $\tilde{c}(r, R)$ and $\tilde{l}(r, R)$ are continuous and monotone decreasing in R, and

$$\lim_{R\to 0}\tilde{c}(r, R) = 1, \quad \lim_{R\to\infty}\tilde{c}(r, R) = 0, \quad \lim_{R\to 0}\tilde{l}(r, R) = 1 - \frac{K_D(1)}{K_M(1)}, \quad \lim_{R\to\infty}\tilde{l}(r, R) = 1 - \frac{K_D(0)}{K_M(0)}.$$

It follows that the function

$$F(R) = \int_0^1 K_M(\tilde{c}(r, R))\tilde{l}(r, R) r^2 dr$$

is continuous and monotone decreasing for $R > 0$, and

$$\lim_{R\to 0} F(R) = \frac{1}{3}(K_M(1) - K_D(1)) > \frac{1}{3}\mu, \quad \lim_{R\to\infty} F(R) = \frac{1}{3}(K_M(0) - K_D(0)) < \frac{1}{3}\mu.$$

Thus there exists a unique number $R^* > 0$ such that $F(R^*) = \frac{1}{3}\mu$. We define $\tilde{c}(r) = \tilde{c}(r, R^*)$, $\tilde{l}(r) = \tilde{l}(r, R^*)$, and set $S(c, l) = (\tilde{c}, \tilde{l})$. The mapping S is then defined. We shall prove that this mapping satisfies all conditions of the Schauder fixed point theorem when restricted to certain bounded closed convex subset of $C[0,1] \times C[0,1]$ satisfying the conditions (1.25), so that it has a fixed point. The fixed point of S clearly corresponds to a solution of the problem (1.19)–(1.24), so that the problem is solved.

The main result of this paper is the following:

Theorem 1.1 *Under the assumptions (A_1)–(A_5), the problem (1.9)–(1.14) has a solution $(c(r), l(r), v(r), R)$ $(R > 0)$, satisfying the following properties:*

$$
\begin{gathered}
0 < c(r) < 1 \quad (0 \le r < R), \quad c'(r) > 0 \quad (0 < r \le R), \\
0 < l(r) < 1 \quad (0 \le r < R), \quad l'(r) > 0 \quad (0 < r < R), \\
v(r) < 0 \quad (0 < r < R).
\end{gathered}
$$

The plan of the rest part is as follows: In §2 we study a singular differential equation arising from the equation (1.22). We shall investigate all solutions of it, choose one special solution which meets our later requirement, and establish some comparison and limit properties for this special solution. In §3 we make rigorous definition of the mapping S and establish some estimates to ensure that it is a continuous and compact mapping when restricted to some bounded closed convex subset of $C[0,1] \times C[0,1]$, which yields Theorem 1.1 by the Schauder fixed point theorem.

2 A Singular Differential Equation

In this section we study the following singular ordinary differential equation:

$$
v(r)y'(r) = [K_M(c(r)) - K_D(c(r))]y(r) - K_M(c(r))y^2(r) \quad (0 < r < 1), \tag{2.1}
$$

where $K_M(c)$, $K_D(c)$ are as before, $v(r)$, $c(r)$ are given functions satisfying the following conditions:

(B_1) $v \in C^1[0,1]$, $v(r) < 0$ for $0 < r < 1$, $v(0) = v(1) = 0$, $v'(0) < 0$, and $v'(1) > 0$.

(B_2) $c \in C^1[0,1]$, $0 < c(r) \le 1$ for $0 \le r \le 1$, $c'(r) > 0$ for $0 < r < 1$, and $c(1) = 1$.

We recall that $K_M(c) = K_B(c) + \mu$, where μ is a positive constant, and $K_B(c)$, $K_D(c)$ satisfy the conditions (A_2)–(A_5). Note that now $a = 1$ in the condition (A_4) by normalization.

For any $0 < r_0 < 1$ and any $-\infty < y_0 < \infty$, by the standard ODE theory we know that the equation (2.1) imposed with the initial condition $y(r_0) = y_0$ has a unique local solution $y(r)$ defined in some small interval $(r_0 - \delta, r_0 + \delta)$, where δ is a positive number. We extend this solution into a maximal open interval $(r_{min}, r_{max}) \subset (0,1)$. Since the equation (2.1) is non-singular at every point $0 < r < 1$, there are four possibilities:

(i) $r_{min} = 0$, $r_{max} = 1$;

(ii) $r_{min} = 0$, $0 < r_{max} < 1$;

(iii) $0 < r_{min} < 1$, $r_{max} = 1$;

(iv) $0 < r_{min} < r_{max} < 1$.

Later on we shall see that the fourth situation cannot occur, but temporally we still put it in our consideration. From the standard ODE theory we can further infer that the following assertions hold:

(a) $\lim_{r\to r_{max}^-} |y(r)| = \infty$ if (ii) occurs.

(b) $\lim_{r\to r_{min}^+} |y(r)| = \infty$ if (iii) occurs.

(c) $\lim_{r\to r_{min}^+} |y(r)| = \infty$ and $\lim_{r\to r_{max}^-} |y(r)| = \infty$ if (iv) occurs.

A solution of (2.1) defined in a maximal open interval will be called an *entire solution*. Note that $y(r) \equiv 0$ $(0 < r < 1)$ is an entire solution, which will be called a *trivial solution*. Any other entire solution will be called a *nontrivial entire solution*. In the sequel we study profiles of all nontrivial entire solutions of the equation (2.1).

We introduce a function m on $[0, 1]$ by defining

$$m(c) = 1 - \frac{K_D(c)}{K_M(c)} \quad (0 \le c \le 1). \tag{2.2}$$

The conditions (A_2)–(A_5) ensure the following properties of this function:

$$0 < m(c) < 1, \quad m'(c) > 0 \quad \text{for } \ 0 \le c < 1. \tag{2.3}$$

Lemma 2.1 *Let assumptions be as above and let $y(r)$ $(r_{min} < r < r_{max})$ be a nontrivial entire solution of* (2.1). *Then for any $r_0 \in (r_{min}, r_{max})$ the following assertion holds: $y'(r_0) > 0$ if either $y(r_0) > m(c(r_0))$ or $y(r_0) < 0$, while $y'(r_0) < 0$ if $0 < y(r_0) < m(c(r_0))$.*

Proof: Clearly,

$$[K_M(c(r_0))-K_D(c(r_0))]y(r_0)-K_M(c(r_0))y^2(r_0) \begin{cases} < 0 & \text{if either } \ y(r_0) > m(c(r_0)) \ \text{ or } \ y(r_0) < 0, \\ > 0 & \text{if } \ 0 < y(r_0) < m(c(r_0)). \end{cases}$$

Since $v(r_0) < 0$, by the equation (2.1) we see easily that the desired assertion holds. □

Using Lemma 2.1, the fact that $y(r) \equiv 0$ is a solution of (2.1) and the fact that different integral curves of (2.1) do not intersect each other (following from uniqueness of solutions), we can easily deduce that if $y(r)$ $(r_{min} < r < r_{max})$ is a nontrivial entire solution then it possesses one of the following four properties:

(i') $y(r) > m(c(r))$ and $y'(r) > 0$ for all $r \in (r_{min}, r_{max})$.

(ii') There exists a unique $r^* \in (r_{min}, r_{max})$ such that $y(r) > m(c(r))$ and $y'(r) > 0$ for $r \in (r_{min}, r^*)$, $0 < y(r) < m(c(r))$ and $y'(r) < 0$ for $r \in (r^*, r_{max})$, $y(r^*) = m(c(r^*))$, and $y'(r^*) = 0$.

(iii') $0 < y(r) < m(c(r))$ and $y'(r) < 0$ for all $r \in (r_{min}, r_{max})$.

(iv') $y(r) < 0$ and $y'(r) > 0$ for all $r \in (r_{min}, r_{max})$.

Thus, $\lim_{r\to r_{min}^+} y(r) = -\infty$ if $r_{min} > 0$, and $\lim_{r\to r_{max}^-} y(r) = \infty$ if $r_{max} < 1$. This further implies that the situation (iv) cannot occur.

Lemma 2.2 *Let assumptions be as before and let $y(r)$ $(r_{min} < r < r_{max})$ be a nontrivial entire solution of* (2.1). *Then the following assertions hold:*

(1) *If $r_{min} = 0$ then $\lim_{r\to 0^+} y(r) = m(c(0))$.*

(2) *If $r_{max} = 1$ then either $\lim_{r\to 1^-} y(r) = m(1)$ or $\lim_{r\to 1^-} y(r) = 0$.*

(3) *The solution satisfying $r_{max} = 1$ and $\lim_{r\to 1^-} y(r) = m(1)$ exists and is unique.*

Proof: Clearly, if $r_{min} = 0$ then we have either $\lim_{r\to 0^+} y(r) = -\infty$ or $\lim_{r\to 0^+} y(r)$ exists. We first prove that the case $\lim_{r\to 0^+} y(r) = -\infty$ cannot occur. Indeed, if this case occurs then we can find a $\delta > 0$ sufficiently small, such that the right-hand side of (2.1) is not larger than $-\frac{1}{2}K_M(c(r))y^2(r)$ for $r \in (0, \delta]$, implying that

$$|v(r)|y'(r) \geq \frac{1}{2}K_M(c(r))y^2(r) \quad \text{for } \ r \in (0, \delta].$$

This further implies, by the condition (B_1), that

$$\frac{y'(r)}{y^2(r)} \geq Cr^{-1} \quad \text{for } \ r \in (0, \delta],$$

for some constant $C > 0$. Integrating this inequality yields

$$\frac{1}{y(r)} - \frac{1}{y(\delta)} \geq C \log \frac{\delta}{r} \quad \text{for } \ r \in (0, \delta].$$

Letting $r \to 0^+$, we get a contradiction. Hence, $\lim_{r\to 0^+} y(r)$ exists. In the sequel we prove that

$$\lim_{r\to 0^+} y(r) = m(c(0)). \tag{2.4}$$

We denote $A = \lim_{r\to 0^+} y(r)$. Since $y(r)$ is monotone decreasing if $0 < y(r) < m(c(r))$ and monotone increasing if $y(r) < 0$, we see that $A \neq 0$. If (2.4) is not true then

$$\lim_{r\to 0^+} \{[K_M(c(r)) - K_D(c(r))] - K_M(c(r))y(r)\} = [K_M(c(0)) - K_D(c(0))] - K_M(c(0))A \equiv b \neq 0.$$

From the condition (B_1) we know that $v(r) = -ar(1 + o(1))$ as $r \to 0^+$, where $a = |v'(0)| > 0$. It follows from (2.1) that

$$\frac{y'(r)}{y(r)} = -\frac{b}{a}\frac{(1 + o(1))}{r} \quad \text{as } \ r \to 0^+.$$

Integrating this equality over the interval $[r, \delta]$ gives us the relation

$$\log|y(\delta)| - \log|y(r)| = -\frac{b}{a}\int_r^\delta \frac{1 + o(1)}{\rho}d\rho \quad \text{for } \ r \in (0, \delta].$$

Letting $r \to 0^+$ we get a contradiction. This proves the assertion (1). The proof of the assertion (2) is similar.

Next we prove the assertion (3). We need to prove that if $\delta > 0$ is sufficiently small then the initial value problem

$$\begin{cases} v(r)y'(r) = [K_M(c(r)) - K_D(c(r))]y(r) - K_M(c(r))y^2(r) \quad (1 - \delta \leq r < 1), \\ y(1) = m(1) \end{cases} \tag{2.5}$$

has a unique solution $y \in C[1 - \delta] \cap C^1[1 - \delta, 1)$. We make the transformation of unknown functions $y \to z$ by letting

$$y(r) = m(1) + z(r).$$

Then the problem (2.5) is transformed into the following equivalent problem:

$$\begin{cases} z'(r) - \frac{\alpha}{1-r} z(r) = f_0(r) \cdot \frac{z(r)}{1-r} + f_1(r) \cdot \frac{z^2(r)}{1-r} + f_2(r) z(r) + f_3(r) \quad (1-\delta \le r < 1), \\ z(1) = 0, \end{cases} \tag{2.6}$$

where

$$\alpha = \frac{K_M(1) - K_D(1)}{v'(1)} > 0, \tag{2.7}$$

$$f_0(r) = -\frac{\alpha v'(1)(1-r)}{v(r)} - \alpha, \quad f_1(r) = \frac{K_M(c(r))(1-r)}{v(r)}.$$

$$f_2(r) = \frac{[K_M(c(r)) - K_D(c(r))] - 2K_M(c(r))m(1) + \alpha v'(1)}{v(r)},$$

and

$$f_3(r) = \frac{[K_M(c(r)) - K_D(c(r))]m(1) - K_M(c(r))m^2(1)}{v(r)},$$

It is immediate to see that $f_j \in C[1-\delta, 1]$ (for small $\delta > 0$, $j = 0, 1, 2, 3$) and

$$f_0(1) = \lim_{r \to 1^-} f_0(r) = 0. \tag{2.8}$$

Since $\alpha > 0$ (by (2.7)), one can easily verify that the problem (2.6) is equivalent to the following integral equation:

$$z(r) = -\frac{1}{(1-r)^\alpha} \int_r^1 f_0(\rho) z(\rho)(1-\rho)^{\alpha-1} d\rho - \frac{1}{(1-r)^\alpha} \int_r^1 f_1(\rho) z^2(\rho)(1-\rho)^{\alpha-1} d\rho$$
$$-\frac{1}{(1-r)^\alpha} \int_r^1 f_2(\rho) z(\rho)(1-\rho)^{\alpha} d\rho - \frac{1}{(1-r)^\alpha} \int_r^1 f_3(\rho)(1-\rho)^{\alpha} d\rho. \tag{2.9}$$

Using the facts $f_j \in C[1-\delta, 1]$ ($j = 0, 1, 2, 3$) and in particular the relation (2.8), we can use the Banach fixed point theorem to prove that the equation (2.9) has a unique solution in $C[1-\delta, 1]$ when δ is sufficiently small. Hence the desired assertion follows. □

From Lemmas 2.1 and 2.2 we get the following result:

Theorem 2.3 *Let assumptions be as before. Then any nontrivial entire solution of the equation (2.1) belongs to one of the following five classes:*

(1) *The maximal existence open interval is of the form* $(0, r_{max})$, *where* $0 < r_{max} < 1$, $y'(r) > 0$, $y(r) > m(c(r))$ *for* $0 < r < r_{max}$, *and*

$$\lim_{r \to 0^+} y(r) = m(c(0)), \qquad \lim_{r \to r_{max}^-} y(r) = \infty.$$

(2) *The maximal existence open interval is of the form* $(0, 1)$, $y'(r) > 0$, $y(r) > m(c(r))$ *for* $0 < r < 1$, *and*

$$\lim_{r \to 0^+} y(r) = m(c(0)), \quad \lim_{r \to 1^-} y(r) = m(1).$$

(3) *The maximal existence open interval is of the form* $(0, 1)$, *and there exists a* $r_0 \in (0, 1)$ *such that* $y'(r) > 0$, $y(r) > m(c(r))$ *for* $0 < r < r_0$, $y'(r) < 0$, $0 < y(r) < m(c(r))$ *for* $r_0 < r < 1$, $y'(r_0) = 0$ *and* $y(r_0) = m(c(r_0))$. *Besides,*

$$\lim_{r \to 0^+} y(r) = m(c(0)), \quad \lim_{r \to 1^-} y(r) = 0. \tag{2.10}$$

(4) *The maximal existence open interval is of the form* $(0,1)$, $y'(r) < 0$, $0 < y(r) < m(c(r))$ *for* $0 < r < 1$, *and the relations (2.10) hold.*

(5) *The maximal existence open interval is of the form* $(r_{min}, 1)$, *where* $0 < r_{min} < 1$, $y'(r) > 0$, $y(r) < 0$ *for* $r_{min} < r < 1$, *and*

$$\lim_{r\to r_{min}^+} y(r) = -\infty \quad \text{and} \quad \lim_{r\to 1^-} y(r) = 0.$$

Moreover, all the five classes of entire solutions exist, and the entire solution of the class (2) *is unique, whereas all the other four classes of entire solutions are infinite, with each class forming an ordered 1-parameter family.*

Existence and uniqueness of an entire solution of the class (2) follows immediately from Lemma 2.2. Existence of the other classes of entire solutions and the assertion that each of these classes forms an ordered 1-parameter family can be proved by making rigorous analysis of initial value problems of (2.1) at initial points $r = 0$ and $r = 1$, following a similar argument as that in [5, §5]. Since we shall only use solutions of the class (2), we omit the proof here.

Later on we shall call the unique entire solution of the class (2) as *the admissible entire solution*, or simply *the admissible solution*.

The following comparison result will play an important role in our later analysis:

Theorem 2.4 *Let* $(v_1(r), c_1(r))$, $(v_2(r), c_2(r))$ $(0 \le r \le 1)$ *be two pairs of functions satisfying the conditions* (B_1) *and* (B_2), *and let* $y_1(r)$, $y_2(r)$ *be respectively the unique admissible solutions of the equations*

$$v_1(r)y_1'(r) = [K_M(c_1(r)) - K_D(c_1(r))]y_1(r) - K_M(c_1(r))y_1^2(r) \quad (0 < r < 1), \tag{2.11}$$

$$v_2(r)y_2'(r) = [K_M(c_2(r)) - K_D(c_2(r))]y_2(r) - K_M(c_2(r))y_2^2(r) \quad (0 < r < 1). \tag{2.12}$$

Assume that

$$v_1(r) \ge v_2(r), \quad c_1(r) \le c_2(r) \quad \textit{for} \quad 0 \le r \le 1.$$

Then $y_1(r) \le y_2(r)$ *for* $0 < r < 1$.

Proof: Since $y_1(0+) = m(c_1(0))$ and $y_2(0+) = m(c_2(0))$, we see that $y_1(0+) \le y_2(0+)$. Similarly we also have $y_1(1-) \le y_2(1-)$. Thus if the inequality $y_1(r) \le y_2(r)$ does not hold for all $0 < r < 1$, then we can find an open interval $(r_1, r_2) \subseteq (0,1)$ such that $y_1(r) > y_2(r)$ for $r_1 < r < r_2$, and $y_1(r_1) = y_2(r_1)$, $y_1(r_2) = y_2(r_2)$. Let $r_0 \in (r_1, r_2)$ be a point such that $y_1(r_0) - y_2(r_0) = \sup_{r_1<r<r_2}(y_1(r) - y_2(r))$. Then

$$y_1'(r_0) = y_2'(r_0) \quad \text{and} \quad y_1(r_0) > y_2(r_0).$$

The latter inequality implies that the right-hand side of (2.11) is less than the right-hand side of (2.12) at the point $r = r_0$, so that $v_1(r_0)y_1'(r_0) < v_2(r_0)y_2'(r_0)$. On the other hand, since $v_1(r_0) \ge v_2(r_0)$ and $y_1'(r_0) = y_2'(r_0) > 0$, we have $v_1(r_0)y_1'(r_0) \ge v_2(r_0)y_2'(r_0)$, which is a contradiction. Hence the desired assertion follows. □

We shall also need the following result:

Theorem 2.5 *Let* $(v_j(r), c_j(r))$ $(j = 1, 2, \cdots,\ 0 \le r \le 1)$ *be a sequence of function pairs satisfying the conditions* (B_1) *and* (B_2), *and for every* j *let* $y_j(r)$ *be the unique admissible solution of the equation*

$$v_j(r)y_j'(r) = [K_M(c_j(r)) - K_D(c_j(r))]y_j(r) - K_M(c_j(r))y_j^2(r) \quad (0 < r < 1). \tag{2.13}$$

Assume that

$$\lim_{j\to\infty} v_j(r) = v(r), \quad \lim_{j\to\infty} c_j(r) = c(r) \quad \text{uniformly for } 0 \le r \le 1. \tag{2.14}$$

Assume further that there exist two pairs of functions $(\bar{v}(r), \bar{c}(r))$, $(\hat{v}(r), \hat{c}(r))$ *satisfying the conditions* (B_1) *and* (B_2), *such that for every* j *there hold*

$$\bar{v}(r) \le v_j(r) \le \hat{v}(r), \quad \bar{c}(r) \le c_j(r) \le \hat{c}(r) \quad (0 \le r \le 1). \tag{2.15}$$

Then

$$\lim_{j\to\infty} y_j(r) = y(r) \quad \text{uniformly for } 0 < r < 1, \tag{2.16}$$

where $y(r)$ *is the unique admissible solution of the equation*

$$v(r)y'(r) = [K_M(c(r)) - K_D(c(r))]y(r) - K_M(c(r))y^2(r) \quad (0 < r < 1). \tag{2.17}$$

Proof: By a similar argument as in the proof of [5, Theorem 8.1] we can prove that $\lim_{j\to\infty} y_j(r) = y(r)$ pointwisely for $0 < r < 1$. What we need to prove is that this limit relation actually holds uniformly. The argument is given below.

First, from (2.15) we see that for every $\delta > 0$ sufficiently small, there holds $|v_j(r)| \ge c_\delta > 0$ for all $r \in [\delta, 1-\delta]$ and every j. It follows from (2.14) and a standard argument as in the classical ODE theory that

$$\lim_{j\to\infty} y_j(r) = y(r) \quad \text{uniformly for } \delta \le r \le 1-\delta. \tag{2.18}$$

Next, from (2.14) it follows that for every $\varepsilon > 0$ we can find a corresponding integer $N > 0$ such that

$$c(r) - \varepsilon \le c_j(r) \le c(r) + \varepsilon \quad \text{for } 0 \le r \le 1 \text{ and } j \ge N.$$

We denote $\bar{c}_\varepsilon(r) = \max\{c(r) - \varepsilon, \bar{c}(r)\}$, $\hat{c}_\varepsilon(r) = \min\{c(r) + \varepsilon, \hat{c}(r)\}$. Then we have

$$\bar{c}_\varepsilon(r) \le c_j(r) \le \hat{c}_\varepsilon(r) \quad \text{for } 0 \le r \le 1 \text{ and } j \ge N. \tag{2.19}$$

Using Theorem 2.4 we get

$$\bar{y}_\varepsilon(r) \le y_j(r) \le \hat{y}_\varepsilon(r) \quad \text{for } 0 \le r \le 1 \text{ and } j \ge N, \tag{2.20}$$

where $\bar{y}_\varepsilon(r)$ and $\hat{y}_\varepsilon$ are respectively the unique admissible solutions of the following two equations

$$\hat{v}(r)\bar{y}_\varepsilon'(r) = [K_M(\bar{c}_\varepsilon(r)) - K_D(\bar{c}_\varepsilon(r))]\bar{y}_\varepsilon(r) - K_M(\bar{c}_\varepsilon(r))\bar{y}_\varepsilon^2(r) \quad (0 < r < 1),$$

$$\bar{v}(r)\hat{y}_\varepsilon'(r) = [K_M(\hat{c}_\varepsilon(r)) - K_D(\hat{c}_\varepsilon(r))]\hat{y}_\varepsilon(r) - K_M(\hat{c}_\varepsilon(r))\hat{y}_\varepsilon^2(r) \quad (0 < r < 1).$$

Since

$$\lim_{r\to 0^+} \bar{y}_\varepsilon(r) = m(\bar{c}_\varepsilon(0)), \quad \lim_{r\to 1^-} \bar{y}_\varepsilon(r) = m(\bar{c}_\varepsilon(1)) = m(1),$$

$$\lim_{r\to 0^+} \hat{y}_\varepsilon(r) = m(\hat{c}_\varepsilon(0)), \quad \lim_{r\to 1^-} \hat{y}_\varepsilon(r) = m(\hat{c}_\varepsilon(1)) = m(1),$$

and $\lim_{\varepsilon\to 0} \bar{c}_\varepsilon(0) = \lim_{\varepsilon\to 0} \hat{c}_\varepsilon(0) = c(0)$, the desired assertion follows easily from (2.18) and (2.20): For any given $\sigma > 0$ we first find an $\varepsilon > 0$ and a $\delta > 0$ such that

$$\hat{y}_\varepsilon(r) - \bar{y}_\varepsilon(r) \le \sigma \quad \text{for } r \in (0, 2\delta] \cup [1 - 2\delta, 1).$$

By (2.20), this implies that there exists a positive integer N such that for any $j \geq N$ there holds

$$|y_j(r) - y(r)| \leq \sigma \quad \text{for} \quad r \in (0, 2\delta] \cup [1-2\delta, 1).$$

By (2.18), we can replace N with another larger integer so that the above inequality holds for any $r \in (0,1)$. This completes the proof. □

Later on we shall regard the admissible solution of (2.1) as defined on the closed interval $[0,1]$, with $y(0) = m(c(0))$ and $y(1) = m(1$.

3 The Proof of Theorem 1.1

In this section we prove Theorem 1.1 by using the idea introduced in §1.

3.1 Definition of the mapping S

Let $f(c)$, $K_M(c)$ be as before. Given a pair of functions $(c, l) \in C[0,1] \times C[0,1]$ satisfying the condition (1.26), we denote by $\tilde{c}(r, R)$ the unique solution of the boundary value problem (1.27) (for any given $R \geq 0$), and by $v(r)$ the unique solution of the non-local initial value problem (1.28).

Lemma 3.1 *Under the assumptions in (1.26), for any given $R \geq 0$ the problem (1.27) has a unique solution $\tilde{c} = \tilde{c}(r, R)$, satisfying the following conditions:*

(1) *$\tilde{c}(\cdot, R) \in C^2[0,1]$ for fixed $R \geq 0$, and*

$$0 < \tilde{c}(r, R) \leq 1 \quad \text{for} \quad 0 \leq r \leq 1, \qquad \frac{\partial \tilde{c}(r,R)}{\partial r} > 0 \quad \text{for} \quad 0 < r \leq 1. \tag{3.1}$$

(2) *$\tilde{c}(r, \cdot) \in C^1[0, \infty)$ for fixed $0 \leq r \leq 1$, or more precisely, $\tilde{c} \in C^1([0,\infty), C^2[0,1])$, and*

$$\frac{\partial \tilde{c}(r,R)}{\partial R} < 0 \quad \text{for} \quad 0 \leq r < 1 \quad \text{and} \quad R \geq 0, \tag{3.2}$$

$$\tilde{c}(r, 0) = 1 \quad \text{for} \quad 0 \leq r \leq 1, \qquad \lim_{R \to \infty} \tilde{c}(r, R) = 0 \quad \text{for} \quad 0 \leq r < 1. \tag{3.3}$$

Proof: See [4, Lemma 2.1]. □

Lemma 3.2 *Assume that (1.26) holds and $c(r)$, $l(r)$ are not simultaneously constant. Then the unique solution $v \in C^1[0,1]$ of the problem (1.28), given by (1.29), satisfying the following properties:*

$$v(r) < 0 \quad \text{for} \quad 0 < r < 1, \quad v(0) = v(1) = 1, \quad v'(0) < 0, \quad \text{and} \quad v'(1) > 0. \tag{3.4}$$

Proof: It is clear that $v \in C^1[0,1]$, $v(0) = 0$, and $v(1) = 0$. A simple computation shows that

$$v'(0) = \frac{1}{3} K_M(c(0)) l(0) - \int_0^1 K_M(c(\rho)) l(\rho) \rho^2 d\rho < 0$$

and

$$v'(1) = K_M(c(1)) l(1) - 3 \int_0^1 K_M(c(\rho)) l(\rho) \rho^2 d\rho > 0,$$

because the function $K_M(c(r))l(r)$ is monotone nondecreasing and nonconstant. The assertion that $v(r) < 0$ for $0 < r < 1$ follows from the following preliminary result:

Lemma 3.3 *Assume that $u \in C[0,1]$ and it is monotone nondecreasing and nonconstant. Then the function*

$$g(r) = \frac{1}{r^3}\int_0^r u(\rho)\rho^2 d\rho - \int_0^1 u(\rho)\rho^2 d\rho \quad (0 < r \le 1)$$

is negative for all $0 < r < 1$.

Proof: Since $u(r)$ is monotone nondecreasing and nonconstant, a simple computation shows that $g'(r) \ge 0$ for $0 < r < 1$ and $g'(1) > 0$. Since $g(1) = 0$, the desired assertion follows. □

Having obtained the solutions $\tilde{c}(r,R)$, $v(r)$ of the problems (1.27) and (1.28), we now consider the equation (3.5). We first assume that $c(r)$ and $l(r)$ are not simultaneously constant. Then Lemmas 3.1 and 3.2 ensure that the conditions (B_1) and (B_2) are respectively satisfied by $v(r)$ and $\tilde{c}(r,R)$ (for fixed $R \ge 0$). Thus the equation (1.30) has a unique admissible solution which we denote as $\tilde{l}(r,R)$, satisfying the following properties:

$$m(\tilde{c}(r,R)) < \tilde{l}(r,R) < 1 \quad \text{and} \quad \frac{\partial \tilde{l}(r,R)}{\partial r} > 0 \quad \text{for} \quad 0 < r < 1, \tag{3.5}$$

$$\lim_{r\to 0^+} \tilde{l}(r,R) = m(\tilde{c}(0,R)), \qquad \lim_{r\to 1^-} \tilde{l}(r,R) = m(1). \tag{3.6}$$

In the special case that both $c(r)$ and $l(r)$ are constant, we still let $\tilde{c}(r,R)$ be the unique solution of (1.27), but define $\tilde{l}(r,R) = m(\tilde{c}(r,R))$. Then (1.30) is still satisfied because $v(r) \equiv 0$.

Lemma 3.4 *For any fixed $0 < r < 1$, $\tilde{l}(r,R)$ is continuous and monotone non-increasing for $R \in [0,\infty)$, $\tilde{l}(r,0) = m(1)$, and*

$$\lim_{R\to\infty} \tilde{l}(r,R) = m(0). \tag{3.7}$$

Proof: The equation (1.30) is a regular ordinary differential equation with a parameter R when restricted to the interval $[\delta, 1-\delta]$ for any given $0 < \delta < \frac{1}{2}$. Thus it follows immediately from the standard ODE theory that $\tilde{l}(r,R)$ is continuous in R for any fixed $0 < r < 1$. The monotonicity of $\tilde{l}(r,R)$ in R follows from Theorem 2.4 and Lemma 3.1. The relation $\tilde{l}(r,0) = m(1)$ for all $0 < r < 1$ follows from the fact that $\tilde{c}(r,0) \equiv 1$ and from similar arguments as in the proofs of Lemmas 2.1, 2.2 and Theorem 2.3. In the sequel we give the proof of the relation (3.7).

Multiplying the equation (1.30) with r^2, integrating over $[0,r]$ for an arbitrary $0 < r < 1$, and using the equation (1.28), we get the integral equation

$$\begin{aligned} v(r)\tilde{l}(r,R) =& \tfrac{1}{r^2}\textstyle\int_0^r [K_M(\tilde{c}(\rho,R)) - K_D(\tilde{c}(\rho,R))]\tilde{l}(\rho,R)\rho^2 d\rho - \tfrac{1}{r^2}\int_0^r K_M(\tilde{c}(\rho,R))\tilde{l}^2(\rho,R)\rho^2 d\rho \\ &+\tfrac{1}{r^2}\textstyle\int_0^r K_M(c(\rho))l(\rho)\tilde{l}(\rho,R)\rho^2 d\rho - \tfrac{3}{r^2}\int_0^r \tilde{l}(\rho,R)\rho^2 d\rho \cdot \int_0^1 K_M(c(\rho))l(\rho)\rho^2 d\rho. \end{aligned} \tag{3.8}$$

Conversely, it is immediate to verify that if (for fixed R) a function $\tilde{l}(\cdot,R) \in C[0,1]$ satisfies the integral equation (3.8), then it also satisfies the differential equation (1.30). More generally, we can use a similar argument as in the proof of [5, Theorem 8.1] to show that if a function $\tilde{l}(\cdot,R) \in L^\infty[0,1]$ satisfies the equation (3.8), then $\tilde{l}(\cdot,R) \in C[0,1]$, so that it also satisfies the equation (1.30). Hence equations (1.30) and (3.8) are equivalent. Now, since $\tilde{l}(r,R)$ is monotone in R and uniformly bounded, we infer that, for any fixed $0 < r < 1$, the limit $\tilde{l}(r, \infty) \equiv \lim_{R\to\infty} \tilde{l}(r, R)$ exists. By letting $R \to \infty$ in (3.8), using (3.3) and the equivalence assertion we have just stated, we see that $\tilde{l}(r, \infty)$ is a solution of the equation

$$v(r)\frac{d\tilde{l}(r,\infty)}{dr} = [K_M(0) - K_D(0)]\tilde{l}(r,\infty) - K_M(0)\tilde{l}^2(r,\infty).$$

Moreover, since $\tilde{l}(r, R) \geq m(\tilde{c}(r, R)) \geq m(0)$ for any $0 \leq r \leq 1$ and $R \geq 0$, we have $\tilde{l}(r, \infty) \geq m(0)$ for any $0 \leq r \leq 1$. By using similar arguments as in the proofs of Lemmas 2.1, 2.2 and Theorem 2.3, it is not hard to verify that the unique monotone non-decreasing solution of the above equation defined for all $0 < r < 1$ and not less than $m(0)$ is the constant solution $\tilde{l}(r, \infty) = m(0)$. Hence (3.7) holds. This completes the proof. □

We define

$$F(R) = \int_0^1 K_M(\tilde{c}(r, R))\tilde{l}(r, R)r^2 dr \quad \text{for} \quad R \geq 0. \tag{3.9}$$

By Lemmas 3.1 and 3.4, $F(R)$ is continuous and monotone non-increasing for $R \geq 0$, and

$$F(0) = \int_0^1 K_M(1)m(1)r^2 dr = \frac{1}{3}(K_B(1) + \mu - K_D(1)) > \frac{1}{3}\mu,$$

$$\lim_{R\to\infty} F(R) = \int_0^1 K_M(0)m(0)r^2 dr = \frac{1}{3}(\mu - K_D(0)) < \frac{1}{3}\mu.$$

Thus there exists a unique $\tilde{R} > 0$ such that $F(\tilde{R}) = \frac{1}{3}\mu$. We denote

$$\tilde{c}(r) = \tilde{c}(r, \tilde{R}), \quad \tilde{l}(r) = \tilde{l}(r, \tilde{R}).$$

Then we get a mapping S: $(c, l) \to (\tilde{c}, \tilde{l})$. Note that, by (1.30), we have

$$v(r)\tilde{l}'(r) = [K_M(\tilde{c}(r)) - K_D(\tilde{c}(r))]\tilde{l}(r) - K_M(\tilde{c}(r))\tilde{l}^2(r) \quad (0 < r < 1), \tag{3.10}$$

and the equation $F(\tilde{R}) = \frac{1}{3}\mu$ can be rewritten as

$$\int_0^1 K_M(\tilde{c}(r))\tilde{l}(r)r^2 dr = \frac{1}{3}\mu. \tag{3.11}$$

It is clear that S maps the subset of $C[0, 1] \times C[0, 1]$ consisting of functions satisfying the condition (1.26) into itself. From (1.27), (1.28), (3.4), (3.10) and (3.11) it is also clear that a fixed point of S corresponds to a solution of the problem (1.20)–(1.25).

3.2 Preliminary estimates

For any given $R \geq 0$, we denote by $\bar{c}(r, R)$ $(0 \leq r \leq 1)$ the solution of the boundary value problem

$$\begin{cases} \bar{c}'' + \frac{2}{r}\bar{c}' = R^2 f(\bar{c}) \quad (0 < r < 1), \\ \bar{c}'|_{r=0} = 0, \quad \bar{c}|_{r=1} = 1. \end{cases} \tag{3.12}$$

We define

$$\bar{F}(R) = \int_0^1 K_M(\bar{c}(r, R))m(\bar{c}(r, R))r^2 dr \quad \text{for} \quad R \geq 0. \tag{3.13}$$

By Lemma 3.1, $\bar{F}(R)$ is continuous and monotone non-increasing in R, and

$$\bar{F}(0) = \int_0^1 K_M(1)m(1)r^2dr = \frac{1}{3}(K_B(1) + \mu - K_D(1)) > \frac{1}{3}\mu,$$

$$\lim_{R\to\infty} \bar{F}(R) = \int_0^1 K_M(0)m(0)r^2dr = \frac{1}{3}(\mu - K_D(0)) < \frac{1}{3}\mu.$$

Thus there exists a unique number $R_0 > 0$ such that $\bar{F}(R_0) = \frac{1}{3}\mu$. Note that R_0 depends only on the functions f, K_B, K_D and the number μ; it does not depends on any other variables.

Since $l(r) \leq 1$, by a standard comparison argument we have

$$\tilde{c}(r,R) \geq \bar{c}(r,R) \tag{3.14}$$

(for any $0 \leq r \leq 1$ and $R \geq 0$). This implies that

$$\tilde{l}(r,R) \geq m(\tilde{c}(r,R)) \geq m(\bar{c}(r,R)) \tag{3.15}$$

(also for any $0 \leq r \leq 1$ and $R \geq 0$). Hence, for any $0 \leq R < R_0$ we have

$$F(R) > F(R_0) \geq \bar{F}(R_0) = \frac{1}{3}\mu.$$

Thus the following lemma holds:

Lemma 3.5 $\tilde{R} \geq R_0$.

Next, we have

$$\begin{aligned} v(r) &= \tfrac{1}{r^2}\textstyle\int_0^r K_M(c(\rho))l(\rho)\rho^2d\rho - r\int_0^1 K_M(c(\rho))l(\rho)\rho^2d\rho \\ &= (\tfrac{1}{r^2} - r)\textstyle\int_0^r K_M(c(\rho))l(\rho)\rho^2d\rho - r\int_r^1 K_M(c(\rho))l(\rho)\rho^2d\rho \\ &\geq -r\textstyle\int_r^1 K_M(c(\rho))l(\rho)\rho^2d\rho \geq -r\int_r^1 K_M(1)\rho^2d\rho \geq -K_M(1)r(1-r). \end{aligned}$$

Hence, defining

$$v_0(r) = -K_M(1)r(1-r), \tag{3.16}$$

we get

Lemma 3.6 $v(r) \geq v_0(r)$ *for* $0 \leq r \leq 1$.

We now assume that

$$l(r) \geq m(0) = 1 - \frac{K_D(0)}{\mu} \quad \text{for} \quad 0 \leq r \leq 1. \tag{3.17}$$

Note that $m(0) > 0$ (by the assumption (A_5)). By a standard comparison argument we have

$$\tilde{c}(r,R) \leq \hat{c}(r,R) \quad \text{for} \quad 0 \leq r \leq 1, \quad R \geq 0, \tag{3.18}$$

where, for fixed $R \geq 0$, $\hat{c}(r,R)$ is the solution of the boundary value problem

$$\begin{cases} \hat{c}'' + \frac{2}{r}\hat{c}' = R^2 m(0) f(\hat{c}) \quad (0 < r < 1), \\ \hat{c}'|_{r=0} = 0, \quad \hat{c}|_{r=1} = 1. \end{cases} \tag{3.19}$$

By Lemma 3.1 we know that $\hat{c}(r,R)$ is monotone decreasing in R, and

$$\lim_{R\to\infty}\hat{c}(r,R)=0,\quad \lim_{R\to 0^+}\hat{c}(r,R)=1$$

(for fixed $0\le r<1$). We denote by $\hat{l}(r,R)$ the admissible solution of the equation

$$v_0(r)\hat{l}'=[K_M(\hat{c}(r,R))-K_D(\hat{c}(r,R))]\hat{l}-K_M(\hat{c}(r,R))\hat{l}^2\quad (0<r<1). \tag{3.20}$$

Then by (3.18), Lemma 3.6 and Theorem 2.4 we get

$$\tilde{l}(r,R)\le\hat{l}(r,R)\quad\text{for}\ \ 0\le r\le 1,\ \ R\ge 0. \tag{3.21}$$

Hence, defining

$$\hat{F}(R)=\int_0^1 K_M(\hat{c}(r,R))\hat{l}(r,R)r^2dr\quad\text{for}\ \ R\ge 0, \tag{3.22}$$

we infer that $F(R)\le\hat{F}(R)$ for all $R\ge 0$. Similarly as before we have

$$\lim_{R\to 0^+}\hat{F}(R)=\frac{1}{3}K_M(1)m(1)>\frac{1}{3}\mu,\quad \lim_{R\to\infty}\hat{F}(R)=\frac{1}{3}K_M(0)m(0)<\frac{1}{3}\mu,$$

and $\hat{F}(R)$ is continuous and monotone decreasing. Thus there exists a unique number $R_*>0$ such that $\hat{F}(R_*)=\frac{1}{3}\mu$. It follows that $F(R_*)\le\frac{1}{3}\mu$, so that $\tilde{R}\le R_*$. Hence we have proved the following result:

Lemma 3.7 *Assume that (3.17) holds. Then* $\tilde{R}\le R_*$.

Note that similarly as for R_0, R_* also depends only on the functions F, K_B, K_D and the number μ. We denote

$$c_0(r)=\bar{c}(r,R_*),\quad c_*(r)=\hat{c}(r,R_0).$$

Then from (3.14), (3.18) and Lemmas 3.5, 3.7 we get the following result:

Lemma 3.8 *Assume that (3.17) holds. Then there hold estimates:*

$$c_0(r)\le\tilde{c}(r)\le c_*(r)\quad (0\le r\le 1). \tag{3.23}$$

Let $h(r)$ $(0\le r\le 1)$ be the solution of the boundary value problem

$$\begin{cases} h''(r)+\frac{2}{r}h'(r)=R_0m(0)f(c_0(r)) \quad (0<r<1),\\ h'(0)=0,\quad h(1)=1. \end{cases} \tag{3.24}$$

It is clear that $0<h(r)\le 1$ for $0\le r\le 1$, and $h'(r)>0$ for $0<r\le 1$. Let

$$\kappa=\min_{0\le c\le 1}K_M'(c)=\min_{0\le c\le 1}K_B'(c),$$

and let $v_*(r)$ $(0\le r\le 1)$ be the solution of the initial value problem

$$\begin{cases} v_*'(r)+\frac{2}{r}v_*(r)=\kappa m(0)h(r)-3\kappa m(0)\int_0^1 h(\rho)\rho^2d\rho \quad (0<r<1),\\ v_*(0)=0. \end{cases} \tag{3.25}$$

Obviously,

$$v_*(r) = \kappa m(0) r \Big\{ \frac{1}{r^3} \int_0^r h(\rho)\rho^2 d\rho - \int_0^1 h(\rho)\rho^2 d\rho \Big\}.$$

Thus using Lemma 3.3 we see readily that v_* satisfies the condition (B_1).

Lemma 3.9 *Assume that $c(r) - h(r)$ is monotone nondecreasing. Then $v(r) \le v_*(r)$ for $0 \le r \le 1$.*

Proof: We denote

$$u(r) = K_M(c(r))l(r) - \kappa m(0)h(r) \quad (0 \le r \le 1).$$

Then $u(r)$ is clearly nonconstant. We write

$$\begin{aligned} u(r) =& [K_M(c(r))l(r) - m(0)K_M(c(r))] + [m(0)K_M(c(r)) \\ & - \kappa m(0)c(r)] + [\kappa m(0)c(r) - \kappa m(0)h(r)]. \end{aligned}$$

From this expression we see immediately that $u(r)$ is also monotone nondecreasing. By Lemma 3.3, it follows that the function

$$g(r) = \frac{1}{r^3} \int_0^r u(\rho)\rho^2 d\rho - \int_0^1 u(\rho)\rho^2 d\rho$$

is negative for $0 < r < 1$. Since $v(r) = v_*(r) + rg(r)$, we get the desired assertion. □

3.3 Existence of a fixed point

We introduce a set E in $C[0,1] \times C[0,1]$ as follows:

$$\begin{aligned} E =& \{(c,l) \in C[0,1] \times C[0,1] : \ 0 \le c(r) \le 1, \ 0 \le l(r) \le 1 \ (0 \le r \le 1), \ c(1) = 1, \ l(1) = m(1), \\ & c(r) \ge c_0(r), \ l(r) \ge m(0) \ (0 \le r \le 1) \text{ and } c(r) - h(r), \ l(r) \text{ are monotone nondecreasing}\}. \end{aligned}$$

Clearly, E is a bounded closed convex subset of the Banach space $C[0,1] \times C[0,1]$.

Lemma 3.10 *The mapping S maps the set E into itself.*

Proof: Let $(c,l) \in E$ and let $(\tilde{c}, \tilde{l}) = S(c,l)$. It is clear that $(\tilde{c}, \tilde{l})$ satisfies the following conditions:

$$0 \le \tilde{c}(r) \le 1, \ 0 \le \tilde{l}(r) \le 1 \ (0 \le r \le 1), \ \tilde{c}(1) = 1, \ \tilde{l}(1) = m(1).$$

From (3.23) we see that $\tilde{c}(r) \ge c_0(r)$, and from the inequality $\tilde{l}(r) \ge m(\tilde{c}(r))$ (see (3.15)) we have $\tilde{l}(r) \ge m(0)$. Besides, it is clear that $\tilde{l}(r)$ is monotone nondecreasing. From (1.27) and (3.24) we respectively have

$$\tilde{c}'(r) = \frac{\tilde{R}^2}{r^2} \int_0^r l(\rho) f(\tilde{c}(\rho))\rho^2 d\rho \quad \text{and} \quad h'(r) = \frac{R_0^2}{r^2} \int_0^r m(0) f(c_0(\rho))\rho^2 d\rho.$$

Since $\tilde{R} \ge R_0$, $l(r) \ge m(0)$ and $\tilde{c}(r) \ge c_0(r)$, we see that $\tilde{c}'(r) - h'(r) \ge 0$. Hence $\tilde{c}(r) - h(r)$ is monotone nondecreasing. This proves that $(\tilde{c}, \tilde{l}) \in E$. □

Lemma 3.11 *The mapping S is continuous when restricted on E.*

Proof: Let $(c_j, l_j) \in E$ $(j = 1, 2, \cdots)$ and assume that $\lim_{j\to\infty}(c_j, l_j) = (c, l)$ in $C[0,1] \times C[0,1]$. Let $(\tilde{c}_j, \tilde{l}_j) = S(c_j, l_j)$, $j = 1, 2, \cdots$, and $(\tilde{c}, \tilde{l}) = S(c,l)$. We need to show that also

$\lim_{j\to\infty}(\tilde{c}_j, \tilde{l}_j) = (\tilde{c}, \tilde{l})$ in $C[0,1] \times C[0,1]$. In the sequel we use the same notation of variables as before to denote those variables related to (c, l), and use the notation with subscript j to denote the corresponding variables related to (c_j, l_j). By Lemmas 3.5–3.9 we have

$$R_0 \le \tilde{R}_j \le R_*, \quad c_0(r) \le \tilde{c}_j(r) \le c_*(r), \quad v_0(r) \le v_j(r) \le v_*(r) \quad (j = 1, 2, \cdots).$$

Clearly,

$$\lim_{j\to\infty} v_j(r) = v(r) \quad \text{uniformly for } 0 \le r \le 1,$$

and

$$\lim_{j\to\infty} \tilde{c}_j(r, R) = \tilde{c}(r, R) \quad \text{uniformly for } 0 \le r \le 1 \text{ and } R_0 \le R \le R_*.$$

This further implies, by a similar argument as in the proof of Theorem 2.5, that also

$$\lim_{j\to\infty} \tilde{l}_j(r, R) = \tilde{l}(r, R) \quad \text{uniformly for } 0 \le r \le 1 \text{ and } R_0 \le R \le R_*.$$

Thus

$$\lim_{j\to\infty} F_j(R) = F(R) \quad \text{uniformly for } R_0 \le R \le R_*.$$

It follows that $\lim_{j\to\infty} \tilde{R}_j = \tilde{R}$. Clearly, $\tilde{c}(r, R)$ is continuous in R uniformly for $0 \le r \le 1$. By Theorem 2.5, this further implies that also $\tilde{l}(r, R)$ is continuous in R uniformly for $0 \le r \le 1$. Using these facts and the expressions $\tilde{c}_j(r) = \tilde{c}_j(r, \tilde{R}_j)$, $\tilde{l}_j(r) = \tilde{l}_j(r, \tilde{R}_j)$, $\tilde{c}(r) = \tilde{c}(r, \tilde{R})$ and $\tilde{l}(r) = \tilde{l}(r, \tilde{R})$, one readily get the desired assertion. □

Lemma 3.12 *$S(E)$ is a pre-compact subset of $C[0,1] \times C[0,1]$.*

Proof: Let $(c_j, l_j) \in E$, $j = 1, 2, \cdots$. As before we use the notation with subscript j to denote various variables related to (c_j, l_j). For instance, $(\tilde{c}_j, \tilde{l}_j) = S(c_j, l_j)$, and v_j is the solution of the problem (1.28) when $c(r)$ and $l(r)$ there are respectively replaced with $c_j(r)$ and $l_j(r)$. By definition, $\tilde{c}_j$ satisfies the equation

$$\tilde{c}_j''(r) + \frac{2}{r}\tilde{c}_j'(r) = \tilde{R}_j^2 l_j(r) f(\tilde{c}_j(r)) \quad (0 < r < 1).$$

Since $R_0 \le \tilde{R}_j \le R_*$, $j = 1, 2, \cdots$, we see easily that $\tilde{c}_j''(r) + \frac{2}{r}\tilde{c}_j'(r)$ is uniformly bounded for $0 < r < 1$ and $j = 1, 2, \cdots$. It follows that $\{\tilde{c}_j\}$ has a subsequence which is uniformly convergent in $[0,1]$; for simplicity we use the same notation $\{\tilde{c}_j\}$ to denote this subsequence, and the corresponding subsequence of $\{v_j\}$ is also denoted by the same notation $\{v_j\}$. From (1.28) it is clear that $v_j'(r) + \frac{2}{r}v_j(r)$ is uniformly bounded for $0 < r < 1$ and $j = 1, 2, \cdots$. Thus $\{v_j\}$ has a subsequence which is uniformly convergent in $[0,1]$. From (3.10) we see that $\tilde{l}_j$ is the admissible solution of the equation

$$v_j(r)\tilde{l}_j'(r) = [K_M(\tilde{c}_j(r)) - K_D(\tilde{c}_j(r))]\tilde{l}_j(r) - K_M(\tilde{c}_j(r))\tilde{l}_j^2(r) \quad (0 < r < 1).$$

Thus by Theorem 2.5 we infer that the corresponding subsequence of $\{\tilde{l}_j\}$ is also uniformly convergent in $[0,1]$. Hence, we have proved that $\{(\tilde{c}_j, \tilde{l}_j)\}$ has a subsequence which is convergent in $C[0,1] \times C[0,1]$, so that $S(E)$ is pre-compact. □

From Lemma 10 – Lemma 12 and the Schauder fixed point theorem, we conclude that the mapping S has a fixed point in the set E, which, by what we pointed out earlier, corresponds to a solution of the problem (1.20)–(1.25). This proves Theorem 1.1.

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