GEOMETRIC INVARIANT THEORY AND BIRATIONAL GEOMETRY

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ABSTRACT. In this paper we will survey some recent developments in the last decade or so on variation of Geometric Invariant Theory and its applications to Birational Geometry such as the weighted Weak Factorization Theorems of nonsingular projective varieties and more generally projective varieties with finite quotient singularities. Along the way, we will also mention some progresses on birational geometry of hyperKähler manifolds as well as certain open problems and conjectures.

1. INTRODUCTION

Quotients of projective varieties by reductive algebraic groups arises naturally in many situations. The existence of many important moduli spaces, for example, is proved by expressing them as quotients. There are several quotient theories, among them, the Mumford geometric invariant theory (GIT) is the systematic one. A basic observation there is that many moduli functors can be, at least coarsely, represented by quotient varieties in the sense of GIT.

A remarkable discovery in the last decade is the deep connection and fruitful interaction between GIT and birational geometry (e.g., [12],[29],[33],[34],[66], [73]).

Moduli spaces, or GIT quotients are not unique in general. Hence a natural fundamental problem in this theory is to parameterize all the GIT quotients and describe exactly how a quotient changes as one varies the underlying parameter.

This fundamental question was first raised and answered in the joint work [12], which overlaps with the paper [66]. Here, one may wonder why the founder of the theory, in his famous work [57], did not mention such a fundamental question. But, it is really not that surprising given that the higher dimensional birational geometry witnessed a great advance only when Mori revolutionized the field many years after GIT was invented. Our solution to this fundamental problem is a successful mixture of the great works of the two Fields medalists.

Roughly speaking, the solution given by the theory of Variation of Geometric Invariant Theory quotients (VGIT) can be summarized as follows. For a given action, there are finitely many GIT quotients and they are parameterized by some natural chambers in the effective equivariant ample cone. Moreover, when crossing a wall in the chamber system, the corresponding quotient undergoes a birational transformation similar to a Mori flip.

This, for the first time, provided a significant link between VGIT and Mori program. Later ([33], [34],[29]), we found that the connection between the two is in fact deeper. To be short, we discovered that not only VGIT wall-crossing transformations provide special examples of transformations in birational geometry, but remarkably all the birational transformations in birational geometry can also be decomposed as sequences of VGIT wall-crossing transformations. This establishes a useful philosophy that began in the paper [33] and reconfirmed in [34] and [29]: Birational geometry is a special case of Variation of Geometric Invariant Theory.

The philosophy has two versions: global and local.

The global version says that for a given $\mathbb{Q}$-factorial projective variety and all its Mori flip images, they all can be realized as GIT quotients of certain action. If this holds, we will call the projective variety a *Mori dream space*. Not all $\mathbb{Q}$-factorial projective varieties are Mori dream space. In [33], we give a criterion for a Mori dream space in terms of the Cox section ring.

The local version says that any two smooth projective varieties, or more generally, any two projective varieties with finite quotient singularities related by a birational morphism can be realized as GIT quotients of a smooth projective variety by a reductive algebraic group. This turns out to be always true. For the smooth case, the GIT realization of a birational morphism was stated and proved by Hu and Keel in [34]. We were informed afterwards that it may also follow from Wlodarczyk's paper [73] (see also [2]). A consequence of the above is the GIT Weak Factorization Theorem which asserts that any two birational smooth projective varieties are related by a sequence of VGIT wall-crossing birational transformations. Note here that the (stronger) Weak Factorization Theorems for the smooth cases were proved by Wlodarczyk et. al., see [2] and the reference therein. A proof for singular cases was given in [29]. The singular version of the Weak Factorization Theorems asserts that any two birational projective varieties with finite quotient singularities are related by a sequence of VGIT wall-crossing birational transformations ([29]). Here, by a VGIT wall-crossing birational transformation, we mean the birational transformation as described in Theorem 4.2.7 of [12].

Besides the GIT approach, another way of compactification in moduli problem is Chow quotient. This approach to moduli problems is different and may be harder than GIT, but it has some advantages and potentially has several significant applications. Understanding the boundary of Chow quotient is the key to this approach. In [28], we give a characterization regarding the boundaries of Chow quotient, using a very intuitive and computable method, namely the Perturbation-Translation-Specialization relation. This characterization, for example, can be used to give symplectic and topological interpretations of Chow quotient.

Turning away from the birational geometry of general varieties, we now would like to focus on special varieties. HyperKähler Varieties form a sepcial and very important class of Calabi-Yau manifolds. Its birational geometry has received some considerable attentions recently ([8], [9], [37], [20], [72], among others). Being very restrictive in structures, birtaional transformations among HyperKähler Varieties are among simplest kinds. It was proved in [8], coupled with an improvement of [72], that any two birational HyperKähler varieties of dimension 4 are related by a sequence of Mukai elementary transformations. In higher dimensions, Hu and Yau ([37]) obtained several structure theorems for symplectic contractions, completing important initial steps toward the Hu-Yau conjecture on decomposing symplectic birational maps. We remark here that some of our results in this direction overlap with the work of [9]

2. Moduli, GIT, and Chow Quotient

2.1. **What are Moduli Spaces?** A moduli space is, roughly speaking, a parameter space for equivalence classes of certain geometric objects of fixed topological type. Depending on purposes, there are two types of moduli: fine moduli and coarse moduli. The former often does not exist as a (quasi-) projective scheme but lives naturally as a stack, while the latter frequently exists as a (quasi-) projective scheme.

A naive way to think of a moduli space M is

$$M = \{\text{Geometric Objects}\}/\sim$$

as a collection of the geometric objects of the interest modulo equivalence relation. This way of thinking is "coarse". Such a moduli space M often exists as a projective scheme.

To describe fine moduli, one needs moduli functor, a categorical language for the problem. A moduli functor is covariant functor

$$\mathcal{F} : \{\text{schemes}\} \to \{\text{sets}\}$$

which sends any scheme X to a family of the geometric objects parameterized by X. The moduli space M is fine, or the moduli functor $\mathcal{F}$ is represented by M if there is a universal family over M

$$U \to M$$

such that for any scheme X and a family of the geometric objects

$$Z \to X$$

parameterized by X, it corresponds to a morphism $f : X \to M$ so that the family $Z \to X$ is the pullback of the universal family via the morphism f

$$\begin{array}{ccc} Z = f^*U & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & M \end{array}$$

Fine moduli in general does not exist if some objects possess nontrivial automorphisms. Therefore we are somehow forced to consider coarse moduli if we prefer to work on projective schemes.

Now let us return to a coarse moduli as a parameter space of geometric objects

$$M = \{\text{Geometric Objects}\}/\sim .$$

This way of thinking, as it stands, naturally lead to a quotient theory. First we denote the collection of the objects as

$$X = \{\text{Geometric Objects}\}$$

and then we would introduce a natural group action

$$G \times X \to X$$

such that two geometric objects x, y are equivalent if and only if they, as points in X, are in the same group orbit:

$$x \sim y \text{ if and only if } G \cdot x = G \cdot y.$$

This is what the general idea is. However, taking quotients in algebraic geometry can be very subtle. Indeed, there are several theories about this:

- The Hilbert-Mumford Geometric Invariant Theory;
- Chow quotient varieties
- Hilbert quotient varieties;
- Artin, Kollár, Keel-Mori quotient spaces.

We will only focus on the first two: GIT quotients and Chow quotients.

2.2. **What are GIT quotients?**

2.2.1. *A toy example.* To give the reader some intuitive ideas about GIT quotients, let us *informally* consider a simple, yet quite informative "toy" example.

Let $G = \mathbb{C}^*$ act on $\mathbb{P}^2$ by

$$\lambda \cdot [x : y : z] = [\lambda x : \lambda^{-1} y : z].$$

Consider a map $\Phi : \mathbb{P}^2 \to \mathbb{R}$ given by

$$\Phi([x : y : z]) = \frac{|x|^2 - |y|^2}{|x|^2 + |y|^2 + |z|^2}.$$

This is the so-called moment map for the induced symplectic S^1-action with respect to the Fubibi-Study metric. Its image is the interval $[-1, 1]$.

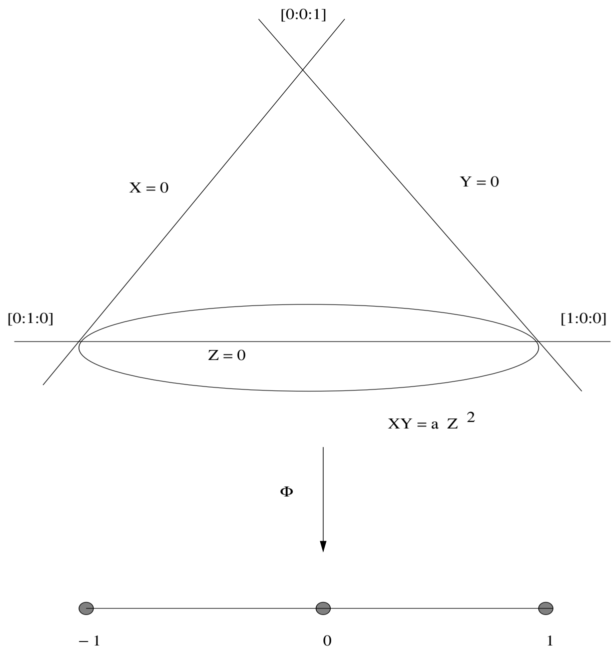

Figure 1. Conics

The $\mathbb{C}^*$ orbits are classified as follows. (See Figure 1 for an illustration.)

- Generic $\mathbb{C}^*$-orbits are conics $XY = aZ^2$ minus two points [1:0:0] and [0:1:0] for $a \neq 0, \infty$. We denote these orbits by $(XY = aZ^2)$.
  The moment map image of the orbit $(XY = aZ^2)$ is $(-1, 1)$.
- Other 1-dimensional orbits are the three coordinate lines $X = 0$, $Y = 0$, and $Z = 0$ minus the coordinate points on them. We denote these orbits by $(X = 0)$, $(Y = 0)$, and $(Z = 0)$.
  The moment map images of the orbits $(X = 0)$, $(Y = 0)$, and $(Z = 0)$ are $(-1, 0)$, $(0, 1)$, and $(-1, 1)$, respectively.
- Finally, the fixed points are the three coordinate points, [1:0:0], [0:1:0], and [0:0:1].
  Their moment map images are $1, -1$, and 0, respectively.

The reader can easily check that the ordinary topological orbit space $\mathbb{P}^2/\mathbb{C}^*$ is a nasty non-Hausdorff space. For example, the orbits $(X = 0)$ and $(Y = 0)$ are in the limits of the generic orbits $(XY = aZ^2)$ when a approaches 0, thus can not be separated.

The idea of GIT is to find some open subset $U \subset \mathbb{P}^2$, called the set of semistable points, such that a good quotient in a suitable sense, $U//\mathbb{C}^*$, exists. In this particular example, such open subsets are selected as follows.

The moment map Φ has three critical values $-1, 0$, and 1 which divide the interval into two top chambers $[-1, 0]$ and $[0, 1]$, and three 0-dimensional chambers $\{-1\}, \{0\}, \{1\}$. Each chamber C defines a GIT stability: a point $[x : y : z]$ is semi-stable with respect to the chamber C if

$$C \subset \Phi(\overline{\mathbb{C}^* \cdot [x : y : z]}),$$

and it is stable if the (relative) interior C° of C satisfies

$$C^\circ \subset \Phi(\mathbb{C}^* \cdot [x : y : z]) \text{ and } \dim \mathbb{C}^* \cdot [x : y : z] = 1.$$

(For a reference for this, see for example, [24].) Thus, for example, the orbit $(X = 0)$ is stable with respect to $[-1, 0]$, unstable with respect to $[0, 1]$; while the orbit $(Y = 0)$ is stable with respect to $[1, 0]$, unstable with respect to $[-1, 0]$. But, $(X = 0)$, $(Y = 0)$, and $[0 : 0 : 1]$ are all semi-stable with respect to the chamber $\{0\}$. Finally, observe that the generic orbits, namely the conics $(XY = aZ^2)$ are stable with respect to any chamber.

A general philosophy of GIT quotient is that it should parameterize orbits that are *closed* in the semi-stable locus. Here in this example, the GIT quotient $X_{[-1,0]}$ defined by the chamber $[-1, 0]$ parameterizes $(XY = aZ^2)$ $(a \neq 0, \infty)$, $(Z = 0)$, and $(X = 0)$. Thus $X_{[-1,0]}$ is isomorphic to $\mathbb{P}^1$ with $(Z = 0)$ and $(X = 0)$ serve as ∞ and 0, respectively. Likewise, the GIT quotient $X_{[0,1]}$ defined by the chamber $[0, 1]$ parameterizes $(XY = aZ^2)$, $(Z = 0)$, and $(Y = 0)$. And, the GIT quotient $X_{\{0\}}$ defined by the chamber $\{0\}$ parameterizes $(XY = aZ^2)$, $(Z = 0)$, and $[0 : 0 : 1]$.

All these quotients are isomorphic to $\mathbb{P}^1$, and hence they are all isomorphic to each other. This is very special because the dimension is too low (namely 1) to allow any variation. In general, they should be quite different and are only birational to each other (see §2.1). This was a very decisive observation, and the determination to investigate the most general relation among different GIT quotients led to some significant discoveries and applications.

2.2.2. *GIT quotients in general.* In general, GIT quotients are constructed as follows. Consider an action

$$G \times X \to X$$

with X a projective variety and G a reductive algebraic group. To define a GIT quotient, we need to make some choices

$$\begin{array}{ccc} G \times L & \longrightarrow & L \\ \big\downarrow & & \big\downarrow \\ G \times X & \longrightarrow & X \end{array}$$

where L is an ample line bundle over X and the action on X is lifted to L such that the induced map

$$g : L_x \to L_{g \cdot x}$$

is a linear map for any $x \in X$ and $g \in G$. Here L_x denotes the fiber of L over the point x. Such a device is called a linearized ample line bundle or simply a linearization, which we still denote by L for simplicity. Because of the lifted action, the space of sections $\Gamma(X, L)$ becomes a representation of G. That is, the linearization induces a G-action on $\Gamma(X, L)$ as follows: for any $s \in \Gamma(X, L)$ and $g \in G$

$$^g s(x) = g \cdot s(g^{-1} \cdot x).$$

We say a section s is G-invariant if ${}^g s = s$ for all $g \in G$. This simply means that

$$s(g \cdot x) = g \cdot s(x)$$

for all $g \in G$ and $x \in X$. We will denote the fixed point set of $\Gamma(X, L)$ by $\Gamma^G(X, L)$.

The linearization L singles out a distinguished Zariski open subset $X^{ss}(L)$ of X as follows:

$$X^{ss}(L) = \{x \in X | \text{ there is } s \in \Gamma^G(X, L) \text{ such that } s(x) \neq 0\}.$$

Points in $X^{ss}(L)$ are called semistable points with respect to the linearization L. $X^{ss}(L)$ contains a generally smaller subset

$$X^s = \{x \in X^{ss}(L) | \; G \cdot x \text{ is closed in } X^{ss}(L) \text{ and } \dim G \cdot x = \dim G\}$$

whose points are called stable points w.r.t the linearization L.

The quotient of GIT is different than topological quotient in that it is not orbit space in general. GIT introduces the following *new* relation in $X^{ss}(L)$:

$$x \sim y \text{ if and only if } \overline{G \cdot x} \cap \overline{G \cdot y} \cap X^{ss}(L) \neq \emptyset.$$

Topologically, the GIT quotient is

$$X^{ss}(L)//G = X^{ss}(L)/ \sim .$$

From the definition, one checks that $X^{ss}(L)//G$ parameterizes the closed orbits in $X^{ss}(L)$, and the equivalence relation on X^s is the same as the orbit relation, hence the orbit space X^s/G is contained in $X^{ss}(L)/ \sim$.

A basic theorem of the geometric invariant theorem says

Theorem 2.1. ([57]) *The GIT quotient $X^{ss}(L)//G$ exists as a projective variety and contains the orbit space X^s/G as a Zariski open subset.*

2.3. **Relation with Symplectic Reduction.** There is a beautiful link between Geometric Invariant Theory and Symplectic Geometry.

Consider again an action $G \times X \to X$ of an algebraic reductive group on a smooth projective variety X, and let L be a linearized ample line bundle. The line bundle induces an integral symplectic form ω on X, the linearization amounts to a Hamiltonian symplectic action of a compact form K of G on the symplectic manifold (X, ω) and an equivariant differentiable map, called moment map

$$\Phi : X \to \mathfrak{k}^*$$

where $\mathfrak{k}$ is the Lie algebra of the compact Lie group K. The moment map is a solution of the following differential equation:

$$d(\Phi \cdot a) = i_{\xi_a} \omega$$

where $a \in \mathfrak{k}$, ξ_a is the vector field generated by a, and $\Phi \cdot a$ is the component of Φ in the direction of a.

The symplectic reduction or symplectic quotient by definition is

$$\Phi^{-1}(0)/K.$$

It inherits a symplectic form from ω on its smooth part. A beautiful theorem, due to contribution from many people, is that $\Phi^{-1}(0)$ is included in $X^{ss}(L)$ and this inclusion induces a homeomorphism

$$\Phi^{-1}(0)/K \cong X^{ss}(L)//G.$$

This relation can be extended to include Kähler quotients ([12]).

2.4. **Relative GIT and Universal Moduli Spaces.** Stable locus behave nicely under an equivariant morphism.

Let $f : Y \to X$ be a projective morphism between two smooth projective varieties X and Y, acted upon by two reductive algebraic group G' and G, respectively. Assume further that we have an epimorphism $\rho : G' \to G$ with respect to which $f : Y \to X$ is equivariant. Let G_0 be the kernel of ρ. Then we have

Theorem 2.2. (–,[25]) *Let L and M be two linearized ample line bundle over X and Y, respectively. Then for sufficiently large n, we have*

(1) $Y^{ss}(f^*L^n \otimes M) \subset Y^{ss}_{G_0}(M) \cap f^{-1}(X^{ss}(L))$;
(2) $Y^{s}(f^*L^n \otimes M) \supset Y^{s}_{G_0}(M) \cap f^{-1}(X^{s}(L))$.
Assume in addition that $X^{ss}(L) = X^s(L)$, then
(3) $Y^{ss}(f^*L^n \otimes M) = Y^{ss}_{G_0}(M) \cap f^{-1}(X^{ss}(L))$;
(4) $Y^{s}(f^*L^n \otimes M) = Y^{s}_{G_0}(M) \cap f^{-1}(X^{s}(L))$.

When $G' = G$, this is the main theorem of Reichstein ([60]).

The theorem has best applications when the semistable locus and stable locus coincide on the base (i.e., the cases (3) and (4)). For example, this can be easily applied to the universal moduli space of semistable coherent sheaves over the moduli space $\overline{M_g}$ of stable curves of genus $g \geq 2$, which was constructed by Pandharipande ([59]), using relative GIT. This is re-done in [25], combining the above theorem and the work of Simpson ([65]).

2.5. **Chow quotients.** In the "toy" example, observe that the lines $X = 0$ and $Z = 0$, which are of degree 1, have different homology classes than the conic orbits $XY = aZ^2$, which are of degree 2. But the GIT quotient $X_{[-1,0]}$ parameterizes these orbits of different homology classes. Even worst, the two orbits $(X = 0)$ and $(Y = 0)$ are all identified with the closed orbit $[0 : 0 : 1]$ in the quotient $X_{\{0\}}$, and the orbit $[0 : 0 : 1]$ even has smaller dimension than the dimension of the generic conic orbits. These are not desirable or suitable for moduli problems.

This kind of problem can however be overcome by considering Chow quotient which takes completely different approach.

Return to our "toy example", to obtain the Chow quotient, we first consider the closures of the generic $\mathbb{C}^*$-orbits, $XY = aZ^2$ ($a \neq 0, \infty$), and then looks at all their possible degenerations. When $a = 0$, we get the degenerated conic $XY = 0$, two crossing lines; and when $a = \infty$, we obtain $Z^2 = 0$, a double line. They all have the same homology classes (degree 2). And the Chow quotient is *the space of all* $\mathbb{C}^*$*-invariant conics*. (See Figure 1.) Each point of the Chow quotient represents an invariant algebraic cycle. In this case, the generic cycles are $[XY = aZ^2]$ ($a \neq 0, \infty$), and the special cycles are $[X = 0] + [Y = 0]$ ($a = 0$), and $2[Z = 0]$ ($a = \infty$).

From the above example, we see that Chow quotient parameterizes cycles of generic orbit closures and their toric degenerations which are *certain sums of orbit closures of dimension* $\dim G$. We will call these cycles *Chow cycles* or *Chow fibers*.

So, when do two arbitrary points belong to the same Chow cycle?

Consider the example again. We have that $[0 : y : z]$ and $[x : 0 : z]$ ($xyz \neq 0$) belong to the same Chow cycle $XY = 0$. To get $[x : 0 : z]$ from $[0 : y : z]$, we first *perturb* $[0 : y : z]$ to a general position

$$\varphi(t) = [tx : y : z](t \neq 0),$$

then *translate* it by $g(t) = t^{-1} \in \mathbb{C}^*$ to

$$g(t) \cdot \varphi(t) = [x : ty : z],$$

and then $g(t) \cdot \varphi(t)$ *specializes* to $[x : 0 : z]$ when t goes to 0. We will call this process *perturbing-translating-specializing* (PTS). It turns out this simple relation holds true in general. That is, we prove in general that two points x and y of X, with

$$\dim G \cdot x = \dim G \cdot y = \dim G,$$

belong to the same Chow cycle if and only if x can be perturbed (to general positions), translated along G-orbits (to positions close to y), and then specialized to the point y.

We will return to this in §4.

3. Variation of GIT and Factorization Theorem

3.1. **Variation of GIT Quotients.** From the earlier discussion, we see that GIT quotients are not unique. Hence a natural fundamental problem in this theory is to parameterize all the GIT quotients and describe exactly how a quotient changes as one varies the underlying parameter. Here we will explain the solution to this problem as given in [12]. To motivate it, we begin with an example.

3.1.1. *An Example.* In the "toy" example, we worked out all the GIT and Chow quotients, but they are all isomorphic to $\mathbb{P}^1$, the unique compactification of $\mathbb{C}^*$, because we insist an example that are very simple to describe. Here it should be fair to at least point out to the reader a workable example where a nontrivial wall crossing phenomenon and different quotients do occur. Hence we take the liberty to include the following example with details left to the reader.

Consider the action of $\mathbb{C}^*$ on $\mathbb{P}^3$ by

$$\lambda \cdot [x : y : z : w] = [\lambda x : \lambda y : \lambda^{-1} z : w].$$

The moment map is

$$\Phi([x : y : z : w]) = \frac{|x|^2 + |y|^2 - |z|^2}{|x|^2 + |y|^2 + |z|^2 + |w|^2}.$$

The image $\Phi(X)$ is $[-1, 1]$ with three critical values $-1, 0, 1$. So, we consider the level sets $\Phi^{-1}(-\frac{1}{2}), \Phi^{-1}(0), \Phi^{-1}(\frac{1}{2})$. In Figure 2, we illustrate the *real parts*

$$\Phi^{-1}(-\frac{1}{2})_{\mathbb{R}}, \Phi^{-1}(0)_{\mathbb{R}}, \Phi^{-1}(\frac{1}{2})_{\mathbb{R}}$$

of the level sets restricted to $\mathbb{C}^3 \subset \mathbb{P}^3$ (the $\mathbb{C}^3$ is defined by setting $w = 1$). It turns out this real picture preserves all the topological information we need.

To understand this picture, note that the real points of S^1 are

$$S^1 \cap \mathbb{R} = \{-1, 1\} = \mathbb{Z}_2.$$

So, the real parts of the symplectic quotients (or the GIT quotients)

$$\Phi^{-1}(-\frac{1}{2})/S^1, \Phi^{-1}(0)/S^1, \Phi^{-1}(\frac{1}{2})/S^1$$

are

$$\Phi^{-1}(-\frac{1}{2})_{\mathbb{R}}/\mathbb{Z}_2, \Phi^{-1}(0)_{\mathbb{R}}/\mathbb{Z}_2, \Phi^{-1}(\frac{1}{2})_{\mathbb{R}}/\mathbb{Z}_2.$$

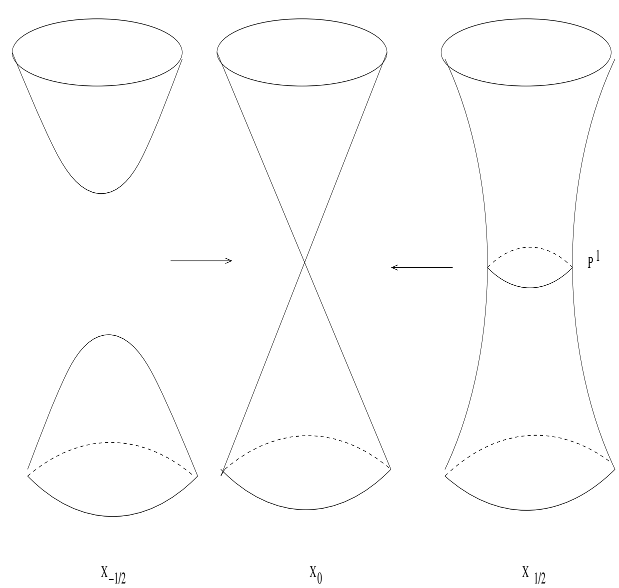

Figure 2. Wall-Crossing Maps

Note that $\mathbb{Z}_2$ acts on each level set by identifying the lower part with the upper part. Hence, the quotient can be naturally identified with the upper part.

There are two natural collapsing maps, as shown in the picture.

Now observe that the left map happens to be an isomorphism, but the right map is a (real) blowup along the origin so that the special fiber is $\mathbb{RP}^1$. Now complexifying this picture and compactifying the results, we obtain three quotients: $X_{[-1,0]} \cong \mathbb{P}^2$, $X_{\{0\}} \cong \mathbb{P}^2$, and $X_{[0,1]}$ isomorphic to the blowup of $\mathbb{P}^2$ along a point.

The reader may try to classify all the generic $\mathbb{C}^*$-orbits and study their degenerations. He can verify that the Chow quotient is also isomorphic to the blowup of $\mathbb{P}^2$ along a point.

This example suggests that when crossing a critical value of a moment map, the corresponding GIT quotient changes by a blowdown followed by a blowup. As it turns out, this simple wall-crossing phenomenon exhibited in this simple example reveals a general nature of GIT quotients and birational geometry.

3.1.2. *The Main Theorem of VGIT* [12]. Some of the main results of [12] may be roughly summarized as

Theorem 3.1. (Dolgachev-Hu, [12]) *Let X be a smooth projective variety acted upon by an algebraic reductive group G. Then there is a convex cone $C^G(X)$, the G-ample cone, which admits natural finite wall and chamber structure such that under very general conditions, we have*

1. *all GIT quotients are parameterized by rational points of the cone $C^G(X)$. (Other points of $C^G(X)$ parameterize Kähler quotients;)*
2. *points in the same chamber give rise to identical quotient;*
3. *when crossing a wall, the quotient undergoes a birational transformation similar to a Mori flip, VGIT wall-crossing flip.*

The main part of this theorem was also proved by Thaddeus ([66]). The special case of this theorem when the group G is Abelian was done earlier in [4] and independently in the author's MIT thesis ([23]). Theorem 3.1 can be applied to a number of moduli problems, some of which, e.g., moduli of parabolic bundles over a curve, were done in the author's joint works ([7]). Many other nice applications, especially to Donaldson invariants, appeared in the works of several other mathematicians. The entire VGIT project, the Abelian and the general cases, followed an observation of Goresky and MacPherson, made in 80's ([17]), which alluded that GIT morphisms should have implications for birational geometry.

3.2. **VGIT and the Mori Program.** Theorem 3.1 shows that VGIT provides natural examples of factorizing general birational transformations into sequences of simple ones. In my joint works [33] and [34], we found that the connection between the two is in fact deeper. These two works were motivated, apart from others, by a natural question that basically asks the converse of Theorem 3.1:

Question 3.2. *Given any general birational map, can it be realized as a sequence of VGIT wall-crossing flips?*

In the special case when the birational map is made of a single flip, this was answered by M. Reid, see the "tabernacle" lecture by Reid, see also Theorem 1.7 of [66] (these were pointed out by Abramovich). Note here that the existence of flips is not known in higher dimensions.

In general, the answer may be divided into two cases: (1). VGIT $\Leftrightarrow$ Mori: the global version; (2) VGIT $\Leftrightarrow$ Mori: the local version. One direction of VGIT $\Leftrightarrow$ Mori is Theorem 3.1. We only need to explain the other.

We begin with the global version. It roughly goes as follows. Consider a $\mathbb{Q}$-factorial projective variety X_0 and all of its Mori flip images $\{X_i\}_i$. The best thing one can possibly hope here is that there exists a group action $G \times \widetilde{X} \to \widetilde{X}$ such that all $\{X_i\}_i$ are GIT quotients of the action and all the flips are VGIT wall-crossing flips. If this is indeed the case, we will call X_0 a *Mori dream space*. We have a criterion for a Mori dream space.

Theorem 3.3. (Hu-Keel, [33]) *Let X be a $\mathbb{Q}$-factorial quasi-projective variety with $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$. Then X is a Mori dream space if and only if the Cox ring*

$$\mathrm{Cox}(X) := \bigoplus_{(n_1,\ldots,n_r)\in\mathbb{N}^r} H^0(X, L_1^{n_1} \otimes \cdots \otimes L_r^{n_r})$$

is finitely generated, where $L_1, \ldots, L_r$ is a basis of $N^1(X)$. In this case, X is naturally a quotient of the affine variety $\mathrm{Spec}(\mathrm{Cox}(X))$ by the group $H = \mathrm{Hom}(\mathbb{N}^r, \mathbb{C}^*)$.

Toric varieties are special cases of Mori dream spaces when $\mathrm{Cox}(X)$ are polynomial rings ([10]). Other examples of Mori dream spaces include quotients of Grassmanians by maximal tori, or equivalently the moduli spaces of configuration of points in projective spaces, and GIT quotients of some affine varieties. Of course, most varieties cannot be Mori dream spaces, for example, when the ample cone of X is not polyhedral.

3.3. **VGIT and Weak Factorization Theorem.** However, the local version, as we will explain now, is basically always valid. This version basically asks if VGIT $\Leftrightarrow$ Mori is true for some part of birational transformations. Indeed, instead of putting all of birational transformations of X_0 in the framework of just one group action (the Mori dream case), we may use possibly different group actions for different birational morphisms.

To this end, we have

Theorem 3.4. (Hu-Keel, [34]) *Let $X \to X'$ be a birational morphism between two smooth projective varieties. Then there is a $\mathbb{C}^*$- smooth projective variety W such that X and X' are geometric GIT quotients of two open subsets of W by $\mathbb{C}^*$. Consequently, by VGIT, any birational map $f : X - -- > Y$ between two smooth projective varieties can be factorized as a sequence of GIT wall-crossing flips.*

Here, by a GIT wall-crossing flip, we mean the birational transformation as described in Theorem 4.2.7 of [12]. The GIT realization of the birational morphism $X \to X'$ was stated and proved in [34]. We were informed afterwards that the same may also be done with Wlodarczyk's construction in [73] (see also [2]).

It would be nice that the *weighted* blowups and blowdowns resulted from applying VGIT can be improved to be just *ordinary* blowups and blowdowns so that the un-weighted factorization theorem may also have an easy and short GIT proof. The problem boils down to the problem of what I call "resolving singular $\mathbb{C}^*$-action". Namely, suppose we have a $\mathbb{C}^*$-action over the union of two nonsingular open varieties $U \cup V$ such that $\mathbb{C}^*$ acts freely with two smooth projective quotients $U/\mathbb{C}^*$ and $V/\mathbb{C}^*$. Then we would like to find a smooth equivariant compactification W of $U \cup V$ such that every isotropy subgroup of $\mathbb{C}^*$ on W is either the identity subgroup or the full group $\mathbb{C}^*$. Such an action is called quasi-free. We do not know whether this can be done.

If this can be answered affirmatively, the method of [34] combined with [12] will imply that $f : X - -- > X'$ can factorized as a sequence of (ordinary) blowup and blow downs along smooth centers. The key is the so-called Bialynicky-Birula decomposition theorem.

3.4. **VGIT and Weak Factorization Theorem for Projective Orbifolds.** A more exciting problem about VGIT $\Leftrightarrow$ Mori is to generalize it to the category of Projective Varieties with Finite Quotient Singularities. This is more compelling because, apart from the intrinsic reasons, there is recently a surge of research interests in orbifolds/stacks, for example, see the stringy geometry and topology of orbifolds ([64]), among others. These varieties occur naturally in many fields such as the Mirror Symmetry Conjectures.

Problem 3.5. Let $M - -- > M'$ be a birational map between two *projective* varieties with *finite quotient singularities*. Decompose it as a composition of explicit *simple* birational transformations.

To this end, we proved

Theorem 3.6. (–, [29]) *Let $\phi : X \to Y$ be a birational morphism between two projective varieties with at worst finite quotient singularities. Then there is a smooth polarized projective $(\mathrm{GL}_n \times \mathbb{C}^*)$-variety $(M, \mathcal{L})$ such that*

(1) *$\mathcal{L}$ is a very ample line bundle and admits two (general) linearizations $\mathcal{L}_1$ and $\mathcal{L}_2$ with $M^{ss}(\mathcal{L}_1) = M^s(\mathcal{L}_1)$ and $M^{ss}(\mathcal{L}_2) = M^s(\mathcal{L}_2)$.*
(2) *The geometric quotient $M^s(\mathcal{L}_1)/(\mathrm{GL}_n \times \mathbb{C}^*)$ is isomorphic to X and the geometric quotient $M^s(\mathcal{L}_2)/(\mathrm{GL}_n \times \mathbb{C}^*)$ is isomorphic to Y.*
(3) *The two linearizations $\mathcal{L}_1$ and $\mathcal{L}_2$ differ only by characters of the $\mathbb{C}^*$-factor, and $\mathcal{L}_1$ and $\mathcal{L}_2$ underly the same linearization of the GL_n-factor. Let $\underline{\mathcal{L}}$ be this underlying GL_n- linearization. Then we have $M^{ss}(\underline{\mathcal{L}}) = M^s(\underline{\mathcal{L}})$.*

The idea of the proof is as follows. First, we use two line bundles over X, resulting from the pullback of an ample line bundle over Y, to construct a $\mathbb{C}^*$-variety Z (possibly very singular) such that X and Y are identified with two geometric quotients $Z^s(L_1)/\mathbb{C}^*$ and

$Z^s(L_2)/\mathbb{C}^*$, respectively. Let $V = Z^s(L_1) \cup Z^s(L_2)$. Then it has at most finite quotient singularities because X and Y are such varieties and the $\mathbb{C}^*$ action on V is free. This implies that there is a quasi-projective scheme U with $\mathrm{GL}_n \times \mathbb{C}^*$ action such that $V = U/\mathrm{GL}_n$ and X and Y are two geometric quotients of U by $\mathrm{GL}_n \times \mathbb{C}^*$. Then take an equivariant compactification of U and resolve singularities eqivariantly, we finally arrive at our theorem above.

Next, we can verify that the action in the above theorem satisfies the condition of the wall-crossing theorem (Theorem 4.2.7) of [12], hence as a consequence, we obtain

Theorem 3.7. (–, [29]) *Let X and Y be two birational projective varieties with at worst finite quotient singularities. Then Y can be obtained from X by a sequence of VGIT wall-crossing birational transformations.*

In Theorem 1.2 of [29], as in this theorem, by a GIT weighted blowup and a weighted blowdown, we mean the VGIT wall-crossing birational transformation (or VGIT wall-crossing flip for short) as described in Theorem 4.2.7 of [12].

As a consequence of the proof, we also obtain the following corollary to render the above suitable and useful for some applications.

Corollary 3.8. (–, [29]) *Let $X \to Y$ be a birational morphism between two projective varieties with finite quotient singularities. Then X and Y can be realized as two geometric quotients of a projective variety with finite quotient singularities by a $\mathbb{C}^*$ action.*

The difference between Theorem 3.6 and Corollary 3.8 is that the former involves a reductive group with a non-Abelian factor, while the latter uses only $\mathbb{C}^*$-action which might be simpler from some perspectives, but for this, one has to sacrifice the smoothness of the ambient variety.

Establishing all the above, a natural next question is

Problem 3.9. Describe how orbifold cohomology, orbifold Chow group, and other orbifold invariants change when crossing a wall.

Obviously, a deeper problem is to compute how orbifold quantum cohomology changes when crossing a codimensional one wall (see [5], [64], and methods of [51], [52]), however, even at the "classical" level (without quantum corrections), this problem is already very interesting and highly non-trivial.

To this end, it may be useful to bring a new cohomology theory into the picture: the orbifold intersection cohomology for singular orbispces, for which the very singular GIT quotient defined by a polarization on the wall, which is also the common blowdown of two adjacent smooth GIT quotient orbifolds, might be a natural example. This is, of course, interesting in its own right. The total orbifold intersection cohomology may be defined as the total intersection cohomology of the twisted sectors, or stack-theoretically, "inertia stack". To make it truly useful for our purpose, a BBD type decomposition theorem for orbifold cohomology may be needed ([6]). Also to better track how orbifold invariants change, we need

Problem 3.10. Describe explicitly the twisted sectors of GIT quotient orbifolds using the raw data from the group action, and describe explicitly how it changes when crossing a wall.

The last two problems are, in fact, already interesting in the settings of [24], in that even if the ambient variety is smooth, the GIT quotients may still have quotient singularities, and then, according to stringy geometry, orbifold invariants are the right ones we should study. In other words, we expect to "refine", in the case of the presence of (finite) quotient singularities, some of our intersection cohomology calculations in [24], in terms of the new orbifold invariants.

4. Chow Quotients: Perturbation-Translation-Specialization

GIT quotient (or stability) can be very difficulty to handle in practice. But Chow quotient in general is even harder. The difficulty of Chow quotient is that it is in general very singular. However, understanding Chow quotient well may lead to some significant applications ([32]).

4.1. **Definition.** Consider a reductive algebraic group action on a projective variety over the field of complex numbers

$$G \times X \to X.$$

The Chow quotient of this action is defined as follows.

There is a small open subset, $U \subset X$, such that $\overline{G \cdot x}$ represents the same Chow cycle δ for all $x \in U$. Let $\text{Chow}_\delta(X)$ be the component of the Chow variety of X containing δ. Then there is an embedding

$$\iota : U/G \to \text{Chow}_\delta(X)$$

$$[G \cdot x] \to [\overline{G \cdot x}] \in \text{Chow}_\delta(X).$$

The Chow quotient, denoted by $X//^{ch}G$, is defined to be the closure of $\iota(U/G)$. This definition is independent of the choice of the open subset U. That is, the Chow quotient is canonical.

4.2. **Chow Family.** There is a "universal" family over the Chow quotient. Let

$$F \subset X \times (X//^{ch}G)$$

be the family of algebraic cycles over the Chow quotient $X//^{ch}G$ defined by

$$F = \{(x, Z) \in X \times (X//^{ch}G) | x \in Z\}.$$

Then, we have a diagram

$$\begin{array}{ccc} F & \xrightarrow{\mathrm{ev}} & X \\ {\scriptstyle f}\downarrow & & \\ X//^{ch}G & & \end{array}$$

where ev and f are the projections to the first and second factor, respectively. For any point $q \in X//^{ch}G$, we will call the fiber, $f^{-1}(q)$, the Chow fiber over the point q. Sometimes we identify $f^{-1}(q)$ with its embedding image $\mathrm{ev}(f^{-1}(q))$ in X.

4.3. **Perturbation-Translation-Specialization.** As we mentioned before, the Chow quotient approach to moduli problems is different and may be harder than GIT, but it has some advantages and potentially has several significant applications. Understanding the boundary of Chow quotient is the key to this approach.

In our paper [28], we showed that for a torus G action on a smooth projective variety X, the boundaries of the Chow quotients, or equivalently, the special fibers of the Chow family, admit a computable characterization. This is very desirable for applications. Roughly, we prove that

Theorem 4.1. (–, [28]) *Two points x and y of X, with*

$$\dim G \cdot x = \dim G \cdot y = \dim G,$$

are in the same Chow fiber if and only if x can be perturbed (to general positions), translated along G-orbits (to positions close to y), and then specialized to the point y. (See Figure 3 for an illustration.)

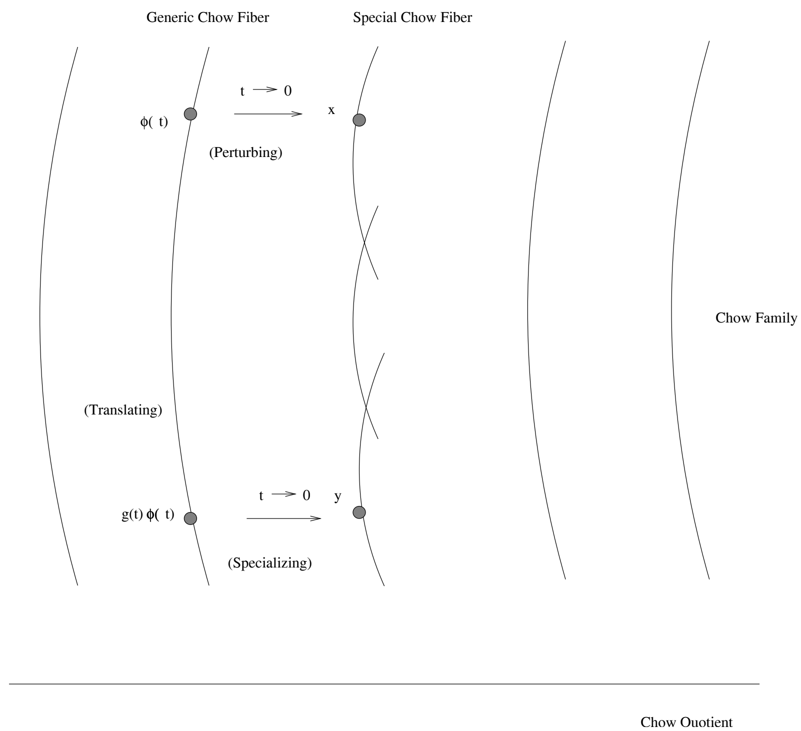

Figure 3. Perturbing-Translating-Specializing

This relation, which we call *perturbing-translating-specializing* (P.T.S.) relation, was first discovered and studied intensively by Neretin in the case of symmetric spaces. In addition to the above, we also proved in [28] that over the field of complex numbers, the Chow quotient admits symplectic and other topological interpretations, namely,

- symplectically, the moduli spaces of stable orbits with prescribed momentum charges; and
- topologically, the moduli space of stable action-manifolds.

Moduli spaces of what I call stable polygons are symplectic interpretations of the Chow quotient $\overline{M}_{0,n}$, the moduli space of stable n-pointed rational curves. For detailed account of this and beyond, see [26].

An upshot of PTS relation is that, comparing to the nondescriptive definition of special Chow fibers, it is computable, and thus provides some much needed information on boundary cycles of the Chow quotient. As an application, we have appllied Theorem 4.1 to the case of point configurations on $\mathbb{P}^n$ ($n > 1$), and propose a *geometric interpretation* of the Chow quotients of $(\mathbb{P}^n)^m$ (equivalently, the Chow quotients of higher Grassmannians).

In [49], Lafforgue provides some toric compactifications of matroid strata of Grassmannians. His compactifications are reducible in general. Our P.T.S theorem may be applied to single out the main irreducible components.

The PTS relation can be used to explain more naturally the moduli space $\overline{M}_{0,n}$ as the Chow quotient of $(\mathbb{P}^1)^n$ ([30]). It may also be useful for its toric degeneration ([31]).

5. Birational Geometry of HyperKähler Varieties

HyperKähler Varieties form a sepcial and very important class of Calabi-Yau manifolds. Its birational geometry has received some considerable attentions recently ([8], [37], [20], [72],among others). Being very restrictive in structures, birtaional transformations among HyperKähler Varieties are among simplest kinds. We will be mainly interested in projective symplectic varieties. This means a nonsingular projective variety X of dimension $2n$, together with a holomorphic two form ω such that $\omega^n \neq 0$.

Let $f : X - -- > Y$ be a birational map between two symplectic varieties. By the weak Factorization Theorem, it can be factorized as blowup and blowdowns. However, we have little control on the varieties in between, in particular, we do not know whether they are symplectic or not. This causes difficulty to keep track the changes of symplectic varieties. Hence, special efforts are need to factorize f in the category of symplectic varieties.

5.1. Factorizations in Dimension 4.

The simplest known birational transformation among symplectic varieties are the so-called Mukai elementary transformations (MET). For a symplectic 4-fold X, this means the following. Let $\mathbb{P}^2$ be embedded in X, then it is a Lagrangian subvariety, hence its normal bundle is isomorphic to its cotangent bundle. Blowup X along this $\mathbb{P}^2$, then its exceptional divisor D is the incident variety

$$D = \{(p, l) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2 |\ p \in l\}$$

where $\check{\mathbb{P}}^2$ is the space of lines in $\mathbb{P}^2$. Mukai showed that it can be contracted along another ruling, producing another symplectic variety X'. This process is called a MET.

The same can be done in general, when X contains a projective bundle whose rank is the same as its codimension.

In dimension 4, we have the following structure theorem.

Theorem 5.1. (Burns-Luo-Hu, [8]) *Let $f : X - -- > Y$ be a birational map between two symplectic varieties of dimension 4. Assume that every component of the degenerate locus of f is normal. Then $f : X - -- > Y$ can be factorized as a sequence of Mukai elementary transformations.*

The normality condition in the theorem was later removed by Wierzba and Wisniewski ([72]).

5.2. Symplectic Contractions in Higher Dimensions.

The general case is much more difficult. As an important initial step, Yau and I obtained the following structure theorems on small symplectic contractions.

Theorem 5.2. (Hu-Yau, [37]) *Let X be a smooth projective symplectic variety and $\pi : X \to Z$ be a small contraction. Let B be an irreducible component of the degenerate locus of π and F is a generic fiber of the restricted map $\pi : B \to S = \pi(B)$. Then we have*

(1) *$T_bF = (T_bB)^{\perp}$ for a generic smooth point $b \in F$ and the orthogonal space is taken with respect to the symplectic form ω.*
(2) *the inclusion $j : B \hookrightarrow X$ is a coisotropic embedding.*
(3) *the null foliation of $\omega|_B$ coincides the generic fibers of $\pi : B \to S$.*

As a special case, we showed

Theorem 5.3. (Hu-Yau, [37]) *Assume that $\pi : B \to S$ is a smooth fiberation, then it must be a projective bundle with rank $r = \operatorname{codim} B$. In particular, the Mukai transformation can be performed along this projective bundle to get another symplectic variety.*

Corollary 5.4. (Hu-Yau, [37]) *If B is smooth and can be contracted to a point, then $B \cong \mathbb{P}^n$, in particular, it is a Lagrangian. In fact, whenever B can be contracted to a point, smooth or not, it must be a Lagrangian.*

Proposition 5.5. (Hu-Yau, [37]) *Any small contraction $\pi : X \to Z$ must be perverse semi-small in the sense that there is a stratification $Z = \cup Z_\alpha$ such that*

$$\dim \pi^{-1}(z) \leq \frac{1}{2} \operatorname{codim} Z_\alpha$$

for all $z \in Z_\alpha$ and non-open strata Z_α.

When all the inequalities are equalities, the map is said to be strictly semi-small. We believe

Conjecture 5.6. (Hu-Yau, [37]) *Any small contraction $\pi : X \to Z$ must be perverse strictly semi-small.*

This helps to convince us the following general conjecture

Conjecture 5.7. (Hu-Yau, [37]) *Let $f : X - - - > Y$ be a birational map between two symplectic varieties. Then after removing subvarieties of codimension greater than 2, X and Y are related by a sequence of Mukai elementary transformations.*

This conjecture was recently confirmed by B. Fu ([20]) in the case of the closures of nilpotent orbits.

We remark that some of our results in this section overlap with the work of [9]

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REFERENCES

[1] M. Atiyah and R. Bott, *The moment map and equivariant cohomology.* Topology 23 (1984), no. 1, 1–28. MR0721448.

[2] Dan Abramovich, Kalle Karu, Kenji Matsuki, Jaroslaw Wlodarczyk *Torification and Factorization of Birational Maps*, J. Amer. Math. Soc. 15 (2002), no. 3, 531–572. MR1896232, Zbl1032.14003.

[3] H. Bass and W. Haboush, *Linearizing certain reductive group actions,* Trans. Amer. Math. Soc. **292** (1985), no. 2, 463–482.

[4] M Brion and C. Procesi, *Action d'un tore dans une variété,* in "Operator algebra, unitary representations, enveloping algebras, and invariant theory", Progress in Mathematics 192 (1990), Birkhäuser, 509-539.

[5] K. Behrend, *Cohomology of Stacks,* University of British Colombia (2002).

[6] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux Pervers.* In: Analyse et topologie sur les espaces singuliers, I. Astérisque #100, Soc. Math. de France (1983).

[7] H. Boden and Y. Hu, *Variation of Moduli of Parabolic Bundles,* Mathematische Annalen Vol. 301 No. 3 (1995), 539 – 559.

[8] D. Burns, Y. Hu and T. Luo, *HyperKähler manifolds and birational transformations in dimension 4,* In: Vector bundles and representation theory, Contemporary Mathematics Volume **322** (2003), 141 – 149.

[9] K. Cho, Y. Miyaoka, and N. Sheperd-Barron, *Characterizations of projective space and applications to complex symplectic manifolds.* Higher dimensional birational geometry (Kyoto, 1997), 1–88, Adv. Stud. Pure Math., 35, Math. Soc. Japan, Tokyo, 2002. MR1929792

[10] D. Cox, *The homogeneous coordinate ring of a toric variety.* J. Algebraic Geom. **4** (1995), no. 1, 17–50.

[11] S. D. Cutkosky, *Strong Toroidalization of Birational Morphisms of 3-Folds.* math.AG/0412497

[12] I. Dolgachev and Y. Hu, *Variation of Geometric Invariant Theory Quotients*, Publ. Math. I.H.E.S. **87** (1998), 5 – 56. MR1659282, Zbl 1001.14018.

[13] S. Donaldson, *Scalar curvature and projective embeddings. I,* J. Differential Geom. **59** (2001), no. 3, 479–522.

[14] S. Donaldson, *Scalar curvature and stability of toric varieties* . J. Differential Geom. Volume **62** (2002), 289–349.

[15] D. Edidin, B. Hassett, A. Kresch, A. Vistoli, *Brauer Groups and Quotient Stacks*. Amer. J. Math. 123 (2001), no. 4, 761–777. MR1844577, Zbl 1036.14001.

[16] V. Guillemin and S. Sternberg, *Birational equivalence in the symplectic category.* Invent. Math. 97 (1989), no. 3, 485–522. MR1005004.

[17] M. Goresky and R. MacPherson, *On the topology of torus actions,* in "Algebraic Groups" (1985).

[18] G. Faltings and G. Wüstholz, *Diophantine approximations on projective spaces,* Invent. Math **116** (1994), 109–138.

[19] P. Foth and Y. Hu, *Toric degenerations of weight varieties and applicationsmath,* Travaux Mathématiques (2005), math.AG/0406329.

[20] B. Fu, *Birational geometry in codimension 2 of symplectic resolutions.* math.AG/0409224

[21] A. Givental, *Equivariant Gromov-Witten invariants.* Internat. Math. Res. Notices 1996, no. 13, 613–663. MR1408320

[22] P. Hacking, *Compact moduli of hyperplane arrangements,* math.AG/0310479.

[23] Y. Hu, *Geometry and topology of quotient varieties,* Ph.d Thesis, MIT , 1991.

[24] Y. Hu, *Geometry and topology of quotient varieties of torus actions,* Duke Math. Journal **68** (1992), 151 – 183. MR1185821.

[25] Y. Hu, *Relative geometric invariant theory and universal moduli spaces.* Internat. J. Math. **7** (1996), no. 2, 151–181. MR1382720, Zbl 0889.14005.

[26] Y. Hu, *Moduli Spaces of Stable Polygons and Symplectic Structures on* $\overline{M}_{0,n}$, Compositio Mathematica Vol. 118 (1999), 159 – 187. MR1713309

[27] Y. Hu, *Stable Configurations of Linear Subspaces and Vector Bundles.* Quarterly Journal of Pure and Applied Mathematics Volume 1 No. 1 (2005), 127 – 164.

[28] Y. Hu, *Topological Aspects of Chow Quotients.* Preprint, submitted (2003). math.AG/0308027

[29] Y. Hu, *Factorization Theorem for Projective Varieties with Finite Quotient Singularities*, Journal of Differential Geometry Volume **68** (2004), 587 – 593.

[30] Y. Hu, $\overline{M}_{0,n}$ *as Chow Quotient Revisited,* In preparation.

[31] Y. Hu, *Toric Degeneration of* $\overline{M}_{0,n}$. In preparation.

[32] Y. Hu, In preparation.

[33] Y. Hu and S. Keel, *Mori dream spaces and GIT,* Dedicated to William Fulton on the occasion of his 60th birthday. Michigan Math. J. **48** (2000), 331–348. MR1786494, Zbl pre01700876.

[34] Y. Hu and S. Keel, *A GIT proof of the weighted weak factorization theorem.* unpublished short notes, available at Math ArXiv, math.AG/9904146.

[35] D. Huybrechts, *Compact HyperKähler manifolds: basic results,* Invent. Math. **135** (1999), 63-113.

[36] Y. Hu and W. Li, *Variation of the Gieseker and Uhlenbeck Compactifications,* International Journal of Mathematics Vol. 6 No. 3 (1995), 397 – 418.

[37] Y. Hu and S.-T. Yau, *HyperKḧler Manifolds and Birational Transformations,* Adv. Theor. Math. Phys. Volume **6** Number 3 (2002), 557-576.

[38] Y. Hu, C.-H. Liu, and S.-T. Yau, *Toric Morphisms and Fibrations of Toric Calabi-Yau hypersufaces,* Adv. Theor. Math. Phys. Volume 6 Number 3 (May 2002), 457 – 506.

[39] M. Kapranov, *Chow quotients of Grassmannian,* I.M. Gelfand Seminar Collection, 29–110, Adv. Soviet Math., 16, Part 2, Amer. Math. Soc., Providence, RI, 1993.

[40] M. Kapranov, B. Sturmfels and A. Zelevinsky, *Quotients of toric varieties*, Math. Ann. 290 (1991), no. 4, 643–655.

[41] S. Keel and S. Mori, *Quotients by groupoids.* Ann. of Math. (2) 145 (1997), no. 1, 193–213.

[42] S. Keel and J. Tevelev, *Chow Quotients of Grassmannians II,* math.AG/0401159.

[43] G. Kempf and L. Ness, *The length of vectors in representation spaces.* Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), pp. 233–243, Lecture Notes in Math., 732, Springer, Berlin, 1979. MR0555701.

[44] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry.* Mathematical Notes, 31. Princeton University Press, Princeton, NJ, 1984. MR0766741

[45] F. Kirwan, *Partial desingularisations of quotients of nonsingular varieties and their Betti numbers.* Ann. of Math. (2) 122 (1985), no. 1, 41–85. MR0799252.

[46] A. Klyachko, *Stable bundles, representation theory and Hermitian operators,* Selecta Math. **4** (1998), 419-445.

[47] J. Kollár, *Quotient spaces modulo algebraic groups.* Ann. of Math. (2) 145 (1997), no. 1, 33–79.

[48] J. Kollár and S. Mori, *Birational geometry of algebraic varieties,* Cambridge University Press, 1998.

[49] L. Lafforgue, *Chirurgie des grassmanniennes.* (French) [Surgery on Grassmannians] CRM Monograph Series, 19. American Mathematical Society, Providence, RI, 2003.

[50] L. Lafforgue, *Pavages des simplexes, schémas de graphes recollés et compactification des* $\mathrm{PGL}_r^{n+1}/\mathrm{PGL}_r$, Invent. Math. 136 (1999), no. 1, 233–271. Erratum: Invent. Math. 145 (2001), no. 3, 619–620.

[51] Jun Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Diff. Geom. **57** (2001), no. 3, 509–578.

[52] Jun Li, *A degeneration formula of GW-invariants*, J. Diff. Geom. **60** (2002), no. 2, 199–293.

[53] B, Liang, K. Liu and S. T. Yau, *Mirror Principles, I* Asian J. of Math. **1** (1997). no. 4, 729–763. MR1621573

[54] B, Lian, K. Liu and S. T. Yau, *Mirror Principles, II, III,* Asian J. of Math. (1999) no. 1, 109–146 and no. 4, 771–800, MR1701925 and MR1797578

[55] B, Lian, K. Liu and S. T. Yau, *Mirror principle. IV* Surveys in differential geometry, 433–474, Surv. Differ. Geom., VII, Int. Press, Somerville, MA, 2000.

[56] H.-Z. Luo, *Geometric criterion for Gieseker-Mumford stability of polarized manifolds,* Journal of Diff. Geom. **49** (1998), no. 3, 577–599.

[57] D. Mumford, *Geometric Invariant Theory,* 1962. Berlin, New York. MR1304906, Zbl 0797.14004.

[58] L. Ness, *Stratification of the null cone via the moment map.* With an appendix by David Mumford. Amer. J. Math. 106 (1984), no. 6, 1281–1329. MR0765581

[59] R. Pandharipande, *A compactification over* $\overline{M}_g$ *of the universal moduli space of slope-semistable vector bundles,* J. Amer. Math. Soc. 9 (1996), no. 2, 425–471. MR1308406

[60] Z. Reichstein, *Stability and equivariant maps*, Inven. Math. **96** (1989), 349-383.

[61] D.H. Phong and J. Sturm, *Stability, Energy Functions, and Kähler-Einstein Metrics*, arXiv:math.DG/0203254.
[62] J. Ross and R. P. Thomas, *A study of the Hilbert-Mumford criterion for the stability of projective varieties.* math.AG/0412519
[63] J. Ross and R. P. Thomas, *An obstruction to the existence of constant scalar curvature Kähler metrics*, math.DG/0412518
[64] Y. Ruan, *Stringy geometry and topology of orbifolds.* Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000), 187–233, Contemp. Math., 312, Amer. Math. Soc., Providence, RI, 2002.
[65] C. Simpson, *Moduli spaces of representations of the fundamental group of a smooth projective variety, I and II*, Inst. Hautes Études Sci. Publ. Math. No. 79 (1994), 47–129. MR1307297. Inst. Hautes Études Sci. Publ. Math. No. 80 (1994), 5–79 (1995). MR1320603.
[66] M. Thaddeus, *Geometric invariant theory and flips,* J. Amer. Math. Soc. 9 (1996), no. 3, 691–723. MR1333296, Zbl 0874.14042.
[67] G. Tian, *Kähler-Einstein metrics on algebraic manifolds,* LNM **1646**, Spinger, Berlin, 1996.
[68] G. Tian, *Kähler-Einstein metrics with positive scalar curvature.* Invent. Math. **130** (1997), 1–39.
[69] B. Totaro, *Tensor products of semistables are semistable,* Geometry and Analysis on complex manifolds, World Sci. Publ. 1994, pp. 242-250.
[70] X. W. Wang, *Balanced point, stability and vector bundles over projective manifolds.* Math. Res. Lett. **9** (2002), no. 2-3, 393–411.
[71] E. Vieweg, *Quasi-projective moduli of polarized manifolds,* Springer-Verlag, 1995.
[72] J. Wierzba and J. Wiśniewski, *Small contractions of symplectic 4-folds.* Duke Math. J. 120 (2003), no. 1, 65–95. MR2010734.
[73] J. Wlodarczyk, *Birational cobordisms and factorization of birational maps.* J. Algebraic Geom. 9 (2000), no. 3, 425–449. MR1752010, Zbl 1010.14002.
[74] J. Wlodarczyk, *Toroidal varieties and the weak factorization theorem.* Invent. Math. 154 (2003), no. 2, 223–331. MR2013783, Zbl pre02021105.
[75] J. Wlodarczyk, *Decomposition of birational toric maps in blow-ups & blow-downs.* Trans. Amer. Math. Soc. 349 (1997), no. 1, 373–411. MR1370654
[76] S.-T. Yau, *Open problems in geometry,* Proc. Symp. Pure Math. **54** (1993) 1-28.
[77] S.-T. Yau, *Mathematical aspects of string theory.* Proceedings of the conference held at the University of California, San Diego, California, July 21–August 1, 1986. Edited by S.-T. Yau. Advanced Series in Mathematical Physics, 1. World Scientific Publishing Co., Singapore, 1987. MR0915812
[78] S. Zhang, *Heights and reductions of semi-stable varieties,* Compositio Mathematica **104** (1996) 77-105.

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