On the Vanishing Viscosity Limit for the 3D Navier-Stokes Equations with a Slip Boundary Condition *

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1 Introduction

Let $\Omega \subset R^3$ be a bounded smooth domain satisfying the topological condition $H_1(\Omega, R) = 0$. We investigate the solvability, regularity and the vanishing viscosity limit of the incompressible Navier-Stokes equations

$$\partial_t u - \varepsilon \Delta u + \omega \times u + \nabla p = 0; \text{ in } \Omega; \tag{1.1}$$

$$\nabla \cdot u = 0; \text{ in } \Omega; \tag{1.2}$$

$$\omega = \nabla \times u; \text{ in } \Omega; \tag{1.3}$$

with the following slip boundary conditions

$$u \cdot n = 0, \quad \omega \cdot \tau = 0, \text{ on } \partial\Omega, \tag{1.4}$$

where and below $\nabla\cdot$ and $\nabla\times$ denote the *div* and *curl* operators respectively, n is the outward normal, and τ is the unit tangential vector of $\partial\Omega$.

The investigation of vanishing viscosity limit of solutions of the Navier-Stokes equations both in the two and three spacial dimensional cases is a classical issue. There are two related questions arising from here: one is how to describe the inviscid limiting behavior of the Navier-Stokes equation; and the other is that does the Euler equation can be approximated by the Navier-Stokes equations. In the case that the solution to the

*This research is supported in part by Zheng Ge Ru Foundation, and Hong Kong RGC Earmarked Research Grants CUHK-4028/04P, CUHK-4040/02P and CUHK-4279/00P.

ideal inviscid Euler system is sufficiently regular and there are no physical boundaries, the answers to both questions are positive ([12-14]). However, these questions become more subtle in the presence of physical boundaries. The most common boundary condition is the classical no-slip boundary condition, $u = 0$ on $\partial\Omega$, which gives rise to the phenomena of strong boundary layers in general as formally derived by Prandtl [30]. However, the rigorous analysis for such boundary layer is still far way from complete except in the case of a half-space and analytical initial data by Sammartino-Califish [32], and the linearized problems by Teman-Wang [39], Xin-Yanagisava [44] and Wang-Xin [42], see also [45]. For various sufficient conditions to ensure the convergence of viscous solutions to the ones of the Euler system, see Kato [23] and [41] and the references therein.

Another commonly used boundary conditions are Navier-type slip boundary conditions, which say that there is a stagnant layer of fluid close to the wall allowing a fluid to slip, and the slip velocity is proportional to the shear stress, i.e.,

$$u \cdot n = 0, \qquad (D(u)n)_\tau = \alpha_\tau\, u_\tau \qquad \text{on} \quad \partial\Omega, \tag{1.5}$$

where $D(u) = \nabla u + (\nabla u)^\tau$ is the shear stress. Such boundary conditions can be induced by effects of free capillary boundaries, (see [3]), or a rough boundary as in [2, 20], or a perforated boundary, which is then called Beavers-Joseph's law, see [4, 31, 34, 19], or an exterior electric field as in [9]. This type of boundary conditions were first introduced by Navier in [29], which was followed by great many applications, numerical studies and analysis for various fluid mechanical problems, see, for instance [2-5, 8, 12, 15, 17-21, 24-28, 31-32, 34-36, 43] and the references therein. In particular, we mention that such type slip boundary conditions are used in the large eddy simulations of turbulent flows, which seeks to compute the large eddies of a turbulent flow accurately neglecting small flow structure, for which the slip boundary conditions are more suitable than the Dirichlet boundary conditions [17].

For the mathematical rigorous analysis of the Navier-Stokes equations with Naiver-type slip boundary conditions, the first pioneering paper is due to Solonnikov and Ščadilov [38] for the stationary linearized Navier-Stokes system under the boundary conditions:

$$u \cdot n = 0, \qquad (D(u)n)_\tau = 0 \qquad \text{on} \quad \partial\Omega, \tag{1.6}$$

and the existence of weak solution and regularity for the stationary Navier-Stokes equations with the Navier slip boundary condition (1.5) has been obtained by Beirăo da Veiga [5] for half-space. In the case of two dimensional simply connected bounded domains, the vanishing viscosity problem has been rigorously justified by J. L. Lions [25] and P. L. Lions [26] for the free boundary condition:

$$u \cdot n = 0, \quad w = 0 \qquad \text{on} \quad \partial\Omega, \tag{1.7}$$

and by Clopeau-Mikelic-Robert [10] for the Navier-slip boundary condition (1.6). Similar results have also been obtained by Mucha [28] under some geometrical contraints on the shape of the domains. It should be noted all these 2-dimensional results are based on the fact the vorticity is a scalar and satisfies a maximum principle under the appropriate slip boundary conditions.

In the case of 3-dimensional domains, very little is known on the existence of strong solutions to Navier-Stokes system with a slip boundary condition. One cannot extend the techniques used in 2-dimensional case easily due to the lack of maximal principle for the vorticity in this case. Furthermore, a standard Sobolev type a priori estimate seems to require some compatibility of the nonlinear convection term with the slip boundary condition in order to obtain a ε-independent estimate.

The main purpose of this paper is to establish the well-posedness and asymptotic behavior as $\varepsilon \to 0^+$ of strong solutions to the initial-boundary value problem, (1.1)-(1.4) for a 3-dimensional smooth bounded domain with the constraint $H_1(\Omega, R) = 0$. In particular, for a 3-dimensional smooth bounded simply-connected domain, we would like to show that the $\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = u^0(x,t)$ exists and u^0 solves the following initial boundary value problem for the ideal Euler system:

$$\begin{cases} \partial_t u^0 + w^0 \times u^0 + \nabla p^0 = 0 & \text{in } \Omega \\ \nabla \cdot u^0 = 0 & w^0 = \nabla \times u^0 \end{cases} \tag{1.8}$$

with the boundary condition

$$u^0 \cdot n = 0 \qquad \text{on } \partial\Omega \tag{1.9}$$

and the same initial data as for the Navier-Stokes system (1.1). It should be noted that our boundary condition (1.4) is indeed a Navier-type slip boundary condition. In fact, we will show (see Proposition 4.1) that the boundary condition (2.4) is equivalent to

$$u \cdot n = 0, \qquad \text{and} \quad \partial_n u_\tau = 0, \qquad \text{on } \partial\Omega. \tag{1.10}$$

However, it was shown by Watanake [43] that on $\partial\Omega$,

$$2(D(u)n)_\tau = \partial_n u_\tau - k_\tau u_\tau + (\nabla(u \cdot n))_\tau, \tag{1.11}$$

where k_τ is the corresponding principal curvature of $\partial\Omega$. It follows that the slip boundary condition (1.4) is equivalent to

$$u \cdot n = 0, \quad (D(u)n)_\tau = -k_\tau u_\tau, \qquad \text{on } \partial\Omega. \tag{1.12}$$

Hence, it is a Navier-type slip boundary condition, and furthermore, it is a geometrical one. In the particular case that Ω is the half space $z > 0$, then the boundary condition (1.4) becomes exactly the boundary condition (1.6), which has been studied for some interesting cases. For example, Scadilov and Solonnikov ([38]) studied the stationary linearized Navier-Stokes system with the boundary condition (1.6) based on the following variational formulation:

$$(D(u), D(v)) = (f, v) \tag{1.13}$$

for the corresponding Stokes problem, and Watanake ([43]) studied the linearized evolutionary problem with the boundary condition (1.6) for an axi-symmetric domain. The major difficulty here is that the bilinear form on the left hand side of (1.13) may not be positive on the spaces they were studying, which leads to some difficulties to the well-posedness of the associated problem due to the compatibility condition on f. Our approach in this paper is motivated more by the work of Busuioc and Ratiu [7] on the second grade Navier-Stokes equations with the slip condition (1.6), where the local well-posedness theory is established by using the modified functional

$$\alpha(D(u), D(v)) + \int_\Omega u \cdot v, \tag{1.14}$$

with $\alpha > 0$. However, their results and analysis depend crucially on $\alpha > 0$ so that one cannot modify their approach directly to study the initial-boundary value problem (1.1)-(1.4) with initial data.

It is clear from (1.10)-(1.12) that the boundary condition (1.4) is a Navier-type slip condition and a natural generalization of the boundary conditions (1.6) and (1.7). We hope that our study on the asymptotic behavior of solutions to the Navier-Stokes system with such a slip boundary conditions will share light on the studies for more general Navier-type slip conditions for general 3-dimensional domains. Our main strategy in this paper is to formulate the associated boundary value problem by the theory of Hodge decomposition so that the corresponding Stokes operator is well-behaved. Furthermore, the nonlinearity in the Navier-Stokes systems is shown to match with the boundary condition (1.4) smoothly. These allow us to show the existence of solutions by a Galerkin approximation, and the regularity, uniqueness, and asymptotic behavior of solutions by a priori uniform estimates (independent of the viscosity). Consequently, we can derive the vanishing viscosity limit of the Navier-Stokes solutions.

The rest of the paper is organized as follows: in next section, we begin with introduction of functional spaces which will be used later and provide some basic Poincare type estimates for elements in such space. These estimates are based on the well-known Helmholtz-Wyle decomposition and the vector calculus formed by Cantarella, De Turk and Gluck [8]. Though most of these results are well-known or simple corollaries of the general Hodge decomposition theorem for differential forms (see Schwarz [34]), there are some new estimates (Theorem 2.1, Theorem 2.2, and Lemma 2.3), which cannot be found in the literatures easily.

Based on these estimates, we study the Stokes problem corresponding to the boundary condition (1.4) in Section 3. The well-posedness of the associated Stokes operator, its eigenvalue problem, and the regularity of the eigenvectors are proved. In Section 4, we investigate the compatibility of the nonlinearity in the Navier-Stokes system and the boundary condition (1.4), and the geometrical characteristics of the boundary condition. One of the main observation is that nonlinear convection term matches smoothly with the slip boundary condition (1.4), which is the key to obtain the ε-independent a priori estimates by the technique of Sobolev spaces. The formulation of a Galerkin approximation and some energy estimates for such approximate solutions are given in Section 5. In Section 6, the global existence of a weak solution is obtained, which is a consequence of the self-adjointness of the Stokes-operator defined in Section 3 (see Theorem 6.1) and the estimates in Section 5. While the local well-posedness of the strong solution and some further regularity estimates, which are basis for the studying of vanishing viscosity limit, are given in Section 7. Then in Section 8, we prove the desired results that the solutions to the initial boundary value of Navier-Stokes system, (1.1)-(1.4), converge strongly to that of the Euler system, (1.8)-(1.9), and a rate of convergence is also obtained. Finally, we give some general remarks is Section 9.

2 Some Preliminaries

In this section, we will introduce some functional spaces and provide basic estimates which will be useful for our later analysis. All of these are based on the Hodge decomposition. Let $\Omega \subset R^3$ be a bounded smooth domain, with the homology group

$H_1(\Omega, R) = 0$. Set

$$D(\Omega) = C^{\infty}(\overline{\Omega}),$$
$$D_\tau(\Omega) = \{u \in D(\Omega); \nabla \cdot u = 0,\ u \cdot n = 0\},$$
$$D_n(\Omega) = \{u \in D(\Omega); \nabla \cdot u = 0,\ u \cdot \tau = 0\},$$
$$D_0(\Omega) = \{u \in C_0^{\infty}(\Omega); \nabla \cdot u = 0\}.$$

Since $H_1(\Omega, R) = 0$, it holds the following four-fold Hodge decomposition (see Cantarella- De Turck-Gluck [8])

$$D(\Omega) = D_\tau(\Omega) \oplus G_c \oplus G_h \oplus G_g, \tag{2.1}$$

with the property that

$$\text{ker curl} = \text{image grad} = G_c \oplus G_h \oplus G_g; \tag{2.2}$$
$$\text{image curl} = D_\tau(\Omega) \oplus G_c; \tag{2.3}$$
$$\text{ker div} = D_\tau(\Omega) \oplus G_c \oplus G_h, \tag{2.4}$$

where

$$G_c = \{u = \nabla\varphi; \nabla \cdot u = 0,\ \int_{\partial\Omega_i} u \cdot n = 0 \qquad \forall\ i\};$$
$$G_h = \{u = \nabla\varphi; \nabla \cdot u = 0,\ \varphi\ = C(i) \quad \text{on } \partial\Omega_i\};$$
$$G_g = \{u = \nabla\varphi; \varphi = 0 \quad \text{on } \partial\Omega\};$$

here $\partial\Omega_i$ denotes the i-th component of $\partial\Omega$.

Define

$$H_\tau^s(\Omega) = \{u \in H^s(\Omega); \nabla \cdot u = 0,\ u \cdot n = 0\},$$

for $s \geq 0$, and

$$H_n^s(\Omega) = \{u \in H^s(\Omega) \cap H; \nabla \cdot u = 0,\ u \cdot \tau = 0\},$$

for $s \geq 1$ the closed subspaces of the Hilbert space $H^s(\Omega)$ of 3-vector valued functions; and

$$W = \{u \in H_\tau^2(\Omega); (\nabla\times)u \in H_n^1(\Omega)\}.$$

Where $u \cdot n$ and $u \cdot \tau$ denote the corresponding normal and tangential component of the velocity u on the boundary respectively, which make sense in the sense of trace. Throughout this paper $\|\cdot\|$ and $\|\cdot\|_s$ denote the norm of $L^2(\Omega)$ and $H^s(\Omega)$ respectively; while $|\cdot|_s$ denotes the norm of $H^s(\partial\Omega)$; and C and $C(\cdot)$ are some uniform constants.

Note that $H_\tau^0(\Omega)$ is the $L^2(\Omega)$ closure of $D_\tau(\Omega) \cap C_0^\infty(\Omega)$ (see for example Galdi [16]). It then holds

$$L^2(\Omega) = H_\tau^0 \oplus \{\nabla p; p \in H^1(\Omega)\}. \tag{2.5}$$

It follows from $H_1(\Omega, R) = 0$ and the Lemma 1 in Cantarella-De Turch-Gluck [8] that

$$D_\tau(\Omega) = \nabla \times D_n(\Omega). \tag{2.6}$$

Since $G_h \subset D_n(\Omega)$, the following decomposition holds

$$D_n(\Omega) = G_h \oplus D_{nc}(\Omega), \tag{2.7}$$

with $D_{nc}(\Omega) = G_h^{\perp} \cap D_n(\Omega)$. Thus $u \in D_n(\Omega)$ can be written as

$$u = v + \Sigma(u, h_j)h_j, \tag{2.8}$$

where $\{h_j\}$ is an orthogonal basis of $G_h \cong H_2(\Omega, R)$ (see Lemma 3 in Cantarella-De Turck-Gluck [8]). It is easily seen from (2.6) and (2.7) that

$$D_\tau(\Omega) = \nabla \times (D_n(\Omega) \cap G_h^{\perp}). \tag{2.9}$$

We also use the notation:

$$H^s_{nc}(\Omega) = H^s_n(\Omega) \cap G_h^{\perp}.$$

Let us recall some well known estimates:

Lemma 2.1 *Let $u \in H^s(\Omega)$ be a vector valued function. Then*

$$\|u\|_s \le C(\|\nabla \times u\|_{s-1} + \|div\ u\|_{s-1} + |n \cdot u|_{s-\frac{1}{2}} + \|u\|_{s-1}), \tag{2.10}$$

for $s \ge 0$, where $H^{-s}(\Omega)$ is the dual of $H^s_0(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, $|n \cdot u|_{s-\frac{1}{2}}$ is the norm of $n \cdot u$ in $H^{s-\frac{1}{2}}(\partial\Omega)$.

Proof: See [6] and [12].

Lemma 2.2 *Let $u \in H^1_\tau(\Omega)$. Then the following Poincare type inequality holds*

$$\|u\| \le C\|\nabla \times u\|, \tag{2.11}$$

Proof: See [46].

These two Lemmas yield

$$\|u\|_1 \le C\|\nabla \times u\|, \tag{2.12}$$

for $u \in H^1_\tau(\Omega)$. Moreover, inductively, one can deduce

Corollary 2.1 *Let $u \in H^s_\tau(\Omega)$, $s \in N$. Then*

$$\|u\|_s \le C\|\nabla \times u\|_{s-1}. \tag{2.13}$$

It was shown by von Walh [40] (see also Schwarz [34] for more general form) that the following estimate

$$\|\nabla u\| \le C(\|\nabla \times u\| + \|\nabla \cdot u\|) \tag{2.14}$$

is valid for $u \in H^1_\tau(\Omega)$ if $H_1(\Omega, R) = 0$; or $u \in H^1_n(\Omega)$ if $H_2(\Omega, R) = 0$. Noting (2.9), we would like to obtain similar estimate without assuming $H_2(\Omega, R) = 0$.

First, we claim that an estimate similar to (2.10) is also valid in terms of the corresponding tangential component, which is of independent interests and can not be found in literatures.

Theorem 2.1 *Let $u \in \mathrm{D}(\Omega)$ and $s \in N$. Then it holds that*

$$\|u\|_s \leq C(\|\nabla \times u\|_{s-1} + \|\nabla \cdot u\|_{s-1} + |u_\tau|_{s-\frac{1}{2}} + \|u\|_{s-1}). \tag{2.15}$$

Proof: By the Holge decomposition, one may write

$$u = v + \nabla p + \nabla q, \tag{2.16}$$

with v, p, and q satisfying respectively

$$-\Delta q = \nabla \cdot u, \qquad \text{in } \Omega, \tag{2.17}$$
$$q = 0, \qquad \text{on } \partial\Omega \tag{2.18}$$

$$\nabla \cdot v = 0, \qquad \text{in } \Omega, \tag{2.19}$$
$$v \cdot n = 0, \qquad \text{on } \partial\Omega \tag{2.20}$$

$$\Delta p = 0, \qquad \text{in } \Omega, \tag{2.21}$$
$$(\nabla p)_\tau = (u - v)_\tau, \qquad \text{on } \partial\Omega. \tag{2.22}$$

By the standard elliptic regularity, one has

$$\|\nabla q\|_s \leq C\|\nabla \cdot u\|_{s-1}. \tag{2.23}$$

It follows from Corollary 2.1 that

$$\|v\|_s \leq C\|\nabla \times v\|_{s-1} = C\|\nabla \times u\|_{s-1}. \tag{2.24}$$

On the other hand, Lemma 2.3 below shows

$$\|\nabla p\|_s \leq C(\|\nabla p\|_{s-1} + |(u - v)_\tau|_{s-\frac{1}{2}}). \tag{2.25}$$

Note that

$$|(u - v)_\tau|_{s-\frac{1}{2}} \leq |(u)_\tau|_{s-\frac{1}{2}} + |(v)_\tau|_{s-\frac{1}{2}}, \tag{2.26}$$

and

$$|(v)_\tau|_{s-\frac{1}{2}} \leq C\|v\|_s. \tag{2.27}$$

Thus we conclude

$$\|u\|_s = \|v + \nabla p + \nabla q\|_s \leq C(\|\nabla \times u\|_{s-1} + \|\nabla \cdot u\|_{s-1} + |n \times u|_{s-\frac{1}{2}} + \|u\|_{s-1}). \tag{2.28}$$

It remains to show

Lemma 2.3 *Let Ω be a smooth bounded domain and $p \in C^\infty(\Omega)$ be a scalar function which solves the following problem*

$$\Delta p = 0, \qquad \text{in } \Omega, \tag{2.29}$$
$$(\nabla p)_\tau = \phi, \qquad \text{on } \partial\Omega \tag{2.30}$$

with $s \in N$ being an integer. Then for $w = \nabla p$, it holds that

$$\|w\|_{s+1} \leq C(\|w\|_s + |\phi|_{s+\frac{1}{2}}). \tag{2.31}$$

Proof: For simplicity, we assume $\Omega = B(0,1) \subset R^3$ to be the unit ball (in general case, we may replace $\tilde{\Omega}$ in the following by a tubular neighborhood of $\partial\Omega$. Indeed, one can see from below that it is just an estimate for a harmonic function which is independent of the topology), and denote by $(\vec{e}_\theta, \vec{e}_\varphi, \vec{e}_r)(x)$ the standard moving frame at $x \in B(0,1) \setminus \{0\}$ corresponding to the spherical coordinates. Note that

$$\Delta w = 0, \text{ in } B(0,1), \tag{2.32}$$

so from the interior regularity of the Laplace equation implies

$$\|w\|_{s+1,B(0,\frac{1}{2})} \leq C\|w\|_{0,B(0,1)}. \tag{2.33}$$

The trace theorem and (2.33) show that

$$\|w\|_{s+\frac{1}{2},\partial B(0,\frac{1}{2})} \leq C\|w\|_{0,B(0,1)} \tag{2.34}$$

holds for $s \in N$. Now set $w_\theta(x) = w \cdot \vec{e}_\theta$ and $\tilde{\Omega} = B(0,1) \setminus \bar{B}(0, \frac{1}{2})$. Then (2.32) implies that

$$\Delta w_\theta = 2\partial_j w \cdot \partial_j \vec{e}_\theta + w \cdot \Delta \vec{e}_\theta, \qquad \text{in } \tilde{\Omega}, \tag{2.35}$$

$$w_\theta = \phi \cdot \vec{e}_\theta, \qquad \text{on } \partial B(0,1), \tag{2.36}$$

$$w_\theta = w \cdot \vec{e}_\theta, \qquad \text{on } \partial B(0, \frac{1}{2}). \tag{2.37}$$

By the elliptic regularity of Dirichlet problem, (2.34), and noting that

$$|D^k w_\theta(x)|_R \leq C\Sigma_0^k |D^j w(x)|_{R^3}, \tag{2.38}$$

one has

$$\|w_\theta\|_{s+1,\tilde{\Omega}} \leq C(\|w_\theta\|_{s,\tilde{\Omega}} + |w_\theta|_{s+\frac{1}{2},\partial B(0,\frac{1}{2})} + |\phi|_{s+\frac{1}{2}}) \leq C(\|w\|_{s,B(0,1)} + |\phi|_{s+\frac{1}{2}}). \tag{2.39}$$

Similarly, for $w_\varphi(x) = w \cdot \vec{e}_\varphi$, it also holds that

$$\|w_\varphi\|_{s+1,\tilde{\Omega}} \leq C(\|w_\varphi\|_{s,B(0,1)} + |\phi|_{s+\frac{1}{2}}). \tag{2.40}$$

Next, direct computations show that $w_r = w(x) \cdot \vec{e}_r(x)$ satisfies

$$\Delta w_r = 2\partial_j w \cdot \partial_j \vec{e}_r + w \cdot \Delta \vec{e}_r, \qquad \text{in } \tilde{\Omega}; \tag{2.41}$$

$$\nabla w_r \cdot n = -(\partial_\theta \phi + \partial_\varphi \phi), \qquad \text{on } \partial B(0,1); \tag{2.42}$$

$$\nabla w_r \cdot n = D_\tau w_\tau \qquad \text{on } \partial B(0, \frac{1}{2}), \tag{2.43}$$

where

$$D_\tau w_\tau = -(\partial_\theta w_\theta + \partial_\varphi w_\varphi).$$

It follows from the elliptic regularity that

$$\|w_r\|_{s+1,\tilde{\Omega}} \leq C(\|w_r\|_{s,\tilde{\Omega}} + \|w\|_{s,\tilde{\Omega}} + |D_\tau w_\tau|_{s-\frac{1}{2},\partial B(0,\frac{1}{2})} + |\partial_\theta \phi + \partial_\varphi \phi|_{s-\frac{1}{2}}). \tag{2.44}$$

Note that

$$|\partial_\theta \phi + \partial_\varphi \phi|_{s-\frac{1}{2}} \leq |\phi|_{s+\frac{1}{2}}, \tag{2.45}$$

and
$$|D^k w_r(x)|_R \leq C\Sigma_0^k |D^j w(x)|_{R^3}. \tag{2.46}$$

Hence, we have arrived at

$$\|w\|_{s+1,\Omega} = \|w\|_{s+1,B(0,\frac{1}{2})} + \|w_\theta \vec{e}_\theta + w_\varphi \vec{e}_\varphi + v_r \vec{e}_r\|_{s+1,\tilde{\Omega}} \leq C(\|w\|_{s,\Omega} + |\phi|_{s+\frac{1}{2}}), \tag{2.47}$$

and the lemma is proved.

Remark: The estimates in Lemma 2.3 and Theorem 2.1 also hold in the corresponding Sobolev spaces by density argument.

As a direct consequence, we have:

Corollary 2.2 *Let $u \in \mathrm{D}_\tau(\Omega)$ or $u \in \mathrm{D}_n(\Omega)$ and $s \in N$. Then*

$$\|u\|_s \leq C(\|\nabla \times u\|_{s-1} + \|u\|_{s-1}). \tag{2.48}$$

Next, we show the following Poincare type inequality for $u \in H^1_{nc}(\Omega)$.

Theorem 2.2 *Let $u \in H^1_{nc}(\Omega)$. Then it holds that*

$$\|u\| \leq C\|\nabla \times u\|. \tag{2.49}$$

Proof: If not, then there exists a sequence $u_m \in H^1_{nc}(\Omega)$ such that

$$\|u_m\| = 1, \qquad \text{and} \quad \|\nabla \times u_m\| \to 0, \quad m \to \infty.$$

It follows from (2.5) that

$$u_m = v_m + \nabla p_m, v_m \in H^0_\tau(\Omega), \tag{2.50}$$

where p_m solves the following problem:

$$-\Delta p_m = \nabla \cdot u_m; \tag{2.51}$$
$$\nabla p_m \cdot n = u_m \cdot n, \tag{2.52}$$

which implies v_m and $\nabla p_m \in H^1(\Omega)$. This, together with (2.12) and Theorem 2.1, shows that there exist subsequences of (u_m, v_m, p_m), denoted still by (u_m, v_m, p_m), such that

$$u_m \to u \text{ weakly in } H^1(\Omega); \text{strongly in } L^2(\Omega); \tag{2.53}$$
$$v_m \to 0, \text{ strongly in } H^1(\Omega); \tag{2.54}$$
$$\nabla p_m \to \nabla p \text{ weakly in } H^1(\Omega); \text{ strongly in } L^2(\Omega), \tag{2.55}$$

for some $u \in H^1_n(\Omega)$ with $\|u\| = 1$ and $\nabla p \in H^1(\Omega)$. Passing to the limit, we find that

$$-\Delta p = 0, \text{ in } \Omega, \tag{2.56}$$
$$(\nabla p)_n = u \cdot n, \text{ on } \partial\Omega. \tag{2.57}$$

Consequently, $p \in H^2(\Omega)$ is uniquely determined up to an additive constant and satisfies

$$-\Delta p = 0, \text{ in } \Omega, \tag{2.58}$$
$$(\nabla p)_\tau = 0, \text{ on } \partial\Omega. \tag{2.59}$$

However, $(\nabla p)_\tau = 0$ implies that on $\partial\Omega$, p is locally constant. Hence, $u \in G_h$, which contradicts to the fact that u is the weak limit of $u_m \in H^1_{nc}(\Omega)$ in $H^1(\Omega)$. The proof is complete.

It follows from Theorem 2.1 and Theorem 2.2 that (2.12) holds also for $u \in H^1_{nc}(\Omega)$. Moreover, we deduce inductively that

Corollary 2.3 *Let $u \in H^s_{nc}(\Omega)$ and $s \in N$. Then it holds that*

$$\|u\|_s \leq C\|\nabla \times u\|_{s-1}. \tag{2.60}$$

3 The Stokes Operator

In this section, we investigate the properties of the Stokes operator $-\Delta$ under the given slip boundary condition (1.4). First, we show that

Theorem 3.1 *The linear operator $(\nabla\times)$: $H^1_{nc}(\Omega) \to H^0_\tau(\Omega)$ is bijective and bounded.*

Proof: It is easy to check that $(\nabla\times)$ maps $H^1_{nc}(\Omega)$ to $H^0_\tau(\Omega)$ and is a bounded linear operator, see [46]. Let $h \in H^0_\tau(\Omega)$. There exists a sequence $h_m \in D_0(\Omega)$ such that $h_m \to h$ in $L^2(\Omega)$. Then (2.9) shows that there exists a sequence $v_m \in G_h^\perp \cap D(\Omega)$ such that

$$\nabla \times v_m = h_m.$$

Corollary 2.3 now shows that $\{v_m\}$ is a Cauchy sequence in $H^1(\Omega)$. Passing to the limit shows the surjectiveness. The injectiveness also follows from theorem 2.1.

By the Banach theorem of inverse operator, one can then define a bounded linear operator $R : H^0_\tau(\Omega) \to H^1_n(\Omega)$ as

$$h \to u, \text{ if } \nabla \times u = h,$$

i.e.

$$\nabla \times R(h) = h, \ \forall\, h \in H^0_\tau(\Omega).$$

Next, we have

Theorem 3.2 *The linear operator $\nabla\times$: $W \to H^1_{nc}(\Omega)$ is bijective and bounded.*

Proof: We first note that

$$(\nabla \times u, h) = (u, \nabla \times h) = 0, \tag{3.1}$$

holds for all $u \in W$, $h \in G_h$ since $h \times n = 0$ on the boundary. It follows from (3.1) and the definition of W that $\nabla\times$ maps W to $H^1_{nc}(\Omega)$ and is bounded. Let $h \in H^1_{nc}(\Omega)$. Since $D_{nc}(\Omega)$ is dense in $H^1_{nc}(\Omega)$ as in Theorem 3.1, so that there exists a subsequence $h_m \in D_{nc}(\Omega)$, such that $h_m \to h$ in $H^1(\Omega)$. Note that $D_{nc}(\Omega) \cap G_h = \{0\}$ and $D_{nc}(\Omega) \cap G_g = \{0\}$.

It follows from the Hodge decomposition (2.1) that there exists a sequence $v_m \in D_\tau(\Omega)$, such that

$$\nabla \times v_m = h_m.$$

Then $\{v_m\}$ is a Cauchy sequence in $H^2(\Omega)$ due to Corollary 2.1. Passing to the limit shows that $\nabla\times : W \mapsto H^1_{nc}(\Omega)$ is surjective. The injectiveness follows from the fact that the problem

$$\nabla \times v = 0, \tag{3.2}$$

$$v \cdot n = 0, \tag{3.3}$$

has unique solution $v = 0$ by Lemma 2.2.

Based on Theorem 3.2 and the Banach theorem of inverse operators, one can define a bounded linear operator $S : H^1_{nc}(\Omega) \to W$ by

$$u = S(h) \qquad \text{for} \quad h \in H^1_{nc}(\Omega) \qquad \text{if} \quad \nabla \times u = h.$$

Thus one has that

$$\nabla \times S(h) = h, \ \forall\, h \in H^1_{nc}(\Omega).$$

Combining both Theorems 3.1 and 3.2, we conclude that

Theorem 3.3 *The Stokes operator $-\Delta$: $W \mapsto H^0_\tau(\Omega)$ is bijective and bounded with its inverse given by $(-\Delta)^{-1} = S \circ R$ which is positive, symmetric and compact in $H^0_\tau(\Omega)$. Consequently, the eigenvalues of the Stokes operator can be listed as*

$$0 < \lambda_1 \le \lambda_2 \le \cdots \to +\infty,$$

with the corresponding eigenvectors $\{e_j\}_{j=1}^\infty \subset W$ which form a complete orthogonal basis in $H^0_\tau(\Omega)$.

Proof: It remains to check that $S \circ R$ is symmetric and positive. Indeed, noting that $\nabla \times (S \circ R(u)) \times n = \nabla \times (S \circ R(v)) \times n = 0$, for all u and v in $H^0_\tau(\Omega)$, we can obtain after integration by parts that for all u, $v \in H^0_\tau(\Omega)$,

$$(S \circ R(u), v) = (S \circ R(u), -\Delta(S \circ R)(v))$$

$$= (S \circ R(u), -\Delta(S \circ R(v))) \tag{3.4}$$

$$= \int_{\partial\Omega} S \circ R(u) \cdot \nabla \times (S \circ R(v)) \times n + (\nabla \times (S \circ R(u)), \nabla \times (S \circ R(v))) \tag{3.5}$$

$$= (\nabla \times (S \circ R(u)), \nabla \times (S \circ R(v))) = (R(u), R(v)) \tag{3.6}$$

$$= (u, S \circ R(v)). \tag{3.7}$$

Thus, $S \circ R$ is symmetric. The positiveness follows from

$$(S \circ R(u), u) = (R(u), R(u)) \ge C\|u\|^2 \tag{3.8}$$

for some uniform positive constant C. The proof of Theorem 3.3 is complete.

It follows from Theorem 3, Corollary 2.1 and Corollary 2.3 that the following regularity results hold.

Theorem 3.4

$$\begin{aligned} R(H^s_\tau(\Omega)) &= H^{s+1}_{nc}(\Omega) \quad \text{for} \quad s \ge 0, \\ S(H^s_{nc}(\Omega)) &= H^{s+1}_\tau(\Omega) \quad \text{for} \quad s \ge 1, \\ (-\Delta)^{-1}(H^s_\tau(\Omega)) &= H^{s+2}_\tau(\Omega) \quad \text{for} \quad s \ge 0. \end{aligned}$$

As an immediate consequence, one has from the Sobolev's imbedding theorem that

Corollary 3.1 *All the eigenvectors of the Stokes operator are smooth, i.e.,*

$$e_j \in \mathrm{D}(\Omega), \quad j \in \mathrm{N}. \tag{3.9}$$

4 The Boundary Condition And The Nonlinearity

In this section, we investigate some properties of the slip boundary condition (1.4). In particular, we will show that the nonlinearity in the Navier-Stokes system matches smoothly with the slip boundary condition (1.4). We begin with describing a geometrical representation of the boundary conditions (1.4).

Proposition 4.1 *Let $u \in D(\Omega)$. Then the slip boundary condition (1.4) is equivalent to*

$$u \cdot n = 0, \quad \text{and} \quad \partial_n u_\tau = 0 \quad \text{on} \quad \partial\Omega. \tag{4.1}$$

Proof: Let $x_0 \in \partial\Omega$. Since all the quantities involved are independent of choice of the coordinates, we may assume $x_0 = (0,0,0)$ and the coordinates frame $(\vec{i},\vec{j},\vec{k})$ with $\vec{k}$ being the outward normal direction. Then, it follows from $u \cdot n = 0$ that

$$\partial_j u_3 = 0; \; j = 1, 2 \tag{4.2}$$

at x_0. Hence,

$$\omega_2 = \partial_3 u_1 - \partial_1 u_3, \tag{4.3}$$

implies that

$$\partial_3 u_1 = 0 \tag{4.4}$$

at x_0. Similarly, we have

$$\partial_3 u_2 = 0 \tag{4.5}$$

at x_0. Note that this argument can be reversed. Thus the proposition follows.

Now, for $u \in D(\Omega)$, we denote the nonlinearity in the Navier-Stokes system by

$$B(u) = \omega \times u + \nabla p, \tag{4.6}$$

with

$$\omega = \nabla \times u, \tag{4.7}$$

and ∇p determined by

$$\Delta p = \nabla \cdot (\omega \times u), \qquad \text{in } \Omega, \tag{4.8}$$

$$\nabla p \cdot n = (u \times \omega) \cdot n \qquad \text{on } \partial\Omega. \tag{4.9}$$

Then our main result in this section is that $B(u)$ matches smoothly with the boundary condition (1.4), i.e.,

Theorem 4.1 *For $u \in \mathrm{D}(\Omega) \cap W$, $B(u) \in \mathrm{D}(\Omega) \cap W$.*

Proof: It follows from the definitions, (4.6)-(4.9) and elliptic regularity that $B(u) \in D(\Omega)$ and $B(u) \cdot n = 0$ on the boundary. It remains is to show $(\nabla \times B(u)) \cdot \tau = 0$ on $\partial\Omega$. Let $x_0 \in \partial\Omega$. Note that

$$\nabla \times B(u) = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u \equiv [u, \omega] \tag{4.10}$$

is independent of choices of the coordinates. We may assume that $x_0 = (0,0,0)$ and the normal direction of $\partial\Omega$ at x_0 coincides with $\vec{k}$ of the standard chart $(\vec{i},\vec{j},\vec{k})$. Then, $(\nabla \times B(u)) \cdot \tau$ can be calculated by

$$(\nabla \times B(u), \vec{i}) = u_1 \partial_1 \omega_1 + u_2 \partial_2 \omega_1 + u_3 \partial_3 \omega_1 - (\omega_1 \partial_1 u_1 + \omega_2 \partial_2 u_1 + \omega_3 \partial_3 u_1) \tag{4.11}$$

at x_0, where u_i and ∂_i denote the corresponding i-th components of u and ∇ respectively. It follows from $\omega \cdot \tau = 0$ on $\partial\Omega$ that

$$\omega_1 = 0, \quad \omega_2 = 0, \tag{4.12}$$

and

$$\partial_i \omega_j = 0, \quad i, j = 1, 2, \tag{4.13}$$

at x_0. Since $u_3 = 0$ at x_0, we get from (1.10)-(1.13) that

$$(\nabla \times B(u), \vec{i}) = -\omega_3 \, \partial_3 \, u_1 \tag{4.14}$$

at x_0. (4.13), together with Proposition 4.1, shows that

$$(\nabla \times B(u), \vec{i}) = 0 \tag{4.15}$$

at x_0. Similarly, we have

$$(\nabla \times B(u), \vec{j}) = 0, \tag{4.16}$$

at x_0 and the theorem is proved.

5 The Galerkin Approximations

In this section, we study a Galerkin approximation for the initial-boundary value problem (1.1)-(1.4) based on the orthogonal basis given in Theorem 3.3. Let $u_0 \in H^0_\tau(\Omega)$. We consider the following system of ordinary differential equations

$$u_j'(t) + \varepsilon\lambda_j \, u_j(t) + g_j(U) = 0, \tag{5.1}$$
$$u_j(0) = (u_0, e_j), \tag{5.2}$$

$j = 1, \cdots m$, where $U = (u_j)$ and

$$g_i(U) = (B(\Sigma_1^m u_j e_j), e_i).$$

It is clear that $(g_j(U))$ is Lipshitz continuous in U and thus the initial problem, (5.1)-(5.2), is locally well posed, and is equivalent to the following initial value problem

$$u_m'(t, x) - \varepsilon\Delta u_m(t, x) + P_m B(u_m)(t, x) = 0, \tag{5.3}$$
$$u_m(0) = P_m(u_0), \tag{5.4}$$

where $u_m(t, x) = \Sigma_1^m u_j(t) e_j(x)$ and P_m is the orthogonal projection of H^0_τ onto the finite dimensional space spanned by $\{e_j\}_1^m$.

Taking the inner product with u_m and noting that

$$(P_m B(u_m), u_m) = (B(u_m), u_m) = \int_\Omega \nabla \times u_m \cdot (u_m \times u_m) dx = 0, \tag{5.5}$$

we get

$$\frac{d}{dt}\|u_m\|^2 + 2\|\nabla \times u_m\|^2 = 0, \tag{5.6}$$

where we have used the fact that $e_j \in W$.

Set $\omega_m(t,x) = \nabla \times u_m(t,x)$. Then it follows from (5.3) and (5.4) that

$$\omega_m'(t,x) - \varepsilon\Delta\omega_m(t,x) + \Sigma g_j \nabla \times e_j = 0, \tag{5.7}$$
$$\omega_m(0) = \nabla \times u_m(0). \tag{5.8}$$

Taking the inner product with ω_m, and noting that

$$(\nabla \times e_i, \nabla \times e_j) = \lambda_j(e_i, e_j), \tag{5.9}$$

we get

$$\frac{d}{dt}\|\omega_m\|^2 + 2\varepsilon\|\nabla \times \omega_m\|^2 + 2(\nabla \times B(u_m), \omega_m) = 0. \tag{5.10}$$

Set $v_m(t,x) = -\Delta u_m(t,x)$. Then it follows from (5.3) and (5.4) that

$$v_m'(t,x) - \varepsilon\Delta v_m(t,x) + \Sigma g_j\, \lambda_j\, e_j = 0, \tag{5.11}$$
$$v_m(0) = P_m(v_0). \tag{5.12}$$

Due to Theorem 4.1, $B(u_m) \in W$, which implies that

$$(-\Delta P_m B(u_m), e_i) = (\Sigma g_j \lambda_j e_j, e_i) = (B(u_m), -\Delta e_i) = (-\Delta B(u_m), e_i). \tag{5.13}$$

Thus,

$$-\Delta P_m B(u_m) = P_m(-\Delta B(u_m)). \tag{5.14}$$

Here we have used the facts that $\nabla \times e_i \times n = 0$, $\nabla \times e_j \times n = 0$ and $\nabla \times B(u_m) \times n = 0$ on the boundary.

Taking the inner product of (5.11) with v_m and $-\Delta v_m$ respectively, and noting (5.14), we find

$$\frac{d}{dt}\|v_m\|^2 + 2\varepsilon\|\nabla \times v_m\|^2 + 2(-\Delta B(u_m), v_m) = 0, \tag{5.15}$$

and

$$\frac{d}{dt}\|\nabla \times v_m\|^2 + 2\varepsilon\|\Delta v_m\|^2 + 2((\nabla\times)^3 B(u_m), \nabla \times v_m) = 0. \tag{5.16}$$

6 The Weak Solutions

In this section, we will obtain a weak solution to the initial-boundary value problem (1.1)-(1.4) by using the Galerkin approximation given in Section 5. This argument is similar to the argument in [12]. We begin by showing

Theorem 6.1 *The Stokes operator $-\Delta$ with $D(-\Delta) = W$ defined in Section 3 is self-adjoint on $H^0_\tau(\Omega)$.*

Proof: We have shown that $-\Delta$ is symmetric in Section 3. To show that $-\Delta$ is self-adjoint on $H^0_\tau(\Omega)$, we introduce a bilinear from

$$a(u,v) = (\nabla \times u, \nabla \times v); \quad u, v \in D(a), \tag{6.1}$$

with $D(a)$ being the closure of W under the inner product $((u,v)) = a(u,v)$. First, we claim that $D(a) = H^1_\tau(\Omega)$. It is clear that $D(a) \subset H^1_\tau(\Omega)$ is a closed subspace. If $D(a) \neq H^1_\tau(\Omega)$, by the orthogonal decomposition theorem in Hilbert spaces, there exists a $v \in H^1_\tau(\Omega) \setminus D(a)$ which is orthogonal to $D(a)$ under the inner product (6.1). Since $e_j \in D(a)$, it follows that

$$0 = (\nabla \times e_j, \nabla \times v) = (-\Delta e_j, v) = \lambda_j (e_j, v) \tag{6.2}$$

and then $v = 0$ that is a contradiction. Next, it is clear that $a(u,v)$ is symmetric, positive and closed due to Lemma 2.2 and (2.12). Since $D_0(\Omega) \subset D(a)$ which is dense in $H^0_\tau(\Omega)$, thus $a(u,v)$ is densely defined. Hence there exists an operator $A : D(A) \to H^0_\tau(\Omega)$ which is the Friedrichs self-adjoint extension of $-\Delta$ with the domain $D(A)$, $W \subset D(A) \subset D(a)$ such that for any $h \in H^0_\tau(\Omega)$ there exists a unique $u \in D(A)$ with $Au = h$ satisfying

$$(Au, v) = a(u,v) = (\nabla \times u, \nabla \times v); \quad \forall v \in D(a). \tag{6.3}$$

Now let $u \in D(A)$, and set $h = Au$. It then follows from Theorem 3.3 that there exists a $\bar{u} \in W$ such that $-\Delta \bar{u} = h$ and then

$$(-\Delta \bar{u}, v) = (\nabla \times \bar{u}, \nabla \times v); \quad \forall v \in D(a). \tag{6.4}$$

Combining (6.3) with (6.4) shows that

$$(\nabla \times (u - \bar{u}), \nabla \times v); \quad \forall\, v \in D(a). \tag{6.5}$$

Set $v = u - \bar{u}$. Note that u and $\bar{u}$ $\in D(a)$. It follows from (6.5) that

$$\nabla \times (u - \bar{u}) = 0. \tag{6.6}$$

Since $H_1(\Omega, R) = 0$, (6.6) implies there exists a $p \in L^2(\Omega)$ such that

$$(u - \bar{u}) = \nabla p. \tag{6.7}$$

Finally, noting $(u - \bar{u}) \cdot n = 0$, we show that $u = \bar{u}$ since

$$\Delta p = 0; \tag{6.8}$$

$$\nabla p \cdot n = 0 \tag{6.9}$$

has only constant solution which implies $D(A) = W$ and $A = -\Delta$.

We now can define the weak solutions for the initial-boundary value problem (1.1)-(1.4).

Definition 6.1 *u is a weak solution of (1.1)-(1.4) with the initial data $u_0 \in H^0_\tau(\Omega)$ on the time interval $[0,T)$ if $u \in L^2(0,T;V) \cap C_w[0,T;H^0_\tau(\Omega))$ satisfying $u' \in L^1(0,T;V^*)$ and*

$$(u', v) + \varepsilon(\nabla \times u, \nabla \times v) + (\omega \times u, \nabla \times v) = 0, \ \text{a.e. } t \tag{6.10}$$

for all $v \in V$, and

$$u(0) = u_0. \tag{6.11}$$

where V^* denotes the dual space of $V = \mathrm{D}(a) = H^1_\tau(\Omega)$. And a weak solution u is said to be strong solution to (1.1)-(1.4) with the initial data $u_0 \in H^1_\tau(\Omega)$ if $u \in C([0,T], H^1(\Omega))$ and $u^1 \in L^2((0,T), L^2(\Omega))$.

Our main result in this section in the following existence of weak solution:

Theorem 6.2 *For $u_0 \in H^0_\tau(\Omega)$ and $T > 0$, there exists at least one weak solution u of (1.1)-(1.4) which satisfies the energy inequality*

$$\frac{d}{dt}\|u\|^2 + 2\varepsilon\|\nabla \times u\|^2 \leq 0 \tag{6.12}$$

in the sense of distribution.

Proof: Let $u_0 \in H^0_\tau(\Omega), T > 0$, and $\{u_m\}$ be the Galerkin approximations constructed in Section 5. It follows from the energy equation (5.6) and the Gronwall inequality that

$$\{u_m\} \text{ bounded in } L^\infty(0,T; H^0_\tau(\Omega)),$$

$$\{u_m\} \text{ bounded in } L^2(0,T; H^1_\tau(\Omega)).$$

Note that for $v \in H^1_\tau(\Omega)$, one has

$$|(-\Delta u_m, v)| = |(\omega_m, \nabla \times v)|, \tag{6.13}$$

which implies that

$$\{-\Delta u_m\} \text{ bounded in } Ł^2(0,T; V^*); \tag{6.14}$$

and

$$|(P_m B(u_m), v)| = |(\omega_m \times u_m, v_m)| = |(u_m \cdot \nabla)v_m, u_m)| \leq C \int_\Omega |u_m|^2 |\nabla v_m| dx. \tag{6.15}$$

Since $H^1_\tau(\Omega) \subset L^6(\Omega)$, (6.15) implies that

$$\{B(u_m)\} \text{ bounded in } Ł^{\frac{4}{3}}(0,T; V^*). \tag{6.16}$$

Hence,

$$\{u'_m\} \text{ bounded in } Ł^{\frac{4}{3}}(0,T; V^*). \tag{6.17}$$

Now, the rest of argument is similar to the standard one as in [12], thus the theorem is proved.

Remark: It should be noted that for a weak solution u, boundary condition $\omega \cdot \tau = 0$ is missed somehow since it makes no sense by a tangential trace of a vector-valued function in $L^2(\Omega)$. However, it can be shown that there exists a $T_0 > 0$ such that $u(t) \in W$ for $t \in (0, T_0]$ for a weak solution u. Thus the boundary condition $\omega \cdot \tau = 0$ is recovered. We will not pursue this issue in this paper.

7 The Strong Solutions

In this section, we investigate the local well posed-ness of strong solution and its regularities on the time interval.

Let $u_0 \in H^1_\tau(\Omega)$ and u_m be the Galerkin approximate solutions. It follows from the energy equation (5.10), Corollary 2.1, and the inequality

$$\|v\|^2_{L^\infty(\Omega)} \leq C\|v\|_1\|v\|_2, \ \forall \ v \ \in \ H^2(\Omega), \tag{7.1}$$

that

$$(\nabla \times B(u_m), \omega_m) = (\omega_m \times u_m, -\Delta u_m) \le C\|\omega_m\|^{\frac{3}{2}}\|\Delta u_m\|^{\frac{3}{2}}, \tag{7.2}$$

and

$$\frac{d}{dt}\|\omega_m\|^2 + \varepsilon\|\Delta u_m\|^2 \le C\|\omega_m\|^6. \tag{7.3}$$

Consequently, there is time $T_0 > 0$ such that, for any fixed $T \in (0, T_0)$,

$$\{u_m\} \text{ bounded in } L^\infty(0,T;H^1(\Omega)), \text{and} \tag{7.4}$$
$$\{u_m\} \text{ bounded in } L^2(0,T;H^2(\Omega)). \tag{7.5}$$

Note that

$$\|P_m(v \times u)\| \le \|v \times u\| \le C\|v\|\|u\|_{L^\infty(\Omega)}. \tag{7.6}$$

It follows from (7.4)-(7.6) and (5.3) that

$$\{u_m'\} \text{ bounded in } L^2(0,T;L^2(\Omega)). \tag{7.7}$$

Then the standard compactness arguments show that there exists a subsequence of u_m, denoted still by u_m, and a u such that

$$u_m \to u \text{ in } L^\infty(0,T;H^1(\Omega)) \text{ weak} - \text{star}, \tag{7.8}$$
$$u_m \to u \text{ in } L^2(0,T;H^2(\Omega)) \text{ weakly}, \tag{7.9}$$
$$u_m \to u \text{ in } L^2(0,T;H^1(\Omega)) \text{ strongly}. \tag{7.10}$$

Passing to the limit shows that u is a weak solution and such that $u \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$.

Furthermore, standard arguments based on (7.7)-(7.11) show that $u' \in L^2(0,T;L^2(\Omega))$ and $u \in C([0,T];H^1(\Omega))$. Hence, u is a strong solution. Now, let u and v be two strong solutions. Then $w = u - v$ satisfies the following equation

$$w' - \varepsilon\Delta w + B(u) - B(v) = 0. \tag{7.11}$$

Taking the inner of (7.11) with w and noting (7.6) and (7.1), we find

$$\frac{d}{dt}\|w\|^2 \le C(T)\|w\|^2. \tag{7.12}$$

Then, $u = v$ follows from $w(0) = 0$ and the Gronwall inequality. Then by the standard continuation method, we can conclude that

Theorem 7.1 *Let $u_0 \in H^1_\tau(\Omega)$. Then there is a time $T^* = T^*(u_0) > 0$, such that the problem (1.1)-(1.4) with initial data u_0 has a unique strong solution of u on the interval $[0,T^*)$ satisfying*

$$u \in L^2(0,T^*;W) \cap C([0,T^*);H^1_\tau(\Omega)); \tag{7.13}$$
$$u' \in L^2(0,T^*;H^0_\tau(\Omega)); \tag{7.14}$$
$$\|u\|_1 \to \infty, \qquad \text{as} \quad t \to T^*, \qquad \text{if} \quad T^* < \infty, \tag{7.15}$$

for any $T \in (0,T^*)$. Furthermore, the following identity

$$\frac{d}{dt}\|\omega\|^2 + 2\varepsilon\|\Delta u\|^2 + 2(\nabla \times B(u), \omega) = 0, \tag{7.16}$$

holds and (6.12) also becomes the energy identity.

Remark: Let $u_0 \in H^1_\tau$. Our previous arguments show that there exists a unique strong solution, $u(t)$, to the initial-boundary value problem (1.1)-(1.4) with initial data u_0, which can be obtained as a limit of the Galerkin approximations constructed in Section 5. In fact, the uniqueness of the strong solution implies that the whole sequence of the Galerkin approximate solutions converges. Furthermore, the boundary condition, $w \cdot \tau = 0$, is satisfied in the sense of trace.

Next, we study further regularity of the strong solution if the initial data is more regular. First, we assume that $u_0 \in W$. Let u_m be the Galerkin approximate solution in Section 5, and $v_m = -\Delta u_m$. Note that

$$|(\Delta B(u_m), v_m)| = |(\nabla \times B(u_m), \nabla \times v_m)| \leq C(\|u_m\|_{L^\infty} \|u_m\|_2 + \|\nabla u_m\|^2_{L^4(\Omega)})\|\nabla \times v_m\|, \tag{7.17}$$

due to Theorem 4.1. It follows from (5.15), Sobolev's inequality and Corollary 2.1 that

$$\frac{d}{dt}\|v_m\|^2 \leq C\|v_m\|^4. \tag{7.18}$$

Hence, one can show that

$$u_m \text{ is uniformly bounded in } L^\infty(0,T;H^2(\Omega)), \tag{7.19}$$
$$u_m \text{ is uniformly bounded in } L^2((0,T);H^3(\Omega)), \tag{7.20}$$

which imply the following regularity result:

Theorem 7.2 *The unique strong solution u also belongs to $C((0,T^*);W)$. Moreover, if $u_0 \in W$, then*

$$u \in L^2(0,T^*;H^3(\Omega)) \cap C([0,T^*);W); \tag{7.21}$$
$$u' \in L^2(0,T^*;H^1_\tau(\Omega)). \tag{7.22}$$

and the energy equation

$$\frac{d}{dt}\|v\|^2 + 2\varepsilon\|\nabla \times v\|^2 + 2(-\Delta B(u), v) = 0, \tag{7.23}$$

holds for $v = -\Delta u$ on the time interval $(0,T^*)$.

Similarly, if $u_0 \in W \cap H^3(\Omega)$, one can conclude from (5.16) that

$$\{u_m\} \text{ bounded in } L^\infty(0,T;H^3(\Omega)), \tag{7.24}$$
$$\{u_m\} \text{ bounded in } L^2(0,T;H^4(\Omega)), \tag{7.25}$$
$$\{u'_m\} \text{ bounded in } L^2(0,T;W) \tag{7.26}$$

and the following further regularity result can be obtained:

Theorem 7.3 *The unique strong solution u also belong to $C((0,T^*);H^3(\Omega))$. If $u_0 \in W \cap H^3(\Omega)$, then*

$$u \in L^2(0,T^*;H^4(\Omega)) \cap C([0,T^*);H^3(\Omega)); \tag{7.27}$$
$$u' \in L^2(0,T^*;W), \tag{7.28}$$

and the energy equation

$$\frac{d}{dt}\|\nabla \times v\|^2 + 2\varepsilon\|\Delta v\|^2 + 2(\Delta B(u), \Delta v) = 0, \tag{7.29}$$

holds for $v = -\Delta u$ in the sense of distribution. Moreover v satisfies

$$(\nabla \times v) \cdot \tau = 0. \tag{7.30}$$

for a.e.$t \in (0, T^*)$.

Remark: Indeed, we have shown that $\omega = \nabla \times u$ satisfies

$$\partial_t \omega - \varepsilon \Delta \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0; \text{in } \Omega; \tag{7.31}$$
$$\nabla \cdot \omega = 0; \text{in } \Omega; \tag{7.32}$$
$$\omega \cdot \tau = 0;\ (\Delta\omega) \cdot \tau = 0, \tag{7.33}$$

and $v = -\Delta u$ satisfies

$$\partial_t v - \varepsilon \Delta v + \nabla \times (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0; \text{in } \Omega; \tag{7.34}$$
$$\nabla \cdot v = 0; \text{in } \Omega; \tag{7.35}$$
$$v \cdot n = 0;\ (\nabla \times v) \cdot \tau = 0, \tag{7.36}$$

for the corresponding solutions with initial data $u_0 \in W \cap H^3(\Omega)$.

8 The Vanishing Viscosity Limit

In this section, we investigate the asymptotic behavior of the solutions to the Navier-Stokes systems with the Navier-type boundary condition to the solution to the Euler system (1.8) with the boundary condition (1.9) as the viscosity $\varepsilon \to 0$. We begin by recalling the classical result of local smooth solution to the Euler equations:

$$\partial_t u + \omega \times u + \nabla \bar{p} = 0; \quad \text{in } \Omega; \tag{8.1}$$
$$\nabla \cdot u = 0; \quad \text{in } \Omega; \tag{8.2}$$
$$\omega = \nabla \times u; \quad \text{in } \Omega; \tag{8.3}$$

with the following boundary condition

$$u \cdot n = 0, \text{ on } \partial\Omega. \tag{8.4}$$

Proposition 8.1 *Let $u_0 \in H^3_\tau(\Omega)$. Then there is a $T_0 > 0$ and a unique vector-valued function*

$$u \in C([0, T_0]; H^3_\tau(\Omega))$$

satisfying (8.1)-(8.4) and $u(0) = u_0$.

For the proof, we refer to the references [6, 14, 22].

Next, we prove

Theorem 8.1 *Let $u_0 \in W \cap H^3(\Omega)$. Then there is a $T_0 > 0$ such that the strong solution $u(\varepsilon)$ to the problem (1.1)-(1.4) with the initial data u_0 converges to the unique solution u of the problem (8.1)-(8.4) with the same initial data u_0 in the following sense*

$$u(\varepsilon) \to u \text{ in } L^p(0, T_0; H^3(\Omega)); \tag{8.5}$$
$$u(\varepsilon) \to u \text{ in } C([0, T_0]; H^2(\Omega)), \tag{8.6}$$

$1 \le p < \infty$, as $\varepsilon \to 0$.

Proof: Let $u_0 \in W \cap H^3(\Omega)$ be given. It follows from Theorem 7.1 and Theorem 7.3 that for any $\varepsilon > 0$ there is time $T^* = T^*(\varepsilon) > 0$ such that the solution $u(\varepsilon)$ satisfies

$$u(\varepsilon) \in L^2(0, T^*; H^4(\Omega)) \cap C([0, T^*); H^3(\Omega)), \tag{8.7}$$
$$\|u(\varepsilon)\|_1 \to \infty, \ as\ t \to T^*, \tag{8.8}$$
$$\frac{d}{dt}\|\nabla \times v(\varepsilon)\|^2 + 2\varepsilon\|\Delta v(\varepsilon)\|^2 + 2(\Delta B(u(\varepsilon)), \Delta v(\varepsilon)) = 0, \tag{8.9}$$
$$(\nabla \times v) \cdot \tau = 0, \ a.e.\ t \in [0, T^*), \tag{8.10}$$

where $v(\varepsilon) = -\Delta u(\varepsilon)$.

First, we claim that $T^*(\varepsilon)$ is bounded from below for all $\varepsilon > 0$. Indeed, (7.30) implies

$$(\Delta B(u(\varepsilon)), \Delta v(\varepsilon)) = ((\nabla\times)^3 B(u(\varepsilon)), \nabla \times v(\varepsilon)). \tag{8.11}$$

Director calculations show

$$(\nabla\times)^3 B(u(\varepsilon)) = -(u(\varepsilon) \cdot \nabla)\nabla \times v(\varepsilon) + \Sigma_{i,j=1,2,3;i+j=4},\ F_{i,j}(D^i u(\varepsilon), D^j u(\varepsilon)), \tag{8.12}$$

where $F_{i,j}(D^i u, D^j)$ are bilinear forms and D^i is the $i - order$ differential operator. Note

$$((u(\varepsilon) \cdot \nabla)\nabla \times v(\varepsilon), \nabla \times v(\varepsilon)) = 0, \tag{8.13}$$

and

$$\|\Sigma_{i,j=1,2,3;i+j=4}, F_{i,j}(D^i u(\varepsilon), D^j u(\varepsilon))\| \leq C\|u(\varepsilon)\|_3^2. \tag{8.14}$$

It follows from (8.11)-(8.14) and Corollary 2.1 that

$$|(\Delta B(u(\varepsilon)), \Delta v(\varepsilon))| \leq C\|\nabla \times v(\varepsilon)\|^3, \tag{8.15}$$

and

$$\frac{d}{dt}\|\nabla \times v(\varepsilon)\|^2 + 2\varepsilon\|\Delta v(\varepsilon)\|^2 \leq C\|\nabla \times v(\varepsilon)\|^3. \tag{8.16}$$

Comparing (8.16) with the following problem

$$y'(t) = Cy(t)^{\frac{3}{2}}, \tag{8.17}$$
$$y(0) = \|\nabla \times v(0)\|^2, \tag{8.18}$$

and denoting by $\bar{T}$ a time before the blow up time for (8.17)-(8.18), one can show that

$$T^*(\varepsilon) \geq \bar{T}, \forall \varepsilon > 0. \tag{8.19}$$

Next, for any $T < \bar{T}$, it follows from the energy inequality (8.16) and the equation (1.1)-(1.4) that

$$u(\varepsilon) \text{ uniforly bounded in } C([0, T]; H^3(\Omega)); \tag{8.20}$$
$$u'(\varepsilon) \text{ uniformly bounded in } L^2(0, T; W), \tag{8.21}$$

for all $\varepsilon > 0$. By using the standard compactness result (see [37] for example) there is a subsequence ε_n of ε and a vector function u such that

$$u_n \to u \text{ in } L^p(0, T; H^3(\Omega)), \tag{8.22}$$
$$u_n \to u \text{ in } C([0, T]; H^2(\Omega)) \tag{8.23}$$

for all $1 \leq p < \infty$, as $\varepsilon_n \to 0$, where $u_n = u(\varepsilon_n)$ denotes the corresponding solution of the equation (1.1)-(1.4) with the initial data u_0.

Passing to the limit shows that u solves the Euler equation (8.1)-(8.4) and satisfies the following extra boundary condition

$$(\nabla \times u)_\tau = 0, \text{on } \partial\Omega, \tag{8.24}$$

with p satisfying (4.7)-(4.9) corresponding to u. Since the solution to the initial boundary value problem of the Euler equation is unique, we get the desired convergence results.

Consequently, we have

Corollary 8.1 *Let $u_0 \in H^3(\Omega) \cap W$. Then the unique solution u of the Euler equation with initial data u_0 and the slip boundary condition $u \cdot n = 0$ on $\partial\Omega$ satisfies an extra boundary condition $(\nabla \times u)_\tau = 0$ on $\partial\Omega$ on its maximum existent interval $[0, \bar{T})$.*

Finally, we can obtain the following rate of convergence.

Theorem 8.2 *Let $u_0 \in H^3(\Omega) \cap W$, and $T, \bar{T}$ be defined as above. Then we have*

$$\|u(\varepsilon) - u\|_2^2 \leq C(T)\varepsilon, \tag{8.25}$$

on the interval $[0, \min\{T, \bar{T}\}]$.

Proof: Set $w = u(\varepsilon) - u$. Then $\tilde{w} = -\Delta w \in H^1(\Omega)$ solves equations

$$\partial_t \tilde{w} - (\Delta)(B(u(\varepsilon)) - B(u)) = -\varepsilon(\Delta)^2 u(\varepsilon); \text{ in } \Omega; \tag{8.26}$$

$$\nabla \cdot \tilde{w} = 0, \text{ in } \Omega; \tag{8.27}$$

$$\tilde{w} \cdot n = 0, \text{ on } \partial\Omega, \tag{8.28}$$

and $\nabla \times \ u(\varepsilon) \times n = 0,\ \nabla \times \ u \times n = 0,\ (\nabla \times)^3 u(\varepsilon) \times n = 0$ on $\partial\Omega$. Taking the $L^2(\Omega)$ inner product of (8.26) with $\tilde{w}$ and integrating by part, one gets that

$$\frac{d}{dt}\|\tilde{w}\|^2 - 2(\Delta(B(u(\varepsilon)) - B(u)), \tilde{w}) = -2\varepsilon((\nabla \times)^3 u(\varepsilon), \nabla \times \tilde{w}). \tag{8.29}$$

By noting $((\psi \cdot \nabla)w, w) = 0$ for any $\psi \in H^1_\tau(\Omega)$, and

$$\Delta B(u(\varepsilon) - B(u)) = (\psi \cdot \nabla)\tilde{w} + \Sigma_{j=1,2}, F_{i,j}(D^i u, D^j w), \tag{8.30}$$

for some $\psi \in H^1_\tau(\Omega)$, where F_{ij} are bilinear forms, one can get the following estimate

$$|(\Delta(B(u(\varepsilon)) - B(u)), \tilde{w})| \leq C(\|u(\varepsilon)\|_3 + \|u\|_3)\|\nabla \times \tilde{w}\|^2, \tag{8.31}$$

with C a uniform constant independent of ε.

On the other hand,

$$|((\nabla \times)^3 u(\varepsilon), \nabla \times \tilde{w})| \leq C\|(\nabla \times)^3 u(\varepsilon)\|(\|(\nabla \times)^3 u(\varepsilon)\| + \|(\nabla \times)^3 u\|), \tag{8.32}$$

for a uniform constant C independent of ε.

It follows from (8.29), (8.31) and (8.32) that

$$\frac{d}{dt}\|\tilde{w}\|^2 \leq C(T)(\|\tilde{w}\|^2 + \varepsilon). \tag{8.33}$$

Since $w(0) = 0$, then the Gronwall's inequality shows

$$\|\tilde{w}\|^2 \leq C(T)\varepsilon, \tag{8.34}$$

and the theorem is proved.

9 Further remark on general domains

If Ω is a general bounded smooth domain, the subspace of harmonic knots

$$D_{\tau,h}(\Omega) = \{u \in D(\Omega); \nabla \times u = 0,\ \nabla \cdot u = 0,\ u \cdot n = 0\},$$

of $D_\tau(\Omega)$ may be not empty. By using the following Hodge decomposition

$$D(\Omega) = D_{\tau,f}(\Omega) \oplus D_{\tau,h}(\Omega) \oplus G_c \oplus G_h \oplus G_g, \tag{9.1}$$

where $D_{\tau,f}(\Omega) = \{u \in D_{\tau,f}(\Omega); \text{all interior fluxes} = 0\}$, and replacing $H^s_\tau(\Omega)$ by $H^s_{\tau,f}(\Omega)$, we can consider the topological Navier-Stokes equations

$$\partial_t u - \varepsilon\Delta u + \omega \times u + \nabla p = \Sigma f_k(u)h_k^F; \quad \text{in } \Omega; \tag{9.2}$$
$$\nabla \cdot u = 0; \quad \text{in } \Omega; \tag{9.3}$$
$$\int_\Sigma u \cdot n_\Sigma = 0, \forall\ \Sigma; \tag{9.4}$$
$$\omega = \nabla \times u; \quad \text{in } \Omega; \tag{9.5}$$

with the following slip boundary conditions

$$u \cdot n = 0, \quad \omega \cdot \tau = 0, \quad \text{on } \partial\Omega, \tag{9.6}$$

and obtain similar results as in previous sections, where $\{h_k^F\}$ is a basis of $D_{\tau,h}(\Omega) \cong H_2(\Omega, R)$, $f_k(u) = (\omega \times u, h_k^F)$ and Σ be an interior section. It is helpful to understand the term $\Sigma f_k(u)h_k^F$ as a topological force by recalling the derivation of the Newton fluids (see for example [26]). However, it is clear at this moment that the above formula describes a natural fluid motion, and we leave it to be studied further in the future. By taking the curl of these equations, it takes the following form

$$\partial_t \omega - \varepsilon\Delta\omega + \nabla \times (\omega \times u) = 0; \quad \text{in } \Omega; \tag{9.7}$$
$$\nabla \cdot u = 0; \quad \text{in } \Omega; \tag{9.8}$$
$$\int_\Sigma u \cdot n_\Sigma = 0 \tag{9.9}$$
$$\omega = \nabla \times u; \quad \text{in } \Omega, \tag{9.10}$$

the boundary conditions may be assumed as above.

We also note that an external force F can be also considered under assumptions such that $(\nabla \times F) \cdot \tau = 0$ make sense, all results of this paper are still true.

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