GLOBAL MULTIDIMENSIONAL SHOCK WAVE FOR THE STEADY SUPERSONIC FLOW PAST A THREE-DIMENSIONAL CURVED CONE

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Abstract

In this paper, we establish the global existence and stability of a multidimensional conic shock wave for three dimensional steady supersonic flow past an infinite cone. The flow is assumed to be hypersonic and described by a steady potential flow equation. Under an appropriate boundary condition on the curved cone, we show that a pointed shock attached at the vertex of the cone will exist globally in the whole space.

Keywords: Steady potential flow equation, supersonic flow, hypersonic flow, multidimensional shock wave, global existence

Mathematical Subject Classification: 35L70, 35L65, 35L67, 76N15

§1. Introduction

In this paper we are concerned with the inviscid and isentropic steady supersonic gas flow in three space dimensions. The steady flow is defined by a motion in which flow velocity, pressure and density remain unchanged in time. It is described by the following compressible Euler equations:

$$\begin{cases} \displaystyle\sum_{j=1}^{3} \partial_j(\rho u_j) = 0, \\ \displaystyle\sum_{j=1}^{3} \partial_j(\rho u_i u_j) + \partial_i P = 0, \qquad i = 1, 2, 3, \end{cases} \tag{1.1}$$

where ρ, $u = (u_1, u_2, u_3)$ and P stand for the density, the velocity and the pressure respectively. For the polytropic gas, $P(\rho) = A\rho^\gamma$ with the constants $A > 0$ and $1 < \gamma < 2$, here γ is the adiabatic exponent (especially, $\gamma \approx 1.4$ with respect to the air).

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The system (1.1) is multidimensional quasilinear hyperbolic for the supersonic flow (namely, $|u| = \sqrt{u_1^2+u_2^2+u_3^2} > c(\rho) = \sqrt{P'(\rho)}$). In the general case, it can contain the shock waves, the rarefaction waves, the contact discontinuities and even more complicated singularities (see [1-2], [6], [15-16], [21-25], [30-35] and the references therein). So the global existence results of weak solutions with the explicit structures are very rare for the multidimensional conservation law system (1.1). In this paper, we are interested in the global existence and stability of a conic shock wave, which is formed by the supersonic flow past a sharp curved cone. As it is described in the book [6], if a supersonic flow hits a circular cone with an axis being parallel to the velocity of the upstream flow and the vertex angle being less than a critical value, then there appears a circular conical shock attached at the tip of the cone, moreover the flow field between the shock front and the surface of the cone can be determined by solving a second order ordinary differential equation with two boundary conditions. If the infinite cone is symmetrically curved, in [5] we have proved the global existence and stability of a shock attached at the tip of the cone. In addition, W.C.Lien and T.P.Liu in [19] obtained the global existence of a weak solution and long distance asymptotic behavior in the symmetric cone case under suitable conditions on the Mach number, the vertex angle and the shock strength by using Glimm's scheme. In this paper, under an appropriate boundary condition on the cone surface, we focus on establishing the global existence of a multidimensional conic shock as observed in physical experiments and numerical computations. This boundary condition on the cone surface is plausible from the physical point of view for the permeable (porous or perforated) surface of the airfoil (one can see the more explanations in [17-18] for the physical senses with respect to some boundary conditions).

We will restrict ourselves to the irrotational and isentropic case for (1.1). In this case, by introducing a velocity potential one can reduce (1.1) to a second order quasilinear equation, which has been used and strongly recommended in many bibliographies (see [21-25] and so on).

Set $u_i = \partial_i\varphi$. Then from the second equation in (1.1) we have

$$\frac{1}{2}\partial_i(|\nabla_x\varphi|^2) + \partial_i h(\rho) = 0, \tag{1.2}$$

where $h(\rho)$ is the specific enthalpy satisfying $h'(\rho) = \frac{P'(\rho)}{\rho} > 0$ for $\rho > 0$. For the polytropic gas,

$$P(\rho) = A\rho^\gamma, \gamma > 1, h(\rho) = \frac{A\gamma}{\gamma-1}\rho^{\gamma-1}. \tag{1.3}$$

Hence, it holds the Bernoulli's law

$$\frac{1}{2}|\nabla_x\varphi|^2 + h(\rho) \equiv C_0. \tag{1.4}$$

here C_0 is the Bernoulli constant of fluid.

Let $H(q)$ be the inverse function of $h(\rho)$. Then

$$\rho = h^{-1}(C_0 - \frac{1}{2}|\nabla\varphi|^2) \equiv H(\nabla\varphi). \tag{1.5}$$

Substituting (1.5) into the first equation in (1.1), one can get

$$\sum_{j=1}^{3}(H(\nabla\varphi)\partial_j\varphi) = 0. \tag{1.6}$$

More precisely, for the smooth solution φ, (1.6) is equivalent to the following second order quasilinear equation

$$
\begin{aligned}
&((\partial_3\varphi)^2-c^2)\partial_3^2\varphi+((\partial_1\varphi)^2-c^2)\partial_1^2\varphi+((\partial_2\varphi)^2-c^2)\partial_2^2\varphi+2\partial_1\varphi\partial_2\varphi\partial_{12}^2\varphi\\
&+2\partial_1\varphi\partial_3\varphi\partial_{13}^2\varphi+2\partial_2\varphi\partial_3\varphi\partial_{23}^2\varphi=0,
\end{aligned}
\tag{1.7}
$$

where $c^2(\rho)=P'(\rho)=\dfrac{H(\nabla\varphi)}{H'(\nabla\varphi)}$.

It is easy to verify that (1.6) or (1.7) is strictly hyperbolic with respect to x_3-direction if $\partial_3\varphi>c(\rho)$, which means that the third component u_3 of velocity u is supersonic.

Suppose that there is a uniform supersonic flow $(u_1,u_2,u_3)=(0,0,q_0)$ with constant density $\rho_0>0$ which comes from minus infinity. The flow hits a circular cone, whose surface is denoted by $\sqrt{x_1^2+x_2^2}=b_0x_3$ for $x_3\geq 0$. When the vertex angle $\theta_0=arctg b_0$ is less than a given value θ^*, which is determined by the parameters of the coming flow, then there will be a conic shock $\sqrt{x_1^2+x_2^2}=s_0x_3$ $(s_0>b_0)$ attached at the tip of the cone, moreover the solution of (1.1) is self-similar. Under the cylindrical coordinates (r,θ,z) with $r=\sqrt{x_1^2+x_2^2},\theta=arctg\frac{x_2}{x_1}$ and $x_3=z$, the potential function $\bar\varphi(x)$ has such a form: $\bar\varphi(x)=z\psi(s)$, here $s=\frac{r}{z}$. In addition, by the equation (1.7), the Rankine-Hugoniot condition on the shock $s=s_0$ and the fixed boundary condition on $s=b_0$, one can arrive at

$$
\begin{cases}
c^2((1+s^2)\psi_{ss}+\frac{1}{s}\psi_s)-((1+s^2)\psi_s-s\psi)^2\psi_{ss}=0, & b_0<s<s_0,\\
((1+s^2)\psi_s-s\psi)H+s\rho_0q_0=0 \quad on \quad s=s_0,\\
\psi=q_0 \quad on \quad s=s_0,\\
(1+b_0^2)\psi_s-s\psi=0 \quad on \quad s=b_0.
\end{cases}
\tag{1.8}
$$

And Lax's geometric entropy condition on $s=s_0$ implies

$$
\begin{cases}
\lambda_1(s_0)<s_0<\lambda_2(s_0)\\
\frac{c(\rho_0)}{\sqrt{q_0^2-c^2(\rho_0)}}<s_0
\end{cases}
\tag{1.9}
$$

where $\lambda_{1,2}(s)=\frac{\partial_r\bar\varphi\partial_z\bar\varphi\mp c\sqrt{(\partial_r\bar\varphi)^2+(\partial_z\bar\varphi)^2-c^2}}{(\partial_z\bar\varphi)^2-c^2}$.

It follows from [6] (or [14] for more details) that the equation (1.8) with (1.9) can be uniquely solved for the supersonic shock by use of the apple curve. In this paper, we will call this solution $\bar\varphi(x)$ as the background solution.

Our main purpose is to study the case that the supersonic flow $(0,0,q_0)$ with density $\rho_0>0$ hits the three-dimensional curved cone with the surface $\sqrt{x_1^2+x_2^2}=b(x)$, here $b(x)\in C^{k_0}$ and satisfies

$$
b(x)\geq b_0x_3,\qquad b(x)-b_0x_3\in C_0^{k_0}(\omega)\qquad \text{and } |\nabla^\alpha(b(x)-b_0x_3)|\leq\varepsilon
$$

with $\omega=\{x: b_0<\sqrt{x_1^2+x_2^2}<3b_0, 1<x_3<2\}$, k_0 a suitably large positive integer and $0\leq|\alpha|\leq k_0$.

Under the cylindrical coordinates (r,θ,z), the cone surface can be rewritten as $r=b_1(\theta,z)$ satisfying

$$
b_1(\theta,z)\geq b_0z,\qquad b_1(\theta,z)-b_0z\in C_0^{k_0}(\tilde\omega)\qquad \text{and } |\nabla^\alpha_{\theta,z}(b_1(\theta,z)-b_0z)|\leq\varepsilon,\qquad 0\leq|\alpha|\leq k_0. \tag{1.10}
$$

with $\omega=\{(r,\theta,z): b_0<r<3b_0, 0<\theta<2\pi, 1<z<2\}$.

Suppose that the equation of possible shock attached at the curved cone is denoted by $r=\chi(\theta,z)$ with $\chi(\theta,0)=0$. For $b_1(\theta,z)\le r\le\chi(\theta,z)$, the equation (1.7) can be written as

$$\begin{aligned}&((\partial_z\varphi)^2-c^2)\partial_z^2\varphi+2\partial_z\varphi\bigg(\partial_r\varphi\partial_{zr}^2\varphi+\frac{\partial_\theta\varphi}{r^2}\partial_{z\theta}^2\varphi\bigg)+((\partial_r\varphi)^2-c^2)\partial_r^2\varphi+\frac{2}{r^2}\partial_r\varphi\partial_\theta\varphi\partial_{r\theta}^2\varphi\\&+\frac{(\partial_\theta\varphi)^2}{r^4}\partial_\theta^2\varphi+\frac{\partial_r\varphi}{r}\bigg(\frac{(\partial_\theta\varphi)^2}{r^2}-c^2\bigg)-\frac{2}{r^3}\partial_r\varphi(\partial_\theta\varphi)^2=0,\qquad b_0z<r<\chi(\theta,z).\end{aligned}\tag{1.11}$$

In addition, it follows from the Rankine-Hugoniot condition and the continuity condition of the potential on the shock that

$$[H(\nabla\varphi)\partial_r\varphi]-[H(\nabla\varphi)\partial_z\varphi]\partial_z\chi=\frac{1}{r^2}[H(\nabla\varphi)\partial_\theta\varphi]\partial_\theta\chi,\quad on\quad r=\chi(\theta,z)\tag{1.12}$$

and

$$\varphi(\chi(\theta,z),\theta,z)=q_0z.\tag{1.13}$$

Finally, on the fixed boundary, we impose the following Dirichlet boundary condition for the potential

$$\varphi=\bar{\varphi}+\phi(\theta,z)\qquad on\qquad r=b_1(\theta,z)\tag{1.14}$$

with

$$\phi(\theta,z)\in C^{k_0},\qquad \phi(\theta,z)\equiv0\quad\text{ for } z\le1,\qquad\text{and}\qquad |(1+z)^{1+\alpha_2}\partial_\theta^{\alpha_1}\partial_z^{\alpha_2}\phi(\theta,z))|\le\varepsilon,\tag{1.15}$$

here $0\le|\alpha|=\alpha_1+\alpha_2\le k_0$.

Generally speaking, we do not derive from (1.14) that $(\partial_1\varphi,\partial_2\varphi,\partial_3\varphi)\cdot(\frac{x_1}{b}-\partial_1 b,\frac{x_2}{b}-\partial_2 b,-\partial_3 b)=0$ holds on $\sqrt{x_1^2+x_2^2}=b(x)$. Namely, there are inflows or outflows through the surface. Therefore, from the physical viewpoint, the condition (1.14) means that the fixed boundary is permeable (in the engineering design, this is often done). For more related explanations, one can see the references [17-18] and so on.

Our main results in this paper can be stated as:

Theorem 1.1. Suppose that a supersonic polytropic gas flow parallel to the z-axis comes from minus infinity with velocity $(0,0,q_0)$, density $\rho_0>0$ satisfying $q_0>c_0=\sqrt{A\gamma}\rho_0^{\frac{\gamma-1}{2}}$ and it hits a three-dimensional curved cone with the surface $r=b_1(\theta,z)$. When $b_0>0$ is suitably small and the assumptions (1.10) and (1.15) hold, then for the large q_0 and sufficiently small ε depending on q_0,ρ_0,b_0 and γ, the problem (1.11)-(1.14) admits a global weak entropy solution with a pointed shock front attached at the origin.

Remark 1.1. It should be emphasized that there are no other discontinuities in our solution besides the main curved shock. This is in consistent with the result for the supersonic flow past a symmetrically curved cone in [5]. The condition (1.15) especially gives a restriction on the perturbed potential function $\phi(\theta,z)$ for small z and large z. Since the perturbation is sufficiently small and the corresponding tangent velocity on the fixed boundary is controlled, any possible compression of the flow will be absorbed by the main shock. This is the mechanism to prevent the formation of any new shock inside the flow field (if there is no the main shock, then in the general case, the new shocks or other complicated singularities can be formed. For instance, one can see [1-2], [10], [12-13], [27-28], [32-33], [35] and the references therein). Thus our result demonstrates that the self-similar shock solution with an appropriate boundary condition on the fixed boundary is multidimensional structurally stable in a global sense.

Remark 1.2. If the Dirichlet boundary condition (1.14) is replaced by the Neumann boundary condition: $(\partial_1\varphi,\partial_2\varphi,\partial_3\varphi)\cdot(\frac{x_1}{b}-\partial_1 b,\frac{x_2}{b}-\partial_2 b,-\partial_3 b)=0$ on $\sqrt{x_1^2+x_2^2}=b(x)$, i.e., the boundary is not perforated, then the lifespan on the existence of a multidimensional shock is discussed in [34]. But so far it is not known whether the global shock exists or not for the Neumann type boundary condition. On the other hand, it follows from the proof procedure on Theorem 1.1 that the global shock exists for the Neumann boundary condition when the disturbed tangent velocity on the fixed boundary is sufficiently small and decayed with an appropriate decay rate for large z.

Remark 1.3. Since BV spaces fail for the multidimensional hyperbolic system as shown in [26], then the method (i.e., the Glimm scheme) in [19] can not be used to treat our multidimensional problem in this paper.

Remark 1.4. Under the appropriate compatibility conditions, near the vertex of the curved cone, the compact support properties on $b(x) - b_0x_3$ and $\phi(\theta, z)$ can be removed. Besides, the compact support property for large z can be substituted by the appropriate decay for large z.

Remark 1.5. Since the velocity q_0 is large enough, then the coming flow is hypersonic. The famous independence principle on the high Mach numbers (that is, there exists a stable limit state for the hypersonic gas) will be implied in linearizing the equation (1.11), namely, the coefficients of the resulted linear equation are almost constants for large q_0. With respect to more properties on the hypersonic flow, one can see [3], [7], [29] and the references therein.

In order to prove the global existence of solution in Theorem 1.1, we intend to use the continuity method and establish some global uniform estimates. To achieve this goal, we need to derive some global uniform weighted energy estimates for the corresponding linearized problem to (1.11)-(1.14). This can be done by looking for appropriate multipliers as in [5] or [9]. But compared to the analysis in [9], we need to give more elaborate estimates because our background solution depends crucially on the position of boundary of the cone and the Mach number, however, the background solution in [9] is created and the fixed boundary is artificially chosen so that the shock and the fixed boundary is arbitrarily close meanwhile the background solution is invariable. It should be noted that the arbitrary closeness of the boundaries plays a key role in the analysis of [9], which is not the case for our problem. In addition, in this paper the procedure to search the multipliers is also more delicate than that in [5] for $1 < \gamma < 2$ because the symmetric property of the solution and the related domain is lost. Thanks to the special properties on the background solution, we can overcome the difficulties to find the needed multipliers.

The rest of the paper is organized as follows. In §2, we reformulate the problem (1.11)-(1.14) by decomposing its solution as a sum of the background solution and a small perturbation so that its linearization can be studied further. In §3, we establish the weighted energy estimate for the linearized problem, where the appropriate multipliers are given. Based on this energy estimate, Theorem 1.1 is proved in §4 in the special case when the body is a standard circular cone. Finally, in §5, we show that Theorem 1.1 holds for the general curved cone with some minor modifications in §4.

In what follows, we will use the following conventions:

$O(b_0^j)$ $(j \geq 1)$ means that there exists a generic constant M_0 such that $|O(b_0^j)| \leq M_0 b_0^j$, where M_0 depends only on γ.

$O(q_0^{-\nu})$ $(\nu > 0)$ denotes a bounded quantity, which admits the bound $|O(q_0^{-\nu})| \leq M_1 q_0^{-\nu}$, where the generic constant M_1 depends only on b_0 and γ.

$O(\varepsilon)$ denotes a generic quantity, which is bounded by $M_2\varepsilon$ with M_2 depending only on b_0, q_0 and γ.

§2. The reformulation of the main problem (1.11)-(1.14)

In this section, we reformulate the problem (1.11)-(1.14) so that we can derive a linearized equation and the corresponding linearized boundary conditions on the fixed boundary and the shock wave. To this end, we first recall some detailed properties on the background solution of (1.8) with (1.9) when the

Mach number is sufficiently large. These properties can be summarized in the following two lemmas. For their proofs, the readers are referred to the reference [5].

Lemma 2.1. If q_0 is large, that is, the Mach number of the incoming flow is large, and $1 < \gamma < 2$, $0 < b_0 < \sqrt{2} - 1$, then

(i). $s_0 = b_0 + O(q_0^{-\frac{2}{\gamma-1}})$,

(ii). $0 \le s\psi(s) - (1+s^2)\psi'(s) \le O(q_0^{\frac{\gamma-3}{\gamma-1}})$,

(iii). $\psi'(s) = \frac{b_0 q_0}{1+b_0^2} + O(q_0^{\frac{\gamma-3}{\gamma-1}})$,

(iv). $c^2(\rho(s)) = \frac{b_0^2(\gamma-1)q_0^2}{2(1+b_0^2)}(1 + O(\frac{1}{q_0^2}) + O(q_0^{-\frac{2}{\gamma-1}}))$.

(v). $\lambda_2(s) > s_0$.

Lemma 2.2. Under the assumptions of Lemma 2.1, we have

(i). $\lambda_1(s) < s$,

(ii). $\psi''(s) = -\frac{q_0}{(1+b_0^2)^2} + O(\frac{1}{q_0}) + O(q_0^{\frac{\gamma-3}{\gamma-1}})$,

(iii). $|\rho'(s)| \le C$,

where C is a constant independent of q_0.

By using of the assumptions (1.10) and (1.15) and the finite propagation speed property for the wave equation, for large q_0 we know that $\varphi(r,\theta,z) \equiv \bar{\varphi}(r,z) = z\psi(\frac{r}{z})$ and $\chi(\theta,z) = s_0 z$ for $z \le \frac{1}{2}$. Thus we can study the global existence by solving an initial boundary value problem with initial data on $z = \frac{1}{2}$. The initial data on $z = \frac{1}{2}$ can be regarded as the background solution. Moreover, the initial data also satisfy the compatibility conditions on the intersection curve of $z = \frac{1}{2}$ with the shock front and the surface $r = b_0 z$.

As in [5], we can extend the potential $\bar{\varphi}(r,z)$, the function $\psi(s)$ and the density $\rho(s)$ in $[b_0, s_0]$ to the interval $[b_0, s_0+\eta_0]$ for small η_0 satisfying $0 < \eta_0 \le q_0^{-\frac{2}{\gamma-1}}(s_0 - b_0)$. Later on we will denote the extension of $\bar{\varphi}$, ψ and $\rho(s)$ in the domain $\{(r,z) : z \ge \frac{1}{2}, b_0 z \le r \le (s_0+\eta_0)z\}$ by $\hat{\varphi}$, $\hat{\psi}$ and $\hat{\rho}$ respectively.

Set $\dot{\varphi} = \varphi - \hat{\varphi}$. Then by a direct computation the equation, (1.11) can be changed into:

$$\begin{aligned}
&\partial_z^2\dot{\varphi} + 2P_1(\frac{r}{z})\partial_{zr}^2\dot{\varphi} + P_2(\frac{r}{z})\partial_r^2\dot{\varphi} + \tilde{P}_2(\frac{r}{z})(\partial_1^2\dot{\varphi} + \partial_2^2\dot{\varphi}) + P_3(r,z)\partial_z\dot{\varphi} + P_4(r,z)\partial_r\dot{\varphi} \\
&\quad = \sum_{i,j=1}^{3} f_{ij}(\theta, \frac{r}{z}, \nabla_x\dot{\varphi})\partial_{ij}^2\dot{\varphi} + \frac{1}{r} f_0(\theta, \frac{r}{z}, \nabla_x\dot{\varphi}), \qquad z \ge \frac{1}{2}, \qquad b_1(\theta,z) \le r \le \chi(\theta,z)
\end{aligned} \tag{2.1}$$

where $f_{ij}(\theta, s, 0, 0, 0) = 0$, $f_0(\theta, s, 0, 0, 0) = \nabla_q f_0(\theta, s, q_1, q_2, q_3)|_{q=0} = 0$. Moreover

$$
\begin{aligned}
P_1(s) &= \frac{(\hat{\psi}(s) - s\hat{\psi}'(s))\hat{\psi}'(s)}{(\hat{\psi}(s) - s\hat{\psi}'(s))^2 - \hat{c}^2(s)},\\
P_2(s) &= \frac{(\hat{\psi}'(s))^2}{(\hat{\psi}(s) - s\hat{\psi}'(s))^2 - \hat{c}^2(s)},\\
\tilde{P}_2(s) &= -\frac{\hat{c}^2(s)}{(\hat{\psi}(s) - s\hat{\psi}'(s))^2 - \hat{c}^2(s)},\\
P_3(r,z) &= \frac{1}{r((\hat{\psi}(s) - s\hat{\psi}'(s))^2 - \hat{c}^2(s))}\bigg(-2(\hat{c}(s)\hat{c}'(s)+1)(\hat{\psi}(s) - s\hat{\psi}'(s))(\hat{\psi}(s) - s\hat{\psi}'(s))'s^2\\
&\quad + 2\hat{c}(s)\hat{c}'(s)(\hat{\psi}(s) - s\hat{\psi}'(s))\hat{\psi}''(s)s + 2\hat{c}(s)\hat{c}'(s)(\hat{\psi}(s) - s\hat{\psi}'(s))\hat{\psi}'(s) - 2s^2\hat{\psi}'(s)\hat{\psi}''(s)\bigg) \equiv \frac{1}{r}\tilde{P}_3(s),\\
P_4(r,z) &= \frac{1}{r((\hat{\psi}(s) - s\hat{\psi}'(s))^2 - \hat{c}^2(s))}\bigg(-2s^2\hat{c}(s)\hat{c}'(s)\hat{\psi}'(s)(\hat{\psi}(s) - s\hat{\psi}'(s))' + 2s(1+\hat{c}(s)\hat{c}'(s))\hat{\psi}'(s)\hat{\psi}''(s)\\
&\quad + 2\hat{c}(s)\hat{c}'(s)(\hat{\psi}')^2(s) - 2s^2(\hat{\psi}(s) - s\hat{\psi}'(s))\hat{\psi}''(s)\bigg) \equiv \frac{1}{r}\tilde{P}_4(s).
\end{aligned}
$$

with

$$\hat{c}(s) = c(\hat{\rho}(s)), \qquad \hat{c}'(s) = \hat{c}'(\hat{\rho}(s))\hat{\rho}'(s).$$

On the boundary $r = b_1(\theta, z)$, we have

$$\dot{\varphi} = \phi(\theta, z). \tag{2.2}$$

On the free boundary $r = \chi(\theta, z)$, by use of the continuity condition (1.13), we can rewrite (1.12) as

$$H(\nabla\varphi)\bigg((\partial_1\varphi)^2 + (\partial_2\varphi)^2 + (\partial_z\varphi)^2 + q_0\partial_z\varphi\bigg) - \rho_0 q_0 \partial_z\varphi = 0 \qquad on \qquad r = \chi(\theta, z). \tag{2.3}$$

Using $\varphi = \hat{\varphi} + \dot{\varphi}$, and introducing the notation

$$\xi(\theta, z) = \frac{\chi(\theta, z) - s_0 z}{z},$$

which describes the perturbation of the slope of the shock front, one can rewrite (2.3) as

$$B_1\partial_r\dot{\varphi} + B_2\partial_z\dot{\varphi} + B_3\xi = \kappa_0(\xi, \nabla_x\dot{\varphi}) \qquad on \qquad r = \chi(\theta, z) \tag{2.4}$$

where

$$
\begin{aligned}
B_1 &= -\bigg\{\frac{\rho(s)}{c^2(\rho(s))}\bigg((\psi')^2(s) + (\psi(s) - s\psi'(s))^2 + q_0(\psi(s) - s\psi'(s) - q_0)\bigg)\psi'(s) + 2\rho(s)\psi'(s)\bigg\}\bigg|_{s=s_0},\\
B_2 &= -\bigg\{\frac{\rho(s)}{c^2(\rho(s))}\bigg((\psi')^2(s) + (\psi(s) - s\psi'(s))^2 + q_0(\psi(s) - s\psi'(s) - q_0)\bigg)(\psi(s) - s\psi'(s))\\
&\qquad + 2\rho(s)(\psi(s) - s\psi'(s) - q_0) + (\rho(s) - \rho_0)q_0\bigg\}\bigg|_{s=s_0},\\
B_3 &= \bigg\{\rho(s)\bigg(2\psi'(s)\hat{\psi}''(s) + 2(\psi(s) - s\psi'(s))(\hat{\psi}(s) - s\hat{\psi}'(s))' + q_0(\hat{\psi}(s) - s\hat{\psi}'(s))'\bigg) + \hat{\rho}'(s)\bigg((\psi')^2(s)\\
&\qquad + (\psi(s) - s\psi'(s))^2 + q_0(\psi(s) - s\psi'(s) - q_0)\bigg) - \rho_0 q_0(\hat{\psi}(s) - s\hat{\psi}'(s))'\bigg\}\bigg|_{s=s_0},
\end{aligned}
$$

and

$$\kappa_0(\xi, \nabla_x \dot\varphi) \le C(|(\xi, \nabla_x \dot\varphi)|^2)$$

Later on the function $\kappa_j(\xi, \nabla_x \dot\varphi)$ will be used to denote any quantity dominated by $C|(\xi, \nabla_x \dot\varphi)|^2$, here the generic constant C does not depend on ε.

By using Lemma 2.1 and Lemma 2.2, as in [5], we have the following estimates for large q_0.

Lemma 2.3

$$\begin{aligned}
B_1 &= \frac{2b_0}{1+b_0^2}\left(\frac{(\gamma-1)b_0^2}{2A\gamma(1+b_0^2)}\right)^{\frac{1}{\gamma-1}} q_0^{\frac{\gamma+1}{\gamma-1}}(1+O(q_0^{-\frac{2}{\gamma-1}})),\\
B_2 &= \frac{1-b_0^2}{1+b_0^2}\left(\frac{(\gamma-1)b_0^2}{2A\gamma(1+b_0^2)}\right)^{\frac{1}{\gamma-1}} q_0^{\frac{\gamma+1}{\gamma-1}}(1+O(q_0^{-\frac{2}{\gamma-1}})),\\
B_3 &= -\frac{b_0}{(1+b_0^2)^2}\left(\frac{(\gamma-1)b_0^2}{2A\gamma(1+b_0^2)}\right)^{\frac{1}{\gamma-1}} q_0^{\frac{2\gamma}{\gamma-1}}(1+O(q_0^{-\frac{2}{\gamma-1}})).
\end{aligned}$$

Dividing (2.4) by B_1 yields

$$\partial_r \dot\varphi + \mu_1 \partial_z \dot\varphi + \mu_2 \xi = \kappa_1(\xi, \nabla_x \dot\varphi) \qquad on \qquad r = \chi(\theta, z), \tag{2.5}$$

where

$$\begin{aligned}
\mu_1 &= \frac{1-b_0^2}{2b_0}(1+O(q_0^{-\frac{2}{\gamma-1}})),\\
\mu_2 &= -\frac{q_0}{2(1+b_0^2)}(1+O(q_0^{-\frac{2}{\gamma-1}})).
\end{aligned}$$

In addition, (1.13) implies $\partial_z \chi(\theta, z) = -\frac{\partial_z \varphi}{\partial_r \varphi}$ on $r = \chi(\theta, z)$, it follows from Taylor's expansion that

$$\partial_z(z\xi) + \frac{1}{\psi'(s_0)}\Big((\partial_z \dot\varphi)(\chi(\theta,z),\theta,z) + s_0(\partial_r \dot\varphi)(\chi(\theta,z),\theta,z)\Big) = \kappa_2(\xi, \nabla\dot\varphi).$$

Since

$$\begin{aligned}
\partial_z(\dot\varphi(\chi(\theta,z),\theta,z)) &= (\partial_z \dot\varphi)(\chi(\theta,z),\theta,z) + \partial_z\chi(\theta,z)(\partial_r \dot\varphi)(\chi(\theta,z),\theta,z)\\
&= (\partial_z \dot\varphi)(\chi(\theta,z),\theta,z) + s_0(\partial_r \dot\varphi)(\chi(\theta,z),\theta,z) + \kappa_3(\xi, \nabla\dot\varphi),
\end{aligned}$$

then by substituting it into the above equation we arrive at

$$\partial_z\left(z\xi + \frac{1}{\psi'(s_0)}\dot\varphi(\chi(\theta,z),\theta,z)\right) = \kappa_4(\xi, \nabla_x \dot\varphi) \tag{2.6}$$

(2.5) and (2.6) are the new forms of the Rankine-Hugoniot condition (1.12) and the continuity condition (1.13) on the shock front.

After such a reformulation on the problem (1.12)-(1.14), to prove the main theorem we only need to solve the problem (2.1), (2.2), (2.5) and (2.6) with the initial data $\dot\varphi(r,\theta,z)|_{z=\frac{1}{2}} = 0$, $\partial_z \dot\varphi(r,\theta,z)|_{z=\frac{1}{2}} = 0$, $\xi(\theta,z)|_{z=\frac{1}{2}} = 0$ in the domain $\{(r,\theta,z) : z \ge \frac{1}{2}, \theta \in [0,2\pi], b_1(\theta,z) \le r \le \chi(\theta,z)\}$.

When the surface of the cone is curved, then it is convenient to change the boundary into a straight one by the following coordinates transformation:

$$\begin{cases} \tilde{z} = \frac{b_1(\theta,z)}{b_0}, \\ \tilde{\theta} = \theta, \\ \tilde{r} = r. \end{cases} \tag{2.7}$$

Under the transformation (2.7), we use the notations $\tilde{\varphi}, \dot{\tilde{\varphi}}$ and $\tilde{\chi}$ instead of $\varphi, \dot{\varphi}$ and χ. By a direct computation, for $\tilde{z} \geq \frac{1}{2}$ and $b_0\tilde{z} \leq \tilde{r} \leq \tilde{\chi}(\tilde{\theta}, \tilde{z})$, the equation (2.1) can be rewritten as:

$$\begin{aligned} &\partial_{\tilde{z}}^2\dot{\tilde{\varphi}} + 2P_1(\frac{\tilde{r}}{\tilde{z}})\partial_{\tilde{z}\tilde{r}}^2\dot{\tilde{\varphi}} + P_2(\frac{\tilde{r}}{\tilde{z}})\partial_{\tilde{r}}^2\dot{\tilde{\varphi}} + \tilde{P}_2(\frac{\tilde{r}}{\tilde{z}})(\partial_1^2\dot{\tilde{\varphi}} + \partial_2^2\dot{\tilde{\varphi}}) + P_3(\tilde{r},\tilde{z})\partial_{\tilde{z}}\dot{\tilde{\varphi}} + P_4(\tilde{r},\tilde{z})\partial_{\tilde{r}}\dot{\tilde{\varphi}} \\ &\quad = \sum_{i,j=1}^{3} f_{ij}(\tilde{\theta}, \frac{\tilde{r}}{\tilde{z}}, \nabla_{\tilde{x}}\dot{\tilde{\varphi}})\partial_{ij}^2\dot{\tilde{\varphi}} + (1 - \frac{\partial_z b_1}{b_0})\sum_{i,j=1}^{3} f_{ij}^1(\nabla_{\tilde{x}}\tilde{\varphi}, \frac{\partial_z b_1}{b_0}, \partial_\theta b_1)\partial_{ij}^2\tilde{\varphi} \\ &\quad + \frac{\partial_\theta b_1}{r}\sum_{i,j=1}^{3} f_{ij}^2(\nabla_{\tilde{x}}\tilde{\varphi}, \frac{\partial_z b_1}{b_0}, \partial_\theta b_1)\partial_{ij}^2\tilde{\varphi} + f_1^3(\nabla_{\tilde{x}}\tilde{\varphi}, \frac{\partial_z b_1}{b_0}, \partial_\theta b_1)\partial_z^2 b_1\partial_{\tilde{z}}\tilde{\varphi} \\ &\quad + f_2^3(\nabla_{\tilde{x}}\tilde{\varphi}, \frac{\partial_z b_1}{b_0}, \partial_\theta b_1)\frac{\partial_\theta^2 b_1}{r^2}\partial_{\tilde{\theta}}\tilde{\varphi} + \frac{1}{\tilde{r}}f_3^3(\nabla_{\tilde{x}}\tilde{\varphi}, \frac{\partial_z b_1}{b_0}, \partial_\theta b_1)(1 - \frac{\partial_z b_1}{b_0})\partial_{\tilde{r}}\tilde{\varphi} + \frac{1}{\tilde{r}}f_0(\tilde{\theta}, \frac{\tilde{r}}{\tilde{z}}, \nabla_{\tilde{x}}\dot{\tilde{\varphi}}), \end{aligned} \tag{2.8}$$

where $P_1, P_2, \tilde{P}_2, P_3, P_4, f_{ij}$ and f_0 are the same as in (2.1), f_{ij}^k and f_i^k are smooth on its arguments.

Set $\tilde{\xi}(\tilde{\theta}, \tilde{z}) = \frac{\tilde{\chi}(\tilde{\theta},\tilde{z}) - s_0\tilde{z}}{\tilde{z}}$. Then the boundary conditions (2.5), (2.6) and (2.2) take the forms

$$\partial_{\tilde{r}}\dot{\tilde{\varphi}} + \mu_1\partial_{\tilde{z}}\dot{\tilde{\varphi}} + \mu_2\tilde{\xi} = \kappa_5(\tilde{\xi}, \nabla_{\tilde{x}}\dot{\tilde{\varphi}}) + \tilde{f}_1(\tilde{\xi}, \nabla_{\tilde{x}}\tilde{\varphi}, \frac{\partial_z b_1}{b_0}, \partial_\theta b_1)(1 - \frac{\partial_z b_1}{b_0}) + \tilde{f}_2(\tilde{\xi}, \nabla_{\tilde{x}}\tilde{\varphi}, \frac{\partial_z b_1}{b_0}, \partial_\theta b_1)\frac{\partial_{\tilde{\theta}} b_1}{r} \quad \text{on} \quad \tilde{r} = \tilde{\chi}(\tilde{\theta}, \tilde{z}), \tag{2.9}$$

$$\partial_{\tilde{z}}\left(\tilde{z}\tilde{\xi} + \frac{1}{\psi'(s_0)}\dot{\tilde{\varphi}}(\tilde{\chi}(\tilde{\theta},\tilde{z}), \tilde{\theta}, \tilde{z})\right) = \kappa_6(\tilde{\xi}, \nabla_{\tilde{x}}\dot{\tilde{\varphi}}(\tilde{\chi}(\tilde{\theta},\tilde{z}), \tilde{\theta}, \tilde{z})) \quad \text{on} \quad \tilde{r} = \tilde{\chi}(\tilde{\theta}, \tilde{z}), \tag{2.10}$$

$$\dot{\tilde{\varphi}} = \tilde{\phi}(\tilde{\theta}, \tilde{z}), \quad \text{on} \quad \tilde{r} = b_0\tilde{z}, \tag{2.11}$$

here $\tilde{\phi}(\tilde{\theta}, \tilde{z})$ has the same property as $\phi(\theta, z)$ in (1.15).

In addition, $\dot{\tilde{\varphi}}$ and $\tilde{\xi}$ have the following initial data

$$\dot{\tilde{\varphi}}(\tilde{r}, \tilde{\theta}, \tilde{z})|_{\tilde{z}=\frac{1}{2}} = 0, \qquad \partial_{\tilde{z}}\dot{\tilde{\varphi}}(\tilde{r}, \tilde{\theta}, \tilde{z})|_{\tilde{z}=\frac{1}{2}} = 0, \qquad \tilde{\xi}(\tilde{\theta}, \tilde{z})|_{\tilde{z}=\frac{1}{2}} = 0. \tag{2.12}$$

In next section, we will focus on the uniform estimates on the solution $\dot{\tilde{\varphi}}$ to the equation (2.8) with the boundary conditions (2.9)-(2.11) and the initial data (2.12).

§3. Uniform estimate on the linearized operator

Now we derive an energy estimate for the linear parts in the equation (2.8) with the initial-boundary conditions (2.9)-(2.12). For simplicity in presentation, we neglect all the notations "∼" in (2.8)-(2.12) except $\tilde{P}_2$ in (2.8). The following conclusion plays the key role to derive the decay estimate of $\dot{\varphi}$ with respect to large z.

Theorem 3.1. Set $D_T = \{(x_1, x_2, z) : \frac{1}{2} \le z \le T, 0 \le \theta \le 2\pi, b_0 z \le r \le \chi(\theta, z)\}$ for any $T > \frac{1}{2}$. Let $\Gamma_T = \{(x_1, x_2, z) : \frac{1}{2} \le z \le T, 0 \le \theta \le 2\pi, r = \chi(\theta, z)\}$ and $B_T = \{(x_1, x_2, z) : \frac{1}{2} \le z \le T, 0 \le \theta \le 2\pi, r = b_0 z\}$ be the lateral boundaries of D_T. If $\dot{\varphi} \in C^\infty(D_T)$ satisfies the initial-boundary conditions (2.9)-(2.12), moreover, $|\xi(\theta, z)| + |z\partial_z \xi(z, \theta)| + |\partial_\theta \xi(z, \theta)| \le C\varepsilon$ holds in $(\theta, z) \in [0, 2\pi; \frac{1}{2}, T]$ for sufficiently small ε. Then there exists a multiplier $\mathcal{M}\dot{\varphi} = ra(\frac{r}{z})\partial_z \dot{\varphi} + zb(\frac{r}{z})\partial_r \dot{\varphi} + E(\frac{r}{z})\dot{\varphi}$, such that

$$\begin{aligned}
&\frac{C_1}{\sqrt{T}} \iint_{b_0 T \le r \le \chi(\theta, T)} |\nabla_{x_1, x_2} \dot{\varphi}(x_1, x_2, T)|^2 dx_1 dx_2 + C_2 \iiint_{D_T} z^{-\frac{3}{2}} |\nabla \dot{\varphi}|^2 dx_1 dx_2 dz \\
&+ C_3 \iint_{B_T} z^{-\frac{1}{2}} |\partial \dot{\varphi}|^2 dS + C_4 \iint_{\Gamma_T} z^{-\frac{1}{2}} (|\partial_z \dot{\varphi}|^2 + +|\frac{\partial_\theta \dot{\varphi}}{r}|^2) dS \le \iiint_{D_T} z^{-\frac{3}{2}} \mathcal{L}\dot{\varphi} \mathcal{M}\dot{\varphi} dx_1 dx_2 dz \\
&+ C_5 \iint_{\Gamma_T} z^{-\frac{1}{2}} (\mathcal{B}_0 \dot{\varphi})^2 dS + C_6 \varepsilon^2,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
&\mathcal{L}\dot{\varphi} = \partial_z^2 \dot{\varphi} + 2P_1(\frac{r}{z})\partial_{zr}^2 \dot{\varphi} + P_2(\frac{r}{z})\partial_r^2 \dot{\varphi} + \tilde{P}_2(\frac{r}{z})(\partial_1^2 \dot{\varphi} + \partial_2^2 \dot{\varphi}) + P_3(r, z)\partial_z \dot{\varphi} + P_4(r, z)\partial_r \dot{\varphi}, \\
&\mathcal{B}_0 \dot{\varphi} = (\partial_r + \mu_1 \partial_z)\dot{\varphi},
\end{aligned}$$

and $C_i (1 \le i \le 6)$ are positive constants independent of T, q_0 and ε.

Remark 3.1. Although the estimate on the integral $\iint_{\Gamma_T} z^{-\frac{1}{2}} |\partial_r \dot{\varphi}|^2 dS$ is not given in (3.1) directly, it can be estimated when $\iint_{\Gamma_T} z^{-\frac{1}{2}} |\partial_z \dot{\varphi}|^2$ and $\iint_{\Gamma_T} z^{-\frac{1}{2}} (\mathcal{B}_0 \dot{\varphi})^2 dS$ are known. Indeed, one has

$$\iint_{\Gamma_T} z^{-\frac{1}{2}} |\partial_r \dot{\varphi}|^2 dS \le C(\mu_1^2 \iint_{\Gamma_T} z^{-\frac{1}{2}} |\partial_r \dot{\varphi}|^2 dS + \iint_{\Gamma_T} z^{-\frac{1}{2}} (\mathcal{B}_0 \dot{\varphi})^2 dS.$$

Remark 3.2. It follows from (3.1) that we can obtain the all weighted L^2 estimates (including the interior of the domain D and boundaries) on $\nabla \dot{\varphi}$ in terms of the equation (2.8) itself and the boundary condition on the shock. Thus, roughly speaking, the hyperbolic equation (2.8) has some analogous properties as does a second order elliptic equation in the domain D which is formed by the shock surface and the fixed boundary.

Proof. Let $A = A(r, z)$ and $B = B(r, z)$ be determined later. Setting $\mathcal{M}\dot{\varphi} = A(r, z)\partial_z \dot{\varphi} + B(r, z)\partial_r \dot{\varphi} + E(\frac{r}{z})\dot{\varphi}$, we have through integration by parts that

$$\begin{aligned}
&\iiint_{D_T} z^{-\frac{3}{2}} \mathcal{L}\dot{\varphi} \mathcal{M}\dot{\varphi} dx_1 dx_2 dz = \iiint_{D_T} z^{-\frac{3}{2}} \Big\{ K_0 (\partial_z \dot{\varphi})^2 + K_1 (\partial_r \dot{\varphi})^2 + K_2 \partial_z \dot{\varphi} \partial_r \dot{\varphi} + K_3 ((\partial_1 \dot{\varphi})^2 \\
&+ (\partial_2 \dot{\varphi})^2) + (\frac{E}{z} + \frac{r}{z^2} E') \partial_z \dot{\varphi} \dot{\varphi} - \Big(\partial_r ((P_2 + \tilde{P}_2) E) + 2(\partial_z (P_1 E) - \frac{P_1 E}{z}) + \frac{(P_2 + \tilde{P}_2) E}{r} \Big) \partial_r \dot{\varphi} \\
&\times \dot{\varphi} \Big\} dx_1 dx_2 dz + T^{-\frac{3}{2}} \iint_{b_0 T \le r \le \chi(\theta, T)} K_4 (x_1, x_2, T) dx_1 dx_2 + \iint_{B_T} z^{-\frac{3}{2}} \Big(b_0 K_4 - \frac{x_1}{r} K_5 - \frac{x_2}{r} K_6 \Big) dS \\
&+ \iint_{\Gamma_T} z^{-\frac{3}{2}} \Big((\frac{x_1}{r} - cos\theta \partial_\theta \chi) K_5 + (\frac{x_2}{r} - sin\theta \partial_\theta \chi) K_6 - \partial_z \chi K_4 \Big) dS.
\end{aligned} \tag{3.2}$$

where

$$
\begin{aligned}
K_0 &= -\frac{\partial_z A}{2} + \frac{\partial_r B}{2} - \partial_r(P_1 A) + \frac{B}{2r} - \frac{P_1 A}{r} + P_3 A + \frac{3A}{4z} - E,\\
K_1 &= -\partial_z(P_1 B) + \frac{1}{2}\partial_z(P_2 A) - \frac{1}{2}\partial_r((P_2+\tilde{P}_2)B) - \frac{P_2 B}{2r} + P_4 B + \frac{3}{4z}(2P_1 B - P_2 A)\\
&\quad - P_2 E,\\
K_2 &= -\partial_z B - \partial_r((P_2+\tilde{P}_2)A) - \frac{P_2 A}{r} + P_3 B + P_4 A + \frac{3B}{2z} - 2P_1 E,\\
K_3 &= \frac{1}{2}\partial_z(\tilde{P}_2 A) + \frac{1}{2}\partial_r(\tilde{P}_2 B) - \frac{\tilde{P}_2 B}{2r} - \frac{3}{4z}\tilde{P}_2 A - \tilde{P}_2 E,\\
K_4 &= \frac{A}{2}(\partial_z\dot\varphi)^2 + B\partial_z\dot\varphi\partial_r\dot\varphi + (P_1 B - \frac{P_2 A}{2})(\partial_r\dot\varphi)^2 - \frac{\tilde{P}_2 A}{2}((\partial_1\dot\varphi)^2 + (\partial_2\dot\varphi)^2)\\
&\quad + E\partial_z\dot\varphi\dot\varphi + 2P_1 E\partial_r\dot\varphi\dot\varphi,\\
K_5 &= \frac{x_1}{r}(P_1 A - \frac{B}{2})(\partial_z\dot\varphi)^2 + \frac{x_1}{r}P_2 A\partial_z\dot\varphi\partial_r\dot\varphi + \frac{x_1}{2r}P_2 B(\partial_r\dot\varphi)^2 + \tilde{P}_2 A\partial_1\dot\varphi\partial_z\dot\varphi\\
&\quad + \tilde{P}_2 B\partial_1\dot\varphi\partial_r\dot\varphi - \frac{x_1}{2r}\tilde{P}_2 B((\partial_1\dot\varphi)^2 + (\partial_2\dot\varphi)^2) + \frac{x_1}{r}(P_2+\tilde{P}_2)E\partial_r\dot\varphi\dot\varphi - \frac{x_2}{r^2}\tilde{P}_2 E\partial_\theta\dot\varphi\dot\varphi,\\
K_6 &= \frac{x_2}{r}(P_1 A - \frac{B}{2})(\partial_z\dot\varphi)^2 + \frac{x_2}{r}P_2 A\partial_z\dot\varphi\partial_r\dot\varphi + \frac{x_2}{2r}P_2 B(\partial_r\dot\varphi)^2 + \tilde{P}_2 A\partial_2\dot\varphi\partial_z\dot\varphi\\
&\quad + \tilde{P}_2 B\partial_2\dot\varphi\partial_r\dot\varphi - \frac{x_2}{2r}\tilde{P}_2 B((\partial_1\dot\varphi)^2 + (\partial_2\dot\varphi)^2) + \frac{x_2}{r}(P_2+\tilde{P}_2)E\partial_r\dot\varphi\dot\varphi + \frac{x_1}{r^2}\tilde{P}_2 E\partial_\theta\dot\varphi\dot\varphi.
\end{aligned}
$$

Our purpose is to choose suitable coefficients $A(r,z), B(r,z)$ and $E(\frac{r}{z})$ so that all integrals on D_T, B_T and $t = T$ in the right hand side of (3.2) are definitely positive and the integral on Γ_T gives an appropriate control on $\partial_z\dot\varphi$, $\frac{\partial_\theta\dot\varphi}{r}$ and $\mathcal{B}_0\dot\varphi$. We will derive some sufficient conditions for $A(r,z), B(r,z)$ and $E(\frac{r}{z})$ in the process of studying each integral. Assume $A(r,z) = ra(\frac{r}{z})$ and $B(r,z) = zb(\frac{r}{z})$ with $a(s) > 0$ and $b(s) > 0$. Then $a(s)$, $b(s)$ and $E(s)$ will be determined by the following six steps. In what follows, we will denote by C a generic positive constant independent of q_0 and ε, it may take different value in different places.

Step 1. Estimate on $\iint_{B_T} z^{-\frac{3}{2}}(b_0 K_4 - \frac{x_1}{r}K_5 - \frac{x_2}{r}K_6)dS$.

Since

$$
\begin{aligned}
b_0 K_4 - \frac{x_1}{r}K_5 - \frac{x_2}{r}K_6 &= \Big(\frac{b_0}{2}A - (P_1 A - \frac{B}{2})\Big)(\partial_z\dot\varphi)^2 + \Big(b_0 B - (P_2+\tilde{P}_2)A\Big)\partial_r\dot\varphi\partial_z\dot\varphi\\
&\quad + \Big(b_0(P_1 B - \frac{(P_2+\tilde{P}_2)A}{2}) - \frac{(P_2+\tilde{P}_2)B}{2}\Big)(\partial_r\dot\varphi)^2 + \frac{\tilde{P}_2}{2r}(B - b_0 A)(\partial_\theta\dot\varphi)^2\\
&\quad + b_0 E\partial_z\dot\varphi\dot\varphi + \Big(2b_0 P_1 - (P_2+\tilde{P}_2)\Big)E\partial_r\dot\varphi\dot\varphi,
\end{aligned}
$$

then using the boundary condition (2.11) on $r = b_0 z$, one has

$$
\begin{aligned}
b_0 K_4 - \frac{x_1}{r}K_5 - \frac{x_2}{r}K_6 &= z(\partial_r\dot\varphi)^2\Big(P_1(b_0)b_0 - \frac{b_0^2}{2} - \frac{(P_2+\tilde{P}_2)(b_0)}{2})(b(b_0) - b_0^2 a(b_0) - \delta_0\Big)\\
&\quad + O((\partial_\theta\phi)^2 + \frac{|\phi|^2}{z}),
\end{aligned}
$$

here $\delta_0 > 0$ is an appropriate small constant.

By the expressions of $P_1(b_0)$, $P_2(b_0)$ and $\tilde{P}_2(b_0)$, one then can obtain

$$P_1(b_0)b_0 - \frac{b_0^2}{2} - \frac{(P_2+\tilde{P}_2)(b_0)}{2} = \frac{(1+b_0^2)c^2(\rho(b_0))}{2((\psi(b_0)-b_0\psi'(b_0))^2 - c^2(\rho(b_0)))} > 0.$$

Hence by assuming that

$$b(b_0) > b_0^2 a(b_0), \tag{3.3}$$

one has

$$b_0K_4 - \frac{x_1}{r}K_5 - \frac{x_2}{r}K_6 \geq Cz(\partial_r\dot{\varphi})^2 - C(|\partial_\theta\phi|^2 + \frac{|\phi|^2}{z}).$$

This implies

$$\iint_{B_T} z^{-\frac{3}{2}}(b_0K_4 - \frac{x_1}{r}K_5 - \frac{x_2}{r}K_6)dS \geq C\iint_{B_T} z^{-\frac{1}{2}}(\partial_r\dot{\varphi})^2 dS - C\varepsilon^2. \tag{3.4}$$

In fact, (3.3) gives the first constraint for $a(s)$ and $b(s)$.

Step 2. Estimate on $\iint_{b_0T\leq r\leq\chi(\theta,T)} T^{-\frac{3}{2}}K_4(x_1,x_2,T)dx_1dx_2$.

On $z=T$, we have

$$\begin{aligned}K_4(x_1,x_2,T) = {}& z\left(\frac{sa(s)}{2}(\partial_z\dot{\varphi})^2 + b(s)\partial_z\dot{\varphi}\partial_r\dot{\varphi} + (P_1b(s) - \frac{(P_2+\tilde{P}_2)s}{2}a(s))(\partial_r\dot{\varphi})^2\right)\\ &- \frac{\tilde{P}_2a(s)}{2r}(\partial_\theta\dot{\varphi})^2 + E\partial_z\dot{\varphi}\dot{\varphi} + 2P_1E\partial_r\dot{\varphi}\dot{\varphi}.\end{aligned}$$

To ensure the positivity of quadratic terms on $(\partial_r\dot{\varphi},\partial_z\dot{\varphi},\partial_\theta\dot{\varphi})$ in K_4, one requires that the coefficient of $(\partial_\theta\dot{\varphi})^2$ is positive and the discriminant of the quadratic form on $\partial_z\dot{\varphi}$ and $\partial_r\dot{\varphi}$ should be negative, namely,

$$\begin{cases} -\tilde{P}_2a(s) > 0,\\ \Delta = b^2(s)(1-2P_1\frac{sa(s)}{b(s)} + (P_2+\tilde{P}_2)(\frac{sa(s)}{b(s)})^2) < 0.\end{cases}$$

Since $-\tilde{P}_2 > 0$, then $-\tilde{P}_2a(s) > 0$ is obvious for $a(s) > 0$.

Let

$$D_1 = P_1^2 - P_2 - \tilde{P}_2.$$

Then the second inequality above leads to

$$\frac{sa(s)}{b(s)} > \frac{P_1 - \sqrt{D_1}}{P_2+\tilde{P}_2} = \frac{1}{P_1+\sqrt{D_1}} = \frac{1}{\lambda_2(s)}.$$

Therefore, $\iint_{b_0T\leq r\leq\chi(\theta,T)} T^{-1}K_4(x_1,x_2,T)dx_1dx_2 \geq C\iint_{b_0T\leq r\leq\chi(\theta,T)}(|\nabla_{x_1,x_2}\dot{\varphi}|^2 - E^2|\dot{\varphi}|^2)dx_1dx_2$ as long as $a(s)$ and $b(s)$ are chosen to satisfy

$$0 < \frac{b(s)}{sa(s)} < \lambda_2(s). \tag{3.5}$$

In this case, we have

$$\iint_{b_0T\le r\le \chi(\theta,T)} T^{-\frac{3}{2}}K_4(x_1,x_2,T)dx_1dx_2 \ge \frac{C}{\sqrt{T}}\iint_{b_0T\le r\le \chi(\theta,T)}|\nabla\dot{\varphi}|^2dx_1dx_2$$
$$-\frac{C}{T^{\frac{5}{2}}}\iint_{b_0T\le r\le \chi(\theta,T)}|\dot{\varphi}|^2dx_1dx_2. \tag{3.6}$$

Step 3. Positivity of the integral on D_T.
We will choose $a(s)$, $b(s)$ and $E(s)$ such that

$$K_0(\partial_z\dot{\varphi})^2+K_1(\partial_r\dot{\varphi})^2+K_2\partial_z\dot{\varphi}\partial_r\dot{\varphi}+K_3((\partial_1\dot{\varphi})^2+(\partial_2\dot{\varphi})^2)\ge C((\partial_z\dot{\varphi})^2+(\partial_1\dot{\varphi})^2+(\partial_2\dot{\varphi})^2)-C|\frac{\dot{\varphi}}{z}|^2.$$

In fact, the constant C will be explicitly computed as in the Lemma A.9 of Appendix in [5].
The above estimate holds if the coefficients K_0, K_1 and K_2 satisfy

$$K_0>0, K_2^2-4K_0(K_1+K_3)<0, K_3>0. \tag{3.7}$$

Substituting $a(s)$ and $b(s)$ into the expressions of K_0, K_1, K_2 and K_3 yields

$$\begin{aligned}
K_0 &= (\frac{s^2}{2}-P_1s)a'(s)+\frac{b'(s)}{2}-P_1'sa(s)-2P_1a(s)+\tilde{P}_3a(s)+\frac{3}{4}sa(s)+\frac{b(s)}{2s}-E(s),\\
K_2^2-4K_0(K_1+K_3) &= \Big(-(P_2+\tilde{P}_2)sa'(s)+sb'(s)-(P_2+\tilde{P}_2)a(s)-(P_2+\tilde{P}_2)'sa(s)-P_2a(s)+\frac{b(s)}{2}\\
&\quad+\frac{\tilde{P}_3}{s}b(s)+\tilde{P}_4a(s)-2P_1E(s)\Big)^2-4\Big((\frac{s^2}{2}-P_1s)a'(s)+\frac{b'(s)}{2}-P_1'sa(s)-2P_1a(s)+\tilde{P}_3a(s)\\
&\quad+\frac{3}{4}sa(s)+\frac{b(s)}{2s}-E(s)\Big)\Big(-\frac{P_2+\tilde{P}_2}{2}s^2a'(s)+(P_1s-\frac{P_2+\tilde{P}_2}{2})b'(s)+P_1'sb(s)\\
&\quad-\frac{1}{2}(P_2+\tilde{P}_2)'s^2a(s)-\frac{1}{2}(P_2+\tilde{P}_2)'b(s)-\frac{P_2+\tilde{P}_2}{2s}b(s)+\frac{P_1b(s)}{2}+\frac{\tilde{P}_4}{s}b(s)\\
&\quad-\frac{3}{4}(P_2+\tilde{P}_2)sa(s)-(P_2+\tilde{P}_2)E(s)\Big),\\
K_3 &= -\frac{s^2}{2}\tilde{P}_2a'(s)+\frac{\tilde{P}_2}{2}b'(s)-\frac{s^2}{2}\tilde{P}_2'a(s)+\frac{\tilde{P}_2'}{2}b(s)-\frac{3}{4}\tilde{P}_2sa(s)-\frac{\tilde{P}_2}{2s}b(s)-\tilde{P}_2E(s).
\end{aligned}$$

Denote by Q_0, Q_1 and Q_2 the terms which involve only $a(s)$ and $b(s)$, but not their derivatives in K_0, K_1+K_3 and K_2, namely,

$$\begin{cases}
Q_0=(\frac{3}{4}s-P_1's-2P_1+\tilde{P}_3)a(s)+\frac{b(s)}{2s},\\
Q_1=P_1'sb(s)-\frac{1}{2}(P_2+\tilde{P}_2)'s^2a(s)-\frac{1}{2}(P_2+\tilde{P}_2)'b(s)-\frac{P_2+\tilde{P}_2}{2s}b(s)+\frac{P_1b(s)}{2}\\
\qquad -\frac{3}{4}(P_2+\tilde{P}_2)sa(s)+\frac{\tilde{P}_4}{s}b(s),\\
Q_2=-(P_2+\tilde{P}_2)a(s)-(P_2+\tilde{P}_2)'sa(s)-P_2a(s)+\frac{\tilde{P}_3}{s}b(s)+\tilde{P}_4a(s)+\frac{b(s)}{2}.
\end{cases}$$

Then

$$K_2^2-4K_0(K_1+K_3)=\Big(-(P_2+\tilde{P}_2)sa'(s)+sb'(s)\Big)^2-4\Big((\frac{s^2}{2}-P_1s)a'(s)+\frac{b'(s)}{2}\Big)$$
$$\times\Big(-\frac{P_2+\tilde{P}_2}{2}s^2a'(s)+(P_1s-\frac{P_2+\tilde{P}_2}{2})b'(s)\Big)+2Q_2\Big(-(P_2+\tilde{P}_2)sa'(s)+sb'(s)\Big)$$
$$+4Q_0\Big(\frac{P_2+\tilde{P}_2}{2}s^2a'(s)-(P_1s-\frac{P_2+\tilde{P}_2}{2})b'(s)\Big)-4Q_1\Big((\frac{s^2}{2}-P_1s)a'(s)$$
$$+\frac{b'(s)}{2}\Big)+Q_2^2-4Q_0Q_1+4(P_1^2-P_2-\tilde{P}_2)E^2(s)+4E(s)\Big(Q_1-P_1Q_2+(P_2+\tilde{P}_2)Q_0\Big).$$

The right hand side is a quadratic form of $a'(s)$, $b'(s)$ and $E(s)$. Denoting the coefficients of $a'(s)$ and $b'(s)$ by $2a_1(s)$ and $2a_2(s)$ respectively, then

$$\begin{cases} a_1=-(P_2+\tilde{P}_2)Q_2s+(P_2+\tilde{P}_2)Q_0s^2-Q_1(s^2-2P_1s),\\ a_2=Q_2s-Q_0(2P_1s-P_2-\tilde{P}_2)-Q_1.\end{cases}\tag{3.8}$$

Thus we have

$$K_2^2-4K_0(K_1+K_3)=(P_2+\tilde{P}_2+s^2-2P_1s)\Big((P_2+\tilde{P}_2)s^2a'(s)^2-2P_1sa'(s)b'(s)+b'(s)^2\Big)$$
$$+2a_1a'(s)+2a_2b'(s)+Q_2^2-4Q_0Q_1+4(P_1^2-P_2-\tilde{P}_2)E^2(s)+4E(s)\Big(Q_1-P_1Q_2$$
$$+(P_2+\tilde{P}_2)Q_0\Big).\tag{3.9}$$

The coefficient $P_2+\tilde{P}_2+s^2-2P_1s$, which will be denoted by $-\tilde{A}$, is equal to $-(\lambda_2(s)-s)(s-\lambda_1(s))<0$ in $[b_0,s_0+\eta_0]$ due to Lemma 2.1 and Lemma 2.2.

To transform (3.9) to a standard quadratic form, we introduce

$$\begin{cases} Y_1=a'(s)+\frac{a_1+a_2P_1s}{\tilde{A}s^2D_1},\\ Y_2=-P_1sa'(s)+b'(s)-\frac{a_2}{\tilde{A}},\\ \tilde{E}=E(s)+\frac{Q_1-P_1Q_2+(P_2+\tilde{P}_2)Q_0}{2D_1}\end{cases}$$

Substituting them into the expressions of K_0 and $K_2^2-4K_0(K_1+K_3)$ yields

$$\begin{cases} K_0=\frac{s^2-P_1s}{2}Y_1+\frac{Y_2}{2}-\tilde{E}+Q_0+\frac{a_2}{2\tilde{A}}-\frac{(s-P_1)(a_1+a_2P_1s)}{2\tilde{A}sD_1}+\frac{Q_1-P_1Q_2+(P_2+\tilde{P}_2)Q_0}{2D_1},\\ K_2^2-4K_0(K_1+K_3)=\tilde{A}s^2D_1Y_1^2-\tilde{A}Y_2^2+4(P_1^2-P_2-\tilde{P}_2)\tilde{E}^2+Q_2^2-4Q_0Q_1+\frac{a_2^2}{\tilde{A}}\\ \qquad -\frac{(a_1+a_2P_1s)^2}{\tilde{A}s^2D_1}-\frac{(Q_1-P_1Q_2+(P_2+\tilde{P}_2)Q_0)^2}{D_1},\\ K_3=\frac{P_1s-s^2}{2}\tilde{P}_2Y_1+\frac{\tilde{P}_2}{2}Y_2-\tilde{P}_2\tilde{E}+\tilde{P}_2\Big(\frac{(s-P_1)(a_1+a_2P_1s)}{2\tilde{A}sD_1}+\frac{a_2}{2\tilde{A}}+\frac{Q_1-P_1Q_2+(P_2+\tilde{P}_2)Q_0}{2D_1}\\ \qquad +\frac{\tilde{P}_2'}{2\tilde{P}_2}(b(s)-s^2a(s))-\frac{s}{2}a(s)-\frac{b(s)}{2s}\Big).\end{cases}$$

A key observation is the fact:

$$\begin{cases} Q_0 + \frac{a_2}{2\tilde{A}} - \frac{(s-P_1)(a_1+a_2P_1s)}{2\tilde{A}sD_1} + \frac{Q_1-P_1Q_2+(P_2+\tilde{P}_2)Q_0}{2D_1} = 0, \\ Q_2^2 - 4Q_0Q_1 + \frac{a_2^2}{\tilde{A}} - \frac{(a_1+a_2P_1s)^2}{\tilde{A}s^2D_1} - \frac{(Q_1-P_1Q_2+(P_2+\tilde{P}_2)Q_0)^2}{D_1} = 0. \end{cases}$$

Hence $K_0 > 0$, $K_2^2 - 4K_0(K_1+K_3) < 0$ and $K_3 > 0$ are equivalent to

$$\begin{cases} (s^2 - P_1s)Y_1 + Y_2 - 2\tilde{E} > 0, \\ \tilde{A}s^2D_1Y_1^2 - \tilde{A}Y_2^2 + 4D_1\tilde{E}^2 < 0, \\ (P_1s - s^2)Y_1 + Y_2 - 2\tilde{E} + 2Q < 0, \end{cases} \tag{3.10}$$

where $2Q = -Q_0 + \frac{(s-P_1)(a_1+a_2P_1s)}{\tilde{A}sD_1} + \frac{\tilde{P}_2'}{2\tilde{P}_2}(b(s) - s^2a(s)) - \frac{s}{2}a(s) - \frac{b(s)}{2s}$.

Step 4. Construction of $a(s)$, $b(s)$ and $E(s)$.

To solve the system (3.10), we require to study the solvability of the following equation system:

$$\begin{cases} (P_1s - s^2)Y_1 - Y_2 + 2\tilde{E} = -\delta_0, \\ \tilde{A}s^2D_1Y_1^2 - \tilde{A}Y_2^2 + 4D_1\tilde{E}^2 = -\delta_0, \\ (P_1s - s^2)Y_1 + Y_2 - 2\tilde{E} = -2Q - \delta_0, \end{cases} \tag{3.11}$$

with $\delta_0 > 0$ an appropriate constant to be determined, which is only dependent on b_0 and γ for large q_0 and small b_0.

It follows from (3.11) that

$$\begin{cases} Y_1 = -\frac{Q(s)+\delta_0}{P_1s-s^2}, \\ Y_2 = 2\tilde{E} - Q(s), \end{cases} \tag{3.12}$$

here $\tilde{E}$ satisfies

$$4(D_1 - \tilde{A})\tilde{E}^2 + 4\tilde{A}Q\tilde{E} + \frac{\tilde{A}D_1(Q+\delta_0)^2}{(P_1-s)^2} + \delta_0 - \tilde{A}Q^2 = 0. \tag{3.13}$$

In order to solve (3.13), one requires that the discriminant Δ satisfies

$$\Delta = 16\bigg(\tilde{A}^2Q^2 + (\tilde{A} - D_1)(\frac{\tilde{A}D_1Q^2}{(P_1-s)^2} - \tilde{A}Q^2 + \frac{\tilde{A}D_1}{(P_1-s)^2}\delta_0^2 + (1 + \frac{2\tilde{A}D_1Q}{(P_1-s)^2})\delta_0)\bigg) > 0. \tag{3.14}$$

Next we choose the constant δ_0 such that (3.14) holds.

By Lemma 2.1 and Lemma 2.2 (or see Lemma A.1 and Lemma A.3 in [5]), we can obtain

$$\begin{aligned} \tilde{A} &= \frac{(\gamma-1)b_0^2(1+b_0^2)^2}{2-(\gamma-1)b_0^2(1+b_0^2)} + O(q_0^{-\frac{2}{\gamma-1}}) + O(q_0^{-2}), \\ D_1 &= \frac{(\gamma-1)b_0^2(1+b_0^2)^2(1-\frac{1}{2}(\gamma-1)b_0^2)}{2(1-\frac{1}{2}(\gamma-1)b_0^2(1+b_0^2))^2} + O(q_0^{-\frac{2}{\gamma-1}}) + O(q_0^{-2}). \end{aligned}$$

Thus

$$\tilde{A} - D_1 = -\frac{(\gamma-1)^2b_0^6(1+b_0^2)^2}{(2-(\gamma-1)b_0^2(1+b_0^2))^2} + O(q_0^{-\frac{2}{\gamma-1}}) + O(q_0^{-2}).$$

Now we start to analyze the troublesome term Q.

In fact, it follows from Lemma 2.1 and Lemma 2.2 that (or see the Appendix in [5])

$$
\begin{aligned}
&P_1=\frac{b_0}{1-\frac{1}{2}(\gamma-1)b_0^2(1+b_0^2)}+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2}),\\
&P_2+\tilde P_2=\frac{b_0^2(1-\frac{1}{2}(\gamma-1)(1+b_0^2))}{1-\frac{1}{2}(\gamma-1)b_0^2(1+b_0^2)}+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2}),\\
&\lambda_2(s)=s+\frac{\sqrt{\gamma-1}b_0(1+b_0^2)(\sqrt{2-(\gamma-1)b_0^2})+\sqrt{\gamma-1}b_0^2}{2-(\gamma-1)b_0^2(1+b_0^2)}+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2}),\\
&\tilde P_2=\frac{(\gamma-1)b_0^2(1+b_0^2)}{2-(\gamma-1)b_0^2(1+b_0^2)}+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2}),\\
&P_1-s=\frac{(\gamma-1)b_0^3(1+b_0^2)}{2-(\gamma-1)b_0^2(1+b_0^2)}+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2}),\\
&P_1'=-\frac{1}{(1+b_0^2)(1-\frac{1}{2}(\gamma-1)b_0^2(1+b_0^2))}-\frac{(1+\frac{1}{2}(\gamma-1)b_0^2(1+b_0^2))b_0^2}{(1-\frac{1}{2}(\gamma-1)b_0^2(1+b_0^2))(1+b_0^2)}+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2}),\\
&|P_2'|\le C,\qquad |\tilde P_2'|\le C,\\
&|\tilde P_3|\le Cq_0^{\frac{2(\gamma-2)}{\gamma-1}},\qquad |\tilde P_4|\le Cq_0^{\frac{2(\gamma-2)}{\gamma-1}}.
\end{aligned}
$$

Set

$$b(s)=s^2\tilde b(s).$$

Then it follows from the expressions of Q_1 and Q_2 and the computations above that

$$
\begin{aligned}
Q_1&=b_0O(q_0^{\frac{2(\gamma-2)}{\gamma-1}})+O(b_0^2)+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2}),\\
Q_2&=O(b_0)+O(q_0^{\frac{2(\gamma-2)}{\gamma-1}})+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2})
\end{aligned}
$$

and

$$
\begin{aligned}
a_1&=O(b_0^4)+b_0^3O(q_0^{\frac{2(\gamma-2)}{\gamma-1}})+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2}),\\
a_2&=O(b_0^2)+b_0O(q_0^{\frac{2(\gamma-2)}{\gamma-1}})+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2})
\end{aligned}
$$

This leads to

$$
\begin{aligned}
2Q&=-Q_0+\frac{\tilde P_2'}{2\tilde P_2}(b(s)-s^2a(s))-\frac{s}{2}a(s)-\frac{b(s)}{2s}+O(b_0^2)+b_0O(q_0^{\frac{2(\gamma-2)}{\gamma-1}})+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2})\\
&=(2P_1+P_1's-\frac{5}{4}s)a(s)-s\tilde b(s)+\frac{\tilde P_2's^2}{2\tilde P_2}(\tilde b(s)-a(s))+O(b_0^2)+O(q_0^{\frac{2(\gamma-2)}{\gamma-1}})+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2})\\
&=-\frac{b_0}{4}a(s)-b_0\tilde b(s)+\frac{\tilde P_2's^2}{2\tilde P_2}(\tilde b(s)-a(s))+O(b_0^2)+O(q_0^{\frac{2(\gamma-2)}{\gamma-1}})+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2}).
\end{aligned}
\tag{3.15}
$$

In addition, a direct computation yields

$$\Delta=16(\tilde A-D_1)\delta_0\Big(\frac{\tilde AD_1}{(P_1-s)^2}\delta_0+\frac{2\tilde AD_1Q}{(P_1-s)^2}+1\Big).$$

Thus, in order to ensure that $\Delta > 0$, we need

$$\frac{\tilde{A}D_1}{(P_1-s)^2}\delta_0 + \frac{2\tilde{A}D_1 Q}{(P_1-s)^2} + 1 < 0. \tag{3.16}$$

We now assume that

$$\tilde{b}(s) = a(s) + O(b_0^2) + O(q_0^{-\frac{2}{\gamma-1}}), \qquad a(s) = 2 + O(q_0^{-\frac{2}{\gamma-1}}), \tag{3.17}$$

and

$$\delta_0 = b_0^2. \tag{3.18}$$

Then for small b_0 and large q_0, (3.16) is equivalent to

$$Q + b_0^2 < 0. \tag{3.19}$$

It follows from (3.15) and (3.17) that

$$Q = -\frac{5}{4}b_0 + O(b_0^2), \tag{3.20}$$

which implies (3.19).

Finally, we determine $a(s)$, $b(s)$ and $E(s)$ such that (3.3), (3.5) and (3.17) hold.

Set

$$a(b_0) = 2, \qquad \tilde{b}(b_0) = 2 + b_0^2.$$

Then it follows from (3.10) that

$\tilde{E}(b_0)$ can be determined. Subsequently, $Y_1(b_0)$ and $Y_2(b_0)$ are also known.

By use of the expressions of $Y_1, Y_2, \tilde{E}$ and $\tilde{b}'(b_0) = \frac{1}{b_0^2}(b'(b_0) - 2b_0\tilde{b}(b_0))$, we can determine $a'(b_0), \tilde{b}'(b_0)$ and $E(b_0)$.

Therefore, define

$$a(s) = 2 + a'(b_0)(s-b_0), \qquad b(s) = s^2(2 + b_0^2 + \tilde{b}'(b_0)(s-b_0)), \qquad E(s) = E(b_0).$$

Then they satisfy the all requirements above. Consequently, we arrive at

$$\begin{aligned}
&\iiint_{D_T} z^{-\frac{3}{2}}\Big\{K_0(\partial_z\dot{\varphi})^2 + K_1(\partial_r\dot{\varphi})^2 + K_2\partial_z\dot{\varphi}\partial_r\dot{\varphi} + K_3((\partial_1\dot{\varphi})^2 + (\partial_2\dot{\varphi})^2) + (\frac{E}{z} + \frac{r}{z^2}E')\partial_z\dot{\varphi}\dot{\varphi} \\
&\qquad - \Big(\partial_r((P_2+\tilde{P}_2)E) + 2(\partial_z(P_1E) - \frac{P_1E}{z}) + \frac{(P_2+\tilde{P}_2)E}{r}\Big)\partial_r\dot{\varphi}\dot{\varphi}\Big\}dx_1dx_2dz \\
&\geq C\Big(\iiint_{D_T} z^{-\frac{3}{2}}|\nabla\dot{\varphi}|^2dx_1dx_2dz - \iiint_{D_T} z^{-\frac{7}{2}}|\dot{\varphi}|^2dx_1dx_2dz\Big).
\end{aligned} \tag{3.21}$$

Step 5. The estimate on $\iint_{\Gamma_T} z^{-\frac{3}{2}}\Big((\frac{x_1}{r} - cos\theta\partial_\theta\chi)K_5 + (\frac{x_2}{r} - sin\theta\partial_\theta\chi)K_6 - \partial_z\chi K_4\Big)dS.$

By the assumptions on $\xi(z,\theta)$ in Theorem 3.1 and $\eta_0 \leq q_0^{-\frac{2}{\gamma-1}}(s_0-b_0)$, it follows from the expressions of K_4, K_5 and K_6 that,

$$
\begin{aligned}
&(\frac{x_1}{r} - cos\theta\partial_\theta\chi)K_5 + (\frac{x_2}{r} - sin\theta\partial_\theta\chi)K_6 - \partial_z\chi K_4 = \left(P_1A - \frac{B}{2} - \partial_z\chi\frac{A}{2}\right)(\partial_z\dot{\varphi})^2 \\
&\quad + ((P_2+\tilde{P}_2)A - \partial_z\chi B)\partial_r\dot{\varphi}\partial_z\dot{\varphi} + \left(\frac{P_2+\tilde{P}_2}{2}B - \partial_z\chi(P_1B - \frac{(P_2+\tilde{P}_2)A}{2})\right)(\partial_r\dot{\varphi})^2 \\
&\quad - \frac{\tilde{P}_2}{2r^2}(B-\partial_z\chi A)(\partial_\theta\dot{\varphi})^2 + (\frac{P_2+\tilde{P}_2}{r} - 2\partial_z\chi P_1)E\partial_r\dot{\varphi}\dot{\varphi} - E\partial_z\chi\partial_z\dot{\varphi}\dot{\varphi} \\
&\quad + (O(\varepsilon) + O(q_0^{-\frac{2}{\gamma-1}}))\frac{1}{z}\left(|\nabla_{r,z}\dot{\varphi}|^2 + |\frac{\partial_\theta\dot{\varphi}}{r}|^2 + |\partial_r\dot{\varphi}\dot{\varphi}| + |\partial_z\dot{\varphi}\dot{\varphi}|\right) \\
&= ra(s_0)\Big\{\left(\beta_0(\partial_z\dot{\varphi})^2 + \beta_1\partial_r\dot{\varphi}\partial_z\dot{\varphi} + \beta_2(\partial_r\dot{\varphi})^2 - \frac{\tilde{P}_2}{2r^2}(\frac{b(s_0)}{s_0a(s_0)} - s_0)(\partial_\theta\dot{\varphi})^2\right) \\
&\quad + \frac{1}{z}\left((O(\varepsilon) + O(b_0^2) + O(q_0^{-\frac{2}{\gamma-1}}))(|\nabla_{r,z}\dot{\varphi}|^2 + |\frac{\partial_\theta\dot{\varphi}}{r}|^2 + |\partial_r\dot{\varphi}\dot{\varphi}| + |\partial_z\dot{\varphi}\dot{\varphi}|\right)\Big\} \\
&\equiv zb_0a(s_0)(I+II).
\end{aligned}
\tag{3.22}
$$

here

$$
\begin{aligned}
\beta_0 &= P_1(s_0) - \frac{b(s_0)}{2s_0a(s_0)} - \frac{s_0}{2}, \\
\beta_1 &= (P_2+\tilde{P}_2)(s_0) - \frac{b(s_0)}{a(s_0)}, \\
\beta_2 &= \frac{(P_2+\tilde{P}_2)(s_0)b(s_0)}{2s_0a(s_0)} - s_0\left(\frac{P_1(s_0)b(s_0)}{s_0a(s_0)} - \frac{(P_2+\tilde{P}_2)(s_0)}{2}\right).
\end{aligned}
$$

Noting $\partial_r\dot{\varphi} = B_0\dot{\varphi} - \mu_1\partial_z\dot{\varphi}$, one has

$$
I = \{\beta_0 - \mu_1\beta_1 + \mu_1^2\beta_2\}(\partial_z\dot{\varphi})^2 + \{\beta_1 - 2\mu_1\beta_2\}\partial_z\dot{\varphi}B_0\dot{\varphi} + \beta_2(B_0\dot{\varphi})^2 - \frac{\tilde{P}_2}{2r^2}\left(\frac{b(s_0)}{s_0a(s_0)} - s_0\right)(\partial_\theta\dot{\varphi})^2. \tag{3.23}
$$

A direct computation yields

$$
\begin{aligned}
\beta_0 &= O(b_0^3) + O(q_0^{-\frac{2}{\gamma-1}}) + O(q_0^{-2}), \\
\beta_1 &= -\frac{\gamma+1}{2}b_0^2 + O(b_0^3) + O(q_0^{-\frac{2}{\gamma-1}}) + O(q_0^{-2}), \\
\beta_2 &= -\frac{\gamma+1}{2}b_0^3 + O(b_0^5) + O(q_0^{-\frac{2}{\gamma-1}}) + O(q_0^{-2}), \\
-\tilde{P}_2\left(\frac{b(s_0)}{s_0a(s_0)} - s_0\right) &= \frac{(\gamma-1)b_0^5}{4} + O(b_0^5) + O(q_0^{-\frac{2}{\gamma-1}}) + O(q_0^{-2}).
\end{aligned}
$$

Thus, one has

$$
\beta_0 - \mu_1\beta_1 + {\mu_1}^2\beta_2 = \frac{\gamma+1}{8}b_0 + O(b_0^2) + O(q_0^{-\frac{2}{\gamma-1}}) + O(q_0^{-2}), \tag{3.24}
$$

$$
\beta_1 - 2\mu_1\beta_2 = O(b_0^3) + O(q_0^{-\frac{2}{\gamma-1}}) + O(q_0^{-2}). \tag{3.25}
$$

Using $\partial_z\dot\varphi\mathcal{B}_0\dot\varphi \geq -\frac{1}{2}(b_0(\mathcal{B}_0\dot\varphi)^2+\frac{1}{b_0}(\partial_z\dot\varphi)^2)$ and then substituting (3.24) and (3.25) into (3.23), we get

$$\begin{aligned}I \geq&(\frac{\gamma+1}{8}b_0+O(b_0^2)+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2}))(\partial_z\dot\varphi)^2\\&-(\frac{\gamma+1}{2}b_0^3+O(b_0^4)+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2}))(\mathcal{B}_0\dot\varphi)^2\\&+(\frac{(\gamma-1)b_0^5}{8}+O(b_0^5)+O(q_0^{-\frac{2}{\gamma-1}})+O(q_0^{-2}))|\frac{\partial_\theta\dot\varphi}{r}|^2.\end{aligned}\tag{3.26}$$

Consequently, we have

$$\begin{aligned}&\iint_{\Gamma_T} z^{-\frac{3}{2}}\Big((\frac{x_1}{r}-cos\theta\partial_\theta\chi)K_5+(\frac{x_2}{r}-sin\theta\partial_\theta\chi)K_6-\partial_z\chi K_4\Big)dS \geq \frac{\gamma+1}{16}b_0\iint_{\Gamma_T} z^{-\frac{1}{2}}(\partial_z\dot\varphi)^2dS\\&\quad+\frac{(\gamma-1)b_0^5}{16}\iint_{\Gamma_T} z^{-\frac{1}{2}}|\frac{\partial_\theta\dot\varphi}{r}|^2dS-\frac{2(\gamma+1)}{3}b_0^3\iint_{\Gamma_T} z^{-\frac{1}{2}}(\mathcal{B}_0\dot\varphi)^2dS\\&\quad+((O(\varepsilon)+O(b_0^2)+O(q_0^{-\frac{2}{\gamma-1}}))\iint_{\Gamma_T} z^{-\frac{3}{2}}(|\partial_r\dot\varphi\dot\varphi|+|\partial_z\dot\varphi\dot\varphi|)dS.\end{aligned}\tag{3.27}$$

Step 6. The estimates on $\frac{1}{T^{\frac{5}{2}}}\iint_{b_0T\leq r\leq\chi(\theta,T)}|\dot\varphi(x_1,x_2,T)|^2dx_1dx_2$, $\iint_{\Gamma_T} z^{-\frac{3}{2}}(|\partial_r\dot\varphi\dot\varphi|+|\partial_z\dot\varphi\dot\varphi|)dS$ and $\iiint_{D_T} z^{-\frac{7}{2}}|\dot\varphi|^2dx$.

We estimate only $\frac{1}{T^{\frac{5}{2}}}\iint_{b_0T\leq r\leq\chi(\theta,T)}|\dot\varphi(x_1,x_2,T)|^2dx_1dx_2$, the other integrals can be treated similarly.

Since

$$\dot\varphi(r,\theta,z)=\phi(\theta,z)+\int_{b_0T}^r\partial_r\dot\varphi(r,\theta,z)dr,$$

then

$$\dot\varphi^2(r,\theta,T)\leq 2\Big(\phi^2(\theta,T)+ln\frac{\chi(\theta,T)}{b_0T}\int_{b_0T}^r r|\partial_r\dot\varphi(r,\theta,T)|^2dr\Big)$$

and

$$\int_{b_0T}^{\chi(\theta,T)} r(\dot\varphi)^2(r,\theta,T)dr\leq C(\chi^2(\theta,T)-b_0^2T^2)\Big(\phi^2(\theta,T)+ln\frac{\chi(\theta,T)}{b_0T}\int_{b_0T}^{\chi(\theta,T)}|\partial_r\dot\varphi(r,\theta,T)|^2dr\Big).$$

Thus

$$\begin{aligned}&\frac{1}{T^{\frac{5}{2}}}\iint_{b_0T\leq r\leq\chi(\theta,T)}|\dot\varphi(x_1,x_2,T)|^2dx_1dx_2\leq C\Big(\frac{(O(\varepsilon)+O(q_0^{-\frac{2}{\gamma-1}}))}{\sqrt{T}}\iint_{b_0T\leq r\leq\chi(\theta,T)}|\partial_r\dot\varphi(x_1,x_2,T)|^2dS\\&\quad+\varepsilon^2\Big).\end{aligned}\tag{3.28}$$

Analogously, using the boundary value of $\dot\varphi$ on B_T as above (or see Lemma 1 in [9]), one can obtain from Lemma 2.1 and $s_0+\eta_0-b_0\leq Cq_0^{-\frac{2}{\gamma-1}}$ that

$$\begin{aligned}\iint_{\Gamma_T} z^{-\frac{5}{2}}(|\partial_r\dot\varphi\dot\varphi|+|\partial_z\dot\varphi\dot\varphi|)dS\leq C\Big\{&(O(\varepsilon)+O(q_0^{-\frac{2}{\gamma-1}}))\Big(\iint_{\Gamma_T} z^{-\frac{1}{2}}|\nabla\dot\varphi|^2dS\\&+\iiint_{D_T} z^{-\frac{3}{2}}|\nabla\dot\varphi|^2dx\Big)+\varepsilon^2\Big\},\end{aligned}\tag{3.29}$$

$$\iiint_{D_T} z^{-\frac{7}{2}}|\dot\varphi|^2dx\leq C\Big((O(\varepsilon)+O(q_0^{-\frac{2}{\gamma-1}}))\iiint_{D_T} z^{-\frac{3}{2}}|\nabla\dot\varphi|^2dx+\varepsilon^2\Big).\tag{3.30}$$

Substituting (3.4), (3.6), (3.21) and (3.27)-(3.30) into (3.2) yields

$$\frac{C_1}{\sqrt{T}}\iint_{b_0T\le r\le\chi(\theta,T)}|\nabla\dot\varphi|^2dS+C_2\iiint_{D_T}z^{-\frac{3}{2}}|\nabla\dot\varphi|^2dx+C_3\iint_{\Gamma_T}z^{-\frac{1}{2}}|\nabla\dot\varphi|^2dS+C_4\iint_{B_T}z^{-\frac{1}{2}}|\nabla\dot\varphi|^2dS$$
$$\le\iiint_{D_T}z^{-\frac{3}{2}}L\dot\varphi M\dot\varphi dx+C_5\iint_{\Gamma_T}z^{-\frac{1}{2}}(\mathcal{B}_0\dot\varphi)^2dS+C_6\varepsilon^2 \tag{3.31}$$

where the constants $C_i(1\le i\le 6)$ are independent of q_0 and ε (but depends on b_0) thanks to the appropriate choices of $a(s)$ and $b(s)$. Therefore Theorem 3.1 is proved.

§4. The proof of Theorem 1.1 for the case $b_1(\theta,z)=b_0z$

In order to prove Theorem 1.1 for the case $b_1(\theta,z)=b_0z$, we first derive the following higher order energy estimates so that one can derive the decay properties of $\nabla\dot\varphi$ and ξ for large z.

Theorem 4.1. Assume that $\dot\varphi\in C^{k_0}(D_T)$ and $\xi(\theta,z)\in C^{k_0}([0,2\pi;\frac{1}{2},T]$ with $k_0\ge 6$ is a solution of (2.1) with (2.2), (2.5) and (2.6). In addition, $|\xi(\theta,z)|+|z\partial_z\xi(\theta,z)|+|\partial_\theta\xi(\theta,z)|\le C\varepsilon$ and $\sum\limits_{0\le l\le[\frac{k_0}{2}]+1}z^l|\nabla^{l+1}\dot\varphi(r,z)|\le C\varepsilon$ hold for $(\theta,z)\in[0,\pi;\frac{1}{2},T]$. Then for sufficiently small ε, we have

$$\iint_{b_0T\le r\le\chi(\theta,T)}\sum_{0\le l\le k_0-1}T^{2l-\frac{1}{2}}|\nabla^{l+1}\dot\varphi(r,\theta,T)|^2dS+\iint_{D_T}\sum_{0\le l\le k_0-1}z^{2l-\frac{3}{2}}|\nabla^{l+1}\dot\varphi|^2dx$$
$$+\iint_{\Gamma_T}\sum_{0\le l\le k_0-1}z^{2l-\frac{1}{2}}|\nabla^{l+1}\dot\varphi|^2dS+\iint_{B_T}\sum_{0\le l\le k_0-1}z^{2l-\frac{1}{2}}|\nabla^{l+1}\dot\varphi|^2dS$$
$$\le C\varepsilon^2, \tag{4.1}$$

here and below $C>0$ denotes a generic constant depending on q_0,b_0 and γ.

Next, we turn to the main arguments for the proof of Theorem 4.1. As in [8-9] or [15-16], we will use the vector fields which are tangent to the surface of the cone and nearly tangential to the shock front so that we can raise the order of the energy estimate by the standard commutation argument. The difference from the usual commutation argument is that the vector field is only nearly tangential to the shock front boundary, and thus there will appear some error terms caused by the perturbation of the shock front to be treated. We first state an elementary estimate.

Lemma 4.2. Assume that $\dot\varphi$ is a C^{k_0} solution, then there is a constant C independent of $\dot\varphi$ and T, so that

$$\sum_{0\le l\le k_0-1}z^l|\nabla^{l+1}\dot\varphi|\le C\sum_{0\le l\le k_0-1}|\nabla S^l\dot\varphi|\quad in\quad D_T, \tag{4.2}$$

where $S=z\partial_z+r\partial_r$ or ∂_θ.

Proof. This lemma can be found in [9] and [16]. So we omit the proof here.

Return to the proof of Theorem 4.1. Since the vector field S is tangent to the boundary $r=b_0z$, then $S^m\dot\varphi=S^m\phi$ on $r=b_0z$ in view of the boundary condition (2.2). Thus one can apply Theorem 3.1 and the Remark 3.1 to $S^m\dot\varphi(0\le m\le k_0-1)$ (at this moment, one can contemporarily neglect the concrete expressions of the constants in (3.1)) to obtain

$$
\begin{aligned}
&\frac{1}{\sqrt{T}}\iint_{b_0T\le r\le\chi(\theta,T)}\sum_{0\le m\le k_0-1}|\nabla S^m\dot\varphi(r,\theta,T)|^2dS+\iiint_{D_T}z^{-\frac32}\sum_{0\le m\le k_0-1}|\nabla S^m\dot\varphi|^2dx\\
&+\iint_{\Gamma_T}z^{-\frac12}\sum_{0\le m\le k_0-1}|\nabla S^m\dot\varphi|^2dS+\iint_{B_T}z^{-\frac12}\sum_{0\le m\le k_0-1}|\nabla S^m\dot\varphi|^2dS\\
&\le C(q_0,b_0,\gamma)\Big(\iiint_{D_T}z^{-\frac32}\sum_{0\le m\le k_0-1}\mathcal{L}S^m\dot\varphi\mathcal{M}S^m\dot\varphi dx+\int_{\Gamma_T}z^{-\frac12}\sum_{0\le m\le k_0-1}(\mathcal{B}_0S^m\dot\varphi)^2dS+\varepsilon^2\Big)
\end{aligned}
\tag{4.3}
$$

To estimate the first term in the right hand side of (4.3), we need an explicit representation of $\mathcal{L}S^m\dot\varphi$. Thanks to $SP_1(\frac{r}{z})=SP_2(\frac{r}{z})=0$ and $S(\frac1r)=-\frac1r$ or $S(\frac1r)=0$, we have $\mathcal{L}S\dot\varphi=S\mathcal{L}\dot\varphi-2\mathcal{L}\dot\varphi$. It follows from the equation (2.1) that

$$
\mathcal{L}S^m\dot\varphi=\sum_{i,j=1}^{3}\sum_{0\le l\le m}C_l\Big\{\sum_{l_1+l_2\le l}C_{l_1l_2}\Big(S^{l_1}(f_{ij})\partial_{ij}^2S^{l_2}\dot\varphi+\frac{(-1)^{l_1}}{r}S^{l_2}(f_0)\Big)\Big\},\tag{4.4}
$$

where $f_{i,j}$ and f_0 are the functions appeared in (2.1). By the properties of f_{ij} and f_0 and the assumptions in Theorem 4.1, one can show that for $m\le k_0-1$

$$
|S^{l_1}f_{ij}|\le C\sum_{m\le k_0-1}|\nabla S^m\dot\varphi|,\qquad |S^{l_1}(f_0)|\le C\sum_{m\le k_0-1}|\nabla S^m\dot\varphi|^2.\tag{4.5}
$$

We will treat $\iiint_{D_T}z^{-\frac32}S^{l_1}(f_{33})\partial_z^2S^{l_2}\dot\varphi\mathcal{M}S^m\dot\varphi dx$ only, because the other terms can be disposed similarly. This is divided into two cases:

If $l_2\le m-1$, from Lemma 4.2 and assumptions in Theorem 4.1, as in [9] or [16], one can get

$$
|S^{l_1}(f_{33})\partial_z^2S^{l_2}\dot\varphi\mathcal{M}S^m\dot\varphi|\le C\varepsilon\sum_{m\le k_0-1}|\nabla S^m\dot\varphi|^2.\tag{4.6}
$$

If $l_1=0,l_2=m$, then

$$
\begin{aligned}
S^{l_1}(f_{33})\partial_z^2S^{l_2}\dot\varphi\mathcal{M}S^m\dot\varphi&=\partial_z(f_{33}B\partial_zS^m\dot\varphi\partial_rS^m\dot\varphi-\frac12Af_{33}(\partial_zS^m\dot\varphi)^2)-\frac12\partial_r(f_{33}B(\partial_zS^m\dot\varphi)^2)\\
&+\frac12(\partial_r(f_{33}B)-\partial_z(f_{33}A))(\partial_zS^m\dot\varphi)^2-\partial_z(f_{33}B)\partial_zS^m\dot\varphi\partial_rS^m\dot\varphi.
\end{aligned}
\tag{4.7}
$$

Hence by the integration by parts we get

$$
\begin{aligned}
\iiint_{D_T}z^{-\frac32}\sum_{0\le m\le k_0-1}\mathcal{L}S^m\dot\varphi\mathcal{M}S^m\dot\varphi dx&\le C\varepsilon\Big(\iint_{\Gamma_T}z^{-\frac12}\sum_{0\le m\le k_0-1}|\nabla S^m\dot\varphi|^2dS\\
&+\iiint_{D_T}z^{-\frac32}\sum_{0\le m\le k_0-1}|\nabla S^m\dot\varphi|^2dx+\frac{1}{\sqrt{T}}\iint_{b_0T\le r\le\chi(\theta,T)}\sum_{m\le k_0-1}|\nabla S^m\dot\varphi(r,\theta,T)|^2dS\\
&+\iint_{B_T}z^{-\frac12}\sum_{0\le m\le k_0-1}|\nabla S^m\dot\varphi|^2dS\Big)+C\varepsilon^2.
\end{aligned}
\tag{4.8}
$$

Next, we estimate the second term on the right hand side of (4.3), that is $\iint_{\Gamma_T} z^{-\frac{1}{2}} \sum_m (\mathcal{B}_0 S^m \dot{\varphi})^2 dS$, which is a major term, because it involves the boundary of shock front. Write

$$\mathcal{B}_0 S^m \dot{\varphi} = [\mathcal{B}_0, S^m]\dot{\varphi} + (S^m - S^m_\Gamma)\mathcal{B}_0\dot{\varphi} + S^m_\Gamma \mathcal{B}_0 \dot{\varphi},$$

we estimate each term separately. The first term has the form

$$[\mathcal{B}_0, S^m]\dot{\varphi} = \sum_{0\le l\le m-1} C_l S^l \mathcal{B}_0 \dot{\varphi}. \tag{4.9}$$

To estimate other two terms, we notice that from the equation (2.6)

$$\sum_{0\le m=m_1+m_2\le k_0-1} z^{m_1}|\partial_z^{m_1}\partial_\theta^{m_2}\xi| \le C(\sum_{0\le m\le k_0-2} z^m |\nabla^{m+1}\dot{\varphi}| + |\xi|) \qquad on \qquad r=\chi(\theta,z). \tag{4.10}$$

Hence by the assumptions in Theorem 4.1, we have

$$\sum_{0\le m\le [\frac{k_0}{2}]+1} z^{m_1}|\partial_z^{m_1}\partial_\theta^{m_2}\xi| \le C\varepsilon. \tag{4.11}$$

In addition, the equation (2.5) yields

$$S^m_\Gamma \mathcal{B}_0\dot{\varphi} + \mu_2 S^m_\Gamma \xi = S^m_\Gamma \kappa_1(\xi, \nabla_{r,z}\dot{\varphi}) \qquad on \qquad r=\chi(\theta,z), \tag{4.12}$$

where $S_\Gamma = z\partial_z + z\partial_z\chi(z)\partial_r$ or $\partial_\theta + \partial_\theta\chi\partial_r$ are tangent to the shock surface $r=\chi(\theta,z)$. It should be noted that $|\mu_2|$ is a large constant with the same order as q_0.

Using (4.11) and (4.12), for $m\le k_0-1$ we have the following estimate:

$$|S^m_\Gamma \mathcal{B}_0\dot{\varphi}| \le C(q_0 \sum_{0\le l\le m} z^{l_1}|\partial_z^{l_1}\partial_\theta^{l_2}\xi| + \varepsilon \sum_{0\le l\le m} z^l|\nabla^{l+1}\dot{\varphi}|). \tag{4.13}$$

As in the Lemma 10 in [9], one can prove that

$$|(S^m - S^m_\Gamma)\mathcal{B}_0\dot{\varphi}| \le C\varepsilon_0(\sum_{0\le l\le m} z^l|\nabla^{l+1}\dot{\varphi}| + |\xi|). \tag{4.14}$$

Now collecting (4.9), (4.13) and (4.14) and using (4.10) and Lemma 4.2 one can get that

$$\begin{aligned}
&\iint_{b_0T\le r\le\chi(\theta,T)} \sum_{0\le l\le k_0-1} T^{2l-\frac{1}{2}}|\nabla^{l+1}\dot{\varphi}(r,\theta,T)|^2 dS + \iiint_{D_T} \sum_{0\le l\le k_0-1} z^{2l-\frac{3}{2}}|\nabla^{l+1}\dot{\varphi}|^2 dx \\
&+ \iint_{\Gamma_T} \sum_{0\le l\le k_0-1} z^{2l-\frac{1}{2}}|\nabla^{l+1}\dot{\varphi}|^2 dS + \iint_{B_T} \sum_{0\le l\le k_0-1} z^{2l-\frac{1}{2}}|\nabla^{l+1}\dot{\varphi}|^2 dS \\
&\le C(q_0,b_0,\gamma)\Big(\iint_{\Gamma_T} \sum_{0\le l\le k_0-2} z^{2l-\frac{1}{2}}|\nabla^{l+1}\dot{\varphi}|^2 dS + \iint_{\Gamma_T} z^{-\frac{1}{2}}|\xi|^2 dS + \varepsilon^2\Big).
\end{aligned} \tag{4.15}$$

In particular, for $k_0 = 1$, due to the estimate (3.1) and the equation (2.5) and the inequality (4.8), then (4.15) becomes

$$\frac{C_1}{\sqrt{T}} \iint_{b_0 T \le r \le \chi(\theta,T)} |\nabla \dot{\varphi}(r,\theta,T)|^2 dS + C_2 \iiint_{D_T} z^{-\frac{3}{2}} |\nabla \dot{\varphi}|^2 dS + C_3 \iint_{\Gamma_T} z^{-\frac{1}{2}} |\nabla \dot{\varphi}|^2 dS$$
$$+ C_4 \iint_{B_T} z^{-\frac{1}{2}} |\nabla \dot{\varphi}|^2 dS \le C_5 q_0^2 \iint_{\Gamma_T} z^{-\frac{1}{2}} |\xi|^2 dS + C_6 \varepsilon^2. \tag{4.16}$$

Here $C_i (1 \le i \le 6)$ are generic constants independent of q_0 and ε_0.

It follows from (4.15) and the inductive argument that the crucial step to prove (4.1) is to estimate the first term in the right hand side of (4.16). Note that the first term in the right side of (4.16) has a large factor q_0^2. We will try to absorb this term into the left hand side of (4.16).

In fact, by the assumption on $\xi(\theta, z)$, we have

$$\iint_{\Gamma_T} z^{-\frac{1}{2}} |\xi(\theta,z)|^2 dS = (1 + O(b_0^2) + O(\varepsilon_0^2) + O(q_0^{-\frac{2}{\gamma-1}})) \int_0^{2\pi} d\theta \int_{\frac{1}{2}}^T z^{-\frac{1}{2}} |\xi(\theta,z)|^2 dz. \tag{4.17}$$

In addition, by the Hardy inequality (see [11]) the term $\int_{\frac{1}{2}}^T z^{-\frac{1}{2}} |\xi(\theta,z)|^2 dz$ can be treated as follows

$$\int_{\frac{1}{2}}^T z^{-\frac{1}{2}} |\xi(\theta,z)|^2 dz = \int_{\frac{1}{2}}^T z^{-\frac{5}{2}} |z\xi(\theta,z)|^2 dz$$
$$\le 2 \int_{\frac{1}{2}}^T z^{-\frac{1}{2}} |z\xi(\theta,z) + \frac{1}{U_+} \dot{\varphi}(\chi(\theta,z),z)|^2 dz + 2 \int_{\frac{1}{2}}^T z^{-\frac{5}{2}} |\frac{1}{U_+} \dot{\varphi}(\chi(\theta,z),\theta,z)|^2 dz$$
$$\equiv I + II. \tag{4.18}$$

Here and below we will use the inequality $(x+y)^2 \le 2x^2 + 2y^2$ repeatedly.

By the Hardy type inequality again, the equation (2.6) and the assumptions in Theorem 4.1, as in [5], we have

$$|I| \le C \int_{\frac{1}{2}}^T z^{-\frac{1}{2}} |\partial_z (z\xi(\theta,z) + \frac{1}{U_+} \dot{\varphi}(\chi(\theta,z),\theta,z))|^2 dz$$
$$\le C(b_0,\gamma) \varepsilon^2 \int_{\frac{1}{2}}^T z^{-\frac{1}{2}} (|\xi(\theta,z)|^2 + |\nabla \dot{\varphi}(\chi(\theta,z),\theta,z)|^2) dz. \tag{4.19}$$

Now we dominate II by $II_1 + II_2$ so that II can be bounded by some integrals on $r = b_0 z$ and the interior of D_T, where

$$II_1 = \frac{4}{U_+^2} \int_{\frac{1}{2}}^T z^{-\frac{5}{2}} |\dot{\varphi}(\chi(\theta,z),\theta,z) - \dot{\varphi}(b_0 z,\theta,z)|^2 dz,$$
$$II_2 = \frac{4}{U_+^2} \int_{\frac{1}{2}}^T z^{-\frac{5}{2}} |\dot{\varphi}(b_0 z,\theta,z)|^2 dz.$$

II_1 can be treated as follows

$$|II_1| \le \frac{C(b_0,\gamma)}{q_0^2} (1 + O(q_0^{-\frac{2}{\gamma-1}})) \int_{\frac{1}{2}}^T z^{-\frac{5}{2}} \left(\int_{b_0 z}^{\chi(\theta,z)} \partial_r \dot{\varphi}(r,\theta,z) dr \right)^2 dz$$
$$\le \frac{C(b_0,\gamma)}{q_0^2} (1 + O(q_0^{-\frac{2}{\gamma-1}})) \int_{\frac{1}{2}}^T z^{-\frac{3}{2}} \left(\int_{b_0 z}^{\chi(\theta,z)} |\partial_r \dot{\varphi}(r,\theta,z)|^2 dr \right) \frac{\chi(\theta,z) - b_0 z}{z} dz$$
$$\le \frac{1}{q_0^2} (O(\varepsilon) + O(q_0^{-\frac{2}{\gamma-1}})) \iint_{D_T} z^{-\frac{3}{2}} |\partial_r \dot{\varphi}(r,\theta,z)|^2 dr dz. \tag{4.20}$$

Using the boundary condition (2.2), we have

$$|II_2| \le C(b_0,\gamma)\varepsilon^2. \tag{4.21}$$

Substituting (4.21), (4.20), (4.19) into (4.18), (4.17) and (4.16), for the fixed b_0 and $\frac{1}{q_0}$ which are very small but $\frac{1}{q_0}$ is much smaller than b_0, for sufficiently small ε, we have

$$\begin{aligned}&\frac{C_1}{\sqrt{T}}\iint_{b_0T\le r\le\chi(\theta,T)}|\nabla\dot\varphi(r,\theta,T)|^2dS + C_2\iiint_{D_T} z^{-\frac{3}{2}}|\nabla\dot\varphi|^2dx + C_3\iint_{\Gamma_T} z^{-\frac{1}{2}}|\nabla\dot\varphi|^2dS\\ &+C_4\iint_{B_T}|\nabla\dot\varphi(b_0z,\theta,z)|^2dS \le C(b_0,\gamma)\varepsilon^2.\end{aligned} \tag{4.22}$$

Where $C_i(1\le i\le 4)$ depend only on b_0 and γ.

Hence we obtain (4.1) for $k_0=1$.

Furthermore, (4.15) shows that the higher order derivatives of $\dot\varphi$ can be dominated by its lower order derivatives, then by inductive argument we obtain (4.1). This completes the proof of Theorem 4.1.

The proof of Theorem 1.1

Based on the the energy estimate of higher order we can easily prove the global existence of the shock wave by using the local existence theorem and the standard continuity extension method. The local existence of the solution of (1.11) with (1.12)-(1.14) can be achieved as in [20], while for any given $z_0>\frac{1}{2}$, the solution of (1.11) with the initial data given on $z=z_0$ and the boundary conditions (1.12)-(1.14) in $[z_0,z_0+\eta_0]$ for some $\eta_0>0$ can be obtained (see [8]), provided that the initial data is smooth and satisfies the compatibility conditions. Moreover, if the perturbation of the initial data given on $z=z_0$ is small as $O(\varepsilon)$, the lifespan of the solution is at least as large as $C\varepsilon^{-1}$ with $C>0$. Therefore, as long as we can establish that the maximum norm of $\dot\varphi,\xi$ and their derivatives decays with a rate in z, then the solution can be extended continuously to the whole domain. That is, by using the local existence theorem and the property of decay of the solution we can obtain the uniform bound of $\dot\varphi,\xi$ and their derivatives, and then extend the solution continuously from $z_0<z<z_1$ to $z_0<z<z_1+\eta_0$ with η_0 being independent of z_1. Hence the key point to prove Theorem 1.1 is to give the decay rate of the maximum norm of $\dot\varphi,\xi$ and their derivatives.

It follows from the Sobolev's imbedding theorem (or see Lemma 14 in [9]) and the assumptions of Theorem 4.1 that for $b_0z\le r\le\chi(\theta,z)$ and $\frac{1}{2}\le z\le T$, one has

$$\sum_{0\le l\le k_0-2}|z^l\nabla^{l+1}\dot\varphi|^2 \le Cz^{-1}\iint_{b_0z\le r\le\chi(\theta,z)}\sum_{0\le l\le k_0-1}|z^l\nabla^{l+1}\dot\varphi(r,\theta,z)|^2dS. \tag{4.22}$$

On the other hand, (4.1) shows that

$$\iint_{b_0z\le r\le\chi(\theta,z)}\sum_{0\le l\le k_0-1}|z^l\nabla^{l+1}\dot\varphi(r,\theta,z)|^2dS \le C\varepsilon^2z^{\frac{1}{2}}. \tag{4.23}$$

Hence $\displaystyle\sum_{0\le l\le k_0-2}|z^l\nabla^{l+1}\dot\varphi|^2 \le C\varepsilon^2z^{-1/2}$ for $b_0z\le r\le\chi(\theta,z)$ and $\frac{1}{2}\le z\le T$.

For $k_0\ge 6$, one has

$$\sum_{l\le[\frac{k_0}{2}]+1}|z^l\nabla^{l+1}\dot\varphi| \le C\varepsilon z^{-\frac{1}{4}}. \tag{4.24}$$

In addition, due to $k_0 - 2 \geq [\frac{k_0}{2}] + 1$, the equations (2.5) and (2.6) yield

$$|\xi(\theta, z)| + |z\partial_z \xi(\theta, z)| + |\partial_\theta \xi(\theta, z)| \leq C\varepsilon z^{-\frac{1}{4}}. \tag{4.25}$$

(4.24) and (4.25) imply that the assumptions on ξ and $\dot{\varphi}$ hold for any $T > \frac{1}{2}$. Thus, noting that the constant C is independent of T in Theorem 4.1, we complete the proof on Theorem 1.1 under the additional assumption $b_1(\theta, z) = b_0 z$.

§5. **The general boundary case**

In this section, we discuss the equation (2.8) with (2.9)-(2.12). In order to prove Theorem 1.1 for general boundary, as in [5], we have to analyze the contributions of the perturbed boundary. It turns out that we can modify the arguments in the proof of Theorem 4.1 slightly to deal this general case. As in (4.3), we first estimate the term $\iiint_{D_T} \tilde{z}^{-\frac{3}{2}} \sum\limits_{0\leq m\leq k_0-1} \mathcal{L}\tilde{S}^m \dot{\varphi} \mathcal{M}\tilde{S}^m \dot{\varphi} d\tilde{x}$, where $\tilde{S} = \tilde{r}\partial_{\tilde{r}} + \tilde{z}\partial_{\tilde{z}}$ or $\partial_{\tilde{\theta}}$.

Note that the first four terms on the right side of (2.8) have been estimated in §4. Without loss of generality, we only estimate $\iiint_{D_T} \tilde{z}^{-\frac{3}{2}} \tilde{S}^m((1 - \frac{\partial_z b_1(\theta,z)}{b_0}) f_{33}^1(\nabla\tilde{\varphi}, \frac{\partial_z b_1(\theta,z)}{b_0}, \partial_\theta b_1)\partial_{\tilde{z}\tilde{z}}^2 \tilde{\varphi}) \mathcal{M}\tilde{S}^m \dot{\varphi} d\tilde{x}$ and $\iiint_{D_T} \tilde{z}^{-\frac{3}{2}} \tilde{S}^m (f_1^3(\nabla\tilde{\varphi}, \frac{\partial_z b_1(\theta,z)}{b_0}, \partial_{\tilde{\theta}} b_1)\partial_z^2 b_1(\theta, z)\partial_{\tilde{z}}\tilde{\varphi})\ \mathcal{M}\tilde{S}^m \dot{\varphi} d\tilde{x}$, the other terms can be analyzed similarly.

To estimate the integrals, as in [5], we use the following decomposition:

$$\begin{aligned}
&(1 - \frac{\partial_z b_1(\theta, z)}{b_0}) f_{33}^1(\nabla\tilde{\varphi}, \frac{\partial_z b_1(\theta, z)}{b_0}, \partial_\theta b_1)\partial_{\tilde{z}\tilde{z}}^2 \tilde{\varphi} = I_1 + II_1 + III_1, \\
&f_1^3(\nabla\tilde{\varphi}, \frac{\partial_z b_1(\theta, z)}{b_0}, \partial_\theta b_1)\partial_z^2 b_1(\theta, z)\partial_{\tilde{z}}\tilde{\varphi} = I_2 + II_2 + III_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= (1 - \frac{\partial_z b_1(\theta, z)}{b_0})\Big(f_{33}^1(\nabla\tilde{\varphi}, \frac{\partial_z b_1(\theta, z)}{b_0}, \partial_\theta b_1) - f_{33}^1(\nabla\hat{\varphi}, \frac{\partial_z b_1(\theta, z)}{b_0}, \partial_\theta b_1)\Big)\partial_{\tilde{z}\tilde{z}}^2 \hat{\varphi}, \\
II_1 &= (1 - \frac{\partial_z b_1(\theta, z)}{b_0}) f_{33}^1(\nabla\hat{\varphi}, \frac{\partial_z b_1(\theta, z)}{b_0}, \partial_\theta b_1)\partial_{\tilde{z}\tilde{z}}^2 \hat{\varphi}, \\
III_1 &= (1 - \frac{\partial_z b_1(\theta, z)}{b_0}) f_{33}^1(\nabla\tilde{\varphi}, \frac{\partial_z b_1(\theta, z)}{b_0}, \partial_\theta b_1)\partial_{\tilde{z}\tilde{z}}^2 \dot{\tilde{\varphi}}, \\
I_2 &= \partial_z^2 b_1(\theta, z)(f_1^3(\nabla\tilde{\varphi}, \frac{\partial_z b_1(\theta, z)}{b_0}, \partial_\theta b_1) - f_1^3(\nabla\hat{\varphi}, \frac{\partial_z b_1(\theta, z)}{b_0}), \partial_\theta b_1)\partial_{\tilde{z}}\hat{\varphi}, \\
II_2 &= \partial_z^2 b_1(\theta, z) f_1^3(\nabla\hat{\varphi}, \frac{\partial_z b_1(\theta, z)}{b_0}, \partial_\theta b_1)\partial_{\tilde{z}}\hat{\varphi}, \\
III_2 &= \partial_z^2 b_1(\theta, z) f_1^3(\nabla\tilde{\varphi}, \frac{\partial_z b_1(\theta, z)}{b_0}, \partial_\theta b_1)\partial_{\tilde{z}}\dot{\tilde{\varphi}}.
\end{aligned}$$

Note also that $|b_1(\theta, z) - b_0 z| \leq \varepsilon$ and $|z(z\partial_z)^{k_1}\partial_\theta^{k_2}(\partial_z b_1(\theta, z) - b_0)| \leq \varepsilon$ for $0 \leq k \leq k_2 - 1$ with $k_2 \geq k_0 + 1$ due to (1.10), here k_0 is the number appeared in Theorem 4.1. Additionally, $\partial_{\tilde{r}}\hat{\varphi}$ and $\partial_{\tilde{z}}\hat{\varphi}$

are positively homogeneous of degree 0. Hence we have the following estimates for $m \le k_0 - 1$

$$|\tilde{S}^m I_1 \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}}| \le \frac{C\varepsilon}{\tilde{z}} \sum_{l \le m} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^l \dot{\tilde{\varphi}}|^2,$$

$$|\tilde{S}^m II_1 \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}}| \le \frac{C\varepsilon}{\tilde{z}} \sum_{l \le m} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^l \dot{\tilde{\varphi}}|,$$

$$|\tilde{S}^m I_2 \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}}| \le \frac{C\varepsilon}{\tilde{z}} \sum_{l \le m} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^l \dot{\tilde{\varphi}}|^2,$$

$$|\tilde{S}^m II_2 \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}}| \le \frac{C\varepsilon}{\tilde{z}} \sum_{l \le m} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^l \dot{\tilde{\varphi}}|,$$

$$|\tilde{S}^m III_2 \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}}| \le \frac{C\varepsilon}{\tilde{z}} \sum_{l \le m} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^l \dot{\tilde{\varphi}}|^2.$$

In addition, the term $\tilde{S}^m III_1 \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}}$ can be treated similarly as in (4.5), (4.6) and (4.7) of §4. Using the inequality $\frac{\varepsilon}{\tilde{z}}|g| \le \eta |g|^2 + C(\eta)\frac{\varepsilon^2}{\tilde{z}^2}$, here $\eta > 0$ is an appropriate small constant, then these estimates and the integration by parts lead to

$$\begin{aligned}
&\iiint_{D_T} \tilde{z}^{-\frac{3}{2}} \sum_{0 \le m \le k_0 - 1} \mathcal{L} \tilde{S}^m \dot{\tilde{\varphi}} \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}} d\tilde{x} \le O(\varepsilon)\Big(\frac{1}{\sqrt{T}} \iint_{b_0 T \le r \le \tilde{\chi}(\theta, T)} \sum_{m \le k_0 - 1} |\nabla \tilde{S}^m \dot{\tilde{\varphi}}(r,\theta,T)|^2 dS \\
&\quad + \iiint_{D_T} \tilde{z}^{-\frac{3}{2}} \sum_{0 \le m \le k_0 - 1} |\nabla \tilde{S}^m \dot{\tilde{\varphi}}|^2 d\tilde{x} + \iint_{\Gamma_T} \tilde{z}^{-\frac{1}{2}} \sum_{0 \le m \le k_0 - 1} |\nabla \tilde{S}^m \dot{\tilde{\varphi}}|^2 dS \\
&\quad + \iint_{B_T} \tilde{z}^{-\frac{1}{2}} \sum_{0 \le m \le k_0 - 1} |\nabla \tilde{S}^m \dot{\tilde{\varphi}}|^2 dS + \varepsilon \Big) + \frac{1}{2C(q_0, b_0, \gamma)} \iiint_{D_T} \tilde{z}^{-\frac{3}{2}} \sum_{0 \le m \le k_0 - 1} |\nabla \tilde{S}^m \dot{\tilde{\varphi}}|^2 d\tilde{x},
\end{aligned}$$

here $C(q_0, b_0, \gamma)$ is the constant in (4.3).

Secondly, as in §4 we need to estimate the term $\iint_{\Gamma_T} \tilde{z}^{-\frac{1}{2}} \sum_{0 \le m \le k_0 - 1} |\mathcal{B}_0 \tilde{S}^m \dot{\tilde{\varphi}}|^2 dS$. Since the equations (2.9) and (2.10) are very similar to (2.5) and (2.6) respectively, then this term can be estimated by the same method in §4.

Therefore, Theorem 1.1 is proved in the general case.

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