Vacuum State for Spherically Symmetric Solutions of the Compressible Navier-Stokes Equations[1]

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⋇ ⋇ ⋇ *Dedicated to Professor Tai-Ping Liu for his 60th birthday* ⋇ ⋇ ⋇

Abstract

In this paper, we study the properties of the vacuum states for weak solutions to the compressible Navier-Stokes system with spherical symmetry. It is shown that vacuum states cannot develop later on in time in the region far away from the center of symmetry if there is no vacuum state initially, and two initially non-interacting vacuum regions will never meet each other in the future. Furthermore, a sufficient condition on the regularity of the velocity to ensure no formulation of vacuum is given.

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§1. Introduction and Main Results

We consider spherically symmetric solutions to the isentropic compressible Navier-Stokes equations, which satisfy the following system

$$
\left\{
\begin{array}{l}
\partial_t \rho + \partial_x(\rho u) + \dfrac{m\rho u}{x} = 0, \\
\partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) + \dfrac{m\rho u^2}{x} = \mu\, \partial_x^2 u + \mu m \partial_x \left(\dfrac{u}{x}\right),
\end{array}
\right.
\tag{1.1}
$$

where $m = n - 1$ (for $n = 2, 3$), $p(\rho) = a\rho^\gamma$, $\gamma > 1$, $a > 0$, and $\mu > 0$ are given constant. Here ρ, u, p denote the density, velocity and pressure respectively, and x is the radial variable so $x \geq 0$. We impose the following initial-boundary conditions:

$$
\left\{
\begin{array}{ll}
\rho(x, 0) = \rho_0(x), & x \geq 0, \\
\rho(x, 0)\, u(x, 0) = v_0(x), & x \geq 0,
\end{array}
\right.
\tag{1.2}
$$

$$
u(0, t) = 0, \qquad t \geq 0. \tag{1.3}
$$

Our main purpose is to study the properties of vacuum states in weak solutions to the initial-boundary value problem (1.1) - (1.3).

As it is well-known, the formation and dynamics of the vacuum states are key issues in the studies of the existence, regularity and long time behavior of strong and weak solutions for both inviscid and viscous compressible fluids (see [1], [4], [9], [10], [5], [3], [11]). This is particularly so for multi-dimensional problems. Indeed, the well-posedness of and stability of strong solutions near a non-vacuum state have been well understood (see [14], [1], [6] and the references therein). However, the situation becomes much more complicated in the presence of vacuum and less is known except the existence of weak solutions and long time dynamics for some special cases (see [6], [7], [10] and the references therein). The uniqueness and regularity of weak solutions of compressible Navier-Stokes system obtained in [10, 8] depend crucially on the understanding of the dynamics of vacuum states in such weak solutions. In fact, it has been shown by Xin in [15, 16] any smooth solutions to the compressible Navier-Stokes systems with compactly supported initial densities will blow-up in finite time. This indicates that the dynamics of vacuum states will play a key role in the theory of regularity of weak solutions of the compressible Navier-Stokes system.

There have been a lot recent studies on the various topics involving vacuum states for viscous compressible fluids, see [2, 3, 4, 10, 12, 13, 15, 16, 5, 9, 7], etc. for instances.

In particular, in [5], Hoff and Smoller studied the dynamics of vacuum states for weak solutions to the one-dimensional compressible Navier-Stokes equations:

$$(1.1)' \qquad \begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho))_x = \mu\, u_{xx} + \rho f \end{cases}$$

and proved the important conclusion that if for any given open set $E \subset \mathbb{R}^1$, it holds that

$$\int_E \rho(x,0)dx > 0,$$

then for any $t \in [0,T]$, and any open set $E \subset \mathbb{R}^1$, it holds that

$$\int_E \rho(x,t)dx > 0,$$

provided (ρ, u) (x,t) is a suitable weak solution to $(1.1)'$. For spherical symmetric Navier-Stokes system (1.1) in the case that $\gamma = 1$ and with bounded initial density definitely away from vacuum states, in [2], Hoff has constructed a global weak solution with initial data in the class of bounded total variations, and proved such solution contains no vacuum state far way from the center of symmetry and pointed out there might appear vacuum state near the center of the symmetry. These results have been generalized in [3].

One of the motivations of this paper is to generalize the one-dimensional results of Hoff-Smoller in [5] to weak solutions to the initial-boundary value problem (1.1) - (1.3). To state our main results, we first need a definition of weak solutions to the problem (1.1) - (1.3).

Definition 1.1 $(\rho,u) = (\rho(x,t), u(x,t))$ is said to be a weak solution to the initial-boundary value problem (1.1)-(1.3) on $\mathbb{R}^1_+ \times [0,T)$ if

(1) $$\rho(x,t) \geq 0 \qquad \text{a.e. on} \qquad (x,t) \times \mathbb{R}^1_+ \times (0,T),$$

$$\rho, \rho u^2 \in L^\infty(0,T; L^1_{loc}(\mathbb{R}^1_+)),$$

$$\partial_x\, u, \frac{u}{x} \in L^2(0,T; L^2_{loc}(\mathbb{R}^1_+));$$

(2) for all t_1 and t_2 such that $0 \leq t_1 < t_2 < T$, $\varphi \in C^1(0,T; C^1_0(\mathbb{R}^1_+))$, it holds that

$$\int_{\mathbb{R}^1_+} \rho\, \varphi\, x^m\, dx \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\mathbb{R}^1_+} (\rho\, \partial_t\, \varphi + \rho\, u\, \partial_x\, \varphi) x^m\, dx\, dt = 0,$$

and

$$\int_{\mathbb{R}^1_+} \rho\, u\, \varphi\, x^m\, dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\mathbb{R}^1_+} \left\{ \rho\, u\, \partial_t \varphi + \rho\, u^2\, \partial_x \varphi + p(\rho) \left(\partial_x \varphi + \frac{m\,\varphi}{x} \right) \right\} x^m\, dx\, dt$$

$$= \; -\mu \int_{t_1}^{t_2} \int_{\mathbb{R}^1_+} \left(\partial_x u\; \partial_x \varphi + \frac{m\, u\, \varphi}{x^2} \right) x^m\, dx\, dt.$$

Set

$$m(t) = \int_{\mathbb{R}^1_+} \rho(x,t) x^m\, dx, \tag{1.4}$$

$$E(t) = \int_{\mathbb{R}^1_+} \left(\frac{1}{2} \rho\, u^2 + \frac{a}{\gamma - 1}\, \rho^\gamma \right)(x,t) x^m\, dx + \mu \int_0^t \int_{\mathbb{R}^1_+} \left((\partial_x\, u)^2 + \frac{m\, u^2}{x^2} \right) x^m\, dx\, d\tau, \tag{1.5}$$

$$E_0 = \int_{\mathbb{R}^1_+} \left(\frac{v_0^2}{2\rho_0} + \frac{a}{\gamma - 1}\, \rho_0^\gamma \right)(x) x^m\, dx. \tag{1.6}$$

We will assume that the weak solutions satisfy the following conditions:

$$m(t) = m_0 \equiv \int_{\mathbb{R}^1_+} \rho_0(x) x^m\, dx, \quad \text{for all} \quad t \in (0,T), \tag{1.7}$$

and

$$E(t) \le E_0, \quad \text{for all} \quad t \in (0,T). \tag{1.8}$$

It should be noted that by following the analysis in [8], one can constructed weak solutions to the problem (1.1)-(1.3) such that both (1.7) and (1.8) hold, see Theorem 1.4 below.

We now define for all $t \in (0,T)$,

$$w(t) = \sup \left\{ x \in \mathbb{R}^1_+ ; \int_0^x \rho(y,t) y^m\, dy = 0 \right\}. \tag{1.9}$$

Then one of the main results in this paper is the following theorem:

Theorem 1.1 Let (ρ, u) be a weak solution to the problem (1.1) - (1.3) on the domain $\mathbb{R}^1_+ \times \mathbb{R}^1_+$ such that (1.7) and (1.8) hold. If for any given open set $E \subset \mathbb{R}^1_+$, it holds true that

$$\int_E \rho_0(x) x^m\, dx > 0, \tag{1.10}$$

then for any $t \in \mathbb{R}^1_+$ and any open set E with $E \subset \bar{E} \subset (w(t), +\infty)$, one has that

$$\int_E \rho(x,t) x^m \, dx > 0. \tag{1.11}$$

Furthermore, it can be shown that

$$w(t) \le C\, t^{1/n} \tag{1.12}$$

for some uniform positive constant C.

It should be noted that in Theorem 1.1, it is allowed that there are vacuum states on a set of measure zero initially, which is different from the assumptions in [2,3]. It will also be clear from the proof of the Theorem 1.1 that if the vacuum states appear on an open interval initially, then the interval of vacuum states will persist in time. Then a natural question arises: if the vacuum states appear on two or more open intervals initially, then do these intervals of vacuum states emerge later on in time? Will they interact? Based on the ideas of the proof of Theorem 1.1, one can derive the following definite answer to such a question:

Theorem 1.2 Let (ρ, u) be a weak solution to the initial-boundary value problem (1.1) - (1.3) satisfying (1.7) and (1.8) on $[0,T] \times \mathbb{R}^1_+$ with $T < +\infty$. Assume that there exist constants a_i $(i = 1,2,3,4)$ such that

$$0 < a_1 < a_2 < a_3 < a_4 < +\infty$$

$$\int_{a_1}^{a_2} \rho_0(x) dx = \int_{a_3}^{a_4} \rho_0(x) dx = 0, \tag{1.13}$$

$$\int_{a_1-\epsilon}^{a_2} \rho_0(x) dx > 0, \quad \forall\, \epsilon \in (0, a_1), \tag{1.14}$$

$$\int_{a_1}^{a_2+\epsilon} \rho_0(x) dx > 0, \quad \forall\, \epsilon \in \left(0, \frac{(a_2+a_3)}{2}\right), \tag{1.15}$$

$$\int_{a_3-\epsilon}^{a_3} \rho_0(x) dx > 0, \quad \forall\, \epsilon \in \left(0, \frac{(a_2+a_3)}{2}\right), \tag{1.16}$$

and

$$\int_{a_3}^{a_4+\epsilon} \rho_0(x) dx > 0, \quad \forall\, \epsilon \in (0, 1). \tag{1.17}$$

Then there exist $x_i = x_i(t) \in C^1[0,T]$ $(i = 1,2,3,4)$ such that

$$x_i(0) = a_i, \qquad i = 1,2,3,4, \tag{1.18}$$

$$0 < x_1(t) < x_2(t) < x_3(t) < x_4(t) < +\infty, \qquad \forall\, t \in [0,T], \tag{1.19}$$

with the following properties: for all $t \in [0,T]$,

$$\int_{x_1(t)}^{x_2(t)} \rho(x,t)dx = \int_{x_3(t)}^{x_4(t)} \rho(x,t)dx = 0, \tag{1.20}$$

$$\int_{x_1(t)-\epsilon}^{x_2(t)} \rho(x,t)dx > 0, \qquad \forall\, \epsilon \in (0, x_1(t)), \tag{1.21}$$

$$\int_{x_1(t)}^{x_2(t)+\epsilon} \rho(x,t)dx > 0, \qquad \forall\, \epsilon \in \left(0, \frac{(x_2(t)+x_3(t))}{2}\right), \tag{1.22}$$

$$\int_{x_3(t)-\epsilon}^{x_4(t)} \rho(x,t)dx > 0, \qquad \forall\, \epsilon \in \left(0, \frac{(x_2(t)+x_3(t))}{2}\right), \tag{1.23}$$

and

$$\int_{x_3(t)}^{x_4(t)+\epsilon} \rho(x,t)dx > 0, \qquad \forall\, \epsilon \in (0,1). \tag{1.24}$$

In particular, it holds that

$$x_3(t) - x_2(t) \geq \frac{1}{C} \left\{ \int_{a_2}^{a_3} \rho_0(x) x^m \, dx \right\}^{\gamma/(\gamma-1)} \tag{1.25}$$

for all $t \in [0,T]$, where C is a positive constant depending only on m_0, E_0 and T.

Remark 1.1 It follows from Theorem 1.2 that initially neighboring open intervals of vacuum states are always separated dynamically.

We now turn to another important question: for a given weak solution to (1.1) - (1.3) satisfying (1.7), (1.8) and (1.10), can the vacuum states appear on an open interval near the center $x = 0$? i.e., when do we have

$$w(t) = 0 \qquad \forall\, t \in [0,T]? \tag{1.26}$$

It seems that the answer to this question depends on the further regularity of the weak solution. A sufficient condition to ensure that there will be no open interval near $x = 0$ on which vacuum states appear is the following:

$$\overline{\lim}_{\delta \to 0^+} \left| \frac{1}{\delta} \int_0^T \int_\delta^{2\delta} \frac{u}{x} \, dx \, dt \right| < \infty, \tag{1.27}$$

or

$$\overline{\lim}_{\delta \to 0^+} \left| \frac{1}{\delta} \int_0^T \int_\delta^{2\delta} u_x \, dx \, dt \right| < \infty. \tag{1.28}$$

Indeed, we have

Theorem 1.3 Let (ρ, u) be a weak solution to the initial-boundary value problem (1.1) - (1.3) such that (1.7), (1.8), and (1.10) hold. If either (1.27) or (1.28) holds true, then

$$\int_E \rho(x,t) x^m \, dx > 0 \tag{1.29}$$

holds for all $t \in (0, T)$ and arbitrary open set in $\mathbb{R}^1_+$.

It should be pointed out that in the Theorems 1.1 - 1.3 above, we have assumed that the mass is conserved and the total energy is finite, i.e., (1.7) and (1.8) hold. In fact, the existence of such weak solutions to the initial-boundary value problem can be proved by following the analysis in [19]. Indeed, we have

Theorem 1.4 Assume that the initial data, (1.2), are given such that both m_0 and E_0 are finite. Then there exists a weak solution (ρ, u) to the initial-boundary value problem (1.1) - (1.3) such that (1.7) and (1.8) hold.

The rest of the paper is organized as follows: we start with some basic lemmas in §2. Since the interfaces between vacuum states and non-vacuum states are governed by particle paths, so the basic lemmas concern the time-integrability of the velocity and the estimates on the progation of the interfaces as in [5], see Lemmas 2.2 and 2.3. In §3, we show that there are no vacuum states far away from the center under the assumptions of Theorem 1.1, see Theorem 3.1. The main part of the proof of Theorem 1.1 is given in §4, where we will show the non-formation of vacuum states in some intermediate region. This is achieved by studying the evolution of an open interval of vacuum state. Through careful estimate of the interfaces separating vacuum and non-vacuum states, we can prove Theorem 1.1 by generalizing some of ideas in [5]. Based on the ideas in the analysis in §4, the Theorems 1.2, 1.3 and 1.4 are proved in §5, §6 and §7 respectively.

§2. Some Preliminary Lemmas

In this section, we list some elementary facts which are simple but useful for our later on analysis.

First, we will use the following simple fact often.

Lemma 2.1 For any $\lambda \in (0, +\infty)$ and any $k \in (0, +\infty)$, it holds that

$$x^k e^{-\lambda x} \le C_1(k, \lambda)$$

for all $x \in [0, +\infty)$, where $C_1(k, \lambda)$ is some positive constant depending only on k and λ.

Next, we estimate the time-integrability of the velocity field for a weak solution.

Lemma 2.2 Let (ρ, u) be a weak solution to (1.1) - (1.3). Then

$$\int_s^t |u(\cdot, \lambda)|_{L^\infty(R,2R)}\, d\lambda \le C_2(t-s)^{1/2}\, R^{(2-n)/2}, \qquad \forall \quad 0 < s < t,$$

for all $R \in (0, +\infty)$, where C_2 is a positive constant depending only on E_0.

Proof We compute

$$\int_s^t |u(\cdot, \lambda)|_{L^\infty(R,2R)}\, d\lambda = \int_s^t \operatorname{ess} \sup_{x \in (R,2R)} |\xi_R(x) u(x, \lambda)|_{L^\infty(R,2R)}\, d\lambda, \tag{2.1}$$

where $\xi_R \in C_0^\infty(\mathbb{R})$ such that

$$\begin{cases} 0 \le \xi_R \le 1, & |\xi_R'| \le CR^{-1}, \\ \xi_R = 1 \quad \text{on} & (R, 2R) \\ \xi_R = 0 \quad \text{on} & \mathbb{R} \setminus (R/2, 3R). \end{cases}$$

Note that for almost all $\lambda \in (0, T)$,

$$\begin{aligned} & \operatorname{ess\,sup}_{R \le x \le 2R} |\xi_R(x) u(x, \lambda)| \\ = \ & \operatorname{ess} \sup_{R \le x \le 2R} \left| \int_{R/2}^x \frac{\partial}{\partial y}[\xi_R(y) u(y, \lambda)] dy \right| \\ \le \ & \frac{C}{R} \int_{R/2}^{2R} |u(x, \lambda)| dx + \int_{R/2}^{2R} |u_x(x, \lambda)| dx. \end{aligned}$$

Hence, by the regularity assumptions on a weak solution,

$$
\begin{aligned}
& \int_s^t |u(\cdot,\lambda)|_{L^\infty(R,2R)}\, d\lambda \\
\leq\ & \frac{C}{R}\int_s^t\int_{R/2}^{2R} |u(x,\lambda)| dx\, d\lambda + \int_s^t\int_{R/2}^{2R} |u_x(x,\lambda)| dx\, d\lambda \\
\leq\ & \frac{C}{R}\left\{\int_s^t\int_{R/2}^{2R} \frac{u^2}{x^2}\cdot x^m\, dx\, d\lambda\right\}^{1/2}\left\{\int_s^t\int_{R/2}^{2R} x^{2-m}\, dx\, d\lambda\right\}^{1/2} \\
& + \left\{\int_s^t\int_{R/2}^{2R} u_x^2\, x^m\, dx\, d\lambda\right\}^{1/2}\left\{\int_s^t\int_{R/2}^{2R} x^{-m}\, dx\, d\lambda\right\}^{1/2} \\
\leq\ & C E_0^{1/2} R^{(1-m)/2} (t-s)^{1/2}.
\end{aligned}
$$

Thus the proof is completed.

As an immediate consequence, one has

Corollary 2.1 For $0 < s < t$ and $r > 0$, it holds that

$$\int_s^t |u(\cdot,\lambda)|_{L^\infty(r,R)}\, d\lambda \leq C_2 (t-s)^{1/2}\ r^{(2-n)/2}$$

for all $R \in (r,+\infty)$, where C is a positive constant depending only on E_0.

Proof For $R \in (r,+\infty)$ with $r > 0$, there exists a positive integer k such that $R \in [2^{k-1} r, 2^k r)$. It then follows from Lemma 2.2 that

$$\int_s^t |u(\cdot,\lambda)|_{L^\infty(2^{j-1}\ r, 2^j r)}\, d\lambda \leq C_2 (t-s)^{1/2}\, (2^{j-1} r)^{(2-n)/2} \leq C_2 (2^{(2-n)/2})^{j-1}\, (t-s)^{1/2}\, r^{(2-n)/2}$$

for $j = 1, 2, \cdots, k$. Therefore, we get

$$\int_s^t |u(\cdot,\lambda)|_{L^\infty(r,2^k r)}\, d\lambda \leq C_2 (t-s)^{1/2}\ r^{(2-n)/2}.$$

So the proof is completed.

Next lemma gives an estimate on the evolution of an open interval of vacuum states.

Lemma 2.3 Let $t_0 \in (0,T)$ and suppose that

$$\rho(\cdot, t_0) = 0 \quad a. \quad e. \quad (a,b)$$

with $0 < a < b < +\infty$. Let

$$t_1 = \inf \left\{ t \in [0, t_0] : \int_t^{t_0} |u(\cdot, \lambda)|_{L^\infty(a,b)}\, d\lambda < \frac{b-a}{2} \right\}$$

and

$$t_2 = \sup \left\{ t \in [t_0, T] : \int_{t_0}^{t} |u(\cdot, \lambda)|_{L^\infty(a,b)}\, d\lambda < \frac{b-a}{2} \right\}.$$

Then $t_1 < t_0 < t_2$, and for any $t \in (t_1, t_2)$, $\rho(\cdot, t) = 0$ a. e. on

$$\left(a + \left| \int_{t_0}^{t} |u(\cdot, \lambda)|_{L^\infty(a,b)}\, d\lambda \right|, \quad b - \left| \int_{t_0}^{t} |u(\cdot, \lambda)|_{L^\infty(a,b)}\, d\lambda \right| \right).$$

Proof Lemma 2.2 shows that

$$t_1 < t_0 < t_2.$$

Now suppose that $t > t_0$. The proof for $t < t_0$ is similar. We prove Lemma 2.3 by studying the approximate particle pathes.

Fix $\delta > 0$ satisfying

$$0 < \delta < \frac{b-a}{6}.$$

For small $\epsilon > 0$, let u^ϵ denote the usual spatial regularization of u. Then for a. e. $t \in [t_0, T]$, we have

$$|u^\epsilon(\cdot, t)|_{L^\infty(a+\delta, b-\delta)} \le |u(\cdot, t)|_{L^\infty(a,b)}.$$

Define a smooth function $w^{\epsilon\delta}(\cdot, t)$ by

$$w^{\epsilon\delta}(x,t) = \begin{cases} |u^\epsilon(\cdot, t)|_{L^\infty(a+\delta, b-\delta)}, & \text{if} \quad x < \dfrac{a+b}{2} - \delta \\ -|u^\epsilon(\cdot, t)|_{L^\infty(a+\delta, b-\delta)}, & \text{if} \quad x > \dfrac{a+b}{2} + \delta \end{cases}$$

and $w^{\epsilon\delta}$ is decreasing on $\left(\frac{a+b}{2} - \delta, \frac{a+b}{2} + \delta \right)$.

Next, set

$$\psi^\delta(x) = \begin{cases} 1, & \text{if} \quad a + 2\delta < x < b - 2\delta \\ 0, & \text{if} \quad x < a + \delta \quad \text{or} \quad x > b - \delta, \end{cases}$$

and ψ^δ is increasing on $(a + \delta, a + 2\delta)$, and decreasing on $(b - 2\delta, b - \delta)$.

Let $\phi^{\epsilon\delta}$ be the solution to the problem

$$\begin{cases} \phi_t + w^{\epsilon\delta} \phi_x = 0, \quad t > t_0 \\ \phi|_{t=t_0} = \psi^\delta. \end{cases}$$

Consider the curves

$$x_r^\epsilon = x_r^\epsilon(t)$$

defined by

$$\begin{cases} \dfrac{dx}{dt} = w^{\epsilon\delta}, & t > t_0 \\ x|_{t=t_0} = r, & r \in [0, +\infty). \end{cases}$$

Set

$$\begin{aligned} V_1 &= \left\{(x,t) : 0 < x < x_{a+\delta}^\epsilon(t), t \in [t_0, T]\right\}, \\ V_2 &= \left\{(x,t) : x_{a+\delta}^\epsilon(t) < x < x_{a+2\delta}^\epsilon(t), t \in [t_0, T]\right\}, \\ V_3 &= \left\{(x,t) : x_{a+2\delta}^\epsilon(t) < x < x_{b-2\delta}^\epsilon(t), t \in [t_0, T]\right\}, \\ V_4 &= \left\{(x,t) : x_{b-2\delta}^\epsilon(t) < x < x_{b-\delta}^\epsilon(t), t \in [t_0, T]\right\}, \\ V_5 &= \left\{(x,t) : x_{b-\delta}^\epsilon(t) < x < +\infty, t \in [t_0, T]\right\}. \end{aligned}$$

Then,

$$\phi^{\epsilon\delta}(x,t) = \begin{cases} 0, & \text{if} \quad (x,t) \in V_1 \cap V_5 \\ 1, & \text{if} \quad (x,t) \in V_3 \end{cases}$$

and

$$\phi_x^{\epsilon\delta}(x,t) \begin{cases} > 0, & \text{if} \quad (x,t) \in V_2 \\ < 0, & \text{if} \quad (x,t) \in V_4. \end{cases}$$

Denote

$$T^{\epsilon\delta} = \sup\left\{t \in [t_0, T] : x_{a+2\delta}^\epsilon(s) < \frac{a+b}{2} - \delta,\ x_{b-2\delta}^\epsilon(s) > \frac{a+b}{2} + \delta, \quad \forall\, s \in [t_0, t]\right\}.$$

Without loss of generality, we may assume that

$$x_{a+2\delta}^\epsilon(T^{\epsilon\delta}) = \frac{a+b}{2} - \delta.$$

Then,

$$x_{a+2\delta}^\epsilon(T^{\epsilon\delta}) - x_{a+2\delta}^\epsilon(t_0) = \int_{t_0}^{T^{\epsilon\delta}} w^{\epsilon\delta}\, dt,$$

which yields

$$\left(\frac{a+b}{2} - \delta\right) - (a + 2\delta) \le \int_{t_0}^{T^{\epsilon\delta}} |u(\cdot,\lambda)|_{L^\infty(a,b)}\, d\lambda.$$

Therefore,

$$\int_{t_0}^{T^{\epsilon\delta}} |u(\cdot,\lambda)|_{L^\infty(a,b)}\, d\lambda \ge \frac{b-a}{2} - 3\delta. \tag{2.5}$$

Define

$$T^{\delta}=\sup\left\{t\in[t_0,T]:\int_{t_0}^{t}|u(\cdot,\lambda)|_{L^\infty(a,b)}\,d\lambda<\frac{b-a}{2}-3\delta\right\}.$$

It then follows from the definition and (2.5) that

$$T^{\epsilon\delta}\geq T^{\delta}.$$

By the definition of weak solution, one may obtain that for $t\in(t_0,T^\delta)$,

$$\begin{aligned}&\int_0^\infty \rho\,\phi^{\epsilon\delta}\,x^m\,dx\Big|_{t_0}^{t}\\
=\;&\int_{t_0}^{t}\int_0^\infty \rho(u-u^\epsilon)\phi_x^{\epsilon\delta}\,x^m\,dx\,d\tau+\int_{t_0}^{t}\int_0^\infty \rho(u^\epsilon-w^{\epsilon\delta})\phi_x^{\epsilon\delta}\,x^m\,dx\,d\tau\\
\leq\;&\int_{t_0}^{t}\int_0^\infty \rho(u-u^\epsilon)\phi_x^{\epsilon\delta}\,x^m\,dx\,d\tau\end{aligned}$$

which implies that

$$\int_{a+\delta}^{b-\delta}\rho\,\phi^{\epsilon\delta}\,x^m\,dx\leq\int_{t_0}^{t}\int_{a+\delta}^{b-\delta}\rho(u-u^\epsilon)\phi_x^{\epsilon\delta}\,x^m\,dx\,d\tau.$$

Consequently, it follows from the definition of the particle path that

$$\int_{I_\delta(t)}\rho(x,t)x^m\,dx\leq\int_{t_0}^{t}\int_{a+\delta}^{b-\delta}\rho(u-u^\epsilon)\phi_x^{\epsilon\delta}\,x^m\,dx\,d\tau,\tag{2.6}$$

where

$$I_\delta(t)=\left(a+2\delta+\int_{t_0}^{t}|u(\cdot,\lambda)|_{L^\infty(a,b)}\,d\lambda,\ b-2\delta-\int_{t_0}^{t}|u(\cdot,\lambda)|_{L^\infty(a,b)}\,d\lambda\right).$$

Letting $\epsilon\to0^+$ in (2.6) yields

$$\int_{I_\delta(t)}\rho(x,t)x^m\,dx\leq 0$$

and therefore,

$$\rho(\cdot,t)=0\quad\text{a.e.}\quad\text{on}\quad I_\delta(t).$$

On the other hand, for $t\in(t_1,t_2)$ we have

$$\int_{t_0}^{t}|u(\cdot,\lambda)|_{L^\infty(a,b)}\,d\lambda<\frac{b-a}{2}$$

There exists a $\delta_0 \in \left(0, \dfrac{b-a}{6}\right)$ such that if $\delta \in (0, \delta_0)$ then

$$\int_{t_0}^{t} |u(\cdot, \lambda)|_{L^\infty(a,b)}\, d\lambda < \frac{b-a}{2} - 4\delta.$$

Therefore we have

$$t \le T^\delta$$

and then

$$\rho(\cdot, t) = 0$$

for a. e. $x \in I_\delta(t)$. Letting $\delta \to 0$ we get

$$\rho(\cdot, t) = 0$$

a. e. on

$$\left(a + \left|\int_{t_0}^{t} |u(\cdot, \lambda)|_{L^\infty(a,b)}\, d\lambda\right|,\ b - \left|\int_{t_0}^{t} |u(\cdot, \lambda)|_{L^\infty(a,b)}\, d\lambda\right|\right).$$

Thus the proof is completed.

Finally, we also need the following lemma.

Lemma 2.4 Let (ρ, u) be a weak solution to (1.1) - (1.3). Then it holds that

$$\int_0^T \operatorname{ess}\sup_{0<x<1} |x^{m/2}\, u(x,t)|^2\, dt \le C_3$$

for all $T \in (0, +\infty)$, where C_3 is a positive constant depending only on E_0.

Proof In a similar way as in the Proof of Lemma 2.2, one can show that

$$\int_0^T \operatorname{ess}\sup_{R<x<1} |x^{m/2}\, u(x,t)|^2\, dt \le C \int_0^T \int_{R/2}^1 \big(|u_y(y,t)|^2 + y^2\, u^2(y,t)\big)\, y^m\, dy\, dt,$$

which implies that

$$\int_0^T \operatorname{ess}\sup_{R<x<1} |x^{m/2}\, u(x,t)|^2 dt \le C\, E_0$$

for all $R \in (0,1)$. Letting $R \to 0^+$ yields the desired estimate. Thus the proof is complete.

§3. Non-formation of Vacuum near $x = +\infty$

In this section, we will show that if there is no vacuum initially, there will be no formation of vacuum state near infinity. More precisely,

Theorem 3.1 Let (ρ, u) be a global weak solution of (1.1) - (1.3) satisfying (1.7) and (1.8). If

$$\int_E \rho_0(x)dx > 0 \tag{3.1}$$

for every open set $E \subset (0, +\infty)$, then for any $t \in (0, +\infty)$, it holds that

$$\int_R^{2R} \rho(x,t)dx > 0 \tag{3.2}$$

for all $R \in (R^*(t), +\infty)$, where

$$R^*(t) = C_3\, t^{1/n}, \tag{3.3}$$

and C_3 is a positive constant depending only on E_0.

Proof of Theorem 3.1 It follows from Lemma 2.2 that

$$\int_0^t |u(\cdot,\lambda)|_{L^\infty(R,2R)}\, d\lambda \le C_2\, t^{1/2}\, R^{(2-n)/2} \tag{3.4}$$

for all $t \in (0, +\infty)$ and all $R \in (0, +\infty)$, where C_2 is defined in Lemma 2.2.

Define

$$R^*(t) = (4C_2)^{2/n}\, t^{1/n}. \tag{3.5}$$

Then

$$C_2\, t^{1/2}[R^*(t)]^{(2-n)/2} = \frac{R^*(t)}{4} \tag{3.6}$$

for all $t \in (0, +\infty)$.

We claim that for any $t \in (0, +\infty)$,

$$\int_R^{2R} \rho(x,t)dx > 0 \tag{3.7}$$

for all $R \in (R^*(t), +\infty)$ with $R^*(t)$ defined by (3.5).

In fact, if (3.7) is not true for some $t_0 \in (0, +\infty)$ and some $R_0 \in (R^*(t_0), +\infty)$ then

$$\rho(x, t_0) = 0 \tag{3.8}$$

for a. e. $x \in (R_0, 2R_0)$. Then Lemma 2.3 implies that

$$\rho(\cdot, t) = 0 \tag{3.9}$$

a. e. on

$$\left(R_0 + \int_t^{t_0} |u(\cdot, \lambda)|_{L^\infty(R_0, 2R_0)}\, d\lambda,\ 2R_0 - \int_t^{t_0} |u(\cdot, \lambda)|_{L^\infty(R_0, 2R_0)}\, d\lambda\right)$$

for all $t \in [t_*, t_0]$, where

$$t_* = \inf\left\{t \in [0, t_0] : \int_t^{t_0} |u(\cdot, \lambda)|_{L^\infty(R_0, 2R_0)}\, d\lambda < \frac{R_0}{4}\right\}. \tag{3.10}$$

It follows from (3.4) and (3.6) that

$$\int_0^{t_0} |u(\cdot, \lambda)|_{L^\infty(R_0, 2R_0)}\, d\lambda \le C_2\, t_0^{1/2}\, R_0^{(2-n)/2} < \frac{R_0}{4} \tag{3.11}$$

for all $R \in (R^*(t_0), +\infty)$. Combining (3.10) - (3.11) with (3.9) one may conclude that

$$\rho_0(x) = \rho(x, 0) = 0$$

for a. e. $x \in (R_0 + R/4, 2R_0 - R_0/4)$. This contradicts to the assumption (3.1). Thus the proof is completed.

§4. Vacuum at the Interior

Next, we turn to the Proof of Theorem 1.1. We assume that

$$\rho(\cdot, t_0) = 0 \quad \text{a. e.} \quad \text{on} \quad (a, b) \tag{4.1}$$

with

$$\omega(t_0) < a < b < +\infty, \tag{4.2}$$

and

$$y(t) = \inf\left\{x : \rho(\cdot, t) = 0 \quad \text{a. e.} \quad \text{on} \quad \left(x, \frac{a+b}{2}\right)\right\}, \tag{4.3}$$

$$z(t) = \sup\left\{x : \rho(\cdot, t) = 0 \quad \text{a. e.} \quad \text{on} \quad \left(\frac{a+b}{2}, x\right)\right\}, \tag{4.4}$$

and

$$y(t_0) = a, \quad z(t_0) = b. \tag{4.5}$$

Then Theorem 3.1 implies

$$b < +\infty. \tag{4.6}$$

We start with some elementary estimates on the curves $x = y(t)$ and $x = z(t)$.

Lemma 4.1 There exists a constant $h_0 = h_0(a, b) > 0$ such that $y(t)$ and $z(t)$ are absolutely continuous functions on $[t_0 - h_0, t_0]$, and

$$\frac{a}{2} \leq y(t) \leq a + \frac{a}{4} < b - \frac{b-a}{8} \leq z(t) \leq a_0$$

for some positive constant a_0 depending only on a, b, E_0 and t_0.

Proof Lemma 2.3 shows

$$\rho(\cdot, t) = 0 \quad \text{a. e.} \quad \text{on} \tag{4.7}$$

$$\left(a + \left|\int_{t_0}^{t} |u(\cdot, \lambda)|_{L^\infty(a,b)}\right|, \; b - \left|\int_{t_0}^{t} |u(\cdot, \lambda)|_{L^\infty(a,b)}\right|\right).$$

for $t \in [t_1, t_0]$, where

$$t_1 = \inf\left\{t \in [0, t_0] : \int_{t}^{t_0} |u(\cdot, \lambda)|_{L^\infty(a,b)}\, d\lambda < \frac{b-a}{2}\right\}. \tag{4.8}$$

Then (4.7) and (4.8) lead to

$$y(t) < a + \frac{b-a}{2} \tag{4.9}$$

for all $t \in [t_1, t_0]$. Similarly, one can show that

$$z(t) > b - \frac{b-a}{2} = \frac{b+a}{2}.$$

Combining (4.9) with Theorem 3.1 one concludes that

$$z(t) \leq 2\max\left\{b - \frac{b-a}{2}, R^*(t_0)\right\} \tag{4.10}$$

for all $t \in [t_1, t_0]$, where $R^*(t_0)$ is defined by (3.3). It follows from (4.9), (4.10) and (4.7) that

$$0 \leq y(t) < \frac{a+b}{2} < z(t) \leq a_0 \tag{4.11}$$

for all $t \in [t_0 - h_1, t_0]$, where

$$a_0 = 2\max\left\{a + \frac{b-a}{2}, R^*(t_0)\right\}, \tag{4.12}$$

$$h_1 = t_0 - t_1 > 0. \tag{4.13}$$

Then Corollary 2.1 shows that

$$\int_{t_0-h}^{t_0} |u(\cdot,\lambda)|_{L^\infty(a/2,(a+b)/2)}\, d\lambda \leq Ch^{1/2}(a/2)^{(2-n)/2} \tag{4.14}$$

for all $h \in (0, h_1)$, where C is a positive constant depending only on E_0. Choose $h_0 \in (0, h_1)$ such that

$$Ch^{1/2}(a/2)^{(2-n)/2} \leq \min\left\{\frac{a}{4}, \frac{b-a}{8}\right\}$$

so that

$$\int_{t_0-h_0}^{t_0} |u(\cdot,\lambda)|_{L^\infty(a/2,(a+b)/2)}\, d\lambda \leq \min\left\{\frac{a}{4}, \frac{b-a}{8}\right\} \tag{4.15}$$

Then we claim that

$$y(t) \geq \frac{a}{2} \tag{4.16}$$

for all $t \in [t_0 - h_0, t_0]$.

In fact, if (4.16) fails then

$$y(t_*) < \frac{a}{2}$$

for some $t_* \in [t_0 - h, t_0)$. Due to (4.11), one gets that

$$\rho(x, t_*) = 0$$

for a. e. $x \in (a/2, (a+b)/2)$. Applying Lemma 2.3 we obtain $\rho(\cdot, t) = 0$ a. e. on

$$\left(\frac{a}{2} + \int_{t_*}^{t} |u(\cdot,\lambda)|_{L^\infty(a/2,(a+b)/2)}, \frac{a+b}{2} - \int_{t_*}^{t} |u(\cdot,\lambda)|_{L^\infty(a/2,(a+b)/2)}\, d\lambda\right)$$

for all $t \in (t_*, t^*)$, where

$$t^* = \sup\left\{t \in [t_*, T] : \int_{t_*}^{t} |u(\cdot,\lambda)|_{L^\infty(a/2,(a+b)/2)}\, d\lambda < \frac{b-a}{4}\right\}.$$

Using (4.15) we have $\rho(\cdot, t) = 0$ a. e. on

$$\left(\frac{a}{2} + \frac{a}{4}, \frac{b+a}{2} - \frac{b-a}{8}\right).$$

This contradicts to (4.3) and (4.5). Therefore, (4.16) holds and then (4.11) gives

$$\frac{a}{2} \le y(t) < z(t) \le a_0 \tag{4.17}$$

for all $t \in [t_0 - h_0, t_0]$.

Next, we prove that y and z are absolutely continuous functions on $[t_0 - h_0, t_0]$.

Let s and t be such that

$$t_0 - h_0 \le s < t \le t_0.$$

It follows from Lemma 2.3 and Corollary 2.1 that $\rho(\cdot, s) = 0$ a. e. on

$$\left(y(t) + \int_s^t |u(\cdot,\lambda)|_{L^\infty(a/2,a_0)}\, d\lambda, z(t) - \int_s^t |u(\cdot,\lambda)|_{L^\infty(a/2,a_0)}\, d\lambda\right),$$

since $\rho(\cdot, t) = 0$ a. e. on $(y(t), z(t))$. So that

$$z(s) \ge z(t) - \int_s^t |u(\cdot,\lambda)|_{L^\infty(a/2,a_0)}\, d\lambda. \tag{4.18}$$

Similarly, if $\rho(\cdot, s) = 0$ a. e. on $(y(s), z(s))$ then

$$z(t) \ge z(s) - \int_s^t |u(\cdot,\lambda)|_{L^\infty(a/2,a_0)} d\lambda. \tag{4.19}$$

Hence (4.18) and (4.19) give, for $t_0 - h_0 \le s < t \le t_0$,

$$|z(t) - z(s)| \le \int_s^t |u(\cdot,\lambda)|_{L^\infty(a/2,a_0)}\, d\lambda. \tag{4.20}$$

Then the absolute continuity of $z(t)$ follows from this and

$$\int_0^T |u(\cdot,\lambda)|_{L^\infty(a/2,a_0)} d\lambda \le C, \tag{4.21}$$

where C is a positive constant depending only on a and T, which follows from Corollary 2.1.

Similarly, y is also a absolutely continuous function. Thus the proof is completed.

Let S be defined as the set of all $t \geq 0$ such that there exists extensions of y and z to $[t, t_0]$ with the following three properties:

(i) y and z are absolutely continuous on $[t, t_0]$;

(ii) $y < z$ on $[t, t_0]$;

(iii)

$$\begin{cases} \displaystyle\int_{y(s)-\epsilon}^{z(s)} \rho(x,s)dx > 0, & \forall\, \epsilon \in (0, y(s)), \quad \forall\, s \in [t, t_0], \\ \displaystyle\int_{y(s)}^{z(s)+\epsilon} \rho(x,s)dx > 0, & \forall\, \epsilon > 0, \qquad \forall\, s \in [t, t_0], \\ \displaystyle\int_{y(s)}^{z(s)} \rho(x,s)dx = 0, & \forall\, s \in [t, t_0]. \end{cases}$$

It follows from Lemma 4.1 that $[t_0 - h_0, t_0] \subset S$ and so,

$$S \neq \emptyset.$$

Thus one can set

$$\tau = \inf S. \tag{4.22}$$

Lemma 4.2 It holds that

$$0 < a_* \leq y(t) < z(t) \leq b_* < +\infty \tag{4.23}$$

for all $t \in (\tau, t_0]$, where

$$\begin{aligned} a_* &= \left[\frac{1}{C} \int_0^a \rho(x, t_0) x^m \, dx \right]^{\gamma/[n(\gamma-1)]}, \\ b_* &= \left[\frac{C}{\displaystyle\int_b^{+\infty} \rho(x, t_0) e^{-x} \, x^m \, dx} \right]^2, \end{aligned}$$

where C is a positive constant depending only on m_0, E_0, a, b and t_0.

Proof Note that (4.2), (4.5) and (1.9) shows that

$$\int_0^a \rho(x, t_0) x^m \, dx > 0$$

for $a > \omega(t_0)$. So we have

$$a_* > 0.$$

On the other hand, $b = z(t_0) < +\infty$ implies that

$$\int_b^{+\infty} \rho(x, t_0) e^{-x} x^m \, dx > 0.$$

And so,

$$b_* < +\infty.$$

For $s \in (\tau, t_0]$, $y = y(t)$ and $z = z(t)$ are absolutely continuous functions on $[s, t_0]$, and

$$y(t) < z(t)$$

for all $t \in [s, t_0]$. Then we have

$$d \equiv d(s) = \inf_{t \in [s, t_0]} |z(t) - y(t)| > 0. \tag{4.24}$$

Set

$$R(t) = \frac{1}{2}(y(t) + z(t)). \tag{4.25}$$

Therefore, for any $t \in [s, t_0 - h_0/2]$, one can find a small positive constant $h(t) > 0$ such that

$$|y(t_1) - y(t)| + |z(t_1) - z(t)| < \frac{d}{4} \tag{4.26}$$

for all $t_1 \in [t - h(t), t + h(t)]$. Thus

$$\Omega(t) \subset V,$$

where

$$\begin{aligned} \Omega(t) &\equiv \left[R(t) - \frac{d}{4}, R(t) + \frac{d}{4} \right] \times [t - h(t), t + h(t)], \\ V &\equiv \{(x, t) : y(t) < x < z(t), \tau < t \leq t_0\}. \end{aligned}$$

Clearly,

$$[s, t_0 - h_0] \subset \bigcup_{t \in [s, t_0 - h_0/2]} (t - h(t), t + h(t)).$$

Therefore there exist $\{t_j\}_{j=1}^N$ and $\{h_j\}_{j=1}^N$ such that

$$\begin{cases} t_1 > t_2 > \cdots > t_N, \\ [s, t_0 - h_0] \subset \cup_{j=1}^N (t_j - h_j, t_j + h_j), \\ s \in (t_N - h_N, t_N + h_N), \\ t_0 - h_0/2 \in (t_1 - h_1, t_1 + h_1), \end{cases}$$

and

$$\cup_{j=1}^N \Omega_j \subset V, \quad \Omega_j \cap \Omega_{j+1} \neq \emptyset, \tag{4.27}$$

where

$$\Omega_j = \left[R_j - \frac{d}{4}, R_j + \frac{d}{4}\right] \times [t_j - h_j, t_j + h_j], \tag{4.28}$$

with

$$R_j = R(t_j), j = 1, 2, \cdots, N. \tag{4.29}$$

Denote by

$$\Omega_0 = \left[a + \frac{a}{4}, b - \frac{b-a}{8}\right] \times [t_0 - h_0, t_0]. \tag{4.30}$$

Choose $\phi^0 \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} 0 \leq \phi^0 \leq 1, \\ \phi^0 = 0 & \text{on} \quad [0, \bar{a}] \\ \phi^0 = 1 & \text{on} \quad \left[b - \dfrac{b-a}{8}, +\infty\right) \end{cases} \tag{4.31}$$

where

$$\bar{a} = \frac{1}{2}\left[\left(a + \frac{a}{4}\right) + \left(b - \frac{b-a}{8}\right)\right].$$

It follows from Definition 1.1 that

$$\int_0^\infty \rho\, \phi^0\, x^m\, dx\Big|_t^{t_0} - \int_t^{t_0}\int_0^\infty (\rho\, \phi_t^0 + \rho\, u\, \phi_x^0) x^m\, dx\, dt = 0$$

for $t \in [t_0 - h_0, t_0]$. Thus

$$\int_0^\infty \rho(x,t_0)\phi^0(x)x^m dx = \int_0^\infty \rho(x,t)\phi^0(x)x^m dx$$

which implies that

$$\int_{z(t)}^\infty \rho(x,t)x^m\,dx = \int_b^\infty \rho(x,t_0)x^m\,dx \tag{4.32}$$

for all $t \in [t_0 - h_0, t_0]$.

In addition, we claim that

$$\int_{z(t)}^\infty \rho(x,t)x^m\,dx = \int_b^\infty \rho(x,t_0)x^m\,dx \tag{4.33}$$

for all $t \in [s, t_0] \subset \cup_{j=0}^N I_j$, where

$$I_0 = (t_0 - h_0, t_0), \quad I_j = (t_j - h_j, t_j + h_j), \quad j = 1, 2, \cdots, N.$$

For $j = 1$, it follows from (4.32), and $t_0 - h_{0/2} \in I_1$ that

$$\int_{z(t)}^\infty \rho(x,t)x^m\,dx = \int_b^\infty \rho(x,t_0)x^m\,dx \tag{4.34}$$

for $t = t_1 + h_1$.

Define $\phi^1 \in C^\infty(\mathbb{R})$ by

$$\begin{cases} 0 \le \phi^1 \le 1, \\ \phi^1 = 0 & \text{on} \quad [0, R_1] \\ \phi^1 = 1 & \text{on} \quad [R_1 + d/4, +\infty). \end{cases}$$

Then Definition 1.1 shows:

$$\int_0^\infty \rho\,\phi^1\,x^m\,dx\bigg|_t^{t_1+h_1} - \int_t^{t_1+h_1}\int_0^\infty (\rho\,\phi^1_t + \rho\,u\,\phi^1_x)x^m\,dx\,dt = 0$$

for $t \in [t_1 - h_1, t_1 + h_1]$. This implies that

$$\int_0^\infty \rho(x,t)\phi^1(x)x^m\,dx = \int_0^\infty \rho(x,t_1+h_1)\phi^1(x)x^m\,dx$$

and so,

$$\int_{z(t)}^\infty \rho(x,t)x^m\,dx = \int_{z(t_1+h_1)}^\infty \rho(x,t_1+h_1)x^m\,dx$$

for all $t \in [t_1 - h_1, t_1 + h_1]$. This, together with (4.34), shows that (4.33) holds for $t \in I_1$.

Repeating the above process shows that

$$\int_{z(t)}^{\infty} \rho(x,t)x^m \, dx = \int_b^{\infty} \rho(x,t_0)x^m \, dx$$

for all $t \in [s, t_0]$ with $s \in (\tau, t_0]$. Therefore,

$$\int_{z(t)}^{\infty} \rho(x,t)x^m \, dx = \int_b^{\infty} \rho(x,t_0)x^m \, dx \tag{4.35}$$

for all $t \in (\tau, t_0]$. Note that

$$\int_0^{\infty} \rho(x,t)x^m \, dx = \int_0^{\infty} \rho(x,t_0)x^m \, dx$$

for all $t \in (0, +\infty)$ due to (1.7). One gets from (4.35) that

$$\int_0^{y(t)} \rho(x,t)x^m \, dx = \int_0^{a} \rho(x,t_0)x^m \, dx \tag{4.36}$$

for all $t \in (\tau, t_0]$.

On the other hand, it follows from (1.5) and (1.8) that

$$\int_0^{y(t)} \rho(x,t)x^m \, dx \le C y^{n(1-1/\gamma)}(t),$$

where C is a positive constant depending only on E_0. This, together with (4.36), shows

$$C y^{n(1-1/\gamma)}(t) \ge \int_0^{a} \rho(x,t_0)x^m \, dx.$$

So that

$$y(t) \ge \left(\frac{1}{C} \int_0^{a} \rho(x,t_0)x^m \, dx \right)^{\gamma/[n(\gamma-1)]} \equiv a_*.$$

for all $t \in (\tau, t_0]$.

Next, we prove that

$$z(t) \le b_* \tag{4.37}$$

for all $t \in (\tau, t_0]$.

Set

$$l = z(t^*) \equiv \sup_{t \in [s,t_0]} z(t)$$

for $s \in (\tau, t_0]$. Without loss of generality we may assume that $l \geq 1$.

Assume that $t^* \in I_k$ for some $k \in \{0, 1, 2, \cdots, N\}$. For $\lambda \in (0, 1)$, it follows from Definition 1.1 that

$$\int_0^\infty \rho\,\phi^0\, e^{-\lambda x}\, x^m\, dx\Big|_t^{t_0} - \int_t^{t_0}\int_0^\infty (\rho(\phi^0\, e^{-\lambda x})_t + \rho\, u(\phi^0\, e^{-\lambda x})_x)x^m\, dx\, dt = 0,$$

where ϕ^0 is defined in (4.31). Thus,

$$\int_{z(t)}^\infty \rho(x,t)e^{-\lambda x}\, x^m\, dx = \int_b^\infty \rho(x,t_0)e^{-\lambda x}\, x^m\, dx + \lambda\int_t^{t_0}\int_{z(s)}^\infty \rho\, u\, e^{-\lambda x}\, x^m\, dx\, ds$$

for all $t \in [t_0 - h_0, t_0]$. In particular, since $t_1 + h_1 = t_0$, it holds that

$$\int_{z(t_1+h_1)}^\infty \rho(x,t_1+h_1)e^{-\lambda x}\, x^m\, dx = \int_b^\infty \rho(x,t_0)e^{-\lambda x}\, x^m\, dx + \lambda\int_{t_1+h_1}^{t_0}\int_{z(s)}^\infty \rho\, u\, e^{-\lambda x}\, x^m\, dx\, ds.$$

Repeating the above process we conclude that

$$\begin{aligned}
&\int_{z(t_2+h_2)}^\infty \rho(x,t_2+h_2)e^{-\lambda x}\, x^m\, dx \\
=\;& \int_{z(t_1+h_1)}^\infty \rho(x,t_1+h_1)e^{-\lambda x}\, x^m\, dx + \lambda\int_{t_2+h_2}^{t_1+h_1}\int_{z(s)}^\infty \rho\, u\, e^{-\lambda x}\, x^m\, dx\, ds \\
&\cdots \\
&\int_{z(t_k+h_k)}^\infty \rho(x,t_k+h_k)e^{-\lambda x}\, x^m\, dx \\
=\;& \int_{z(t_{k-1}+h_{k-1})}^\infty \rho(x,t_{k-1}+h_{k-1})e^{-\lambda x}\, x^m\, dx + \lambda\int_{t_{k-1}+h_{k-1}}^{t_k+h_k}\int_{z(s)}^\infty \rho\, u\, e^{-\lambda x}\, x^m\, dx\, ds
\end{aligned}$$

and

$$\int_{z(t^*)}^\infty \rho(x,t^*)e^{-\lambda x}\, x^m\, dx = \int_{z(t_k+h_k)}^\infty \rho(x,t_k+h_k)e^{-\lambda x}\, x^m\, dx + \lambda\int_{t^*}^{t_k+h_k}\int_{z(s)}^\infty \rho\, u\, e^{-\lambda x}\, x^m\, dx\, ds.$$

Consequently,

$$\int_{z(t^*)}^\infty \rho(x,t^*)e^{-\lambda x}\, x^m\, dx = \int_b^\infty \rho(x,t_0)e^{-\lambda x}\, x^m\, dx + \lambda\int_{t^*}^{t_0}\int_{z(s)}^\infty \rho\, u\, e^{-\lambda x}\, x^m\, dx\, ds$$

i.e.,

$$\int_l^\infty \rho(x,t_*)e^{-\lambda x}\, x^m\, dx = \int_b^\infty \rho(x,t_0)e^{-\lambda x}\, x^m\, dx + \lambda\int_{t^*}^{t_0}\int_{z(s)}^\infty \rho\, u\, e^{-\lambda x}\, x^m\, dx\, ds. \tag{4.38}$$

By (1.7) and the lower bounds on $y(t)$, one gets

$$\begin{aligned} &\left|\int_{t^*}^{t_0}\int_{z(s)}^{\infty} \rho\, u\, e^{-\lambda x}\, x^m\, dx\, ds\right| \\ \leq\quad & \int_{t^*}^{t_0} |u(\cdot,s)|_{L^\infty(a_*,+\infty)} \left(\int_{a_*}^{\infty} \rho(x,s) x^m\, dx\right) ds \\ \leq\quad & m_0 \int_{t^*}^{t_0} |u(\cdot,s)|_{L^\infty(a_*,+\infty)} ds. \end{aligned}$$

Then Corollary 2.1 implies

$$\left|\int_{t^*}^{t_0}\int_{z(s)}^{\infty} \rho\, u\, e^{-\lambda x}\, x^m\, dx\, ds\right| \leq m_0\, a_*^{(2-n)/2} (t_0 - t^*)^{1/2} \leq m_0\, a_*^{(2-n)/2} t_0^{1/2},$$

consequently,

$$\left|\int_{t^*}^{t_0}\int_{z(s)}^{\infty} \rho\, u\, e^{-\lambda x}\, x^m\, dx\, ds\right| \leq m_0\, a_*^{(2-n)/2} t_0^{1/2}. \tag{4.39}$$

On the other hand, direct computations show that

$$\begin{aligned} & \int_l^\infty \rho(x,t) e^{-\lambda x}\, x^m\, dx \\ \leq\quad & \left(\int_l^\infty \rho^\gamma(x,t) x^m\, dx\right)^{1/\gamma} \left(\int_l^\infty (e^{-\lambda x})^{\gamma/(\gamma-1)}\, x^m\, dx\right)^{1-1/\gamma} \\ \leq\quad & C \left(\frac{1}{\lambda^n} \int_{\lambda l}^\infty e^{-\gamma y/(\gamma-1)}\, y^m\, dy\right)^{1-1/\gamma} \\ \leq\quad & C\, \lambda^{-n(\gamma-1)/\gamma} e^{-\lambda\, l(\gamma-1)/\gamma}, \end{aligned}$$

where we have used Lemma 2.1.

It follows from this and (4.38) that

$$\int_b^\infty \rho(x,t_0) e^{-\lambda x}\, x^m\, dx \leq \lambda\, m_0\, a_*^{(2-n)/2} t_0^{1/2} + C\, \lambda^{-n(\gamma-1)/\gamma} e^{-\lambda(\gamma-1)l/\gamma}, \tag{4.40}$$

which implies that

$$\int_b^\infty \rho(x,t_0) e^{-\lambda x}\, x^m\, dx \leq \frac{C}{\sqrt{l}}$$

for $\lambda = l^{-1/2} \leq 1$, where C is a positive constant depending only on m_0, E_0, γ and t_0. So that

$$\int_b^\infty \rho(x, t_0) e^{-x} x^m \, dx \leq \frac{C}{\sqrt{l}}$$

and then

$$\sup_{t \in [s, t_0]} z(t) = l \leq \left(\frac{C}{\displaystyle\int_b^\infty \rho(x, t_0) e^{-x} x^m \, dx} \right)^2 \equiv b_*$$

Therefore we obtain

$$z(t) \leq b_*$$

for all $t \in (\tau, t_0]$. Thus the Proof of Lemma 4.2 is completed.

Next, we describe the qualitative of the velocity field on the interval of vacuum.

Lemma 4.3 It holds that

$$u(x, t) = x^{-m}[x^{m+1}\alpha(t) + \beta(t)]$$

for all $x \in (y(t), z(t))$ and a. e. $t \in (\tau, t_0]$, where $\alpha \in L^2_{loc}(\tau, t_0]$ and $\beta \in L^2_{loc}(\tau, t_0]$.

The proof is similar to [5], so we omit details.

Based on Lemma 4.2 and Lemma 4.3, the growth of $x = y(t)$ and $x = z(t)$ can be estimated more precisely as follows:

Lemma 4.4 It holds that

$$\frac{dz}{dt} \leq z^{-m}(t)(\alpha(t) z^{m+1}(t) + \beta(t))$$

and

$$\frac{dy}{dt} \geq y^{-m}(t)(\alpha(t) y^{m+1}(t) + \beta(t))$$

for almost all $t \in (\tau, t_0]$.

Proof We will only prove

$$\frac{dz}{dt} \leq z^{-m}(t)(\alpha(t) z^{m+1}(t) + \beta(t))$$

for almost all $t \in (\tau, t_0]$.

For any $t_1 \in (\tau, t_0)$, we shall prove

$$\frac{dz}{dt} \leq z^{-m}(t)(\alpha(t)z^{m+1}(t) + \beta(t)) \tag{4.41}$$

for almost all $t \in [t_1, t_0]$.

By Lemma 4.2 and Lemma 4.3, for any $t \in [t_1, t_0]$, there exists a positive number $h(t)$ such that

$$\left| \int_{t-h(t)}^{t+h(t)} z^{-m}(s)(\alpha(s)z^{m+1}(s) + \beta(s))ds \right| < \frac{a_*}{8}.$$

Clearly, we have

$$[t_1, t_0] \subset \bigcup_{t \in [t_1, t_0]} (t - h(t), t + h(t)).$$

There exist $t_j^* (j = 1, 2, \cdots, n)$ such that

$$[t_1, t_0] \subset \bigcup_{j=1}^{n} (t_j^* - h_j, t_j^* + h_j),$$

where $h_j \equiv h(t_j^*)$ for $j = 1, 2, \cdots, n$. In particular, it holds that

$$\left| \int_{t_j^*-h_j}^{t_j^*+h_j} z^{-m}(s)(\alpha(s)z^{m+1}(s) + \beta(s))ds \right| < \frac{a_*}{8}$$

for $j = 1, 2, \cdots, n$. Therefore, we only prove that (4.41) holds a. e. in $(t_j^* - h_j, t_j^* + h_j)$ for $j = 1, 2, \cdots, n$.

Without loss of generality we may assume that

$$|\theta(t)| < \frac{a_*}{8}$$

for all $t \in [t_1, t_0]$, where

$$\theta(t) \equiv \int_t^{t_0} z^{-m}(s)(\alpha(s)z^{m+1}(s) + \beta(s))ds.$$

Define

$$\begin{cases} h(\xi, t) = \rho(x, t) \\ v(\xi, t) = u(x, t) + \theta'(t) \\ \xi = x + \theta(t). \end{cases}$$

Then

$$\frac{\partial}{\partial t}[(\xi - \theta(t))^m h] + \frac{\partial}{\partial \xi}[(\xi - \theta(t))^m hv] = 0$$

in the sense of distribution due to (1.1).

Note that

$$\frac{\partial v}{\partial \xi} = \frac{\partial u}{\partial x}$$

and

$$\begin{aligned} & \int_{t_1}^{t_0} |v(\cdot,t)|_{L^\infty(a_*/4,R)} dt \\ \leq \quad & \int_{t_1}^{t_0} \text{ess} \sup_{a_*/4-\theta(t)<x<R-\theta(t)} |u(x,t)| dt + \int_{t_1}^{t_0} \text{ess} \sup_{a_*/4-\theta(t)<x<R-\theta(t)} |\theta'(t)| dt \\ \leq \quad & \int_{t_1}^{t_0} |u(\cdot,t)|_{L^\infty(a_*/8,R+a_*/8)} dt + \int_{t_1}^{t_0} |z^{-m}(t)[\alpha(t) z^{m+1}(t) + \beta(t)]| dt \\ \leq \quad & \int_{t_1}^{t_0} |u(\cdot,t)|_{L^\infty(a_*/8,R+a_*/8)} dt + C \left\{ \int_{t_1}^{t_0} [\alpha^2(t) + \beta^2(t)] dt \right\}^{1/2} \end{aligned}$$

for all $R \geq a_*/4$, where C is a positive constant depending only on a_* and b_*. In addition, we also have

$$\int_{t_1}^{t_0} \int_{a_*/4}^{R} \left| \frac{\partial v(\xi,t)}{\partial \xi} \right|^2 d\xi \, dt \leq \int_{t_1}^{t_0} \int_{a_*/8}^{R+a_*/8} \left| \frac{\partial u(x,t)}{\partial x} \right|^2 dx \, dt$$

for all $R \geq a_*/4$.

Clearly,

$$h(\xi,t) = 0$$

for a. e. $\xi \in (y_\xi(t), z_\xi(t))$ with $t \in (\tau, t_0]$, where

$$y_\xi(t) \equiv y(t) + \theta(t)$$

and

$$z_\xi(t) \equiv z(t) + \theta(t).$$

It holds that

$$\begin{cases} \xi - \theta(t) \geq \dfrac{a_*}{4} - \dfrac{a_*}{8} = \dfrac{a_*}{8} > 0, \quad \text{f} \quad \xi \in (a_*/4, +\infty) \\ |\theta(t)| \leq \dfrac{a_*}{8} \\ y_\xi(t) = y(t) - \theta(t) \geq a_* - \dfrac{a_*}{8} = \dfrac{7a_*}{8} > \dfrac{a_*}{4} \end{cases}$$

for all $t \in [t_1, t_0]$, and then for any $\phi \in C^1(0,T; C_0^1(a_*/4, +\infty))$ we also have

$$\begin{aligned} & \int_{a_*/4}^{+\infty} (\xi - \theta(t_3))^m h(\xi, t_3)\phi(\xi, t_3) d\xi - \int_{a_*/4}^{+\infty} (\xi - \theta(t_2))^m h(\xi, t_2)\phi(\xi, t_2) d\xi \\ = & \int_{t_2}^{t_3} \int_{a_*/4}^{+\infty} h \left(\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial \xi} \right) (\xi - \theta(s))^m \, d\xi \, ds \end{aligned}$$

for $t_1 \leq t_2 < t_3 \leq t_0$.

Now, arguing in a similar way as in the Proof of Lemma 2.3, one can show that if there exists $s_0 \in (t_1, t_0)$ such that

$$h(\cdot, s_0) = 0 \quad \text{a. e.} \quad \text{on} \quad (a, b)$$

where

$$a = y_\xi(s_0), \qquad b = z_\xi(s_0),$$

then for any $t \in (s_1, s_2)$, it holds that

$$h(\cdot, t) = 0 \tag{4.42}$$

a. e. on

$$\left(a + \left| \int_{s_0}^{t} |v(\cdot, \lambda)|_{L^\infty(a,b)} d\lambda \right|, b - \left| \int_{s_0}^{t} |v(\cdot, \lambda)|_{L^\infty(a,b)} d\lambda \right| \right),$$

where

$$\begin{aligned} s_1 &= \inf \left\{ t \in [0, s_0] : \int_{t}^{s_0} |v(\cdot, \lambda)|_{L^\infty(a,b)} d\lambda < \frac{b-a}{2} \right\} \\ s_2 &= \sup \left\{ t \in [s_0, t_0] : \int_{s_0}^{t} |v(\cdot, \lambda)|_{L^\infty(a,b)} d\lambda < \frac{b-a}{2} \right\}. \end{aligned}$$

Define

$$V_\xi \equiv \{(\xi, t) : y_\xi(t) < \xi < z_\xi(t), \ t \in (\tau, t_0]\}.$$

Set also

$$\begin{aligned} A &= \left\{ t \in [t_1, t_0] : \frac{\partial v}{\partial \xi}(\cdot, t) \notin L^2[a_*/8, b_* + a_* + 1] \right\}, \\ B &= \{t \in [t_1, t_0] : (x, t) \in V_\xi, v(\xi, t) \neq (\xi - \theta(t))^{-m}[(\xi - \theta(t))^{m+1}\alpha(t) + \beta(t)] + \theta'(t)\}, \\ C &= \{t \in [t_1, t_0] : z_\xi = z_\xi(t) \text{ is not differentiable at } t\}, \\ D &= \cup F_{jk}, \\ F_{jk} &= \{t \in [t_1, t_0] : t \text{ is not a Lebesgue point of } |v(\cdot, t)|_{L^\infty(B_{jk})}\}, \\ B_{jk} &= \left\{ x : |x - r_k| < \frac{1}{j} \right\}, \ j, k = 1, 2, \cdots \end{aligned}$$

where $\{r_k\}$ is the set of rational numbers.

It follows from Lemma 4.3 that

$$\text{meas } (A \cup B \cup C \cup D) = 0.$$

For any $t \in [t_0 - h, t_0] \setminus (A \cup B \cup C \cup D)$ we now show that (4.41) holds. By the definition of $z_\xi(t)$, we need only to prove

$$\frac{dz_\xi(t)}{dt} \leq 0, \tag{4.43}$$

since

$$\frac{dz_\xi(t)}{dt} = \frac{dz(t)}{dt} + \theta'(t).$$

In fact, if there exists $\bar{t} \in [t_1, t_0] \setminus (A \cup B \cup C \cup D)$ such that (4.43) is not true, then there exits $\epsilon > 0$, for any n, there exists a t_n such that $0 < |\bar{t} - t_n| < 1/n$ and

$$\frac{z_\xi(t_n) - \bar{z}_\xi}{t_n - t_0} \geq \epsilon,$$

where

$$\bar{z}_\xi \equiv z_\xi(\bar{t}).$$

We may assume that

$$\bar{t} < t_n < \bar{t} + \frac{1}{n}, \quad n = 1, 2, \cdots,$$

and then,

$$z_\xi(t_n) \geq \bar{z}_\xi + \epsilon(t_n - \bar{t}) \tag{4.44}$$

for $n = 1, 2, \cdots$.

Since $v(\cdot, t)$ is in H^1_{loc}, we can find $\delta > 0$ such that if $|\xi - \bar{z}_\xi| \leq \delta$,

$$|v(\xi, \bar{t})| = |v(\xi, \bar{t}) - v(\bar{z}_\xi, \bar{t})| \leq \frac{\epsilon}{2} \tag{4.45}$$

due to $\bar{t} \in [t_1, t_0] \setminus (A \cup B \cup C \cup D)$ so that $v(\bar{z}_\xi, \bar{t}) = 0$, and

$$y_\xi(\bar{t}) < \bar{z}_\xi - \delta.$$

Then choose B_{jk} such that

$$\bar{z}_\xi \in B_{jk} \subset [\bar{z}_\xi - \delta, \bar{z}_\xi + \delta].$$

Let $B_{jk} = (c, d)$ and choose e such that

$$\bar{z}_\xi - \delta < c < e < \bar{z}_\xi < d < \bar{z}_\xi + \delta.$$

We can find $N_1 > 1$ such that

$$|t - \bar{t}| < \frac{1}{N_1}$$

and

$$y_\xi(t) < c, \qquad e \le z_\xi(t) \le d;$$

Then if $|t - \bar{t}| < 1/N_1$

$$h(\cdot, t) = 0$$

a. e. on $(y_\xi(t), z_\xi(t)) \supset (c, e)$. By (4.42), there exists a $N_2 > 1$ such that

$$h(\cdot, s) = 0$$

a. e. on

$$\left(c + \left| \int_s^t |v(\cdot, \lambda)|_{L^\infty(c, z_\xi(t))} d\lambda \right|, z_\xi(t) - \left| \int_s^t |v(\cdot, \lambda)|_{L^\infty(a, z_\xi(t))} \right| d\lambda \right),$$

if

$$|t - s| < \frac{1}{N_2}, \qquad |t - \bar{t}| < \frac{1}{N_1}.$$

Thus for these s and t,

$$\begin{aligned} z_\xi(s) &\ge z_\xi(t) - \left| \int_s^t |v(\cdot, \lambda)|_{L^\infty(c, z_\xi(t))} d\lambda \right| \\ &\ge z_\xi(t) - \left| \int_s^t |v(\cdot, \lambda)|_{L^\infty(B_{jk})} d\lambda \right|. \end{aligned}$$

Let $s = \bar{t}$. Then for $|t_n - \bar{t}| < \dfrac{1}{N}$ with $N = N_1 + N_2$, $t_n > \bar{t}$, one gets

$$z_\xi(\bar{t}) \ge z_\xi(t_n) - \int_{\bar{t}}^{t_n} |v(\cdot, \lambda)|_{L^\infty(B_{jk})} d\lambda$$

It follows from this and (4.44) that

$$\bar{z}_\xi + \epsilon(t_n - \bar{t}) \le \bar{z}_\xi + \int_{\bar{t}}^{t_n} |v(\cdot, \lambda)|_{L^\infty(B_{jk})} d\lambda.$$

So that

$$\epsilon \le \frac{1}{t_n - \bar{t}} \int_{\bar{t}}^{t_n} |v(\cdot, \lambda)|_{L^\infty(B_{jk})},$$

which contradicts to (4.44) since $B_{jk} \subset [\bar{z}_\xi - \delta, \bar{z}_\xi + \delta]$. Thus the proof is completed.

Lemma 4.5 It holds that

$$\left| \int_t^{t_0} \alpha(\lambda) d\lambda \right| \leq C$$

for all $t \in (\tau, t_0]$, where C is a positive constant depending only on E_0, a_* and b_*.

Proof Fix a $s \in (\tau, t_0]$. Define $\theta \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} 0 \leq \theta \leq 1 \\ \quad \theta = 0 \qquad \text{on} \quad [0, a_*/2] \cup [2b_*, +\infty) \\ \quad \theta = 1 \qquad \text{on} \quad [a_*, b_*], \end{cases}$$

and

$$w^\epsilon(x,t) = x^{-m}(\alpha^\epsilon(t) x^{m+1} + \beta^\epsilon(t))\theta(x),$$

where $\alpha^\epsilon(t)$ and $\beta^\epsilon(t)$ are regularizations of $\alpha(t)$ and $\beta(t)$,respectively.

We consider

$$\begin{cases} \phi_t^\epsilon + w^\epsilon(x,t)\phi_x^\epsilon = 0 \\ \phi^\epsilon|_{t=t_0} = \psi, \end{cases}$$

where $\psi \in C_0^\infty(\mathbb{R})$ such that

$$\begin{cases} 0 \leq \psi \leq 1 \\ \quad \psi = 0 \qquad \text{on} \quad [0, r_*] \\ \quad \psi = 1 \qquad \text{on} \quad [R_*, +\infty), \end{cases}$$

and

$$\begin{cases} r_* &= r + \frac{1}{3}(R - r) \\ R_* &= r + \frac{2}{3}(R - r) \\ r &= a + \frac{1}{3}(b - a) \\ R &= a + \frac{2}{3}(b - a). \end{cases}$$

Using Lemma 4.2 and Lemma 4.3, one can define $r(t)$ and $R(t)$ such that

$$0 < a_* \leq y(t) \leq r(t) \leq R(t) \leq z(t) \leq b_* < +\infty$$

for a. e. $t \in (\tau, t_0]$. Indeed, we can define $r(t)$ and $R(t)$ to be the solutions of the following problems respectively:

$$\begin{cases} \dfrac{dr(t)}{dt} = r^{-m}(t)[r^{m+1}(t)\alpha(t) + \beta(t)] \\ r(t)|_{t=t_0} = r, \end{cases} \tag{4.46}$$

and

$$\begin{cases} \dfrac{dR(t)}{dt} = R^{-m}(t)[R^{m+1}(t)\alpha(t) + \beta(t)] \\ R(t)|_{t=t_0} = R, \end{cases} \tag{4.47}$$

The regularized problems associated with (4.46) and (4.47) are given respectively as

$$\begin{cases} \dfrac{dr_*^\epsilon(t)}{dt} = [r_*^\epsilon]^{-m}(t)[[r_*^\epsilon]^{m+1}\alpha^\epsilon(t) + \beta^\epsilon(t)] \\ r_*^\epsilon(t)|_{t=t_0} = r_*, \end{cases} \tag{4.48}$$

and

$$\begin{cases} \dfrac{dR_*^\epsilon(t)}{dt} = [R_*^\epsilon]^{-m}(t)[[R_*^\epsilon]^{m+1}\alpha^\epsilon(t) + \beta^\epsilon(t)] \\ R_*^\epsilon(t)|_{t=t_0} = R_*. \end{cases} \tag{4.49}$$

There exists a $\epsilon_0 > 0$ such that

$$r(t) \le r_*^\epsilon(t) \le R_*^\epsilon(t) \le R(t) \tag{4.50}$$

for all $t \in [s, t_0]$, and $0 < \epsilon < \epsilon_0$.

Set

$$\begin{cases} D_1 = \{(x,t) : 0 \le x \le r(t), s \le t \le t_0\} \\ D_2 = \{(x,t) : r(t) \le x \le R(t), s \le t \le t_0\} \\ D_3 = \{(x,t) : x > R(t), s \le t \le t_0\}. \end{cases} \tag{4.51}$$

It follows from the definition that

$$\phi^\epsilon(x,t) = \begin{cases} 0, & \text{if } (x,t) \in D_1 \\ 1, & \text{if } (x,t) \in D_3, \end{cases} \tag{4.52}$$

with

$$0 \le \phi^\epsilon(x,t) \le 1 \tag{4.53}$$

for all $(x,t) \in D_2$. Using $x^{-m}\,\varphi^\epsilon\,\theta$ as a test function in Definition 1.1 and noting that $\rho\,\theta\,\partial_x\,\phi^\epsilon \equiv 0$, one gets

$$
\begin{aligned}
&\int_0^\infty \rho u(x^{-m}\phi^\epsilon\theta)x^m dx\Big|_s^{t_0} \\
&-\int_s^{t_0}\int_0^\infty (\rho u(x^{-m}\phi^\epsilon\theta)_t + \rho u^2(x^{-m}\phi^\epsilon\theta)_x)x^m\,dx\,dt \\
&+\int_s^{t_0}\int_0^\infty P(\rho)\left[(x^{-m}\phi^\epsilon\theta)_x + \frac{m(x^{-m}\phi^\epsilon\theta)}{x}\right]x^m\,dx\,dt \\
= \;& -\mu\int_s^{t_0}\int_0^\infty x^{-m}(x^m\,u)_x\;(x^m\,x^{-m}\,\phi^\epsilon\,\theta)dx\,dt
\end{aligned}
\tag{4.54}
$$

Direct estimate shows that

$$
\begin{aligned}
&\left|\int_0^\infty \rho\,u\,\phi^\epsilon(x,t)\theta(x)dx\right| \\
\le\;& \int_{a_*/2}^{3b^*} \rho|u(x,t)|\theta(x)dx \\
\le\;& \left(\int_{a_*/2}^{3b^*} \rho u^2(x,t)x^m dx\right)^{1/2}\left(\int_{a_*/2}^{3b^*} \rho x^{-m}dx\right)^{1/2} \;\le\; C,
\end{aligned}
$$

and so,

$$
\left|\int_0^\infty \rho u(x,t)(x^{-m}\phi^\epsilon\theta)x^m dx\right| \le C \tag{4.55}
$$

for all $t \in [s,t_0]$, where C is a constant depending only on m_0, E_0, a, b, a_* and b_*. In addition,

$$
\begin{aligned}
&\left|\int_s^{t_0}\int_0^\infty (\rho u(x^{-m}\phi^\epsilon\theta)_t + \rho u^2(x^{-m}\phi^\epsilon\theta)_x)x^m\,dx\,dt\right| \\
=\;& \left|\int_s^{t_0}\int_0^\infty \rho\,u\,\theta(u-w^\epsilon)\phi^\epsilon_x\,dx\,dt + \int_s^{t_0}\int_0^\infty \rho\,u^2(x^{-m}\theta)_x\phi^\epsilon\,x^m\,dx\,dt\right| \\
\le\;& \left|\int_s^{t_0}\int_{a_*/2}^{2b_*} \rho\,u^2(x^{-m}\theta)_x\,x^m\,dx\,dt\right| \\
\le\;& |(x^{-m}\theta)_x|_{L^\infty(a_*/2,2b_*)}\int_s^{t_0}\int_{a_*/2}^{2b_*} \rho\,u^2\,x^m\,dx\,dt \;\le\; C,
\end{aligned}
$$

which implies that

$$
\left|\int_s^{t_0}\int_0^\infty (\rho u(x^{-m}\phi^\epsilon\theta)_t + \rho u^2(x^{-m}\phi^\epsilon\theta)_x)x^m\,dx\,dt\right| \le C. \tag{4.56}
$$

In addition,

$$
\begin{aligned}
&\left|\int_s^{t_0}\int_0^\infty P(\rho)\left[(x^{-m}\phi^\epsilon\theta)_x+\frac{m(x^{-m}\phi^\epsilon\theta)}{x}\right]x^m\,dx\,dt\right|\\
= &\left|\int_s^{t_0}\int_0^\infty P(\rho)(x^{-m}\theta)\phi_x^\epsilon\,x^m\,dx\,dt+\int_s^{t_0}\int_0^\infty P(\rho)\left[(x^{-m}\theta)_x\phi^\epsilon+\frac{m(x^{-m}\phi^\epsilon\theta)}{x}\right]x^m\,dx\,dt\right|\\
\le &\; C\int_s^{t_0}\int_{a_*/2}^{2b_*}P(\rho)x^m\,dx\,dt \;\le\; C.
\end{aligned}
$$

Therefore,

$$
\left|\int_s^{t_0}\int_0^\infty P(\rho)\left[(x^{-m}\phi^\epsilon\theta)_x+\frac{m(x^{-m}\phi^\epsilon\theta)}{x}\right]x^m dxdt\right|\le C. \tag{4.57}
$$

Finally,

$$
\begin{aligned}
&\int_s^{t_0}\int_0^\infty\left[x^{-m}(x^m\,u)_x(x^m\cdot x^{-m}\phi^\epsilon\theta)_x\right]dx\,dt\\
= &\int_s^{t_0}\int_0^\infty x^{-m}(x^m\,u)_x\phi_x^\epsilon\theta\,dx\,dt+\int_s^{t_0}\int_0^\infty x^{-m}(x^m\,u)_x\phi^\epsilon\theta_x\,dx\,dt\\
= &\;(m+1)\int_s^{t_0}\alpha(\lambda)d\,\lambda+\int_s^{t_0}\int_0^\infty x^{-m}(x^m\,u)_x\phi^\epsilon\theta_x\,dx\,dt
\end{aligned} \tag{4.58}
$$

and

$$
\left|\int_s^{t_0}\int_0^\infty x^{-m}(x^m\,u)_x\phi^\epsilon\theta_x\,dx\,dt\right|\le\int_s^{t_0}\int_{a_*/2}^{2b_*}x^{-m}|(x^m\,u)_x||\theta_x|dx\,dt\le C \tag{4.59}
$$

Now the conclusion of the lemma follows from (4.55) - (4.59) and (4.54).

Thus the proof is completed.

Lemma 4.6 $y=y(t)$ and $z=z(t)$ are uniformly continuous on the interval $(\tau,t_0]$.

Proof Let $\epsilon>0$ be given. Choose $\delta>0$ such that if $0\le s<t\le T$,$|s-t|\le\delta$, then

$$
\int_s^t|u(\cdot,\lambda)|_{L^\infty(a_*/2,2b_*)}d\lambda\le\epsilon.
$$

By Lemma 2.3, there exists $h=h(t)>0$ with $h<\delta$, such that $\rho(\cdot,s)=0$ a. e. on

$$
\left(y(t)+\int_s^t|u(\cdot,\lambda)|_{L^\infty(a_*/2,2b_*)}d\lambda,\,z(t)-\int_s^t|u(\cdot,\lambda)|_{L^\infty(a_*/2,2b_*)}d\lambda\right)
$$

if $|t-s|<h$.

So that

$$z(s) \geq z(t) - \int_s^t |u(\cdot,\lambda)|_{L^\infty(a_*/2,2b_*)} d\lambda.$$

Similarly, if $\rho(\cdot,s)=0$ a.e. on $(y(s),z(s))$ then $\rho(\cdot,t)=0$ a.e. on

$$\left(y(s) + \int_s^t |u(\cdot,\lambda)|_{L^\infty(a_*/2,2b_*)} d\lambda, z(s) - \int_s^t |u(\cdot,\lambda)|_{L^\infty(a_*/2,2b_*)} d\lambda\right)$$

if $|t-s|<h$.

So that

$$z(t) \geq z(s) - \int_s^t |u(\cdot,\lambda)|_{L^\infty(a_*/2,2b_*)} d\lambda.$$

Therefore, we get

$$|z(t)-z(s)| \leq \int_s^t |u(\cdot,\lambda)|_{L^\infty(a_*/2,2b_*)} d\lambda. \tag{4.60}$$

if $|t-s|<h$.

Now, fix $s<t$ with $|s-t|<\delta$ and $s,t\in(\tau,t_0]$,then the interval $[s,t]$ is covered by $\cup_{k=1}^q B_{h_k/2}(s_k)$, where $s_1<s_2<\cdots<s_q$, and

$$s_j + \frac{h_j}{2} > s_{j+1} - \frac{h_{j+1}}{2}, \quad h_j<\delta, \quad j=1,2,\cdots,q.$$

Then

$$|s_{j+1}-s_j| \leq \frac{h_j+h_{j+1}}{2} \leq \max\{h_j,h_{j+1}\} < \delta.$$

Thus by (4.60),

$$|z(s_j)-z(s_{j+1})| \leq \int_{s_j}^{s_{j+1}} |u(\cdot,\lambda)|_{L^\infty(a_*/2,2b_*)} d\lambda.$$

For some j and k ,

$$s \in B_{h_k/2}(s_k), \quad t \in B_{h_j/2}(s_j).$$

we have

$$\begin{aligned}
& |z(t)-z(s)| \\
\leq\ & |z(s)-z(s_j)| + |z(s_j)-z(s_{j-1})| + \cdots + |z(s_k)-z(t)| \\
\leq\ & \int_s^{s_j} |u(\cdot,\lambda)|_{L^\infty(a_*/2,2b_*)} d\lambda + \cdots + \int_{s_k}^t |u(\cdot,\lambda)|_{L^\infty(a_*/2,2b_*)} d\lambda \\
\leq\ & \int_s^t |u(\cdot,\lambda)|_{L^\infty(a_*/2,2b_*)} d\lambda \ \leq\ \varepsilon
\end{aligned}$$

Thus the proof is completed.

Proof of Theorem 1.1 Assume that (1.13) is not true, then

$$\rho(\cdot, t_0) = 0$$

a. e. on (a, b) with $\omega(t_0) < a < b < +\infty$. Then we have $y(\tau) = z(\tau)$ if $\tau > 0$. Then in this case, by (4.46) and (4.47) we have

$$\frac{d(R^{m+1}(t) - r^{m+1}(t))}{dt} = (m+1)\alpha(t)(R^{m+1}(t) - r^{m+1}(t)).$$

So

$$R^{m+1}(t) - r^{m+1}(t) = (R - r)\exp\left(-\int_t^{t_0} \alpha(s)ds\right).$$

Thus,

$$\lim_{t \to \tau} \int_t^{t_0} \alpha(s)ds = +\infty$$

which contradicts to Lemma 4.5. Therefore, the only possibility is

$$y(\tau) < z(\tau), \quad \tau = 0,$$

which contradicts to (1.10). Therefore, (1.13) holds true. Thus the Proof of Theorem 1.1 is completed.

§5. Proof of Theorem 1.2

It follows from the Proof of Theorem 1.1 that

$$0 < x_1(t) < x_2(t) < x_3(t) < x_4(t) < +\infty \tag{5.1}$$

and

$$\int_{x_2(t)}^{x_3(t)} \rho(x,t)x^m\, dx = \int_{a_2}^{a_3} \rho_0(x)x^m\, dx. \tag{5.2}$$

Since

$$\begin{aligned}\int_{x_2(t)}^{x_3(t)} \rho(x,t)x^m\, dx &\le \left(\int_{x_2(t)}^{x_3(t)} \rho^\gamma(x,t)x^m\, dx\right)^{1/\gamma}\left(\int_{x_2(t)}^{x_3(t)} x^m dx\right)^{1-1/\gamma}\\ &\le C\left(x_3(t) - x_2(t)\right)^{1-1/\gamma}\end{aligned}$$

This, together with (5.1) and (5.2), implies

$$x_3(t) - x_2(t) \geq \frac{1}{C}\left(\int_{a_2}^{a_3} \rho_0(x)x^m\, dx\right)^{(\gamma-1)/\gamma}$$

for all $t \in (0, +\infty)$. Thus the Proof of Theorem 1.2 is completed.

§6. Proof of Theorem 1.3

In this section we shall prove Theorem 1.3.

We assume that

$$\rho(\cdot, t_0) = 0 \tag{6.1}$$

a. e. on $(0, a)$ with $a > 0$, and

$$a = \sup\{x > 0 : \int_0^x \rho(y, t_0)dy = 0\} > 0.$$

It follows from Lemma 2.3 that there exists a positive number $h_0 = h_0(a) > 0$ such that

$$\rho(\cdot, t) = 0 \tag{6.2}$$

a. e. on $(a/4, a/2)$ and all $t \in (t_0 - h_0, t_0)$. Similar to the proof of (4.35) one has

$$\int_{a/2}^{\infty} \rho(x,t)x^m dx = \int_a^{\infty} \rho(x, t_0)x^m dx$$

for all $t \in [t_0 - h_0, t_0]$. On the other hand it follows from (1.7) that

$$\int_0^{\infty} \rho(x,t)x^m dx = \int_0^{\infty} \rho(x, t_0)x^m dx$$

for all $t \in [t_0 - h_0, t_0]$. Therefore, we conclude that

$$\int_0^{a/2} \rho(x,t)x^m dx = 0$$

for all $t \in [t_0 - h_0, t_0]$. Thus

$$\rho(\cdot, t) = 0 \tag{6.3}$$

a. e. on $(0, a/2)$ and all $t \in (t_0 - h_0, t_0)$.

Denote

$$z(t) = \sup\{x > 0 : \int_0^x \rho(y,t)dy = 0\} > 0$$

and assume that $z(t_0) = a$. Similarly to the proof of Lemma 4.1 we can show that

$$z = z(t) \tag{6.4}$$

is a absolutely continuous functions in $[t_0 - h_0, t_0]$.

Let S be defines as the set of all $t \geq 0$ such that there are extensions of z to $[t, t_0]$ such that the following three properties hold:

(i) z is a absolutely continuous on $[t, t_0]$;

(ii) $z > 0$ on $[t, t_0]$;

(iii) For all $s \in [t, t_0]$, we have

$$\int_0^{z(s)} \rho(x,t)dx = 0$$

and

$$\int_0^{z(s)+\epsilon} \rho(x,t)dx > 0 \quad \forall \epsilon > 0.$$

By (6.3) and (6.4), $S \neq \emptyset$. So

$$\tau = \inf S \tag{6.5}$$

is well-defined.

Lemma 6.1 It holds true that

$$u(x,t) = x\alpha(t)$$

for all $x \in (0, z(t))$ and a. e. $t \in (\tau, t_0]$.

Proof It follows from Lemma 4.3 that

$$u(x,t) = x^{-m}[x^{m+1}\alpha(t) + \beta(t)]$$

for all $x \in (0, z(t))$ and a. e. $t \in (\tau, t_0]$. Note that

$$\begin{aligned} \int_s^{t_0} \int_0^1 u_x^2 \, x^m \, dx \, dt &= \int_s^{t_0} \int_0^1 [\alpha(t) - mx^{-m-1}\beta(t)]^2 \, x^m \, dx \, dt \\ &\geq \frac{1}{2} \int_s^{t_0} \int_0^1 m^2 \, x^{-m-2} \, \beta^2(t) dx \, dt - \int_s^{t_0} \int_0^1 \alpha^2(t) x^m \, dx \, dt, \end{aligned}$$

which implies that

$$\frac{1}{2}\int_s^{t_0}\int_0^1 m^2\, x^{-m-2}\, \beta^2(t) dx\, dt \le \int_s^{t_0}\int_0^1 \alpha^2(t) x^m\, dx\, dt + \int_s^{t_0}\int_0^1 u_x^2\, x^m\, dx\, dt.$$

Therefore we conclude that

$$\beta(t) \equiv 0$$

for a. e. $t \in [s, t_0]$ with $s \in (\tau, t_0]$. Thus Lemma 6.1 is proved.

Lemma 6.2 It holds that

$$\frac{dz}{dt} \le z\alpha(t)$$

for almost all $t \in (\tau, t_0]$.

The proof is similarly to Lemma 4.4. Therefore the details are omitted.

We are now ready to show Theorem 1.3.

Proof of Theorem 1.3 Assume that the conclusion of Theorem 1.3 is not true. Then

$$\rho(\cdot, t_0) = 0 \quad \text{a.e.} \quad \text{on} \quad (0, a)$$

for some $a \in (0, +\infty)$ and some $t_0 \in (0, \infty)$. By (6.3) we define $z(t_0) = a$ and

$$z(t) = \sup\left\{ x > 0 : \int_0^x \rho(y, t) dy = 0 \right\}.$$

Define

$$V = \{(x, t) : 0 \le x \le z(s), \tau < s < t_0\}$$

where τ is defined by (6.5).

Denote by

$$\overline{\lim}_{t \to \tau^+} z(t) = \bar{z}, \quad \underline{\lim}_{t \to \tau^+} z(t) = \underline{z}.$$

If $\underline{z} > 0$ and $\tau > 0$, then there exists a small $\delta \in (0, 1)$ such that

$$z(t) \ge \underline{z}/2 > 0$$

for all $t \in [\tau, \tau + \delta]$. Therefore we get

$$\rho(x, \tau) = 0$$

for a. e. $x \in (0, \underline{z}/2)$.By Lemma 2.3 there exists a positive $h_* > 0$ such that

$$\rho(x, t) = 0$$

for a. e. $x \in (0, \underline{z}/4)$, for all $t \in [\tau - h_*, \tau + h_*]$. By the proof of Lemma 4.6, $z(t)$ is uniformly continuous on $[\tau - h_*, \tau + h_*]$. In particular,

$$\bar{z} = \underline{z} = \lim_{t \to \tau} z(t) = z(\tau) > 0$$

then τ is not minimal. Therefore, we conclude that either

$$\underline{z} = 0 \tag{6.6}$$

or

$$\underline{z} > 0, \tau = 0. \tag{6.7}$$

The case (6.7) is impossible due to the assumption (1.10). We now show that (6.6) is also impossible. Then it follows from (1.12) or (1.13) that

$$\left| \int_{\tau}^{t_0} \alpha(s) ds \right| < +\infty. \tag{6.8}$$

Choose $R \in (0, a)$ and define $R(t)$ such that

$$\frac{dR(t)}{dt} = R(t)\alpha(t), \quad R(t_0) = R.$$

Then

$$R(t) = R \exp\left(- \int_t^{t_0} \alpha(s) ds \right).$$

On the other hand,

$$0 < R(t) < z(t), \quad \forall\, t \in (\tau, t_0].$$

Therefore

$$R \exp\left(- \int_t^{t_0} \alpha(s) ds \right) < z(t).$$

Letting $t \to \tau^+$ and using (6.8), we get

$$\underline{z} \geq R \exp\left(- \int_t^{t_0} \alpha(s) ds \right) > 0$$

which contradicts to (6.6). Thus the Proof of Theorem 1.3 is completed.

§7. Proof of Theorem 1.4

We consider the approximate solutions $(\rho_\varepsilon, u_\varepsilon)$ defined to be solutions of the following equations

$$\text{(7.1)} \qquad \begin{cases} \rho_{\epsilon t} + (\rho_\epsilon\, u_\epsilon)_x + \dfrac{m\,\rho_\epsilon\, u_\epsilon}{x} = 0 \\ (\rho_\epsilon\, u_\epsilon)_t + (\rho_\epsilon\, u_\epsilon^2 + P(\rho_\epsilon))_x + \dfrac{m\,\rho_\epsilon\, u_\epsilon^2}{x} = \mu\, u_{\epsilon xx} + \mu\, m \left(\dfrac{u_\epsilon}{x}\right)_x \end{cases}$$

with initial conditions

$$\text{(7.2)} \qquad \begin{cases} \rho_\epsilon(x,t)|_{t=0} = \rho_0^\epsilon(x) \\ \rho_\epsilon(x,t) u_\epsilon(x,t)|_{t=0} = m_0^\epsilon(x) \end{cases}$$

and the boundary conditions

$$\text{(7.3)} \qquad u_\epsilon(x,t)|_{x=\epsilon} = 0, t \geq 0,$$

where $\rho_0^\epsilon(x)$ and $m_0^\epsilon(x)$ are defined by [8].

Choose $\xi_R \in C^\infty(0, +\infty)$ such that

$$\text{(7.4)} \qquad \begin{cases} 0 \leq \xi_R \leq 1 \\ \quad \xi_R = 1 \qquad \text{on} \quad (0, R) \\ \quad \xi_R = 0 \qquad \text{on} \quad (2R, +\infty) \end{cases}$$

for $R \in (1, +\infty)$, and $|\xi_R| \leq C R^{-1}$. By (7.1) we have

$$\text{(7.5)} \qquad \int_\epsilon^\infty \xi_R^2\, \rho_\epsilon(x,t) x^m\, dx - \int_\epsilon^\infty \xi_R^2\, \rho_0^\epsilon(x) x^m\, dx = -\int_0^t \int_\epsilon^\infty 2\xi_R\, \xi_R'\, \rho_\epsilon\, u_\epsilon\, x^m\, dx\, ds.$$

It has been shown in [8] that

$$\text{(7.6)} \qquad \int_\epsilon^\infty \psi_\epsilon(\rho_\epsilon(x,t)) x^m\, dx + \int_\epsilon^\infty \rho_\epsilon u_\epsilon^2(x,t) x^m\, dx \leq C_0$$

for all $t \geq 0$, where C_0 is a positive constant depending only on m_0 and E_0, and

$$\psi_\epsilon(\rho) \equiv \frac{A\rho^\gamma}{\gamma - 1} - \frac{A\epsilon^{\gamma-1}\rho}{\gamma - 1} - A\epsilon^{\gamma-1}\rho + A\epsilon^\gamma.$$

It follows from (7.5) and (7.6) that

$$
\begin{aligned}
& \left| \int_\epsilon^\infty \xi_R^2\, \rho_\epsilon(x,t) x^m\, dx - \int_\epsilon^\infty \xi_R^2\, \rho_0^\epsilon(x) x^m\, dx \right| \\
= \;& \left| - \int_0^t \int_\epsilon^\infty 2\xi_R\, \xi_R'\, \rho_\epsilon\, u_\epsilon\, x^m\, dx\, ds \right| \\
\le \;& \left(\int_0^t \int_\epsilon^\infty \rho_\epsilon\, u_\epsilon^2\, x^m\, dx\, ds \right)^{1/2} \left(\int_0^t \int_\epsilon^\infty 4\xi_R^2 |\xi_R'|^2 \rho_\epsilon\, x^m\, dx\, ds \right)^{1/2} \\
\le \;& \frac{Ct^{1/2}}{R} \left(1 + \int_0^t \int_\epsilon^\infty \xi_R^2\, \rho_\epsilon\, x^m\, dx\, ds \right).
\end{aligned}
\tag{7.7}
$$

On the other hand we compute

$$
\begin{aligned}
& \int_\epsilon^\delta \rho_\epsilon^\gamma\, x^m\, dx\, dt \\
= \;& \frac{\gamma - 1}{a} \int_\epsilon^\delta \frac{a\rho_\epsilon^\gamma x^m}{\gamma - 1} dx\, dt \\
= \;& \frac{\gamma - 1}{a} \left\{ \int_\epsilon^\delta \psi_\epsilon(\rho_\epsilon(x,t)) x^m\, dx + \int_\epsilon^\delta \left(\frac{a\epsilon^{\gamma-1}\rho_\epsilon}{\gamma - 1} + a\epsilon^{\gamma-1}\, \rho_\epsilon - a\epsilon^\gamma \right) x^m\, dx \right\} \\
\le \;& C \left\{ \int_\epsilon^\delta \rho_\epsilon^\gamma(x,t) x^m\, dx \right\}^{1/\gamma} \left\{ \int_\epsilon^\delta x^m\, dx \right\}^{1-1/\gamma} + C,
\end{aligned}
$$

which implies that

$$
\int_\epsilon^\delta \rho_\epsilon^\gamma x^m\, dx\, dt \le C \tag{7.8}
$$

for all $\delta \in (\epsilon, 1)$, where C is a positive constant depending only on C_0.

By (7.7) and (7.8), we can estimate that

$$
\begin{aligned}
& \left|\int_\delta^{+\infty} \xi_R^2\, \rho_\epsilon(x,t) x^m\, dx - \int_\delta^{+\infty} \xi_R^2\, \rho_0(x) x^m\, dx\right| \\
\leq\ & \left|\int_\delta^{+\infty} \xi_R^2 (\rho(x,t) - \rho_\epsilon(x,t)) x^m\, dx\right| \\
& + \left|\int_\delta^{+\infty} \xi_R^2 (\rho_0(x) - \rho_0^\epsilon(x)) x^m\, dx\right| \\
& + \left|\int_\delta^{+\infty} \xi_R^2\, \rho_\epsilon(x,t) x^m\, dx - \int_\delta^{+\infty} \xi_R^2\, \rho_0^\epsilon(x) x^m\, dx\right| \\
\leq\ & \left|\int_\delta^{+\infty} \xi_R^2 (\rho(x,t) - \rho_\epsilon(x,t)) x^m\, dx\right| \\
& + \left|\int_\delta^{+\infty} \xi_R^2 (\rho_0(x) - \rho_0^\epsilon(x)) x^m\, dx\right| \\
& + \frac{Ct^{1/2}}{R} \left(1 + \int_0^t \int_\epsilon^\infty \xi_R^2\, \rho_\epsilon(x,\tau) x^m\, dx\, d\tau + 1\right) \\
& + \left|\int_\epsilon^\delta \xi_R^2\, \rho_\epsilon(x,t) x^m\, dx\right| + \left|\int_\epsilon^\delta \xi_R^2\, \rho_0^\epsilon(x) x^m\, dx\right| \\
\leq\ & \left|\int_\delta^{+\infty} \xi_R^2 (\rho(x,t) - \rho_\epsilon(x,t)) x^m\, dx\right| \\
& + \left|\int_\delta^{+\infty} \xi_R^2 (\rho_0(x) - \rho_0^\epsilon(x)) x^m\, dx\right| \\
& + \frac{Ct^{1/2}}{R} \left(1 + \int_0^t \int_\delta^\infty \xi_R^2\, \rho_\epsilon(x,\tau) x^m\, dx\, d\tau + \int_0^t \int_\epsilon^\delta \xi_R^2\, \rho_\epsilon(x,\tau) x^m\, dx\, d\tau\right) \\
& + \left|\int_\epsilon^\delta \xi_R^2\, \rho_\epsilon(x,t) x^m\, dx\right| + \left|\int_\epsilon^\delta \xi_R^2\, \rho_0^\epsilon(x) x^m\, dx\right| \\
\leq\ & \left|\int_\delta^{+\infty} \xi_R^2 (\rho(x,t) - \rho_\epsilon(x,t)) x^m\, dx\right| \\
& + \left|\int_\delta^{+\infty} \xi_R^2 (\rho_0(x) - \rho_0^\epsilon(x)) x^m\, dx\right| \\
& + \frac{Ct^{1/2}}{R} \left(1 + \int_0^t \int_\delta^\infty \xi_R^2\, \rho_\epsilon(x,\tau) x^m\, dx\, d\tau + C\delta^{n(1-1/\gamma)}\right) \\
& + C\delta^{n(1-1/\gamma)} + \int_\epsilon^\delta \xi_R^2 |\rho_0^\epsilon(x) - \rho_0 - \epsilon| x^m\, dx + \int_\epsilon^\delta \xi_R^2 (\rho_0(x) + \epsilon) x^m\, dx
\end{aligned}
$$

Letting $\epsilon \to 0^+$ and using (3.1) in [8], we get

$$\begin{aligned} & \left| \int_\delta^{+\infty} \xi_R^2 \, \rho(x,t) x^m \, dx - \int_\delta^{+\infty} \xi_R^2 \, \rho_0(x) x^m \, dx \right| \\ \leq \; & \frac{Ct^{1/2}}{R} \left(1 + \int_0^t \int_\delta^\infty \xi_R^2 \, \rho(x,\tau) x^m \, dx \, d\tau + C\delta^{n(1-1/\gamma)} \right) \\ & + C\delta^{n(1-1/\gamma)} + \int_0^\delta \xi_R^2 \, \rho_0(x) x^m \, dx \end{aligned}$$

Letting $R \to +\infty$ and $\delta \to 0+$ in the above estimate, we get

$$\int_0^\infty \rho(x,t) x^m \, dx = \int_0^\infty \rho_0(x) x^m \, dx$$

which is (1.7). Similarly, (1.8) follows from letting $\varepsilon \to 0^+$ in (7.6).

Thus the proof is completed.

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