Convergence to Equilibria and Blowup Behavior of Global Strong Solutions to the Stokes Approximation Equations for Two-Dimensional Compressible Flows with Large Data

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Abstract. This paper concerns the large time behavior of strong and classical solutions to the two-dimensional Stokes approximation equations for the compressible flows. We consider the unique global strong solution or classical solution to the two-dimensional Stokes approximation equations for the compressible flows with large external potential force, together with a Navier-slip boundary condition, for arbitrarily large initial data. Under the conditions that the corresponding steady state exists uniquely with the steady state density away from vacuum, we prove that the density is bounded from above independent of time, consequently, it converges to the steady state density in L^p and the velocity u converges to the steady state velocity in $W^{1,p}$ for any $1 \le p < \infty$ as time goes to infinity; furthermore, we show that if the initial density contains vacuum at least at one point, then the derivatives of the density must blow up as time goes to infinity.

Key words. Stokes approximation equations, large external potential forces, large-time behavior, uniform upper bound, vacuum, blowup

AMS subject classifications. 35Q30, 76N15

1 Introduction

The compressible isentropic Navier-Stokes equations, which are the basic models describing the evolution of a viscous compressible gas, read as follows

$$\begin{cases} \rho_t + div(\rho u) = 0, \\ (\rho u)_t + div(\rho u \otimes u) - \mu \Delta u - \nabla(\xi div u) + \nabla P(\rho) = \rho F, \end{cases} \tag{1.1}$$

where $x \in \Omega \subset R^N, t \in (0,T)$ and $P(\rho) = a\rho^\gamma, a > 0, \gamma > 1$, F is the external forces density, and the viscosity coefficients μ, ξ are assumed to satisfy $\mu > 0$ and $\xi + \mu \ge 0$.

There is huge literature on the studies on the global existence and large time behavior of solutions to (1.1) (see [4, 10, 11, 14, 21–23, 29, 30]). For the existence of weak solutions

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for arbitrarily large data(which may include vacuum states), the major breakthrough is due to P. L. Lions [18–20] (see also Feireisl et al [5]), where he obtains global existence of weak solutions - defined as solutions with finite energy - when the exponent γ is suitably large. The only restriction on initial data is that the initial energy is finite, so that the density is allowed to vanish. Despite this progress, the regularity and behavior of these weak solutions remains open in many important cases. As emphasized in many papers related to compressible fluid dynamics [3, 10, 12, 14, 28, 29, 32, 37], the possible appearance of vacuum and uniform upper bound estimate on the density is one of the major difficulties in the theory of global existence and regularity of solutions. In particular, the results of Xin [37] show that there is no global smooth solution (ρ, u) to Cauchy problem for (1.1) ($F \equiv 0$) with a nontrivial compactly supported initial density, which gives results for finite time blow-up in the presence of vacuum.

There are many results concerning the large-time dynamics of solutions to problem (1.1). For 1D case, see [33, 34] and the references therein. In several space dimensions, Matsumura and Nishida [21, 22] first prove the stability of a constant steady state $(\overline{\rho}, 0)$ in H^3-framework with respect to small initial disturbances in the case $F \equiv 0$. For $F \equiv \nabla \cdot F_1 + F_2$ small enough, Shibata and Tanaka [31] obtain the stability of steady flows with respect to initial disturbances, provided the H^3 norm of the initial disturbance is small enough. For large $F = \nabla f$ and $\gamma > N/2$, Feireisl and Petzeltová [6], Novotny and Straškraba [27] prove that for different boundary conditions, the density of any global weak solution converge to the steady steady state density in L^p space for some p as time goes to infinity if there exists a unique steady state. As soon as the unique steady state with density away from vacuum exists, under the conditions that the initial data are close enough to the steady state with the steady state density away from vacuum, Matsumura and Padula [24] obtain both the existence of the unique classical solution to problem (1.1) in H^3-framework and the stability of the steady state.

The major difficulties in analysis of the compressible Navier-Stokes equations (1.1) are the nonlinearities in both the convection and the pressure and their interactions. To study the well-posedness of solutions and gain understanding of the key issues, one has been looking into various simplified models of the Navier-Stokes systems. One of the prototype simplifications of the Navier-Stokes system (1.1) is the Stokes approximation

$$\begin{cases} \rho_t + div(\rho u) = 0, \\ \overline{\rho} u_t - \mu \Delta u - \xi \nabla (div u) + \nabla P = \rho F, \end{cases} \tag{1.2}$$

where $\overline{\rho} = \text{const.} > 0$ is the mean density, and $P = a\rho^\gamma, a > 0, \gamma > 1$. This is a good approximation for strongly viscous fluids when the convection is unimportant.

For simplicity, we take $\overline{\rho} = 1, \mu = 1, \xi = 0, a = 1$, and study the following system with large potential force

$$\rho_t + div(\rho u) = 0, \tag{1.3}$$

$$u_t - \Delta u + \nabla P = \rho \nabla f, \tag{1.4}$$

in a bounded domain Ω in R^N, where $P = \rho^\gamma, \gamma > 1$. As usual we impose the initial conditions

$$\rho(0) = \rho_0, u(0) = u_0, \tag{1.5}$$

and the no-stick boundary condition:

$$u \cdot n = 0 \text{ and } \begin{cases} curlu = 0 \text{ on } \partial\Omega & \text{if } N = 2, \\ curlu \times n = 0 \text{ on } \partial\Omega & \text{if } N = 3, \end{cases} \tag{1.6}$$

where n is the unit outward normal to $\partial\Omega$. The first condition in (1.6) is the non-penetration boundary condition, while the second one is also known in the form

$$(D(u) \cdot n)_\tau = 0, \tag{1.7}$$

where $D(u)$ is the stress tensor with components

$$D_{ij}(u) = \frac{1}{2}\left(\partial_{x_i} u^j + \partial_{x_j} u^i\right).$$

Condition (1.7) means the tangential component of $D(u) \cdot n$ vanishes on the boundary $\partial\Omega$. This is known as a Navier-type slip boundary condition.

In this paper, we consider the two-dimensional case, i.e. $N = 2$. The corresponding steady problem to the initial-boundary-value problem (1.3)-(1.6) is

$$\begin{cases} div(\rho_s u_s) = 0, \\ -\Delta u_s + \nabla \rho_s^\gamma = \rho_s \nabla f, \\ u_s \cdot n|_{\partial\Omega} = 0, curlu_s|_{\partial\Omega} = 0. \\ \displaystyle\int_\Omega \rho_s dx = \int_\Omega \rho_0 dx. \end{cases} \tag{1.8}$$

In [24], Matsumura and Padula prove

Lemma 1.1 *Let $f \in C^k\left(\overline{\Omega}\right), k \geq 1$. Assume further that*

$$\int_\Omega \left(\frac{\gamma - 1}{\gamma}\left(f - \inf_{\overline{\Omega}} f\right)\right)^{1/(\gamma-1)} dx < \int_\Omega \rho_0 dx. \tag{1.9}$$

Then problem (1.8)has a unique solution $(\rho_s, 0), 0 < \rho_s \in C^k\left(\overline{\Omega}\right)$.

Remark 1.1 *Condition (1.9) means that the steady problem (1.8) has a unique solution provided the total mass exceeds some critical value(which depends on the potential). If the total mass is less than the critical value then the solution does not always exist, see the counterexamples in [1].*

It should be noted that the 2D initial-boundary-value problem (1.3)-(1.6) has been thoroughly studied by many people. In particular, the existence of classical solutions to the 2D initial-boundary-value problem on any finite interval $[0, T](T > 0)$ for arbitrarily large smooth initial data has been proved by [2, 15, 17, 20, 25]. However, several important physical questions still remain unsolved. In particular, there have been no results on the uniform estimates and the large-time behavior of the solutions for "large external forces".

In this paper, our main aims are to derive some uniform time-independent estimates on the strong solution to problem (1.3)-(1.6) and study the large-time behavior of the solution with arbitrarily large potential force and initial data. As a byproduct, we have obtained the appropriate asymptotic stability of steady state under general perturbations. First, we derive a uniform time-independent upper bound for the density to the problem (1.3)-(1.6) for arbitrary large smooth initial data; then, as a consequence of the uniform estimate on the bound of density, we show the large time asymptotic behavior of the strong solutions. Our first result is

Theorem 1.1 *Suppose that $N = 2$ and that for some $q > 2, l \geq 1$,*

$$\rho_0 \in W^{l,q}(\Omega), u_0 \in W^{l+1,q}(\Omega), f \in C^l\left(\overline{\Omega}\right). \tag{1.10}$$

Assume further that (1.9) holds. Then problem (1.3)-(1.6) has a unique solution (ρ, u) such that for any $T > 0$,

$$\frac{\partial^k \rho}{\partial t^k} \in L^\infty(0,T; W^{l-k,q}(\Omega)), \frac{\partial^k u}{\partial t^k} \in L^\infty(0,T; W^{l-k+1,q}(\Omega)), \tag{1.11}$$

for any $k, 0 \leq k \leq l$, and moreover, there exists some C independent of T such that

$$\sup_{0 \leq t \leq T} \|\rho(\cdot,t)\|_{L^\infty(\Omega)} \leq C, \tag{1.12}$$

and

$$\lim_{t \to \infty} \left(\|\rho - \rho_s\|_{L^\alpha(\Omega)} + \|u(\cdot,t)\|_{W^{1,\beta}(\Omega)} \right) = 0, \tag{1.13}$$

for any $\alpha, \beta \in [1, \infty)$.

Remark 1.2 *If $l = 1$, the unique solution is the so-called strong solution; if $l \geq 2$, the unique solution is also a classical one. In this paper, by a strong solution, we mean a pair of functions ρ and u satisfying the equations (1.3) (1.4) almost everywhere in $\Omega \times (0,\infty)$; and a classical solution means a pair of functions $(\rho, u), \rho \in C^1(\Omega \times (0,\infty)), u \in C^2(\Omega \times (0,\infty))$ satisfying (1.3) and (1.4) everywhere in $\Omega \times (0,\infty)$.*

Remark 1.3 *In contrast to [24], we require neither that the initial data are closed enough to steady state nor that the initial density is away from vacuum.*

Remark 1.4 *In both [6] and [27], it has been shown that for problem (1.1), if the steady state is unique then the density of any weak solutions to problem (1.1), whose $L^\gamma(\Omega)$-norm is bounded independent of time, must converge to the steady state density in $L^p(\Omega)$ for $1 \leq p \leq \gamma$. Theorem 1.1 shows that for large external potential force and large smooth initial data, where the initial density may contain vacuum, there exists a unique strong (or classical) solution (ρ, u) on $[0, T]$ to the 2D problem (1.3)-(1.6) for any $T > 0$. Furthermore, if the steady problem (1.8) has a unique solution $(\rho_s, 0)$ with ρ_s away from vacuum, under the conditions that the initial data are smooth and that the mean value of the initial density is equal to that of ρ_s, then not only the density must converge to the steady state density in $L^p(\Omega)$ for any $1 \leq p < \infty$ as time goes to infinity and must be bounded from above independent of time but the velocity u must converge to 0 in $W^{1,p}(\Omega)$ for any $1 \leq p < \infty$.*

A natural question rises from the consequence result (1.13): Can one obtain the large time asymptotic convergence of the density in stronger norms? It will be shown that if the initial density contains vacuum at least at a point then the derivatives of the density has to blow up as time goes to infinity, that is

Theorem 1.2 *In addition to the conditions of Theorem 1.1, assume further that there exists some point $x_0 \in \overline{\Omega}$ such that $\rho_0(x_0) = 0$. Then the unique global strong (or classical) solution (ρ, u) to problem (1.3)-(1.6) obtained in Theorem 1.1 has to blow up as $t \to \infty$; that is*

$$\lim_{t \to \infty} \|\nabla \rho(\cdot,t)\|_{L^q(\Omega)} = \infty.$$

Remark 1.5 *It would be interesting to study the existence and large time asymptotic behavior of solutions for the case $q = 2$. This is left for the future.*

We now comment on the analysis of this paper. By using the space-time higher power estimate on the density due to P. L. Lions and the theory of compensated compactness as in [6], one can derive the L^1-convergence of the density to the steady state density. Thus, the key step to prove (1.13) is to derive the uniform time-independent L^∞-estimate on the density, (1.12). To this end, we try to modify our analysis in [17]. However, due to the arbitrariness of the size of the potential force, we cannot generalize our approach in [17] directly to our case, where the key step is to estimate the deviation of the density from the steady state density. To overcome this difficulty, we first normalize the momentum equation by dividing it by ρ_s and by making full use of the structure of the steady states; then we can show that the power of the deviation of the pressure from the steady state pressure are smaller than that of the deviation of the density ρ from the steady state ρ_s of the other terms. The combination of these facts, together with some careful estimates on the deviation of the pressure from the steady state pressure and the difference between the divergence of the velocity field and the deviation of the pressure from the steady state pressure, then yields the desired estimates.

This paper is organized as follows. In Section 2 we collect some elementary facts which are helpful for our analysis in the future. The main results, Theorem 1.1 and Theorem 1.2, are proved in Section 3 and Section 4 respectively.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used and play important roles later.

Consider the following parabolic problem in a bounded smooth domain $\Omega \subset R^N$,

$$\begin{cases} \varphi_t - \Delta\varphi = f, \\ \varphi(x,0) = 0, \\ \dfrac{\partial\varphi}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

We denote by $\overline{f}$ the average of f over Ω, i.e.,

$$\overline{f} = \frac{1}{|\Omega|}\int_\Omega f(x)dx.$$

Then the following facts are well-known (see [8, 9]):

Lemma 2.1 *Assume that $p \in (1,\infty)$ and $T \in (0,\infty]$. Then for*

$$f \in \left\{f \in L^p(\Omega \times (0,T)), \overline{f} = 0\right\},$$

the problem (2.1) has a unique solution φ such that

$$\varphi_t, D^2\varphi \in L^p(0,T;L^p(\Omega)), \text{ and } \overline{\varphi} = 0;$$

moreover, there exists a positive constant A independent of T such that

$$\int_0^T \|\varphi_t(t)\|_{L^p}^p dt + \int_0^T \|\Delta\varphi(t)\|_{L^p}^p dt \le A\int_0^T \|f\|_{L^p}^p dt.$$

Lemma 2.1 and the Hodge decomposition lead to the following simple derivative estimate.

Lemma 2.2 *Let $r \in (1, \infty)$ and $f \in \left\{ f \in L^r(\Omega \times (0,T)) \,\middle|\, \overline{f} = 0 \right\}$. Then the solution of the following parabolic problem:*

$$\begin{cases} v_t - \Delta v = \nabla f, \\ v(x,0) = 0, \end{cases}$$

supplemented with (1.6), satisfies

$$\|Dv\|_{L^r(\Omega\times(0,T))} \le A\|f\|_{L^r(\Omega\times(0,T))}$$

with A independent of T.

Also, the following estimate will be used later to get the uniform upper bound for the density.

Lemma 2.3 ([39]) *Let the function y satisfy*

$$y'(t) \le g(y) + b'(t) \text{ on } [0,T], y(0) = y^0,$$

with $g \in C(R)$ and $y, b \in W^{1,1}(0,T)$. If $g(\infty) = -\infty$ and $b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1)$ for all $0 \le t_1 < t_2 \le T$ with some $N_0 \ge 0$ and $N_1 \ge 0$, then

$$y(t) \le \max\left\{ y^0, \overline{\zeta} \right\} + N_0 < \infty \text{ on } [0,T],$$

where $\overline{\zeta}$ such that $g(\zeta) \le -N_1$ for $\zeta \ge \overline{\zeta}$.

The following well-known inequality is due to Ladyzhenskaya.

Lemma 2.4 ([16]) *Assume that $N = 2$, Ω is a bounded domain in R^2 with piecewise smooth boundary, and that*

$$u \in \left\{ u \in H^1(\Omega), \overline{u} = 0 \right\} \text{ or } u \in \left\{ \left(H^1(\Omega)\right)^2, u \cdot n|_{\partial\Omega} = 0 \right\}.$$

Then there exists a constant C independent of u such that

$$\|u\|_{L^4} \le C\|u\|_{L^2}^{1/2}\|Du\|_{L^2}^{1/2}.$$

To get the space-time estimate for the pressure P, we need the following lemma concerning the solution to the problem

$$\begin{cases} divv = f, \\ v|_{\partial\Omega} = 0. \end{cases} \tag{2.2}$$

Lemma 2.5 ([7]) *There exists a linear operator*

$$\mathcal{B} = [\mathcal{B}_1, \mathcal{B}_2] : \left\{ f \in L^p(\Omega) \,\middle|\, \overline{f} = 0 \right\} \to [W_0^{1,p}(\Omega)]^2$$

such that $v = \mathcal{B}(f)$ satisfies (2.2), and

$$\|\mathcal{B}(f)\|_{W_0^{1,p}(\Omega)} \le C(p)\|f\|_{L^p(\Omega)}, \text{ for any } 1 < p < \infty;$$

moreover, if $f = div\overrightarrow{g}$ for a certain $\overrightarrow{g} \in [L^r(\Omega)]^2$, $\overrightarrow{g} \cdot n|_{\partial\Omega} = 0$, then

$$\|\mathcal{B}(f)\|_{L^r(\Omega)} \le C(r)\left\|\overrightarrow{g}\right\|_{L^r(\Omega)},$$

for any $1 < r < \infty$.

3 Proof of Theorem 1.1.

Due to the existence and uniqueness results established in [15, 17, 20], we need only to show that both (1.12) and (1.13) hold.

Let $T \in (0, \infty)$ be fixed and C denote a generic positive constant independent of T.

Integrating (1.3) over $\Omega \times (0, t)$ leads to

$$\|\rho(\cdot, t)\|_{L^1} = \|\rho_0\|_{L^1}, \text{ for all } t \geq 0. \tag{3.1}$$

Standard energy estimates for (1.3)-(1.6) yield that

$$\frac{d}{dt}E(t) + \|Du\|_{L^2}^2 \leq 0, \tag{3.2}$$

with the total energy $E(t)$ being defined by

$$E(t) \triangleq \int_\Omega \left(\frac{1}{2}|u|^2 + \frac{1}{\gamma - 1}\rho^\gamma - \rho f\right) dx.$$

Consequently, both (3.1) and (3.2) give that

$$\begin{aligned}
&\frac{1}{2}\sup_{0\leq s\leq t}\|u(\cdot, s)\|_{L^2}^2 + \frac{1}{\gamma-1}\sup_{0\leq s\leq t}\|P(\cdot, s)\|_{L^1} + \int_0^t \|Du\|_{L^2}^2 ds \\
&\leq \frac{1}{2}\|u_0\|_{L^2}^2 + \frac{1}{\gamma-1}\|P_0\|_{L^1} + \int_\Omega \rho f dx - \int_\Omega \rho_0 f dx \\
&\leq C.
\end{aligned} \tag{3.3}$$

Thus, we use Lemma 2.4, (3.3), (1.6) and Poincaré's inequality to derive that for any $1 < p < \infty$,

$$\int_0^t \left(\|Du\|_{L^2}^2 + \|u\|_{L^4}^4 + \|u\|_{L^p}^2\right) ds \leq C. \tag{3.4}$$

Using (3.3) and (3.4), similar to [6], we can deduce

Lemma 3.1 *Let (ρ, u) be the unique strong solution to problem (1.3)-(1.6). Then*

$$\lim_{t\to\infty} \|\rho(\cdot, t) - \rho_s(\cdot)\|_{L^1} = 0. \tag{3.5}$$

Remark 3.1 *Since (3.5) holds, without loss of generality, we can assume that for any $\delta > 0$,*

$$\sup_{0\leq t<\infty} \|(\rho - \rho_s)(\cdot, t)\|_{L^1} < \delta. \tag{3.6}$$

To prove Lemma 3.1, we need the following stronger space-time estimates on P.

Lemma 3.2 *Let $0 < \theta < 1/2$. Then there exists a constant $C(\theta, \Omega)$ such that*

$$\int_{\tau-1}^{\tau+2} \int_\Omega P^{1+\theta} dxds \leq C(\theta, \Omega), \text{ for all } \tau > 1. \tag{3.7}$$

Proof. Multiplying (1.4) by $\mathcal{B}\left(P^\theta-\overline{P^\theta}\right)$, where $\mathcal{B}$ is as Lemma 2.5, integrating the result over $\Omega\times(\tau-1,\tau+2)$, we get after integration by parts

$$\begin{aligned}
&\int_{\tau-1}^{\tau+2}\int_\Omega P^{1+\theta}dxds\\
&\quad=\overline{P^\theta}\int_{\tau-1}^{\tau+2}\int_\Omega Pdxds+\int_\Omega u\mathcal{B}\left(P^\theta-\overline{P^\theta}\right)dx\\
&\quad\quad-\int_\Omega u_0\mathcal{B}\left(P_0^\theta-\overline{P_0^\theta}\right)dx+\int_{\tau-1}^{\tau+2}\int_\Omega u\mathcal{B}\left(div\left(P^\theta u\right)\right)dxds\\
&\quad\quad+(\gamma\theta-1)\int_{\tau-1}^{\tau+2}\int_\Omega u\mathcal{B}\left(P^\theta divu-\overline{P^\theta divu}\right)dxds\\
&\quad\quad+\int_{\tau-1}^{\tau+2}\int_\Omega \nabla u^i\nabla\mathcal{B}^i\left(P^\theta-\overline{P^\theta}\right)dxds\\
&\quad\quad-\int_{\tau-1}^{\tau+2}\int_\Omega \rho\nabla f\mathcal{B}\left(P^\theta-\overline{P^\theta}\right)dxds. \qquad (3.8)
\end{aligned}$$

We can estimate each of the terms on the right hand side of (3.8) as follows:
First, we use Lemma 2.5 and (3.4) to derive that

$$\begin{aligned}
&\left|\int_{\tau-1}^{\tau+2}\int_\Omega u\mathcal{B}\left(div\left(P^\theta u\right)\right)dxds\right|\\
&\quad\le C\int_{\tau-1}^{\tau+2}\left\|P^\theta\right\|_{L^{1/\theta}}\|u\|^2_{L^{2/(1-\theta)}}ds\\
&\quad\le C, \qquad (3.9)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\tau-1}^{\tau+2}\int_\Omega u\mathcal{B}\left(P^\theta divu-\overline{P^\theta divu}\right)dxds\right|\\
&\quad\le C\int_{\tau-1}^{\tau+2}\|u\|_{L^{1/(1-\theta)}}\|P\|^\theta_{L^1}\|Du\|_{L^2}ds\\
&\quad\le C; \qquad (3.10)
\end{aligned}$$

Next, (3.3), Lemma 2.5 and Hölder's inequality lead to

$$\begin{aligned}
&\left|\int_{\tau-1}^{\tau+2}\int_\Omega \nabla u^i\nabla\mathcal{B}^i\left(P^\theta-\overline{P^\theta}\right)dxds\right|\\
&\quad\le C\int_{\tau-1}^{\tau+2}\|Du\|_{L^2}\|P\|^\theta_{L^{2\theta}}ds\\
&\quad\le C, \qquad (3.11)
\end{aligned}$$

owing to $\theta<1/2$. Finally, (3.1) and (3.3) yield that

$$\begin{aligned}
&\left|\int_{\tau-1}^{\tau+2}\int_\Omega \rho\nabla f\mathcal{B}\left(P^\theta-\overline{P^\theta}\right)dxds\right|\\
&\quad\le C\int_{\tau-1}^{\tau+2}\|\rho\|_{L^1}\left\|P^\theta\right\|_{L^{1/\theta}}ds\\
&\quad\le C. \qquad (3.12)
\end{aligned}$$

(3.7) thus follows easily from (3.8)-(3.12).

Proof of Lemma 3.1. We deduce from (3.3) and (3.4) that

$$\begin{aligned}\int_0^\infty \|\rho u\|^2_{L^{(\gamma+1)/2}(\Omega)} ds &\le C\int_0^\infty \|\rho\|^2_{L^\gamma(\Omega)}\|u\|^2_{L^{\gamma(\gamma+1)/(\gamma-1)}(\Omega)} ds\\ &\le C\int_0^\infty \|Du\|^2_{L^2(\Omega)} ds\\ &\le C.\end{aligned}$$

Thus, for any $1 < p < \infty$, it holds that

$$\lim_{\tau\to\infty}\int_{\tau-1}^{\tau+2}\left(\|\rho u\|^2_{L^{(\gamma+1)/2}(\Omega)} + \|u\|^2_{L^p(\Omega)} + \|Du\|^2_{L^2}\right) ds = 0. \tag{3.13}$$

Consider a sequence $\tau_n \to \infty$, and define

$$\rho_n(x,t) = \rho(x, t+\tau_n), u_n(x,t) = u(x, t+\tau_n), t \in (-1, 2), x \in \Omega.$$

We shall prove that (3.5) holds in two steps. The first one is to show

$$\lim_{n\to\infty} \|\rho_n - \rho_s\|_{L^\gamma(\Omega\times(0,1))} = 0. \tag{3.14}$$

By virtue of (3.3), there is a subsequence $\tau_n \to \infty$ such that

$$\rho_n \to \widehat{\rho} \text{ weakly in } L^\gamma(\Omega \times (-1, 2)). \tag{3.15}$$

Moreover, (3.7) leads to

$$P(\rho_n) \to \widehat{P} \text{ weakly in } L^{p_1}(\Omega \times (-1, 2)) \text{ for } 1 < p_1 < (\gamma + \theta)/\gamma.$$

In view of (3.13), it is easy to pass to the limit in the continuity equation (1.3) to deduce that $\widehat{\rho}$ must be independent of t. Moreover, passing to the limit in (1.4) and using (3.13), we get

$$\nabla\widehat{P} = \widehat{\rho}\nabla f \text{ in } \mathcal{D}'(\Omega), \int_\Omega \widehat{\rho}dx = \int_\Omega \rho_0 dx. \tag{3.16}$$

Consequently, since P is a strictly increasing function of ρ, it is enough to show that the convergence in (3.15) is, in fact, strong.

To this end, we set

$$G(z) = z^\alpha \text{ for } 0 < \alpha < \min\left\{\frac{1}{2\gamma}, \frac{\theta}{\theta+\gamma}\right\}.$$

Consider the vector functions

$$[G(P(\rho_n)), 0, 0] \text{ and } [P(\rho_n), 0, 0]$$

of the time variable t and the spatial coordinates x. Noticing that $G(P(\rho_n))$ satisfy

$$G(P(\rho_n))_t = -div\,(G(P(\rho_n))u_n) - (\gamma\alpha - 1)G(P(\rho_n))divu_n,$$

we can use (3.7) to get

$$Div_{t,x}\,[G(P(\rho_n)), 0, 0] \text{ precompact in } W^{-1,q_1}(\Omega \times (-1, 2))$$

with $q_1 - 1 > 0$ small enough, where

$$Div_{t,x}(V_0, V_1, V_2) \triangleq (V_0)_t + \partial_{x_1} V_1 + \partial_{x_2} V_2.$$

Similarly, we get from (1.4) and (3.13) that

$$Curl_{t,x}\left[P(\rho_n), 0, 0\right] \text{ precompact in } \left[W^{-1,q_2}(\Omega \times (-1,2))\right]^9,$$

where

$$(Curl_{t,x}(V_0, \cdots, V_3))_{ij} \triangleq \partial_{x_i} V_j - \partial_{x_j} V_i, x_0 \triangleq t, i, j = 0, 1, 2.$$

Finally, we can assume that

$$G(P(\rho_n)) \to \widehat{G(P)} \text{ weakly in } L^{p_2}(\Omega \times (-1,2)),$$

and

$$G(P(\rho_n))P(\rho_n) \to \widehat{G(P)P} \text{ weakly in } L^r(\Omega \times (-1,2)),$$

with

$$p_2 = \frac{1}{\alpha}, \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} < 1.$$

Using the L^p-version of div-curl lemma of Murat [26] and Tartar [35] (see also Zhou [38]), we deduce that

$$\widehat{G(P)}\widehat{P} = \widehat{G(P)P}. \tag{3.17}$$

As G is strictly monotone, (3.17) yields that

$$\widehat{G(P)} = G\left(\widehat{P}\right).$$

Thus, we get the strong convergence in (3.15) easily. Moreover, we infer from (3.16) that

$$\widehat{\rho} \equiv \rho_s.$$

This finishes the proof of (3.14).

The second step is to show that (3.5) holds. Since ρ_s is unique, we get that for any $\tau \to \infty$, the shifts $\rho_\tau(t) = \rho(t+\tau)$ converge to the steady state ρ_s, specifically,

$$\rho_\tau \to \rho_s \text{ strongly in } L^\gamma(\Omega \times (0,1)) \text{ as } \tau \to \infty.$$

On the other hand, since $E'(t) \leq 0$, the energy $E(t)$ converges to a finite contant for large time:

$$E_\infty \triangleq ess \lim_{t \to \infty} E(t).$$

Thus,

$$E_\infty = \lim_{\tau_n \to \infty} \int_{\tau_n}^{\tau_n + 1} \int_\Omega \left(\frac{1}{\gamma - 1}\rho^\gamma - f\rho\right) dxdt = \int_\Omega \left(\frac{1}{\gamma - 1}\rho_s^\gamma - f\rho_s^\gamma\right) dx.$$

Moreover, the continuity equation (1.3) easily yields that

$$\rho(t) \to \rho_s \text{ weakly in } L^\gamma(\Omega) \text{ as } t \to \infty.$$

Consequently,

$$
\begin{aligned}
E_\infty &= \int_\Omega \left(\frac{1}{\gamma-1}\rho_s^\gamma - f\rho_s \right) dx \\
&\leq \liminf_{t\to\infty} \int_\Omega \left(\frac{1}{\gamma-1}\rho^\gamma(t) - f\rho(t) \right) dx \\
&\leq \limsup_{t\to\infty} \int_\Omega \left(\frac{1}{\gamma-1}\rho^\gamma(t) - f\rho(t) \right) dx \\
&\leq ess \limsup_{t\to\infty} \int_\Omega \left(\frac{1}{2}|u|^2 + \frac{1}{\gamma-1}\rho^\gamma(t) - f\rho(t) \right) dx \\
&= ess \lim_{t\to\infty} E(t) \\
&= E_\infty.
\end{aligned}
$$

The proof of Lemma 3.1 is completed.

Next, our main analysis is the following estimates on the density and pressure deviation, η and Q, which are defined by

$$
\eta \triangleq \rho - \rho_s, Q \triangleq \rho^\gamma - \rho_s^\gamma
$$

respectively. We get from (1.8) that $\nabla f = \gamma \rho_s^{\gamma-2}\nabla\rho_s$. Thus (1.4) is equivalent to

$$
u_t - \Delta u + \nabla\left(\rho^\gamma - \rho_s^\gamma\right) - \gamma(\rho-\rho_s)\rho_s^{\gamma-2}\nabla\rho_s = 0. \tag{3.18}
$$

Noticing that

$$
\begin{aligned}
&\rho_s^{-1}\left(\nabla\left(\rho^\gamma - \rho_s^\gamma\right) - \gamma(\rho-\rho_s)\rho_s^{\gamma-2}\nabla\rho_s\right) \\
&\quad = \nabla\left(\rho_s^{-1}(\rho^\gamma-\rho_s^\gamma)\right) - (\rho^\gamma-\rho_s^\gamma)\nabla\rho_s^{-1} + \gamma\rho_s^{\gamma-3}\left(\rho_s-\rho\right)\nabla\rho_s \\
&\quad = \nabla\left(\rho_s^{-1}(\rho^\gamma-\rho_s^\gamma)\right) + \left((\rho^\gamma-\rho_s^\gamma) - \gamma\rho_s^{\gamma-1}(\rho-\rho_s)\right)\rho_s^{-2}\nabla\rho_s \\
&\quad = \nabla\left(\rho_s^{-1}(\rho^\gamma-\rho_s^\gamma)\right) + \gamma(\gamma-1)\rho_s^{-2}\nabla\rho_s\eta^2\int_0^1\int_0^1 \sigma(\rho_s+\sigma\lambda\eta)^{\gamma-2}d\sigma d\lambda,
\end{aligned}
$$

multiplying (3.18) by $\rho_s^{-1}\mathcal{B}(\eta)$, integrating over $\Omega\times(0,t)$, we get after integration by parts and using Lemma 2.5 that

$$
\begin{aligned}
&\int_0^t\int_\Omega \rho_s^{-1}Q\eta dxds \\
&\quad = \int_\Omega \rho_s^{-1}u\mathcal{B}(\eta)dx - \int_\Omega \rho_s^{-1}u_0\mathcal{B}(\rho_0-\rho_s)dx \\
&\qquad + \int_0^t\int_\Omega \rho_s^{-1}u\left(\mathcal{B}(div(\eta u)) + \mathcal{B}(div(\rho_s u))\right)dxds \\
&\qquad + \int_0^t\int_\Omega \nabla u^i\left(\rho_s^{-1}\nabla\mathcal{B}^i(\eta) + \mathcal{B}^i(\eta)\nabla\rho_s^{-1}\right)dxds \\
&\qquad + \gamma(\gamma-1)\int_0^t\int_\Omega \frac{\nabla\rho_s}{\rho_s^2}\eta^2\int_0^1\int_0^1 \sigma(\rho_s+\sigma\lambda\eta)^{\gamma-2}d\sigma d\lambda\mathcal{B}(\eta)dxds.
\end{aligned} \tag{3.19}
$$

The terms on the right hand side of (3.19) can be estimated separately as follows:

Lemma 2.5 and (3.3) give that

$$
\left|\int_\Omega \rho_s^{-1}u\mathcal{B}(\eta)dx\right| \leq C\|u\|_{L^2}\|\eta\|_{L^\gamma} \leq C; \tag{3.20}
$$

Lemma 2.5 and (3.4) lead to

$$\begin{aligned}
&\left|\int_0^t\int_\Omega \rho_s^{-1}u\left(\mathcal{B}(div(\eta u))+\mathcal{B}(div(\rho_s u))\right)dxds\right|\\
&\quad\le C\int_0^t\left(\|u\|_{L^4}^2\|\eta\|_{L^2}+\|u\|_{L^2}^2\right)ds\\
&\quad\le C_\varepsilon+\varepsilon\int_0^t\|\eta\|_{L^2}^2ds,
\end{aligned}\tag{3.21}$$

and

$$\begin{aligned}
&\left|\int_0^t\int_\Omega \nabla u^i\left(\rho_s^{-1}\nabla\mathcal{B}^i(\eta)+\mathcal{B}^i(\eta)\nabla\rho_s^{-1}\right)dxds\right|\\
&\quad\le C\int_0^t\|Du\|_{L^2}\|\eta\|_{L^2}ds\\
&\quad\le C_\varepsilon+\varepsilon\int_0^t\|\eta\|_{L^2}^2ds.
\end{aligned}\tag{3.22}$$

It follows easily from Lemma 2.5 that

$$\begin{aligned}
\|\mathcal{B}(\eta)\|_{L^\infty} &\le C\|\eta\|_{L^{1+\gamma}}\\
&\le C\|\eta\|_{L^1}^{1-\theta}\|\eta\|_{L^{2\gamma}}^{\theta}\\
&\le C\|\eta\|_{L^1}^{1-\theta}\left\||Q|^{1/\gamma}+\rho_s\right\|_{L^{2\gamma}}^{\theta}\\
&\le C\delta^{1-\theta}\left(\|Q\|_{L^2}^2+1\right),
\end{aligned}$$

with $\theta=2\gamma^2/((\gamma+1)(2\gamma-1))\in(0,1)$. This yields that

$$\begin{aligned}
&\left|\int_0^t\int_\Omega \rho_s^{-2}\nabla\rho_s\eta^2\int_0^1\int_0^1\sigma(\rho_s+\sigma\lambda\eta)^{\gamma-2}d\sigma d\lambda\mathcal{B}(\eta)dxds\right|\\
&\quad\le C\int_0^t\|\mathcal{B}(\eta)\|_{L^\infty}\int_\Omega Q\eta dxds\\
&\quad\le C\delta^{1-\theta}\left(\sup_{0\le s\le t}\|Q\|_{L^2}^2+1\right)\int_0^t\int_\Omega Q\eta dxds.
\end{aligned}\tag{3.23}$$

We deduce from (3.19)-(3.23) by letting δ be small enough that

$$\int_0^t\int_\Omega\left(Q\eta+\eta^2\right)dxds\le C+C\delta^{1-\theta}\sup_{0\le s\le t}\|Q\|_{L^2}^2\int_0^t\int_\Omega Q\eta dxds,\tag{3.24}$$

where we use the fact $Q\eta\ge\rho_s^{\gamma-1}\eta^2\ge C\eta^2$.

Thus, we have to estimate the term $\sup_{0\le s\le t}\|Q\|_{L^2}^2$. Note that Q satisfies

$$Q_t+div(Qu)+div(\rho_s^\gamma u)+(\gamma-1)Qdivu+(\gamma-1)\rho_s^\gamma divu=0.\tag{3.25}$$

We multiply the equation (3.25) by Q, then integrate the result over both space and time to obtain

$$\|Q(\cdot,t)\|_{L^2}^2\le C\left(\int_0^t\|Q\|_{L^4}^4ds\right)^{1/2}+\frac{1}{2}\int_0^t\|Q\|_{L^2}^2ds+C.\tag{3.26}$$

In order to estimate the second term on the right hand side of (3.26), we denote by $S \triangleq divu - Q$ and integrate (3.25) over $\Omega \times (0,t)$ to get

$$\begin{aligned}
&\int_\Omega \rho^\gamma dx + (\gamma-1)\int_0^t\int_\Omega Q^2 dxds \\
&\quad = \int_\Omega \rho_0^\gamma dx - (\gamma-1)\int_0^t\int_\Omega QS dxds - (\gamma-1)\int_0^t\int_\Omega \eta u\cdot\nabla f dxds \\
&\quad\quad -(\gamma-1)\int_\Omega f\rho_0 dx + (\gamma-1)\int_\Omega f\rho dx \\
&\quad \le \frac{\gamma-1}{2}\int_0^t\int_\Omega Q^2 dxds + C\int_0^t\int_\Omega S^2 dxds + C\int_0^t\int_\Omega \eta^2 dxds + C \\
&\quad \le \frac{\gamma-1}{2}\int_0^t\int_\Omega Q^2 dxds + C\int_0^t \|DS\|_{L^2}^2 ds + C\int_0^t\int_\Omega \left(Q\eta + \eta^2\right) dxds + C, \quad (3.27)
\end{aligned}$$

where the first equality is due to

$$\begin{aligned}
&\int_0^t\int_\Omega \rho_s^\gamma divu dxds \\
&\quad = -\int_0^t\int_\Omega u\cdot\nabla\rho_s^\gamma dxds \\
&\quad = -\int_0^t\int_\Omega \rho_s u\cdot\nabla f dxds \\
&\quad = \int_0^t\int_\Omega \eta u\cdot\nabla f dxds - \int_0^t\int_\Omega \rho u\cdot\nabla f dxds \\
&\quad = \int_0^t\int_\Omega \eta u\cdot\nabla f dxds + \int_0^t\int_\Omega f div(\rho u) dxds \\
&\quad = \int_0^t\int_\Omega \eta u\cdot\nabla f dxds - \int_0^t\int_\Omega f\rho_t dxds \\
&\quad = \int_0^t\int_\Omega \eta u\cdot\nabla f dxds - \int_\Omega f\rho dx + \int_\Omega f\rho_0 dx,
\end{aligned}$$

and in the third inequality we have used the following two facts:

$$\|S\|_{L^2}^2 \le C\|DS\|_{L^2}^2 + C\overline{Q}^2 \tag{3.28}$$

since $\overline{S} = -\overline{Q}$, and

$$\begin{aligned}
\int_0^t \overline{Q}^2 ds &= |\Omega|^{-2}\int_0^t\left(\int_\Omega Q\eta^{-1}\eta dx\right)^2 ds \\
&\le C\int_0^t\int_\Omega Q\eta^{-1}dx\int_\Omega Q\eta dxds \\
&\le C\int_0^t\int_\Omega\int_0^1 (\lambda\rho + (1-\lambda)\rho_s)^{\gamma-1} d\lambda dx\int_\Omega Q\eta dxds \\
&\le C\int_0^t\left(\|\rho\|_{L^\gamma}^{\gamma-1} + \|\rho_s\|_{L^\gamma}^{\gamma-1}\right)\int_\Omega Q\eta dxds \\
&\le C\int_0^t\int_\Omega Q\eta dxds. \quad (3.29)
\end{aligned}$$

Thus, (3.27) leads to

$$\begin{aligned}&\int_\Omega \rho^\gamma dx+\int_0^t\int_\Omega Q^2dxds\\&\qquad\le C+C\int_0^t\|DS\|_{L^2}^2ds+C\int_0^t\int_\Omega\left(Q\eta+\eta^2\right)dxds.\end{aligned}\tag{3.30}$$

Adding (3.30) to (3.26) yields that

$$\begin{aligned}&\|Q\|_{L^2}^2+\int_0^t\int_\Omega Q^2dxds\\&\qquad\le C\left(\int_0^t\|Q\|_{L^4}^4ds\right)^{1/2}+C\int_0^t\int_\Omega\left(Q\eta+\eta^2\right)dxds\\&\qquad\quad+C\int_0^t\|DS\|_{L^2}^2ds+C.\end{aligned}\tag{3.31}$$

It remains to estimate the terms on the right hand side of (3.31).

First, multiplying (3.25) by Q^2 and integrating the result over $\Omega\times(0,t)$, one can obtain after integrating by parts and Hölder's inequality and (3.3) that

$$\|P(t)\|_{L^3}^3+\int_0^t\|Q(s)\|_{L^4}^4ds\le C+C\int_0^t\|S\|_{L^4}^4ds,\tag{3.32}$$

where one has used the simple fact that $Q^3\ge\rho^{3\gamma}/2-C\rho_s^{3\gamma}$.

Next, (3.31) and (3.32) show that we have to estimate both S and DS. Using the notations $curlu=\partial_{x_1}u^2-\partial_{x_2}u^1$ and $\nabla^\perp=(\partial_{x_2},-\partial_{x_1})^{\mathrm{T}}$, one can rewrite (1.4) as

$$u_t-\nabla S+\nabla^\perp curlu=\eta\nabla f.\tag{3.33}$$

We multiply (3.33) by $-\nabla S$ and integrate the result over Ω to get after integration by parts

$$\begin{aligned}&\left(\|S\|_{L^2}^2\right)_t+\|DS\|_{L^2}^2\\&\quad\le-2\int_\Omega Q_tSdx+C\|\eta\|_{L^2}^2\\&\quad\le-2\int_\Omega Qu\nabla Sdx-2\int_\Omega\rho_s^\gamma u\nabla Sdx\\&\qquad+C\int_\Omega\left(|Q||Du||S|+|Du||S|\right)dx+C\|\eta\|_{L^2}^2\\&\quad\le-2\int_\Omega Quu_tdx+C\int_\Omega|Qu|\left(|Dcurlu|+|\eta|\right)dx\\&\qquad+C\int_\Omega\left(|u||DS|+|Du||S|\right)dx+C\int_\Omega|Q||Du||S|dx+C\|\eta\|_{L^2}^2\\&\quad\le-2\int_\Omega Quu_tdx+\varepsilon\|Qu\|_{L^2}^2+C_\varepsilon\|Dcurlu\|_{L^2}^2+C_\varepsilon\|\eta\|_{L^2}^2+\frac14\|DS\|_{L^2}^2\\&\qquad+C\left(\|u\|_{L^2}^2+\|Du\|_{L^2}^2\right)+C\overline{Q}^2+C\int_\Omega\left(|Du|^2|S|+|Du|S^2\right)dx,\end{aligned}\tag{3.34}$$

where we have used (3.25), (3.33), and (3.28) for the second, third, and last inequality respectively.

By using (3.25), the first term on the right hand side of (3.34) can be estimated by

$$\begin{aligned}
&-2\int_\Omega Quu_t dx\\
&=-\left(\int_\Omega Q|u|^2dx\right)_t+\int_\Omega Q_t|u|^2dx\\
&\le -\left(\int_\Omega Q|u|^2dx\right)_t+C\int_\Omega |Du|^2|u|^2dx+C\int_\Omega |S||u|^2|Du|dx\\
&\quad +C\|Du\|_{L^2}^2+C\|u\|_{L^4}^4.
\end{aligned}\tag{3.35}$$

To estimate the third one, we notice that $curlu$ satisfies

$$\begin{cases} curlu_t-\Delta curlu=curl(\eta\nabla f),\\ curlu|_{\partial\Omega}=0.\end{cases}\tag{3.36}$$

Multiplying (3.36) by $curlu$, and then integrating the result over $\Omega\times(0,t)$ lead to

$$\|curlu\|_{L^2}^2+\int_0^t\|Dcurlu\|_{L^2}^2ds\le C+C\int_0^t\|\eta\|_{L^2}^2ds. \tag{3.37}$$

To estimate the second term on the right hand side of (3.35), we multiply the equation (1.4) by $|u|^2u$ and integrating the result over $\Omega\times(0,t)$ to get

$$\begin{aligned}
&\sup_{0\le s\le t}\|u\|_{L^4}^4+\int_0^t\|D(|u|u)\|_{L^2}^2ds\\
&\le C\int_0^t\int_\Omega\left(|u|^2|Du||S|+|Du|^2|u|^2\right)dxds+C\lambda\left(\int_0^t\|u\|_{L^8}^8ds\right)^{1/2}+C_\lambda\int_0^t\|\eta\|_{L^2}^2ds\\
&\le C\int_0^t\int_\Omega\left(|u|^2|Du||S|+|Du|^2|u|^2\right)dxds\\
&\quad +C\lambda\left(\sup_{0\le s\le t}\|u\|_{L^4}^4+\int_0^t\|D(|u|u)\|_{L^2}^2ds\right)+C_\lambda\int_0^t\|\eta\|_{L^2}^2ds,
\end{aligned}\tag{3.38}$$

due to the following Gagliardo-Nirenberg inequality

$$\left(\int_0^t\|u\|_{L^8}^8ds\right)^{1/2}\le C\sup_{0\le s\le t}\|u\|_{L^4}^4+C\int_0^t\|D(|u|u)\|_{L^2}^2ds,$$

since $|u|^2u\cdot n|_{\partial\Omega}=0$. Choosing λ small enough in (3.38) leads to

$$\begin{aligned}
&\sup_{0\le s\le t}\|u\|_{L^4}^4+\int_0^t\|D(|u|u)\|_{L^2}^2ds\\
&\quad\le C\int_0^t\int_\Omega\left(|u|^2|Du||S|+|Du|^2|u|^2\right)dxds+C\int_0^t\|\eta\|_{L^2}^2ds.
\end{aligned}\tag{3.39}$$

We derive from (3.34), (3.35), (3.37), (3.39) and (3.31) that

$$\begin{aligned}
&\int_\Omega\left(|u|^4+|S|^2+\rho^\gamma|u|^2+\varepsilon|Q|^2\right)dx+\int_0^t\left(\varepsilon\|Q\|_{L^2}^2+\|DS\|_{L^2}^2\right)ds\\
&\le C_\varepsilon\int_0^t\int_\Omega\left(|Du|S^2+|u|^2|Du||S|+|Du|^2|S|+|Du|^2|u|^2\right)dxds+C_\varepsilon\\
&\quad +C\varepsilon\left(\int_0^t\|Q\|_{L^4}^4ds\right)^{1/2}+C\varepsilon\int_0^t\|DS\|_{L^2}^2ds+C_\varepsilon\int_0^t\int_\Omega\left(Q\eta+\eta^2\right)dxds.
\end{aligned}\tag{3.40}$$

The first term on the right hand side of (3.40) can be estimated as follows: First, since $\overline{S} = -\overline{Q}$, Lemma 2.4 yields that

$$\|S\|_{L^4}^2 \le C\|S\|_{L^2}\|DS\|_{L^2} + C\|DS\|_{L^2} + C\overline{Q}^2. \tag{3.41}$$

Thus, Hölder's inequality and (3.29) lead to

$$\begin{aligned}
&\int_0^t \left\|DuS^2(\cdot,s)\right\|_{L^1} ds \\
&\quad \le \int_0^t \|Du\|_{L^2}\|S\|_{L^4}^2 ds \\
&\quad \le C\int_0^t \|Du\|_{L^2}\left(\|S\|_{L^2}\|DS\|_{L^2} + \|DS\|_{L^2} + \overline{Q}^2\right) ds \\
&\quad \le \varepsilon\int_0^t \|DS\|_{L^2}^2 ds + C_\varepsilon\int_0^t \|Du\|_{L^2}^2\|S\|_{L^2}^2 ds + C\int_0^t\int_\Omega Q\eta dxds + C.
\end{aligned} \tag{3.42}$$

Next, using (3.41) and (3.3), we have

$$\begin{aligned}
&\int_0^t\int_\Omega \left(|Du|^2|S| + |u|^2|Du||S|\right) dxds \\
&\quad \le \int_0^t \left(\|Du\|_{L^2}\|Du\|_{L^4}\|S\|_{L^4} + \|u\|_{L^4}^2\|Du\|_{L^4}\|S\|_{L^4}\right) ds \\
&\quad \le C\int_0^t (\|Du\|_{L^2}\|Du\|_{L^4}\|S\|_{L^4} + \|u\|_{L^2}\|Du\|_{L^2}\|Du\|_{L^4}\|S\|_{L^4}) ds \\
&\quad \le C\int_0^t \|Du\|_{L^2}\|Du\|_{L^4}\left(\|S\|_{L^2}^{1/2}\|DS\|_{L^2}^{1/2} + \|DS\|_{L^2}^{1/2} + \left|\overline{Q}\right|\right) ds \\
&\quad \le C\left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/4}\left(\int_0^t \|Du\|_{L^2}^{4/3}\left(\|S\|_{L^2}^{2/3}\|DS\|_{L^2}^{2/3} + \overline{Q}^{4/3}\right) ds\right)^{3/4} \\
&\qquad +\varepsilon\left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/2} + \varepsilon\int_0^t \|DS\|_{L^2}^2 ds + C_\varepsilon \\
&\quad \le C\left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/4}\left(\int_0^t \|DS\|_{L^2}^2 ds\right)^{1/4}\left(\int_0^t \|Du\|_{L^2}^2\|S\|_{L^2} ds\right)^{1/2} \\
&\qquad +C\left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/4}\left(\int_0^t \|Du\|_{L^2}^2 ds\right)^{1/2}\left(\int_0^t \overline{Q}^4 ds\right)^{1/4} \\
&\qquad +\varepsilon\left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/2} + \varepsilon\int_0^t \|DS\|_{L^2}^2 ds + C_\varepsilon \\
&\quad \le C\varepsilon\left(\int_0^t \|Du\|_{L^4}^4 ds\right)^{1/2} + C\varepsilon\int_0^t \|DS\|_{L^2}^2 ds + C_\varepsilon\int_0^t \|Du\|_{L^2}^2\|S\|_{L^2}^2 ds \\
&\qquad +C_\varepsilon\int_0^t\int_\Omega Q\eta dxds + C_\varepsilon.
\end{aligned} \tag{3.43}$$

And also,

$$
\begin{aligned}
&\int_0^t \int_\Omega |u|^2 |Du|^2 dxds \\
&\quad \le \int_0^t \||u|u\|_{L^4} \|Du\|_{L^4} \|Du\|_{L^2} ds \\
&\quad \le C \int_0^t \||u|u\|_{L^2}^{1/2} \|uDu\|_{L^2}^{1/2} \|Du\|_{L^4} \|Du\|_{L^2} ds \\
&\quad \le \frac{1}{2} \int_0^t \|uDu\|_{L^2}^2 ds + C \int_0^t \|u\|_{L^4}^{4/3} \|Du\|_{L^4}^{4/3} \|Du\|_{L^2}^{4/3} ds,
\end{aligned}
$$

consequently,

$$
\begin{aligned}
&\int_0^t \int_\Omega |u|^2 |Du|^2 dxds \\
&\quad \le C \int_0^t \|u\|_{L^4}^{4/3} \|Du\|_{L^4}^{4/3} \|Du\|_{L^2}^{4/3} ds \\
&\quad \le C \left(\int_0^t \|Du\|_{L^4}^4 ds \right)^{1/3} \left(\int_0^t \|u\|_{L^4}^2 \|Du\|_{L^2}^2 ds \right)^{2/3} \\
&\quad \le \varepsilon \left(\int_0^t \|Du\|_{L^4}^4 ds \right)^{1/2} + C_\varepsilon \left(\int_0^t \|u\|_{L^4}^2 \|Du\|_{L^2}^2 ds \right)^{2} \\
&\quad \le \varepsilon \left(\int_0^t \|Du\|_{L^4}^4 ds \right)^{1/2} + C_\varepsilon \int_0^t \|u\|_{L^4}^4 \|Du\|_{L^2}^2 ds + C_\varepsilon. \qquad (3.44)
\end{aligned}
$$

Finally, Lemma 2.2 yields that

$$
\left(\int_0^t \|Du\|_{L^4}^4 ds \right)^{1/2} \le C + C \left(\int_0^t \|Q\|_{L^4}^4 ds \right)^{1/2} + C \int_0^t \|\eta\|_{L^2}^2 ds. \qquad (3.45)
$$

It follows from (3.40), (3.42)-(3.45) that

$$
\begin{aligned}
&\int_\Omega \left(|u|^4 + |S|^2 + \rho^\gamma |u|^2 + \varepsilon |Q|^2 \right) dx + \int_0^t \left(\varepsilon \|Q\|_{L^2}^2 + \|DS\|_{L^2}^2 \right) ds \\
&\quad \le C\varepsilon \left(\int_0^t \|Q\|_{L^4}^4 ds \right)^{1/2} + C\varepsilon \int_0^t \|DS\|_{L^2}^2 ds + C_\varepsilon \int_0^t \int_\Omega \left(Q\eta + \eta^2 \right) dxds \\
&\qquad + C_\varepsilon \int_0^t \|Du\|_{L^2}^2 \left(\|u\|_{L^2}^2 + \|u\|_{L^4}^4 \right) ds + C_\varepsilon \\
&\quad \le C\varepsilon \left(\int_0^t \|S\|_{L^2}^2 \|DS\|_{L^2}^2 ds \right)^{1/2} + C_\varepsilon \int_0^t \int_\Omega \left(Q\eta + \eta^2 \right) dxds \\
&\qquad + C\varepsilon \int_0^t \|DS\|_{L^2}^2 ds + C_\varepsilon \int_0^t \|Du\|_{L^2}^2 \left(\|u\|_{L^2}^2 + \|u\|_{L^4}^4 \right) ds + C_\varepsilon. \qquad (3.46)
\end{aligned}
$$

Choosing ε small enough in (3.46) and using (3.24) yield that

$$\begin{aligned}
&\int_\Omega \left(|u|^2+|u|^4+|S|^2+|Q|^2\right)dx+\int_0^t\int_\Omega \left(Q\eta+Q^2+|DS|^2\right)dxds\\
&\quad\le C\int_0^t\int_\Omega \left(Q\eta+\eta^2\right)dxds+C\int_0^t \|Du\|_{L^2}^2\left(\|u\|_{L^2}^2+\|u\|_{L^4}^4\right)ds+C\\
&\quad\le C+C\delta^{1-\theta}\sup_{0\le s\le t}\|Q\|_{L^2}^2\int_0^t\int_\Omega Q\eta dxds\\
&\qquad+C\int_0^t \|Du\|_{L^2}^2\left(\|u\|_{L^2}^2+\|u\|_{L^4}^4\right)ds+C.
\end{aligned}$$

Gronwall's inequality thus gives that

$$\sup_{0\le s\le t}\left(\|S\|_{L^2}^2+\|Q\|_{L^2}^2+\|u\|_{L^4}^4\right)(s)+\int_0^t\left(\|DS\|_{L^2}^2+\|Q\|_{L^2}^2\right)ds\le C, \tag{3.47}$$

due to (3.3). This estimate, together with (3.41), (3.32) and (3.45), yields that

$$\sup_{0\le s\le T}\|P(\cdot,s)\|_{L^3}^3+\int_0^T\left(\|Q\|_{L^2}^2+\|Q\|_{L^4}^4+\|Du\|_{L^4}^4\right)ds\le C. \tag{3.48}$$

With (3.48) at hand, we can use Lemma 2.3 to get the uniform upper bound for ρ. To this end, we need some elementary estimates first. Denote by φ and ψ the unique functions such that $u=\nabla\varphi+\nabla^\perp\psi$. Then

$$\begin{cases}\Delta\varphi=divu,\\ \frac{\partial\varphi}{\partial n}\Big|_{\partial\Omega}=0,\displaystyle\int_\Omega\varphi dx=0;\end{cases}$$

$$\begin{cases}\Delta\psi=-curlu,\\ \psi|_{\partial\Omega}=0;\end{cases}$$

Similarly, we can define the unique functions φ_0 and ψ_0 with respect to u_0.

Thus,

$$\|D\varphi\|_{L^p}\le C\|u\|_{L^p},$$

for $1<p<\infty$. Since $\overline{\varphi}=0$, this estimate, together with (3.47) and Poincáre's inequality, gives that

$$\sup_{0\le s\le t}\|\varphi\|_{L^\infty}\le C\sup_{0\le s\le t}\|u\|_{L^4}\le C. \tag{3.49}$$

We use (3.47) and the Gagliardo-Nirenberg inequality to get

$$\begin{aligned}
\|\nabla\varphi(t)\|_{L^\infty}&\le C\|D\varphi(t)\|_{L^4}^{1/2}\|D^2\varphi(t)\|_{L^4}^{1/2}\\
&\le C\|u(t)\|_{L^4}^{1/2}\|Du(t)\|_{L^4}^{1/2}\\
&\le C\|Du(t)\|_{L^4}^{1/2},
\end{aligned}$$

due to $\nabla\varphi\cdot n|_{\partial\Omega}=0$. This estimate and (3.48) thus yield that

$$\int_0^T\|D\varphi\|_{L^\infty}^8 ds\le C.$$

Since $u \cdot n|_{\partial\Omega} = 0$, it follows from (3.47), (3.48) and the Poincaré-Sobolev inequality that

$$\int_0^T \|u\|_{L^\infty}^8 ds \le C.$$

Consequently,

$$\int_0^T \|u^j \partial_j \varphi\|_{L^\infty}^4 ds \le C. \tag{3.50}$$

(3.33) is equivalent to

$$\nabla\left(\varphi_t - S\right) + \nabla^\perp\left(\psi_t + curl u\right) = \eta \nabla f. \tag{3.51}$$

Hence, $g \triangleq \psi_t + curl u$ satisfies

$$\begin{cases} \Delta g = -curl(\eta \nabla f), \\ g|_{\partial\Omega} = 0. \end{cases}$$

Standard L^p-theory of elliptic equations leads to

$$\int_0^T \|Dg\|_{L^4}^4 ds \le C \int_0^T \|\eta\|_{L^4}^4 ds.$$

This estimate, (3.51) and (3.48) yield that

$$\int_0^T \|D(\varphi_t - S)\|_{L^4}^4 ds \le C \int_0^T \|\eta\|_{L^4}^4 ds \le C.$$

Since $\overline{\varphi} = 0$, this estimate and Poincáre's inequality give that

$$\int_0^T \left\|\varphi_t - S + \overline{S}\right\|_{L^\infty}^4 ds \le C \int_0^T \|D(\varphi_t - S)\|_{L^4}^4 ds \le C. \tag{3.52}$$

Set $D_t w = w_t + u \cdot \nabla w$. Using (3.3), we conclude from (1.3) that

$$\begin{aligned} D_t\left(\log P + \gamma\varphi\right) &= -\gamma P + \gamma\left(\varphi_t - S + \overline{S}\right) + \gamma\overline{Q} + \gamma u \cdot \nabla\varphi + \gamma\rho_s^\gamma \\ &\le -\gamma P + \gamma\left(\varphi_t - S + \overline{S}\right) + \gamma u \cdot \nabla\varphi + C. \end{aligned} \tag{3.53}$$

Now, we express (3.53) in terms of the Lagrangian coordinates and take $y = \log P$, $g(y) = -\gamma e^y$, and $b(t) = b_1(t) - b_0(t)$ where

$$b_1(t) = \gamma \int_0^t \left(\left(\varphi_t - S + \overline{S}\right) + u^j \partial_j \varphi\right) ds + Ct \text{ and } b_0(t) = \gamma\varphi.$$

Thus, (3.50) and (3.52) yield that for $0 \le t_1 < t_2 \le T$,

$$\begin{aligned} &|b_1(t_2) - b_1(t_1)| \\ &\quad \le \gamma \int_{t_1}^{t_2} \left(\left\|\varphi_t - S + \overline{S}\right\|_{L^\infty} + \left\|u^j \partial_j \varphi\right\|_{L^\infty}\right) ds + C(t_2 - t_1) \\ &\quad \le C \int_0^t \left(\left\|\varphi_t - S + \overline{S}\right\|_{L^\infty}^4 + \left\|u^j \partial_j \varphi\right\|_{L^\infty}^4\right) ds + C(t_2 - t_1) \\ &\quad \le C + C(t_2 - t_1). \end{aligned}$$

while (3.49) gives that

$$\sup_{0\le s\le T} |b_0(s)| \le C.$$

Hence, we have

$$|b(t_2) - b(t_1)| \le C + C(t_2 - t_1). \tag{3.54}$$

Due to estimate (3.54), the uniform upper bounds for $\log P$ and consequently for ρ follow from Lemma 2.3. This finishes the proof of (1.12).

The combination of (1.12) with (3.5) shows that the first part of (1.13) holds.

Next, we shall prove the second part of (1.13), i.e.,

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{W^{1,p}} = 0, \text{ for any } p \in [1,\infty).$$

It suffices to show that

$$\lim_{t\to\infty} \|Du(\cdot,t)\|_{L^p} = 0, \text{ for any } p \in [2,\infty), \tag{3.55}$$

due to (3.3) and Hölder's and Poincaré-Sobolev's inequalities.

One deduces from Lemma 2.2, (1.12) and (3.48) that

$$\int_0^\infty \left(\|u\|_{L^p}^p + \|Du\|_{L^p}^p + \|S\|_{L^p}^p\right) ds \le C, \tag{3.56}$$

for any $p \in [2,\infty)$.

This estimate, together with Lemma 2.2 and (3.36), leads to

$$\int_0^\infty \left(\|curlu\|_{L^p}^p + \|Dcurlu\|_{L^p}^p\right) ds \le C, \tag{3.57}$$

for arbitrary $p \in [2,\infty)$.

It follows easily from (3.36) that

$$\begin{aligned} &\left(\|curlu(\cdot,t)\|_{L^p}^p\right)' \\ &\quad = -p\int_\Omega |curlu|^{p-2}\Big((p-2)\,|D\left(|curlu|\right)|^2 + |Dcurlu|^2\Big)\,dx \\ &\qquad +p\int_\Omega \eta\nabla f\nabla^\perp\left(|curlu|^{p-2}curlu\right)dx, \end{aligned} \tag{3.58}$$

for any $p \in [2,\infty)$. Using (3.57) and (3.56), we deduce from (3.58) that

$$\int_0^\infty \left|\left(\|curlu\|_{L^p}^p\right)'\right| ds \le C.$$

This estimate, together with (3.57), gives that

$$\lim_{t\to\infty} \|curlu(\cdot,t)\|_{L^p} = 0, \tag{3.59}$$

for any $p \in [2,\infty)$.

Next, we show

$$\lim_{t\to\infty} \|S(\cdot,t)\|_{L^p} = 0, \text{ for all } p \in [2,\infty). \tag{3.60}$$

For $p \geq 2$, multiplying (3.33) by $-\nabla\left(|S|^{p-2}S\right)$, then integrating the resulting identity over space, one gets after integration by parts

$$\begin{aligned}
&\frac{1}{p}\left(\|S(\cdot,t)\|_{L^p}^p\right)' + \int_\Omega |S|^{p-2}\left(|DS|^2 + (p-2)|D(|S|)|^2\right)dx \\
&= \int_\Omega Pu\cdot\nabla\left(|S|^{p-2}S\right)dx - (\gamma-1)\int_\Omega P divu|S|^{p-2}Sdx \\
&\quad - \int_\Omega \eta\nabla f\nabla\left(|S|^{p-2}S\right)dx \\
&\triangleq I_1 + I_2 + I_3.
\end{aligned} \tag{3.61}$$

We use (1.12) to estimate $I_i (i = 1,2,3)$ as follows:

$$\begin{aligned}
|I_1| &\leq C\int_\Omega P|u||S|^{p-2}|DS|dx \\
&\leq \lambda\int_\Omega |S|^{p-2}|DS|^2dx + C_\lambda\int_\Omega \left(|u|^p + |S|^p\right)dx,
\end{aligned} \tag{3.62}$$

$$|I_2| \leq C\int_\Omega \left(|Du|^p + |S|^p\right)dx, \tag{3.63}$$

and

$$\begin{aligned}
|I_3| &\leq C\int_\Omega |\eta||S|^{p-2}|DS|dx \\
&\leq \lambda\int_\Omega |S|^{p-2}|DS|^2dx + C_\lambda\int_\Omega \left(|S|^p + |\eta|^p\right)dx.
\end{aligned} \tag{3.64}$$

Collecting all these estimates (3.62)-(3.64), using (3.61) and (3.56), we choose λ small enough to deduce that

$$\int_0^\infty \left|\left(\|S(\cdot,t)\|_{L^p}^p\right)'\right| dt \leq C.$$

The combination of this estimate with (3.56) yields (3.60).

(1.6) implies that $\|Du(\cdot,t)\|_{L^p}$ can be estimated by

$$\begin{aligned}
\|Du(\cdot,t)\|_{L^p} &\leq C\left(\|curlu(\cdot,t)\|_{L^p} + \|divu(\cdot,t)\|_{L^p}\right) \\
&\leq C\left(\|curlu(\cdot,t)\|_{L^p} + \|S(\cdot,t)\|_{L^p} + \|Q(\cdot,t)\|_{L^p}\right).
\end{aligned} \tag{3.65}$$

It follows from (1.12), (3.5), (3.59) and (3.60) that the right hand side of (3.65) goes to 0 as $t \to \infty$. Hence, (3.55) holds.

4 Proof of Theorem 1.2

With the basic facts (1.12) and (1.13) in Theorem 1.1 at hand, we can establish the Theorem 1.2 easily in this section.

Proof of Theorem 1.2. Otherwise, there exist some $C_0 > 0$ and a subsequence $\{t_{n_j}\}_{j=1}^{\infty}$, $t_{n_j} \to \infty$ such that $\left\|\nabla\rho\left(\cdot, t_{n_j}\right)\right\|_{L^q(\Omega)} \leq C_0$.

Hence, the Poincaré-Sobolev inequality yields that there exists some positive constant C independent of t_{n_j} such that for $a = q/(2(q-1)) \in (0,1)$,

$$\begin{aligned}
&\left\|\rho(x, t_{n_j}) - \rho_s\right\|_{C(\overline{\Omega})} \\
&\quad \leq C \left\|\left|\nabla\rho(x, t_{n_j})\right| + |\nabla\rho_s(x)|\right\|_{L^q(\Omega)}^{a} \left\|\rho(x, t_{n_j}) - \rho_s\right\|_{L^2(\Omega)}^{1-a} \\
&\quad \leq C C_0^a \left\|\rho(x, t_{n_j}) - \rho_s\right\|_{L^2(\Omega)}^{1-a}, \qquad (4.1)
\end{aligned}$$

due to the basic fact that $\overline{\rho}(t) \equiv \overline{\rho_s}$, for all $t \geq 0$. We deduce from (1.13) that the right hand side of (4.1) goes to 0 as $t_{n_j} \to \infty$. Hence,

$$\left\|\rho(x, t_{n_j}) - \rho_s\right\|_{C(\overline{\Omega})} \to 0 \text{ as } t_{n_j} \to \infty. \qquad (4.2)$$

On the other hand, for $T > 0$, we introduce the Lagrangian coordinates which are defined as initial data to the Cauchy problem:

$$\begin{cases} \frac{\partial}{\partial s} X(s;t,x) = u(X(s;t,x), s) & 0 \leq s \leq T, \\ X(t;t,x) = x & 0 \leq t \leq T, x \in \overline{\Omega}. \end{cases} \qquad (4.3)$$

(1.11) shows that the transformation (4.3) is well-defined. Consequently, on the one hand, we have

$$\rho(x,t) = \rho_0(X(0;t,x)) \exp\left\{-\int_0^t divu(X(s;t,x), s)ds\right\}; \qquad (4.4)$$

on the other hand, since, by assumption, there exists some point $x_0 \in \overline{\Omega}$ such that $\rho_0(x_0) = 0$, we get that there exists a $x_0(t) \in \overline{\Omega}$ such that $X(0;t,x_0(t)) = x_0$. Using (4.4), we deduce from (1.11) that

$$\rho(x_0(t), t) \equiv 0 \text{ for all } t \geq 0.$$

So, we conclude from this identity that

$$\begin{aligned}
\left\|\rho(x, t_{n_j}) - \rho_s\right\|_{C(\overline{\Omega})} &\geq \left|\rho\left(x_0\left(t_{n_j}\right), t_{n_j}\right) - \rho_s\left(x_0\left(t_{n_j}\right)\right)\right| \\
&= \rho_s\left(x_0\left(t_{n_j}\right)\right) \geq \inf_{x\in\overline{\Omega}} \rho_s(x) > 0,
\end{aligned}$$

which contradicts (4.2).

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