Vortex sheets with reflection symmetry in exterior domains

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Abstract

In this paper we prove the existence of a weak solution of the incompressible 2D Euler equations in the exterior of a reflection symmetric smooth bluff body with symmetric initial flow corresponding to vortex sheet type data whose vorticity is of distinguished sign on each side of the symmetry axis. This work extends a result proved for full plane flow by the authors in [6].

1 Introduction and Preliminaries

Let $D \subseteq \mathbb{R}^2$ be a smooth, bounded, simply connected domain with boundary $\partial D = \Gamma$. We assume that D is symmetric with respect to the horizontal coordinate axis. We will be studying the initial-boundary value problem for the incompressible 2D Euler equations in the exterior of D, denoted by $\Omega \equiv \mathbb{R}^2 \setminus D$. We will prove the existence of a weak solution of the incompressible 2D Euler equations in Ω with initial flow symmetric with respect with the horizontal axis, with distinguished sign vorticity on each side of the symmetry axis and with vortex sheet initial data. This work extends a similar result proved for full plane flow by the authors in [6].

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In [6] the authors proved the existence of a weak solution to the incompressible 2D Euler equations in the full plane with initial vorticity odd with respect to a straight line and with a distinguished sign on each side of the symmetry line. This is the only extension of Delort's existence theorem which includes flows with vortex sheets without distinguished sign.

The main ingredients in proving the existence result in [6] were the following facts:

(F1) An L^2_{loc} *a priori* estimate on velocity restricted to the symmetry axis (estimate (4) in [6]),

(F2) An estimate on the mass of vorticity near the symmetry axis in terms of the integral of velocity at the symmetry axis (estimate (5) in [6]),

(F3) Persistence of cancellation in the weak form of the nonlinearity up to the symmetry axis (expressed in identity (10) in [6]).

The proofs of (F1), (F2) and (F3), which ultimately concern half-plane flow, relied heavily on the fact that the boundary of the half-plane is a straight line.

Each one of these facts, especially (F3), has a certain independent interest, when regarded as information on the behavior of incompressible, ideal 2D flows near a straight rigid boundary. One of the motivations behind the present work is to show that these facts can be generalized to domains with curved boundaries. The existence result may be thus regarded as an application of (F1), (F2) and (F3).

We will consider the 2D Euler equations in vorticity form in an exterior domain. We must contend, however, with the fact that the system coupling velocity to vorticity, namely div $u = 0$, curl $u = \omega$, $u \cdot \hat{n} = 0$ and $|u(x,t)| \to 0$ at ∞, does not determine u uniquely in terms of ω. This is due to the nonvanishing homology of the exterior domain. This issue was examined in detail in [3], where it was shown that the velocity field is determined by the vorticity up to a harmonic vector field, called the harmonic part. In our problem we will assume that the initial velocity u_0 is mirror symmetric with respect to the horizontal axis. This implies two facts: (1) the vorticity is odd with respect to the variable x_2 and therefore its integral in Ω vanishes, and (2) the circulation of the initial velocity around $\partial\Omega$ vanishes. These facts, together with Lemma 3.1 in [3] imply that the harmonic part of the velocity must vanish. Consequently, we can write the Biot-Savart law expressing velocity in terms of vorticity in the following manner. Let $G_\Omega = G_\Omega(x, y)$ be the Greens function for the Laplacian in Ω, and set $K_\Omega \equiv \nabla_x^\perp G_\Omega$. With this notation the Biot-Savart law is given by:

$$u = u(x,t) = K_\Omega[\omega](x,t) \equiv \int_\Omega K_\Omega(x,y)\omega(y,t)\,dy. \tag{1}$$

We now write the vortex sheet initial data problem as:

$$\begin{cases} \omega_t + u \cdot \nabla\omega = 0 & \text{in } \Omega \times (0,\infty) \\ u = K_\Omega[\omega] & \text{in } \Omega \times (0,\infty) \\ \omega(x,0) = \omega_0(x) & \text{on } \Omega \times \{t = 0\}, \end{cases} \tag{2}$$

where $\hat{n}$ is the unit exterior normal to the boundary Γ.

Our main result will be the existence of weak solutions to (2) for ω_0 a bounded measure, odd with respect to mirror symmetry, nonnegative in the upper half-plane outside of D. We call $\mu \in \mathcal{BM}(\Omega)$ *nonnegative mirror symmetric* (*NMS*) if it is odd with respect to reflection about the horizontal axis and if it is nonnegative in $\Omega \cap \{x_2 \geq 0\}$.

Let us first define what we mean by weak solution in this context. We introduce $\mathcal{A}$, the set of admissible test functions, defined by:

$$\mathcal{A} \equiv \left\{\varphi \in C_c^\infty([0,\infty) \times \overline{\Omega}) \mid \varphi \equiv 0 \text{ on } \Gamma\right\}.$$

Definition 1 *The function $\omega \in L^\infty([0,\infty); \mathcal{BM}(\Omega))$ is called a weak solution of the incompressible 2D Euler equations with initial data ω_0 if:*

(a) the velocity $u \equiv K_\Omega[\omega]$ belongs to $L^\infty_{loc}([0,\infty); (L^2(\Omega))^2)$, and

(b) for any test function $\varphi \in \mathcal{A}$, it holds that

$$\begin{aligned} \mathcal{W}[\omega,\varphi] \equiv & \int_0^\infty \int_\Omega \varphi_t \omega(x,t) dx dt \\ & + \int_0^\infty \int_\Omega \int_\Omega H_\varphi^\Omega(x,y,t)\omega(x,t)\omega(y,t) dy dx dt \\ & + \int_\Omega \varphi(x,0)\omega_0(x) dx = 0, \end{aligned} \tag{3}$$

where

$$H_\varphi^\Omega(x,y,t) \equiv \frac{1}{2}(\nabla\varphi(x,t) \cdot K_\Omega(x,y) + \nabla\varphi(y,t) \cdot K_\Omega(y,x)). \tag{4}$$

Remark: This is stronger than the standard definition of weak solution, because usually test functions are required to be compactly supported inside Ω. The bounded domain version of Delort's Theorem guaranteed the existence of such a standard weak solution, see [2]. This stronger notion of weak solution was introduced by the authors in [6]. We used the expression *boundary coupled weak solutions* to designate solutions in the sense of Definition 1 in [6] to distinguish these weak solutions from the standard ones, but this will not be necessary here.

The strategy for obtaining a weak solution is to pass to the weak limit along a suitably constructed approximate solution sequence. The methods of constructing such a sequence of approximations in the context of the initial-value problem for the incompressible 2D Euler equations involve: smoothing out or truncating the initial vorticity, approximation by vanishing viscosity and the use of several numerical methods. Here we will obtain an approximate solution sequence by smoothing out initial data and we will use the available global well-posedness theory which can be found in [4]. Next we will observe that the symmetry of the problem is preserved under smooth flows. We denote the reflection about the horizontal axis by $x = (x_1, x_2) \mapsto \overline{x} = (x_1, -x_2)$.

Proposition 1 *Let $\omega_0 \in C_c^\infty(\Omega)$ be NMS and let $\omega = \omega(x,t)$ be the unique solution of the incompressible 2D Euler equations in Ω. Then ω is NMS for all $t \geq 0$.*

Proof: Define $\widetilde{\omega}(x,t) = -\omega(\overline{x},t)$. Then $\widetilde{\omega}(x,0) = \omega_0(x)$. As the Euler equations are covariant with respect to mirror symmetry it follows that $\widetilde{\omega}$ also satisfies the Euler equations in Ω. It follows from the uniqueness that $\widetilde{\omega}(x,t) = \omega(x,t)$ for all t. The sign condition is a consequence of the fact that vorticity is transported by the flow, that each half-plane is invariant under symmetric flow and of the hypothesis on the initial data.

∎

2 Non-concentration of vorticity at the boundary

We will begin with a reasonably straightforward generalization of the argument used in [6] to show non-concentration in mass of vorticity all the way up to the boundary. This argument consists of two Lemmas which are given below. In fact, it will be shown that versions of (F1) and (F2) hold on certain domains with curved boundaries.

Let $\Omega_+ \equiv \Omega \cap \{x_2 > 0\}$ and $\Gamma_+ = \partial\Omega_+$.

Lemma 1 *Let $\omega = \omega(x,t)$ be the solution of (2) with smooth, compactly supported, and NMS initial vorticity. Let $\varphi = \varphi(x)$ be a smooth function on Ω_+ with bounded derivatives up to second order. Then the following identity holds:*

$$\frac{d}{dt}\int_{\Omega_+} \varphi(x)\omega(x,t)\,dx = \frac{1}{2}\int_{\Gamma_+} |u\cdot\hat{n}^\perp|^2 \nabla\varphi\cdot\hat{n}^\perp\,dS$$

$$+\int_{\Omega_+} \left[\left((u_1)^2-(u_2)^2\right)\varphi_{x_1x_2} - u_1u_2(\varphi_{x_1x_1}-\varphi_{x_2x_2})\right]\,dx.$$

Proof: We will prove this identity by direct computations. It holds that

$$\mathcal{I} \equiv \frac{d}{dt}\int_{\Omega_+} \varphi(x)\omega(x,t)\,dx$$

$$= \int_{\Omega_+} \varphi\omega_t\,dx = -\int_{\Omega_+} \varphi \text{ div } (u\omega)\,dx = \int_{\Omega_+} (\nabla\varphi\cdot u)\omega\,dx,$$

where the boundary terms have disappeared since u is tangent to Γ_+ (due to symmetry) and bounded everywhere and hence ω has compact support at each fixed time. Re-write $\omega = -\text{div } u^\perp$ and integrate by parts once more to obtain:

$$\mathcal{I} = \int_{\Omega_+} \nabla(\nabla\varphi\cdot u)\cdot u^\perp\,dx - \int_{\Gamma_+} (\nabla\varphi\cdot u)(u^\perp\cdot\hat{n})dS,$$

where again the boundary terms at infinity have vanished, this time because $|u|$ decays sufficiently fast at infinity. Indeed, $|u| = \mathcal{O}(|x|^{-2})$ for large $|x|$, see the discussion in section 2.2 of [3] for a proof. Next, observe that

$$\nabla(\nabla\varphi\cdot u)\cdot u^\perp = \nabla\left(\frac{|u|^2}{2}\right)\cdot\nabla^\perp\varphi + \left((u_1)^2-(u_2)^2\right)\varphi_{x_1x_2} - u_1u_2(\varphi_{x_1x_1}-\varphi_{x_2x_2}).$$

Therefore, this vector calculus identity yields

$$\mathcal{I} = \int_{\Omega_+}\left[\nabla\left(\frac{|u|^2}{2}\right)\cdot\nabla^\perp\varphi + \left((u_1)^2-(u_2)^2\right)\varphi_{x_1x_2} - u_1u_2(\varphi_{x_1x_1}-\varphi_{x_2x_2})\right]\,dx$$

$$- \int_{\Gamma_+} (\nabla\varphi \cdot u)(u^\perp \cdot \hat{n})\, dS$$

$$= \int_{\Omega_+} \left[\left((u_1)^2 - (u_2)^2 \right) \varphi_{x_1 x_2} - u_1 u_2 (\varphi_{x_1 x_1} - \varphi_{x_2 x_2}) \right] \, dx$$

$$+ \int_{\Gamma_+} \left(\frac{|u|^2}{2} \nabla^\perp \varphi - (\nabla\varphi \cdot u) u^\perp \right) \cdot \hat{n}\, dS,$$

where, once again, we have used the decay of $|u|$ at infinity. Finally, using the fact that the velocity u is tangent to the boundary, one can compute that there is a simpler expression for the boundary term:

$$\left(\frac{|u|^2}{2} \nabla^\perp \varphi - (\nabla\varphi \cdot u) u^\perp \right) \cdot \hat{n} = \frac{1}{2} |u \cdot \hat{n}^\perp|^2 \nabla\varphi \cdot \hat{n}^\perp.$$

This concludes the proof. In fact, since kinetic energy is finite initially, conserved exactly for smooth flows and φ has been assumed to have bounded derivatives up to second order, it follows that the expression on the right-hand-side of the identity we have just proved is finite and integrable in time. ■

We now use this identity to deduce an *a priori* estimate for the L^2_{loc}-norm (on $\Gamma_+ \times (0, \infty)$) of the tangential component of velocity, namely, a generalization of (F1). For the sake of convenience, we assume that Γ_+ is the graph of a piecewise smooth, compactly supported function $\gamma = \gamma(x_1)$. In this case we will use in Lemma 1 the function $\varphi(x) = \arctan(x_1)$. Note that, for this test function, for each compact subset $\mathcal{K}$ of Γ_+ there exists $\widetilde{C} > 0$ such that $\nabla\varphi \cdot \hat{n}^\perp \geq \widetilde{C}$ a.e. on $\mathcal{K}$. Indeed, this follows easily from the observations that $\nabla\varphi = ((1 + x_1^2)^{-1}, 0)$ and that $\widehat{n}^\perp = (1, 0)$ on the straight portion of Γ_+ and $\widehat{n}^\perp = (1 + (\gamma'(x_1))^2)^{-1/2}(1, \gamma'(x_1))$ on the curved portion of Γ_+. We then obtain that, for every $\mathcal{K} \subset\subset \Gamma_+$ and for every $T > 0$ there exists $C > 0$, depending only on $\mathcal{K}$, T, $\|\omega_0\|_{L^1(\Omega)}$ and $\|u_0\|_{L^2(\Omega)}$ such that:

$$\int_0^T \int_{\mathcal{K}} |u|^2 \, dS dt \leq C. \tag{5}$$

This is (the generalization of) (F1). To verify (5) we estimate directly:

$$\int_0^T \int_{\mathcal{K}} |u|^2 \, dS dt = \int_0^T \int_{\mathcal{K}} |u \cdot \hat{n}^\perp|^2 \, dS dt$$

$$
\begin{aligned}
&\leq \frac{1}{\widetilde{C}} \int_0^T \int_{\mathcal{K}} |u \cdot \hat{n}^\perp|^2 \nabla\varphi \cdot \hat{n}^\perp \, dSdt, \\
&= \frac{2}{\widetilde{C}} \left(\int_{\Omega_+} \varphi(x)\omega(x,T)\, dx - \int_{\Omega_+} \varphi(x)\omega_0(x)\, dx \right. \\
&\left. + \int_0^T \int_{\Omega_+} \left[\left((u_2)^2 - (u_1)^2 \right) \varphi_{x_1x_2} + u_1u_2(\varphi_{x_1x_1} - \varphi_{x_2x_2}) \right] \, dx \right) \\
&\leq \frac{2}{\widetilde{C}} \left(2T\|\varphi\|_{L^\infty(\Omega_+)}\|\omega_0\|_{L^1(\Omega_+)} + T\|D^2\varphi\|_{L^\infty(\Omega_+)}\|u_0\|^2_{L^2(\Omega_+)} \right),
\end{aligned}
$$

where $D^2\varphi$ stands for a generic second derivative of φ. In the last inequality we have used the fact that smooth incompressible Euler flows preserve the mass of vorticity and kinetic energy. It follows also from the symmetry that $\|\omega_0\|_{L^1(\Omega)} = 2\|\omega_0\|_{L^1(\Omega_+)}$ and $\|u_0\|^2_{L^2(\Omega)} = 2\|u_0\|^2_{L^2(\Omega_+)}$.

Finally, we obtain the nonconcentration result on the mass of vorticity up to the boundary.

Lemma 2 *Let $\omega_0 \in C_c^\infty(\Omega)$ be NMS and let $\omega = \omega(x,t)$, $u = K_\Omega[\omega]$ be the solution to (2) with initial data ω_0. For each $T > 0$ and each compact set $\mathcal{K} \subseteq \overline{\Omega}$ there exists a constant $C > 0$ such that for any $0 < \delta < 1$,*

$$
\int_0^T \left(\sup_{x\in\mathcal{K}} \int_{B(x;\delta)\cap\Omega} |\omega(y,t)|\, dy \right) dt \leq C|\log\delta|^{-1/2}.
$$

Proof:

Fix $\mathcal{K} \subseteq \overline{\Omega}$ and $0 < \delta < 1$. We make use of the following cut-off function, also used by S. Schochet in [11]:

$$
\eta_\delta(z) = \begin{cases} 1, \text{ if } |z| \leq \delta \\[2ex] \frac{\log(|z|)-\log(\sqrt{\delta})}{\log(\sqrt{\delta})}, \text{ if } \delta \leq |z| \leq \sqrt{\delta} \\[2ex] 0, \text{ if } |z| \geq \sqrt{\delta}. \end{cases}
$$

Note that for $x \in \mathcal{K}$:

$$
\int_{B(x;\delta)\cap\Omega} |\omega(y,t)|\, dy = \int_{B(x;\delta)\cap\Omega_+} \omega(y,t)\, dy - \int_{B(x;\delta)\cap(\Omega\setminus\Omega_+)} \omega(y,t)\, dy.
$$

Each integral above can be estimated by using the fact that ω has a distinguished sign in each of Ω_+ and $\Omega \setminus \Omega_+$. Indeed, for the first integral, one has

$$\int_{B(x;\delta)\cap\Omega_+} \omega(y,t)\,dy \le \int_{B(x;\sqrt{\delta})\cap\Omega_+} \eta_\delta(x-y)\omega(y,t)\,dy$$

$$= \int_{B(x;\sqrt{\delta})\cap\Omega_+} \nabla_y \eta_\delta(x-y)\cdot u^\perp(y,t)\,dy + \int_{B(x;\sqrt{\delta})\cap\Gamma_+} \eta_\delta(x-y) u(y,t)\cdot \hat{n}^\perp(y)\,dS,$$

by integrating by parts. Note that the other boundary terms vanish since $\eta_\delta(x-y) = 0$ for $y \in \partial B(x;\sqrt{\delta})$. Therefore,

$$\int_{B(x;\delta)\cap\Omega_+} \omega(y,t)\,dy \le C|\log\delta|^{-1/2}\|u_0\|^2_{L^2(\Omega_+)}$$

$$+ \left(\int_{B(x;\sqrt{\delta})\cap\Gamma_+} |u\cdot\hat{n}^\perp|^2\,dS\right)^{1/2} |B(x;\sqrt{\delta})\cap\Gamma_+|^{1/2}.$$

Finally, since the boundary Γ_+ was assumed to be piecewise smooth it follows easily that $|B(x;\sqrt{\delta})\cap\Gamma_+| \le C\sqrt{\delta}$, which, together with (5), yields the desired estimate of the first integral.

The estimate of the second integral follows in an analogous way.

∎

It should be noted that Γ_+ was assumed to be a graph of a piecewise smooth, compactly supported function. This simplified the derivation of (5) by allowing us to explicitly produce an appropriate test function φ. However, this hypothesis is not needed and the derivation of (5) can be obtained, for example, through deformation of Γ_+ into a graph. Lemma 2 is the curved domain generalization of (F2).

3 Desingularization of the nonlinearity

Let $\omega_0 \in \mathcal{BM}_c(\Omega)$ be NMS and assume that $u_0 \equiv K_\Omega[\omega_0] \in L^2(\Omega)$. To produce a weak solution to (2) with initial data ω_0, it is a key step to study the concentration-cancellation effects of the nonlinearity in the Euler equations. To this end, we will show the persistence of cancellation in the weak form of the nonlinearity up to the symmetry axis, (F3). We begin by considering a smooth approximation of the initial data. Let ω_0^n be a sequence in $C_c^\infty(\Omega)$ such that

1. $\omega_0^n \rightharpoonup \omega_0$ weak-$*$ in $\mathcal{BM}(\Omega)$,
2. $\|\omega_0^n\|_{L^1(\Omega)}$ and $\|u_0^n \equiv K_\Omega[\omega_0^n]\|_{L^2(\Omega)}$ are uniformly bounded with respect to n,
3. ω_0^n is NMS.

One way of building such an sequence of approximations is to solve the heat equation in Ω with ω_0 as initial data for time $1/n$ and then smoothly truncate near infinity.

Let $\omega^n = \omega^n(x,t)$ be the smooth solution of (2) with $u^n = K_\Omega[\omega^n]$ and initial vorticity ω_0^n, given by Kikuchi's Theorem, see [4]. Of course, since ω_0^n is NMS we have that ω^n is NMS as well for all n and t. Therefore the conclusion of Lemma 2 can be re-formulated as a uniform *a priori* estimate on the mass of vorticity in small balls. For any $T > 0$ and any compact set $\mathcal{K} \subset\subset \overline{\Omega}$ there exists a constant $C > 0$ such that, for all n,

$$\int_0^T \left(\sup_{x\in\mathcal{K}} \int_{B(x;\delta)\cap\Omega} |\omega^n(y,t)|\, dy \right) dt \leq C|\log\delta|^{-1/2}. \tag{6}$$

We wish to pass to the limit in the weak formulation of (2) given in Definition 1 for this approximate solution sequence. The crucial step is to pass the limit in the nonlinearity. To do so we will need to establish the boundedness of the auxiliary function H_φ^Ω, for $\varphi \in \mathcal{A}$, where H_φ^Ω was defined in (4). This is the content of the Theorem below.

Theorem 1 *Let $\varphi \in \mathcal{A}$. Then there exists $C > 0$ such that*

$$|H_\varphi^\Omega(x,y,t)| \leq C,$$

for all x, $y \in \overline{\Omega}$ and $t \in [0,\infty)$.

Proof: Note that, as φ has compact support in $\overline{\Omega} \times [0,\infty)$, it is enough to prove the boundedness of H_φ^Ω in a compact set $\mathcal{K} \subset\subset \overline{\Omega}$ and on a finite interval $[0,T]$. Re-write H_φ^Ω as:

$$H_\varphi^\Omega(x,y,t) = \frac{1}{2}(\nabla\varphi(x,t) - \nabla\varphi(y,t)) \cdot K_\Omega(x,y)$$

$$+\frac{1}{2}\nabla\varphi(y,t) \cdot (K_\Omega(x,y) + K_\Omega(y,x)) \equiv \mathcal{I} + \mathcal{J}.$$

We will use the basic framework developed in Sections 2.1 and 2.2 of [3]. Let $U = \{|x| > 1\}$ and let $T : \Omega \to U$ be the biholomorphic mapping given in Lemma 2.1 of [3]. The mapping T induces a diffeomorphism between Γ and $\{|x| = 1\}$. Recall that the Biot-Savart kernel K_Ω can be explicitly expressed using this mapping in the following manner:

$$K_\Omega(x,y) = \frac{((T(x)-T(y))DT(x))^\perp}{2\pi|T(x)-T(y)|^2} - \frac{((T(x)-(T(y))^*)DT(x))^\perp}{2\pi|T(x)-(T(y))^*|^2}, \tag{7}$$

where $z \mapsto z^* = z/|z|^2$ is the inversion with respect to the unit circle. Note that $K_\Omega(x,y) = (DT(x))^t K_U(T(x), T(y))$. Next we recall an estimate obtained in Section 2.2 of [3], namely,

$$|K_\Omega(x,y)| \le C \frac{|T(y)-(T(y))^*|}{|T(x)-T(y)||T(x)-(T(y))^*|},$$

for some constant $C > 0$. It is easy to see that, for each $z \in U$ fixed, we have, for any $w \in U$,

$$\frac{|z-z^*|}{|w-z^*|} \le \frac{|z-z^*|}{|(z/|z|)-z^*|} = |z| + 1.$$

Hence,

$$|K_\Omega(x,y)| \le \frac{C}{|T(x)-T(y)|} \le \frac{C}{|x-y|},$$

for all $(x,y) \in \mathcal{K} \times \mathcal{K}$, since DT and its inverse are bounded. This implies that $\mathcal{I}$ is bounded.

Next we re-write $\mathcal{J}$ in the following manner:

$$\mathcal{J} = \frac{1}{2}\nabla\varphi(y,t)[(DT(x))^t - (DT(y))^t]K_U(T(x),T(y))]$$

$$+\frac{1}{2}\nabla\varphi(y,t)(DT(y))^t[K_U(T(x),T(y)) + K_U(T(y),T(x))] \equiv \mathcal{J}_1(x,y,t) + \mathcal{J}_2(x,y,t).$$

As before we find that

$$|K_U(T(x),T(y))| \le \frac{C}{|x-y|},$$

for x, y in $\mathcal{K}$, so that, since D^2T is also bounded, we conclude that $\mathcal{J}_1$ is bounded in $\mathcal{K} \times \mathcal{K}$.

We are left with the estimate of $\mathcal{J}_2$, which is the heart of the matter. We will need the following claim.

Claim: If $y \in \Gamma$ then $\mathcal{J}_2(x, y, t) \equiv 0$.

Proof of Claim: Let $y \in \Gamma$. For each $\theta \in [0, 2\pi)$ let $\mathcal{C}_\theta \equiv T^{-1}(\{re^{i\theta} \,|\, r \in (1, \infty)\})$. Of course, $\mathcal{C}_\theta$ is a smooth curve in Ω, naturally parametrized by $r \in (1, \infty)$. Let $A(z) \equiv \arg(T(z))$. Note that $A(z) = \theta$ for $z \in \mathcal{C}_\theta$. Therefore, ∇A is orthogonal to the family of curves $\mathcal{C}_\theta$ and we have that

$$\nabla A(z) = (DT(z))^t \frac{(T(z))^\perp}{|T(z)|^2}. \tag{8}$$

As φ is admissible, the boundary Γ is a level curve of φ and hence $\nabla\varphi(y, t)$ is orthogonal to Γ. On the other hand, the curves $\mathcal{C}_\theta$ are also orthogonal to Γ because T is conformal and $T(\mathcal{C}_\theta)$ is a straight ray perpendicular to $T(\Gamma) = \{|z| = 1\}$. Therefore,

$$\nabla\varphi(y, t) \cdot \nabla A(y) = 0. \tag{9}$$

Let z and w be points in the plane such that $|z| \geq 1$ and $|w| \geq 1$. We use (7), with T being the identity, and a straightforward calculation to obtain

$$K_U(z, w) + K_U(w, z) = -\frac{1}{2\pi}\left\{\frac{(|w|^2 - 1)z^\perp}{|w|^2|z - w^*|^2} + \frac{(|z|^2 - 1)w^\perp}{|z|^2|w - z^*|^2}\right\}. \tag{10}$$

Since $y \in \Gamma$, we have that $|T(y)| = 1$, and therefore,

$$K_U(T(x), T(y)) + K_U(T(y), T(x)) = -\frac{1}{2\pi}\frac{(|T(x)|^2 - 1)(T(y))^\perp}{|T(x)|^2|T(y) - (T(x))^*|^2}. \tag{11}$$

Putting together (9), (11) and (8) it follows that $\mathcal{J}_2(x, y, t) \equiv 0$ for $y \in \Gamma$, which concludes the proof of the Claim.

For $x \neq 0$ in the plane, we write $\widehat{x} = x/|x|$. Let $(x, y) \in \mathcal{K} \times \mathcal{K}$. First we observe that, for z and w with $|z| \geq 1$ and $|w| \geq 1$ we have the following elementary fact

$$|w||z - w^*| = |z||w - z^*|. \tag{12}$$

Next, using (10) and (12) we write

$$-4\pi\mathcal{J}_2(x,y,t) = \nabla\varphi(y,t)(DT(y))^t \frac{(|T(y)|^2-1)(T(x))^\perp}{|T(x)|^2|T(y)-(T(x))^*|^2}$$

$$+\nabla\varphi(y,t)(DT(y))^t \frac{(|T(x)|^2-1)(T(y))^\perp}{|T(y)|^2|T(x)-(T(y))^*|^2} \equiv \mathcal{M}_1 + \mathcal{M}_2.$$

Let us first estimate $\mathcal{M}_1$. Using the Claim above we find:

$$\mathcal{M}_1 = \nabla\varphi(y,t)(DT(y))^t \frac{(|T(y)|^2-1)(T(x))^\perp}{|T(x)|^2|T(y)-(T(x))^*|^2}$$

$$= [\nabla\varphi(y,t) - \nabla\varphi(T^{-1}(\widehat{T(x)}),t)](DT(y))^t \frac{(|T(y)|^2-1)(T(x))^\perp}{|T(x)|^2|T(y)-(T(x))^*|^2}$$

$$+\nabla\varphi(T^{-1}(\widehat{T(x)}),t)[(DT(y))^t - (DT(T^{-1}(\widehat{T(x)}))^t]\frac{(|T(y)|^2-1)(T(x))^\perp}{|T(x)|^2|T(y)-(T(x))^*|^2},$$

as $T^{-1}(\widehat{T(x)}) \in \Gamma$. Therefore, there exists $C > 0$, depending only on φ and T and their derivatives up to second order, such that

$$|\mathcal{M}_1| \leq C|y - T^{-1}(\widehat{T(x)})| \left| \frac{(|T(y)|^2-1)(T(x))^\perp}{|T(x)|^2|T(y)-(T(x))^*|^2} \right|.$$

Next we note that $|T(y)|^2 - 1 = (T(y) - \widehat{T(x)})(T(y) + \widehat{T(x)})$, so that, since $|T(x)|$ and $|T(y)|$ are bounded for x and y in $\mathcal{K}$, it follows that

$$|\mathcal{M}_1| \leq C\frac{|y - T^{-1}(\widehat{T(x)})||T(y) - \widehat{T(x)}|}{|T(y)-(T(x))^*|^2}$$

$$\leq C\frac{|T(y) - \widehat{T(x)}|^2}{|T(y)-(T(x))^*|^2},$$

as T^{-1} is a diffeomorphism with bounded derivative in $\mathcal{K}$. We conclude by observing that

$$|T(y) - (T(x))^*| \geq |\widehat{T(x)} - (T(x))^*|, \tag{13}$$

as $\widehat{T(x)}$ is the point in $\overline{U}$ closest to $(T(x))^*$, and

$$|T(y) - \widehat{T(x)}|^2 \leq 2(|T(y) - (T(x))^*|^2 + |(T(x))^* - \widehat{T(x)}|^2).$$

These inequalities allow us to conclude that $|\mathcal{M}_1| \leq 4C$.

Next we estimate $\mathcal{M}_2$. We have, once again using the Claim proved above,

$$\mathcal{M}_2 = \nabla\varphi(y,t)(DT(y))^t \frac{(|T(x)|^2-1)(T(y))^\perp}{|T(y)|^2|T(x)-(T(y))^*|^2}$$

$$= [\nabla\varphi(y,t) - \nabla\varphi(T^{-1}(\widehat{T(y)}),t)](DT(y))^t \frac{(|T(x)|^2-1)(T(y))^\perp}{|T(y)|^2|T(x)-(T(y))^*|^2}$$

$$+\nabla\varphi(T^{-1}(\widehat{T(y)}),t)[DT(y))^t - (DT(T^{-1}(\widehat{T(y)}))^t] \frac{(|T(x)|^2-1)(T(y))^\perp}{|T(y)|^2|T(x)-(T(y))^*|^2},$$

as $T^{-1}(\widehat{T(y)}) \in \Gamma$. Therefore, as before, we find

$$|\mathcal{M}_2| \leq C \frac{|T(y) - \widehat{T(y)}||T(x) - \widehat{T(y)}|}{|T(x)-(T(y))^*|^2},$$

for some constant $C > 0$ depending on φ, T, T^{-1}, their derivatives up to second order, and the diameter of $T(\mathcal{K})$. Using again (13) we obtain

$$|\mathcal{M}_2| \leq C \frac{|T(y) - \widehat{T(y)}||T(x) - \widehat{T(y)}|}{|\widehat{T(y)} - (T(y))^*||T(x)-(T(y))^*|}$$

$$\leq C \frac{|T(x)-(T(y))^*| + |(T(y))^* - \widehat{T(y)}|}{|T(x)-(T(y))^*|} \leq 2C.$$

This concludes the proof.

∎

Remark: We observe that the proof of Theorem 1 can easily be adapted to a simply connected bounded domain.

Next we discuss the relation between this result and (F3). Let us begin by considering the case $\Omega = \mathbb{R}^2$. The nonlinearity in the vorticity equation (2) has the form $u\,\omega = K_{\mathbb{R}^2}[\omega]\,\omega$, where

$$K_{\mathbb{R}^2}[\omega] = K * \omega \equiv \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)\,dy.$$

If $\omega \in L^p$, $1 < p < 2$, then, by the Hardy-Littlewood-Sobolev inequality, u belongs to L^{p^*}, with $p^* = 2p/(p-2)$. The naïve condition needed to make sense of $u\,\omega$ is therefore $p \geq 4/3$. However, due to the antisymmetry of the kernel $K_{\mathbb{R}^2}$, we easily deduce that, for any test function $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R}^2)$, the auxiliary function $H_\varphi^{\mathbb{R}^2}$ is smooth away from $x = y$ and globally bounded, see [2], Proposition 1.2.3 and see [11] for an alternative proof. The weak form of the nonlinearity, which is

$$\int_{\mathbb{R}^2} \nabla\varphi(x)u(x)\omega(x)\,dx = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_\varphi^{\mathbb{R}^2}(x,y)\omega(x)\omega(y)\,dxdy,$$

clearly makes sense for any $\omega \in L^1$, and even for ω a continuous measure in $\mathcal{BM}$. It is the boundedness of $H_\varphi^{\mathbb{R}^2}$ which we refer to as cancellation in the weak form of the nonlinearity.

For domains with boundary, the kernel K_Ω is no longer antisymmetric. Nevertheless, in the case of bounded domains, Delort observed that H_φ^Ω is still bounded if φ is compactly supported in the interior of Ω, see the comment following identity (2.3.12) in [2]. In [6] we proved that, if Ω is the half-plane $\mathbb{H}$, then $H_\varphi^{\mathbb{H}}$ is bounded for all $\varphi \in C_c^\infty(\overline{\mathbb{H}})$, with $\varphi = 0$ on $\partial\mathbb{H}$. This is what we are calling *persistence* of cancellation in the weak form of the nonlinearity up to the boundary, i.e. (F3). Theorem 1 is thus a generalization of (F3) to domains with curved boundaries.

The boundedness of H_φ^Ω is a key ingredient in the proof of existence of weak solution of (2) with vortex sheet initial data, see [2, 5, 7, 11, 12], and the stronger version where φ is not required to be compactly supported was used in [6] for the same purpose. The boundedness of H_φ^Ω can be also be useful in other problems regarding (2), see for example, the proof of Theorem 2.1 in [3] and the discussion following Definition 1.1 in [9]. Moreover, a very similar idea applies to the Vlasov-Poisson system and was used in [8, 10]. In conclusion, there is broad potential applicability for Theorem 1, beyond the existence result which we will present in the next section.

4 Existence of a weak solution and concluding remarks

We are now ready to state and prove our main result, extending the half-plane existence result in [6] to exterior domains.

Theorem 2 *Let D be a closed bounded region of the plane with smooth boundary and symmetric with respect to reflection about the horizontal coordinate axis and let $\Omega = \mathbb{R}^2 - D$. Let $\omega_0 \in \mathcal{BM}_c^+(\Omega)$, be NMS and such that $u_0 = K_\Omega[\omega_0] \in (L^2(\Omega))^2$. Then there exists a weak solution of the 2D incompressible Euler equations in Ω with initial data ω_0.*

Proof: Let $\omega_0^n \in C_c^\infty(\Omega)$ be such that $\omega_0^n \rightharpoonup \omega_0$ weak-$*$ $\mathcal{BM}(\Omega)$. Let ω^n, $u^n = K_\Omega[\omega^n]$, be the unique global smooth solutions of (2) with initial data ω_0^n. The existence of such solutions follow from the well-posedness of 2D Euler with smooth initial data in exterior domains, due to K. Kikuchi in [4]. It is an easy calculation to verify that ω^n satisfies Definition 1.

The sequence ω^n satisfies the following *a priori* estimates:

1. $\|\omega^n\|_{L^\infty((0,\infty);L^1(\Omega))} \leq C < \infty$;
2. $\|u^n\|_{L^\infty((0,\infty);(L^2(\Omega))^2)} \leq C < \infty$;
3. $\{\omega^n\}$ is *equicontinuous* from $(0,T)$, for any $T > 0$, to $H^{-M}(\Omega)$ for some $M > 0$.

Indeed, the first estimate follows from conservation in time of L^1 norm of vorticity for smooth solutions and the second follows from the standard energy estimate. The third estimate is a bit more complicated. To prove it we consider a test function $\varphi \in C_c^\infty((0,\infty)\times\Omega)$. We use this test function in identity (3), noting that the initial data term disappears because $\varphi(x,0) \equiv 0$, to get:

$$\int_0^\infty \int_\Omega \varphi_t \omega^n(x,t)dxdt + \int_0^\infty \int_\Omega \int_\Omega H_\varphi^\Omega(x,y,t)\omega^n(x,t)\omega^n(y,t)dydxdt = 0.$$

Hence,

$$\left| \int_0^\infty \int_\Omega \varphi_t \omega^n(x,t)dxdt \right| \leq \|H_\varphi^\Omega\|_{L^1((0,\infty);L^\infty(\Omega))} \|\omega^n\|^2_{L^\infty((0,\infty);L^1(\Omega))}.$$

It follows from the first *a priori* estimate above and Lemma 1 that

$$\left| \int_0^\infty \int_\Omega \varphi_t \omega^n(x,t)dxdt \right| \leq C_\varphi.$$

The dependence of C_φ on φ comes from Lemma 1. Examining the proof of Lemma 1 it is possible to infer that $C_\varphi = C\|\varphi\|_{W^{2,\infty}(\Omega)}$. This, together with the Sobolev imbedding theorem, gives, by duality, an estimate of ω_t in $L^\infty((0,T);H^{-M}(\Omega))$, for any $T>0$ and some $M>3$. This clearly implies the third *a priori* estimate.

It follows, by the Aubin-Lions Lemma and the Banach-Alaoglu Theorem, that there exists $\omega \in L^\infty((0,\infty);\mathcal{BM}(\Omega)) \cap C((0,T);H^{-L}(\Omega))$, for any $T>0$ and some $L<M$, such that, passing to a subsequence if necessary, we have $\omega^n \rightharpoonup \omega$ weak-$*$ in $L^\infty((0,\infty);\mathcal{BM}(\Omega))$ and $\omega^n \to \omega$ strongly in $C((0,T);H^{-L}(\Omega))$. We will observe that ω is a weak solution with initial data ω_0. Let $\varphi \in \mathcal{A}$. As usual, the only difficulty in passing to the limit in each of the terms in (3) is the nonlinear term,

$$\mathcal{W}_{NL}[\omega^n,\varphi] \equiv \int_0^\infty \int_\Omega \int_\Omega H_\varphi^\Omega(x,y,t)\omega^n(x,t)\omega^n(y,t)dydxdt.$$

By Lemma 2 we have that there are no time-averaged concentrations, i.e. $|\omega^n|$ does not form Diracs when $n\to\infty$. This fact, together with the boundedness of H_φ^Ω derived in Lemma 1, and the fact that H_φ^Ω is continuous off of the diagonal $x=y$, allows us to deduce that $\mathcal{W}_{NL}[\omega^n,\varphi] \to \mathcal{W}_{NL}[\omega,\varphi]$ as $n\to\infty$. The proof of this last convergence follows precisely the same argument of the proof of Theorem 1 in [6] so we choose not to repeat it.

∎

One issue that was discussed at length in [6] was the method of images. The relevant formulation states that smooth flow on a half-plane is a solution of the the incompressible 2D Euler equations if and only if its symmetric extension is a solution in the full plane. This is not true for weak solutions if one uses the standard definition of weak solution in domains with boundary, as in [2], but if one uses the definition as it was stated here, this equivalence was proved in [6] for the half plane. The extension of the method of images to weak solutions also works in the present case, mirror symmetric flow in the exterior of a bluff body. More precisely we have

Theorem 3 *The function $\omega=\omega(x,t) \in L^\infty([0,\infty);\mathcal{BM}(\Omega_+)$ is a weak solution of the incompressible 2D Euler equations in the sense of Definition 1 in Ω_+ if and only if its odd extension is a weak solution in the sense of Definition 1 in Ω.*

The proof is somewhat involved, but it is a straightforward adaptation of the proof of Theorem 2 in [6]. One immediate consequence of Theorems 2 and 3 is the following

Corollary 1 *Let $\omega_0 \in \mathcal{BM}_c^+(\Omega_+) \cap H^{-1}(\Omega_+)$. There exists a weak solution ω of the incompressible 2D Euler equations in the sense of Definition 1 in Ω_+ with initial data ω_0.*

A few remarks are in order. First, All the conclusions discussed above remain true if the initial vorticity is perturbed by an integrable function with reflection symmetry as we did in [6]. Second, the weak solution in the Theorem 2 is a limit of the approximate solutions obtained by regularizing the initial data, it would be extremely interesting to study the limits of approximate solutions generated by either Navier-Stokes approximations as in [7] or vortex methods as in [5]in this case.

Finally, It is natural to investigate the problem of existence of weak solutions in the sense of Definition 1 in a general domain. We already understand the special cases of the half-plane (see [6]) and certain compactly supported perturbations of the half plane, as noted above. The argument we presented here can be used to prove such a result for domains in the plane with simply connected boundary (plus technical assumptions on the behavior of said boundary at infinity), such as curved boundary half planes and channels. In the case of a bounded domain, a version of Corollary 1 would be a slight improvement of Delort's Theorem for bounded domains. However, our argument cannot be adapted to prove existence for bounded domain flow.

This is somewhat surprising, and the difficulty stems from the derivation of the *a priori* L^2_{loc} bound on the tangential velocity at the boundary, given by (5). To derive (5), we need to exhibit a test function with derivatives bounded up to second order which is monotonic when restricted to each connected component of the boundary. Otherwise, the identity obtained in Lemma 1 does not lead to an actual *a priori* estimate. Such a test function cannot exist on a domain with compact boundary components. It may be that this is just a technical difficulty, and that a "boundary coupled" weak solution does exist for distinguished signed vortex sheet initial data in a bounded domain, but we would like to argue that this might not be the case. In fact we observe that this restriction might be the result of a meaningful physical distinction between compact and noncompact boundary components with respect to concentrations of vorticity. For a noncompact boundary component, vorticity that concentrates, forming a Dirac, and at the same time approaching the boundary, tends to move with large velocity and leave the compact parts of the flow domain. Since the test functions involved in the definition of weak solutions are compactly supported this kind of concentration ends up being irrelevant. However, with a compact boundary component, concentration of vorticity near the boundary leads to this vorticity moving faster and faster around this boundary component, without disappearing. Such concentration behavior would be entirely consistent with Lemma 1 and would require a substantially different approach to handle existence.

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