MULTIDIMENSIONAL TRANSONIC SHOCK IN A NOZZLE II, 3-D CASE

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Abstract

In this paper, we study the following problem proposed by Courant and Friedrichs in [6] on transonic flow phenomena in a 3-dimensional nozzle: Given the appropriately large receiver pressure p_r, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle a shock front intervenes and the gas is compressed and slowed down to subsonic speed. The position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes p_r. The flow is assumed to be described by the inviscid steady potential equation. The transonic shock is a free boundary dividing the hyperbolic region and the elliptic region in the nozzle. The potential equation is hyperbolic upstream where the flow is supersonic, and elliptic in the downstream subsonic region. Our results indicate that the conjecture of Courant and Friedrichs cannot be true for the arbitrarily given and appropriately large pressure at the exit of a slowly-varying nozzle, namely, the transonic shock problem is ill-posed for the general given pressure at the exit. Furthermore, we find a class of pressures which are induced by the appropriate boundary conditions at the exit such that the transonic shock problem is stable.

Keywords: Transonic flow, ill-posedness, well-posedness, potential equation, multidimensional shock wave, nozzle

Mathematical Subject Classification: 35L70, 35L65, 35L67, 76N15

§1. Introduction and the main results

This is a continuation of our study on the problem of the well-posedness of a multidimensional transonic shock to the steady flow through a general curved nozzle [24]. Our focus is on transonic flows with shocks in a general 3-dimensional nozzle, which is an important subject in gas dynamics ([2, 6, 7]). In particular, we are concerned with the following transonic phenomena in a De Laval nozzle as conjectured by Courant

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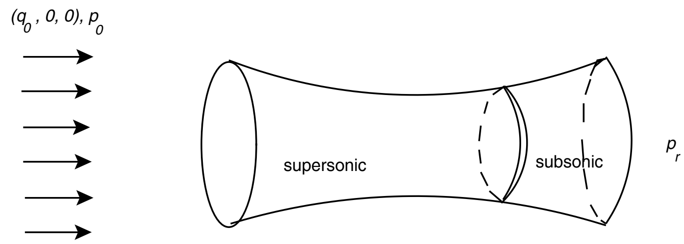
and Friedrichs in [6]: Given the appropriately large receiver pressure p_r, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle a shock front intervenes and the gas is compressed and slowed down to subsonic speed. The position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes p_r. See Picture 1.



Picture 1

In [24], we have established the well-posedness of the above-mentioned structure of the transonic flow with shocks in a general 2-dimensional nozzle for a class of pressures which are induced by appropriate boundary conditions at the exit of the nozzle. However, as indicated by Courant-Friedrichs (pages 377 in [6]), a flow through a duct should be considered as a steady, isentropic, irrotational flow with cylindrical symmetry and should be determined by solving the 3-D potential flow equations with appropriate boundary conditions. Thus, one of the major goals of this paper is to treat the existence, stability and well-posedness of such a transonic flow pattern with a multidimensional shock in a general 3-D nozzle with slowly-varying sections. As to other discussions on transonic flows and transonic flows with shocks and references on recent studies on multidimensional transonic shocks, we refer to [24] and the references cited therein, see also [3, 4, 21, 22].

Suppose that there is a uniform supersonic flow $(u_1, u_2, u_3) = (q_0, 0, 0)$ with constant density $\rho_0 > 0$ which comes from minus infinity, and the flow enters the nozzle from the entrance. In this paper, we always assume that the nozzle wall is of a small perturbation of a cylindrical surface $\{x : x_2^2 + x_3^2 = 1, -1 \leq x_1 \leq 1\}$. Thus, the flow in the nozzle can be approximately assumed to be irrotational and isentropic (see [1-2, 6, 17, 21-22] and so on).

Let $\varphi(x)$ be the potential of velocity, i.e. $(\partial_1\varphi, \partial_2\varphi, \partial_3\varphi) = (u_1, u_2, u_3)$, then the Bernoulli's law implies

$$\frac{1}{2}|\nabla\varphi|^2 + h(\rho) \equiv C_0 = \frac{1}{2}q_0^2 + h(\rho_0), \tag{1.1}$$

where $h(\rho)$ is the specific enthalpy. For the given equation of state $P = P(\rho)$ with $P'(\rho) = c^2(\rho) > 0$ for $\rho > 0$, then $h'(\rho) = \frac{c^2(\rho)}{\rho}$.

Since $h'(\rho) > 0$, one then can define the inverse function of $h(\rho)$ as $H(s)$, namely,

$$\rho = H(C_0 - \frac{1}{2}|\nabla\varphi|^2). \tag{1.2}$$

Then the continuity equation becomes

$$\sum_{i=1}^{3} \partial_i(\partial_i\varphi H) = 0, \tag{1.3}$$

which can be rewritten as

$$\sum_{i=1}^{3}((\partial_i\varphi)^2 - c^2)\partial_i^2\varphi + 2\sum_{1\le i<j\le 3}\partial_i\varphi\partial_j\varphi\partial_{ij}^2\varphi = 0. \tag{1.4}$$

It is easy to verify that (1.4) is strictly hyperbolic for $|\nabla\varphi| > c(\rho)$ and strictly elliptic for $|\nabla\varphi| < c(\rho)$.

Suppose that the wall of the nozzle is given by

$$\sqrt{x_2^2 + x_3^2} = f(x), \qquad -1 \le x_1 \le 1, \tag{1.5}$$

such that

$$|\nabla_x^\alpha(f(x) - 1)| \le \varepsilon \qquad \text{for} \qquad -1 \le x_1 \le 1, \qquad |\alpha| \le k_0, \tag{1.6}$$

here $k_0 \in \mathbb{N}$ and $k_0 \ge 7$.

Without loss of generality and for the convenience to study, we assume that

$$f(-1, x_2, x_3) = 1, \qquad f(1, x_2, x_3) = 1, \qquad \nabla_x^\alpha f(x)|_{x_1=-1} = 0 \qquad \text{for} \qquad 1 \le |\alpha| \le k_0. \tag{1.7}$$

When the uniform supersonic flow $(q_0, 0, 0)$ enters the entry of the nozzle, then the potential $\varphi_-(x)$ in the nozzle will be determined by the following initial boundary value problem for a quasilinear wave equation

$$\begin{cases} \sum_{i=1}^{3}((\partial_i\varphi_-)^2 - c_-^2)\partial_i^2\varphi_- + 2\sum_{1\le i<j\le 3}\partial_i\varphi_-\partial_j\varphi_-\partial_{ij}^2\varphi_- = 0, \\ \varphi_-|_{x_1=-1} = -q_0, \\ \partial_1\varphi_-|_{x_1=-1} = q_0, \\ \partial_1 f\partial_1\varphi_- + \sum_{i=2}^{3}(\partial_i f - \frac{x_i}{f})\partial_i\varphi_- = 0 \qquad \text{on} \qquad \sqrt{x_2^2 + x_3^2} = f(x), \end{cases} \tag{1.8}$$

where $c_- = c(\rho_-)$ and $\rho_- = H(C_0 - \frac{1}{2}|\nabla\varphi_-|^2)$.

It follows from Lemma 2.1 in §2 that (1.8) has a C^5 solution $\varphi_-(x)$ in the nozzle $\{(x_1, x_2, x_3) : -1 \le x_1 \le 1, \sqrt{x_2^2 + x_3^2} \le f(x)\}$, moreover $|\nabla_x^\alpha(\varphi_-(x) - q_0x_1)| \le C\varepsilon$ holds for $|\alpha| \le 5$.

Given an appropriately larger pressure $\tilde{P}_+(x_2, x_3) = P(\tilde{\rho}_+(x_2, x_3))$ at the exit $x_1 = 1$ of the nozzle than that in the entry, where $\tilde{\rho}_+(x_2, x_3) \in C^4(\{(x_2, x_3) : \sqrt{x_2^2 + x_3^2} \le f(1, x_2, x_3)\})$ is a small perturbation of the constant density ρ_+, more precisely,

$$|\nabla_{x_2,x_3}^\alpha(\tilde{\rho}_+(x_2, x_3) - \rho_+)| \le \varepsilon \qquad \text{for} \qquad 0 \le |\alpha| \le 3,$$

here the density ρ_+ and the constant velocity $|\nabla\varphi| = q_+$ satisfy the following relations

$$\frac{1}{2}q_+^2 + h(\rho_+) = C_0, \qquad \rho_+q_+ = \rho_0q_0 \qquad \text{and} \qquad q_+ < c(\rho_+). \tag{1.9}$$

Then it is expected that there appears a transonic shock $\Sigma : x_1 = \xi(x_2, x_3)$ in the nozzle. To assure uniqueness of the flow pattern (as in [4] and [24]), we also require that the shock Σ goes through a specified point, say, $(0, 0, 0)$, namely

$$\xi(0, 0) = 0. \tag{1.10}$$

Denote by $\varphi_+(x)$ the velocity potential across the shock Σ. Then the potential is continuous across the shock Σ [2, 6], i.e.,

$$\varphi_+(x) = \varphi_-(x) \qquad \text{on} \qquad x \in \Sigma \tag{1.11}$$

and $\nabla\varphi$ must satisfy the Rankine-Hugoniot condition

$$[\partial_1 \varphi H] - \sum_{i=2}^{3} \partial_i \xi [\partial_i \varphi H] = 0 \qquad \text{on} \qquad \Sigma. \tag{1.12}$$

Furthermore, the following physical entropy condition should be satisfied:

$$H(C_0 - \frac{1}{2}|\nabla\varphi_-|^2) < H(C_0 - \frac{1}{2}|\nabla\varphi_+|^2) \qquad \text{on} \qquad \Sigma. \tag{1.13}$$

On the exit of the nozzle, the given pressure is equivalent to

$$H(C_0 - \frac{1}{2}|\nabla\varphi_+|^2) = \tilde{\rho}_+(x_2, x_3) \qquad \text{on} \qquad x_1 = 1. \tag{1.14}$$

Finally, the no-flow boundary condition on the wall of the nozzle says

$$\partial_1 f \partial_1 \varphi_+ + \sum_{i=2}^{3} (\partial_i f - \frac{x_i}{f}) \partial_i \varphi_+ = 0 \qquad \text{on} \qquad \sqrt{x_2^2 + x_3^2} = f(x). \tag{1.15}$$

We will use the following notations

$$\Omega = \left\{ (x_1, x_2, x_3) : -1 < x_1 < 1, \sqrt{x_2^2 + x_3^2} < f(x) \right\};$$

$$\Omega_+ = \left\{ (x_1, x_2, x_3) : \xi(x_2, x_3) < x_1 < 1, \sqrt{x_2^2 + x_3^2} < f(x) \right\};$$

$$S = \left\{ (x_2, x_3) : (\xi(x_2, x_3), x_2, x_3) \in \Sigma \right\} \qquad \text{which is the projection of the shock surface } \Sigma$$

$$\text{on the } (x_2, x_3)\text{−plane};$$

$$\tilde{\Gamma}_1 = \Sigma \cap \left\{ (x_1, x_2, x_3) : \sqrt{x_2^2 + x_3^2} = f(x) \right\}, \tilde{\Gamma}_2 = \left\{ (1, x_2, x_3) : \sqrt{x_2^2 + x_3^2} = f(1, x_2, x_3) \right\};$$

$$|d_{\tilde{\Gamma}_1}| = dist(x, \tilde{\Gamma}_1), \qquad x \in \Sigma \qquad \text{and} \qquad (x_2, x_3) \in S;$$

$$|d_x| = min\{dist(x, \tilde{\Gamma}_1), dist(x, \tilde{\Gamma}_2)\} \qquad \text{for} \qquad x \in \Omega_+.$$

The first main result in this paper is on the uniqueness of the solution to the equation (1.4) with the boundary conditions (1.10)-(1.15).

Theorem 1.1. (Uniqueness)
Suppose that (1.6), (1.7) and (1.9) hold. Then for suitably small $\varepsilon > 0$, the equation (1.4) with the boundary conditions (1.10)-(1.15) has no more than one pair of solution $(\varphi_+(x), \xi(x_2, x_3))$ with the following regularities:

(i). For $k=2,3$ and $(x_2,x_3)\in S$,

$$\xi(x_2,x_3)\in C^{1,1-\delta_0}(\bar S)\cap C^3(S),\qquad \|\xi(x_2,x_3)\|_{C^{1,1-\delta_0}(\bar S)}\le C\varepsilon,\qquad |\nabla^k_{x_2,x_3}\xi(x_2,x_3)|\le\frac{C\varepsilon}{|d_{\tilde\Gamma_1}|^{k-2+\delta_0}}.$$

(ii). $\varphi_+(x)\in C^{1,1-\delta_0}(\bar\Omega_+)\cap C^3(\Omega_+)$ such that,

$$\|\varphi_+(x)-q_+x_1\|_{C^{1,1-\delta_0}(\bar\Omega_+)}\le C\varepsilon,\qquad |\nabla^k_x\varphi_+(x)|\le\frac{C\varepsilon}{|d_x|^{k-2+\delta_0}}\qquad for\qquad k=2,3;\qquad x\in\Omega_+$$

Remark 1.1. *It follows from the regularity theory of the second order elliptic equations with the cornered boundaries (one can see the references [14] and so on) that the assumptions on the regularities of solution $(\varphi_+(x),\xi(x_2,x_3))$ in Theorem 1.1 are plausible. See also Theorem 1.3 below.*

Remark 1.2. *If the end pressure $\tilde\rho(x_2,x_3)$ in (1.14) is given on a C^3 smooth surface $x_1=g(x_2,x_3)$ with $|\nabla^\alpha_{x_2,x_3}(g(x_2,x_3)-1)|\le\varepsilon$ for $0\le|\alpha|\le3$ and $(x_2,x_3)\in\{(x_2,x_3):\sqrt{x_2^2+x_3^2}\le f(1,x_2,x_3)\}$, then by an analogous proof, we know that Theorem 1.1 still holds in this case.*

Remark 1.3. *For $A_1=(\xi(x_2^1,x_3^1),x_2^1,x_3^1),...,A_m=(\xi(x_2^m,x_3^m),x_2^m,x_3^m)\in\Sigma$ ($m\in\mathbb{N}$), we suppose that $\gamma_1,...,\gamma_m$ are any smooth curves which lie in $\sqrt{x_2^2+x_3^2}=f(x)$ and start from $A_1,...,A_m$ respectively. Denote by $|d_0|=\min\{|d_{\tilde\Gamma_1}|,dist(x,A_1),...,dist(x,A_m),\sqrt{x_2^2+x_3^2}\}$ for $x\in\Sigma$ and $|d|=\min\{|d_x|,dist(x,\gamma_1),...,\ dist(x,\gamma_m),\sqrt{x_2^2+x_3^2}\}$ for $x\in\Omega_+$. Then it follows from a similar procedure as in §3 that the uniqueness result in Theorem 1.1 continues to hold if the solution $(\varphi_+(x),\xi(x_2,x_3))$ has the following regularities and estimates:*

(i). $\xi(x_2,x_3)\in C^{1,1-\delta_0}(\bar S)\cap C^3(S)$, $\|\xi(x_2,x_3)\|_{C^{1,1-\delta_0}(\bar S)}\le C\varepsilon$ and $|\nabla^k_{x_2,x_3}\xi(x_2,x_3)|\le\frac{C\varepsilon}{|d_0|^{k-2+\delta_0}}$ with $k=2,3$ and $(x_2,x_3)\in S$.

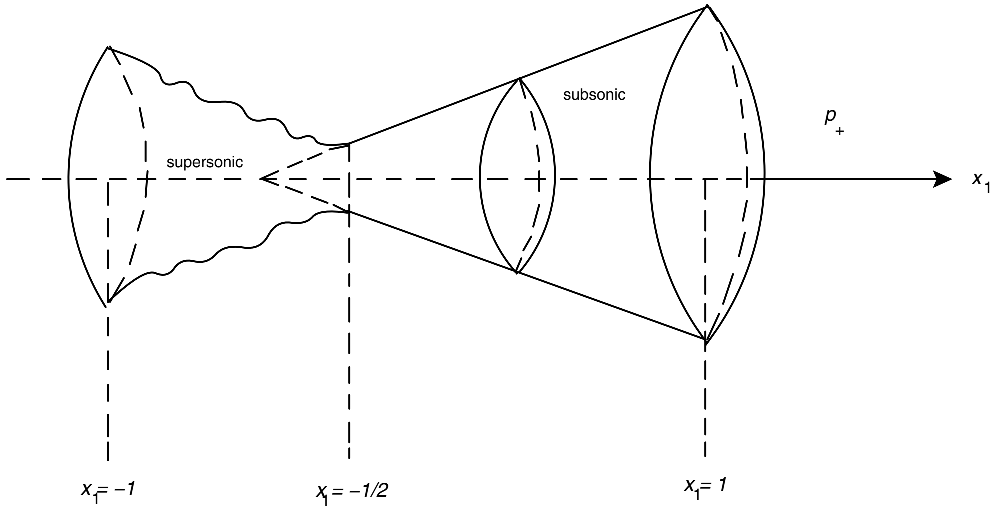
(ii). $\varphi_+(x)\in C^{1,1-\delta_0}(\bar\Omega_+)\cap C^3(\Omega_+)$ and $\|\varphi_+(x)-q_+x_1\|_{C^{1,1-\delta_0}(\bar\Omega_+)}\le C\varepsilon$, $|\nabla^k_x\varphi_+(x)|\le\frac{C\varepsilon}{|d|^{k-2+\delta_0}}$ for $k=2,3$ and $x\in\Omega_+$.

This remark will be useful in proving Theorem 1.2 below.

Next, we turn to the non-existence of solutions to the transonic shock problem with general given pressure $\tilde\rho_+(x)$ at the exit of the nozzle.

Suppose that a nozzle wall is C^5-regular for $-1\le x_1\le1$ and it consists of two surfaces Π_1 and Π_2, here Π_1 is the converging part of the nozzle, Π_2 is a cone surface in $-\frac12\le x_1\le1$ (i.e. the diverging part of the nozzle), whose vertex is $(x_1^0,0,0)$ with $x_1^0<0$ sufficiently small. Moreover Π_1 and Π_2 are very close to the cylindrical surface $\{x:x_2^2+x_3^2=1,-1\le x_1\le1\}$. More precisely, we assume that the equation of Π_2 is represented by $x_2^2+x_3^2=(x_1-x_1^0)^2tg^2\alpha_0(\alpha_0>0)$, here $tg\alpha_0=\frac{1}{1-x_1^0}$ (this condition guarantees that Π_2 is very near $x_2^2+x_3^2=1$ in $-\frac12\le x_1\le1$ for sufficiently small $x_1^0<0$ since one has $x_2^2+x_3^2=\frac{(x_1-x_1^0)^2}{(1-x_1^0)^2}$). Besides, the transonic shock is assumed to go through the origin, and suppose that the supersonic coming flow is symmetric in $-x_1^0-\frac14\le r\le-x_1^0$ with $r=\sqrt{(x_1-x_1^0)^2+x_2^2+x_3^2}$ (namely, the potential $\varphi_-(x)$ depends only on r for $-x_1^0-\frac14\le r\le-x_1^0$) and is of a small perturbation of the constant state $(\rho_0,q_0,0,0)$. By the hyperbolicity, we can obtain a supersonic flow $\varphi_-(x)$ in the global nozzle, which is symmetric in $-x_1^0-\frac14\le r\le(1-x_1^0)\sec\alpha_0$ and very close to q_0x_1. Furthermore, we let the boundary condition (1.14) be replaced by

$$H(C_0-\frac12|\nabla\varphi_+|^2)=\rho_+\qquad\text{on}\qquad r=(1-x_1^0)\sec\alpha_0.\qquad(1.14)'$$



Picture 2

where the constant density ρ_+ is determined by (1.9). See Picture 2.

Then based on Theorem 1.1, Remark 1.2 and Remark 1.3, we can show the following ill-posedness result.

Theorem 1.2. (Ill-posedness) *If the nozzle wall consists of Π_1 and Π_2, then the problem (1.4) with (1.10)-(1.13),* $(1.14)'$ *and (1.15) is ill-posed. More precisely, one can find the supersonic coming flows which are of small perturbations of $(\rho_0, q_0, 0, 0)$ such that the problem (1.4) with (1.10)-(1.13),* $(1.14)'$ *and (1.15) has no transonic shock solution $(\varphi_+(x), \xi(x_2, x_3))$ with the regularities and estimates as stated in Theorem 1.1.*

Remark 1.4. *For the arbitrarily given and appropriately large pressure $\tilde{p}_+(x)$ at the exit, Theorem 1.2 states that the transonic problem (1.4) in the nozzle with a shock can not occur, namely, the conjecture of Courant and Friedrichs is not true for the general given pressure at the exit of a slowly-varying nozzle. Similar conclusions hold for the 2-dimensional case, see §4 for details.*

Remark 1.5. *For the complete Euler equations, if the pressure at the exit of the nozzle is arbitrarily given, it then can be proved that the transonic shock wave pattern as conjectured by Courant-Friedrich's in [6] cannot occur in a slowly-varying nozzle. The details can be found in [27].*

Despite the non-existence results in Theorem 1.2, we can find a class of pressures (although we do not give the pressure directly at the exit) which are induced by the following oblique derivative boundary conditions $(1.14)''$ such that the transonic shock problem (1.4) is stable and satisfies the given boundary conditions

$$\partial_1 \varphi_+ + b_2(x)\partial_2 \varphi_+ + b_3(x)\partial_3 \varphi_+ + b_1(x)\varphi_+ = g(x) \qquad \text{on} \qquad x_1 = 1 \tag{1.14$''$}$$

here $b_i(x) \in C^3(\bar{\Omega})(i = 1, 2, 3), g(x) \in C^3(\bar{\Omega})$ and $\lambda \le b_1(1, x_2, x_3) \le \Lambda$ for $(x_2, x_3) \in \{(x_2, x_3) : \sqrt{x_2^2 + x_3^2} = f(1, x_2, x_3)\}$, here Λ and λ are two positive constants. Besides, $|\nabla^\alpha(g(x) - (1 + b_1(x)x_1)q_+)| + |\nabla^\alpha b_2(x)| + |\nabla^\alpha b_3(x)| \le \varepsilon$ for $0 \le |\alpha| \le 3$ and $x \in \bar{\Omega}$ holds. With the same notations as for Theorem 1.1, the main existence results can be stated as follows:

Theorem 1.3. *Let the assumptions (1.6), (1.7) and (1.9) hold. Then for suitably small $\varepsilon > 0$, there exists a unique transonic pair $(\varphi(x), \xi(x_2, x_3))$ such that $\varphi(x)$ is piecewise smooth, i.e.,*

$$\varphi(x) = \begin{cases} \varphi_-(x), & for \quad x_1 < \xi(x_2, x_3) \\ \varphi_+(x), & for \quad x_1 > \xi(x_2, x_3) \end{cases}$$

and $(\varphi(x), \xi(x_2, x_3))$ solves the problem (1.4), (1.11)-(1.13), $(1.14)''$ and (1.15).

Moreover, for a given constant $0 < \delta_0 < \frac{1}{3}$, there exists a constant C independent of ε with the following properties:

(i). **(Regularity of supersonic flow)** *$\varphi_-(x) \in C^5(\bar{\Omega})$ solves the initial-boundary value problem (1.8) in Ω. Furthermore,*

$$\|\varphi_-(x) - q_0 x_1\|_{C^5(\bar{\Omega})} \leq C\varepsilon$$

(ii). **(Regularity of the shock surface)**

$$\xi(x_2, x_3) \in C^{1,1-\delta_0}(\bar{S}) \cap C^3(S), \qquad and$$

$$\|\xi(x_2, x_3)\|_{C^{1,1-\delta_0}(\bar{S})} \leq C\varepsilon, \qquad |\nabla^k_{x_2,x_3}\xi(x_2, x_3)| \leq \frac{C\varepsilon}{|d_{\tilde{\Gamma}_1}|^{k-2+\delta_0}} \qquad for \qquad k = 2, 3; (x_2, x_3) \in S.$$

(iii). **(Regularity of the subsonic flow)**
$\varphi_+(x) \in C^{1,1-\delta_0}(\bar{\Omega}_+) \cap C^3(\Omega_+)$ admits the following estimates:

$$\|\varphi_+(x) - q_+ x_1\|_{C^{1,1-\delta_0}(\bar{\Omega}_+)} \leq C\varepsilon, \qquad |\nabla^k_x \varphi_+(x)| \leq \frac{C\varepsilon}{|d_x|^{k-2+\delta_0}} \qquad for \qquad k = 2, 3; \qquad x \in \Omega_+.$$

(iv). **(Entropy condition)** *The physical entropy condition (1.13) holds on the shock Σ.*

Remark 1.6. *It should be noted that the transonic shock in theorem is perpendicular to the wall of the nozzle. This fact follows easily from the boundary conditions (1.11) and (1.15), and the ones on the nozzle wall for $\varphi_-(x)$ in (1.8).*

Remark 1.7. *It will follow from the proof of Theorem 1.3 that one can also establish the structural stability of the transonic shock solution $(\varphi(x), \xi(x_2, x_3))$ with respect to small perturbation of the initial state at the entrance. More precisely, for $k_0 \geq 7$ and $0 \leq \alpha \leq k_0$, if $|\nabla^\alpha_{x_2,x_3}(\hat{\varphi}_-(-1, x_2, x_3) + q_0)| \leq \varepsilon$ and $|\nabla^\alpha_{x_2,x_3}(\partial_1\hat{\varphi}_-(-1, x_2, x_3) - q_0)| \leq \varepsilon$ hold, then there exists a unique solution $(\hat{\varphi}_+(x), \hat{\xi}(x_2, x_3))$ to (1.4) with the corresponding perturbed boundary conditions such that*

$$\|\hat{\varphi}_+(x) - q_+ x_1\|_{C^{1,1-\delta_0}(\overline{\hat{\Omega}_+})} \leq C\varepsilon, \qquad \|\hat{\xi}(x_2, x_3)\|_{C^{1,1-\delta_0}(\bar{\hat{S}})} \leq C\varepsilon$$

where $\hat{\Omega}_+ = \{(x_1, x_2, x_3) : \hat{\xi}(x_2, x_3) < x_1 < 1, \sqrt{x_2^2 + x_3^2} < f(x)\}$, $\hat{S}$ is the open projection set of the corresponding shock surface $\hat{\Sigma}$ onto the $(x_2, x_3)-$plane. Besides, the constant C is independent of ε.

Remark 1.8. *It should be noted that the main assumption in Theorem 1.1-1.3 is that the wall of the nozzle is a small perturbation of a straight cylinder, (1.6). This is in general necessary for the existence of such a transonic shock wave pattern as conjectured by Courant-Friedrichs and mentioned at the beginning of this section. Since for the nozzle of large deviation from a flat cylinder, there may be supersonic shocks in the supersonic region, or supersonic bubbles surrounded by subsonic flow, see [2, 6].*

Remark 1.9. *Some ideas developed in this paper can be used to study the problem of existence, uniqueness, well-posedness and structural stability or instability of a transonic shock for the supersonic flow past a three-dimensional wedge. This is reported in [25].*

Remark 1.10. *If, instead of the no-flow boundary condition (1.15), the wall of the nozzle is assumed to be porous (perforated) or curved appropriately large, then the corresponding transonic shock problem can be shown to be well-posed for the arbitrarily given and appropriately large pressure* $\tilde{\rho}_+(x)$ *on* $x_1 = 1$. *This will be reported in a forth coming paper [26].*

Obviously, combining Theorem 1.1 with Theorem 1.3 yields the following result on the existence and uniqueness for a class of pressures at the exit of the nozzle.

Theorem 1.4. (Stability for a class of pressures at the exit)

If $\tilde{\rho}_+(x_2, x_3)$ *in (1.14) and a specified point in (1.10) are determined by Theorem 1.3, then the transonic shock solution to the problem (1.4) with (1.10)-(1.15) exists uniquely.*

We now comment on the proof of the main results. Some of the main difficulties are that (1.4) is a mixed-type quasilinear equation and the shock surface is a free boundary with nonlinear boundary condition (1.12). In order to prove Theorem 1.1, the main strategy of the analysis comes from our treatment on the 2-D problem in [24]. First, we introduce a new partial hodograph transformation which maps the domain $\bar{\Omega}_+$ into the fixed domain $\bar{Q}_+ = \{(X_1, X_2, X_3) : 0 \le X_1 \le 1, X_2^2 + X_3^2 \le 1\}$ as in [24], see also [5, 18, 20]. Under this transformation, the quasilinear potential equation (1.4), whose coefficients contain only the first order derivative of $\varphi(x)$, becomes a new second order nonlinear equation with coefficients and source term containing the unknown function $V(X)$ and its first order derivative $\nabla_X V(X)$. Correspondingly, the boundary conditions (1.12)-(1.15) are also changed into new nonlinear boundary conditions containing $V(X)$ and $\nabla_X V(X)$. It is crucial in our analysis that we can choose the partial-hodograph transformation so that the coefficients of $V(X)$ and $\nabla_X V(X)$ in the second order elliptic equation and the coefficients of $V(X)$ in the boundary conditions are all suitably small in appropriately weighted space, and thus avoid the possibility of negative eigenvalue of the resulted linear equation on $v(X)$ in Theorem 1.1. One of key elements in the proof on Theorem 1.1 is to derive $\|v(X)\|_{H^2} = 0$ to the solution $v(X)$ by the multiplier method instead of establishing $\|v\|_{L^\infty} = 0$ by the maximum principle, since it seems difficult to obtain $\|v\|_{L^\infty}$ by the maximum principle in [13] due to the structures of the equation and boundary conditions on $v(X)$. In order to prove Theorem 1.3, we will use the Schauder fixed point theorem to solve the corresponding nonlinear elliptic equation on $\bar{Q}_+$ which is resulted by the generalized hodograph transformation in §2. Weighted Hölder spaces will be used to treat the possible singularities due to the corners of the domain under discussion [3,8,13-15,24]. In addition, one can use the maximum principle to derive the uniform $L^\infty-$estimate by use of $(1.14)''$. Although the main strategy to prove Theorem 1.1 and Theorem 1.3 is similar to our approach used for 2-dimensional case [24], much more delicate a priori estimates are needed to overcome some main difficulties occurred in the 3-dimensional case. In particular, more complicated and careful analysis are needed for the estimates near shock and fixed boundaries. Finally, based on Theorem 1.1, making full use of the symmetry of nozzle wall Π_2, the supersonic coming flow in the diverging part for $-\frac{1}{4} \le x_1 \le \frac{1}{4}$ and the end pressure condition, we can show that the pressure at the exit should be uniquely determined by the supersonic coming flow for the transonic solution with a shock. The main approach is that we can derive an ordinary differential equation by the symmetry of the nonlinear equation, the nozzle wall and the boundary conditions if we assume that the transonic shock is well-posed with respect to any appropriately large pressure at the exit for any slowly-varying nozzle or any part of the nozzle which is bounded by the nozzle wall and two planes through the x_1-axis (these assumptions are plausible physically if the transonic problem with one shock is indeed well-posed in the nozzle with the end pressure).

Next, we note that there have been many studies on the transonic problems as we mentioned in [24]. See also [1, 3, 4] and the references therein. Here we mention only the study on the existence and uniqueness of axially symmetric compressible subsonic flows of jets and cavities by Alt-Caffarelli-Friedman [1], where they use a variational approach to solve such a free boundary problem. However, we were not able to adapt their analysis due to different conditions on both free and fixed boundaries and our problem is truly 3-dimensional.

The rest of the paper is outlined as follows. In Sect.2, we introduce a generalized partial hodograph transformation and reformulate the original problem (1.4) with the boundary conditions (1.11)-(1.15) in terms of the new variables. Besides, some basic estimates on the coefficients of this resulted problem are carried out too. In Sect.3, the H^2-norm estimates for the solution $v(X)$ to the linear problem induced in §2 are derived. This directly yields $v(X) \equiv 0$, namely, we complete the proof on Theorem 1.1. In Sect.4, based on Theorem 1.1, we show the ill-posedness result in Theorem 1.2. In addition, we list two ill-posedness results on the 2-D transonic shock problem and only give a sketch on the proof. Finally, the proof on Theorem 1.3 is given in Sect.5.

¿From now on, the following convention will be used in this paper:

$O(\varepsilon)$ and $O(M\varepsilon)$ mean that there exists a generic constant C such that $|O(\varepsilon)| \le C\varepsilon$ and $|O(M\varepsilon)| \le CM\varepsilon$ respectively, where C is independent of M and ε.

§2. The reformulation on Theorem 1.1 and the generalized hodograph transformation

With the help of $\varphi_-(x)$ determined by solving the initial-boundary value problem (1.8), the nonlinear mixed-type equation (1.4) with (1.8)-(1.15) can be reduced to a boundary value problem for a second order quasilinear elliptic equation with a free boundary (the transonic shock). In this section, we reduce this free boundary value problem on Ω_+ to a boundary value problem on a fixed domain $Q_+ = \{(X_1, X_2, X_3) : 0 < X_1 < 1, X_2^2 + X_3^2 < 1\}$ by introducing a generalized hodograph transformation and a coordinate transformation. First, we estimate the potential $\varphi_-(x)$ for the supersonic flow.

Lemma 2.1. *Assume that (1.6) and (1.7) hold. Then (1.8) has a unique $C^5(\bar{\Omega})$ solution $\varphi_-(x)$ such that*

$$\|\varphi_-(x) - q_0 x_1\|_{C^5(\bar{\Omega})} \le C\varepsilon.$$

holds for small $\varepsilon > 0$, where C is independent of ε.

Proof. Note that $\tilde{\varphi}(x) = \varphi_-(x) - q_0 x_1$ satisfies

$$\begin{cases} ((q_0 + \partial_1\tilde{\varphi})^2 - c_-^2)\partial_1^2\tilde{\varphi} + 2(q_0 + \partial_1\tilde{\varphi})\displaystyle\sum_{i=2}^{3}\partial_i\tilde{\varphi}\partial_{1i}^2\tilde{\varphi} + \sum_{i=2}^{3}((\partial_i\tilde{\varphi})^2 - c_-^2)\partial_i^2\tilde{\varphi} + 2\partial_2\tilde{\varphi}\partial_3\tilde{\varphi}\partial_{23}^2\tilde{\varphi} = 0, \\ \tilde{\varphi}(x)|_{x_1=-1} = 0, \\ \partial_1\tilde{\varphi}(x)|_{x_1=-1} = 0, \\ \partial_1 f\partial_1\tilde{\varphi} + \displaystyle\sum_{i=2}^{3}(\partial_i f - \frac{x_i}{f})\partial_i\tilde{\varphi} = -q_0\partial_1 f \qquad \text{on} \qquad \sqrt{x_2^2 + x_3^2} = f(x), \end{cases} \tag{2.1}$$

where $c_- = c(H(C_0 - \frac{1}{2}(|q_0 + \partial_1\tilde{\varphi}|^2 + |\partial_2\tilde{\varphi}|^2 + |\partial_3\tilde{\varphi}|^2)))$.

It follows from (1.7) that the initial-boundary values in (2.1) satisfy the compatibility conditions up to $(k_0 - 1)-$th order.

Since $q_0 > c(\rho_0)$, then (2.1) is strictly hyperbolic with respect to x_1-direction for the small perturbations of constant solution. By the standard energy estimate on the linear wave equation with the initial-boundary conditions and the Picard iteration (for example, see [11]), then for small $\varepsilon > 0$, (2.1) has a unique solution $\tilde{\varphi}(x) \in \cap_{i=0}^{k_0} C^i([-1,1], H^{k_0-i}(\bar{\Omega}))$ and there exists a constant C independent of ε such that

$$\sum_{i=0}^{k_0} \|\tilde{\varphi}(x)\|_{C^i([-1,1],H^{k_0-i}(\bar{\Omega}))} \leq C\varepsilon.$$

Hence the Sobolev's imbedding theorem implies that Lemma 2.1 holds.

We now reduce the free boundary value problem (1.4) with (1.9)-(1.15) to a fixed boundary value problem. Rescaling if necessary, we will assume $q_0 - q_+ = 1$ in the rest of the paper unless otherwise stated. Define new independent variables as

$$\begin{cases} X_1 = 1 - \frac{1-x_1}{1-x_1+\varphi_-(x)-\varphi_+(x)}, \\ X_i = \frac{x_i}{f(x)}, \qquad i = 2, 3. \end{cases} \tag{2.2}$$

It is expected that $|\partial_x^\alpha(\varphi_+(x) - q_+x_1)| \leq C\varepsilon$ for $0 \leq |\alpha| \leq 1$. Consequently, one has that $\partial_1(\varphi_-(x) - \varphi_+(x)) = \partial_1(\varphi_-(x) - q_0x_1) - \partial_1(\varphi_+(x) - q_+x_1) + q_0 - q_+ > \frac{1}{2}$ for small ε and all $x \in \Omega_+$. This implies $\varphi_-(x) > \varphi_+(x)$ when $x_1 > \xi(x_2, x_3)$. It follows that (2.2) is an invertible transformation from the domain $\bar{\Omega}_+$ to

$$\bar{Q}_+ = \{(X_1, X_2, X_3) : 0 \leq X_1 \leq 1, X_2^2 + X_3^2 \leq 1\}.$$

Furthermore, the boundaries $x_1 = \xi(x_2, x_3)$, $x_1 = 1$ and $\sqrt{x_2^2 + x_3^2} = f(x)$ are transformed into $X_1 = 0$, $X_1 = 1$ and $X_2^2 + X_3^2 = 1$ respectively. Besides, the origin $(x_1, x_2, x_3) = (0, 0, 0)$ becomes the new origin $O = (0, 0, 0)$ in the coordinates $X = (X_1, X_2, X_3)$.

Now, as in [24], we define the new unknown function as

$$V(X) = 1 - x_1 + \varphi_-(x) - \varphi_+(x). \tag{2.3}$$

One would expect that $V(X) = 1 + O(\varepsilon)$ and $\nabla_X V(X) = O(\varepsilon)$. These properties are important in our later analysis. It now follows from (2.2) and (2.3) that

$$\begin{cases} x_1 = 1 + (X_1 - 1)V(X), \\ x_i = x_i(1 + (X_1 - 1)V(X), X_2, X_3), \qquad i = 2, 3, \end{cases} \tag{2.4}$$

here $x_i(1 + (X_1 - 1)V, X_2, X_3) \in C^{k_0}(i = 1, 2)$ on X and V, which follows from the smoothness of $f(x)$ and the assumption (1.7). By direct calculations, one has

$$\begin{cases} \partial_{x_j} X_1 = D(X, V, \nabla V)\Big(\delta_{ij} - (X_1 - 1)\sum_{i=2}^{3} \partial_{X_i} V \partial_{x_j} X_i\Big), \qquad j = 2, 3, \\ \partial_{x_j} X_i = f^{-2}(f\delta_{ij} - x_i\partial_{x_j} f) = \delta_{ij} + O(\varepsilon), \qquad i = 2, 3; \qquad j = 1, 2, 3, \end{cases} \tag{2.5}$$

and

$$\begin{cases} \partial^2_{x_jx_k} X_1 &= \sum_{i,l=1}^{3} b_{il}^{jk} \partial^2_{X_iX_l} V + b_0^{jk}, \qquad j, k = 1, 2, 3, \\ \partial^2_{x_jx_k} X_i &= -f^{-2}(\delta_{ij}\partial_{x_k} f + \delta_{ik}\partial_{x_j} f + x_i\partial^2_{x_jx_k} f) + 2f^{-3}x_i\partial_{x_j} f\partial_{x_k} f \\ &= O(\varepsilon), \qquad i = 2, 3, \qquad j, k = 1, 2, 3, \end{cases} \tag{2.6}$$

where

$$\begin{cases} D(X,V,\nabla V) = (V + (X_1-1)\partial_{X_1}V)^{-1}, \\ b_{il}^{jk} \equiv b_{il}^{jk}(X,V,\nabla V) = \frac{1}{2}D(X,V,\nabla V)(1-X_1)(\partial_{x_k}X_l\partial_{x_j}X_i + \partial_{x_k}X_i\partial_{x_j}X_l), \\ b_0^{jk} \equiv b_0^{jk}(X,V,\nabla V) = -D(X,V,\nabla V)\Big(2\partial_{x_k}X_1\partial_{x_j}X_1 + \partial_{x_k}X_2\partial_{x_j}X_1 + \partial_{x_j}X_2\partial_{x_k}X_1 \\ \quad +(X_1-1)\partial^2_{x_jx_k}X_2 + \partial_{x_j}X_3\partial_{x_k}X_1 + \partial_{x_k}X_3\partial_{x_j}X_1 + (X_1-1)\partial^2_{x_jx_k}X_3\Big)\nabla_X V. \end{cases} \tag{2.7}$$

It should be noted that for suitably small ε, all the functions $D(X,V,\nabla V)$, $b_{il}^{jk}(X,V,\nabla V)$ and $b_0^{jk}(X,V,\nabla V)$ are smooth functions of X, V and $\nabla_X V$.

In terms of the new variables (2.2) and (2.3), the equation (1.4) becomes

$$\sum_{i,j=1}^{3} a_{ij}(X,V,\nabla_X V)\partial^2_{X_iX_j}V + F_0(X,V,\nabla_X V) = 0, \tag{2.8}$$

where

$$\begin{cases} a_{ij}(X,V,\nabla_X V) = & -\sum_{k,l=1}^{3}\tilde{a}_{kl}(\varphi_+)(\partial_{x_k}X_i\partial_{x_l}X_j + b_{ij}^{kl}\partial_{X_1}V), \\ F_0(X,V,\nabla_X V) = & -\sum_{i,k=1}^{3}\tilde{a}_{ik}(\varphi_+)\Big(b_0^{ik}\partial_{X_1}V + \sum_{j=2}^{3}\partial_{X_j}V\partial^2_{x_ix_k}X_j\Big) \\ & +\sum_{i,k=1}^{3}(\tilde{a}_{ik}(\varphi_+) - \tilde{a}_{ik}(\varphi_-))\partial^2_{x_ix_k}\varphi_-, \end{cases} \tag{2.9}$$

here the matrix $(\tilde{a}_{ik})$ is defined by

$$\Big(\tilde{a}_{ij}(\varphi)\Big) = \begin{pmatrix} (\partial_{x_1}\varphi)^2 - C^2(\nabla\varphi) & \partial_{x_1}\varphi\partial_{x_2}\varphi & \partial_{x_1}\varphi\partial_{x_3}\varphi \\ \partial_{x_1}\varphi\partial_{x_2}\varphi & (\partial_{x_2}\varphi)^2 - C^2(\nabla\varphi) & \partial_{x_2}\varphi\partial_{x_3}\varphi \\ \partial_{x_1}\varphi\partial_{x_3}\varphi & \partial_{x_2}\varphi\partial_{x_3}\varphi & (\partial_{x_3}\varphi)^2 - C^2(\nabla\varphi) \end{pmatrix}$$

with sound speed $C(\nabla\varphi) = c(H(C_0 - \frac{1}{2}|\nabla\varphi|^2))$. In the derivation of (2.8) from (1.4), we have used the equations (1.8). It is important to note that the quasilinear equation (2.8) is uniformly elliptic on Q_+ in the region we are interested provided that ε is suitably small.

This and other important properties of $a_{ij}(X,V,\nabla_X V)$ and $F_0(X,V,\nabla_X V)$ are listed in Lemma 2.4.

Next, we transform the boundary conditions in terms of the new variables. First, it follows from (1.11) that the Rankine-Hugoniot condition (1.12) is equivalent to

$$\sum_{i=1}^{3}[\partial_i\varphi H]\partial_i(\varphi_+ - \varphi_-) = 0 \qquad \text{on} \qquad x_1 = \xi(x_2,x_3),$$

which takes the form in the new coordinates as

$$G(X,V,\nabla_X V) = 0 \qquad \text{on} \qquad X_1 = 0, \tag{2.10}$$

where

$$G(X,V,\nabla_X V) = H(C_0 - \frac{1}{2}((1+\partial_{x_1}V - \partial_1\varphi_-)^2 + \sum_{i=2}^{3}(\partial_{x_i}V - \partial_i\varphi_-)^2))\Big((\partial_1\varphi_- - \partial_{x_1}V - 1)(1+\partial_{x_1}V) + \sum_{i=2}^{3}(\partial_i\varphi_- - \partial_{x_i}V)\partial_{x_i}V\Big) - (\partial_1\varphi_-(1+\partial_{x_1}V) + \sum_{i=2}^{3}\partial_i\varphi_-\partial_{x_i}V)H(C_0 - \frac{1}{2}|\nabla\varphi_-|^2).$$

Analogously, (1.14) and (1.15) are transformed respectively as follows

$$H(C_0 - \frac{1}{2}((1+\partial_{x_1}V - \partial_1\varphi_-)^2 + \sum_{i=2}^{3}(\partial_{x_i}V - \partial_i\varphi_-)^2)) = \tilde{\rho}_+(x) \qquad \text{on} \qquad X_1 = 1, \tag{2.11}$$

$$\sum_{j=1}^{3}\Big(\sum_{i=2}^{3}(\frac{x_i}{f} - \partial_i f)\frac{\partial X_j}{\partial x_i} - \partial_1 f\frac{\partial X_j}{\partial x_1}\Big)\partial_{X_j}V = \partial_1 f(1-\partial_1\varphi_-) + \sum_{i=2}^{3}(\frac{x_i}{f} - \partial_i f)\partial_i\varphi_- \qquad \text{on} \qquad X_2^2 + X_3^2 = 1, \tag{2.12}$$

here the variable $x = (x_1, x_2, x_3)$ is a function of $X = (X_1, X_2, X_3)$ and $V(X)$. It will be clear that (2.10) and (2.11) represent nonlinear uniform oblique derivative boundary conditions for (2.8).

Finally, it follows from (1.10) and the transformation (2.2) that

$$V(0,0,0) = 1. \tag{2.13}$$

Hence our major problem is reduced to study the quasilinear equation (2.8) on the domain Q_+ with nonlinear boundary conditions (2.10)-(2.13).

By the assumptions in Theorem 1.1, we derive that $V(X) \in C^{1,1-\delta_0}(\bar{Q}_+) \cap C^3(\bar{Q}_+ \setminus \cup_{i=1}^{2}\Gamma_i)$ holds and satisfies the following estimates:

$$\|V(X) - 1\|_{C^{1,1-\delta_0}(\bar{Q}_+)} \le C\varepsilon, \qquad |\nabla_X^k V(X)| \le \frac{C\varepsilon}{|R_X|^{k-2+\delta_0}}, \qquad k = 2, 3, \tag{2.14}$$

where

$$\begin{cases} \Gamma_1 = \{(0, X_2, X_3) : X_2^2 + X_3^2 = 1\}, \\ \Gamma_2 = \{(1, X_2, X_3) : X_2^2 + X_3^2 = 1\}, \\ R_X = X_1(1 - X_1) + \sqrt{1 - (X_2^2 + X_3^2)} \end{cases} \tag{2.15}$$

Next we analyze the nonlinear boundary conditions (2.10)-(2.12) so that one can treat the uniqueness result in Theorem 1.1.

Let us consider the boundary condition for V on $X_1 = 0$ first. Note that the boundary condition (2.10) can be rewritten as

$$\sum_{i=1}^{3} B_{1i}(X,V,\nabla_X V)\partial_{X_i}V + B_1(X,V,\nabla_X V)(V-1) = -G(X,1,0,0,0) \qquad \text{on} \qquad X_1 = 0, \tag{2.16}$$

with

$$B_{1i}(X,V,\nabla_X V)=\int_0^1 \partial_{\partial_{X_i}V} G(X,\theta(V-1)+1,\theta\nabla_X V)d\theta, \qquad i=1,2,3,$$
$$B_1(X,V,\nabla_X V)=\int_0^1 \partial_V G(X,\theta(V-1)+1,\theta\nabla_X V)d\theta.$$

The following estimates hold true for the coefficients of (2.16).

Lemma 2.2. *In terms of (2.14), then one has*

$$\sum_{k=0}^{3}|\nabla_X^k G(X,1,0,0,0)|\le C\varepsilon, \tag{2.17}$$
$$B_{11}(X,V,\nabla_X V)=-\frac{\rho_+(q_+-q_0)(c^2(\rho_+)-q_+^2)}{c^2(\rho_+)}(1+O(\varepsilon)), \tag{2.18}$$
$$B_{1i}(X,V,\nabla_X V)=O(\varepsilon), \qquad i=2,3, \tag{2.19}$$
$$B_1(X,V,\nabla_X V)=O(\varepsilon), \tag{2.20}$$
$$\sum_{i=1}^{2}|\nabla_X^k B_{1i}(X,V,\nabla_X V)|+|\nabla_X^k B_1(X,V,\nabla_X V)|=O(\frac{\varepsilon}{R^{k-1+\delta_0}}), \qquad k=1,2. \tag{2.21}$$

Proof. We start with the proof of (2.17). By definition, one has

$$G(X,1,0,0,0)=\Big(H(C_0-\frac{1}{2}((1-\partial_1\varphi_-)^2+\sum_{i=2}^{3}(\partial_i\varphi_-)^2)(\partial_1\varphi_--1)-H(C_0-\frac{1}{2}|\nabla\varphi_-|^2)\partial_1\varphi_-\Big)(\bar{x})$$

with $\bar{x}=(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ given by

$$\begin{cases} \bar{x}_1=X_1, \\ \bar{x}_i=\bar{x}_i(X_1,X_2,X_3), \qquad i=2,3. \end{cases}$$

More precisely, $(\bar{x}_2,\bar{x}_3)$ is determined by

$$\begin{cases} X_2=\bar{x}_2(f(\bar{x}))^{-1}, \\ X_3=\bar{x}_3(f(\bar{x}))^{-1}. \end{cases}$$

Making use of Lemma 2.1 and the assumption $q_0-q_+=1$, one can compute to get

$$\begin{aligned} G(X,1,0,0,0)&=\Big(H\Big(C_0-\frac{1}{2}(q_0-1)^2+O(\varepsilon)\Big)(q_0-1+O(\varepsilon))-H\Big(C_0-\frac{1}{2}q_0^2+O(\varepsilon)(q_0+O(\varepsilon))\Big)(\bar{x})\\ &=\Big(H\Big(C_0-\frac{1}{2}(q_0-1)^2\Big)(q_0-1)-H\Big(C_0-\frac{1}{2}q_0^2\Big)q_0\Big)+O(\varepsilon)\\ &=(\rho_+q_+-\rho_0q_0)+O(\varepsilon)=O(\varepsilon)\end{aligned}$$

where we have used the fact that $\rho_+ q_+ = \rho_0 q_0$ (see (1.9)). Thus

$$|G(X,1,0,0,0)| \le C\varepsilon$$

Similarly, noting also (1.6), one has

$$\sum_{k=1}^{3} |\nabla_X^k G(X,1,0,0,0)| \le C\varepsilon.$$

Thus (2.17) is proved.

Next, we verify the rest of the lemma, i.e., (2.18)-(2.21). Direct calculations based on (2.4)-(2.5) show:

$$\begin{cases} \frac{\partial x_1}{\partial V} = X_1 - 1, \\ \frac{\partial x_i}{\partial V} = (X_1-1)\partial_{X_1} f (1-(X_2\partial_{x_2} f + X_3 \partial_{x_3} f))^{-1} X_i, \qquad i=2,3 \\ \frac{\partial(\partial_{x_i} V)}{\partial(\partial_{X_j} V)} = D(X,V,\partial_X V) V \frac{\partial X_j}{\partial x_i}, \qquad i,j=1,2,3, \\ \frac{\partial(\partial_i \varphi_-)}{\partial V} = \sum_{j=1}^{3} \partial_{ij}^2 \varphi_-(x) \frac{\partial x_j}{\partial V}, \qquad i=1,2,3, \\ \frac{\partial(\partial_i \varphi_+)}{\partial V} = \sum_{j=1}^{3} \Big(\partial_{ij}^2 \varphi_-(x) \frac{\partial x_j}{\partial V} - \partial_{X_j} V \frac{\partial}{\partial V} (\frac{\partial X_j}{\partial x_i}) \Big), \qquad i=1,2,3. \end{cases} \tag{2.22}$$

Define $\bar{G}(\nabla\varphi_+, \nabla\varphi_-) = \sum_{j=1}^{3} [\partial_j \varphi H] \partial_j (\varphi_+ - \varphi_-)$. Then $G(X,V,\nabla_X V) = \bar{G}(\nabla\varphi_+, \nabla\varphi_-)$ and one can compute

$$\begin{cases} \partial_{\partial_{X_i} V} G = -VD(X,V,\nabla_X V) \sum_{k=1}^{3} \partial_{\partial_k \varphi_+} \bar{G} \frac{\partial X_i}{\partial x_k}, \\ \partial_V G = \sum_{k=1}^{3} \Big(\partial_{\partial_k \varphi_+} \bar{G} \frac{\partial(\partial_k \varphi_+)}{\partial V} + \partial_{\partial_k \varphi_-} \bar{G} \frac{\partial(\partial_k \varphi_-)}{\partial V} \Big), \\ \partial_{\partial_i \varphi_+} \bar{G} = \sum_{j=1}^{3} \Big([\partial_j \varphi H] \delta_{ij} + (H_+ \delta_{ij} - \partial_j \varphi_+ \partial_i \varphi_+ H'_+)(\partial_j \varphi_+ - \partial_j \varphi_-) \Big), \qquad i=1,2,3 \\ \partial_{\partial_i \varphi_-} \bar{G} = -\sum_{j=1}^{3} \Big([\partial_j \varphi H] \delta_{ij} + (H_- \delta_{ij} - \partial_j \varphi_- \partial_i \varphi_- H'_-)(\partial_j \varphi_+ - \partial_j \varphi_-) \Big), \qquad i=1,2,3. \end{cases} \tag{2.23}$$

One can obtain from (2.5) and (2.14) that

$$\begin{cases} \partial_{x_i} X_1 = \delta_{1i} + O(\varepsilon), \qquad i=1,2,3 \\ \partial_{x_j} X_i = \delta_{ij} + O(\varepsilon), \qquad i=2,3, \qquad j=1,2,3 \end{cases} \tag{2.24}$$

and it follows from (2.22), (2.24) and Lemma 2.1 that

$$\begin{cases} (\frac{\partial(\partial_i \varphi_-)}{\partial V})(X,V) = O(\varepsilon), \qquad i=1,2,3 \\ (\frac{\partial(\partial_i \varphi_+)}{\partial V})(X,V) = O(\varepsilon), \qquad i=1,2,3 \\ (\frac{\partial(\partial_{x_i} V)}{\partial(\partial_{X_j} V)})(X,V,\nabla_X V) = \delta_{ij} + O(\varepsilon), \qquad i,j=1,2,3, \end{cases} \tag{2.25}$$

Note that

$$\begin{cases} \partial_{\partial_1\varphi_+}\bar{G}=\frac{\rho_+(q_+-q_0)(c^2(\rho_+)-q_+^2)}{c^2(\rho_+)}(1+O(\varepsilon)), \\ \partial_{\partial_1\varphi_-}\bar{G}=-\frac{\rho_+(q_+-q_0)(c^2(\rho_+)-q_+^2)}{c^2(\rho_+)}(1+O(\varepsilon)), \\ \partial_{\partial_i\varphi_\pm}\bar{G}=O(\varepsilon), \qquad i=2,3. \end{cases} \tag{2.26}$$

Indeed, recall that we have assumed that $q_+-q_0=1$, then it follows from $V(X)=1-x_1+\varphi_-(x)-\varphi_+(x)$ that $\partial_i\varphi_+=\partial_i\varphi_--\delta_{1i}-\nabla_X V\frac{\partial X}{\partial x_i}$. So one can derive from Lemma 2.1 and (2.24) that

$$\begin{cases} \partial_1\varphi_-=q_0+O(\varepsilon), \\ \partial_1\varphi_+=q_++O(\varepsilon), \\ \partial_i\varphi_\pm=O(\varepsilon), \qquad i=2,3 \end{cases} \tag{2.27}$$

On the other hand, the Bernoulli's law, (1.1) and (1.2) imply that $c^2(\rho)=\frac{H}{H'}$. Hence, one obtains from (2.23) and (2.27) that

$$\begin{aligned} \partial_{\partial_1\varphi_+}\bar{G} &= [\partial_1\varphi H]+\Big(H_+(\partial_1\varphi_+-\partial_1\varphi_-)-\sum_{j=1}^{3}(\partial_j\varphi_+\partial_1\varphi_+H'_+)(\partial_j\varphi_+-\partial_j\varphi_-)\Big) \\ &= (\partial_1\varphi_+H_+-\partial_1\varphi_-H_-)+H_+(\partial_1\varphi_+-\partial_1\varphi_-)\Big(1-(\partial_1\varphi_+)^2\frac{H'_+}{H_+}\Big)+O(\varepsilon) \\ &= \frac{\rho_+(q_+-q_0)(c^2(\rho_+)-q_+^2)}{c^2(\rho_+)}(1+O(\varepsilon)). \end{aligned}$$

The other estimates in (2.26) can be estimated similarly.

Substituting the computations above into expressions for B_{1i} and B_1, we obtain that

$$\begin{aligned} &B_{11}(X,V,\nabla_X V)=-\frac{\rho_+(q_+-q_0)(c^2(\rho_+)-q_+^2)}{c^2(\rho_+)}(1+O(\varepsilon)), \\ &B_{1i}(X,V,\nabla_X V)=O(\varepsilon), \qquad i=2,3, \\ &B_1(X,V,\nabla_X V)(W-1)=O(\varepsilon). \end{aligned}$$

These prove (2.18)-(2.20). the other property (2.21) can be verified similarly. Hence the proof of Lemma 2.2 is completed.

It follows from $q_+<c(\rho_+)$, $q_+<q_0$ and (2.18) in Lemma 2.2 that $B_{11}(X,V,\nabla_X V)\neq 0$ for small ε. Thus one can rewrite (2.16) as

$$\partial_{X_1}V+\sum_{i=2}^{3}\tilde{B}_{1i}(X,V,\nabla_X V)\partial_{X_i}V+\tilde{B}_1(X,V,\nabla_X V)(V-1)=0 \qquad \text{on} \qquad X_1=0, \tag{2.28}$$

where the coefficients satisfy the estimates

$$\begin{cases} \tilde{B}_{1i}(X,V,\nabla_X V)=O(\varepsilon), \qquad i=2,3, \\ \tilde{B}_1(X,V,\nabla_X V)=O(\varepsilon), \\ \nabla_X^k\tilde{B}_{1i}(X,V,\nabla_X V)=O(\frac{\varepsilon}{R^{k-1+\delta_0}}), \qquad k=1,2; \qquad i=2,3, \\ \nabla_X^k\tilde{B}_1(X,V,\nabla_X V)=O(\frac{\varepsilon}{R^{k-1+\delta_0}}), \qquad k=1,2, \end{cases} \tag{2.29}$$

which follow from Lemma 2.2.

Next we determine the boundary condition for $V(X)$ on $X_1 = 1$.

Set

$$\tilde{G}(X, V, \nabla_X V) = H(C_0 - \frac{1}{2}((1 + \partial_{x_1} V - \partial_1 \varphi_-)^2 + \sum_{i=2}^{3} (\partial_{x_i} V - \partial_i \varphi_-)^2)) - H(C_0 - \frac{1}{2} q_+^2).$$

It follows from $H(C_0 - \frac{1}{2} q_+^2) = \rho_+$ that (2.11) becomes

$$\tilde{G}(X, V, \nabla_X V) = \tilde{\rho}_+(x) - \rho_+ \qquad \text{on} \qquad X_1 = 1.$$

which may be written as

$$\sum_{i=1}^{3} B_{2i}(X, V, \nabla_X V) \partial_{X_i} V + B_2(X, V, \nabla_X V)(V - 1) = \tilde{G}(X, 1, 0, 0, 0) \qquad \text{on} \qquad X_1 = 1,$$

where

$$\begin{aligned} B_{2i}(X, V, \nabla_X V) &= \int_0^1 \partial_{\partial_{X_i} V} \tilde{G}(X, \theta(V - 1) + 1, \theta \nabla_X V) d\theta, \\ B_2(X, V, \nabla_X V) &= \int_0^1 \partial_V \tilde{G}(X, \theta(V - 1) + 1, \theta \nabla_X V) d\theta, \\ \tilde{G}(X, 1, 0, 0) &= \Big(H(C_0 - \frac{1}{2}((1 - \partial_1 \varphi_-)^2 + \sum_{i=2}^{3} (\partial_i \varphi_-)^2)) - H(C_0 - \frac{1}{2} q_+^2) \Big)(\bar{x}). \end{aligned}$$

As for (2.28), a direct computation yields

$$\partial_{X_1} V + \sum_{i=2}^{3} \tilde{B}_{2i}(X, V, \nabla_X V) \partial_{X_i} V + \tilde{B}_2(X, V, \nabla_X V)(V - 1) = \tilde{\rho}_+(x) - \rho_+ \qquad \text{on} \qquad X_1 = 1, \tag{2.30}$$

where

$$\left\{ \begin{array}{l} \tilde{B}_{2i}(X, V, \nabla_X V) = O(\varepsilon), \qquad i = 2, 3, \\ \tilde{B}_2(X, V, \nabla_X V) = O(\varepsilon), \\ \nabla_X^k \tilde{B}_{2i}(X, V, \nabla_X V) = O(\frac{\varepsilon}{R^{k-1+\delta_0}}), \qquad k = 1, 2; \qquad i = 2, 3, \\ \nabla_X^k \tilde{B}_2(X, V, \nabla_X V) = O(\frac{\varepsilon}{R^{k-1+\delta_0}}), \qquad k = 1, 2. \end{array} \right. \tag{2.31}$$

Similarly, we can obtain the boundary condition for $V(X)$ on $X_2^2 + X_3^2 = 1$ as follows

$$\sum_{i=1}^{3} \tilde{B}_{3i}(X, V, \nabla_X V) \partial_{X_i} V + \tilde{B}_3(X, V)(V - 1) = 0 \qquad \text{on} \qquad X_2^2 + X_3^2 = 1. \tag{2.32}$$

The properties of $\tilde{B}_{3i}$ and $\tilde{B}_3$ can be described by the following lemma whose proof will be omitted.

Lemma 2.3. *Under the assumptions in (2.14), it holds that*

$$\begin{aligned}
&\tilde{B}_{31}(X,V,\nabla_X V)=O(\varepsilon),\\
&\tilde{B}_{3i}(X,V,\nabla_X V)-X_i=O(\varepsilon),\qquad i=2,3,\\
&\nabla_X^k\tilde{B}_{31}(X,V,\nabla_X V)=O(\frac{\varepsilon}{R^{k-1+\delta_0}}),\qquad k=1,2,\\
&\nabla_X^k(\tilde{B}_{3i}(X,V,\nabla_X V)-X_i)=O(\frac{\varepsilon}{R^{k-1+\delta_0}}),\qquad k=1,2;\qquad i=2,3,\\
&\nabla_X^k\tilde{B}_3(X,V)=O(\varepsilon),\qquad k=0,1,\\
&\nabla_X^k\tilde{B}_3(X,V)=O(\frac{\varepsilon}{R^{k-2+\delta_0}}),\qquad k=2,3.
\end{aligned}$$

In addition, we need more information on $a_{ij}(X,V,\nabla_X V)$ and $F_0(X,V,\nabla_X V)$. In the following lemma, we list some important estimates on a_{ij} and F_0 which will be used later.

Lemma 2.4. *It follows from (2.14) that*

$$\begin{aligned}
&a_{11}(X,V,\nabla_X V)=-(q_+^2-c_+^2)(1+O(\varepsilon)),\\
&a_{ij}(X,V,\nabla_X V)=O(\varepsilon),\qquad 1\le i<j\le 3,\\
&a_{ii}(X,V,\nabla_X V)=-c_+^2(1+O(\varepsilon)),\qquad i=2,3,\\
&F_0(X,V,\nabla_X V)=O(\varepsilon),\\
&\nabla_X^k a_{ij}(X,V,\nabla_X V)=O(\frac{\varepsilon}{R^{k-1+\delta_0}}),\qquad k=1,2,\\
&\nabla_X^k F_0(X,V,\nabla_X V)=O(\frac{\varepsilon}{R^{k-1+\delta_0}}),\qquad k=1,2.
\end{aligned}$$

Proof. We will only sketch the proof since it mostly involves tedious computations. By (2.9), one has

$$\begin{aligned}
a_{11}(X,V,\nabla_X V)=&-\sum_{k=1}^{3}\Big((\partial_k\varphi_+)^2-c_+^2\Big)\Big((\partial_{x_k}X_1)^2+b_{11}^{kk}\partial_{X_1}V\Big)\\
&-\sum_{k\neq l}\partial_k\varphi_+\partial_l\varphi_+(\partial_{x_k}X_1\partial_{x_l}X_1+b_{11}^{kl}\partial_{X_1}V)
\end{aligned}$$

Taking into account of (2.24), (2.27), (2.5) and (2.14), one can get from above that

$$a_{11}(X,V,\nabla_X V)=-(q_+^2-c_+^2)(1+O(\varepsilon)). \tag{2.33}$$

Similarly, for $i=2,3$,

$$\begin{aligned}
a_{ii}(X,V,\nabla_X V)=&-\sum_{k=1}^{3}\Big((\partial_k\varphi_+)^2-c_+^2\Big)\Big((\partial_{x_k}X_i)^2+b_{ii}^{kk}\partial_{X_1}V\Big)\\
&-\sum_{k\neq l}\partial_k\varphi_+\partial_l\varphi_+(\partial_{x_k}X_i\partial_{x_l}X_i+b_{ii}^{kl}\partial_{X_1}V)\\
=&\,c_+^2\Big(\frac{\partial X_i}{\partial x_i}\Big)^2+O(\varepsilon)=c_+^2(1+O(\varepsilon))
\end{aligned} \tag{2.34}$$

and for $i \neq j$,

$$
\begin{aligned}
a_{ij}(X, V, \nabla_X V) &= -\sum_{k=1}^{3}\Big((\partial_k \varphi_+)^2 - c_+^2\Big)\Big(\partial_{x_k} X_i \partial_{x_k} X_j + b_{ij}^{kk}\partial_{X_1} V\Big) \\
&\quad - \sum_{k\neq l} \partial_k \varphi_+ \partial_l \varphi_+ (\partial_{x_k} X_i \partial_{x_l} X_j + b_{ij}^{kl}\partial_{X_1} V) \\
&= O(\varepsilon)
\end{aligned}
\tag{2.35}
$$

Next, noting (2.5)-(2.7), (2.24) and (2.14), one can estimate b_0^{lk} as

$$
|b_0^{lk}(X, V, \nabla_X V)| = O(\varepsilon).
$$

This, together with (2.9) and Lemma 2.1, leads to

$$
|F_0(X, V, \nabla_X V)| = O(\varepsilon).
$$

The rest of Lemma 2.4 follows from similar argument and direct computations. This proves the lemma.

So far we have outlined the linearization of the equation (2.8) and the boundary conditions (2.10)-(2.12) and derived some estimates on the corresponding coefficients. In the subsequent section, we will focus on the solvability of (2.8) with (2.28), (2.30), (2.32) and (2.13).

§3. The proof of Uniqueness

Based on the preparations in §2, we start to prove Theorem 1.1. Suppose that there are two solutions $(\varphi_+^1(x), \xi^1(x_2, x_3))$ and $(\varphi_+^2(x), \xi^2(x_2, x_3))$ to the equation (1.4) with (1.10)-(1.15), which satisfy the corresponding regularity conditions in Theorem 1.1. Through the general partial hodograph transformation (2.2), then one gets two corresponding solutions $V_1(X)$ and $V_2(X)$ to the equation (2.8) with the boundary conditions (2.28), (2.30), (2.32) and (2.13). Moreover, $V_j(X) \in C^{1,1-\delta_0}(\bar{Q}_+) \cap C^3(\bar{Q}_+ \setminus \cup_{i=1}^2 \Gamma_i)(j = 1, 2)$ and satisfy the estimates in (2.14). Our aim in this section is to prove $V_1(X) \equiv V_2(X)$ in $\bar{Q}_+$.

Set $v(X) = V_1(X) - V_2(X)$, then it follows from the equation (2.8) with (2.28), (2.30), (2.32) and (2.13) that

$$
\begin{cases}
\displaystyle\sum_{i,j=1}^{3} a_{ij}(X, V_1, \nabla_X V_1)\partial^2_{X_i X_j} v + \sum_{i=1}^{3} b_i(X)\partial_{X_i} v + c(X)v = 0, & X \in Q_+, \\
\displaystyle\partial_{X_1} v + \sum_{i=2}^{3} \gamma_{1i}(X)\partial_{X_i} v + d_1(X)v = 0 \qquad \text{on} \qquad X_1 = 0, & \\
\displaystyle\partial_{X_1} v + \sum_{i=2}^{3} \gamma_{2i}(X)\partial_{X_i} v + d_2(X)v = 0 \qquad \text{on} \qquad X_1 = 1, & \\
\displaystyle\sum_{i=1}^{3} \gamma_{3i}(X)\partial_{X_i} v + d_3(X)v = 0 \qquad \text{on} \qquad X_2^2 + X_3^2 = 1, & \\
v(0) = 0, &
\end{cases}
\tag{3.1}
$$

With respect to the regularities and the estimates of $b_i(X), c(X)$ and $\gamma_{ij}(X), d_i(X)$, in terms of (2.14), Lemma 2.1- 2.4 and the assumption on $\tilde{\rho}_+(x)$ we have

Lemma 3.1. *$b_i(X), c(X) \in C^1(\bar{Q}_+ \backslash \cup_{i=1}^2 \Gamma_i)$ and $\gamma_{ij}(X), d_i(X), d_i(X) \in C^1(\bar{Q}_+ \backslash \cup_{i=1}^2 \Gamma_i) \cap C^{1-\delta_0}(\bar{Q}_+)$. Moreover, these coefficients satisfy the following estimates*

$$\sum_{i=1}^3 |\nabla_X^k b_i(X)| + |\nabla_X^k c(X)| \le \frac{C\varepsilon}{R^{k+\delta_0}}, \qquad k = 0, 1,$$

$$\sum_j \|\gamma_{ij}\|_{C^{1-\delta_0}} + \sum_{j=1}^3 \|d_j\|_{C^{1-\delta_0}} \le C\varepsilon, \qquad i = 1, 2,$$

$$\sum_{j=2}^3 \|\gamma_{3j} - X_j\|_{C^{1-\delta_0}} + \sum_{i=1}^3 \|\gamma_{31}\|_{C^{1-\delta_0}} \le C\varepsilon,$$

$$\sum_j |\nabla_X^k \gamma_{ij}| + \sum_{j=1}^3 |\nabla_X^k d_j| \le \frac{C\varepsilon}{R^{k-1+\delta_0}}, \qquad i = 1, 2; \qquad k = 1, 2,$$

$$\sum_{j=2}^3 |\nabla_X^k (\gamma_{3j} - X_j)| + |\nabla_X^k \gamma_{31}| \le \frac{C\varepsilon}{R^{k-1+\delta_0}}, \qquad k = 1, 2.$$

As commented in [24], it seems to be difficult to apply the maximum principle directly to derive $v \equiv 0$. We intend to establish $\|v\|_{H^2(Q_+)} = 0$ to obtain $v \equiv 0$. To this end, we first need an inequality with Poincare's type.

Lemma 3.2. *If $u(X) \in H^2(Q_+)$ and $u(0) = 0$, then there exists a constant C independent of u such that*

$$\int_{Q_+} |u|^2 dX \le C \int_{Q_+} (|\nabla u|^2 + |\nabla^2 u|^2) dX. \tag{3.2}$$

Proof. We use the techniques in [19] to prove (3.2).

If (3.2) does not hold, then for each $m \in \mathbb{N}$, there exists a function $u_m \in H^2(Q_+)$ with $u_m(0) = 0$ such that

$$\int_{Q_+} |u_m|^2 dX > m \int_{Q_+} (|\nabla u_m|^2 + |\nabla^2 u_m|^2) dX.$$

Let $v_m = \frac{u_m}{\|u_m\|_{L^2(Q_+)}}$, then v_m has the following properties

(i). $\|v_m\|_{L^2(Q_+)} = 1$.

(ii). $v_m(0) = 0$.

(iii). $v_m \in H^2(Q_+)$.

(iv). $\int_{Q_+} (|\nabla v_m|^2 + |\nabla^2 v_m|^2) dX < \frac{1}{m}$.

By (i) and (iv) one knows easily that there exist a subsequence $\{v_{m_j}\} \subset \{v_m\}$ and a function $v \in H^2(Q_+)$ such that

$$v_{m_j} \rightharpoonup v, \qquad H^2(Q_+).$$

It follows from (iv) that $v = C, a.e. X \in Q_+$. In addition, $v_{m_j} \rightharpoonup v, H^2(Q_+)$ implies $v_{m_j} \to v$ in $C(\bar{Q}_+)$. Thus $v(0) = 0$ and $v \equiv 0$. But this is contradictory with $\|v\|_{L^2(Q_+)} = 1$. Hence Lemma 3.2 is proved.

Lemma 3.3. *For the problem (3.1), if v has the regularities and estimates in (2.14), then for suitably small ε we have*

$$v(X) \equiv 0.$$

Proof. This proof procedure will be divided into three steps.

Step 1. Estimate on $\|\nabla v\|_{L^2(Q_+)}$.

Multiplying v on two sides of (3.1) and integrating on Q_+ by parts, we have

$$\int_{Q_+} -\sum_{i,j=1}^{3} a_{ij}\partial_i v \partial_j v dX = \sum_{i=1}^{5} I_5, \tag{3.3}$$

where

$$I_1 = \int_{Q_+} \sum_{i,j=1}^{3} \partial_i a_{ij} \partial_j v v dX,$$

$$I_2 = \int_{X_1=0} \sum_{j=1}^{3} a_{1j}\partial_j v v dS,$$

$$I_3 = -\int_{X_1=1} \sum_{j=1}^{3} a_{1j}\partial_j v v dS,$$

$$I_4 = -\int_{X_2^2+X_3^2=1} \sum_{i=2}^{3}(\sum_{j=1}^{3} a_{ij}\partial_j v v)X_i dS,$$

$$I_5 = -\int_{Q_+} \left(\sum_{i=1}^{3} b_i(X)\partial_i v + c(X)v\right) v dX.$$

$I_i (i = 1, ..., 5)$ will be treated respectively.

(i). Estimate on I_1.

For a small constant $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$|\sum_{i,j=1}^{3} \partial_i a_{ij}\partial_j v v| \le \delta |\nabla v|^2 + C_\delta \sum_{i,j=1}^{3} |\nabla a_{ij}|^2 |v|^2.$$

Hence by Lemma 2.4, Sobolev's imbedding theorem and Lemma 3.2, one has

$$\begin{aligned}
|I_1| &\le \delta \int_{Q_+} |\nabla v|^2 dX + C_\delta \int_{Q_+} \frac{\varepsilon^2 |v|^2}{R^{2\delta_0}} dX \\
&\le \delta \int_{Q_+} |\nabla v|^2 dX + C_\delta \varepsilon^2 (\int_{Q_+} |v|^4 dX)^{\frac{1}{2}} \\
&\le \delta \int_{Q_+} |\nabla v|^2 dX + C_\delta \varepsilon \int_{Q_+} (|v|^2 + |\nabla v|^2) dX \\
&\le \delta \int_{Q_+} |\nabla v|^2 dX + C_\delta \varepsilon \int_{Q_+} (|\nabla v|^2 + |\nabla^2 v|^2) dX.
\end{aligned}$$

(ii). Estimate on I_2 and I_3.

We only estimate I_2; the estimate on I_3 is completely parallel.

We rewrite I_2 as

$$I_2 = I_{21} + I_{22},$$

where

$$I_{21} = \int_{X_1=0} a_{11}\partial_1 v v dS,$$
$$I_{22} = \int_{X_1=0} \sum_{j=2}^{3} a_{1j}\partial_j v v dS.$$

By the boundary conditions in (3.1), the trace theorem and Lemma 3.2, one has

$$\begin{aligned} |I_{21}| &\le C\varepsilon \int_{X_1=0} (|v|^2 + |\nabla v|^2) dS \\ &\le C\varepsilon \int_{Q_+} (|\nabla v|^2 + |\nabla^2 v|^2) dX. \end{aligned}$$

Similarly, it follows from Lemma 2.4 that the same estimate holds for I_{22}. Thus

$$|I_2| \le C\varepsilon \int_{Q_+} (|\nabla v|^2 + |\nabla^2 v|^2) dX.$$

(iii). Estimate on I_4.

It follows from the boundary conditions in (3.1), Lemma 2.3 and Lemma 2.4 that

$$\begin{aligned} |I_4| \le & \int_{X_2^2+X_3^2=1} |X_2\Big(\sum_{j\neq 2} a_{2j}\partial_j v\Big) + X_3\Big(\sum_{j\neq 3} a_{3j}\partial_j v\Big)||v| dS \\ &+ \int_{X_2^2+X_3^2=1} |(a_{22}+c_+^2)X_2\partial_2 v + (a_{33}+c_+^2)X_3\partial_3 v||v| dS \\ &+ c_+^2 \int_{X_2^2+X_3^2=1} |(\gamma_{32}-X_2)\partial_2 v + (\gamma_{33}-X_3)\partial_3 v||v| dS + c_+^2 \int_{X_2^2+X_3^2=1} |\gamma_{31}\partial_1 v + d_3 v||v| dS \\ &\le C\varepsilon \int_{X_2^2+X_3^2=1} (|v|^2 + |\nabla v|^2) dS. \end{aligned}$$

Hence by the trace theorem and Lemma 3.2, one gets

$$|I_4| \le C\varepsilon \int_{Q_+} (|\nabla v|^2 + |\nabla^2 v|^2) dX.$$

(iv). Estimate on I_5

$$|I_5| \le C\varepsilon \int_{Q_+} \frac{(|\nabla v| + |v|)|v|}{R^{\delta_0}} dX \le C\varepsilon \int_{Q_+} (|\nabla v|^2 + |\nabla^2 v|^2) dX.$$

Substituting all the above estimates on $I_i (1 \le i \le 5)$ into (3.3), then for small $\delta > 0$ and $\varepsilon > 0$ we obtain

$$\int_{Q_+} |\nabla v|^2 dX \le C\varepsilon \int_{Q_+} |\nabla^2 v|^2 dX. \tag{3.4}$$

Step 2. Estimate on $\|\nabla \partial_1 v\|_{L^2(Q_+)}$.
Set $w_i = \partial_i v (i = 1, 2, 3)$. Then w_1 satisfies

$$\begin{cases} \displaystyle\sum_{i,j=1}^{3} a_{ij}\partial_{ij}^2 w_1 + \sum_{i,j=1}^{3} \partial_1 a_{ij}\partial_i w_j + \partial_1\Big(\sum_{i=1}^{3} b_i(X)\partial_{X_i} v + c(X)v\Big) = 0, \\ \displaystyle w_1 + \sum_{i=2}^{3} \gamma_{1i} w_i + d_1 v = 0 \qquad \text{on} \qquad X_1 = 0, \\ \displaystyle w_1 + \sum_{i=2}^{3} \gamma_{2i} w_i + d_2 v = 0 \qquad \text{on} \qquad X_1 = 1, \\ \displaystyle\sum_{i=1}^{3} \gamma_{3i}\partial_i w_1 + \sum_{i=1}^{3} \partial_1\gamma_{3i} w_i + d_3 w_1 + \partial_1 d_3 v = 0 \qquad \text{on} \qquad X_2^2 + X_3^2 = 1. \end{cases} \tag{3.5}$$

Multiplying w_1 on two sides of (3.5) and integrating by parts in Q_+, we get

$$\int_{Q_+} -\sum_{i,j=1}^{3} a_{ij}\partial_i w_1 \partial_j w_1 dX = \sum_{i=1}^{5} J_i, \tag{3.6}$$

where

$$\begin{aligned} J_1 &= \int_{Q_+} \sum_{i,j=1}^{3} \partial_i a_{ij}\partial_j w_1 w_1 dX - \int_{Q_+} \sum_{i,j=1}^{3} \partial_1 a_{ij}\partial_i w_j w_1 dX, \\ J_2 &= \int_{X_1=0} \sum_{j=1}^{3} a_{1j}\partial_j w_1 w_1 dS, \\ J_3 &= -\int_{X_1=1} \sum_{j=1}^{3} a_{1j}\partial_j w_1 w_1 dS, \\ J_4 &= -\int_{X_2^2+X_3^2=1} \sum_{i=2}^{3}(\sum_{j=1}^{3} a_{ij}\partial_j w_1 w_1) X_i dS, \\ J_5 &= -\int_{Q_+} \partial_1\Big(\sum_{i=1}^{3} b_i(X)\partial_{X_i} v + c(X)v\Big) w_1 dX. \end{aligned}$$

Now we treat $J_i (i = 1, ..., 5)$ separately.
(i). Estimate on J_1.

By Lemma 2.4, Sobolev's imbedding theorem and $\delta_0 < \frac{1}{2}$, one can get

$$\begin{aligned}|J_1| &\le \delta \int_{Q_+} \sum_{j=1}^{3} |\nabla w_j|^2 dX + C_\delta \int_{Q_+} \frac{\varepsilon^2 |w_1|^2}{R^{2\delta_0}} dX \\ &\le \delta \int_{Q_+} \sum_{j=1}^{3} |\nabla w_j|^2 dX + C_\delta \varepsilon^2 \int_{Q_+} (|w_1|^2 + |\nabla w_1|^2) dX.\end{aligned}$$

Substituting (3.4) into the above expression, we get

$$|J_1| \le C(\delta\varepsilon + C_\delta \varepsilon^2) \int_{Q_+} |\nabla^2 v|^2 dX. \tag{3.7}$$

(ii). Estimate on J_2 and J_3.

We decompose J_2 as

$$J_2 = J_{21} + J_{22}$$

with

$$\begin{aligned}J_{21} &= \int_{X_1=0} \sum_{j=2}^{3} a_{1j} \partial_j w_1 w_1 dS, \\ J_{22} &= \int_{X_1=0} a_{11} \partial_1 w_1 w_1 dS.\end{aligned}$$

We treat the term J_{21} first.

Integrating by parts leads to

$$2J_{21} = -\int_{X_1=0} \sum_{j=2}^{3} \partial_j a_{1j} w_1^2 dS + \int_L \sum_{j=2}^{3} a_{1j} X_j w_1^2 dl,$$

here $L = \{X : X_1 = 0, X_2^2 + X_3^2 = 1\}$.

The first term on the right hand side above can be treated by Lemma 2.4 and Sobolev's imbedding theorem as

$$\left|\int_{X_1=0} \sum_{j=2}^{3} \partial_j a_{1j} w_1^2 dS\right| \le C \int_{X_1=0} \frac{\varepsilon w_1^2}{R^{\delta_0}} dS \le C\varepsilon \int_{Q_+} (|w_1|^2 + |\nabla w_1|^2) dX.$$

It is a bit more difficult to treat the second term in $2J_{21}$ because one can not use the trace theorem to control $\int_L |w_1|^2 dl$ directly by $\int_{Q_+} (|w_1|^2 + |\nabla w_1|^2) dX$. To overcome this difficulty, we will use $\int_L |\partial_\theta v|^2 dl$ to control $\int_L |w_1|^2 dl$ since $\int_L |\partial_\theta v|^2 dl$ can be estimated by the trace theorem, here $\partial_\theta = X_2 \partial_3 - X_3 \partial_2$.

Indeed, it follows from the trace theorem that

$$\int_L |\partial_\theta v|^2 dl \le \|v|_L\|^2_{H^1(L)} \le C \int_{Q_+} (|v|^2 + |\nabla v|^2 + |\nabla^2 v|^2) dX. \tag{3.8}$$

Additionally, by the boundary conditions in (3.1) and the expression of $\partial_\theta v$, one gets on L

$$\begin{cases} w_1 + \sum_{i=2}^{3} \gamma_{1i} w_i + d_1 v = 0, \\ \sum_{i=1}^{3} \gamma_{3i} w_i + d_3 v = 0, \\ X_2 w_3 - X_3 w_2 = \partial_\theta v. \end{cases} \tag{3.9}$$

By Lemma 2.2 and Lemma 2.3, we obtain that on the curve L

$$w_i = C_{i1}(X) d_1 v + C_{i2}(X) d_3 v + C_{i3}(X) \partial_\theta v, \qquad i = 1, 2, 3, \tag{3.10}$$

here $|C_{ij}(X)| \le C$.

It follows from (3.10), (3.8), Lemma 3.2 and (3.4) that

$$\begin{aligned} |\int_L \sum_{j=2}^{3} a_{1j} X_j w_1^2 dl| \le C\varepsilon \int_L |w_1|^2 dl \le C\varepsilon \int_L (|\partial_\theta v|^2 + |v|^2) dl \\ \le C\varepsilon \int_{Q_+} |\nabla^2 v|^2 dX. \end{aligned}$$

Hence

$$|J_{21}| \le C\varepsilon \int_{Q_+} |\nabla^2 v|^2 dX. \tag{3.11}$$

We now treat the term J_{22}.

¿From the equation (3.1), we derive that

$$\partial_1 w_1 = -\frac{1}{a_{11}} \Big(\sum_{j=2}^{3} a_{1j} \partial_j w_1 + \sum_{i,j=2}^{3} a_{ij} \partial_i w_j + \sum_{i=1}^{3} b_i(X) \partial_{X_i} v + c(X) v \Big).$$

Then

$$|J_{22}| \le |J_{21}| + |J'_{22}| + \Big| \int_{X_1=0} \Big(\sum_{i=1}^{3} b_i(X) \partial_{X_i} v + c(X) v \Big) w_1 dS \Big|$$

with $J'_{22} = -\int_{X_1=0} \sum_{i,j=2}^{3} a_{ij} \partial_i w_j w_1 dS$.

Substituting the boundary condition in (3.1) into J'_{22} yields

$$J'_{22} = \int_{X_1=0} \sum_{i,j=2}^{3} a_{ij} \partial_i w_j (\sum_{k=2}^{3} \gamma_{1k} w_k + d_1 v) dS.$$

Noting $\partial_i w_j w_k = \frac{1}{2}[\partial_i(w_j w_k) - \partial_k(w_i w_j) + \partial_j(w_i w_k)]$, then analogous to the treatment on J_{21}, we obtain

$$|J'_{22}| \le C\varepsilon \int_{Q_+} |\nabla^2 v|^2 dX. \tag{3.12}$$

Thus

$$|J_2| \leq C\varepsilon \int_{Q_+} |\nabla^2 v|^2 dX. \tag{3.13}$$

(iii). Estimate on J_4.

Set

$$J_4 = J_{41} + J_{42}$$

with

$$J_{41} = -\int_{X_2^2+X_3^2=1} \sum_{i=2}^{3} X_i a_{i1} \partial_1 w_1 w_1 dS$$

and

$$J_{42} = -\int_{X_2^2+X_3^2=1} \sum_{i,j=2}^{3} X_i a_{ij} \partial_j w_1 w_1 dS.$$

Since

$$J_{41} = -\frac{1}{2}\int_{X_2^2+X_3^2=1} \partial_1(\sum_{i=2}^{3} X_i a_{i1} w_1^2) dS + \frac{1}{2}\int_{X_2^2+X_3^2=1} \sum_{i=2}^{3} \partial_1(X_i a_{i1}) w_1^2 dS,$$

then one has

$$|J_{41}| \leq C\varepsilon \int_L |w_1|^2 dl + C\varepsilon(\int_{X_2^2+X_3^2=1} |w_1|^4 dS)^{\frac{1}{2}} \leq C\varepsilon \int_{Q_+} |\nabla^2 v|^2 dX.$$

Next we estimate J_{42}.

In terms of the cylindrical coordinates

$$\begin{cases} X_1 = X_1, \\ X_2 = r\cos\theta, \\ X_3 = r\sin\theta, \end{cases} \tag{3.14}$$

one has

$$J_{42} = -\int_0^1 \int_0^{2\pi} [(a_{22}cos^2\theta + a_{33}sin^2\theta + a_{23}sin2\theta)\partial_r w_1 w_1 + (a_{23}cos2\theta + (a_{33}-a_{22})sin\theta cos\theta)\partial_\theta w_1 w_1] d\theta dX_1.$$

It follows from the third boundary condition in (3.5) that

$$\partial_r w_1 = D_{11}(X)\partial_1 w_1 + D_{12}(X)\partial_\theta w_1 + \sum_{i=1}^{3} D_{13}^i(X) w_i + D_{14}(X) v \qquad \text{on} \qquad X_2^2 + X_3^2 = 1,$$

where

$$|D_{11}(X)| + |D_{12}(X)| \leq C\varepsilon,$$
$$|\nabla D_{11}(X)| + |\nabla D_{12}(X)| \leq \frac{C\varepsilon}{R^{\delta_0}},$$
$$\sum_{i=1}^{3} |D_{13}^i(X)| \leq \frac{C\varepsilon}{R^{\delta_0}},$$
$$|D_{14}(X)| \leq \frac{C\varepsilon}{R^{\delta_0}}.$$

By Lemma 2.4, integration by parts and (3.8), we get

$$|J_{42}| \le C\varepsilon\Big(\int_{Q_+}(|w_1|^2+|\nabla w_1|^2)dX+\int_L|\partial_\theta v|^2dl+\int_L\frac{|v|^2}{R^{2\delta_0}}\Big)$$
$$\le C\varepsilon\int_{Q_+}|\nabla^2 v|^2dX.$$

Hence one has

$$|J_4| \le C\varepsilon\int_{Q_+}|\nabla^2 v|^2dX. \tag{3.15}$$

(iv). Estimate on J_5.

$$|J_5| \le C\varepsilon\Big(\int_{Q_+}\frac{|\nabla v|^2+|v|^2}{R^{1+\delta_0}}dX+\int_{Q_+}\frac{|\nabla^2 v||\nabla v|}{R^{\delta_0}}dX\Big).$$

Since $H^1(\bar{Q}_+)\subset L^6(Q_+)$ and $H^{\frac12}(\bar{Q}_+)\subset L^3(Q_+)$, noting $0<\delta_0<\frac13$, then we have

$$\int_{Q_+}\frac{|\nabla v|^2}{R^{1+\delta_0}}dX \le \Big(\int_{Q_+}\frac{dX}{R^{\frac32(1+\delta_0)}}\Big)^{\frac23}\Big(\int_{Q_+}|\nabla v|^6dX\Big)^{\frac13} \le C\int_{Q_+}\Big(|\nabla v|^2+|\nabla^2 v|^2\Big)dX,$$
$$\int_{Q_+}\frac{|v|^2}{R^{1+\delta_0}}dX \le C\Big(\int_{Q_+}|v|^6dX\Big)^{\frac13} \le C\int_{Q_+}(|v|^2+|\nabla v|^2)dX,$$
$$\int_{Q_+}\frac{|\nabla^2 v||\nabla v|}{R^{\delta_0}}dX \le \int_{Q_+}|\nabla^2 v|^2dX+\int_{Q_+}\frac{|\nabla v|^2}{R^{2\delta_0}}dX \le C\int_{Q_+}(|\nabla v|^2+|\nabla^2 v|^2)dX.$$

It follows that

$$|J_5| \le C\varepsilon\int_{Q_+}|\nabla^2 v|^2dX. \tag{3.16}$$

Finally, substituting the all estimates on $J_i(1\le i\le 5)$ into (3.6) yields

$$\int_{Q_+}|\nabla\partial_1 v|^2dX \le C\varepsilon\int_{Q_+}|\nabla^2 v|^2dX. \tag{3.17}$$

Step 3. Estimate on $\sum_{k=2}^{3}\|\nabla\partial_k v\|_{L^2(Q_+)}$.

Since $w_k=\partial_k v(k=2,3)$ satisfies

$$\begin{cases}\displaystyle\sum_{i,j=1}^{3}a_{ij}\partial_{ij}^2 w_k+\sum_{i,j=1}^{3}\partial_k a_{ij}\partial_i w_j+\partial_k\Big(\sum_{i=1}^{3}b_i(X)\partial_{X_i}v+c(X)v\Big)=0,\\ \displaystyle\partial_1 w_k+\sum_{i=2}^{3}\gamma_{1i}\partial_i w_k++\sum_{i=2}^{3}\partial_k\gamma_{1i}w_i+\partial_k(d_1v)=0 \qquad \text{on} \qquad X_1=0,\\ \displaystyle\partial_1 w_k+\sum_{i=2}^{3}\gamma_{2i}\partial_i w_k++\sum_{i=2}^{3}\partial_k g_{2i}w_i+\partial_k(d_2v)=0 \qquad \text{on} \qquad X_1=1,\\ \displaystyle\sum_{i=1}^{3}\gamma_{3i}w_i+d_3v=0 \qquad \text{on} \qquad X_2^2+X_3^2=1,\end{cases} \tag{3.18}$$

then multiplying w_k on two sides of (3.18) and integrating by parts in Q_+, we get

$$\int_{Q_+} -\sum_{k=2}^{3}\sum_{i,j=1}^{3} a_{ij}\partial_i w_k \partial_j w_k dX = \sum_{i=1}^{5} K_i. \tag{3.19}$$

where

$$K_1 = \int_{Q_+} \sum_{k=2}^{3}\sum_{i,j=1}^{3} \partial_i a_{ij}\partial_j w_k w_k dX - \int_{Q_+} \sum_{k=2}^{3}\sum_{i,j=1}^{3} \partial_k a_{ij}\partial_i w_j w_k dX,$$

$$K_2 = \int_{X_1=0} \sum_{k=2}^{3}\sum_{j=1}^{3} a_{1j}\partial_j w_k w_k dS,$$

$$K_3 = -\int_{X_1=1} \sum_{k=2}^{3}\sum_{j=1}^{3} a_{1j}\partial_j w_k w_k dS,$$

$$K_4 = -\int_{X_2^2+X_3^2=1} \sum_{k=2}^{3}\sum_{i=2}^{3}\Big(\sum_{j=1}^{3} a_{ij}\partial_j w_k w_k\Big) X_i dS,$$

$$K_5 = -\int_{Q_+} \sum_{k=2}^{3} \partial_k \Big(\sum_{i=1}^{3} b_i(X)\partial_{X_i} v + c(X)v\Big) w_k dX.$$

The terms $K_i (i = 1, 2, 3, 5)$ can be treated analogously as for $J_i (i = 1, 2, 3, 5)$ in Step 2. Namely, one has

$$|K_1| + |K_2| + |K_3| + |K_5| \le C\varepsilon \int_{Q_+} |\nabla^2 v|^2 dX. \tag{3.20}$$

However, one can not expect to control $|K_4|$ in terms of $C\varepsilon \int_{Q_+} |\nabla^2 v|^2 dX$ since it contains a term $-\int_{X_2^2+X_3^2=1} |\partial_\theta v|^2 dS$ with no small coefficients. So additional case is needed. Our main observation is that

$$K_4 - \int_{X_2^2+X_3^2=1} a_{22}|\partial_\theta v|^2 dS \ge -C\varepsilon \int_{Q_+} |\nabla^2 v|^2 dX, \tag{3.21}$$

which, together (3.19)-(3.20), yields the derived estimate.

We now verify (3.21). First we decompose K_4 as

$$K_4 = K_{41} + K_{42}$$

with

$$K_{41} = -\int_{X_2^2+X_3^2=1} \sum_{k=2}^{3}\Big(\sum_{i=2}^{3} a_{i1}\partial_1 w_k w_k X_i\Big) dS,$$

$$K_{42} = -\int_{X_2^2+X_3^2=1} \sum_{k=2}^{3}\sum_{i,j=2}^{3} a_{ij} X_i \partial_j w_k w_k dS.$$

In a similar way as for J_{41}, one can show

$$|K_{41}| \le C\varepsilon \int_L |w_k|^2 dl + C\varepsilon \Big(\int_{X_2^2+X_3^2=1} |w_k|^4 dS\Big)^{\frac{1}{2}} \le C\varepsilon \int_{Q_+} |\nabla^2 v|^2 dX. \tag{3.22}$$

Next we estimate K_{42}.

Using the cylindrical coordinate transformation (3.14), one has

$$K_{42} = K'_{42} + K''_{42} \tag{3.23}$$

with

$$K'_{42} = -\int_{X_2^2+X_3^2=1} a_{22}(\partial_r^2 v \partial_r v + \partial_{r\theta}^2 v \partial_\theta v - (\partial_\theta v)^2) dS,$$

$$K''_{42} = \int_{X_2^2+X_3^2=1} \Big(E_{11}(X)\partial_r^2 v + E_{12}(X)\partial_{r\theta}^2 v + E_{13}(X)\partial_\theta^2 v + E_{14}(X)\partial_r v + E_{15}(X)\partial_\theta v\Big)\Big(H_1(\theta)\partial_r v + H_2(\theta)\partial_\theta v\Big) dS,$$

where

$$\sum_{i=1}^{5} |E_{1i(X)}| \le C\varepsilon,$$

$$\sum_{i=1}^{5} |\nabla E_{1i}(X)| \le \frac{C\varepsilon}{R^{\delta_0}},$$

and $H_i(\theta)(i=1,2)$ are smooth functions on θ.

¿From the equation (3.1), we have on $X_2^2 + X_3^2 = 1$

$$\begin{aligned} \partial_r^2 v = {} & E_{21}(X)\partial_1^2 v + E_{22}\partial_\theta^2 v + E_{23}(X)\partial_{1r}^2 v + E_{24}(X)\partial_{1\theta}^2 v + E_{25}(X)\partial_{r\theta}^2 v \\ & + E_{26}(X)\partial_\theta v + E_{27}\partial_r v + E_{28}(X) v, \end{aligned} \tag{3.24}$$

here

$$|E_{21}| + |E_{22}| \le C,$$

$$|\nabla E_{21}| + |\nabla E_{22}| \le \frac{C\varepsilon}{R^{\delta_0}},$$

$$\sum_{j=3}^{5} |E_{2j}| + |E_{28}| \le C\varepsilon,$$

$$\sum_{j=3}^{5} |\nabla E_{2j}| \le \frac{C\varepsilon}{R^{\delta_0}},$$

$$|E_{26}| + |E_{27}| + |E_{28}| \le \frac{C\varepsilon}{R^{\delta_0}}, \qquad |\nabla E_{26}| + |\nabla E_{27}| + |\nabla E_{28}| \le \frac{C\varepsilon}{R^{1+\delta_0}}.$$

Additionally, it follows from the boundary condition in (3.1) that

$$\partial_r v = E_{31}(X)\partial_1 v + E_{32}(X)\partial_\theta v + E_{33}(X) v \qquad \text{on} \qquad X_2^2 + X_3^2 = 1, \tag{3.25}$$

here

$$|E_{31}|+|E_{32}|+|E_{33}|\leq C\varepsilon,$$
$$|\nabla E_{31}|+|\nabla E_{32}|+|\nabla E_{33}|\leq \frac{C\varepsilon}{R^{\delta_0}}.$$

Substituting (3.24) and (3.25) into K'_{42} yields

$$K'_{42}=\int_{X_2^2+X_3^2=1}a_{22}(\partial_\theta v)^2dS+|\int_{X_2^2+X_3^2=1}G(\partial_1 v,\partial_1^2 v,\partial_\theta v,\partial_\theta^2 v,\partial_{1\theta}^2 v)dS|$$

with

$$\begin{aligned}G(\partial_1 v,&\partial_1^2 v,\partial_\theta v,\partial_\theta^2 v,\partial_{1\theta}^2 v)=E_{51}(X)\partial_1^2 v\partial_1 v+E_{52}(X)\partial_1^2 v\partial_\theta v+E_{53}(X)\partial_\theta^2 v\partial_1 v\\&+E_{54}(X)\partial_\theta^2 v\partial_\theta v+E_{55}(X)\partial_{1\theta}^2 v\partial_1 v+E_{56}(X)\partial_{1\theta}^2 v\partial_\theta v+E_{57}(X)(\partial_1 v)^2+E_{58}(X)\partial_1 v\partial_\theta v\\&+E_{59}(X)(\partial_\theta v)^2+N_1(X)\partial_1^2 vv+N_2(X)\partial_\theta^2 vv+N_3(X)\partial_1 vv+N_4(X)\partial_\theta vv+N_5(X)v^2,\end{aligned}$$

where

$$\begin{aligned}&\sum_{i=1}^6|E_{5i(X)}|\leq C\varepsilon,\qquad \sum_{i=1}^6|\nabla E_{5i(X)}|\leq\frac{C\varepsilon}{R^{\delta_0}},\\&\sum_{i=7}^9|E_{5i(X)}|\leq \frac{C\varepsilon}{R^{\delta_0}},\qquad \sum_{i=7}^9|\nabla E_{5i(X)}|\leq\frac{C\varepsilon}{R^{1+\delta_0}},\\&|N_1(X)|+|N_2(X)|\leq C\varepsilon,\qquad |\nabla N_1(X)|+|\nabla N_2(X)|\leq\frac{C\varepsilon}{R^{\delta_0}},\\&|N_3(X)|+|N_4(X)|+|N_5(X)|\leq\frac{C\varepsilon}{R^{\delta_0}},\qquad |\nabla N_3(X)|+|\nabla N_4(X)|+|\nabla N_5(X)|\leq\frac{C\varepsilon}{R^{1+\delta_0}}.\end{aligned}$$

Similar to the treatment on J_{42} in Step 2, we have

$$\begin{aligned}\Big|\int_{X_2^2+X_3^2=1}G(\partial_1 v,\partial_1^2 v,\partial_\theta v,\partial_\theta^2 v,\partial_{1\theta}^2 v)dS\Big|&\leq C\varepsilon\Big(\int_{X_2^2+X_3^2=1}\frac{|\nabla v|^2+|v|^2}{R^{\delta_0}}dS\\&+\int_L(|\nabla v|^2+|v|^2)dl+\int_L\frac{|v|^2}{R^{1+\delta_0}}dl\Big)\\&\leq C\varepsilon\Big(\int_{Q_+}|\nabla^2 v|^2dX+(\int_L|v|^6dl)^{\frac13}\Big)\\&\leq C\varepsilon\int_{Q_+}|\nabla^2 v|^2dX.\end{aligned}\tag{3.26}$$

By the same method, we can conclude that

$$|K''_{42}|\leq C\varepsilon\int_{Q_+}|\nabla^2 v|^2dX.\tag{3.27}$$

Thus, it follows from (3.19)-(3.21) that

$$\int_{Q_+}\sum_{k=2}^{3}|\nabla w_k|^2dX+\int_{X_2^2+X_3^2=1}|\partial_\theta v|^2dS\le C\varepsilon\int_{Q_+}|\nabla^2 v|^2dX. \tag{3.28}$$

Summing up (3.17) and (3.28), we obtain

$$\int_{Q_+}|\nabla^2 v|^2dX+\int_{X_2^2+X_3^2=1}|\partial_\theta v|^2dS\le C\varepsilon\int_{Q_+}|\nabla^2 v|^2dX.$$

Hence if we choose ε such that $C\varepsilon<\frac{1}{2}$, then

$$\int_{Q_+}|\nabla^2 v|^2dX=0. \tag{3.29}$$

In light of Lemma 3.2, we obtain

$$\int_{Q_+}(|v|^2+|\nabla v|^2+|\nabla^2 v|^2)dX=0.$$

Thus $v\equiv 0$ and the proof of Lemma 3.3 is completed.

Proof of Theorem 1.1. Based on Lemma 3.3, it follows from the transformation (2.2) and the definition of $V(X)$ in (2.3) that the uniqueness of solution $(\varphi_+(x),\xi(x_2,x_3))$ in Theorem 1.1 holds.

Remark 3.1. *It follows from the proof of Theorem 1.1 that the same uniqueness still holds true under the weaker assumptions on the regularities of the shock front Σ and the potential $\varphi_+(x)$ as stated in Remark 1.3. Indeed, it can be checked easily that the key ingredients in the proof of the uniqueness are the uniform ellipticity of the equation (1.4) and the appropriate integrability of $\nabla^k\xi$ and $\nabla^k\varphi_+ (k=2,3)$ on S and Q_+ respectively. Thus, the Remark 1.3 holds true. This observation will be useful in the non-existence analysis given in the next section.*

§4. On the non-existence

In this section, we study the non-existence of transonic shock waves patterns conjectured by Courant-Friedrich's as mentioned in the introduction for a class of nozzles. In particular, we will prove the Theorem 1.2. This will be done by contradiction. Thus, we assume that for a supersonic incoming flow, which is of small perturbation of the uniform flow $(\rho_0, q_0, 0, 0)$, passing through a nozzle, a transonic shock pattern as in Theorem 1.1 always exists for any appropriately large pressure (which is a small perturbation of $p_+\equiv p(\rho_+)$) given at the exist of the nozzle for any slowly-varying nozzle. To analyze the nozzle given in the Theorem 1.2, we introduce the following spherical coordinate transformation:

$$\begin{cases} x_1=x_1^0+r\cos\alpha,\\ x_2=r\sin\alpha\cos\theta,\\ x_3=r\sin\alpha\sin\theta \end{cases} \tag{4.1}$$

with $-\alpha_0\le\alpha\le\alpha_0$ and $0\le\theta\le 2\pi$.

In terms of the spherical coordinates (4.1), the equation (1.4) on φ_+ can be written as

$$\begin{aligned} c^2(H_+)\Big\{\frac{1}{r^2}\partial_r(r^2\partial_r\varphi_+)+\frac{1}{r^2\sin\alpha}\partial_\alpha(\sin\alpha\partial_\alpha\varphi_+)+\frac{1}{r^2\sin^2\alpha}\partial_\theta^2\varphi_+\Big\} \\ -\frac{1}{2}\Big(\partial_r\varphi_+\partial_r+\frac{1}{r^2\sin^2\alpha}\partial_\theta\varphi_+\partial_\theta+\frac{1}{r^2}\partial_\alpha\varphi_+\partial_\alpha\Big)(|\nabla\varphi_+|^2)=0 \end{aligned} \tag{4.2}$$

with $H_+=H\Big(C_0-\frac{1}{2}((\partial_r\varphi_+)^2+\frac{1}{r^2\sin^2\alpha}(\partial_\theta\varphi_+)^2+\frac{1}{r^2}(\partial_\alpha\varphi_+)^2)\Big)$.

Suppose that the equation of the shock surface $\Sigma: x_1=\xi(x_2,x_3)$ is expressed by $r=r(\theta,\alpha)-x_1^0$ with $0\le\theta\le 2\pi$ and $-\alpha_0\le\alpha\le\alpha_0$, moreover Σ goes through the origin, namely,

$$r(\theta,0)=0. \tag{4.3}$$

In addition, the corresponding boundary conditions are described as follows

$$\varphi_+=\varphi_- \qquad \text{on} \quad \Sigma, \tag{4.4}$$

$$\begin{aligned} &\Big(\partial_r\varphi_+\partial_r+\frac{1}{r^2\sin^2\alpha}\partial_\theta\varphi_+\partial_\theta+\frac{1}{r^2}\partial_\alpha\varphi_+\partial_\alpha\Big)(\varphi_+-\varphi_-)H_+ \\ &\quad-\Big(\partial_r\varphi_-\partial_r+\frac{1}{r^2\sin^2\alpha}\partial_\theta\varphi_-\partial_\theta+\frac{1}{r^2}\partial_\alpha\varphi_-\partial_\alpha\Big)(\varphi_+-\varphi_-)H_-=0 \qquad \text{on} \quad \Sigma, \end{aligned} \tag{4.5}$$

$$H\Big(C_0-\frac{1}{2}((\partial_r\varphi_+)^2+\frac{1}{r^2\sin^2\alpha}(\partial_\theta\varphi_+)^2+\frac{1}{r^2}(\partial_\alpha\varphi_+)^2)\Big)=\rho_+ \qquad \text{on} \quad r=(1-x_1^0)\sec\alpha_0 \tag{4.6}$$

$$\partial_\alpha\varphi_+=0 \qquad \text{on} \quad \Pi_2 \tag{4.7}$$

Let two planes $\Xi_{\theta_1}: x_3=x_2tg\theta_1$ and $\Xi_{\theta_2}: x_3=x_2tg\theta_2$ $(0\le\theta_1<\theta_2\le\frac{\pi}{2})$, which go through the x_1-axis, be the fixed solid boundaries. Then the appropriate boundary conditions are

$$\partial_2\varphi_+tg\theta_i-\partial_3\varphi_+=0 \qquad \text{on} \quad \Xi_{\theta_i}, \qquad i=1,2,$$

i.e.,

$$\partial_\theta\varphi_+=0 \qquad \text{on} \quad \Xi_{\theta_i}, \qquad i=1,2, \tag{4.8}$$

Let Ω be the domain bounded by $\Pi_1\cup\Pi_2$, $x_1=-1$ and $r=(1-x_1^0)\sec\alpha_0$. The following notations will be used:

$\Omega_{\theta_1,\theta_2}=\Omega\cap\{x:\theta_1\le\theta\le\theta_2\}$;

$\Omega^+_{\theta_1,\theta_2}=\Omega_{\theta_1,\theta_2}\cap\Big\{x: r(\theta,\alpha)-x_1^0\le r\le(1-x_1^0)\sec\alpha_0\Big\}$;

$S_{\theta_1,\theta_2}=\Big\{(x_2,x_3):(\xi(x_2,x_3),x_2,x_3)\in\Sigma\Big\}\cap\{x:\theta_1\le\theta\le\theta_2\}$ which is the projection of the shock surface $\Sigma\cap\{x:\theta_1\le\theta\le\theta_2\}$ on the $(x_2,x_3)-$ plane;

$\Gamma^1_{\theta_1,\theta_2}=\Sigma\cap\Pi_2\cap\{x:\theta_1\le\theta\le\theta_2\}$, $\quad \Gamma^2_{\theta_1,\theta_2}=\Pi_2\cap\{x:r=(1-x_1^0)\sec\alpha_0\}\cap\{x:\theta_1\le\theta\le\theta_2\}$;

$T^1_{\theta_i}=\Sigma\cap\Xi_{\theta_i}$, $\quad T^2_{\theta_i}=\Xi_{\theta_i}\cap\{x:r=(1-x_1^0)\sec\alpha_0\}$, $\quad \gamma_{\theta_i}=\Pi_2\cap\Xi_{\theta_i}$, $\quad i=1,2$;

$P^j_{\theta_i}=\Gamma^j_{\theta_1,\theta_2}\cap T^j_{\theta_i}$, $\quad i,j=1,2$; $\quad M_j=T^j_{\theta_1}\cap T^j_{\theta_2}$ $\quad j=1,2$.

It follows by similar arguments as in §3 that one can obtain following uniqueness result on any part of the nozzle which is bounded by the slowly-varying nozzle wall and two planes through the x_1-axis for any appropriately large pressure at the exit.

Proposition 4.1. *Suppose that (1.6) and (1.9) hold. Then for suitably small $\varepsilon > 0$, if the equation (4.2) with the boundary conditions (4.3)-(4.8) has a pair of solution $(\varphi_+(x), \xi(x_2, x_3))$ with the following regularities and estimates*

(i). $\xi(x_2, x_3) \in Lip(\bar{S}_{\theta_1,\theta_2}) \cap C^{1,1-\delta_0}(\bar{S}_{\theta_1,\theta_2} \setminus P^1_{\theta_1} \cup P^1_{\theta_2} \cup M_1) \cap C^3(S_{\theta_1,\theta_2})$.

For $x \in \Sigma$ and $(x_2, x_3) \in S_{\theta_1,\theta_2}$, define

$$|d_0| = \min\{dist(x, P^1_{\theta_1}), dist(x, P^1_{\theta_2}), dist(x, M_1)\}; \quad |d_1| = \min\{dist(x, T^1_{\theta_1}), dist(x, T^1_{\theta_2}), dist(x, \Gamma^1_{\theta_1\theta_2})\}$$

Then, near the points $P^1_{\theta_1}, P^1_{\theta_2}$ and M_1,

$$|\nabla^k_{x_2,x_3}\xi(x_2, x_3)| \le \frac{C\varepsilon}{|d_0|^{k-1}}, \qquad k = 1, 2, 3;$$

Away from the points $P^1_{\theta_1}, P^1_{\theta_2}$ and M_1,

$$\|\xi(x_2, x_3)\|_{C^{1,1-\delta_0}} \le C\varepsilon, \qquad |\nabla^k_{x_2,x_3}\xi(x_2, x_3)| \le \frac{C\varepsilon}{|d_1|^{k-2+\delta_0}}.$$

(ii). For $x \in \Omega^+_{\theta_1,\theta_2}$, set

$$|\tilde{d}_x| = \min\{dist(x, P^j_{\theta_i}), i, j = 1, 2; dist(x, M_j), j = 1, 2\};$$
$$|d_x| = min\{dist(x, \Gamma^1_{\theta_1,\theta_2}), dist(x, \Gamma^2_{\theta_1,\theta_2}), dist(x, T^j_{\theta_i}), i, j = 1, 2, dist(x, \gamma_{\theta_i}), i = 1, 2, \sqrt{x_2^2 + x_3^2}\},$$

then $\varphi_+(x) \in Lip(\bar{\Omega}^+_{\theta_1,\theta_2}) \cap C^{1,1-\delta_0}(\bar{\Omega}^+_{\theta_1,\theta_2} \setminus \cup T^j_{\theta_i} \cup M_k) \cap C^3(\Omega^+_{\theta_1,\theta_2})$ admits the following estimates: Near the points $P^j_{\theta_i}$ and M_k,

$$|\nabla^k_x\varphi_+(x)| \le \frac{C\varepsilon}{|\tilde{d}_x|^{k-1}}, \qquad k = 1, 2, 3;$$

Away from the points $P^j_{\theta_i}$ and M_k,

$$\|\varphi_+(x) - q_+x_1\|_{C^{1,1-\delta_0}} \le C\varepsilon, \qquad |\nabla^k_x\varphi_+(x)| \le \frac{C\varepsilon}{|d_x|^{k-2+\delta_0}}.$$

Then the corresponding solution $(\varphi_+(x), \xi(x_2, x_3))$ is unique. Furthermore, a corresponding generalization of Remark 1.3 is available in this case also.

Remark 4.1. *By the regularity theory for second order elliptic equations with cornered boundaries (see the references in [23]), we know that the assumptions on the regularities of solution $(\varphi_+(x), \xi(x_2, x_3))$ in Proposition 4.1 are plausible.*

Choose $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$ in Proposition 4.1 and denote the corresponding solution by $(\varphi_1(r,\theta,\alpha), r_1(\theta,\alpha))$ in the spherical coordinates (4.1). We then extend this solution as

$$\tilde{\varphi}_1(x) = \begin{cases} \varphi_1(r,\theta,\alpha), & 0 \le \theta \le \frac{\pi}{2}; \\ \varphi_1(r,\pi-\theta,\alpha), & \frac{\pi}{2} \le \theta \le \pi; \\ \varphi_1(r,\theta-\pi,\alpha), & \pi \le \theta \le \frac{3\pi}{2}; \\ \varphi_1(r,2\pi-\theta,\alpha), & \frac{3\pi}{2} \le \theta \le 2\pi; \end{cases}$$

and

$$\tilde{r}_1(\theta,\alpha) = \begin{cases} r_1(\theta,\alpha), & 0 \le \theta \le \frac{\pi}{2}; \\ r_1(\pi-\theta,\alpha), & \frac{\pi}{2} \le \theta \le \pi; \\ r_1(\theta-\pi,\alpha), & \pi \le \theta \le \frac{3\pi}{2}; \\ r_1(2\pi-\theta,\alpha), & \frac{3\pi}{2} \le \theta \le 2\pi; \end{cases}$$

Return to the $x-$coordinates and denote by $x_1 = \tilde{\xi}_1(x_2,x_3)$ instead of $r = \tilde{r}(\theta,\alpha) - x_1^0$. Then by the symmetry of $\varphi_-(x)$ near the shock and the equation (4.2) with (4.3)-(4.7), one can verify that $(\tilde{\varphi}_1, \tilde{\xi}_1)$ is a solution of (1.4) with (1.10)-(1.13), $(1.14)^{'}$ and (1.15), moreover $(\tilde{\varphi}_1(x), \tilde{\xi}_1(x_2,x_3))$ has the following regularity properties

(i). $\tilde{\xi}_1(x_2,x_3) \in Lip(\bar{S}) \cap C^{1,1-\delta_0}(\bar{S} \setminus \cup P^1_{\frac{l\pi}{2}} \cup M_1) \cap C^3(S)$ with $l = 0,1,2,3$.

Near the points $P^1_{\frac{l\pi}{2}}$ and M_1,

$$|\nabla^k_{x_2,x_3} \tilde{\xi}_1(x_2,x_3)| \le \frac{C\varepsilon}{|d_0^1|^{k-1}}, \qquad k = 1,2,3;$$

Away from the points $P^1_{\frac{l\pi}{2}}$ and M_1,

$$\|\tilde{\xi}_1(x_2,x_3)\|_{C^{1,1-\delta_0}} \le C\varepsilon, |\nabla^k_{x_2,x_3} \tilde{\xi}_1(x_2,x_3)| \le \frac{C\varepsilon}{|d_1^1|^{k-2+\delta_0}}, \qquad k = 2,3$$

here $|d_0^1| = \min\{dist(x, P^1_{\frac{l\pi}{2}}), l = 0,1,2,3; dist(x, M_1)\}$ and $|d_1^1| = \min\{dist(x, T^1_{\frac{l\pi}{2}}), l = 0,1,2,3; dist(x, \Gamma^1_{0,2\pi})\}$.

(ii). $\tilde{\varphi}_1(x) \in Lip(\bar{\Omega}^+) \cap C^{1,1-\delta_0}(\bar{\Omega}^+ \setminus \cup P^j_{\frac{l\pi}{2}} \cup M_j) \cap C^3(\Omega^+)$ admits the following estimates:

Near the points $P^j_{\frac{l\pi}{2}}$ and M_j with $l = 0,1,2,3$ and $j = 1,2$,

$$|\nabla^k_x \tilde{\varphi}_1(x)| \le \frac{C\varepsilon}{|\tilde{d}_x^1|^{k-1}}, \qquad k = 1,2,3;$$

Away from the points $P^j_{\frac{l\pi}{2}}$ and M_j,

$$\|\tilde{\varphi}_1(x) - q_+ x_1\|_{C^{1,1-\delta_0}} \le C\varepsilon, \qquad |\nabla^k_x \tilde{\varphi}_1(x)| \le \frac{C\varepsilon}{|d_x^1|^{k-2+\delta_0}}.$$

here $|\tilde{d}_x^1| = \min\{dist(x, P^j_{\frac{l\pi}{2}}), l = 0,1,2,3; j = 1,2; dist(x, M_j), j = 1,2\}$, $|d_x^1| = \min\{dist(x, \Gamma^1_{\frac{l\pi}{2},\frac{(l+1)\pi}{2}}), dist(x, \Gamma^2_{\frac{l\pi}{2},\frac{(l+1)\pi}{2}}), l = 0,1,2; dist(x, T^j_{\frac{l\pi}{2}}), dist(x, \gamma_{\frac{l\pi}{2}}), l = 0,1,2,3; j = 1,2, \sqrt{x_2^2 + x_3^2}\}$.

It follows from the uniqueness results in Proposition 4.1 with $\theta_1 = 0$ and $\theta_2 = 2\pi$, in particular the results corresponding to Remark 3.1 that $(\tilde{\varphi}_1(x), \tilde{\xi}_1(x_2, x_3)) = (\varphi_+(x),\ \xi(x_2, x_3))$. Therefore, we have shown the assertion that

$$(\varphi_+(r,\theta,\alpha), r(\theta,\alpha)) \textbf{ is symmetric with respect to } \theta = \tfrac{\pi}{2}, \pi, \tfrac{3\pi}{2}. \tag{4.9}$$

Next, choosing $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{4}$ in Proposition 4.1, then one denotes the corresponding solution by $(\varphi_2(r,\theta,\alpha), r_2(\theta,\alpha))$ in $\Omega^+_{0,\frac{\pi}{4}}$.

Set

$$\varphi_{2,1}(r,\theta,\alpha) = \begin{cases} \varphi_2(r,\theta,\alpha), & 0 \le \theta \le \frac{\pi}{4}, \\ \varphi_2(r, \frac{\pi}{2} - \theta, \alpha), & \frac{\pi}{4} \le \theta \le \frac{\pi}{2}, \end{cases}$$

and

$$r_{2,1}(\theta,\alpha) = \begin{cases} r_2(\theta,\alpha), & 0 \le \theta \le \frac{\pi}{4}, \\ r_2(\frac{\pi}{2} - \theta, \alpha), & \frac{\pi}{4} \le \theta \le \frac{\pi}{2}, \end{cases}$$

then we obtain a solution $(\varphi_{2,1}, r_{2,1})$ in the domain $\Omega^+_{0,\frac{\pi}{2}}$ to (4.2) with (4.3)-(4.8). As in the previous step, one can get a solution $(\tilde{\varphi}_2(x), \tilde{\xi}_2(x_2,x_3))$ in Ω_+ with the following properties

(i). $\tilde{\xi}_2(x_2,x_3) \in Lip(\bar{S}) \cap C^{1,1-\delta_0}(\bar{S} \setminus \cup P^1_{\frac{l\pi}{4}} \cup M_1) \cap C^3(S)$ with $l = 0, 1, ..., 7$.

Near the points $P^1_{\frac{l\pi}{4}}$ and M_1,

$$|\nabla^k_{x_2,x_3} \tilde{\xi}_2(x_2,x_3)| \le \frac{C\varepsilon}{|d^2_0|^{k-1}}, \qquad k = 1, 2, 3;$$

Away from the points $P^1_{\frac{l\pi}{4}}$ and M_1,

$$\|\tilde{\xi}_2(x_2,x_3)\|_{C^{1,1-\delta_0}} \le C\varepsilon, |\nabla^k_{x_2,x_3} \tilde{\xi}_2(x_2,x_3)| \le \frac{C\varepsilon}{|d^2_1|^{k-2+\delta_0}}, \qquad k = 2, 3$$

here $|d^2_0| = \min\{dist(x, P^1_{\frac{l\pi}{4}}), l = 0, 1, ..., 7; dist(x, M_1)\}$ and $|d^2_1| = \min\{dist(x, T^1_{\frac{l\pi}{4}}), l = 0, 1, ..., 7;$ $dist(x, \Gamma^1_{0,2\pi})\}$.

(ii). $\tilde{\varphi}_2(x) \in Lip(\bar{\Omega}^+) \cap C^{1,1-\delta_0}(\bar{\Omega}^+ \setminus \cup P^j_{\frac{l\pi}{4}} \cup M_j) \cap C^3(\Omega^+)$ admits the following estimates:

Near the points $P^j_{\frac{l\pi}{4}}$ and M_j with $l = 0, 1, ..., 7$ and $j = 1, 2$,

$$|\nabla^k_x \tilde{\varphi}_1(x)| \le \frac{C\varepsilon}{|\tilde{d}^2_x|^{k-1}}, \qquad k = 1, 2, 3;$$

Away from the points $P^j_{\frac{l\pi}{2}}$ and M_j,

$$\|\tilde{\varphi}_2(x) - q_+ x_1\|_{C^{1,1-\delta_0}} \le C\varepsilon, \qquad |\nabla^k_x \tilde{\varphi}_2(x)| \le \frac{C\varepsilon}{|d^2_x|^{k-2+\delta_0}}.$$

here $|\tilde{d}^2_x| = \min\{dist(x, P^j_{\frac{l\pi}{4}}), l = 0, 1, ..., 7; j = 1, 2; dist(x, M_j), j = 1, 2\}$, $|d^2_x| = \min\{dist(x, \Gamma^1_{\frac{l\pi}{4}, \frac{(l+1)\pi}{2}}),$ $dist(x, \Gamma^2_{\frac{l\pi}{2}, \frac{(l+1)\pi}{2}}), l = 0, 1, ..., 6; dist(x, T^j_{\frac{l\pi}{2}}), dist(x, \gamma_{\frac{l\pi}{2}}), l = 0, 1, ..., 7; j = 1, 2, \sqrt{x_2^2 + x_3^2}\}$.

Thus, one can applying Proposition 4.1 again to conclude that $(\tilde{\varphi}_2(x), \tilde{\xi}_2(x_2,x_3)) = (\varphi_+(x), \xi(x_2,x_3))$, and so

$$(\varphi_+(r,\theta,\alpha), r(\theta,\alpha)) \textbf{ is symmetric with respect to } \theta = \tfrac{\pi}{4}, \tfrac{\pi}{2}, \tfrac{3\pi}{4}, \pi, \tfrac{5\pi}{4}, \tfrac{3\pi}{2}, \tfrac{7\pi}{4}. \tag{4.10}$$

Repeating this procedure shows that

$$(\varphi_+(r,\theta,\alpha), r(\theta,\alpha)) \textbf{ is symmetric with respect to } \theta = \tfrac{k\pi}{2^m}, k = 1,2,...,2^m-1, m \in \mathbb{N}. \tag{4.11}$$

By the continuity of $(\varphi_+(r,\theta,\alpha), r(\theta,\alpha))$, we conclude that

$$(\varphi_+(r,\theta,\alpha), r(\theta,\alpha)) \textbf{ is independent of the invariable } \theta. \tag{4.12}$$

From now on, we will use the notations $(\varphi_+(r,\alpha), r(\alpha))$ instead of $(\varphi_+(r,\theta,\alpha), r(\theta,\alpha))$.

Now, the equation (4.4) with (4.3)-(4.8) can be simplified as follows:

$$\begin{cases} c^2(H_+)\Big(\frac{1}{r^2}\partial_r(r^2\partial_r\varphi_+) + \frac{1}{r^2\sin\alpha}\partial_\alpha(\sin\alpha\partial_\alpha\varphi_+)\Big) - \frac{1}{2}\Big(\partial_r\varphi_+\partial_r + \frac{1}{r^2}\partial_\alpha\varphi_+\partial_\alpha\Big)(|\nabla\varphi_+|^2) = 0, & (4.13)\\ \varphi_+ = \varphi_- \quad \text{on} \quad \Sigma, & (4.14)\\ r(0) = 0, & (4.15)\\ \Big(\partial_r\varphi_+\partial_r + \frac{1}{r^2}\partial_\alpha\varphi_+\partial_\alpha\Big)(\varphi_+ - \varphi_-)H_+ - \Big(\partial_r\varphi_-\partial_r + \frac{1}{r^2}\partial_\alpha\varphi_-\partial_\alpha\Big)(\varphi_+ - \varphi_-)H_- = 0 \text{ on } \Sigma, & (4.16)\\ H\Big(C_0 - \frac{1}{2}((\partial_r\varphi_+)^2 + \frac{1}{r^2}(\partial_\alpha\varphi_+)^2)\Big) = \rho_+ \quad \text{on} \quad r = (1-x_1^0)\sec\alpha_0, & (4.17)\\ \partial_\alpha\varphi_+ = 0 \quad \text{on} \quad \Pi_2 & (4.18) \end{cases}$$

with $H_+ = H\Big(C_0 - \frac{1}{2}((\partial_r\varphi_+)^2 + \frac{1}{r^2}(\partial_\alpha\varphi_+)^2)\Big)$.

To study the problem (4.13)-(4.18), we introduce the following notations: for $\alpha_1, \alpha_2 \in [-\alpha_0, \alpha_0]$,

$$\begin{aligned} &\Omega_{\alpha_1,\alpha_2} = \Omega \cap \{x : \alpha_1 \le \alpha \le \alpha_2\};\\ &\Omega^+_{\alpha_1,\alpha_2} = \Omega_{\alpha_1,\alpha_2} \cap \Big\{x : r(\alpha) - x_1^0 \le r \le (1-x_1^0)\sec\alpha_0\Big\};\\ &S_{\alpha_1,\alpha_2} = \Big\{(x_2,x_3) : (\xi(x_2,x_3), x_2, x_3) \in \Sigma\Big\} \cap \{x : \alpha_1 \le \alpha \le \alpha_2\};\\ &\Pi^2_{\alpha_1,\alpha_2} = \{x : |x_2^2 + x_3^2 = (x_1 - x_1^0)^2\, tg\alpha_i, \qquad i = 1,2, \qquad -\frac{1}{2} \le x_1 \le 1\};\\ &\Gamma^1_{\alpha_1,\alpha_2} = \Sigma \cap \Pi^2_{\alpha_1,\alpha_2}, \qquad \text{and} \quad \Gamma^2_{\alpha_1,\alpha_2} = \Pi^2_{\alpha_1,\alpha_2} \cap \{x : r = (1-x_1^0)\sec\alpha_0\}. \end{aligned}$$

Then, in a similar way as for the proof of Theorem 1.1, one can show the uniqueness result on the transonic problem in half a nozzle.

Proposition 4.2. *Suppose that (1.6) and (1.9) hold. Then for suitably small $\varepsilon > 0$, if the equation (4.13)-(4.18) with the boundary conditions $\partial_\alpha\varphi_+ = 0$ on $\alpha = 0$ and $\alpha = \alpha_1$ has a pair of solution $(\varphi_+(x), \xi(x_2,x_3))$ with the following regularities and estimates*

(i). $\xi(x_2,x_3) \in C^{1,1-\delta_0}(\bar{S}_{0,\alpha_1}) \cap C^3(S_{0,\alpha_1})$, $\|\xi(x_2,x_3)\|_{C^{1,1-\delta_0}(\bar{S}_{0,\alpha_1})} \le C\varepsilon$, $|\nabla^k_{x_2,x_3}\xi(x_2,x_3)| \le \frac{C\varepsilon}{|d_1|^{k-2+\delta_0}}$, here $k = 2,3$; $(x_2,x_3) \in S_{0,\alpha_1}$ and $|d_1| = dist(x, \Gamma^1_{0,\alpha_1})$ for $x \in \Sigma$ and $(x_2,x_3) \in S_{0,\alpha_1}$.

(ii). $\varphi_+(x) \in C^{1,1-\delta_0}(\bar{\Omega}^+_{0,\alpha_1}) \cap C^3(\Omega^+_{0,\alpha_1})$ *admits the following estimates:*

$$\|\varphi_+(x) - q_+x_1\|_{C^{1,1-\delta_0}(\bar{\Omega}^+_{0,\alpha_1})} \leq C\varepsilon, \qquad |\nabla^k_x \varphi_+(x)| \leq \frac{C\varepsilon}{|d_x|^{k-2+\delta_0}} \qquad for \qquad k = 2,3; \qquad x \in \Omega^+_{0,\alpha_1}$$

here $|d_x| = min\{dist(x, \Gamma^1_{0,\alpha_1}), dist(x, \Gamma^2_{0,\alpha_1})\}$ *for* $x \in \Omega^+_{0,\alpha_1}$.

Then the corresponding solution $(\varphi_+(x), \xi(x_2, x_3))$ *is unique. Furthermore, a corresponding generalization of Remark 1.3 holds true in this case.*

We choose $\alpha_1 = \alpha_0$ in Proposition 4.2 and denote the corresponding solution by $(\phi_1(r,\alpha),\ R_1(\alpha))$ in the spherical coordinates (4.1).

Set

$$\bar{\phi}_1(x) = \begin{cases} \phi_1(r,\alpha), & 0 \leq \alpha \leq \alpha_0; \\ \phi_1(r,-\alpha), & -\alpha_0 \leq \alpha \leq 0 \end{cases}$$

and

$$\bar{R}_1(\alpha) = \begin{cases} R_1(\alpha), & 0 \leq \alpha \leq \alpha_0; \\ R_1(-\alpha), & -\alpha_0 \leq \alpha \leq 0 \end{cases}$$

Denote by $x_1 = \bar{\xi}_1(x_2, x_3)$ instead of $r = \bar{R}_1(\alpha) - x^0_1$. Then one can verify that $(\bar{\phi}_1, \bar{\xi}_1)$ is a solution of (4.13)-(4.18), moreover $(\bar{\phi}_1(x), \bar{\xi}_1(x_2, x_3))$ has the following regularity and satisfies the estimates

(i). $\bar{\xi}_1(x_2, x_3) \in C^{1,1-\delta_0}(\bar{S}) \cap C^3(S), \|\bar{\xi}_1(x_2, x_3)\|_{C^{1,1-\delta_0}(\bar{S})} \leq C\varepsilon, |\nabla^k_{x_2,x_3}\bar{\xi}_1(x_2, x_3)| \leq \frac{C\varepsilon}{|\bar{d}_1|^{k-2+\delta_0}}$,
here $k = 2, 3$; $(x_2, x_3) \in S$ and $|\bar{d}_1| = min\{dist(x, \Gamma^1_{0,\alpha_0}), dist(x, \Gamma^1_{0,-\alpha_0})\}$ for $x \in \Sigma$ and $(x_2, x_3) \in S$.

(ii). $\bar{\phi}_1(x) \in C^{1,1-\delta_0}(\bar{\Omega}_+) \cap C^3(\Omega_+)$ admits the following estimates:

$$\|\bar{\phi}_1(x) - q_+x_1\|_{C^{1,1-\delta_0}(\bar{\Omega}_+)} \leq C\varepsilon, \qquad |\nabla^k_x \bar{\phi}_1(x)| \leq \frac{C\varepsilon}{|\bar{d}^1_x|^{k-2+\delta_0}} \qquad \text{for} \qquad k = 2,3; \qquad x \in \Omega_+$$

here $|\bar{d}^1_x| = min\{dist(x, \Gamma^i_{0,-\alpha_0}), dist(x, \Gamma^i_{0,\alpha_0}) : i = 1, 2\}$.

It follows from a similar uniqueness results as in Theorem 1.1 and Remark 1.3 that $(\bar{\phi}_1(x), \bar{\xi}_1(x_2, x_3)) = (\varphi_+(x), \xi(x_2, x_3))$ and thus

$$(\varphi_+(r,\alpha), r(\alpha)) \textbf{ is symmetric with respect to } \alpha = 0. \tag{4.19}$$

Next, choosing $\alpha_1 = \frac{\alpha_0}{2}$ in Proposition 4.2, and denoting the corresponding solution in $\Omega^+_{0,\frac{\alpha_0}{2}}$ by $(\phi_2(r,\alpha), R_2(\alpha))$, we may extend it as follow

Set

$$\phi_{2,1}(r,\alpha) = \begin{cases} \phi_2(r,\alpha), & 0 \leq \alpha \leq \frac{\alpha_0}{2}, \\ \phi_2(r,\alpha_0 - \alpha), & \frac{\alpha_0}{2} \leq \alpha \leq \alpha_0, \end{cases}$$

and

$$R_{2,1}(\alpha) = \begin{cases} R_2(\alpha), & 0 \leq \alpha \leq \frac{\alpha_0}{2}, \\ R_2(\alpha_0 - \alpha), & \frac{\alpha_0}{2} \leq \alpha \leq \alpha_0, \end{cases}$$

Then $(\phi_{2,1}, R_{2,1})$ is defined on the domain Ω^+_{0,α_0}, which can extend evenly to Ω_+ as above to obtain a solution $(\bar{\phi}_2(x), \bar{\xi}_2(x_2, x_3))$ in Ω_+ with the following regularities and estimates

(i). $\xi_2(x_2, x_3) \in C^{1,1-\delta_0}(\bar{S}) \cap C^3(S), \|\xi_2(x_2, x_3)\|_{C^{1,1-\delta_0}(\bar{S})} \leq C\varepsilon, |\nabla^k_{x_2,x_3}\xi_2(x_2, x_3)| \leq \frac{C\varepsilon}{|\bar{d}_2|^{k-2+\delta_0}}$,

here $k=2,3$; $(x_2,x_3)\in S$ and
$min\left\{dist(x,\Gamma^1_{\alpha_1,\alpha_2}),(\alpha_1,\alpha_2)=(0,\frac{\alpha_0}{2}),(\frac{\alpha_0}{2},\alpha_0),(-\frac{\alpha_0}{2},0),\ (-\alpha_0,-\frac{\alpha_0}{2})\right\}$ for $x\in\Sigma$ and $(x_2,x_3)\in S$.

(ii). $\bar{\phi}_2(x)\in C^{1,1-\delta_0}(\bar{\Omega}_+)\cap C^3(\Omega_+)$ admits the following estimates:

$$\|\bar{\phi}_2(x)-q_+x_1\|_{C^{1,1-\delta_0}(\bar{\Omega}_+)}\le C\varepsilon,\qquad |\nabla_x^k\bar{\phi}_2(x)|\le\frac{C\varepsilon}{|\bar{d}_x^2|^{k-2+\delta_0}}\qquad\text{for}\qquad k=2,3;\qquad x\in\Omega_+$$

here $|\bar{d}_x^2|=min\{dist(x,\Gamma^i_{\alpha_1,\alpha_2});\qquad i=1,2,\qquad(\alpha_1,\alpha_2)=(0,\frac{\alpha_0}{2}),(\frac{\alpha_0}{2},\alpha_0),(-\frac{\alpha_0}{2},0),(-\alpha_0,-\frac{\alpha_0}{2})\}$.

It follows from a similar argument as in the proof in §3, one can show that $(\bar{\phi}_2(x),\bar{\xi}_2(x_2,x_3))=(\varphi_+(x),\xi(x_2,x_3))$. Thus

$$(\varphi_+(r,\alpha),r(\alpha))\ \textbf{is symmetric with respect to}\ \alpha=0,\pm\tfrac{\alpha_0}{2}.\tag{4.20}$$

Continue this procedure repeatedly to get that

$$(\varphi_+(r,\alpha),r(\theta,\alpha))\ \textbf{is symmetric with respect to}\ \alpha=\pm\tfrac{k\alpha_0}{2^m},k=0,1,2,...,2^m-1,m\in\mathbb{N}.\tag{4.21}$$

By the continuity of $(\varphi_+(r,\alpha),r(\alpha))$, we conclude that

$$(\varphi_+(r,\alpha),r(\alpha))\ \textbf{is independent of the invariable}\ \alpha.\tag{4.22}$$

Subsequently, we will use the notations $(\varphi_+(r),0)$ instead of $(\varphi_+(r,\alpha),r(\alpha))$. Then the equation (4.13)-(4.18) can be rewritten as

$$\begin{cases} c^2(H_+)\partial_r(r^2\partial_r\varphi_+)-\frac{r^2}{2}\partial_r\varphi_+\partial_r(|\partial_r\varphi_+|^2)=0, & (4.23)\\ \varphi_+=\varphi_-\qquad\text{on}\qquad\Sigma & (4.24)\\ \partial_r\varphi_+H_+-\partial_r\varphi_-H_-=0\qquad\text{on}\qquad\Sigma, & (4.25)\\ H(C_0-\frac{1}{2}(\partial_r\varphi_+)^2)=\rho_+\qquad\text{on}\qquad r=(1-x_1^0)\sec\alpha_0, & (4.26)\end{cases}$$

with $H_+=H(C_0-\frac{1}{2}(\partial_r\varphi_+)^2)$.

Since $\frac{1}{2}(\partial_r\varphi_+(r))^2+h(\rho_+(r))\equiv C_0$, then it follows from (4.23)-(4.26) that

$$\begin{cases}\Big(2(C_0-h(\rho_+(r)))-c^2(\rho_+(r))\Big)\partial_r\rho_+(r)+\frac{4\rho_+(r)}{r}(C_0-h(\rho_+(r)))=0,-x_1^0\le r\le(1-x_1^0)\sec\alpha_0,\\ \rho_+^2(r)(C_0-h(\rho_+(r)))=\rho_-^2(r)(C_0-h(\rho_-(r)))\qquad\text{on}\qquad r=-x_1^0,\\ \rho_+(r)=\rho_+\qquad\text{on}\qquad r=(1-x_1^0)\sec\alpha_0.\end{cases}\tag{4.27}$$

Let $\rho_-(-x_1^0)=\rho_0$. Then we obtain $\rho_+(-x_1^0)=\rho_+$ in terms of the Rankine-Hugoniot condition in (4.27). Thus the problem (4.27) can be reduced as follows

$$\begin{cases}\Big(2(C_0-h(\rho_+(r)))-c^2(\rho_+(r))\Big)\partial_r\rho_+(r)+\frac{4\rho_+(r)}{r}(C_0-h(\rho_+(r)))=0,\\ \rho_+(-x_1^0)=\rho_+,\\ \rho_+((1-x_1^0)\sec\alpha_0)=\rho_+\end{cases}\tag{4.28}$$

Obviously, it follows from the first equation in (4.28) that $\partial_r\rho_+(r) > 0$ holds for the subsonic flow in the domain $\{r : -x_1^0 \le r \le (1-x_1^0)\sec\alpha_0\}$. Hence the problem (4.28) has no solution.

Proof of Theorem 1.2. If we choose $\Pi_1 \cup \Pi_2$ as the nozzle wall, and the supersonic coming flow is determined by solving the following hyperbolic equation

$$\begin{cases} \sum_{i=1}^{3}((\partial_i\varphi_-)^2 - c_-^2)\partial_i^2\varphi_- + 2\sum_{1\le i<j\le 3}\partial_i\varphi_-\partial_j\varphi_-\partial_{ij}^2\varphi_- = 0, \\ \varphi_-|_{\sqrt{(x_1-x_1^0)^2+x_2^2+x_3^2}=-x_1^0} = 0, \\ \partial_r\varphi_-|_{\sqrt{(x_1-x_1^0)^2+x_2^2+x_3^2}=-x_1^0} = q_0, \\ \partial_1 f\partial_1\varphi_- + \sum_{i=2}^{3}(\partial_i f - \frac{x_i}{f})\partial_i\varphi_- = 0 \qquad \text{on} \qquad \Pi_1\cup\Pi_2. \end{cases}$$

here $\sqrt{x_2^2+x_3^2} = f(x)$ represents the equation of $\Pi_1 \cup \Pi_2$.

Then for any given constant pressure $p_r \ne p_r^0$ (here p_r^0 is determined by the first two equations in (4.28)), by (4.28) we have shown that the problem (1.4) with (1.10)-(1.15) has no transonic shock solution $(\varphi_+(x), \xi(x_2,x_3))$ with the regularities and estimates as stated in Theorem 1.1. Thus we complete the proof on Theorem 1.2.

Remark. 4.2. *It follows from the equation (4.28), we know that the pressure $P_+(x)$ at the exit of the nozzle cannot be given arbitrarily, otherwise the problem is over-determined. Even if we adjust the position of the shock at the diverging part in $-\frac{1}{4} \le x_1 \le 1$, the corresponding transonic shock problem on the potential equation is still ill-posed for the arbitrarily given pressure at the exit of a slowly-varying nozzle.*

In the rest of this section, we will present some non-existence results for the transonic shock problem in a 2-D nozzle.

Suppose that there is a uniform supersonic flow $(u_1,u_2) = (q_0, 0)$ with constant density $\rho_0 > 0$ which comes from negative infinity, and the flow enters the $2-D$ nozzle from the entrance. We assume that the two nozzle walls are a small perturbation of two straight line segments $x_2 = -1$ and $x_2 = 1$ with $-1 \le x_1 \le 1$ respectively. The flow in the nozzle is assumed to be irrotational and isentropic.

Let $\varphi(x)$ be the potential of velocity, i.e. $(\partial_1\varphi, \partial_2\varphi) = (u_1,u_2)$, then it follows from the Bernoulli's law and mass conservation that

$$((\partial_1\varphi)^2 - c^2)\partial_1^2\varphi + 2\partial_1\varphi\partial_2\varphi\partial_{12}^2\varphi + ((\partial_2\varphi)^2 - c^2)\partial_2^2\varphi = 0 \tag{4.29}$$

with $c = c(\rho)$ and $\rho = H(C_0 - \frac{1}{2}|\nabla\varphi|^2)$.

Suppose that the two walls of the nozzle are given respectively by

$$x_2 = f_2(x_1) \qquad \text{and} \qquad x_2 = f_1(x_1) \tag{4.30}$$

where $f_i(x_1)(i=1,2)$ satisfies

$$|\frac{d^k}{dx_1^k}(f_2(x_1)-1)| \le \varepsilon, \qquad |\frac{d^k}{dx_1^k}(f_1(x_1)+1)| \le \varepsilon \qquad \text{for} \qquad -1 \le x_1 \le 1, k \le 4, k \in \mathbb{N}\cup\{0\} \tag{4.31}$$

Without loss of generality and for the convenience to write, it will be assumed that

$$f_1(-1) = f_1(1) = -1, \qquad f_2(-1) = f_2(1) = 1, \qquad f_i^{(k)}(-1) = 0 \qquad \text{for} \qquad i = 1, 2; \qquad 1 \le k \le 4 \tag{4.32}$$

When the uniform supersonic flow $(q_0, 0)$ enters the entry of the nozzle, then the potential $\varphi_-(x)$ in the nozzle will be determined by the equation (4.29) with the corresponding initial-boundary conditions. One can show that $\varphi_-(x) \in C^4$ holds in the whole nozzle $\{(x_1, x_2) : -1 \le x_1 \le 1, f_1(x_1) \le x_2 \le f_2(x_1)\}$, moreover $|\nabla_x^\alpha(\varphi_-(x) - q_0 x_1)| \le C\varepsilon$ holds for $|\alpha| \le 4$.

Given an appropriate pressure $\tilde{P}_+(x_2) = P(\tilde{\rho}_+(x_2))$ at the exit $x_1 = 1$ of the nozzle, where $\tilde{\rho}_+(x_2)$ is a small perturbation of the constant density ρ_+.

For the definiteness, we assume that the shock $x_1 = \xi(x_2)$ goes through the point $(0, 0)$,

$$\xi(0) = 0 \tag{4.33}$$

Across the shock $x_1 = \xi(x_2)$, we denote the potential by $\varphi_+(x)$. Then

$$\varphi_+(x) = \varphi_-(x) \qquad \text{on} \qquad x_1 = \xi(x_2) \tag{4.34}$$

and the derivative of $\varphi_+(x)$ must satisfy the Rankine-Hugoniot condition

$$[\partial_1 \varphi_+ H] - \xi'(x_2)[\partial_2 \varphi_+ H] = 0 \qquad \text{on} \qquad x_1 = \xi(x_2) \tag{4.35}$$

In addition, φ_+ should satisfy the physical entropy condition (see [6]):

$$H(C_0 - \frac{1}{2}|\nabla \varphi_-|^2) < H(C_0 - \frac{1}{2}|\nabla \varphi_+|^2) \qquad \text{on} \qquad x_1 = \xi(x_2) \tag{4.36}$$

At the exit of the nozzle, the density $\rho(x)$ is given by

$$H(C_0 - \frac{1}{2}|\nabla \varphi_+|^2) = \tilde{\rho}_+(x_2) \qquad \text{on} \qquad x_1 = 1 \tag{4.37}$$

Finally, the velocity of the flow is tangent to the nozzle walls, hence

$$\partial_2 \varphi_+ = f_i'(x_1)\partial_1 \varphi_+ \qquad \text{on} \qquad x_2 = f_i(x_1), \qquad i = 1, 2 \tag{4.38}$$

Completely similar to Theorem 1.2, we have

Theorem 4.3.(Ill-posedness)

(i). If the walls of the nozzle are straight, namely, $f_1(x_1) \equiv -1$ *and* $f_2(x_1) \equiv 1$*. Let* $\tilde{\rho}_+(x_2) = \rho_+$ *in (4.38), then for the constant supersonic coming flow* $(\tilde{\rho}_0, \tilde{q}_0, 0)$ *with* $(\tilde{\rho}_0, \tilde{q}_0) \ne (\rho_0, q_0)$*, the problem (4.29) with (4.33)-(4.38) has no transonic shock solution* $(\varphi_+(x), \xi(x_2))$ *such that* $(\varphi_+(x), \xi(x_2))$ *have the following regularities*

(I). Denote by $\tilde{P}_i = (x_1^i, x_2^i)(i = 1, 2)$ *the intersection points of* $x_1 = \xi(x_2)$ *with* $x_2 = f_i(x_1)$ *and define* $|d_{x_2}| = min\{dist(x, \tilde{P}_1), dist(x, \tilde{P}_2)\}$ *with* $x = (\xi(x_2), x_2)$*. Then for* $k = 2, 3$ *and* $x_2 \in (x_2^1, x_2^2)$

$$\xi(x_2) \in C^{1,1-\delta_0}[x_2^1, x_2^2] \cap C^3(x_2^1, x_2^2), \qquad \|\xi(x_2)\|_{C^{1,1-\delta_0}} \le C\varepsilon, \qquad |\frac{d^k \xi(x_2)}{dx_2^k}| \le \frac{C\varepsilon}{|d_{x_2}|^{k-2+\delta_0}}.$$

here and below $\delta_0 \in (0, \frac{1}{3})$ is any fixed constant.

(II). Denote by $\Omega_+ = \{(x_1, x_2) : \xi(x_2) < x_1 < 1, f_1(x_1) < x_2 < f_2(x_1)\}$. For $x \in \Omega_+$, write $|d_x| = \min_{1\le i\le 4}\{dist(x, \tilde{P}_i)\}$ with $\tilde{P}_3 = (1,1)$ and $\tilde{P}_4 = (1,-1)$. We assume that $\varphi_+(x) \in C^{1,1-\delta_0}(\bar{\Omega}_+) \cap C^3(\bar{\Omega}_+ \setminus \cup_{i=1}^4 \tilde{P}_i)$ satisfies

$$\|\varphi_+(x) - q_+x_1\|_{C^{1,1-\delta_0}} \le C\varepsilon, \qquad |\nabla_x^k \varphi_+(x)| \le \frac{C\varepsilon}{|d_x|^{k-2+\delta_0}} \qquad for \qquad k = 2, 3; \qquad x \in \Omega_+$$

(ii). If a diverging exhaust section of the nozzle is a two-dimensional angular section, then for the given pressure p_r at the exit , which is appropriately larger than that in the entry and is of a small perturbation of ρ_+, then the problem (4.29) with (4.33)-(4.38) is ill-posed. More concretely, one can find the supersonic coming flows which are of small perturbations of (ρ_0, q_0) such that the problem (4.29) with (4.33)-(4.38) has no transonic shock solution $(\varphi_+(x), \xi(x_2))$ with the corresponding regularities and estimates as stated in (i).

Sketch of Proof:

(i). As in Theorem 1.1, if a pair of solution $(\varphi_+(x), \xi(x_2))$ of (4.29) with (4.33)-(4.38) has the regularities and estimates stated in (i), then such a solution $(\varphi_+(x), \xi(x_2))$ is unique. Based on this uniqueness result, we can arrive at

$(I)_1$. $(\varphi_+(x), \xi(x_2))$ is symmetric with respect to $x_2 = 0$.

$(I)_2$. $(\varphi_+(x), \xi(x_2))$ is symmetric with respect to $x_2 = 0, \pm\frac{1}{2}$.

More generally, for any $m \ge 2$ and $m \in \mathbb{N}$ one has

$(I)_m$. $(\varphi_+(x), \xi(x_2))$ is symmetric with respect to $x_2 = \pm\frac{k}{2^m}$, $\qquad k = 0, 1, ..., 2^m - 1$.

Thus we get

$$\varphi_+(x) = \varphi_+(x_1), \qquad \xi(x_2) \equiv 0.$$

It follows from (4.29) with (4.33)-(4.38) that

$$\begin{cases} \partial_1^2 \varphi_+ = 0, & x_1 > 0 \\ \partial_1 \varphi_+ H_+ = \tilde{\rho}_0 \tilde{q}_0 & \text{on} \quad x_1 = 0 \\ H_+ = \rho_+ & \text{on} \quad x_1 = 1 \end{cases} \tag{4.39}$$

The first and the second equations in (4.39) imply that

$$\partial_1 \varphi_+ = \tilde{q}_+, \qquad H_+ = \tilde{\rho}_+$$

here the constants $\tilde{\rho}_+$ and $\tilde{q}_+$ are determined by

$$\tilde{\rho}_+ \tilde{q}_+ = \tilde{\rho}_0 \tilde{q}_0, \qquad \frac{1}{2}\tilde{q}_+^2 + h(\tilde{\rho}_+) = \frac{1}{2}\tilde{q}_0^2 + h(\tilde{\rho}_0), \qquad \tilde{q}_+ < c(\tilde{\rho}_+).$$

Since $(\tilde{\rho}_0, \tilde{q}_0) \ne (\rho_0, q_0)$, then $\tilde{\rho}_+ \ne \rho_+$. This is contradictory with the third equation in (4.39).

(ii). Its proof is completely similar to that for (4.13)-(4.18) (even much simpler), thus we omit it here.

Remark 4.3. *For the complete steady compressible Euler system, the non-existence result still holds for the general given pressure condition at the exit of a slowly-varying nozzle. See [27] for details.*

Furthermore, for the case of 3-D standard cylindrical surface, the ill-posedness problem for the transonic shock with a given pressure at the exit will also be treated in [27]. But if the fixed boundaries of the nozzle are porous(perforated) or curved appropriately large, then we can show that the transonic shock problem for the full Euler system is well-posed for the arbitrarily given and appropriately large pressure $\tilde{P}_+(x)$ at the exit, the details see [26].

§5. **The proof on Theorem 1.3**

In this section, we will prove the existence results, i.e. Theorem 1.3. We will use the notations introduced in §2. Under the transformation (2.2), it follows from the equation (1.4) and the boundary conditions (1.11)-(1.12), (1.13), $(1.14)''$ and (1.15) that the unknown function $V(X)$ defined in (2.3) satisfies the following second order equation with the corresponding nonlinear boundary conditions

$$\begin{cases} \sum_{i,j=1}^{3} a_{ij}(X,V,\nabla_X V)\partial^2_{X_iX_j}V + F_0(X,V,\nabla_X V) = 0 \qquad \text{in} \qquad Q_+, \\ G(X,V,\nabla_X V) = 0 \qquad \text{on} \qquad X_1 = 0, \\ \sum_{j=1}^{3}\Big(\sum_{i=2}^{3}(\frac{x_i}{f} - \partial_i f)\frac{\partial X_j}{\partial x_i} - \partial_1 f\frac{\partial X_j}{\partial x_1}\Big)\partial_{X_j}V = \partial_1 f(1-\partial_1\varphi_-) \\ \qquad + \sum_{i=2}^{3}(\frac{x_i}{f} - \partial_i f)\partial_i\varphi_- \qquad \text{on} \qquad X_2^2 + X_3^2 = 1, \\ \sum_{k=1}^{3}\frac{\partial X_k}{\partial x_1}\partial_{X_k}V + \sum_{l=2}^{3}\sum_{k=1}^{3} b_l(x)\frac{\partial X_k}{\partial x_l}\partial_{X_k}V + b_1(x)V = -1 + \partial_1\varphi_- + b_2(x)\partial_2\varphi_- \\ \qquad + b_3(x)\partial_3\varphi_- + b_1(x)(1 - x_1 + \varphi_-(x)) - g(x) \qquad \text{on} \qquad X_1 = 1. \end{cases} \tag{5.1}$$

For the problem (5.1), we will establish the following existence, uniqueness and regularity results:

Theorem 5.1. *Let $\delta_0 \in (0, \frac{1}{3})$ be a given constant. Assume that (1.6)-(1.9) hold. Then there exist positive constants ε_0 and C depending only on ρ_+, q_+ and δ_0 such that for any $\varepsilon \in (0, \varepsilon_0)$, the problem (5.1) has a unique solution $V(X) \in C^{1,1-\delta_0}(\bar{Q}_+) \cap C^{3,\delta_0}(\bar{Q}_+ \setminus \cup_{i=1}^2 \Gamma_i)$ with the following estimates:*

$$\begin{cases} \|V(X) - 1\|_{C^{1,1-\delta_0}} \le C\varepsilon, \qquad |\nabla_X^k V(X)| \le \frac{C\varepsilon}{|R_X|^{k-2+\delta_0}}, \qquad k = 2,3, \\ \sup_{X,Y \in \bar{Q}_+ \setminus \cup_{i=1}^2 \Gamma_i} \sum_{k=3} |d_{X,Y}|^{1+2\delta_0} \frac{|\nabla^k V(X) - \nabla^k V(Y)|}{|X-Y|^{\delta_0}} \le C\varepsilon, \end{cases} \tag{5.2}$$

and

$$\begin{cases} \|\partial_\theta V(X)\|_{C^{1,1-\delta_0}} \le C\varepsilon, \qquad |\nabla_X^2 \partial_\theta V(X)| \le \frac{C\varepsilon}{|R_X|^{\delta_0}}, \qquad k = 2,3, \\ \sup_{X,Y \in \bar{Q}_+ \setminus \cup_{i=1}^2 \Gamma_i} \sum_{k=2} |d_{X,Y}|^{2\delta_0} \frac{|\nabla^k \partial_\theta V(X) - \nabla^k \partial_\theta V(Y)|}{|X-Y|^{\delta_0}} \le C\varepsilon, \end{cases} \tag{5.3}$$

where

$$\begin{cases} \Gamma_1 = \{(0, X_2, X_3) : X_2^2 + X_3^2 = 1\}, \\ \Gamma_2 = \{(1, X_2, X_3) : X_2^2 + X_3^2 = 1\}, \\ R_X = X_1(1 - X_1) + 1 - (X_2^2 + X_3^2), \\ d_{X,Y} = min\{R_X, R_Y\}, \\ \partial_\theta = X_2\partial_3 - X_3\partial_2. \end{cases} \tag{5.4}$$

Once Theorem 5.1 is proved, then Theorem 1.3 can be deduced easily from Theorem 5.1 and the generalized partial hodograph transformation (2.2) and (2.3). It thus remains to prove Theorem 5.1.

The basic strategy for the proof of Theorem 5.1 is to generalize the one for the 2-dimensional case [24]. We will use the following Schauder fixed point theorem.

Theorem 5.2.(Theorem 11.1 in [9]) *Let $\mathbb{K}$ be a compact, convex subset of a Banach space $\mathbb{B}$, and let J be a continuous mapping from K into itself. Then J has a fixed point in $\mathbb{K}$.*

To prove Theorem 5.1, we choose the Banach space to be the following weighted Hölder space $\mathbb{B}$

$$\mathbb{B} = \{W(X) \in C^{1,1-\tilde{\delta}_0}(\bar{Q}_+) \cap C^{3,\tilde{\delta}_0}(\bar{Q}_+ \setminus \cup_{i=1}^2 \Gamma_i) : \|W\|_{C^{1,1-\tilde{\delta}_0}} \le C, \qquad \sup_X |R_X|^{\tilde{\delta}_0} |\nabla_X^2 W| \le C,$$
$$\sup_X |R_X|^{1+\tilde{\delta}_0} |\nabla_X^3 W| \le C, \sup_{X,Y \in \bar{Q}_+ \setminus \cup_{i=1}^2 \Gamma_i} \sum_{k=3} |d_{X,Y}|^{1+\tilde{\delta}_0 + \frac{\delta_0}{2}} \frac{|\nabla^k V(X) - \nabla^k V(Y)|}{|X-Y|^{\frac{\delta_0}{2}}} \le C, \delta_0 < \tilde{\delta}_0 < \frac{1}{3}\}$$

$\mathbb{B}$ is equipped with the norm

$$\|W\|_{\mathbb{B}} = \|W\|_{C^{1,1-\tilde{\delta}_0}} + \sum_{k=2}^{3} \sup_X |R_X|^{k-2+\tilde{\delta}_0} |\nabla_X^k W|$$
$$+ \sup_{X,Y \in \bar{Q}_+ \setminus \cup_{i=1}^2 \Gamma_i} \sum_{k=3} |d_{X,Y}|^{1+\tilde{\delta}_0 + \frac{\delta_0}{2}} \frac{|\nabla^k V(X) - \nabla^k V(Y)|}{|X-Y|^{\frac{\delta_0}{2}}}$$

It can be shown that $\mathbb{B}$ is a Banach space ([8]). The role of R_X in $\mathbb{B}$ is to measure the loss of regularity of $W(X)$ near the circles Γ_1 and Γ_2. Sometimes we neglect the subscript X in R_X for convenience.

Next we define a subset $\mathbb{K}$ of $\mathbb{B}$ as

$$\mathbb{K} = \{W(X) \in C^{1,1-\delta_0}(\bar{Q}_+) \cap C^{3,\delta_0}(\bar{Q}_+ \setminus \cup_{i=1}^2 \Gamma_i) : \|W - 1\|_{C^{1,1-\tilde{\delta}_0}} \le M\varepsilon, \qquad \|\partial_\theta W\|_{C^{1,1-\tilde{\delta}_0}} \le M\varepsilon,$$
$$\sup_X \sum_{k=2}^{3} |R|^{k-2+\delta_0} |\nabla_X^k W| \le M\varepsilon, \qquad \sup_X |R|^{\delta_0} |\nabla_X^2 \partial_\theta W| \le M\varepsilon,$$
$$\sup_{X,Y \in \bar{Q}_+ \setminus \cup_{i=1}^2 \Gamma_i} \sum_{k=2} |d_{X,Y}|^{2\delta_0} \frac{|\nabla^k \partial_\theta V(X) - \nabla^k \partial_\theta V(Y)|}{|X-Y|^{\delta_0}} \le M\varepsilon,$$
$$\sup_{X,Y \in \bar{Q}_+ \setminus \cup_{i=1}^2 \Gamma_i} \sum_{k=3} |d_{X,Y}|^{1+2\delta_0} \frac{|\nabla^k V(X) - \nabla^k V(Y)|}{|X-Y|^{\delta_0}} \le M\varepsilon\}$$

where $M \ge 1$ is a constant to be determined later.

It is clear that $\mathbb{K}$ is a convex subset of $\mathbb{B}$, and furthermore, $\mathbb{K}$ is also compact in $\mathbb{B}$ (see [8]).

We now define a continuous mapping J, which maps $\mathbb{K}$ into itself, by solving an appropriate boundary value problem for some second order linear elliptic equation on a fixed domain with linear boundary conditions, which is an appropriate linearization of the nonlinear problem of (5.1). More precisely, for any $W \in \mathbb{K}$, we define $J : \mathbb{K} \to \mathbb{K}$ by

$$JW = \tilde{V} + 1 \tag{5.5}$$

here $\tilde{V}$ is required to solve the equation

$$\sum_{i,j=1}^{3} a_{ij}(X, W, \nabla_X W)\partial_{X_i X_j}\tilde{V} + F_0(X, W, \nabla_X W) = 0 \qquad \text{in} \qquad Q_+. \tag{5.6}$$

Motivated by (2.16) in §2, $\tilde{V}$ is required to satisfy the following linear boundary condition on $X_1 = 0$:

$$\sum_{i=1}^{3} B_{1i}(X, W, \nabla_X W)\partial_{X_i}\tilde{V} + B_1(X, W, \nabla_X W)(W - 1) = G(X, 1, 0, 0, 0). \tag{5.7}$$

Since $B_{11}(X, W, \nabla_X W) \neq 0$ for small ε. And so (5.7) can be rewritten as

$$\partial_{X_1}\tilde{V} + \sum_{i=2}^{3} \tilde{B}_{1i}(X, W, \nabla_X W)\partial_{X_i}\tilde{V} + \tilde{B}_1(X, W, \nabla_X W) = 0 \qquad \text{on} \qquad X_1 = 0, \tag{5.8}$$

with the coefficients satisfying

$$\begin{cases} \tilde{B}_{1i}(X, W, \nabla_X W) = O(M\varepsilon), \qquad i = 2, 3 \\ \tilde{B}_1(X, W, \nabla_X W) = O(\varepsilon), \\ \nabla_X^k \tilde{B}_{1i}(X, W, \nabla_X W) = O(\frac{M\varepsilon}{R^{k-1+\delta_0}}), \qquad k = 1, 2; \qquad i = 2, 3 \\ \nabla_X^k \tilde{B}_1(X, W, \nabla_X W) = O(\frac{\varepsilon}{R^{k-1+\delta_0}}), \qquad k = 1, 2, \\ \sup\limits_{X,Y \in \bar{Q}_+ \setminus \cup_{i=1}^2 \Gamma_i} \sum\limits_{k=2} |d_{X,Y}|^{1+2\delta_0} \dfrac{|\nabla^k \tilde{B}_{1i}(X) - \nabla^k \tilde{B}_{1i}(Y)|}{|X - Y|^{\delta_0}} = O(M\varepsilon), \qquad i = 2, 3, \\ \sup\limits_{X,Y \in \bar{Q}_+ \setminus \cup_{i=1}^2 \Gamma_i} \sum\limits_{k=2} |d_{X,Y}|^{1+2\delta_0} \dfrac{|\nabla^k \tilde{B}_1(X) - \nabla^k \tilde{B}_1(Y)|}{|X - Y|^{\delta_0}} = O(\varepsilon) \end{cases}$$

which follow from Lemma 2.2. Here we emphasize that the fact $\tilde{B}_1(X, W, \nabla_X W) = O(\varepsilon)$ will be critical to determine the constant M in $\mathbb{K}$.

Analogously, we require $\tilde{V}$ to satisfy the following boundary conditions on $X_1 = 1$ and $X_2^2 + X_3^2 = 1$ respectively

$$\partial_{X_1}\tilde{V} + \sum_{i=2}^{3} \tilde{B}_{2i}(X, W, \nabla_X W)\partial_{X_i}\tilde{V} + B_0(X, W)\tilde{V} + \tilde{B}_2(X, W, \nabla_X W) = 0 \qquad \text{on} \qquad X_1 = 1, \tag{5.9}$$

and

$$\sum_{i=1}^{3} \tilde{B}_{3i}(X, W, \nabla_X W)\partial_{X_i}\tilde{V} + \tilde{B}_3(X, W, \nabla_X W) = 0 \qquad \text{on} \qquad X_2^2 + X_3^2 = 1, \tag{5.10}$$

where $\tilde{B}_{2i}(X,W,\nabla_X W)$ and $\tilde{B}_2(X,W,\nabla_X W)$ have the same estimates as for $\tilde{B}_{1i}(X,W,\nabla_X W)$ and $\tilde{B}_1(X,W,\nabla_X W)$ respectively, and

$$\begin{cases} \frac{\lambda}{2} < B_0(X,W) < 2\Lambda, \\ \nabla_X \tilde{B}_0(X,W) = O(M\varepsilon), \qquad \nabla_X^2 \tilde{B}_0(X,W) = O(\frac{M\varepsilon}{R^{\delta_0}}), \\ \sup\limits_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^2\Gamma_i} \sum\limits_{k=2} |d_{X,Y}|^{2\delta_0} \dfrac{|\nabla^k \tilde{B}_1(X) - \nabla^k \tilde{B}_1(Y)|}{|X-Y|^{\delta_0}} = O(M\varepsilon) \end{cases}$$

In addition,

$$\begin{cases} \tilde{B}_{31}(X,W,\nabla_X W) = O(M\varepsilon) \\ \tilde{B}_{3i}(X,W,\nabla_X W) - X_i = O(M\varepsilon), \qquad i=2,3 \\ \nabla_X^k \tilde{B}_{31}(X,W,\nabla_X W) = O(\frac{M\varepsilon}{R^{k-1+\delta_0}}), \qquad k=1,2 \\ \nabla_X^k (\tilde{B}_{3i}(X,W,\nabla_X W) - X_i) = O(\frac{M\varepsilon}{R^{k-1+\delta_0}}), \qquad k=1,2; \qquad i=2,3 \\ \nabla_X^k \tilde{B}_3(X,W) = O(\varepsilon), \qquad k=0,1 \\ \nabla_X^k \tilde{B}_3(X,W) = O(\frac{\varepsilon}{R^{k-2+\delta_0}}), \qquad k=2,3 \\ \sup\limits_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^2\Gamma_i} \sum\limits_{k=2} |d_{X,Y}|^{1+2\delta_0} \dfrac{|\nabla^k \tilde{B}_{3i}(X) - \nabla^k \tilde{B}_{3i}(Y)|}{|X-Y|^{\delta_0}} = O(M\varepsilon), \qquad i=1,2,3 \\ \sup\limits_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^2\Gamma_i} \sum\limits_{k=3} |d_{X,Y}|^{1+2\delta_0} \dfrac{|\nabla^k \tilde{B}_3(X) - \nabla^k \tilde{B}_3(Y)|}{|X-Y|^{\delta_0}} = O(\varepsilon) \end{cases}$$

Since $B_0(X,W) > \frac{\lambda}{2}$, then by the maximal principle we arrive at

$$|\tilde{V}| \le C_0\varepsilon \tag{5.11}$$

here the constant $C_0 > 0$ is independent of M and ε.

With the basic L^∞ estimate on $\tilde{V}$ in (5.11), we now can derive the required higher order estimates for $\tilde{V}$ in order to define the mapping J in (5.5). The desired estimates are stated in the following proposition.

Lemma 5.3. *Assume that $W \in \mathbb{K}$. If $\tilde{V}(X) \in C^{1,1-\delta_0}(\bar{Q}_+) \cap C^{3,\delta_0}(\bar{Q}_+ \setminus \cup_{i=1}^2\Gamma_i)$ is a solution of (5.6) with the boundary conditions (5.8)-(5.10), then for small $\varepsilon > 0$, there exists a constant $C_0 > 0$ independent of M and ε such that*

$$\begin{aligned} &\|\tilde{V}\|_{C^{1,1-\delta_0}} \le C_0\varepsilon \\ &\sup_X R^{k-2+\delta_0} |\nabla_X^k \tilde{V}| \le C_0\varepsilon, \qquad k=2,3 \\ &\sup_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^2\Gamma_i} \sum_{k=3} |d_{X,Y}|^{1+2\delta_0} \frac{|\nabla^k \tilde{V}(X) - \nabla^k \tilde{V}(Y)|}{|X-Y|^{\delta_0}} \le C_0\varepsilon, \end{aligned}$$

and

$$\begin{aligned} &\|\partial_\theta \tilde{V}\|_{C^{1,1-\delta_0}} \le C_0\varepsilon, \\ &\sup_X R^{\delta_0} |\nabla_X^2 \partial_\theta \tilde{V}| \le C_0\varepsilon, \qquad k=2,3, \\ &\sup_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^2\Gamma_i} \sum_{k=2} |d_{X,Y}|^{2\delta_0} \frac{|\nabla^2 \partial_\theta \tilde{V}(X) - \nabla^2 \partial_\theta \tilde{V}(Y)|}{|X-Y|^{\delta_0}} \le C_0\varepsilon. \end{aligned}$$

Proof. Without loss of generality and for simplicity, we assume

$$a_{11} = -1 + O(M\varepsilon), \qquad a_{22} = -1 + O(M\varepsilon), \qquad a_{33} = -1 + O(M\varepsilon),$$

Otherwise, we can make a transformation $X_1' = \frac{X_1}{\sqrt{c_+^2 - q_+^2}}, X_2' = \frac{X_2}{c_+}, X_3' = \frac{X_3}{c_+}$ such that the coefficients of the resulting equation satisfy the above requirements.

Denote by $\Sigma_1 = \{X : X_1 = 0, X_2^2 + X_3^2 < 1\}$, $\Sigma_2 = \{X : 0 < X_1 < 1, X_2^2 + X_3^2 = 1\}$, $\Sigma_3 = \{X : X_1 = 1, X_2^2 + X_3^2 < 1\}$. Then $\partial Q_+ = \overline{\cup_{i=1}^3 \Sigma_i}$. Consider a subdomain Q_1 of Q_+ with the property that $\partial Q_1 \cap \partial Q_+$ lies in the interior of ∂Q_+. Then by the classical Schauder estimates on the second order elliptic equation with the uniform oblique derivative boundary conditions (see [12] or [15]), there exists a constant $C(\|\tilde{B}_{ki}\|_{C^{1,1-\delta_0}(\bar{Q}_1)}, \|\tilde{B}_{3j}\|_{C^{1,1-\delta_0}(\bar{Q}_1)})(k = 1, 2; i = 2, 3; j = 1, 2, 3)$ such that

$$\begin{aligned}\|\tilde{V}\|_{C^{2,1-\delta_0}(\bar{Q}_1)} \le C(\|\tilde{B}_{ki}\|_{C^{1,1-\delta_0}(\bar{Q}_1)}, \|\tilde{B}_{3j}\|_{C^{1,1-\delta_0}(\bar{Q}_1)})(\|\tilde{V}\|_{L^\infty(Q_+)} + \|F_0\|_{C^{1-\delta_0}(Q_+)} \\ + \sum_{i=1}^3 \|\tilde{B}_i\|_{C^{1,1-\delta_0}(Q_+)}) \end{aligned} \tag{5.12}$$

Thus our main task is to estimate the derivatives of $\tilde{V}$ near the circles Γ_1 and Γ_2. To this end, without loss of generality we only consider the problem in a small neighborhood $G(r_0) = \{X : |X - P_0| < r_0\}$ of $P_0 = (0, 1, 0)$.

We will use the cylindrical coordinates (3.14) and denote by $Z_1 = X_1, Z_2 = r$ and $Z_3 = \theta$. Then in the domain $G'(r_0) = G(r_0) \cap \{Z : Z_1 \ge 0, 1 - \delta \le Z_2 \le 1, -r_0 \le Z_3 \le r_0\}$, the equation (5.6) and the boundary conditions (5.8), (5.10) can be rewritten as

$$\begin{cases} \sum_{i,j=1}^3 A_{ij}(Z)\partial^2_{Z_iZ_j}\tilde{V} + \sum_{i=1}^3 M_i(Z)\partial_{Z_i}\tilde{V} = F(Z), \\ \partial_{Z_1}\tilde{V} + \sum_{i=2}^3 N_{1i}(Z)\partial_{Z_i}\tilde{V} = G_1(Z) \qquad \text{on} \qquad Z_1 = 0, \\ \partial_{Z_2}\tilde{V} + N_{21}(Z)\partial_{Z_1}\tilde{V} + N_{23}(Z)\partial_{Z_3}\tilde{V} = G_2(Z) \qquad \text{on} \qquad Z_2 = 1, \end{cases} \tag{5.13}$$

where

$$\sum_{i,j=1}^3 |A_{ij}(Z) + \delta_{ij}| \le C(r_0 + M\varepsilon), \qquad \sum_{i=1}^3 |M_i(Z)| \le C(1 + M\varepsilon), \qquad \sum_{i=1}^2 |G_i(Z)| + |F(Z)| \le C\varepsilon,$$

$$|N_{12}(Z)| + |N_{13}(Z)| + |N_{21}(Z)| + |N_{23}(Z)| \le CM\varepsilon,$$

$$\sum_{i=1}^2 |\nabla_Z^k G_i(Z)| + |\nabla_Z^k F(Z)| \le \frac{C\varepsilon}{R_Z^{k-1+\delta_0}}, \qquad k = 1, 2,$$

$$|\nabla_Z^k A_{ij}(Z)| + \sum |\nabla_Z^k N_{ij}(Z)| \le \frac{CM\varepsilon}{R_Z^{k-1+\delta_0}}, \qquad k = 1, 2,$$

$$\begin{aligned}\sup_{Z,Z' \in G'(r_0)} \sum_{k=2} |d_{Z,Z'}|^{1+2\delta_0} \Big(\frac{|\nabla^k A_{ij}(Z) - \nabla^k A_{ij}(Z')|}{|Z - Z'|^{\delta_0}} + \frac{|\nabla^k M_i(Z) - \nabla^k M_i(Z')|}{|Z - Z'|^{\delta_0}} \\ + \frac{|\nabla^k N_{ij}(Z) - \nabla^k N_{ij}(Z')|}{|Z - Z'|^{\delta_0}} \Big) \le CM\varepsilon \end{aligned}$$

with $R_Z = \sqrt{Z_1^2 + (Z_2 - 1)^2}$, $d_{Z,Z'} = min(R_Z, R_{Z'})$ and a generic constant $C > 0$ independent of M, ε and r_0.

Define a C^∞ function $\chi(Z)$ as

$$\chi(Z) = \begin{cases} 1, & \sqrt{|Z_1|^2 + |Z_2 - 1|^2 + |Z_3|^2} \leq \frac{r_0}{2} \\ 0, & \sqrt{|Z_1|^2 + |Z_2 - 1|^2 + |Z_3|^2} \geq \frac{2}{3} r_0 \end{cases}$$

Let $\tilde{V}_1(Z) = \chi(Z)\tilde{V}$. Then it follows from (5.13) that $V_1(Z)$ satisfies the following elliptic equation and the boundary conditions

$$\begin{cases} \sum_{i,j=1}^{3} A_{ij}(Z)\partial^2_{Z_i Z_j}\tilde{V}_1 + \sum_{i=1}^{3} M_i(Z)\partial_{Z_i}\tilde{V}_1 = F'(Z), \\ \partial_{Z_1}\tilde{V}_1 + \sum_{i=2}^{3} N_{1i}(Z)\partial_{Z_i}\tilde{V}_1 = G_1'(Z) \qquad \text{on} \qquad Z_1 = 0, \\ \partial_{Z_2}\tilde{V}_1 + N_{21}(Z)\partial_{Z_1}\tilde{V}_1 + N_{23}(Z)\partial_{Z_3}\tilde{V}_1 = G_2'(Z) \qquad \text{on} \qquad Z_2 = 1, \\ \tilde{V}_1 = 0 \qquad \text{on} \qquad \sqrt{|Z_1|^2 + |Z_2 - 1|^2 + |Z_3|^2} = r_0. \end{cases} \tag{5.14}$$

Combining with the Schauder interior estimate (5.12), one easily knows that $F'(Z)$ and $G_i'(Z)$ have the same properties as $F(Z)$ and $G_i(Z)$.

By use of (5.11) and Lemma 3.1 (3.4) in [14] (more concretely, we choose $\delta = 1 - \delta_0$, $\alpha = 1 - \delta_0$ in (3.4) of [14]. Noting that the angle between $Z_1 = 0$ and $Z_2 = 1$ is $\frac{\pi}{2}$, by a careful check on the proof procedure in Lemma 3.1 of [14], then one can know that Lemma 3.1 still holds for this case, or one can see the details in [24]), for small r_0 and $M\varepsilon$ we obtain

$$\|\tilde{V}_1\|_{C^{1,1-\delta_0}} + \sup_{Z \in G'(r_0)} R_Z^{\delta_0} |\nabla_Z^2 \tilde{V}_1(Z)| + \sup_{Z,Z' \in G'(r_0)} \sum_{k=2} |d_{Z,Z'}| \frac{|\nabla^k \tilde{V}_1(Z) - \nabla^k \tilde{V}_1(Z')|}{|Z - Z'|^{1-\delta_0}} \leq C_0 \varepsilon, \tag{5.15}$$

with C_0 a uniform constant.

Next we improve the estimates on the tangential derivatives of $\tilde{V}_1$.

Set $U = \partial_3 \tilde{V}_1$. It follows from (5.14), (5.15) and the assumption on the tangent regularities of $W(X)$ that

$$\begin{cases} \sum_{i,j=1}^{3} A_{ij}(Z)\partial^2_{Z_i Z_j}U + \sum_{i=1}^{3} M_i(Z)\partial_{Z_i}U = F_1(Z), \\ \partial_{Z_1}U + \sum_{i=2}^{3} N_{1i}(Z)\partial_{Z_i}U = H_1(Z) \qquad \text{on} \qquad Z_1 = 0, \\ \partial_{Z_2}U + N_{21}(Z)\partial_{Z_1}U + N_{23}(Z)\partial_{Z_3}U = H_2(Z) \qquad \text{on} \qquad Z_2 = 1, \\ U = 0 \qquad \text{on} \qquad \sqrt{|Z_1|^2 + |Z_2 - 1|^2 + |Z_3|^2} = r_0, \end{cases} \tag{5.16}$$

here

$$\|F_1(Z)\|_{C^{1-\delta_0}} \leq C\varepsilon, \qquad |\nabla F_1(Z)| \leq \frac{C\varepsilon}{R_Z^{\delta_0}},$$
$$\|H_1(Z)\|_{C^{1-\delta_0}} + \|H_2(Z)\|_{C^{1-\delta_0}} \leq C\varepsilon.$$

By (5.15) and Lemma 3.1 in [14], we obtain

$$\|U\|_{C^{1,1-\delta_0}} + \sup_{Z\in G'(r_0)} R_Z^{\delta_0}|\nabla_Z^2 U(Z)| + \sup_{Z,Z'\in G'(r_0)} \sum_{k=2} |d_{Z,Z'}| \frac{|\nabla^k U(Z) - \nabla^k U(Z')|}{|Z-Z'|^{1-\delta_0}} \le C\varepsilon. \tag{5.17}$$

¿From [8], it is easy to derive from (5.17) that

$$\sup_{Z,Z'\in G'(r_0)} \sum_{k=2} |d_{Z,Z'}|^{2\delta_0} \frac{|\nabla^k U(Z) - \nabla^k U(Z')|}{|Z-Z'|^{\delta_0}} \le C\varepsilon. \tag{5.18}$$

Additionally, in light of (5.17) and the equation (5.13) one has

$$\begin{cases} \sum_{i,j=1}^{2} A_{ij}(Z)\partial^2_{Z_iZ_j}\tilde V_1 + \sum_{i=1}^{2} M_i(Z)\partial_{Z_i}\tilde V_1 = F_2(Z), \\ \partial_{Z_1}\tilde V_1 + N_{12}(Z)\partial_{Z_2}\tilde V_1 = Q_1(Z) \qquad \text{on} \qquad Z_1 = 0, \\ \partial_{Z_2}\tilde V_1 + N_{21}(Z)\partial_{Z_1}\tilde V_1 = Q_2(Z) \qquad \text{on} \qquad Z_2 = 1, \\ \tilde V_1 = 0 \qquad \text{on} \qquad \sqrt{|Z_1|^2 + |Z_2-1|^2 + |Z_3|^2} = r_0 \end{cases}$$

with

$$\sum_{i=1}^{2} \|Q_i(Z)\|_{C^{1-\delta_0}} + \|F_2(Z)\|_{C^{1-\delta_0}} \le C\varepsilon,$$
$$\sum_{i=1}^{2} |\nabla_Z Q_i(Z)| + |\nabla_Z F_2(Z)| \le \frac{C\varepsilon}{R_Z^{\delta_0}},$$
$$\sup_{Z,Z'\in G'(r_0)} |d_{Z,Z'}|^{2\delta_0} \left(\frac{|\nabla F_2(Z) - \nabla F_2(Z')|}{|Z-Z'|^{\delta_0}} + \sum_{i=1}^{2} \frac{|\nabla Q_i(Z) - \nabla Q_i(Z')|}{|Z-Z'|^{\delta_0}} \right) \le C\varepsilon.$$

Analogous to the treatment on 2-D problem in [24], we get

$$\begin{aligned} &\|\tilde V_1(Z_1,Z_2,\cdot)\|_{C^{1,1-\delta_0}} + \sum_{k=2}^{3} \sup_{Z\in G'(r_0)} R_Z^{k-2+\delta_0} |\nabla^k_{Z_1,Z_2}\tilde V_1(Z_1,Z_2,\cdot)| \\ &+ \sup_{(Z_1,Z_2,Z_3),(Z_1',Z_2',Z_3)\in G'(r_0)} \sum_{k=3} |d_{Z,Z'}|^{1+2\delta_0} \frac{|\nabla^k \tilde V_1(Z_1,Z_2,\cdot) - \nabla^k \tilde V_1(Z_1',Z_2',\cdot)|}{|(Z_1-Z_1', Z_2-Z_2')|^{\delta_0}} \le C\varepsilon. \end{aligned} \tag{5.19}$$

Combining (5.15), (5.18) with (5.19) shows that Lemma 5.3 holds.

Based on (5.11) and Lemma 5.3, and by the continuity method as given in [9] (see also [23] or Lemma 2.3 in [14]), the linear equation (5.6) with the boundary conditions (5.8)-(5.10) is solvable in the space $\mathbb{K}$. Furthermore, (5.11) and Lemma 5.3 imply that we can choose the constant C_0 as the constant M in $\mathbb{K}$. Hence the mapping J which is defined in (5.5) maps from $\mathbb{K}$ into $\mathbb{K}$. Moreover we have

Lemma 5.4. *J is a continuous mapping from $\mathbb{K} \to \mathbb{K}$.*

Proof. To prove Lemma 5.4, we need to verify the assertion:

If $W_l(X), W_0(X) \in \mathbb{K}$ and $W_l(X) \to W_0(X)$ in $\mathbb{K}$ as $l \to \infty$, then the corresponding solutions $V_l(X) \to V_0(X)$ in $\mathbb{B}$. (5.20)

At first, it follows from Lemma 5.3 that $\{V_l(X)\}_{l=1}^{\infty}$ and $V_0(X)$ are uniformly bounded in $\mathbb{K}$. To prove (5.20), we need only to show that $\|V_l(X) - V_0(X)\|_{C(\bar{Q}_+)} \to 0$ and $\|\partial_\theta(V_l(X) - V_0(X))\|_{C(\bar{Q}_+)} \to 0$ by the interpolation inequality on the weighted Holder space (see [8]).

Set $\bar{V}_l = V_l(X) - V_0(X)$, then it satisfies

$$\left\{\begin{aligned}
&\sum_{i,j=1}^{3} a_{ij}(X_1, W_l, \nabla_X W_l)\partial^2_{X_i X_j}\bar{V}_l + \sum_{i=1}^{3} F_l^i(X)\partial_{X_i}(W_l - W_0) + F_l^0(X)(W_l - W_0) = 0,\\
&\partial_{X_1}\bar{V}_l + \sum_{i=2}^{3}\tilde{B}_{1i}(X, W_l, \nabla_X W_l)\partial_{X_i}\bar{V}_l + \sum_{i=2}^{3} B_{li}^1(X)\partial_{X_i}(W_l - W_0) + B_{l1}^0(X)(W_l - W_0) = 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \qquad X_1 = 0\\
&\partial_{X_1}\bar{V}_l + \sum_{i=2}^{3}\tilde{B}_{2i}(X, W_l, \nabla_X W_l)\partial_{X_i}\bar{V}_l + \sum_{i=2}^{3} B_{li}^2(X)\partial_{X_i}(W_l - W_0) + B_{l2}^0(X)(W_l - W_0) = 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \qquad X_1 = 1,\\
&\sum_{i=1}^{3}\tilde{B}_{3i}(X, W_l, \nabla_X W_l)\partial_{X_i}\bar{V}_l + \sum_{i=1}^{3} B_{li}^3(X)\partial_{X_i}(W_l - W_0) + B_{l3}^0(X)(W_l - W_0) = 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \qquad X_2^2 + X_3^2 = 1,
\end{aligned}\right. \tag{5.21}$$

where

$$\begin{aligned}
F_l^i(X) &= \sum_{j=1}^{3}\partial^2_{X_i X_j}V_0\int_0^1(\partial_{W_i}a_{ij})(X_1, \theta W_l + (1-\theta)W_0, \theta\nabla W_l + (1-\theta)\nabla W_0)d\theta\\
&\quad + \int_0^1(\partial_{W_i}F_0)(X, \theta W_l + (1-\theta)W_0, \theta\nabla W_l + (1-\theta)\nabla W_0)d\theta, \qquad i = 0,1,2,3,\\
B_{lj}^i(X) &= \partial_{X_j}V_0\int_0^1(\partial_{W_i}\tilde{B}_{ij})(X, \theta W_l + (1-\theta)W_0, \theta\nabla W_l + (1-\theta)\nabla W_0)d\theta\\
&\quad + \int_0^1(\partial_{W_i}\tilde{B}_j)(X, \theta W_l + (1-\theta)W_0, \theta\nabla W_l + (1-\theta)\nabla W_0)d\theta,
\end{aligned}$$

here we use the notations $(W_0, W_1, W_2, W_3) = (W(X), \partial_{X_1}W, \partial_{X_2}W, \partial_{X_3}W)$ for convenience.

¿From the expressions of $F_l^i(X)$ and $B_{lj}^i(X)$, one has the following estimates

$$\begin{aligned}
&|F_l^i(X)| \leq \frac{CM\varepsilon}{R^{\delta_0}}\\
&|\nabla_X F_l^i(X)| \leq \frac{CM\varepsilon}{R^{1+\delta_0}}\\
&\frac{\lambda}{2} < B_{l3}^0(X) < 2\Lambda,\\
&|B_{lj}^i(X)| \leq CM\varepsilon, \qquad (i,j) \neq (0,3),\\
&|\nabla_X B_{lj}^i(X)| \leq \frac{CM\varepsilon}{R^{\delta_0}}
\end{aligned}$$

By the Aleksandrov's maximal principle (see [13]), we arrive at

$$\|V_l(X) - V_0(X)\|_{C(\bar{Q}_+)} \le C\|W_l - W_0\|_{C^{1,1-\delta_0}(\bar{Q}_+)}$$

Namely,

$$\|V_l(X) - V_0(X)\|_{C(\bar{Q}_+)} \to 0 \qquad \text{as} \qquad l \to \infty.$$

In a similar way, one can derive

$$\|\partial_\theta(V_l(X) - V_0(X))\|_{C(\bar{Q}_+)} \to 0 \qquad as \qquad l \to \infty.$$

Hence, the proof on Lemma 5.4 is completed.

Proof for Theorem 5.1.
It follows from Lemma 5.3 and Lemma 5.4 that the mapping J satisfies the all requirements of Theorem 5.2. By the choice of J, the existence of solution in Theorem 5.1 follows. The similar argument as the proof of Lemma 5.4 shows the uniqueness in Theorem 5.1. Finally, we give the proof on Theorem 1.3.

Proof of Theorem 1.3.
(i) in Theorem 1.3 follows from Lemma 2.1 directly.
By the regularity and uniqueness of $V(X)$ in Theorem 5.1, we conclude that the inverse transformation (2.2) has the following properties:

$$x_1(X), x_2(X), x_3(X) \in C^{1,1-\delta_0}(\bar{Q}_+) \cap C^{3,\delta_0}(Q_+)$$

Since the shock Σ: $x_1 = \xi(x_2, x_3)$ corresponds to $X_1 = 0$ in $\bar{Q}_+$, then $\xi(x_2, x_3) \in C^{1,1-\delta_0}(\bar{S}) \cap C^{3,\delta_0}(S)$, where S represents the open projection set of Σ onto the $(x_2, x_3)-$plane. Besides, it is easy to verify the other conclusions, (ii), (iii) and (iv) in Theorem 1.3, by the properties of $V(X)$ in Theorem 5.1. Thus the proof of Theorem 1.3 is completed.

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References

1. H. W. Alt, L. A. Caffarelli, A. Friedman, *Compressible flows of jets and cavities, J. Diff. Eqs. 56, 82-141* (1985).
2. L. Bers, *Existence and uniqueness of subsonic flows past a given profile Comm. Pure Appl. Math. 7, 441-504* (1954).
3. S. Canic, B. L. Keyfitz, G. M. Lieberman, *A proof of existence of perturbed steady transonic shocks via a free boundary problem, Comm. Pure Appl. Math., Vol. LIII, 484-511* (2000).
4. Guiqiang Chen, M. Feldman, *Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type, J.A.M.S., Vol. 16, No. 3, 461-494* (2003).
5. Shuxing Chen, Zhouping Xin, Huicheng Yin, *Unsteady supersonic flow past a wedge, IMS of CUHK, Preprint* (2002).
6. R. Courant, K. O. Friedrichs, *Supersonic flow and shock waves, Interscience Publishers Inc., New York, 1948.*
7. C. A. J. Fletcher, *Computational techniques for fluid dynamics, Vol. I-II, Berlin: Springer-Verlag; HongKong: Springer-Verlag,* (1991).

8. D. Gilbarg, L. Hörmander, *Intermediate Schauder estimates, Arch. Rational Mech. Anal. 74, No. 4, 297-318* (1980).
9. D. Gilbarg, N. S. Tudinger, *Elliptic partial differential equations of second order. Second edition. Grundlehren der Mathematischen Wissenschaften, 224, Springer, Berlin-New York,1983.*
10. J. Goodman, P. Embid, A. Majda, *Multiple steady states for 1-D transonic flow, SIAM J. Sci. Statist. Compt. 5, 21-41* (1984).
11. F. John, *Nonlinear wave equations, formation of singularities, University Lecture Series 2, American Mathematical Society, Providence, RI* (1990).
12. O. A. Ladyzhenskaya, N. N. Ural'tseva, *Linear and quasilinear elliptic equations, English Transl., Academic Press, New York* (1968).
13. G. M. Lieberman, *Local estimates for subsolution and supersolution of oblique derivative problems for general second order elliptic equations, Trans.A.M.S., Vol. 304, No. 1, 343-353* (1987).
14. G. M. Lieberman, *Oblique derivative problems in Lipschitz domains II, J.reine angew.Math.389, 1-21* (1988).
15. G. Lieberman, N. Trudinger, *Nonlinear oblique boundary value problems for nonlinear elliptic equations, Trans. Amer. Math. Soc. 295, 509-546* (1986).
16. T. P. Liu, *Nonlinear stability and instability of transonic flows through a nozzle, Comm. Math. Phys., 83, 243-260* (1982).
17. A. Majda, *One perspective on open problems in multidimensional conservation laws, Multidimensional hyperbolic problems and computation, Springer Verlag, IMA 29, 217-237* (1990).
18. A. Majda, E. Thomann, *Multidimensional shock fronts for second order wave equations, Comm.P.D.E. 12, 777-828* (1987).
19. V. G. Maz'ja, *Sobolev spaces, English Transl., Springer-Verlag, Berlin, Heideberg* (1994).
20. A. M. Meirmanov, *On a classical solution of the multidimensional Stefan problem for quasilinear parabolic equations, Math. Sb. 112, 170-192* (1980).
21. C. S. Morawetz, *Mathematical problems in transonic flow, Canad. Math. Bull. 29(2), 129-139* (1986).
22. C. S. Morawetz, *On the nonexistence of continuous transonic flows past profiles, I, Comm. Pure Appl. Math. 9(1956), 45-68; III 11(1958), 129-144; 17(1964), 357-367; 10(1975), 107-132.*
23. S. A. Nazarov, B. A. Plamenevsky, *Elliptic problems in domains with piecewise smooth boundaries, De Gruyter Expositions in Mathematics, Walter de Gruyter, Berlin, New York* (1994).
24. Zhouping Xin, Huicheng Yin, *Transonic shock in a nozzle I, 2-D case, Comm. Pure Appl. Math., (to appear).*
25. Zhouping Xin, Huicheng Yin, *Transonic flow for the supersonic gas past a wedge, IMS of CUHK, Preprint* (2003).
26. Zhouping Xin, Huicheng Yin, *Transonic shock in a nozzle III: complete Euler flow case, IMS of CUHK, Preprint, 2004.*
27. Zhouping Xin, Wei Yan, Huicheng Yin, *On the conjecture of Courant and Friedrichs for the transonic shock in a nozzle, IMS Preprint, 2004.*