On the Strong Convergence of the Approximate Solutions for the 3-D Steady Euler Equations with Axial-Symmetry

QUANSEN JIU [1] AND ZHOUPING XIN [2]

Abstract: In this paper we are concerned with the convergence properties of the approximate solutions for the three-dimensional steady axisymmetric Euler equations without swirls. Making use of the special structure of the equations for axisymmetric flows and special test functions, we establish a strong convergence criterion for the approximate solutions for general case. Then under the assumption that the vorticity has a distinguished sign, we obtain a sufficient and necessary condition for the strong convergence in $L^2_{loc}(R^3)$ for the approximate solutions. In addition, we show that if the approximate solutions have only one single-point concentration in $(r, z)-$ plane, then the concentration point can appear neither in the region near the axis (including the symmetry axis itself) nor in the region far away from the axis. Some approximate solutions with strong convergence in $L^2_{loc}(R^3)$ are presented.

Key Words: 3-D steady Euler equations, axisymmetric solutions, the approximate solutions, strong convergence, vortex rings

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1 Introduction

Three-dimensional (3-D for convenience) incompressible steady Euler equations in R^3 are

$$\begin{cases} (u\cdot\nabla)u+\nabla p=0, \quad x\in R^3, \\ \text{div } u=0. \end{cases} \tag{1.1}$$

Here $u=(u_1(x,t),u_2(x,t),u_3(x,t))$ represents the velocity fields and $p=p(x,t)$ is the pressure function. The equation div $u=0$ stands for the incompressibility of the flows.

By axisymmetric solutions of (1.1), we mean that, in the cylindrical coordinate system, the unknown functions $u(x,t)$ and $p(x,t)$ do not depend on θ-variable, that is,

$$\begin{aligned} u(x,t) &= u_r(r,z,t)e_r+u_\theta(r,z,t)e_\theta+u_z(r,z,t)e_z, \\ p(x,t) &= p(r,z,t), \end{aligned}$$

where

$$e_r=(\cos\theta,\sin\theta,0), \quad e_\theta=(-\sin\theta,\cos\theta,0), \quad e_z=(0,0,1)$$

form the standard orthogonal bases in the cylindrical coordinate system. Furthermore, when $u_\theta\equiv 0$, which means that the axisymmetric flow has no swirls, the corresponding 3-D steady axisymetric Euler equations (without swirls) can be written as

$$\begin{cases} u_r\partial_r u_r+u_z\partial_z u_r+\partial_r p=0, \\ u_r\partial_r u_z+u_z\partial_z u_z+\partial_z p=0. \end{cases} \tag{1.2}$$

And the incompressibility condition becomes

$$\partial_r(ru_r)+\partial_z(ru_z)=0. \tag{1.3}$$

In this case, the vorticity of the velocity has a simple expression,

$$\omega=\nabla\times u=\omega_\theta e_\theta$$

with $\omega_\theta=\partial_z u_r-\partial_r u_z$.

It is well-known that when the initial data is a vortex-sheets data, *i.e.* , the initial vorticity is a finite Radon measure and the initial velocity is locally square-integrable, the two-dimensional unsteady Euler equations has global (in time) weak solutions when the initial vorticity ω_0 is of one-sign (see [2], [7], [13], [14], [17]). However, for the three-dimensional unsteady axisymmetric flows without swirls, when the initial data is a vortex-sheets data (the initial vorticity is a finite Radon measure and the initial velocity is square-integrable), even when the initial vorticity is of one sign, the global existence is still an outstanding open problem. It was proved in [3] that, for the 3-D unsteady axisymmetric Euler equations without swirls, the sequence of the approximate solutions generated by smoothing the initial data either converges strongly in $L^2_{loc}(R^3 \times (0, +\infty))$ or converges weakly in $L^2_{loc}(R^3 \times (0, +\infty))$ to a limit which is not a classical weak solution to the Euler equations under the additional assumption that the initial vorticity has a distinguished sign. In other words, there is no concentration-cancellation occurring for one-sign axisymmetric flows with no swirls. This is in sharp contrast to the 2-D theory (see [5]). Recently, the authors proved in [11] that the approximate solutions, generated by smoothing the initial data, converge strongly in $L^2([0, T]; L^2_{loc}(R^3))$ provided that they have strong convergence in the region away from the symmetry axis. This means that if there would appear singularity or energy lost in the process of limit for the approximate solutions, it then must happen in the region away from the symmetry axis. It is noted that there is no restriction on the signs of initial vorticity in [11]. The convergence properties of the viscous approximations are studied in [12]. When the initial vorticity is in Orlitz space $L(\log^+ L)^\alpha(R^3)(\alpha > 1/2)$, which includes any $L^1(R^3) \cap L^p(R^3)(p > 1)$ space, the global existence of weak solutions was obtained in [1].

In this paper, we are mainly concerned with the convergence properties of the approximate solutions of the 3-D steady axisymmetric Euler equations without swirls (1.2)-(1.3). Similar to unsteady case, the approximate solutions for the equations (1.2)-(1.3) can be defined in usual way.

Definition 1.1 We call $\{u^\varepsilon\}$ ($\epsilon \in J$ a parameter) **the approximate solutions** of the equations (1.2)-(1.3) if the following conditions are satisfied:

(i) $u^\varepsilon(x)$ is uniformly bounded in $L^2(R^3)$;

(ii) $u^\varepsilon = u^\varepsilon_r e_r + u^\varepsilon_z e_z$;

(iii) $\omega^\varepsilon = \nabla \times u^\varepsilon = \omega^\varepsilon_\theta e_\theta$;

(iv) For $\varphi_r(r, z), \varphi_z(r, z) \in C^\infty_0(\bar{H})$, satisfying

$$\partial_r(r\varphi_r) + \partial_z(r\varphi_z) = 0, \tag{1.4}$$

one has

$$\begin{aligned}&\int_H [(u^\varepsilon_r)^2 \partial_r \varphi_r + (u^\varepsilon_z)^2 \partial_z \varphi_z] r dr dz \\ &= -\int_H u^\varepsilon_r u^\varepsilon_z (\partial_r \varphi_z + \partial_z \varphi_r) r dr dz + h(\varepsilon)\end{aligned} \tag{1.5}$$

with $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Here $H = \{(r, z) | (r, z) \in (0, \infty) \times (-\infty, +\infty)\}$ represents the $(r, z)-$plane.

Formally, multiplying $r\varphi_r$ and $r\varphi_z$ on both side of $(1.2)_1$ and $(1.2)_2$ respectively, integrating the resulted equations on $(0,\infty)\times(-\infty,\infty)$ with respect to r and z and summing over them, one obtains (1.5) with $h(\varepsilon)=0$. There are many ways to get the approximate solutions (see Section 5 of this paper, for example).

For the two-dimensional steady Euler equations, DiPerna and Majda proved that, even though there exist approximate solutions with energy concentration, the weak limit of any approximate solutions is a weak solution, by using the shielding method (see [4]). That is, concentration-cancellation occurs in this case. The reader may refer to [6] for a more concise proof. However, for the three-dimensional steady equations, even for the axisymmetric case, the convergence properties of the approximate solutions are not as clear as those for the two-dimensional case. It is also not known whether or not there exist approximate solutions with energy concentration for the three-dimensional steady Euler equations.

On the other hand, the existence of solutions of the 3-D steady axisymmetric Euler equations without swirls (1.2)-(1.3) has been widely studied. In particular, the vortex rings, which are steady, axisymmetric solutions without swirls of the equations (1.1), propagating with constant speed in the z-direction, has been extensively and systematically investigated, based mainly on the variational approaches (see [8],[9] [15] and references therein). Another approach to seek nontrivial steady, axisymmetric solutions without swirls of the equations (1.1), which is called pseudo-advection method, is referred to [16], [18].

In this paper, we will first obtain a criterion for strong convergence for approximate solutions as defined in Definition 1.1 for general case. We will establish a relation between the energy distributions of the weak limit and the defect measure of the approximate solutions in this case. Then, under the assumptions that the vorticity is of one sign and uniformly bounded in L^1-space, we obtain that the approximate solutions (or its subsequence) converge strongly in L^2_{loc}-space if and only if the corresponding weak limit $u=(u_1,u_2,u_3)$ satisfies an equilibrium energy distributions (see (3.12) in section 3). Furthermore, we will study the case of single-point concentration in (r,z)-plane. We will prove that if a sequence of approximate solutions has only one single-point concentration in $(r,z)-$ plane and no other singularity occurring in the limit process, then the concentration point appears neither in the region near the axis (including the symmetry axis itself) nor in the region far away from the symmetry axis. Finally, we will present some approximate solutions which converge strongly in $L^2_{loc}(R^3)$ based on pseudo-advection method and vortex rings mentioned above. However, it is still open whether there exist approximate solutions with concentrations or whether all the approximate solutions converge strongly in $L^2_{loc}(R^3)$.

The rest of this paper is organized as follows. In Section 2, we give a criterion on the strong convergence for approximate solutions of the three-dimensional steady axisymmetric Euler equations without swirls for general case. In Section 3, we will present a sufficient and necessary condition for the strong convergence $L^2_{loc}(R^3)$ for approximate solutions with one-sign vorticity. In Section 4, we will consider the case of single-point concentration in (r,z)-plane. Finally, in Section 5, we will give some examples of the approximate solutions which converge strongly in $L^2_{loc}(R^3)$ and some concluding remarks.

2 A Criterion on the Strong Convergence

Given a sequence of approximate solutions $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$, which is also expressed by $u^\varepsilon = (u_r^\varepsilon, 0, u_z^\varepsilon)$ in the cylindrical coordinates systems (see Definition 1.1), it is obvious that there exists a subsequence of u^ε, still denoted by itself, converging weakly in $L^2(R^3)$ and in $L^2(H; rdrdz)$. Precisely, as $\varepsilon \to 0^+$, one has

$$u_1^\varepsilon \rightharpoonup u_1, \quad u_2^\varepsilon \rightharpoonup u_2, \quad u_3^\varepsilon \rightharpoonup u_3 \tag{2.1}$$

weakly in $L^2(R^3)$, and, in the cylindrical coordinate systems,

$$u_r^\varepsilon \rightharpoonup u_r, \quad u_z^\varepsilon \rightharpoonup u_z \tag{2.2}$$

weakly in $L^2(H; rdrdz)$.

Moreover, since the square of the approximate solutions $(u^\varepsilon(x))^2$ is uniformly bounded in $L^1(R^3)$ and so there exists a subsequence of the approximate solutions, still denoted by itself, converging weakly in the space of finite Radon measures $M(R^3)$, as $\varepsilon \to 0^+$. More precisely, we suppose that

$$(u_1^\varepsilon)^2 \rightharpoonup u_1^2 + \mu_1, (u_2^\varepsilon)^2 \rightharpoonup u_2^2 + \mu_2, (u_3^\varepsilon)^2 \rightharpoonup u_3^2 + \mu_3 \tag{2.3}$$

weakly in $M(R^3)$, where $\mu_i \geq 0 (i = 1, 2, 3)$ is the defect measure of $(u_i^\varepsilon)^2 (i = 1, 2, 3)$ respectively. The total variation of $\mu_i (i = 1, 2, 3)$, denoted by $|\mu_i| (i = 1, 2, 3)$, is finite. Then we have

Theorem 2.1 There exists a subsequence of the approximate solutions $\{u^\varepsilon\}$, still denoted by itself, satisfying (2.1), (2.2) and (2.3). Moreover, we have

$$\int_{R^3} u_3^2 dx - \frac{1}{2}\int_{R^3} (u_1^2 + u_2^2) dx + |\mu_3| - \frac{1}{2}(|\mu_1| + |\mu_2|) = 0. \tag{2.4}$$

Consequently, if $u^\varepsilon \to u$ strongly in $L^2_{loc}(R^3)$, then

$$\int_{R^3} u_3^2 dx - \frac{1}{2}\int_{R^3} (u_1^2 + u_2^2) dx = 0. \tag{2.5}$$

Proof. The proofs of (2.1), (2.2) and (2.3) are clear. It suffices to prove (2.4).

We start from the fact that the approximate solutions $u^\varepsilon = u_r^\varepsilon e_r + u_z^\varepsilon e_z$ and p^ε satisfy (1.5), as defined in Definition 1.1.

Let $\chi = \chi(s)$ be the usual smooth cutting-off function satisfying

$$\begin{cases} \chi(s) = 1, & |s| \le 1, \\ \chi(s) = 0, & |s| > 2. \end{cases} \tag{2.6}$$

Denote $\chi_+(s) = \chi(s)|_{s\ge 0}$, which is the restriction of $\chi(s)$ on $\{s \ge 0\}$, satisfying

$$\begin{cases} \chi_+(s) = 1, & 0 \le s \le 1, \\ \chi(s) = 0, & s > 2. \end{cases} \tag{2.7}$$

We choose the test functions in (1.5) as

$$\begin{aligned} &\varphi_r = \tfrac{1}{2} r \chi_+(\tfrac{r}{\eta})[\chi(\tfrac{z-z_0}{\eta}) + \tfrac{z-z_0}{\eta}\chi'(\tfrac{z-z_0}{\eta})], \\ &\varphi_z = -[\chi_+(\tfrac{r}{\eta}) + \tfrac{r}{2\eta}\chi_+{}'(\tfrac{r}{\eta})](z-z_0)\chi(\tfrac{z-z_0}{\eta}) \end{aligned} \tag{2.8}$$

for any $\eta > 0$. Then direct calculations lead to

$$\begin{aligned} \tfrac{\varphi_r}{r} &= \tfrac{1}{2}\chi_+(\tfrac{r}{\eta})[\chi(\tfrac{z-z_0}{\eta}) + \tfrac{z-z_0}{\eta}\chi'(\tfrac{z-z_0}{\eta})], \\ \partial_r\varphi_r &= \tfrac{1}{2}(\chi_+(\tfrac{r}{\eta}) + \tfrac{r}{\eta}\chi_+'(\tfrac{r}{\eta}))[\chi(\tfrac{z-z_0}{\eta}) + \tfrac{z-z_0}{\eta}\chi'(\tfrac{z-z_0}{\eta})], \\ \partial_z\varphi_z &= -[\chi_+(\tfrac{r}{\eta}) + \tfrac{r}{2\eta}\chi_+{}'(\tfrac{r}{\eta})][\chi(\tfrac{z-z_0}{\eta}) + \tfrac{z-z_0}{\eta}\chi'(\tfrac{z-z_0}{\eta})], \\ \partial_z\varphi_r &= \tfrac{1}{2} r\chi_+(\tfrac{r}{\eta})[\tfrac{2}{\eta}\chi'(\tfrac{z-z_0}{\eta}) + \tfrac{z-z_0}{\eta^2}\chi''(\tfrac{z-z_0}{\eta})], \\ \partial_r\varphi_z &= -[\tfrac{3}{2\eta}\chi_+'(\tfrac{r}{\eta}) + \tfrac{r}{2\eta^2}\chi_+{}''(\tfrac{r}{\eta})](z-z_0)\chi(\tfrac{z-z_0}{\eta}). \end{aligned} \tag{2.9}$$

Letting $\epsilon \to 0^+$ in (1.5), noting that

$$\begin{aligned} &|\int_H u_r^\varepsilon u_z^\varepsilon(\partial_z\varphi_r + \partial_r\varphi_z) r\,dr\,dz| \\ &\le \int_H [(u_r^\varepsilon)^2 + (u_z^\varepsilon)^2](|\partial_z\varphi_r| + |\partial_r\varphi_z|) r\,dr\,dz, \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{1}{2\pi}\{\int_{R^3}(u_1^2+u_2^2)\partial_r\varphi_r dx + \int_{R^3} u_3^2\partial_z\varphi_z dx \\ &+ \int_{R^3}\partial_r\varphi_r d(\mu_1+\mu_2) + \int_{R^3}\partial_z\varphi_z d\mu_3\} \\ &\le \int_H (u_r^2+u_z^2)(|\partial_z\varphi_r| + |\partial_r\varphi_z|) r\,dr\,dz \\ &+ \int_H (|\partial_z\varphi_r| + |\partial_r\varphi_z|) d(\mu_1+\mu_2+\mu_3). \end{aligned} \tag{2.10}$$

Substitute the test functions (2.9) into (2.10) to get

$$
\begin{aligned}
&\frac{1}{2\pi}\{|\int_{R^3}(\frac{u_1^2}{2}+\frac{u_2^2}{2}-u_3^2)\chi_+(\frac{r}{\eta})\chi(\frac{z-z_0}{\eta})dx \\
&+\int_{R^3}\chi_+(\frac{r}{\eta})\chi(\frac{z-z_0}{\eta})d(\frac{\mu_1}{2}+\frac{\mu_2}{2}-\mu_3)|\} \\
\leq\ & |\int_H(\frac{1}{2}u_r^2+u_z^2)\chi_+(\frac{r}{\eta})\frac{z-z_0}{\eta}\chi'(\frac{z-z_0}{\eta})rdrdz| \\
&+\frac{1}{2}|\int_H u_r^2\frac{r}{\eta}\chi'_+(\frac{r}{\eta})[\chi(\frac{z-z_0}{\eta})+\frac{z-z_0}{\eta}\chi'(\frac{z-z_0}{\eta})]rdrdz| \\
&+|\int_H u_z^2\frac{r}{2\eta}\chi'_+(\frac{r}{\eta})[\chi(\frac{z-z_0}{\eta})+\frac{z-z_0}{\eta}\chi'(\frac{z-z_0}{\eta})]rdrdz| \\
&+|\frac{1}{2}\int_{R^3}\{\chi_+(\frac{r}{\eta})\frac{z-z_0}{\eta}\chi'(\frac{z-z_0}{\eta})+\frac{r}{\eta}\chi'_+(\frac{r}{\eta})\cdot \\
&\quad[\chi(\frac{z-z_0}{\eta})+\frac{z-z_0}{\eta}\chi'(\frac{z-z_0}{\eta})]\}d(\mu_1+\mu_2)| \\
&+|\int_{R^3}\{\chi_+(\frac{r}{\eta})\frac{z-z_0}{\eta}\chi'(\frac{z-z_0}{\eta})+\frac{r}{2\eta}\chi'_+(\frac{r}{\eta}) \\
&\quad[\chi(\frac{z-z_0}{\eta})+\frac{z-z_0}{\eta}\chi'(\frac{z-z_0}{\eta})]\}d\mu_3| \\
&+\int_H(u_r^2+u_z^2)(|\partial_z\varphi_r|+|\partial_r\varphi_z|)rdrdz+ \\
&\int_H(|\partial_z\varphi_r|+|\partial_r\varphi_z|)d(\mu_1+\mu_2+\mu_3).
\end{aligned}
\tag{2.11}
$$

Using the fact that

$$
\int_{R^3}|u|^2dx<\infty,\quad |\mu|=\int_{R^3}d\mu=\sum_{k=1}^{\infty}\int_{\{k-1\leq r^2+z^2<k\}}d\mu<\infty,
$$

and

$$
\chi'(\frac{s}{\eta})\equiv 0\ \text{ for }\ |s|\leq\eta/2,\quad \chi'_+(\frac{s}{\eta})\equiv 0\ \text{ for }\ 0\leq s\leq\eta/2,
$$

we obtain that the all terms on the right side of (2.11) vanish as $\eta\to+\infty$. So letting $\eta\to+\infty$ on both sides of (2.11), using the dominate convergence theorem, we finally get

$$
\int_{R^3}u_3^2dx-\frac{1}{2}\int_{R^3}(u_1^2+u_2^2)dx+|\mu_3|-\frac{1}{2}(|\mu_1|+|\mu_2|)=0.
$$

(2.4) is proved and consequently, if $u^\varepsilon\to u$ strongly in $L^2_{loc}(R^3)$, then

$$
\int_{R^3}u_3^2dx-\frac{1}{2}\int_{R^3}(u_1^2+u_2^2)dx=0.
$$

The proof of the theorem is finished.

3 One-Signed Case

In this section, we assume additionally that the approximate solutions satisfy

(A1) the vorticity $\{\omega^\varepsilon\}$ has a distinguished sign in the sense that $\omega_\theta^\varepsilon \geq 0$ or $\omega_\theta^\varepsilon \leq 0$.

(A2) $\omega_\theta^\varepsilon$ is uniformly bounded in $L^1(\bar{H}, (1+r^2)drdz))$, that is

$$\int_{-\infty}^{+\infty}\int_0^{+\infty} |\omega_\theta^\varepsilon| drdz \leq C,$$
$$\int_{-\infty}^{+\infty}\int_0^{+\infty} |\omega_\theta^\varepsilon| r^2 drdz \leq C,$$

where C is a constant independent of ε.

Then, based on Delot's result in [3], we have the following result which will also be used in next section.

Lemma 3.1 Suppose that the approximate solutions satisfy (**A1**)-(**A2**) additionally. Then there exists a subsequence of $\{u^\varepsilon\}$, denoted still by itself, such that (2.1) and (2.2) hold. Moreover, as $\varepsilon \to 0^+$,

$$\int_{R^3}((u_1^\varepsilon)^2 + (u_2^\varepsilon)^2 - (u_3^\varepsilon)^2)\varphi(x)dx \to \int_{R^3}(u_1^2 + u_2^2 - u_3^2)\varphi(x)dx, \tag{3.1}$$

for all $\varphi(x) \in C_0^\infty(R^3)$.

$$\int_H u_r^\varepsilon u_z^\varepsilon \bar{\varphi}(r,z) rdrdz \to \int_H u_r u_z \bar{\varphi}(r,z) rdrdz \tag{3.2}$$

for all $\bar{\varphi}(r,z) \in C_0^\infty(\bar{H})$ satisfying $\max_{0\leq r\leq\delta, z\in R} |\bar{\varphi}(r,z)| \to 0$ as $\delta \to 0$. Here $\bar{H} = [0,\infty) \times (-\infty, +\infty)$.

Proof. As mentioned above, the convergence (2.1) and (2.2) is easy to get. So it suffices to prove (3.1) and (3.2).

Under the assumptions of the lemma, by [3], one has, as $\varepsilon \to 0^+$,

$$u_1^\varepsilon u_3^\varepsilon \rightharpoonup u_1 u_3, \quad u_2^\varepsilon u_3^\varepsilon \rightharpoonup u_2 u_3, \tag{3.3}$$

and

$$(u_1^\varepsilon)^2 + (u_2^\varepsilon)^2 - (u_3^\varepsilon)^2 \rightharpoonup (u_1)^2 + (u_2)^2 - (u_3)^2 \tag{3.4}$$

in the sense of distributions. We note that the convergence (3.3)-(3.4) were established in [3] for approximate solutions for the unsteady Euler equations and it also holds true for approximate solutions for the steady case. So (3.1) is a direct result of (3.4) after applying the cylindrical coordinate transformations:

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z. \tag{3.5}$$

Now we prove (3.2). By (3.3), we have, as $\varepsilon \to 0^+$,

$$\int_{R^3} u_1^\varepsilon u_3^\varepsilon \psi(x)dx \to \int_{R^3} u_1 u_3 \psi(x)dx, \tag{3.6}$$

$$\int_{R^3} u_2^\varepsilon u_3^\varepsilon \psi(x)dx \to \int_{R^3} u_2 u_3 \psi(x)dx \tag{3.7}$$

for all $\psi(x) \in C_0^\infty(R^3)$. In particular, for any $h(r,z) \in C_0^\infty(H)$, letting $\psi(x) = \frac{x_1}{r}h$ in (3.6) and $\psi(x) = \frac{x_2}{r}h$ in (3.7) respectively, applying the coordinate transformations (3.5), we have, as $\varepsilon \to 0^+$,

$$\int_H \frac{x_1^2}{r^2} u_r^\varepsilon u_z^\varepsilon h(r,z) r dr dz \to \int_H \frac{x_1^2}{r^2} u_r u_z h(r,z) r dr dz, \tag{3.8}$$

and

$$\int_H \frac{x_2^2}{r^2} u_r^\varepsilon u_z^\varepsilon h(r,z) r dr dz \to \int_H \frac{x_2^2}{r^2} u_r u_z h(r,z) r dr dz, \tag{3.9}$$

for all $h(r,z) \in C_0^\infty(H)$. Adding (3.8) to (3.9) yields

$$\int_H u_r^\varepsilon u_z^\varepsilon h(r,z) r dr dz \to \int_H u_r u_z h(r,z) r dr dz \tag{3.10}$$

for all $h(r,z) \in C_0^\infty(H)$.

Let $\chi = \chi(s), \chi^+(s) = \chi(s)|_{s\geq 0}$ be same as in (2.6) and (2.7), respectively. Then for any $\bar{\varphi}(r,z) \in C_0^\infty(\bar{H})$ satisfying $\max_{0\leq r\leq\delta, z\in R} |\bar{\varphi}(r,z)| \to 0$ as $\delta \to 0$, one has

$$\begin{aligned}
& |\textstyle\int_H u_r^\varepsilon u_z^\varepsilon \bar{\varphi}(r,z) dr dz - \int_H u_r u_z \bar{\varphi}(r,z) r dr dz| \\
\leq\ & |\textstyle\int_H u_r^\varepsilon u_z^\varepsilon \chi_+(\frac{r}{\delta})\bar{\varphi}(r,z) r dr dz - \int_H u_r u_z \chi_+(\frac{r}{\delta})\bar{\varphi}(r,z) r dr dz| \\
& +|\textstyle\int_H u_r^\varepsilon u_z^\varepsilon (1-\chi_+(\frac{r}{\delta}))\bar{\varphi}(r,z) r dr dz - \int_H u_r u_z (1-\chi_+(\frac{r}{\delta})\bar{\varphi}(r,z) r dr dz| \\
\equiv\ & I_1 + I_2.
\end{aligned}$$

It is direct to get that for any $\epsilon_0 > 0$, there exists a $\delta = \delta(\epsilon_0) > 0$ small enough such that

$$|\int_H u_r^\varepsilon u_z^\varepsilon \chi_+(\frac{r}{\delta})\bar{\varphi}(r,z) r dr dz| \leq C \max_{0\leq r\leq 2\delta, z\in R} |\bar{\varphi}(r,z)| \leq \frac{\epsilon_0}{2},$$

and

$$|\int_H u_r u_z \chi_+(\frac{r}{\delta})\bar{\varphi}(r,z) r dr dz| \leq C \max_{0\leq r\leq 2\delta, z\in R} |\bar{\varphi}(r,z)| \leq \frac{\epsilon_0}{2},$$

where C is a constant independent of ε. Consequently, we get that $I_1 \leq \varepsilon_0$ as $\delta = \delta(\epsilon_0)$ small enough. Fixing this $\delta = \delta(\epsilon_0)$, in view of (3.10), we have that $I_2 \to 0$ as $\varepsilon \to 0^+$. So

$$\limsup_{\epsilon\to 0^+} |\int_H u_r^\varepsilon u_z^\varepsilon \bar{\varphi}(r,z) r dr dz - \int_H u_r u_z \bar{\varphi}(r,z) r dr dz| \leq \epsilon_0.$$

By the arbitrariness of ϵ_0, we get the desired result. The proof of the Lemma is complete.

By (3.1), we have $|\mu_1| + |\mu_2| = |\mu_3|$. It follows directly from Theorem 2.1 that

Theorem 3.1 There exists a subsequence of the approximate solutions $\{u^\varepsilon\}$, still denoted by itself, satisfying (2.1), (2.2) and (2.3). Moreover, we have

$$\int_{R^3} u_3^2 dx - \frac{1}{2}\int_{R^3}(u_1^2 + u_2^2)dx + \frac{1}{2}|\mu_3| = 0. \tag{3.11}$$

Consequently, the subsequence of the approximate solutions $\{u^\varepsilon\}$ (still denoted by itself) of the equations (1.2)-(1.3) has a strong convergence in $L^2_{loc}(R^3)$ if and only if

$$\int_{R^3} u_3^2 dx - \frac{1}{2}\int_{R^3}(u_1^2 + u_2^2)dx = 0. \tag{3.12}$$

4 Single-point Concentration in H

Theorem 3.1 gives a criterion on the strong convergence of the approximate solutions for the steady Euler equations (1.2)-(1.3) under the assumptions (A1) and (A2). But the possibility of energy concentration is still not excluded for the approximate solutions. Now we consider a special case. We assume that the approximate solutions have only one concentration point occurring in $(r, z)-$plane in the limit process. Then we will prove that the concentration point will appear neither near the symmetry axis (including the symmetry axis itself) nor in the region far away from the symmetry axis. More precisely, we have

Theorem 4.1 Suppose that the assumptions (A1) and (A2) hold. Suppose further that the approximate solutions have only one single-point concentration occurring in $(r, z)-$plane in the limit process. Then, there exists a subsequence of the approximate solutions $\{u^\varepsilon\}$, denoted still by itself, satisfying (2.1), (2.2) and (2.3). Moreover, we have

(i) there exist some $r^* > 0$ small and some $R^* > 0$ large such that a subsequence of the approximate solutions, still denote by itself, converges strongly in $L^2(Q_1)$ and in $L^2(Q_2)$, where $Q_1 = \{x \in R^3 | x_1^2 + x_2^2 < (\frac{r^*}{2})^2\}$ is the domain including the symmetry axis and $Q_2 = \{x \in R^3 | x_1^2 + x_2^2 + x_3^2 > (2R^*)^2\}$ is the domain far away from the symmetry axis;

(ii)

$$\int_{R^3} u_3^2 dx - \frac{1}{2}\int_{R^3}(u_1^2 + u_2^2)dx + \frac{a}{2} = 0.$$

Before we prove the theorem, we recall a result stated in [11], which shows that if a sequence of approximate solutions for the 3-D axisymmetric unsteady Euler equations converges strongly outside the axis, then there will be no energy concentrations on the symmetry axis in the process of the limit.

Lemma 4.2 For the approximate solutions $\{u^\varepsilon(x,t)\}$ of the 3-D axisymmetric unsteady Euler equations, if there exists a subsequence $\{u^{\varepsilon_j}\} \subset \{u^\varepsilon\}$ such that for any $Q \subset\subset R^3\backslash\{x \in R^3|\ r = 0\}$, an open set compactly contained in $R^3\backslash\{x \in R^3|\ r = 0\}$,

$$u^{\varepsilon_j} \longrightarrow u \quad \text{strongly in} \quad L^2([0,T]; L^2(Q)),$$

then there exists a further subsequence of $\{u^{\varepsilon_j}\}$, denoted still by itself, such that, as $\varepsilon_j \to 0$,

$$u^{\varepsilon_j} \longrightarrow u \quad \text{strongly in} \quad L^2([0,T]; L^2_{loc}(R^3)).$$

However, the proof of Lemma 4.2 relies heavily on the following estimate

$$\int_0^T \int_{R^3} \frac{1}{1+x_3^2}(\frac{u_r^\varepsilon}{r})^2 dxdt \leq C(\|u_0^\varepsilon\|_{L^2}^2 + \|\frac{\omega_0^\varepsilon}{r}\|_{L^1}),$$

which is due to Chae and Imanuvilov ([1]). And such an estimate is not available for the steady Euler equations. Thus, to exclude the one-single concentration, we use the following special test functions

$$\begin{aligned}
\Phi_1(x) &= -\tfrac{1}{2}x_1\chi(\tfrac{r}{\delta})[\chi(\tfrac{x_3-x_3^0}{\delta}) + \tfrac{x_3-x_3^0}{\delta}\chi'(\tfrac{x_3-x_3^0}{\delta})],\\
\Phi_2(x) &= -\tfrac{1}{2}x_2\chi(\tfrac{r}{\delta})[\chi(\tfrac{x_3-x_3^0}{\delta}) + \tfrac{x_3-x_3^0}{\delta}\chi'(\tfrac{x_3-x_3^0}{\delta})],\\
\Phi_3(x) &= [\chi(\tfrac{r}{\delta}) + \tfrac{r}{2\delta}\chi'(\tfrac{r}{\delta})](x_3 - x_3^0)\chi(\tfrac{x_3-x_3^0}{\delta}),
\end{aligned}$$

where $(0,0,x_3^0)$ is the possible concentration point on the symmetry axis. After using a slightly modified approach given in [11], we can obtain the following result for the approximate solutions of the steady Euler equations with only one-single concentration point in the limit process.

Lemma 4.3 Suppose that the assumptions of Theorem 4.1 hold. Suppose further that for any $r_0 > 0$ and any $A \subset\subset H_{r_0} = \{(r,z) \in H | 0 < r < r_0\}$,

$$u^\varepsilon \longrightarrow u$$

strongly in $L^2(A; rdrdz)$ as $\varepsilon \to 0^+$. Then there exists a subsequence of $\{u^\varepsilon\}$, denoted still by itself, such that

$$u^\varepsilon \longrightarrow u$$

strongly in $L^2_{loc}(B;dx)$ as $\varepsilon \to 0^+$, where $B = \{x \in R^3 | 0 \le r < r_0\}$ includes the symmetry axis.

Proof of Theorem 4.1. The second part (ii) of the theorem is a direct consequence of Theorem 2.1. We need only to prove the first part (i) of the theorem. Using Lemma 4.3, we suppose that, without loss of generality, there exists a point (r_0, z_0) in H and a nonnegative number $a > 0$ satisfying

$$\int_H (u_z^\varepsilon)^2 \varphi r dr dz \to \int_H u_z^2 \varphi r dr dz + a\varphi(r_0, z_0) \tag{4.1}$$

for all $\varphi(r,z) \in C_0^\infty(H)$.

Now, it suffices to prove that (i)$'$: there exist some $r^* > 0$ small and some $R^* > 0$ large such that the concentration point (r_0, z_0) appears neither in the the region $Q_1' = \{(r,z) \in H | 0 < r < r^*/2\}$ nor in the region $Q_2 = \{x \in R^3 | x_1^2 + x_2^2 + x_3^2 > (2R^*)^2\}$.

By (1.5), one has

$$\begin{aligned}\int_H (u_z^\varepsilon)^2 \frac{\varphi_r}{r} r dr dz &= \int_H ((u_r^\varepsilon)^2 - (u_z^\varepsilon)^2)\partial_r \varphi_r r dr dz \\ &+ \textstyle\int_H u_r^\varepsilon u_z^\varepsilon (\partial_z \varphi_r + \partial_r \varphi_z) r dr dz + h(\varepsilon)\end{aligned} \tag{4.2}$$

for $\varphi_r(r,z), \varphi_z(r,z) \in C_0^\infty(\bar{H})$ satisfying (1.4).

Let $\chi(s)$ be same as in (2.6) with

$$|\chi'(s)| \le K_1, \quad |\chi''(s)| \le K_2,$$

where K_1, K_2 are absolute positive constants.

Let

$$\psi(r,z) = (z - z_0)\chi(\frac{r - r_0}{\eta})\chi(\frac{z - z_0}{\eta}), \ (r,z) \in H,$$

where $0 < |\eta| < r_0/2$. Then $\psi(r,z) \in C_0^\infty(H)$. We choose the test functions in (4.2) as

$$\varphi_r = \frac{1}{r}\partial_z \psi, \quad \varphi_z = -\frac{1}{r}\partial_r \psi.$$

Clearly, $\varphi_r, \varphi_z \in C_0^\infty(H)$ satisfy (1.4).

Applying Lemma 3.1, letting $\epsilon \to 0$ in (4.2), one has

$$\begin{aligned}&\int_H u_z^2 \frac{\varphi_r}{r} r dr dz + \frac{a\phi_r(r_0, z_0)}{r_0} \\ &= \int_H (u_r^2 - u_z^2)\partial_r \varphi_r r dr dz + \int_H u_r u_z (\partial_z \varphi_r + \partial_r \varphi_z) r dr dz.\end{aligned} \tag{4.3}$$

Then the equation (4.3) becomes

$$\begin{aligned}&\textstyle\int_H u_z^2 \frac{1}{r^2} \partial_z \psi r dr dz + \frac{a}{r_0^2}\partial_z \psi(r_0, z_0) \\ = &\textstyle\int_H (u_r^2 - u_z^2)\partial_r(\frac{1}{r}\partial_z \psi) r dr dz \\ &+ \textstyle\int_H u_r u_z (\frac{1}{r}\partial_z^2 \psi - \partial_r(\frac{1}{r}\partial_r \psi)) r dr dz.\end{aligned} \tag{4.4}$$

Direct calculations yield

$$
\begin{aligned}
&\partial_z\psi = \chi(\tfrac{r-r_0}{\eta})\chi(\tfrac{z-z_0}{\eta}) + \tfrac{z-z_0}{\eta}\chi(\tfrac{r-r_0}{\eta})\chi'(\tfrac{z-z_0}{\eta}),\\
&\partial_z^2\psi = \tfrac{2}{\eta}\chi(\tfrac{r-r_0}{\eta})\chi'(\tfrac{z-z_0}{\eta}) + \tfrac{z-z_0}{\eta^2}\chi(\tfrac{r-r_0}{\eta})\chi''(\tfrac{z-z_0}{\eta}),\\
&\partial_r\psi = \tfrac{z-z_0}{\eta}\chi'(\tfrac{r-r_0}{\eta})\chi(\tfrac{z-z_0}{\eta}),\\
&\partial_r^2\psi = \tfrac{z-z_0}{\eta^2}\chi''(\tfrac{r-r_0}{\eta})\chi(\tfrac{z-z_0}{\eta}),\\
&\partial_r(\tfrac{1}{r}\partial_r\psi) = -\tfrac{z-z_0}{r^2\eta}\chi'(\tfrac{r-r_0}{\eta})\chi(\tfrac{z-z_0}{\eta}) + \tfrac{z-z_0}{r\eta^2}\chi''(\tfrac{r-r_0}{\eta})\chi(\tfrac{z-z_0}{\eta}),\\
&\partial_r(\tfrac{1}{r}\partial_z\psi) = -\tfrac{1}{r^2}\partial_z\psi + \tfrac{1}{r\eta}\chi'(\tfrac{r-r_0}{\eta})\chi(\tfrac{z-z_0}{\eta}) + \tfrac{z-z_0}{r\eta^2}\chi'(\tfrac{r-r_0}{\eta})\chi'(\tfrac{z-z_0}{\eta}).
\end{aligned}
\tag{4.5}
$$

So we have

$$
\begin{aligned}
&|\partial_z\psi| \le 1 + K_1, \quad |\partial_r\psi| \le K_1,\\
&|\partial_r^2\psi| \le \tfrac{K_2}{\eta}, \quad |\partial_z^2\psi| \le \tfrac{2K_1}{\eta} + \tfrac{K_2}{\eta},\\
&|\partial_r(\tfrac{1}{r}\partial_r\psi)| \le \tfrac{K_1}{r^2} + \tfrac{K_2}{r\eta},\\
&|\partial_r(\tfrac{1}{r}\partial_z\psi)| \le \tfrac{1+K_1}{r^2} + \tfrac{K_1}{r\eta} + \tfrac{K_1^2}{r\eta}.
\end{aligned}
$$

Noting that the function $\psi = \psi(r, z)$ has support

$$
D = D_\eta(r_0, z_0) = \{(r, z) \in H||r - r_0| \le 2\eta, |z - z_0| \le 2\eta\},
$$

and that $\partial_z\psi(r_0, z_0) = 1$, we obtain from (4.4) and (4.5) that

$$
\begin{aligned}
\tfrac{a}{r_0^2} \;&\le (1 + K_1)\int_D \tfrac{1}{r^2}u_r^2 rdrddz + \tfrac{K_1+K_1^2}{\eta}\int_D \tfrac{1}{r}|u_r^2 - u_z^2|rdrdz\\
&+\tfrac{2K_1}{\eta}\int_D \tfrac{1}{r}|u_ru_z|rdrdz + \tfrac{K_2}{\eta}\int_D \tfrac{1}{r}|u_ru_z|rdrdz\\
&+K_1\int_D \tfrac{1}{r^2}|u_ru_z|rdrdz + \tfrac{K_2}{\eta}\int_D \tfrac{1}{r}|u_ru_z|rdrdz.
\end{aligned}
\tag{4.6}
$$

Furthermore, if we set $\eta = r_0/4$, then we get from (4.6) that

$$
\begin{aligned}
\tfrac{a}{r_0^2} \;&\le \frac{C(K_1, K_2)}{r_0^2}[\int_D u_r^2 rdrddz + \int_D |u_r^2 - u_z^2|rdrdz\\
&+\int_D |u_ru_z|rdrdz + \int_D |u_ru_z|rdrdz\\
&+\int_D |u_ru_z|rdrdz + \int_D |u_ru_z|rdrdz,]
\end{aligned}
\tag{4.7}
$$

where $C(K_1, K_2)$ is an absolute constant depending only on K_1 and K_2.

Set $\delta_1 = a/12C(K_1, K_2)$. Since $u = u(x)$ is uniformly bounded in $L^2(R^3)$, there exists a large number $R^* > 0$ satisfying

$$
\int_{\{(r,z)\in H|r^2+z^2>(R^*)^2\}} (u_r^2 + u_z^2)rdrdz \le \delta_1 = \frac{a}{12C(K_1, K_2)}. \tag{4.8}
$$

We now claim that the concentration point (r_0, z_0) does not appear in the region $Q_2 = \{(r, z) \in H | r^2 + z^2 > (2R^*)^2\}$. Otherwise, if $(r_0, z_0) \in Q_2$, setting $\eta = r_0/4$, we have $D = D_\eta(r_0, z_0) \subset \{(r, z) \in H | r^2 + z^2 > (R^*)^2\}$. It concludes from (4.7) and (4.8) that

$$\frac{a}{r_0^2} \leq \frac{6\delta_1}{r_0^2} C(K_1, K_2) = \frac{a}{2r_0^2}, \tag{4.9}$$

a contradiction.

Using a similar approach, there exists a small number $r^* > 0$ such that

$$\int_{\{(r,z) \in H | 0 < r < r^*, |z| < 2R^* + 1\}} (u_r^2 + u_z^2) r dr dz \leq \delta_1 = \frac{a}{12C(K_1, K_2)},$$

where R^* is same as above. Let $\eta = r_0/4$ in (4.6). If $(r_0, z_0) \in G = \{(r, z) \in H | 0 < r < r^*/2, |z| < 2R^* + 1\}$, a similar arguments as above will give a contradicting inequality (4.9). Thus $(r_0, z_0) \notin G$. Combining the known fact that $(r_0, z_0) \notin \{(r, z) \in H | r^2 + z^2 > (2R^*)^2\}$, we obtain that the concentration point (r_0, z_0) does not appear in the region $Q_1' = \{(r, z) | 0 < r < r^*/2\}$. The desired result (i)$'$ is then proved and the proof of the theorem is finished.

Remark. It is clear that the results of Theorem 4.1 hold true for the case of finite-points concentrations occurring in the limit process.

5 Examples of Strong Convergence, Concluding Remarks

In this section, we give some examples of the approximate solutions which converge strongly in $L^2_{loc}(R^3)$.

The first example lies in the approach to seek nontrivial axisymmetric solutions without swirls of the equations (1.1) in [18], [16] and references therein, which is called the "pseudo-advection method". Using the identity $(u \cdot \nabla)u = \omega \times u + \nabla(|u|^2/2)$ with $\omega = \nabla \times u$, the steady Euler equations (1.1) can be rewritten as

$$\omega \times u = -\nabla(P + |u|^2/2), \quad div\ u = 0.$$

In order to obtain the solutions, Vallis et al [18] proposed the following unsteady Euler equations with an artificial term:

$$v_t + \omega \times u + \alpha\omega \times v_t = -\nabla(P + |u|^2/2), \quad div\ u = 0, \tag{5.1}$$

on which an initial data are imposed, where α is a constant. However, because of the nonlinearity of the term $\alpha\omega \times v_t$ is very strong, it is difficult to prove the global existence of 5.1). In [16], the author proved strictly that a sequence of the solutions to the Galerkin approximations of (5.1), which is actually the approximate solutions of the equations (1.1), converge strongly to a generalized solutions (1.1) by letting the number of basis functions in the Galerkin method go to infinity at the same time as $t \to \infty$. The obtained nontrivial solution holds finite L^2 norm of ω_θ/r with $\omega_\theta = \partial_z u_r - \partial_r u_z$.

Some other examples come from vortex rings. The vortex rings are steady, axisymmetric solutions of (1.2)-(1.3), propagating with constant speed $W > 0$ in the negative $z-$ direction (like smoking circle). They have been extensively and systematically studied by the variational method.

We recall briefly the approach here. For smooth solutions, the Euler equations (1.2) is equivalent to

$$(u \cdot \nabla)\xi = 0 \tag{5.2}$$

with $\xi = \omega_\theta(r, z)/r$. Introducing a stream function Ψ satisfying

$$u_r = -\frac{\Psi_z}{r}, \quad u_z = \frac{\Psi_r}{r}, \tag{5.3}$$

we obtain from (5.2)-(5.3) that

$$\frac{\partial(\Psi, \xi)}{\partial(r, z)} = 0.$$

This is equivalent to the existence of a functional dependence Φ with $\nabla\Phi \neq 0$ such that

$$\Phi(\Psi, \xi) = 0.$$

In particular, it is only needed to seek solutions with

$$\xi = \lambda f(\Psi) \Leftrightarrow \omega_\theta = \lambda r f(\Psi),$$

where λ is a vortex-strength parameter and f is some given function. Noting that

$$\omega_\theta = \partial_z u_r - \partial_r u_z,$$

one has

$$L\Psi \equiv r(\frac{1}{r}\Psi_r)_r + \Psi_{zz} = -\lambda r^2 f(\Psi).$$

Decompose the stream function Ψ by writing

$$\Psi(r, z) = \psi(r, z) - \frac{1}{2}Wr^2 - k$$

where $k \geq 0$ is the flux constant and ψ is the vortex stream function satisfying

$$L\psi \equiv r(\frac{1}{r}\psi_r)_r + \psi_{zz} = -\lambda r^2 f(\Psi). \tag{5.4}$$

The existence of vortex ring reduces to studying the semi-linear elliptic equation (5.4).

Now, for a vortex ring (u_r, u_z) which is a classical solutions of (1.1), setting $\tilde{u}_r = u_r, \tilde{u}_z = u_z + W$, we get

$$\begin{cases} \tilde{u}_r\partial_r\tilde{u}_r + \tilde{u}_z\partial_z\tilde{u}_r + \partial_r(p - W\tilde{u}_z) = \lambda rWf(\Psi), \\ \tilde{u}_r\partial_r\tilde{u}_z + \tilde{u}_z\partial_z\tilde{u}_z + \partial_z(p - W\tilde{u}_z) = 0, \\ \partial_r(r\tilde{u}_r) + \partial_z(r\tilde{u}_z) = 0. \end{cases} \tag{5.5}$$

We denote by $(\tilde{u}_r^\lambda, \tilde{u}_z^\lambda)$ the solutions of (5.5) depending on λ. Then the L^2-norm (the energy) of $(\tilde{u}_r^\lambda, \tilde{u}_z^\lambda)$ is finite. When λ tends to 0, if the right term of the first equation of (5.5) tends to 0 in weak sense, then it is easy to prove that the solutions $(\tilde{u}_r^\lambda, \tilde{u}_z^\lambda)$ are the approximate solutions of (1.2)-(1.3) converging strongly in $L^2_{loc}(R^3)$. In fact, in this case the vorticity ω_θ^λ will vanish in $L^\infty(H; drdz)$ as $\lambda \to 0$. And the strong convergence of $\tilde{u}^\lambda = (\tilde{u}_r^\lambda, \tilde{u}_z^\lambda)$ in $L^2_{loc}(R^3)$ can be derived from the Sobolev compactly imbedding theory. We note that Fraenkel and Berger's global theory [8] of the vortex rings provides us such solutions under some assumptions on the smoothness of f(f is differential continuous, for instance). A trivial case is that in [8] if we fix $W > 0$ and let $\eta = \int_H \frac{1}{r^2}(\psi_r^2 + \psi_z^2) rdrddz$ tend to 0, then we obtain that $\tilde{u}^\lambda$ are the approximate solutions of (1.2)-(1.3) and converge strongly in $L^2_{loc}(R^3)$ to 0.

If the vortex rings (u_r, u_z) are weak solutions of (1.2)-(1.3), that is, for any $\varphi_r(r,z), \varphi_z(r,z) \in C_0^\infty(\bar{H})$, satisfying

$$\partial_r(r\varphi_r) + \partial_z(r\varphi_z) = 0,$$

one has

$$\begin{aligned} &\int_H [(u_r)^2\partial_r\varphi_r + (u_z)^2\partial_z\varphi_z] rdrdz \\ &= -\int_H u_r u_z(\partial_r\varphi_z + \partial_z\varphi_r) rdrdz \end{aligned} \tag{5.6}$$

Then, setting $\tilde{u}_r = u_r, \tilde{u}_z = u_z + W$, one easily derives

$$\begin{aligned} &\int_H [(\tilde{u}_r)^2\partial_r\varphi_r + (\tilde{u}_z)^2\partial_z\varphi_z] rdrdz + \int_H u_r u_z(\partial_r\varphi_z + \partial_z\varphi_r) rdrdz \\ &= W\int_H [\tilde{u}_r\partial_z\varphi_r + \tilde{u}_r\partial_r\varphi_z + 2\tilde{u}_z\partial_z\varphi_z] rdrdz - W^2\int_H \partial_z\varphi_z rdrdz \end{aligned} \tag{5.7}$$

As an example, let's look at Hill's spherical vortex in which the vorticity is confined to the interior of a uniformly translating sphere of radius a (see [10],[8]). The stream function in Hill's solution is

$$\Psi = \begin{cases} \frac{1}{10}\lambda R^2 \sin^2\Theta(a^2 - R^2), R \le a, \\ -\frac{1}{2}WR^2\sin^2\Theta(1 - \frac{a^3}{R^3}), R \ge a, \end{cases}$$

where R, Θ are the spherical coordinates such that $r = R\sin\Theta$ and $z = R\cos\Theta$, and $\lambda a^2/W = 15/2$ (to make $\partial\Psi/\partial R$ continuos on $R = a$). To construct the energy-finite solutions, we define

$$\psi(r,z) = \Psi_H(r,z) + \frac{1}{2}Wr^2 = \begin{cases} \frac{1}{2}Wr^2(\frac{5}{2} - \frac{3}{2}\frac{R^2}{a^2}), R \le a, \\ \frac{1}{2}Wr^2\frac{a^3}{R^3}, R \ge a, \end{cases}$$

which corresponds to the stream function for the flow being at rest at infinity. And the corresponding velocity fields are

$$\tilde{u}_r = -\frac{1}{r}\partial_z\psi = \begin{cases} \frac{3}{2}Wr\frac{z}{a^2}, R \le a, \\ \frac{3}{2}Wr\frac{a^3}{R^5}z, R \ge a, \end{cases}$$

$$\tilde{u}_z = -\frac{1}{r}\partial_r\psi = \begin{cases} W(\frac{5}{2} - \frac{3R^2}{2a^2}) - \frac{3Wr^2}{2a^2}, R \le a, \\ W\frac{a^3}{R^3} - \frac{3}{2}Wr^2\frac{a^3}{R^5}, R \ge a, \end{cases}$$

Direct calculations yield

$$\int_H (\tilde{u}_r^2 + \tilde{u}_z^2) r dr dz = \int_H \frac{1}{r^2}(\partial_r\psi^2 + \partial_z\psi^2) r dr dz = \frac{10}{7}W^2a^3.$$

Letting $W^2a^3 = 1$ to conserve the energy, denoting the solution by $(\tilde{u}_r^\lambda, \tilde{u}_z^\lambda)$ depending on λ, in view of the restriction $\lambda a^2 = \frac{15}{2}W$, we obtain that $(\tilde{u}_r^\lambda, \tilde{u}_z^\lambda)$ converge strongly to 0 in $L^2_{loc}(R^3)$. We note that when $\lambda \to 0$, the radius a will tend to $+\infty$ and the vorticity will tend to 0, and the energy is equal to 10/7 in the limit process. This is a vanishing phenomenon. Anyway, Hill's vortex ring provides us the approximate solutions which converge strongly in $L^2_{loc}(R^3)$.

It should be noted that the interesting case lies in that when $\lambda \to \infty$. In this case, the vorticity will become larger and lager in the vortex ring and the vortex ring will possibly shrink to a point in $(r,z)-$plane, and the concentrations will possibly appear, like Hill's vortex ring and the vortex ring constructed by Friedman-Turkington in [9], for instance. However, we have not yet succeeded in constructing an example of the approximate solutions of (1.1) or (1.2)-(1.3) with the concentrations in the limit process. To make it more clear and as an example, let's look at in a little more detail the results obtained in [9], which is

Proposition 5.1 For any $\lambda > 0$, there exists a velocity $v_\lambda(r,z)$ of which corresponding vorticity is $\omega_\lambda(r,z) = r\xi_\lambda e_\theta$ with $\xi_\lambda = \lambda I_{B_\lambda}$, where B_λ is a connected, bounded domain in $r - z$ plane H and I_{B_λ} is the characteristic function on B_λ. Moreover, one has

i) The volume and the diameter of B_λ satisfy

$$|B_\lambda| \le \frac{1}{2\pi\lambda}, \quad \frac{c}{\lambda^{\frac{1}{2}}} \le d(B_\lambda) \le \frac{C}{\lambda^{\frac{1}{2}}},$$

respectively;

ii) B_λ is connected and asymptotically tends to the point $(\sqrt{2}, 0)$ as $\lambda \to \infty$;

iii)

$$\|\xi_\lambda\|_{L^1(rdrdz)} \le C, \quad c\log\lambda \le \|u_\lambda\|_{L^2(rdrdz)} \le C\log\lambda;$$

iv)

$$|v_\lambda| \le \frac{1}{4\pi}\int_{R^3} \frac{r'\xi_\lambda(x')dx'}{|x-x'|^2}.$$

Here c, C are absolute constants independent of λ.

We omit the proof of Proposition 5.1 here. The existence of v_λ was proved in Theorem 2.1 in [9], and the properties of i)-iii) are from Theorem 7.6 and Theorem 8.1 in [9], and the estimate iv) is a direct consequence of Lemma 1.1 of [9] (see also the estimate (4.17) in [9]). So, clearly, after an appropriate scaling $u_\lambda = (\log\lambda)^{-\frac{1}{2}} v_\lambda$, we will find that the concentration do happen as $\lambda \to \infty$ for the energy of the velocity u_λ. Unfortunately we will show in the following that v_λ and therefore u_λ can not be the approximate solutions of (1.2)-(1.3).

Theorem 5.2 The velocity v_λ presented in Proposition 5.1 is not a weak solution and also not the approximate solution of (1.2)-(1.3).

Proof of Theorem 5.2. Let $\varepsilon = 1/\lambda$. By scaling, we define

$$u^\varepsilon = |\log\varepsilon|^{-\frac{1}{2}} v_{\frac{1}{\varepsilon}}, \tag{5.8}$$

whose vorticity is

$$\omega^\varepsilon = |\log\varepsilon|^{-\frac{1}{2}} r\xi_{\frac{1}{\varepsilon}}(r, z). \tag{5.9}$$

Then by Proposition 5.1, one has that

$$0 < c \le \|u^\varepsilon\|_{L^2(rdrdz)} \le C, \tag{5.10}$$

and

$$\|\frac{\omega^\varepsilon}{r}\|_{L^1(rdrdz)} = O(|\log\varepsilon|)^{-\frac{1}{2}}, \quad \varepsilon \to 0.$$

If the result of Theorem 5.2 is not true, then the functions $\{u^\varepsilon\}$ given in (5.8) are the approximate solutions of the 3-D steady axisymmetric Euler equations with the vorticity ω^ε given by (5.8). Moreover, in view of iv) of Proposition 5.1, we have

$$|u^\varepsilon(x)| \le C|\log\epsilon|^{-\frac{1}{2}} \int_{R^3} \frac{r'\xi_{\frac{1}{\varepsilon}}(x')dx'}{|x-x'|^2},$$

where C is a constant independent of ε. So for any $x_0 = (r_0, \theta_0, z_0) \in R^3$ with $(r_0, z_0) \neq (\sqrt{2}, 0)$, due to i)-iii) of the Proposition 5.1, we obtain

$$|u^\varepsilon(x_0)| \leq C|\log\varepsilon|^{-\frac{1}{2}} \int_{R^3} \frac{r'\xi_{\frac{1}{\varepsilon}}(x')}{(\sqrt{2}-r_0)^2} dx' \leq \frac{C}{(\sqrt{2}-r_0)^2} |\log\varepsilon|^{-\frac{1}{2}}$$

for ε small enough. Thus $u^\varepsilon \to 0$ *a.e.* in R^3 as $\varepsilon \to 0$. It follows from (5.10) that there exists a subsequence of $\{u^\varepsilon\}$, denoted still by itself, such that,

$$u^\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad L^2(R^3) \ \text{ as } \ \varepsilon \to 0.$$

Using the criterion (3.12) established in Theorem 3.1 (noting that the vorticity is of one-sign in this case), we obtain that

$$u^\varepsilon \to 0 \quad \text{strongly in} \quad L^2_{loc}(R^3). \tag{5.11}$$

Moreover, when $|x|$ is large enough, say, $|x| > \sqrt{2}+1$, a direct estimate yields

$$|u^\varepsilon(x)| \leq C|\log\varepsilon|^{-\frac{1}{2}} \cdot \frac{1}{|x|^2 - 2} \cdot \int_{R^3} r'|\xi_{\frac{1}{\varepsilon}}(x')| dx',$$

which implies that

$$u^\varepsilon \to 0 \quad \text{in} \quad L^2(\{|x| > \sqrt{2}+1\}). \tag{5.12}$$

Combining (5.11) with (5.12), we get that

$$u^\varepsilon \to 0 \quad \text{in} \quad L^2(R^3).$$

This contradicts to the estimates (5.10). The proof of Theorem 5.2 is complete.

Finally, we would like to remark that further studies on the approximate solutions of (1.2)-(1.3)are expected. Especially, the following problems remain open: Do there exist approximate solutions with concentrations for the 3-D steady axisymmetric Euler equations without swirls? Or, do all the approximate solutions with L^1-bounded vorticity converge strongly in $L^2_{loc}(R^3)$, at least for one-sign case?

References

[1] D. Chae, O.Y. Imanuvilov, Existence of axisymmetric weak solutions of the 3-D Euler equations for near-vortex-sheets initial data, Elect. J. Diff. Eq. 26, 1-17, 1998.

[2] J. M. Delort, Existence de nappes de tourbillon en dimension deux, J. Amer. Math. Sco., 4, 553-586, 1991.

[3] J. M. Delort, Une remarque sur le probleme des nappes de tourbillon axisymetriques sur R^3, J. Funct. Anal., 108, 274-295, 1992.

[4] R. J. DiPerna and A. Majda, Reduced Hausdorff dimension and concentration-cancellation for 2-D incompressible flow, J. of Amer. Math. Soc., 1, 59-95, 1988.

[5] R. J. DiPerna and A. Majda, Concentrations in regularizations for 2-D incompressible flow, Comm. Pure Appl. Math., 40, 301-345, 1987.

[6] L. C. Evans, Weak Covergence Methods for Nonlinear Partial Differential Equations, CBMS Regional Conf. Ser. in Math. no. 74, Amer. Math. Soc., Providence, RI, 1990.

[7] L. C. Evans and S. Müller, Hardy space and the two-dimensional Euler equations with non-negative vorticity, J. Amer. Math. Soc. 7, 199-219, 1994.

[8] L. E. Fraenkel, M. S. Burger, A Global theory of steady vortex rings in an ideal fluid, Acta Math. 132, 14-51, 1974.

[9] A. Friedman, B. Turkington, Vortex rings: Existence and Asymptotic Estimates, Transactions of the American Math. Soc., 268(1), 1-37, 1981.

[10] M. J. M. Hill, On a spherical vortex, Philos. Trans. Roy. Soc. London Ser. A 185, 213-245, 1894.

[11] Q. S. Jiu, Z. P. Xin, On Strong convergence to 3D axisymmetric vortex sheets, accepted by JDE.

[12] Q. S. Jiu, Z. P. Xin, Viscous approximation and decay rate of maximal vorticity function for 3-D axisymmetric Euler equations, Acta Math. Sinica 20(3), 385-404, 2004.

[13] J. G. Liu and Z. P. Xin, Convergence of vortex methods for weak solutions to the 2-D Euler equations with vortex sheet data, Comm. Pure Appl. Math., 48, 611-628, 1995.

[14] A. Majda, Remarks on weak solutions for vortex sheets with a distinguished sigh, Indiana Univ. Math. J., 42, 921-939, 1993.

[15] W. M. Ni, On the exitence of global vortex rings, J. Anal. Math., 17, 208-247, 1980.

[16] T. Nishiyama, Pseudo-advection method for the axisymmetric stationary Euler equations, Z. Angew. Math. Mech.,81(10), 711-715, 2001.

[17] S. Schochet, The weak vorticity formulation of the 2D Euler equations and concentration-cancellation, Comm. P. D. E., 20, 1077-1104, 1995.

[18] G.K. Vallis, G.F. Carnevale, W.R. Young, Extremal energy properties and construction of stable solutions of the Euler equations, J. Fluid Mech., 207, 133-152, 1989.

Quansen Jiu
Department of Mathematics,
Capital Normal University, Beijing 100037,PRC

e-mail address: jiuqs@mail.cnu.edu.cn

Zhouping Xin
IMS and Department of Mathematics,
The Chinese University of Hong Kong,
Shatin, N.T., Hong Kong
&
Department of Mathematics,
Capital Normal University, Beijing 100037,PRC

e-mail address: zpxin@ims.cuhk.edu.hk