ON THE CONJECTURE OF COURANT AND FRIEDRICHS FOR THE TRANSONIC SHOCK IN A NOZZLE

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Abstract

In the book [8] of Courant and Friedrichs, the following transonic phenomena in a nozzle is illustrated: Given the appropriately large receiver pressure p_r, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle a shock front intervenes and the gas is compressed and slowed down to subsonic speed. The position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes p_r. Motivated by this conjecture, we will study the well-posedness problem on the transonic shock for the steady compressible Euler flow through a two-dimensional slowly-varying nozzle when the pressure at the exit is appropriately given. The transonic shock is a free boundary dividing two regions of C^2 flow in the nozzle. The full Euler system is nonlinear hyperbolic upstream where the flow is supersonic, and coupled hyperbolic-elliptic in the downstream region Ω_+ of the nozzle when the flow is subsonic. Based on the Bernoulli's law, in Ω_+ we can reformulate the 3×3 full Euler system into a weakly coupled second order elliptic equation on the density ρ with the mixed boundary conditions, a 2×2 first order system on u_2 with a value at a point and an algebraic equation on (ρ, u_1, u_2) along the streamline. With respect to the reformulated problem, we can show that the transonic shock solution is unique if it exists and satisfies some regularity assumptions. Based on this uniqueness result, we derive that the conjecture of Courant-Friedrichs [8] on the transonic shock in a very slowly-varying nozzle cannot hold for the C^2 subsonic solution and the arbitrarily given large pressure p_r at the exit, namely, the transonic shock problem is ill-posed with respect to the general given pressure at the exit for the slowly-varying nozzle. Finally, for the large curved nozzle walls, if the diverging part of the nozzle walls are straight and the corresponding supersonic coming flow in the diverging part is symmetric, then we can give an example to illustrate that the conjecture of Courant-Friedrichs is right for the complete Euler equations. More precisely, there exist two constant pressures

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P_1 and P_2 with $P_1 < P_2$ which depend only on the incoming flow and the shape of the nozzle, such that if the end pressure $P_e \in (P_1, P_2)$, then the transonic shock exists, moreover the position and the strength of the shock is uniquely determined by P_e.

Keywords: Steady Euler equation, supersonic flow, subsonic flow, transonic shock, nozzle, entropy.

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§1. Introduction and the main results

In this paper we are concerned with the problem on the well-posedness of a transonic shock to the steady compressible Euler flow through a 2-D slowly variable nozzle with an appropriate end pressure. In [21-22], under the assumptions that the flow is isentropic and irrotational, we use the potential equation to study the existence, uniqueness and well-posedness or ill-posedness of a transonic shock to the steady flow through a general 2-D or 3-D slowly-varying nozzle. Especially, for the slowly-varying nozzle, we establish that the transonic shock problem is ill-posed for the general given pressure at the exit for the irrotational and isentropic steady flow. However, when the strength of the shock is large (for example, if the flow is highly supersonic upstream and becomes subsonic across a shock, then the strength of the shock is rather large), the flow behind the shock is not irrotational and isentropic, thus it is more plausible to use the full Euler system to study the movement of the inviscid steady flow in the nozzle.

The compressible Euler system of steady flow in two dimensional spaces is

$$\begin{cases} \partial_1(\rho u_1) + \partial_2(\rho u_2) = 0 \\ \partial_1(P + \rho u_1^2) + \partial_2(\rho u_1 u_2) = 0 \\ \partial_1(\rho u_1 u_2) + \partial_2(P + \rho u_2^2) = 0 \end{cases} \tag{1.1}$$

where $u = (u_1, u_2)$, P and ρ represent the velocity, pressure and density respectively. Moreover, $P = P(\rho)$ is a smooth function of ρ and $c^2(\rho) = P'(\rho) > 0$ for $\rho > 0$. For the polytropic gas, $P(\rho) = A\rho^\gamma$, here $A > 0$ and $1 < \gamma < 3$ are constants.

Suppose that there is a uniform supersonic flow $(u_1, u_2) = (q_0, 0)$ with constant density $\rho_0 > 0$ which comes from negative infinity, and the flow enters the $2 - D$ nozzle from the entrance. In most of the paper, we assume that the two nozzle walls are of a small perturbation of two straight line segments $x_2 = -1$ and $x_2 = 1$ with $-1 \le x_1 \le 1$.

For simplicity, we assume that the walls of the nozzle are given by

$$x_2 = f_1(x_1) \qquad \text{and} \qquad x_2 = f_2(x_1) \tag{1.2}$$

satisfying

$$|\frac{d^k}{dx_1^k}(f_1(x_1) + 1)| \le \varepsilon \quad \text{and} \quad |\frac{d^k}{dx_1^k}(f_2(x_1) - 1)| \le \varepsilon \quad \text{for} \quad -1 \le x_1 \le 1, k \le 4, \quad k \in \mathbb{N} \cup \{0\} \tag{1.3}$$

with $\varepsilon > 0$ suitably small.

Without loss of generality and for the convenience to write, we assume that

$$f_1(-1) = f_1(1) = -1, f_2(-1) = f_2(1) = 1; \quad f_i^{(k)}(-1) = 0 \quad \text{for} \quad i = 1, 2; 1 \le k \le 4. \tag{1.4}$$

When the uniform supersonic flow with the velocity $(q_0, 0)$ enters the entry of the nozzle, then the supersonic flow field (ρ^-, u_1^-, u_2^-) in the nozzle will be determined by the following quasilinear hyperbolic system with the initial-boundary value conditions

$$\begin{cases} \partial_1(\rho^- u_1^-) + \partial_2(\rho^- u_2^-) = 0 \\ \partial_1(P(\rho^-) + \rho^-(u_1^-)^2) + \partial_2(\rho^- u_1^- u_2^-) = 0 \\ \partial_1(\rho^- u_1^- u_2^-) + \partial_2(P(\rho^-) + \rho^-(u_2^-)^2) = 0, \\ \rho^-|_{x_1=-1} = \rho_0 \\ u_1^-|_{x_1=-1} = q_0, \\ u_2^-|_{x_1=-1} = 0, \\ u_2^- = f_i'(x_1) u_1^- \qquad \text{on} \qquad x_2 = f_i(x_1), \qquad i = 1, 2. \end{cases} \tag{1.5}$$

It follows from Lemma 2.1 in §2 that (1.5) has a C^3 supersonic solution $(\rho^-(x), u_1^-(x), u_2^-(x))$ in the whole nozzle $\Omega = \{(x_1, x_2) : -1 < x_1 < 1, f_1(x_1) < x_2 < f_2(x_1)\}$, moreover $|\nabla_x^\alpha(\rho^-(x) - \rho_0)| + |\nabla_x^\alpha(u_1^-(x) - q_0)| + |\nabla_x^\alpha u_2^-(x)| \le C\varepsilon$ holds for $|\alpha| \le 3$ and $x \in \bar{\Omega}$.

Let an appropriate large pressure $\tilde{P}_+(x_2) = P(\tilde{\rho}_+(x_2))$ be given at the exit of the nozzle. More concretely, $\tilde{\rho}_+(x_2) \in C^2[f_1(1), f_2(1)] \cap C^3(f_1(1), f_2(1))$ satisfies $|\frac{d^k}{dx_2^k}(\tilde{\rho}_+(x_2) - \rho_+)| \le \varepsilon$ for $0 \le k \le 2$, here the constant ρ_+ and the related constant velocity $(q_+, 0)$ satisfy the following relations

$$\rho_0 q_0 = \rho_+ q_+, \qquad \rho_0 q_0^2 + P(\rho_0) = \rho_+ q_+^2 + P(\rho_+); \qquad \rho_0 < \rho_+ \qquad \text{and} \qquad q_+ < c(\rho_+). \tag{1.6}$$

In fact, (1.6) follows from the Rankine-Hugoniot conditions and the physical entropy condition for the system (1.1) when the transonic shock is straight and $u_2 \equiv 0$.

In light of statements in the book of [8], one expects that there will appear a transonic shock $\Sigma : x_1 = \xi(x_2)$ in the nozzle. For the definiteness (as in [5-7]), we assume that the shock Σ goes through a fixed point (x_1^1, x_2^1) with $x_2^1 = f_1(x_1^1)$, i.e.,

$$\xi(x_2^1) = x_1^1. \tag{1.7}$$

As indicated in [8](pages 372), it is a question of great importance to know under what circumstances a steady flow involving shocks is uniquely determined by the boundary conditions and by the conditions at the entrance, and when further conditions at the exit are appropriate.

Across the shock Σ, we denote the flow field by $(\rho^+(x), u_1^+(x), u_2^+(x))$. Then the Rankine-Hugoniot conditions on Σ become

$$\begin{cases} [\rho u_1] - \xi'(x_2)[\rho u_2] = 0, \\ [P(\rho) + \rho u_1^2] - \xi'(x_2)[\rho u_1 u_2] = 0, \\ [\rho u_1 u_2] - \xi'(x_2)[P(\rho) + \rho u_2^2] = 0. \end{cases} \tag{1.8}$$

In addition, $\rho^+(x)$ should satisfy the physical entropy condition (see [8]):

$$\rho^+(x) > \rho^-(x) \qquad \text{on} \qquad x_1 = \xi(x_2). \tag{1.9}$$

On the exit of the nozzle, one poses the following boundary condition

$$\rho^+(x) = \tilde{\rho}_+(x_2) \qquad \text{on} \qquad x_1 = 1. \tag{1.10}$$

Finally, since the velocity of the flow is tangent to the nozzle walls $x_2 = f_i(x_1)(i = 1, 2)$, then we have

$$u_2^+ = f_i'(x_1)u_1^+ \qquad \text{on} \qquad x_2 = f_i(x_1). \tag{1.11}$$

The first main result in this paper is on the uniqueness of the solution to the equation (1.1) with the boundary conditions (1.7)-(1.11).

Theorem 1.1 (uniqueness) *Under the assumptions (1.2) - (1.4) and (1.6), for small $\varepsilon > 0$, the equation (1.1) with the boundary conditions (1.7) - (1.11) has no more than one pair of solutions $(\rho^+(x), u_1^+(x), u_2^+(x); \xi(x_2))$ with the following regularities and estimates*

(i). $\xi(x_2) \in C^3[x_2^1, x_2^2]$, here $(x_1^i,\ x_2^i)$ with $x_2^i = f_i(x_1^i)(i = 1, 2)$ stands for the intersection point of $x_1 = \xi(x_2)$ with $x_2 = f_i(x_1)$. Moreover

$$\|\xi(x_2) - x_1^1\|_{C^3[x_2^1, x_2^2]} \leq C\varepsilon.$$

(ii). Denote by $\Omega_+ = \{(x_1, x_2) : \xi(x_2) < x_1 < 1, f_1(x_1) < x_2 < f_2(x_1)\}$. Assume $f_i''(x_1^i) = 0$ and $f_i'(1) = f_i''(1) = \partial_2\tilde{\rho}_+(f_i(1)) = 0$, then $(\rho^+(x), u_1^+(x), u_2^+(x)) \in C^2(\bar{\Omega}_+) \cap C^3(\Omega_+)$ satisfies

$$\|\rho^+(x) - \rho_+\|_{C^2(\bar{\Omega}_+)} + \|u_1^+(x) - q_+\|_{C^2(\bar{\Omega}_+)} + \|u_2^+(x)\|_{C^2(\bar{\Omega}_+)} \leq C\varepsilon.$$

Remark 1.1 $f_i''(x_1^i) = 0$ means that the compatibility condition holds at the intersection point $(x_1^i, f_i(x_1^i))(i = 1, 2)$ for the Rankine-Hugoniot conditions (1.8) and the fixed boundary conditions (1.11) (see Lemma 3.3 for more details). $f_i'(1) = f_i''(1) = \partial_2\tilde{\rho}_+(f_i(1)) = 0$ imply the compatibility conditions on the corner points $P_i = (1, f_i(1))(i = 1, 2)$. It follows from the proof of Theorem 1.1 and the regularity theory of the second order elliptic equations with the cornered boundaries (one can see [2-3], [15-16], [20] and so on) that the assumptions on the regularities of solution $(\rho^+(x), u_1^+(x), u_2^+(x); \xi(x_2))$ are plausible. See §3 for more details. In addition, if the walls of the nozzle are composed by the straight lines, then it is obvious that $f_i''(x_1^i) = 0$ holds.

Remark 1.2 Our method in this paper can be used to treat the well-posedness or ill-posedness problems on the transonic shock for the supersonic flow past a two-dimensional wedge for the full Euler systems when the pressure condition is given at the downstream subsonic domain. This will be treated in a forthcoming paper [23].

Remark 1.3 If the end pressure $\tilde{\rho}_+(x_2)$ in (1.10) is given on a smooth curve $x_1 = g(x_2) \in C^3[f_1(1), f_2(1)]$ with $|\frac{d^k}{dx_2^k}(g(x_2) - 1)| \leq \varepsilon$, $0 \leq k \leq 3$ for suitably small ε, moreover, $x_1 = g(x_2)$ is perpendicular to $x_2 = f_i(x_1)$ at the point $(1, f_i(1))$ and the compatibility condition holds at $(1, f_i(1))$ for $\tilde{\rho}_+(x_2)$ and boundary conditions (1.11), then the unique results as in Theorem 1.1 still holds by a similar analysis.

Remark 1.4 The regularity assumptions on (ρ^+, u_1^+, u_2^+) in Theorem 1.1 can be replaced by

$$(\rho^+(x), u_1^+(x), u_2^+(x)) \in H_{3-\delta_0}^{-(2-\delta_1)}(\Omega_+, P_1 \cup P_2 \cup Q_1 \cup Q_2)$$

satisfying

$$\|\rho^+(x) - \rho_+\|_{3-\delta_0}^{-(2-\delta_1)} + \|u_1^+(x) - q_+\|_{3-\delta_0}^{-(2-\delta_1)} + \|u_2^+(x)\|_{3-\delta_0}^{-(2-\delta_1)} \leq C\varepsilon,$$

here $H_{3-\delta_0}^{-(2-\delta_1)}(\Omega_+, P_1 \cup P_2 \cup Q_1 \cup Q_2)$ is a weighted Sobolev space with $P_i = (1, f_i(1))$, $Q_i = (x_1^i, x_2^i)$ and $0 < \delta_1 < \delta_0 < 1$, which is defined as follows:

$v \in H_{3-\delta_0}^{-(2-\delta_1)}(\Omega_+, P_1 \cup P_2 \cup Q_1 \cup Q_2)$ means that $\|v\|_{3-\delta_0}^{-(2-\delta_1)} = \sup_{\mu>0} \mu^{1-(\delta_0-\delta_1)} |v|_{C^{2,1-\delta_0}(\Omega_+^\mu)} < \infty$ holds, here $|d_x| = \min\{dist(x, P_1), dist(x, P_2), dist(x, Q_1), dist(x, Q_2)\}$ for $x \in \Omega_+$, and $\Omega_+^\mu = \{x \in \Omega_+ : |d_x| > \mu\}$. For more properties on this weighted Hölder space, one can see [5], [11-12], [20] and so on.

Remark 1.5 Theorem 1.1 also applies to the nonisentropic compressible Euler system

$$\begin{cases} \partial_1(\rho u_1) + \partial_2(\rho u_2) = 0, \\ \partial_1(\rho u_1^2 + P) + \partial_2(\rho u_1 u_2) = 0, \\ \partial_1(\rho u_1 u_2) + \partial_2(\rho u_2^2 + P) = 0, \\ \partial_1((\rho e + \frac{1}{2}\rho|u|^2 + P)u_1) + \partial_2((\rho e + \frac{1}{2}\rho|u|^2 + P)u_2) = 0 \end{cases}$$

where $u = (u_1, u_2)$ is the velocity, ρ the density, P the pressure, e the internal energy and S the specific entropy. Moreover, the pressure function $P = P(\rho, S)$ and the internal energy function $e = e(\rho, S)$ are smooth in their arguments. Furthermore, we assume that $\partial_\rho P(\rho, S) > 0$ and $\partial_S e(\rho, S) > 0$ for $\rho > 0$.

In this case, the Bernoulli's law takes the form

$$(u_1\partial_1 + u_2\partial_2)\left(\frac{1}{2}|u|^2 + e + \frac{P}{\rho}\right) \equiv 0.$$

Similar to the proof of Theorem 1.1, in the subsonic region, we can obtain a second order elliptic equation on P, a 2×2 first order system on u_2, a first order partial differential equation on S and an algebraic equation $\frac{1}{2}|u|^2 + e + \frac{P}{\rho} \equiv C$ along the streamline. Thus, if the end pressure is given, as in Theorem 1.1, we can get the uniqueness of a transonic shock solution in the regularity can similar to that Theorem 1.1.

Based on Theorem 1.1, we can show the non-existence results for the transonic shock problem in a 2-D nozzle with two straight walls.

Theorem 1.2 (**Ill-posedness**) *If the walls of the nozzle are straight, namely, $f_1(x_1) \equiv -1$ and $f_2(x_1) \equiv 1$, then for the constant supersonic coming flow $(\tilde\rho_0, \tilde q_0, 0)$ with $(\tilde\rho_0, \tilde q_0) \neq (\rho_0, q_0)$ and the end pressure $\tilde\rho_+(x_2) = \rho_+$, the problem (1.1) with the boundary conditions (1.7)-(1.11) has no transonic shock solution $(\rho^+(x), u_1^+(x), u_2^+(x); \xi(x_2))$ such that $(\rho^+(x), u_1^+(x), u_2^+(x); \xi(x_2))$ has the following regularities and estimates*

(i). $\xi(x_2) \in C^3[-1, 1]$, and

$$\|\xi(x_2) - x_1^1\|_{C^3[-1,1]} \le C\varepsilon.$$

(ii). $(\rho^+(x), u_1^+(x), u_2^+(x)) \in C^2(\bar\Omega_+)$ and satisfies

$$\|\rho^+(x) - \rho_+\|_{C^2} + \|u_1^+(x) - q_+\|_{C^2} + \|u_2^+(x)\|_{C^2} \le C\varepsilon.$$

Next, we turn to the nonexistence of solution to the transonic shock problem in the diverging part of the nozzle with the general given pressure $\tilde\rho_+(x_2)$ at the exit of the nozzle.

Suppose that the nozzle walls Γ_1 and Γ_2 are C^5-regular for $-1 \le x_1 \le 1$ and Γ_i consists of two curves Π_i^1 and Π_i^2, here Π_1^1 and Π_2^1 enclose the converging part of the nozzle, while Π_1^2 and Π_2^2 form a two-dimensional angular section (i.e. the diverging part of the nozzle), whose vertex is $(x_1^0, 0)$ with

$x_1^0 < 0$ sufficiently small. More precisely, we assume that the equation of Π_i^2 is represented by $x_2 = (-1)^i(x_1 - x_1^0)tg\alpha_0$, here $tg\alpha_0 = \frac{1}{1-x_1^0}$ (this condition guarantees that Π_i^2 is very near $x_2 = (-1)^i$ in $-\frac{1}{2} \leq x_1 \leq 1$ for sufficiently small $x_1^0 < 0$). Besides, the transonic shock is assumed to go through the origin, and suppose that the supersonic coming flow is symmetric in $-\frac{1}{4} \leq x_1 \leq 0$ (namely, the solution (ρ^-, u_1^-, u_2^-) depends only on $r = \sqrt{(x_1 - x_1^0)^2 + x_2^2}$) and is of a small perturbation of the constant state $(\rho_0, q_0, 0)$. By the hyperbolicity, we can obtain a supersonic flow (ρ^-, u_1^-, u_2^-) in the global nozzle, which is symmetric in $-\frac{1}{4} \leq x_1 \leq 1$ and very close to $(q_0, 0, 0)$. Furthermore, we let the boundary condition (1.10) be replaced by

$$\tilde{\rho}_+(x_2) = \rho_+ \qquad \text{on} \qquad r = (1 - x_1^0)\sec\alpha_0. \tag{1.10'}$$

where the constant ρ_+ is determined by (1.6).

Then based on Theorem 1.1 and Remark 1.3, we can show the following ill-posedness result.

Theorem 1.3 (Ill-posedness) *If the nozzle walls consist of* Γ_1 *and* Γ_2 *as defined above, then the problem (1.1) with the boundary conditions (1.7)-(1.9), (1.10') and (1.11) is ill-posed. More precisely, one can find the supersonic coming flows are of small perturbations of* $(\rho_0, q_0, 0)$ *such that the problem (1.1) with the boundary conditions (1.7)-(1.9), (1.10') and (1.11) has no transonic shock solution* $(\rho^+(x), u_1^+(x), u_2^+(x); \xi(x_2))$ *with the regularities and estimates as stated in Theorem 1.1.*

We now comment on the proof of the main results. Some of the main difficulties are that (1.1) is hyperbolic-elliptic in the subsonic domain and the shock curve is a free boundary. In order to prove Theorem 1.1, first we take the transformations such that the nozzle walls are straighten and the free boundary is fixed. Second, we apply the Bernoulli's law and the characteristics method to reformulate the 3×3 full Euler system into a weakly coupled second order elliptic equation on the density ρ^+ with the mixed boundary conditions, a 2×2 first order system on u_1^+ or u_2^+ with a value at a point and an algebraic equation on (ρ^+, u_1^+, u_2^+) along the streamline. From this, we can obtain some a priori estimates and obtain the uniqueness.

Finally we note that there have been many works on the transonic problem or subsonic flow in a channel (see [1], [4-10], [13], [18-19] and the references therein). In particular, we mention on several recent works that are related to this paper. Chen-Feldman in [6] prove the existence and stability of a steady transonic shock when the flow is in the channel $\Omega = (0,1)^{n-1} \times (-1,1)$ and the Dirichlet boundary condition on the potential is posed at the end of the channel. However, as described in [8], it is more physical to prescribe the pressure at the end than prescribe the value of the potential function. In [7], the well-posedness of transonic shocks for the steady 2-D compressible Euler system in $(-N_1, N_2) \times (0, b)$ with given exit pressure was treated. However, it seems that the proof of main results in [7] has a gap (more precisely, the linearized system (5.16) in [7] is overdetermined for antisymmetric solutions, and so has no solution satisfying the boundary conditions in [7] in general). Similar difficulties appear when one treats the local transonic stability problems on the the supersonic flow past a wedge or the Mach reflection when the related pressure boundary conditions are given. In fact, Theorem 1.2 in this paper is contrary to the main results (i.e. Theorem 2.1) in [7], namely, we show the ill-posedness for the transonic shock problem when the steady compressible Euler flow goes through a flat nozzle and the pressure at the exit is arbitrarily given. In [21-22], for the steady potential equation and the slowly-varying nozzle walls, we have shown that the conjecture of Courant and Friedrichs cannot be true for the arbitrarily given and appropriately large pressure at the exit. However, for a class of curved nozzles, it has been shown for an appropriate pressure at the exit, the transonic shock exits for the 2D isentropic compressible Euler flow in [25].

Our paper is organized as follows. In §2, we reformulate the problem (1.1) with the boundary conditions (1.7)-(1.11). Subsequently we describe the main approaches to prove Theorem 1.1. In §3, first we reduce the free boundary problem (1.1) to a fixed boundary problem of the Euler system, then by use of the techniques in §2 we decompose the complete Euler equations and establish some a priori estimates on the solution. From this, Theorem 1.1 can be shown directly. In §4, using the uniqueness result in Theorem 1.1 we complete the proof on the ill-posedness results in Theorem 1.2 and Theorem 1.3. Finally, we will give several useful remarks in §5. In particular, for the large curved nozzle walls, if the diverging part of the nozzle walls are straight and the corresponding supersonic coming flow in the diverging part is symmetric, then we will give an example to show that the conjecture of Courant-Friedrichs is right for the complete Euler equations.

In what follows, we will use the following convention in this paper:
$O(Y)$ means that there exists a generic constant C such that $|O(Y)| \le CY$, where C is independent of ε.

§2. The reformulation on problem (1.1) with (1.7)-(1.11)

In this section, we will reformulate the nonlinear problem (1.1) with (1.7)-(1.11) so that we can obtain a second order elliptic equation on $\rho^+(x)$ and a 2×2 system on u_1^+ or u_2^+. Before doing this, we first give an estimate on $(\rho^-(x), u_1^-(x), u_2^-(x))$ in system (1.5).

Lemma 2.1 Under the assumptions (1.2) - (1.4), the system (1.5) has a $C^3(\bar{\Omega})$ solution $(\rho^-(x), u_1^-(x), u_2^-(x))$. Moreover, for small $\varepsilon > 0$, there exists a positive constant C independent of ε such that

$$\|\rho^-(x) - \rho_0\|_{C^3(\bar{\Omega})} + \|u_1^-(x) - q_0\|_{C^3(\bar{\Omega})} + \|u_2^-(x)\|_{C^3(\bar{\Omega})} \le C\varepsilon.$$

Proof We note that the system (1.5) is strictly hyperbolic with respect to the x_1-direction for the supersonic flow $u_1^- > c(\rho^-)$.

Indeed, in this case, (1.5) has three distinct real eigenvalues

$$\lambda_1 = \frac{u_1^- u_2^- - c(\rho^-)\sqrt{(u_1^-)^2 + (u_2^-)^2 - c^2(\rho^-)}}{(u_1^-)^2 - c^2(\rho^-)}, \qquad \lambda_2 = \frac{u_2^-}{u_1^-},$$
$$\lambda_3 = \frac{u_1^- u_2^- + c(\rho^-)\sqrt{(u_1^-)^2 + (u_2^-)^2 - c^2(\rho^-)}}{(u_1^-)^2 - c^2(\rho^-)}.$$

Set

$$\tilde{\rho}(x) = \rho^-(x) - \rho_0, \tilde{u}_1 = u_1^- - q_0, \tilde{u}_2 = u_2^-,$$

then it follows from (1.5) that $(\tilde{\rho}, \tilde{u}_1, \tilde{u}_2)$ satisfies

$$\begin{cases} \partial_1 \begin{pmatrix} \tilde{\rho} \\ \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} + A(\tilde{\rho}, \tilde{u}_1, \tilde{u}_2)\partial_2 \begin{pmatrix} \tilde{\rho} \\ \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = 0 \\ \tilde{\rho}(-1, x_2) = \tilde{u}_1(-1, x_2) = \tilde{u}_2(-1, x_2) = 0, \\ \tilde{u}_2 - f_i'(x_1)\tilde{u}_1 = q_0 f_i'(x_1) \qquad \text{on} \qquad x_2 = f_i(x_1), \end{cases} \tag{2.1}$$

where

$$A(\tilde{\rho}, \tilde{u}_1, \tilde{u}_2) = \begin{pmatrix} \frac{(q_0+\tilde{u}_1)\tilde{u}_2}{(q_0+\tilde{u}_1)^2-c^2(\rho_0+\tilde{\rho})} & -\frac{(\rho_0+\tilde{\rho})\tilde{u}_2}{(q_0+\tilde{u}_1)^2-c^2(\rho_0+\tilde{\rho})} & \frac{(\rho_0+\tilde{\rho})(q_0+\tilde{u}_1)}{(q_0+\tilde{u}_1)^2-c^2(\rho_0+\tilde{\rho})} \\ -\frac{c^2(\rho_0+\tilde{\rho})\tilde{u}_2}{(\rho_0+\tilde{\rho})((q_0+\tilde{u}_1)^2-c^2(\rho_0+\tilde{\rho}))} & \frac{(q_0+\tilde{u}_1)\tilde{u}_2}{(q_0+\tilde{u}_1)^2-c^2(\rho_0+\tilde{\rho})} & -\frac{c^2(\rho_0+\tilde{\rho})}{(q_0+\tilde{u}_1)^2-c^2(\rho_0+\tilde{\rho})} \\ \frac{c^2(\rho_0+\tilde{\rho})}{(\rho_0+\tilde{\rho})(q_0+\tilde{u}_1)} & 0 & \frac{\tilde{u}_2}{q_0+\tilde{u}_1} \end{pmatrix}.$$

The assumption (1.4) implies that initial-boundary values in (2.1) satisfy the compatible conditions up to 3−th order. Moreover, $x_2 = f_i(x_1)$ are the characteristics of (2.1) corresponding to the second eigenvalue λ_2. Furthermore, the matrix $A(\tilde{\rho}, \tilde{u}_1, \tilde{u}_2)$ has three distinct real eigenvalues, so that the system (2.1) is strictly hyperbolic. Hence by the characteristics method and the standard Picard iteration (for example, see [14]), when $\varepsilon > 0$ is suitably small we know that (2.1) has a unique $C^3(\bar{\Omega})$−solution, and there exists a constant C independent of ε such that

$$\|\tilde{\rho}\|_{C^3(\bar{\Omega})} + \|\tilde{u}_1(x)\|_{C^3(\bar{\Omega})} + \|\tilde{u}_2(x)\|_{C^3(\bar{\Omega})} \le C\varepsilon.$$

Hence Lemma 2.1 is proved.

Now we start to reformulate the system (1.1) and its boundary conditions in the subsonic region Ω_+.

First, due to the Bernoulli's law, for any C^1 solutions, the system (1.1) in Ω_+ is equivalent to

$$\begin{cases} \partial_1(\rho^+ u_1^+) + \partial_2(\rho^+ u_2^+) = 0, \\ (u_1^+\partial_1 + u_2^+\partial_2)\Big(\frac{1}{2}(u_1^+)^2 + \frac{1}{2}(u_2^+)^2 + h(\rho^+)\Big) = 0, \\ u_1^+\partial_1 u_2^+ + u_2^+\partial_2 u_2^+ + \frac{c^2(\rho^+)}{\rho^+}\partial_2\rho^+ = 0, \end{cases} \tag{2.2}$$

here $h(\rho^+)$ is the enthalpy with $h'(\rho^+) = \frac{c^2(\rho^+)}{\rho^+}$.

Next, we derive a second order equation on the density ρ^+ from (2.2).

For simplicity, we set $D = u_1^+\partial_1 + u_2^+\partial_2$. Then it follows from the first equation in (2.2) that

$$D^2\rho^+ + \rho^+ D(\partial_1 u_1^+ + \partial_2 u_2^+) - \frac{(D\rho^+)^2}{\rho^+} = 0.$$

This, together with the second equation and the third equation in (2.2), yields

$$\begin{aligned} D^2\rho^+ - \rho^+\Big(\partial_1(\frac{c^2(\rho^+)}{\rho^+}\partial_1\rho^+) + \partial_2(\frac{c^2(\rho^+)}{\rho^+}\partial_2\rho^+)\Big) \\ - \frac{(D\rho^+)^2}{\rho^+} - \Big((\partial_1 u_1^+)^2 + 2\partial_1 u_2^+\partial_2 u_1^+ + (\partial_2 u_2^+)^2\Big)\rho^+ = 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \partial_1\Big(((u_1^+)^2 - c^2(\rho^+))\partial_1\rho^+ + u_1^+u_2^+\partial_2\rho^+\Big) + \partial_2\Big(u_1^+u_2^+\partial_1\rho^+ + ((u_2^+)^2 - c^2(\rho^+))\partial_2\rho^+\Big) \\ + \frac{c^2(\rho^+)}{\rho^+}((\partial_1\rho^+)^2 + (\partial_2\rho^+)^2) - \Big((\partial_1 u_1^+)^2 + 2\partial_1 u_2^+\partial_2 u_1^+ + (\partial_2 u_2^+)^2\Big)\rho^+ = 0. \end{aligned} \tag{2.3}$$

Next, we reduce the boundary conditions (1.8) to obtain a Dirichlet boundary condition of ρ^+ on the shock Σ.

It follows from the third equation in (1.8) and the assumption (1.7) that

$$\begin{cases} \xi'(x_2) = \frac{[\rho u_1 u_2]}{[P(\rho)+\rho u_2^2]}, \\ \xi(x_2^1) = x_1^1. \end{cases} \tag{2.4}$$

Substituting (2.4) into the two other equations in (1.8) yields on Σ

$$\begin{cases} G_1(\rho^+, u_1^+, u_2^+) \equiv [\rho u_1 u_2][\rho u_2] - [\rho u_1][P(\rho) + \rho u_2^2] = 0, \\ G_2(\rho^+, u_1^+, u_2^+) \equiv [\rho u_1 u_2]^2 - [P(\rho) + \rho u_1^2][P(\rho) + \rho u_2^2] = 0. \end{cases} \tag{2.5}$$

Due to the condition (1.6), (2.5) is equivalent to

$$\begin{cases} \rho_+(u_1^+ - q_+) + q_+(\rho^+ - \rho_+) = g_1((u_2^+)^2, u_2^+ u_2^-, (u_2^-)^2, \rho^+ - \rho_+, u_1^+ - q_+, \rho_0 - \rho^-, q_0 - u_1^-), \\ 2\rho_+ q_+(u_1^+ - q_+) + (q_+^2 + c^2(\rho_+))(\rho^+ - \rho_+) = g_2((u_2^+)^2, u_2^+ u_2^-, (u_2^-)^2, \rho^+ - \rho_+, u_1^+ - q_+), \end{cases}$$

where

$$\begin{aligned} & g_1((u_2^+)^2, u_2^+ u_2^-, (u_2^-)^2, \rho^+ - \rho_+, u_1^+ - q_+, \rho_0 - \rho^-, q_0 - u_1^-) = \frac{[\rho u_1 u_2][\rho u_2] - [\rho u_1][\rho u_2^2]}{[P(\rho)]} \\ & \qquad - (u_1^+ - q_+)(\rho^+ - \rho_+) - q_0(\rho_0 - \rho^-) - \rho^-(q_0 - u_1^-), \\ & g_2((u_2^+)^2, u_2^+ u_2^-, (u_2^-)^2, \rho^+ - \rho_+, u_1^+ - q_+) = \frac{[\rho u_1 u_2]^2 - [P(\rho) + \rho u_1^2][\rho u_2^2]}{[P(\rho)]} - \Big(P(\rho^+) - P(\rho_+) \\ & \qquad - c^2(\rho_+)(\rho^+ - \rho_+)\Big) - \Big(\rho^+(u_1^+)^2 - \rho_+ q_+^2 - 2\rho_+ q_+(u_1^+ - q_+) - q_+^2(\rho^+ - \rho_+)\Big) \\ & \qquad + P(\rho^-) + \rho^-(u_1^-)^2 - P(\rho_0) - \rho_0 q_0^2. \end{aligned}$$

Thus, on Σ, it follows from the implicit function theorem and Lemma 2.1 that

$$\begin{cases} u_1^+ - q_+ = \tilde{g}_1(x, u_2^+), \\ \rho^+ - \rho_+ = \tilde{g}_2(x, u_2^+), \end{cases} \tag{2.6}$$

and

$$\tilde{g}_1(x, u_2^+) = O((u_2^+)^2) + O(\varepsilon)O(u_2^+) + O(\varepsilon), \qquad \tilde{g}_2(x, u_2^+) = O((u_2^+)^2) + O(\varepsilon)O(u_2^+) + O(\varepsilon)$$

for suitably small u_2^+ and ε.

Next, we derive the boundary conditions of $\rho^+(x)$ on the fixed boundary $x_2 = f_i(x_1)$. It follows from (1.11) that

$$\partial_1 u_2^+ + f_i'(x_1)\partial_2 u_2^+ = (\partial_1 u_1^+ + f_i'(x_1)\partial_2 u_1^+) f_i'(x_1) + f_i''(x_1) u_1^+ \quad \text{on} \quad x_2 = f_i(x_1), \quad i = 1, 2.$$

This, together with the second equation and the third equation in (2.2), yields

$$\partial_n \rho^+ = -\frac{\rho^+(u_1^+)^2}{c^2(\rho^+)} f_i''(x_1) \qquad \text{on} \qquad x_2 = f_i(x_1), \tag{2.7}$$

here ∂_n represents the derivative along the outer normal direction n.

It follows from the analysis above that ρ^+ in Ω_+ can be determined by the following boundary value problem of a second order equation

$$\begin{cases} \partial_1\Big(((u_1^+)^2 - c^2(\rho^+))\partial_1\rho^+ + u_1^+u_2^+\partial_2\rho^+\Big) + \partial_2\Big(u_1^+u_2^+\partial_1\rho^+ + ((u_2^+)^2 - c^2(\rho^+))\partial_2\rho^+\Big) \\ \qquad + \frac{c^2(\rho^+)}{\rho^+}((\partial_1\rho^+)^2 + (\partial_2\rho^+)^2) - \Big((\partial_1 u_1^+)^2 + 2\partial_1 u_2^+\partial_2 u_1^+ + (\partial_2 u_2^+)^2\Big)\rho^+ = 0, \\ \rho^+ - \rho_+ = \tilde{g}_2(x, u_2^+) \qquad \text{on} \qquad x_1 = \xi(x_2), \\ \partial_n\rho^+ + \frac{\rho^+(u_1^+)^2}{c^2(\rho^+)} f_i''(x_1) = 0 \qquad \text{on} \qquad x_2 = f_i(x_1), \\ \rho^+ = \tilde{\rho}_+(x_2) \qquad \text{on} \qquad x_1 = 1. \end{cases} \tag{2.8}$$

Next, we derive an algebraic relation on ρ^+, u_1^+ and u_2^+ so that one can determine u_1^+ in terms of ρ^+ and u_2^+.

It follows from the second equation in (2.2) and the boundary conditions (1.11) and (2.6) that

$$\begin{cases} (u_1^+\partial_1 + u_2^+\partial_2)\Big(\frac{1}{2}(u_1^+)^2 + \frac{1}{2}(u_2^+)^2 + h(\rho^+)\Big) = 0, \\ u_1^+ = q_+ + \tilde{g}_1(x, u_2^+), \qquad \rho^+ = \rho_+ + \tilde{g}_2(x, u_2^+) \qquad \text{on} \qquad x_1 = \xi(x_2), \\ u_2^+ = f_i'(x_1)u_1^+ \qquad \textit{on} \qquad x_2 = f_i(x_1). \end{cases} \tag{2.9}$$

We define the curve $x_2 = x_2(x_1, \beta)$ as the characteristics starting from the point $(\xi(\beta), \beta)$ for the first order differential operator $u_1^+\partial_1 + u_2^+\partial_2$, that is, $x_2(x_1, \beta)$ satisfies

$$\begin{cases} \frac{dx_2(x_1,\beta)}{dx_1} = (\frac{u_2^+}{u_1^+})(x_1, x_2(x_1, \beta)), \\ x_2(\xi(\beta), \beta) = \beta, \qquad \beta \in [x_2^1, x_2^2]. \end{cases} \tag{2.10}$$

Integrating the first order equation in (2.9) along the characteristics $x_2 = x_2(x_1, \beta)$ and noting that $x_2 = f_i(x_1)$ is the characteristics of $u_1^+\partial_1 + u_2^+\partial_2$ starting from the point (x_1^i, x_2^i), then we have in Ω_+

$$\Big(\frac{1}{2}(u_1^+)^2 + \frac{1}{2}(u_2^+)^2 + h(\rho^+)\Big)(x_1, x_2(x_1, \beta)) = \tilde{g}_0(\xi(\beta), \beta, u_2^+(\xi(\beta), \beta)) \tag{2.11}$$

with

$$\begin{aligned} &\tilde{g}_0(\xi(\beta), \beta, u_2^+(\xi(\beta), \beta)) \\ =&\frac{1}{2}(q_+ + \tilde{g}_1(\xi(\beta), \beta, u_2^+(\xi(\beta), \beta)))^2 + h(\rho_+ + \tilde{g}_2(\xi(\beta), \beta, u_2^+(\xi(\beta), \beta)) + \frac{1}{2}(u_2^+)^2(\xi(\beta), \beta). \end{aligned}$$

It should be noted that $u_2^+(\xi(\beta), \beta)$ is not estimated until now.

Finally, we derive a system governing $u_2^+(x)$.

It follows from (2.10) that

$$\begin{cases} \frac{d}{dx_1}(\frac{\partial x_2}{\partial\beta}) = \partial_2(\frac{u_2^+}{u_1^+})(x_1, x_2(x_1, \beta))\frac{\partial x_2}{\partial\beta}, \\ \frac{\partial x_2}{\partial\beta}(\xi(\beta), \beta) = 1 - \xi'(\beta)(\frac{u_2^+}{u_1^+})(\xi(\beta), \beta), \qquad \beta \in [x_2^1, x_2^2]. \end{cases} \tag{2.12}$$

By (2.11), we obtain

$$(u_1^+\partial_2 u_1^+ + u_2^+\partial_2 u_2^+ + \frac{c^2(\rho^+)}{\rho^+}\partial_2\rho^+\Big)(x_1, x_2(x_1,\beta))\frac{\partial x_2}{\partial\beta} = \frac{d}{d\beta}\tilde{g}_0(\xi(\beta),\beta,u_2^+(\xi(\beta),\beta)) \tag{2.13}$$

and

$$\begin{aligned}
&\Big(u_1^+\partial_1 u_1^+ + u_2^+\partial_1 u_2^+ + \frac{c^2(\rho^+)}{\rho^+}\partial_1\rho^+\Big)(x_1, x_2(x_1,\beta))\frac{\partial x_2}{\partial\beta}\\
=&-\Big(u_1^+\partial_2 u_1^+ + u_2^+\partial_2 u_2^+ + \frac{c^2(\rho^+)}{\rho^+}\partial_2\rho^+\Big)(x_1, x_2(x_1,\beta))\frac{dx_2}{dx_1}\frac{\partial x_2}{\partial\beta}\\
=&-(\frac{u_2^+}{u_1^+})(x_1, x_2(x_1,\beta))\frac{d}{d\beta}\tilde{g}_0(\xi(\beta),\beta,u_2^+(\xi(\beta),\beta)).
\end{aligned} \tag{2.14}$$

It follows from (2.12), (2.14), (2.4) and (2.6) that

$$\begin{aligned}
&\Big(u_1^+\partial_1 u_1^+ + u_2^+\partial_1 u_2^+ + \frac{c^2(\rho^+)}{\rho^+}\partial_1\rho^+\Big)(x_1, x_2(x_1,\beta))\\
=&O(u_2^+)\Big((O(u_2^+)+O(\varepsilon))\partial_1 u_2^+ + (O(u_2^+)+O(\varepsilon))\partial_2 u_2^+ + O(\varepsilon)O(u_2^+) + O(\varepsilon)\Big)(\xi(\beta),\beta).
\end{aligned}$$

In addition, one can rewrite the first equation and the third equation in (2.2) as

$$\begin{cases}
\partial_1 u_1^+ + \partial_2 u_2^+ = -\frac{1}{\rho^+}(u_1^+\partial_1\rho^+ + u_2^+\partial_2\rho^+),\\
u_1^+\partial_1 u_2^+ + u_2^+\partial_2 u_2^+ = -\frac{c^2(\rho^+)}{\rho^+}\partial_2\rho^+.
\end{cases} \tag{2.15}$$

Since u_2^+ is expected to be small, then it follows from (2.14) and (2.15) that

$$\begin{cases}
\partial_1 u_2^+ = h_1(\rho^+, u_1^+, u_2^+, \partial_1\rho^+, \partial_2\rho^+),\\
\partial_2 u_2^+ = h_2(\rho^+, u_1^+, u_2^+, \partial_1\rho^+, \partial_2\rho^+),\\
u_2^+(x_1^1, x_2^1)) = m_2^1,
\end{cases} \tag{2.16}$$

here

$$\begin{aligned}
h_1(\rho^+, u_1^+, u_2^+, \partial_1\rho^+, \partial_2\rho^+) =& \frac{u_2^+((u_1^+)^2 - c^2(\rho^+))\partial_1\rho^+ + u_1^+((u_2^+)^2 - c^2(\rho^+))\partial_2\rho^+}{\rho^+((u_1^+)^2 + (u_2^+)^2)}\\
&-\frac{(u_2^+)^2\frac{d}{d\beta}\Big(\tilde{g}_0(\xi(\beta(x)),\beta(x),u_2^+(\xi(\beta(x)),\beta(x)))\Big)\partial_2\beta(x)}{u_1^+((u_1^+)^2 + (u_2^+)^2)},\\
h_2(\rho^+, u_1^+, u_2^+, \partial_1\rho^+, \partial_2\rho^+) =& \frac{u_1^+(c^2(\rho^+) - (u_1^+)^2)\partial_1\rho^+ - u_2^+((u_1^+)^2 + c^2(\rho^+))\partial_2\rho^+}{\rho^+((u_1^+)^2 + (u_2^+)^2)}\\
&+\frac{u_2^+\frac{d}{d\beta}\Big(\tilde{g}_0(\xi(\beta(x)),\beta(x),u_2^+(\xi(\beta(x)),\beta(x)))\Big)\partial_2\beta(x)}{(u_1^+)^2 + (u_2^+)^2}
\end{aligned}$$

and $\beta(x)$ denotes the inverse function of $x_2 = x_2(x_1, \beta)$. Additionally, the constant m_2^1 is determined by the boundary condition in (2.6) and the boundary condition (1.11), namely m_2^1 satisfies

$$\begin{cases} m_2^1 = u_1^+(x_1^1, x_2^1) f_1'(x_1^1), \\ u_1^+(x_1^1, x_2^1) = q_+ + \tilde{g}_1(m_2^1). \end{cases} \tag{2.17}$$

The solvability of m_2^1 in (2.17) will be shown in Lemma 3.2 in next section.

Obviously, $h_i = O(\partial_1\rho^+) + O(\partial_2\rho^+) + O(u_2^+)\Big((O(u_2^+) + O(\varepsilon))\partial_1 u_2^+ + (O(u_2^+) + O(\varepsilon))\partial_2 u_2^+ + O(\varepsilon)O(u_2^+) + O(\varepsilon)\Big)(\beta(x))$. This implies that only $\nabla\rho^+$ has a more "important" impact on u_2^+.

In subsequent section, we will focus on studying the reformulated problems (2.4), (2.8)-(2.11) and (2.16). For this end, we take a transformation to straighten the nozzle walls

$$\begin{cases} X_1 = x_1, \\ X_2 = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)}. \end{cases} \tag{2.18}$$

This transformation changes the boundaries $x_2 = f_1(x_1)$ and $x_2 = f_2(x_1)$ into $X_2 = 0$ and $X_2 = 1$ respectively. Under the transformation (2.18), the new equation for the shock Σ becomes $X_1 = \zeta(X_2)$. Then a simple computation yields

$$\zeta'(X_2) = \frac{(f_2(X_1) - f_1(X_1))\xi'(x_2)}{1 - \xi'(x_2)(f_1'(X_1) + X_2(f_2'(X_1) - f_1'(X_1)))}. \tag{2.19}$$

For convenience, we still write $(\rho(X), u_1(X), u_2(X))$ instead of $(\rho(x), u_1(x), u_2(x))$ in the new transformation (2.18). Then (2.4) becomes

$$\begin{cases} \zeta'(X_2) = \frac{(f_2(X_1) - f_1(X_1))[\rho u_1 u_2]}{[P(\rho) + \rho u_2^2] - (f_1'(X_1) + X_2(f_2'(X_1) - f_1'(X_1)))[\rho u_1 u_2]}, \\ \zeta(0) = x_1^1. \end{cases} \tag{2.20}$$

Correspondingly, (2.8) can be rewritten as

$$\begin{cases} D_1\Big(((u_1^+)^2 - c^2(\rho^+))D_1\rho^+ + u_1^+u_2^+D_2\rho^+\Big) + D_2\Big(u_1^+u_2^+D_1\rho^+ + ((u_2^+)^2 - c^2(\rho^+))D_2\rho^+\Big) \\ \qquad + \frac{c^2(\rho^+)}{\rho^+}((D_1\rho^+)^2 + (D_2\rho^+)^2) - \Big((D_1u_1^+)^2 + 2D_1u_2^+D_2u_1^+ + (D_2u_2^+)^2\Big)\rho^+ = 0, \\ \rho^+ - \rho_+ = \tilde{g}_2(x, u_2^+) \qquad \text{on} \qquad X_1 = \zeta(X_2), \\ D_n^i\rho^+ + \frac{\rho^+(u_1^+)^2}{c^2(\rho^+)} f_i''(X_1) = 0 \qquad \text{on} \qquad X_2 = i-1, \qquad i = 1, 2; \\ \rho^+ = \tilde{\rho}_+(x_2) \qquad \text{on} \qquad X_1 = 1, \end{cases} \tag{2.21}$$

here $x = (x_1, x_2) = (X_1, f_1(X_1) + X_2(f_2(X_1) - f_1(X_1)))$, $D_1 = \partial_{X_1} - \frac{f_1'(X_1) + X_2(f_2'(X_1) - f_1'(X_1))}{f_2(X_1) - f_1(X_1)}\partial_{X_2}$, $D_2 = \frac{1}{f_2(X_1) - f_1(X_1)}\partial_{X_2}$ and $D_n^1 = \frac{1}{\sqrt{1 + (f'(X_1))^2}}(f_1'(X_1)D_1 - D_2)$, $D_n^2 = \frac{1}{\sqrt{1 + (f_2'(X_1))^2}}(D_2 - f_2'(X_1)D_1)$.

Additionally, the equation in (2.9) has the following form

$$\Big(u_1^+\partial_{X_1} + \frac{1}{f_2 - f_1}(u_2^+ - (f_1' + X_2(f_2' - f_1'))u_1^+)\partial_{X_2}\Big)\Big(\frac{1}{2}(u_1^+)^2 + \frac{1}{2}(u_2^+)^2 + h(\rho^+)\Big) = 0. \tag{2.22}$$

Define the curve $X_2 = X_2(X_1, \beta)$ as the characteristics starting from the point $(\zeta(\beta), \beta)$ for the equation (2.22). Namely, $X_2(X_1, \beta)$ satisfies

$$\begin{cases} \frac{dX_2(X_1,\beta)}{dX_1} = \left(\frac{u_2^+ - (f_1' + X_2(f_2' - f_1'))u_1^+}{(f_2 - f_1)u_1^+}\right)(X_1, X_2(X_1, \beta)), \\ X_2(\zeta(\beta), \beta) = \beta, \qquad \beta \in [0, 1]. \end{cases} \tag{2.23}$$

Thus, it follows from (2.22) and (2.23) that

$$\left(\frac{1}{2}(u_1^+)^2 + \frac{1}{2}(u_2^+)^2 + h(\rho^+)\right)(X_1, X_2(X_1, \beta)) = \tilde{g}_0(x(\beta), u_2^+(\zeta(\beta), \beta)) \tag{2.24}$$

and

$$(u_1^+ \partial_{X_2} u_1^+ + u_2^+ \partial_{X_2} u_2^+ + \frac{c^2(\rho^+)}{\rho^+} \partial_{X_2} \rho^+\Big)(X_1, X_2(X_1, \beta)) \frac{\partial X_2}{\partial \beta} = \frac{d}{d\beta}\Big(\tilde{g}_0(x(\beta), u_2^+(\zeta(\beta), \beta))\Big) \tag{2.25}$$

and

$$\begin{aligned} &(u_1^+ \partial_{X_1} u_1^+ + u_2^+ \partial_{X_1} u_2^+ + \frac{c^2(\rho^+)}{\rho^+} \partial_{X_1} \rho^+\Big)(X_1, X_2(X_1, \beta)) \frac{\partial X_2}{\partial \beta} \\ = &- \left(\frac{u_2^+ - (f_1' + X_2(f_2' - f_1'))u_1^+}{(f_2 - f_1)u_1^+}\right)(X_1, X_2(X_1, \beta)) \frac{d}{d\beta}\Big(\tilde{g}_0(x(\beta), u_2^+(\zeta(\beta), \beta))\Big) \end{aligned} \tag{2.26}$$

with $x(\beta) = (\zeta(\beta), f_1(\zeta(\beta)) + \beta(f_2(\zeta(\beta)) - f_1(\zeta(\beta))))$.

Finally, the first equation and the third equation in (2.2) becomes

$$\begin{cases} D_1 u_1^+ + D_2 u_2^+ = -\frac{1}{\rho^+}(u_1^+ D_1 \rho^+ + u_2^+ D_2 \rho^+), \\ u_1^+ D_1 u_2^+ + u_2^+ D_2 u_2^+ = -\frac{c^2(\rho^+)}{\rho^+} D_2 \rho^+. \end{cases} \tag{2.27}$$

Combining (2.25), (2.26) with (2.27) yields

$$\begin{cases} \partial_{X_1} u_1^+ = H_1(\rho^+, u_1^+, u_2^+, \partial_{X_1}\rho^+, \partial_{X_2}\rho^+), \\ \partial_{X_2} u_1^+ = H_2(\rho^+, u_1^+, u_2^+, \partial_{X_1}\rho^+, \partial_{X_2}\rho^+), \\ u_1^+(x_1^1, 0) = m_1^1. \end{cases} \tag{2.28}$$

and

$$\begin{cases} \partial_{X_1} u_2^+ = H_3(\rho^+, u_1^+, u_2^+, \partial_{X_1}\rho^+, \partial_{X_2}\rho^+), \\ \partial_{X_2} u_2^+ = H_4(\rho^+, u_1^+, u_2^+, \partial_{X_1}\rho^+, \partial_{X_2}\rho^+), \\ u_2^+(x_1^1, 1) = m_2^1. \end{cases} \tag{2.29}$$

here (m_1^1, m_2^1) is determined by the relations $m_2^1 = f_1'(x_1^1)m_1^1$ and $m_1^1 - q_+ = \tilde{g}_1(x_1^1, x_2^1, m_2^1)$, which come from the boundary conditions (1.11) and (2.6). In addition, $H_i = \frac{det(A_i)}{det(A_0)}$ for $1 \le i \le 4$, here $A_0 = \begin{pmatrix} 0 & u_1^+ & 0 & u_2^+ \\ u_1^+ & 0 & u_2^+ & 0 \\ 1 & -\frac{(f_1' + X_2(f_2' - f_1'))}{f_2 - f_1} & 0 & \frac{1}{f_2 - f_1} \\ 0 & 0 & u_1^+ & \frac{u_2^+ - u_1^+(f_1' + X_2(f_2' - f_1'))}{f_2 - f_1} \end{pmatrix}$ with $det(A_0) = \dfrac{q_+^3}{2} + O(\varepsilon) + O(|u_2^+|) \neq 0$ for small ε and $|u_2^+|$ and A_i denotes the 4×4 matrix which is obtained through substituting the $i-$column in A_0 by the following vector L

$$L=\Bigg(\frac{d}{d\beta}\Big(\tilde{g}_0(x(\beta),u_2^+(\zeta(\beta),\beta))\Big)\partial_{X_2}\beta-\frac{c^2(\rho^+)}{\rho^+}\partial_{X_2}\rho^+,-\Big(\frac{u_2^+-(f_1'+X_2(f_2'-f_1'))u_1^+}{(f_2-f_1)u_1^+}\Big)\frac{d}{d\beta}\Big(\tilde{g}_0(x(\beta),\\ u_2^+(\zeta(\beta),\beta))\Big)\partial_{X_2}\beta-\frac{c^2(\rho^+)}{\rho^+}\partial_{X_1}\rho^+,-\frac{1}{\rho^+}(u_1^+D_1\rho^++u_2^+D_2\rho^+),\qquad -\frac{c^2(\rho^+)}{\rho^+}D_2\rho^+\Bigg)^T,$$

where $\beta=\beta(X)$ is an inverse function of $X_2=X_2(X_1,\beta)$.

In order to show Theorem 1.1, one needs only to prove the uniqueness of solutions to the problem (2.20)-(2.21), (2.23)-(2.24) and (2.28) or (2.29). This will be done in the next section.

§3. **The proof of Theorem 1.1.**

To prove the uniqueness of solutions in Theorem 1.1, it is convenient to change the domain Ω_+ including a free boundary Σ into a fixed domain $Q_+=\{y:0<y_1<1, 0<y_2<1\}$. For this end, we take a transformation as follows

$$\begin{cases} y_1=\frac{X_1-\zeta(X_2)}{1-\zeta(X_2)},\\ y_2=X_2.\end{cases}\tag{3.1}$$

For simplicity, in Q_+, we still write (ρ^+,u_1^+,u_2^+) as the state of fluid on the right of the shock in the new coordinates (y_1,y_2).

Noting that

$$\partial_{X_1}=\frac{1}{1-\zeta(y_2)}\partial_{y_1},\qquad \partial_{X_2}=\frac{(1-y_1)\zeta'(y_2)}{\zeta(y_2)-1}\partial_{y_1}+\partial_{y_2}.$$

Then the equations (2.20)-(2.21) become

$$\begin{cases} \zeta'(y_2)=\frac{\Big(f_2(\zeta(y_2)+(1-\zeta(y_2))y_1)-f_1(\zeta(y_2)+(1-\zeta(y_2))y_1)\Big)[\rho u_1u_2]}{[P(\rho)+\rho u_2^2]-\Big(f_1'(\zeta(y_2)+(1-\zeta(y_2))y_1)+y_2(f_2'(\zeta(y_2)+(1-\zeta(y_2))y_1)-f_1'(\zeta(y_2)+(1-\zeta(y_2))y_1)\Big)[\rho u_1u_2]},\\ \zeta(0)=x_1^1.\end{cases}\tag{3.2}$$

and

$$\begin{cases} \tilde{D}_1\Big(((u_1^+)^2-c^2(\rho^+))\tilde{D}_1\rho^++u_1^+u_2^+\tilde{D}_2\rho^+\Big)+\tilde{D}_2\Big(u_1^+u_2^+\tilde{D}_1\rho^++((u_2^+)^2-c^2(\rho^+))\tilde{D}_2\rho^+\Big)\\ \qquad +\frac{c^2(\rho^+)}{\rho^+}((\tilde{D}_1\rho^+)^2+(\tilde{D}_2\rho^+)^2)-\Big((\tilde{D}_1u_1^+)^2+2\tilde{D}_1u_2^+\tilde{D}_2u_1^++(\tilde{D}_2u_2^+)^2\Big)\rho^+=0,\\ \rho^+-\rho_+=\tilde{g}_2(x(y),u_2^+)\qquad \text{on}\qquad y_1=0,\\ \tilde{D}_n^i\rho^++\frac{\rho^+(u_1^+)^2}{c^2(\rho^+)}f_i''(\zeta(y_2)+(1-\zeta(y_2))y_1)=0\qquad\text{on}\qquad y_2=i-1,\qquad i=1,2;\\ \rho^+=\tilde{\rho}_+(x_2(y))\qquad\text{on}\qquad y_1=1\end{cases}\tag{3.3}$$

with

$$
\begin{aligned}
x(y) =&(x_1(y), x_2(y)),\\
x_1(y) =&\zeta(y_2) + y_1(1-\zeta(y_2)),\\
x_2(y) =&f_1(\zeta(y_2) + y_1(1-\zeta(y_2))) + y_2(f_2-f_1)(\zeta(y_2) + y_1(1-\zeta(y_2))),\\
\tilde{D}_1 =&\frac{1}{1-\zeta(y_2)}\left(1+\frac{(1-y_1)\zeta'(y_2)(f_1'(x_1(y)) + y_2(f_2'-f_1')(x_1(y)))}{(f_2-f_1)(x_1(y))}\right)\partial_{y_1}\\
&-\frac{f_1'(x_1(y)) + y_2(f_2'-f_1')(x_1(y))}{(f_2-f_1)(x_1(y))}\partial_{y_2},\\
\tilde{D}_2 =&\frac{(1-y_1)\zeta'(y_2)}{(\zeta(y_2)-1)(f_2-f_1)(x_1(y))}\partial_{y_1} + \frac{1}{(f_2-f_1)(x_1(y))}\partial_{y_2},\\
\tilde{D}_n^1 =&(1+(f_1'(x_1(y)))^2)^{-1/2}\ (f_1'(x_1(y))\tilde{D}_1 - \tilde{D}_2),\\
\tilde{D}_n^2 =&(1+(f_2'(x_1(y)))^2)^{-1/2}\ (f_2'(x_1(y))\tilde{D}_1).
\end{aligned}
$$

Additionally, it follows from the equation (2.22) that

$$
\begin{aligned}
&\Bigg(\left(\frac{u_1^+}{1-\zeta(y_2)} - \frac{(1-y_1)\zeta'(y_2)\big(u_2^+ - (f_1'(x_1(y)) + y_2(f_2'-f_1')(x_1(y)))u_1^+\big)}{(f_2-f_1)(x_1(y))(1-\zeta(y_2))}\right)\partial_{y_1}\\
&+\left(\frac{u_2^+ - \big(f_1'(x_1(y)) + y_2(f_2'-f_1')(x_1(y))\big)u_1^+}{(f_2-f_1)(x_1(y))}\right)\partial_{y_2}\Bigg)\left(\frac{1}{2}(u_1^+)^2 + \frac{1}{2}(u_2^+)^2 + h(\rho^+)\right) = 0.
\end{aligned}
\tag{3.4}
$$

The characteristics $y_2 = y_2(y_1,\beta)$ starting from the point $(0,\beta)$ of (3.4) is defined as

$$
\begin{cases}
\frac{dy_2}{dy_1} = \frac{\left(u_2^+ - (f_1' + y_2(f_2'-f_1'))u_1^+\right)(1-\zeta(y_2))}{(f_2-f_1)u_1^+ - (1-y_1)\zeta'(y_2)\left(u_2^+ - (f_1'+y_2(f_2'-f_1'))u_1^+\right)},\\
y_2(0,\beta) = \beta.
\end{cases}
\tag{3.5}
$$

Thus it follows from (3.4) that

$$
\left(\frac{1}{2}(u_1^+)^2 + \frac{1}{2}(u_2^+)^2 + h(\rho^+)\right)(y_1, y_2(y_1,\beta)) = \tilde{g}_0(\zeta(\beta), f_1(\zeta(\beta)) + \beta(f_2-f_1)(\zeta(\beta)), u_2^+(0,\beta)).
$$

Analogous to (2.29), we can obtain a system on u_2^+ as follows

$$
\begin{cases}
\partial_{y_1}u_2^+ = \tilde{H}_1(\rho^+, u_1^+, u_2^+, \partial_{y_1}\rho^+, \partial_{y_2}\rho^+),\\
\partial_{y_2}u_2^+ = \tilde{H}_2(\rho^+, u_1^+, u_2^+, \partial_{y_1}\rho^+, \partial_{y_2}\rho^+),\\
u_2^+(0,0) = m_2^1,
\end{cases}
\tag{3.6}
$$

here $\tilde{H}_i = \frac{det(\tilde{A}_i)}{det(\tilde{A}_0)}$ for $i = 1,2$, the 4×4 matrix $\tilde{A}_0 = (l_1, l_2, l_3, l_4)$ is defined as

$$
l_1 = \left(0, u_2^+, \frac{(1-y_1)\zeta'(y_2)}{(\zeta(y_2)-1)(f_2-f_1)}, \frac{(f_2-f_1+(1-y_1)\zeta'(y_2)(f_1'+y_2(f_2'-f_1')))u_1^+ - (1-y_1)\zeta'(y_2)u_2^+}{(1-\zeta(y_2))(f_2-f_1)}\right)^T,
$$

$$
l_2 = \left(u_2^+, 0, \frac{1}{f_2-f_1}, \frac{u_2^+ - (f_1'+y_2(f_2'-f_1'))u_1^+}{f_2-f_1}\right)^T,
$$

$$
l_3 = \left(0, u_1^+, \frac{f_2-f_1+(1-y_1)\zeta'(y_2)(f_1'+y_2(f_2'-f_1'))}{(1-\zeta(y_2))(f_2-f_1)}, 0\right)^T
$$

$$
l_4 = \left(u_1^+, 0, -\frac{f_1'+y_2(f_2'-f_1')}{f_2-f_1}, 0\right)^T
$$

and $\tilde{A}_i (i = 1, 2)$ denotes the 4×4 matrix which is obtained from $\tilde{A}_0$ by replacing the $i-$row in $\tilde{A}_0$ with the vector $\tilde{l} = (\tilde{l}_{01}, \tilde{l}_{02}, \tilde{l}_{03}, \tilde{l}_{04})^T$ defined by

$$\begin{aligned}
\tilde{l}_{01} &= \frac{d}{d\beta}\big(\tilde{g}_0(\zeta(\beta), f_1(\zeta(\beta)) + \beta(f_2 - f_1)(\zeta(\beta)), u_2^+(0,\beta))\big)\partial_2\beta(y) - \frac{c^2(\rho^+)}{\rho^+}\partial_2\rho^+,\\
\tilde{l}_{02} &= \frac{d}{d\beta}\big(\tilde{g}_0(\zeta(\beta), f_1(\zeta(\beta)) + \beta(f_2 - f_1)(\zeta(\beta)), u_2^+(0,\beta))\big)\partial_2\beta(y)\frac{dy_2(y_1,\beta)}{dy_1} - \frac{c^2(\rho^+)}{\rho^+}\partial_1\rho^+,\\
\tilde{l}_{03} &= -\frac{1}{\rho^+}(u_1^+\tilde{D}_1\rho^+ + u_2^+\tilde{D}_2\rho^+),\\
\tilde{l}_{04} &= -\frac{c^2(\rho^+)}{\rho^+}\tilde{D}_2\rho^+,
\end{aligned}$$

where $\beta = \beta(y_1, y_2)$ is an inverse function of $y_2 = y_2(y_1, \beta)$. The existence of m_2^1 in (3.6) will follow from Lemma 3.2 below.

To illustrate the validity of regularity to the solution in Theorem 1.1, we now give three lemmas to ensure the compatibility relations of solutions at the cornered points formed by the shock curve and the nozzle walls. We start with the property that the shock curve must be perpendicular to the nozzle walls:

Lemma 3.1 (**Orthogonality**) Under the assumptions on the regularities of solutions in Theorem 1.1, we have

$$\xi'(x_2^i) = -f_i'(x_1^i), \qquad i = 1, 2.$$

Namely, the shock curve is perpendicular to the walls of the nozzle.

Proof Since $u_2^\pm(x_1^i, x_2^i) = f_i'(x_1^i)u_1^\pm(x_1^i, x_2^i)$, it follows from the first equation in (1.8) that

$$[\rho u_1](x_1^i, x_2^i)(1 - \xi'(x_2^i)f_i'(x_1^i)) = 0.$$

Thus

$$[\rho u_1](x_1^i, x_2^i) = 0. \tag{3.7}$$

(3.7) together with the second equation in (1.8), yields

$$[P(\rho)](x_1^i, x_2^i) = -(\rho^+ u_1^+[u_1])(x_1^i, x_2^i)(1 - \xi'(x_2^i)f_i'(x_1^i)). \tag{3.8}$$

In addition, the third equation in (1.8) gives

$$\xi'(x_2^i)[P(\rho)](x_1^i, x_2^i) = f_i'(x_1^i)(\rho^+ u_1^+[u_1])(x_1^i, x_2^i)(1 - \xi'(x_2^i)f_i'(x_1^i)). \tag{3.9}$$

Noting that $[P(\rho)](x_1^i, x_2^i) \neq 0$ and $[u_1](x_1^i, x_2^i) \neq 0$, one can obtain from (3.9) and (3.10) that

$$\xi'(x_2^i) = -f_i'(x_1^i).$$

Next, we show the solvability of m_2^1 in (3.6).

Lemma 3.2 Under the boundary conditions (1.11) and (2.5), then in the small neighborhood of $(q_+, 0, \rho_+)$ there exists a unique solution $(m_1^i, m_2^i, m_3^i) = (u_1^+(x_1^i, x_2^i), u_2^+(x_1^i, x_2^i), \rho^+(x_1^i, x_2^i))(i = 1, 2)$ such that (1.11) and (2.5) hold at the point (x_1^i, x_2^i).

Proof It follows from the boundary conditions in (1.11) and (2.5) that

$$\begin{cases} L_1(m_1^i, m_2^i, m_3^i) \equiv m_2^i - f_i'(x_1^i)(m_1^i - q_+) = q_+ f_i'(x_1^i), \\ L_2(m_1^i, m_2^i, m_3^i) \equiv \rho_+(m_1^i - q_+) + q_+(m_3^i - \rho_+) + (m_1^i - q_+)(m_3^i - \rho_+) = (\rho^- u_1^-)(x_1^i, x_2^i) - \rho_+ q_+, \\ L_3(m_1^i, m_2^i, m_3^i) \equiv (1 + (f_i'(x_1^i))^2)(\rho^- u_1^-)(x_1^i, x_2^i)(m_1^i - q_+) + P(m_3^i) - P(\rho_+) = P(\rho^-(x_1^i, x_2^i)) \\ \qquad - P(\rho_0) - (1 + (f_i'(x_1^i))^2)(\rho^- u_1^-)(x_1^i, x_2^i)(q_+ - u_1^-(x_1^i, x_2^i)) + \rho_+ q_+^2 - \rho_0 q_0^2, \end{cases} \tag{3.10}$$

where we have used Lemma 3.1, (3.7), (1.6) and (3.8). If $f_i'(x_1^i) = 0$, $\rho^-(x_1^i, x_2^i) = \rho_0$, and $u_1^-(x_1^i, x_2^i) = q_0$, then it follows from (1.6) that $(m_1, m_2, m_3) = (q_+, 0, \rho_+)$ is a solution of (3.10). Furthermore, for small $\varepsilon > 0$ one has

$$\begin{aligned} \left(\frac{\partial(L_1, L_2, L_3)}{\partial(m_1^i - q_+, m_2^i, m_3^i - \rho_+)} \right) \Bigg|_{(m_1^i, m_2^i, m_3^i) = (q_+, 0, \rho_+)} &= \det \begin{pmatrix} -f_i'(x_1^i) & 1 & 0 \\ \rho_+ & 0 & q_+ \\ (1 + (f_i'(x_1^i))^2)(\rho^- u_1^-)(x_1^i, x_2^i) & 0 & c^2(\rho_+) \end{pmatrix} \\ &= -\rho_+(c^2(\rho_+) - q_+^2) + O(\varepsilon) \neq 0. \end{aligned}$$

It follows from Lemma 2.1 and the implicit function theorem that Lemma 3.1 holds.

Therefore the proof on Lemma 3.2 is complete.

Finally, we turn to the compatibility conditions at the corners.

Lemma 3.3 (**Compatibility**) Let the regularity assumptions in Theorem 1.1 hold and $f_i''(x_1^i) = 0$. Then

$$\partial_\tau \rho^+(x_1^i, x_2^i) = 0,$$

here $\partial_\tau = \xi'(x_2)\partial_1 + \partial_2$, which represents the directional derivative along the tangent direction of the shock curve $x_1 = \xi(x_2)$.

Remark 3.1 It follows from Lemma 3.2 that $\partial_\tau|_{(x_1^i, x_2^i)} = \partial_{n_i}|_{(x_1^i, x_2^i)}$ holds, here n_i represents the outer normal of $x_2 = f_i(x_1)$. In addition, due to $f_i''(x_1^i) = 0$ and (2.7), one has $\partial_{n_i}\rho^+(x_1^i, x_2^i) = 0$. This, together with Lemma 3.3, shows that the first order compatibility condition for ρ^+ hold at the corner point (x_1^i, x_2^i).

Proof Taking ∂_τ on two sides of the equations (2.5), and noting that $[\rho u_1](x_1^i, x_2^i) = [\rho u_2](x_1^i, x_2^i) = 0$ (see (3.7)), one obtains at the points (x_1^i, x_2^i) that

$$\begin{cases} [\rho u_1 u_2]\partial_\tau[\rho u_2] - [P(\rho) + \rho u_2^2]\partial_\tau[\rho u_1] = 0, \\ 2[\rho u_1 u_2]\partial_\tau[\rho u_1 u_2] - [P(\rho) + \rho u_2^2]\partial_\tau[P(\rho) + \rho u_1^2] - [P(\rho) + \rho u_1^2]\partial_\tau[P(\rho) + \rho u_2^2] = 0. \end{cases}$$

Lemma 3.1 yields

$$\begin{cases} \partial_\tau[\rho u_1](x_1^i, x_2^i) + f_i'(x_1^i)\partial_\tau[\rho u_2](x_1^i, x_2^i) = 0, \\ (f_i'(x_1^i))^2\partial_\tau[P(\rho) + \rho u_2^2](x_1^i, x_2^i) + 2f_i'(x_1^i)\partial_\tau[\rho u_1 u_2](x_1^i, x_2^i) + \partial_\tau[P(\rho) + \rho u_1^2](x_1^i, x_2^i) = 0. \end{cases} \tag{3.11}$$

Thus it follows from a direct computation that at the point (x_1^i, x_2^i)

$$
\begin{cases}
\partial_\tau u_1^+ + f_i'(x_1^i)\partial_\tau u_2^+ = \frac{1}{\rho^+}\{\partial_\tau(\rho^- u_1^-) + f_i'(x_1^i)\partial_\tau(\rho^- u_2^-) - (u_1^+ + f_i'(x_1^i)u_2^+)\partial_\tau\rho^+\}, \\
\partial_\tau u_1^+ + f_i'(x_1^i)\partial_\tau u_2^+ = \frac{1}{2\rho^+(u_1^+ + f_i'(x_1^i)u_2^+)}\Big\{(f_i'(x_1^i))^2\partial_\tau(P(\rho^-) + \rho^-(u_2^-)^2) + 2f_i'(x_1^i)\partial_\tau(\rho^- u_1^- u_2^-) \\
\quad +\partial_\tau(P(\rho^-) + \rho^-(u_1^-)^2) - \Big((f_i'(x_1^i))^2(c^2(\rho^+) + (u_2^+)^2) + 2f_i'(x_1^i)u_1^+u_2^+ \\
\quad +c^2(\rho^+) + (u_1^+)^2\Big)\partial_\tau\rho^+\Big\}.
\end{cases}
\tag{3.12}
$$

Since

$$
u_2^-(x_1^i, x_2^i) = f_i'(x_1^i)u_1^-(x_1^i, x_2^i)), \qquad \partial_\tau\rho^-(x_1^i, x_2^i) = -\left(\frac{\rho^-(u_1^-)^2}{c^2(\rho^-)}\right)(x_1^i, x_2^i)f_i''(x_1^i) = 0,
$$
$$
u_1^-\partial_\tau u_1^- + u_2^-\partial_\tau u_2^- + \frac{c^2(\rho^-)}{\rho^-}\partial_\tau\rho^- \equiv 0,
$$

then

$$
(\partial_\tau(\rho^- u_1^-) + f_i'(x_1^i)\partial_\tau(\rho^- u_2^-))(x_1^i, x_2^i) = \left((1 + (f_i'(x_1^i))^2)u_1^-(x_1^i, x_2^i) - (\frac{c^2(\rho^-)}{u_1^-})(x_1^i, x_2^i)\right)\partial_\tau\rho^-(x_1^i, x_2^i) = 0
$$

and

$$
\begin{aligned}
&\Big((f_i'(x_1^i))^2\partial_\tau(P(\rho^-) + \rho^-(u_2^-)^2) + 2f_i'(x_1^i)\partial_\tau(\rho^- u_1^- u_2^-) + \partial_\tau(P(\rho^-) + \rho^-(u_1^-)^2)\Big)(x_1^i, x_2^i) \\
&= (1 + (f_i'(x_1^i))^2)\Big((u_1^-)^2 + (u_2^-)^2 - c^2(\rho^-)\Big)(x_1^i, x_2^i)\partial_\tau\rho^-(x_1^i, x_2^i) = 0.
\end{aligned}
$$

Substituting the above computations into (3.12) yields

$$
\Big((u_1^+)^2 - c^2(\rho^+) + 2f_i'(x_1^i)u_1^+u_2^+ + (f_i'(x_1^i))^2((u_2^+)^2 - c^2(\rho^+))\Big)\partial_\tau\rho^+(x_1^i, x_2^i) = 0.
$$

Thus, for small ε, we arrive at $\partial_\tau\rho^+(x_1^i, x_2^i) = 0$, which shows Lemma 3.3.

Now we start to prove Theorem 1.1.

Suppose that the problem (3.2)-(3.6) has two solutions $(\rho^{+,1}, u_1^{+,1}, u_2^{+,1}; \zeta_1(y_2))$ and $(\rho^{+,2}, u_1^{+,2}, u_2^{+,2}; \zeta_2(y_2))$ with the corresponding regularities in Theorem 1.1.

Set

$$
W_1(y) = u_1^{+,1}(y) - u_1^{+,2}(y), \qquad W_2(y) = u_2^{+,1}(y) - u_2^{+,2}(y), \qquad W_3(y) = \rho_1^{+,1}(y) - \rho_1^{+,2}(y)
$$

and

$$
\Xi(y_2) = \zeta_1(y_2) - \zeta_2(y_2).
$$

By use of (3.2), Lemma 2.1 and the assumptions in Theorem 1.1, we obtain

$$\begin{cases} \Xi'(y_2) = a_0(y_2)\Xi(y_2) + a_1(y_2)W_1(\zeta^1(y_2), y_2) + a_2(y_2)W_2(\zeta^1(y_2), y_2) + a_3(y_2)W_3(\zeta^1(y_2), y_2), \\ \Xi(0) = 0, \end{cases} \tag{3.13}$$

here $a_0(y_2) \in C^1[0,1], a_i(y_2) \in C^2[0,1](1 \le i \le 3)$ satisfying

$$\|a_0\|_{C^1} + \|a_1\|_{C^2} \le C\varepsilon, \qquad \|a_2\|_{C^2} + \|a_3\|_{C^2} \le C.$$

It follows from the Granwall's inequality that

$$|\Xi(y_2)| \le C\varepsilon\|W_1\|_{L^\infty(Q_+)} + C(\|W_2\|_{L^\infty(Q_+)} + \|W_3\|_{L^\infty(Q_+)}). \tag{3.14}$$

Thus, using (3.13) again, we arrive at

$$\|\Xi(y_2)\|_{C^1[0,1]} \le C\varepsilon\|W_1\|_{L^\infty(Q_+)} + C(\|W_2\|_{L^\infty(Q_+)} + \|W_3\|_{L^\infty(Q_+)})$$

and

$$\|\Xi(y_2)\|_{C^{1,1-\delta_0}[0,1]} \le C\varepsilon\|W_1\|_{C^{1-\delta_0}(Q_+)} + C(\|W_2\|_{C^{1-\delta_0}(Q_+)} + \|W_3\|_{C^{1-\delta_0}(Q_+)}), \tag{3.15}$$

here $0 < \delta_0 < 1$ is a fixed constant.

Next, we estimate W_3 by making use of the equation (3.3), the estimate (3.15) on $\Xi(y_2)$, and the assumptions in Theorem 1.1. Indeed, it follows from (3.3) that

$$\begin{cases} \tilde{D}_1^1\Big(((u_1^{+,1})^2 - c^2(\rho^{+,1}))\tilde{D}_1^1W_3 + u_1^{+,1}u_2^{+,1}\tilde{D}_2^1W_3\Big) + \tilde{D}_2^1\Big(u_1^{+,1}u_2^{+,1}\tilde{D}_1^1W_3 + ((u_2^{+,1})^2 \\ \quad -c^2(\rho^{+,1}))\tilde{D}_2^1W_3\Big) = F(\zeta_1(y_2), \zeta_2(y_2), \zeta_1'(y_2), \zeta_2'(y_2), \zeta_1''(y_2), \zeta_2''(y_2), \rho^{+,1}, \nabla\rho^{+,1}, \rho^{+,2}, \nabla\rho^{+,2}, \\ \qquad u_1^{+,1}, \nabla u_1^{+,1}, u_1^{+,2}, \nabla u_1^{+,2}, u_2^{+,1}, \nabla u_2^{+,1}, u_2^{+,2}, \nabla u_2^{+,2}), \\ W_3 = \tilde{g}_2(x^1(y), u_2^{+,1}) - \tilde{g}_2(x^2(y), u_2^{+,2}) \qquad \text{on} \qquad y_1 = 0, \\ \tilde{D}_{n_i}^1 W_3 = G(\zeta_1(y_2), \zeta_2(y_2), \zeta_1'(y_2), \zeta_2'(y_2), \rho^{+,1}, \nabla\rho^{+,1}, \rho^{+,2}, \nabla\rho^{+,2}) \qquad \text{on} \qquad y_2 = i-1, i = 1, 2, \\ W_3 = \tilde{\rho}_+(x_2^1(y)) - \tilde{\rho}_+(x_2^2(y)) \qquad \text{on} \qquad y_1 = 1, \end{cases} \tag{3.16}$$

here

$$\begin{aligned} x^i(y) &= (x_1^i(y), x_2^i(y)), \\ x_1^i(y) &= \zeta_i(y_2) + y_1(1 - \zeta_i(y_2)), \\ x_2^i(y) &= f_1(\zeta_i(y_2) + y_1(1 - \zeta_i(y_2))) + y_2(f_2 - f_1)(\zeta_i(y_2) + y_1(1 - \zeta_i(y_2))), \\ \tilde{D}_1^i &= \frac{1}{1 - \zeta_i(y_2)}\left(1 + \frac{(1 - y_1)\zeta_i'(y_2)(f_1'(x_1^i(y)) + y_2(f_2' - f_1')(x_1^i(y)))}{(f_2 - f_1)(x_1^i(y))}\right)\partial_{y_1} \\ &\qquad - \frac{f_1'(x_1^i(y)) + y_2(f_2' - f_1')(x_1^i(y))}{(f_2 - f_1)(x_1^i(y))}\partial_{y_2}, \\ \tilde{D}_2^i &= \frac{(1 - y_1)\zeta_i'(y_2)}{(\zeta_i(y_2) - 1)(f_2 - f_1)(x_1^i(y))}\partial_{y_1} + \frac{1}{(f_2 - f_1)(x_1^i(y))}\partial_{y_2}, \\ \tilde{D}_{n_1}^i &= (1 + (f_1'(x_1^i(y)))^2)^{-1/2}\ (f_2'(x_1^i(y))\tilde{D}_1^i - \tilde{D}_2^i), \\ \tilde{D}_{n_2}^i &= (1 + (f_2'(x_1^i(y)))^2)^{-1/2}\ (\tilde{D}_2^i - f_2'(x_1^i(y))\tilde{D}_1^i) \end{aligned}$$

and

$$
\begin{aligned}
F = & \sum_{k=1,2;j=1,2,3} \partial_{y_k}(b_{0j}^k(y)W_j) + \sum_{k=1,2} \partial_{y_k}(b_{04}^k(y)\Xi'(y_2)) + \sum_{k=1,2;j=1,2,3} b_{1j}^k(y)\partial_{y_k}W_j + \sum_{j=1}^{3} b_{2j}(y)W_j \\
& + b_{31}(y)\Xi(y_2) + b_{32}(y)\Xi'(y_2), \\
G = & \, c_0(y)W_3 + c_1(y)W_1 + c_2(y)\Xi(y_2) + c_3(y)\Xi'(y_2)
\end{aligned}
$$

with $b_{ij}^l(y), b_{ij}, c_k(y) \in C^1(\bar{Q}_+)$ and $\|b_{ij}^l(y)\|_{C^1(\bar{Q}_+)} + \|b_{ij}(y)\|_{C^1(\bar{Q}_+)} + \|c_k(y)\|_{C^1(\bar{Q}_+)} \le C\varepsilon$.

Due to Lemma 3.3, one can check easily the compatibility of W_3 at the cornered points $(0,0)$ and $(0,1)$ in (3.16). In addition, by the assumptions in Theorem 1.1, we have also the compatibility at $(1,0)$ and (1.1) in (3.16). It thus follows from the regularity estimates on second order elliptic equations of divergence form with cornered boundary and mixed boundary conditions (see [3], [15] and so on) that the solution to (3.16) admits the following estimate:

$$
\begin{aligned}
\|W_3\|_{C^{1,1-\delta_0}} \le & \, C\Big(\|\tilde{g}_2(x^1(y), u_2^{+,1}) - \tilde{g}_2(x^2(y), u_2^{+,2})\|_{C^{1,1-\delta_0}} + \|\tilde{\rho}_+(x_2^1(y)) - \tilde{\rho}_+(x_2^2(y))\|_{C^{1,1-\delta_0}} \\
& + \|G\|_{C^{1-\delta_0}} + \sum_{k=1,2;j=1,2,3} \|b_{0j}^k W_j\|_{C^{1-\delta_0}} + \sum_{k=1,2;j=1,2,3} \|b_{1j}^k \partial_{y_k} W_j\|_{C^{1-\delta_0}} \\
& + \sum_{k=1,2} \|b_{04}^k(y)\Xi'(y_2)\|_{C^{1-\delta_0}} + \sum_{1\le j\le 3} \|b_{2j}W_j\|_{C^{1-\delta_0}} + \|b_{31}(y)\Xi(y_2) + b_{32}(y)\Xi'(y_2)\|_{C^{1-\delta_0}} \Big) \\
\le & \, C\varepsilon(\|W_1\|_{C^{1,1-\delta_0}} + \|W_2\|_{C^{1,1-\delta_0}} + \|W_3\|_{C^{1,1-\delta_0}} + \|\Xi(y_2)\|_{C^{1,1-\delta_0}}). \qquad (3.17)
\end{aligned}
$$

Substituting (3.15) into (3.17) yields

$$
\|W_3\|_{C^{1,1-\delta_0}} \le C\varepsilon(\|W_1\|_{C^{1,1-\delta_0}} + \|W_2\|_{C^{1,1-\delta_0}}). \qquad (3.18)
$$

Next, we treat W_2 by using the equations (3.5) and (3.6).

Since the characteristics $y_2 = y_2^i(y_1, \beta)(i = 1,2)$ starting from the point $(0, \beta)$ satisfies

$$
\begin{cases}
\dfrac{dy_2^i}{dy_1} = \dfrac{(u_2^{+,i} - (f_1' + y_2(f_2' - f_1'))u_1^{+,i})(1 - \zeta_i(y_2))}{(f_2 - f_1)u_1^{+,i} - (1 - y_1)\zeta_i'(y_2)(u_2^{+,i} - (f_1' + y_2(f_2' - f_1'))u_1^{+,i})}, \\
y_2^i(0, \beta) = \beta,
\end{cases} \qquad (3.19)
$$

then we obtain

$$
\begin{aligned}
\|y_2^1(y_1, \beta) - y_2^2(y_1, \beta)\|_{C^{1,1-\delta_0}[0,1;0,1]} \le & \, C(\|W_1\|_{C^{1,1-\delta_0}} + \|W_2\|_{C^{1,1-\delta_0}} + \|W_3\|_{C^{1,1-\delta_0}} + \|\Xi(y_2))\|_{C^{1,1-\delta_0}}) \\
\le & \, C(\|W_1\|_{C^{1,1-\delta_0}} + \|W_2\|_{C^{1,1-\delta_0}} + \|W_3\|_{C^{1,1-\delta_0}}). \qquad (3.20)
\end{aligned}
$$

It follows from (3.6) that W_2 satisfies

$$
\begin{cases}
\partial_{y_1} W_2 = \bar{H}_1(y), \\
\partial_{y_2} W_2 = \bar{H}_2(y), \\
W_2(0,0) = 0.
\end{cases} \qquad (3.21)
$$

A tedious but direct computation shows that $\bar{H}_i(y)$ has such a form

$$\begin{aligned}\bar{H}_i(y) &= d_1^i(y)\partial_{y_1}W_3 + d_2^i(y)\partial_{y_2}W_3 + d_3^i(y)W_3 + d_4^i(y)W_1 + d_5^i(y)W_2 + d_6^i(y)(y_2^1(y_1,\beta) - y_2^2(y_1,\beta)) \\ &\quad + d_7^i(y)\Xi(y_2) + d_8^i(y)\Xi'(y_2) + d_9^i(y)\partial_\beta(y_2^1(y_1,\beta) - y_2^2(y_1,\beta)) + d_{10}^i(y)\big(\partial_2 u_2^{+,2}(0,\beta_2(y)) \\ &\quad - \partial_2 u_2^{+,2}(0,\beta_1(y))\big),\end{aligned}$$

here $\beta_i(y)$ is the inverse function of $y_2 = y_2^i(y_1,\beta)$, $d_k^i(y) \in C^1$ for $1 \le k \le 10$ and

$$\sum_{k=1}^{2} \|d_k^i\|_{C^1} \le C, \qquad \sum_{k=3}^{10} \|d_k^i\|_{C^1} \le C\varepsilon.$$

Thus, taking into account of (3.20), one can obtain from (3.21) that

$$\|W_2\|_{C^{1,1-\delta_0}} \le C(\|\bar{H}_1\|_{C^{1-\delta_0}} + \|\bar{H}_2\|_{C^{1-\delta_0}}) \le C\|W_3\|_{C^{1,1-\delta_0}} + C\varepsilon(\|W_1\|_{C^{1,1-\delta_0}} + \|W_2\|_{C^{1,1-\delta_0}}).$$

For sufficiently small ε, one has

$$\|W_2\|_{C^{1,1-\delta_0}} \le C\|W_3\|_{C^{1,1-\delta_0}} + C\varepsilon\|W_1\|_{C^{1,1-\delta_0}}. \tag{3.22}$$

Finally, we estimate W_1.

By use of the equation (3.4) with (3.5) and the estimates (3.15), (3.20), we obtain

$$\begin{aligned}\|W_1\|_{C^{1,1-\delta_0}} &\le C\|W_3\|_{C^{1,1-\delta_0}} + C\varepsilon\big(\|W_2\|_{C^{1,1-\delta_0}} + \|y_2^1(y_1,\beta) - y_2^2(y_1,\beta)\|_{C^{1,1-\delta_0}} \\ &\qquad + \|\beta_2(y) - \beta_1(y)\|_{C^{1,1-\delta_0}}\big) \\ &\le C\varepsilon\big(\|W_1\|_{C^{1,1-\delta_0}} + \|W_2\|_{C^{1,1-\delta_0}}) + C\|W_3\|_{C^{1,1-\delta_0}}.\end{aligned} \tag{3.23}$$

Combining (3.18), (3.22) with (3.23) yields

$$\|W_1\|_{C^{1,1-\delta_0}} + \|W_2\|_{C^{1,1-\delta_0}} + \|W_3\|_{C^{1,1-\delta_0}} \le C\varepsilon(\|W_1\|_{C^{1,1-\delta_0}} + \|W_2\|_{C^{1,1-\delta_0}} + \|W_3\|_{C^{1,1-\delta_0}}).$$

Thus, for small ε we arrive at

$$W_1 = W_2 = W_3 = 0.$$

It follows from (3.15) that

$$\Xi(y_2) = 0.$$

Therefore, we can obtain $\rho^{+,1}(y) = \rho^{+,2}(y), u_1^{+,1}(y) = u_1^{+,2}(y), u_2^{+,1}(y) = u_2^{+,2}(y)$ and $\zeta_1(y_2) = \zeta_2(y_2)$ immediately. This leads to the proof of Theorem 1.1.

§4. **The proofs of Theorem 1.2 and Theorem 1.3**

Based on Theorem 1.1, in this section we show the ill-posedness results in Theorem 1.2 and Theorem 1.3. We begin with the proof of Theorem 1.2.

Proof of Theorem 1.2 The proof will be divided into two steps.

Step 1. (ρ^+, u_1^+) **and** u_2^+ **can be extended into a** C^2**-smooth functions for** $x_2 \in [-3, 3]$**, which are symmetric and anti-symmetric with respect to** $x_2 = \pm 1$ **respectively, while can be extended into a** C^3**-smooth symmetric function on** $[-3, 3]$**.**

Indeed, for any C^1 solution (ρ^+, u_1^+, u_2^+), the system (1.1) can be rewritten as

$$\begin{cases} \partial_1(\rho^+ u_1^+) + \partial_2(\rho^+ u_2^+) = 0, \\ u_1^+ \partial_1 u_1^+ + u_2^+ \partial_2 u_1^+ + \frac{c^2(\rho^+)}{\rho^+} \partial_1 \rho^+ = 0, \\ u_1^+ \partial_1 u_2^+ + u_2^+ \partial_2 u_2^+ + \frac{c^2(\rho^+)}{\rho^+} \partial_2 \rho^+ = 0. \end{cases} \tag{4.1}$$

Since $(\rho^+(x_1, \cdot), u_1^+(x_1, \cdot), u_2^+(x_1, \cdot)) \in C^2[-1, 1]$, it then follows from the boundary conditions $u_2^+(x_1, \pm 1) = 0$ and the third equation in (4.1) that

$$\partial_2 \rho^+(x_1, \pm 1) = 0. \tag{4.2}$$

By $u_2^\pm(x_1, \pm 1) = 0$ and the third equality in (1.8), one can get

$$\xi'(\pm 1) = 0. \tag{4.3}$$

Differentiating the first equality in (1.8) implies

$$\partial_1[\rho u_1](\xi(x_2), x_2)\xi'(x_2) + \partial_2[\rho u_1](\xi(x_2), x_2) - \xi''(x_2)[\rho u_2](\xi(x_2), x_2) - \xi'(x_2)\partial_2[\rho u_2](\xi(x_2), x_2) = 0. \tag{4.4}$$

Substituting (4.3) and $u_2^\pm(x_1, \pm 1) = 0$ into (4.4) yields

$$\partial_2[\rho u_1](\xi(\pm 1), \pm 1) = 0. \tag{4.5}$$

This, together with (4.2), shows

$$\partial_2 u_1^+(\xi(\pm 1), \pm 1) = 0. \tag{4.6}$$

Furthermore, the second equation in (4.1) yields

$$u_1^+(x_1, \pm 1)\partial_1(\partial_2 u_1^+(x_1, \pm 1)) + (\partial_1 u_1^+ + \partial_2 u_2^+)(x_1, \pm 1)\partial_2 u_1^+(x_1, \pm 1) = 0. \tag{4.7}$$

It follows from (4.7) and (4.6) that

$$\partial_2 u_1^+(x_1, \pm 1) = 0. \tag{4.8}$$

Noting that

$$\partial_1\partial_2(\rho^+ u_1^+) + \partial_2^2 \rho^+ u_2^+ + 2\partial_2 \rho^+ \partial_2 u_2^+ + \rho^+ \partial_2^2 u_2^+ = 0,$$

one obtains from (4.2), (4.8) and $u_2^+(x_1, \pm 1) = 0$ that

$$\partial_2^2 u_2^+(x_1, \pm 1) = 0. \tag{4.9}$$

Next we show $\xi^{(3)}(\pm 1) = 0$.

By the third equality in (1.8) we obtain

$$\partial_2^2\Big([\rho u_1 u_2](\xi(x_2), x_2)\Big) - \xi^{(3)}(x_2)[P(\rho) + \rho u_2^2](\xi(x_2), x_2) - 2\xi''(x_2)\partial_2\Big([P(\rho) + \rho u_2^2](\xi(x_2), x_2)\Big)$$
$$- \xi'(x_2)\partial_2^2\Big([P(\rho) + \rho u_2^2](\xi(x_2), x_2)\Big) = 0. \tag{4.10}$$

We can conclude from (4.2), (4.3), (4.8), (4.9) and (4.10) that

$$\xi^{(3)}(\pm 1) = 0. \tag{4.11}$$

Set

$$\Xi(x_2) = \begin{cases} \xi(2 - x_2), & x_2 \in [1, 3]; \\ \xi(x_2), & x_2 \in [-1, 1]; \\ \xi(-x_2 - 2), & x_2 \in [-3, -1] \end{cases}$$

and

$$U_1^{\pm}(x_1, x_2) = \begin{cases} u_1^{\pm}(x_1, 2 - x_2), & x_2 \in [1, 3]; \\ u_1^{\pm}(x_1, x_2), & x_2 \in [-1, 1]; \\ u_1^{\pm}(x_1, -x_2 - 2), & x_2 \in [-3, -1] \end{cases}$$

and

$$U_2^{\pm}(x_1, x_2) = \begin{cases} -u_2^{\pm}(x_1, 2 - x_2), & x_2 \in [1, 3]; \\ u_2^{\pm}(x_1, x_2), & x_2 \in [-1, 1]; \\ -u_2^{\pm}(x_1, -x_2 - 2), & x_2 \in [-3, -1] \end{cases}$$

and

$$U_3^{\pm}(x_1, x_2) = \begin{cases} \rho^{\pm}(x_1, 2 - x_2), & x_2 \in [1, 3]; \\ \rho^{\pm}(x_1, x_2), & x_2 \in [-1, 1]; \\ \rho^{\pm}(x_1, -x_2 - 2), & x_2 \in [-3, -1] \end{cases}$$

Then it follows from (4.2), (4.3), (4.8) and (4.9) that $(U_1^{\pm}, U_2^{\pm}, U_3^{\pm}) \in C^2$ and $\Xi(x_2)) \in C^3$ is a transonic solution to (1.1) in the domain $[-1, 1; -3, 3]$. Moreover, the shock goes through the point $(x_1^1, -1)$.

Step 2. It holds that $\xi(x_2) = u_2^+(x_1, x_2) \equiv 0$, and $u_1^+(x), \rho^+(x)$ are independent of x_2.

Based on a similar idea as in Step 1 and the uniqueness result in Theorem 1.1, it can be checked easily

$(I)_1$. $(u_1^+(x), \rho^+(x), \xi(x_2))$ is symmetric and $u_2^+(x)$ is anti-symmetrical with respect to $x_2 = 0$ respectively (where $x_2 = 0$ and $x_2 = -1$ are regarded as the fixed walls of the nozzle).

$(I)_2$. $(u_1^+(x), \rho^+(x), \xi(x_2))$ is symmetric and $u_2^+(x)$ is anti-symmetrical with respect to $x_2 = 0, \pm\frac{1}{2}$ respectively (where we regard $x_2 = -\frac{1}{2}$ and $x_2 = -1$ as the fixed walls of the nozzle).

More generally, for any $m \geq 2$ and $m \in \mathbb{N}$, one can show that

$(I)_m$. $(u_1^+(x), \rho^+(x), \xi(x_2))$ is symmetric and $u_2^+(x)$ is anti-symmetrical with respect to $x_2 = \pm\frac{k}{2^m}$ for $k = 0, 1, ..., 2^m - 1$ (where $x_2 = -(1 - \frac{1}{2^m})$ and $x_2 = -1$ are regarded as the fixed walls of the nozzle).

Thus due to the density of $\pm\frac{k}{2^m}(k=0,1,...,2^m-1)$ in $[-1,1]$, one can conclude that

$$u_2^+(x)\equiv 0, \xi(x_2)\equiv 0 \qquad \text{and } u_1^+(x), \rho^+(x) \text{ are independent of } x_2.$$

We now can complete the proof of Theorem 1.2.

Based on Step 2, it follows from the system (4.1) that

$$\rho^+(x_1)u_1^+(x_1)\equiv C_1, \qquad P(\rho^+(x_1))+\rho^+(x_1)(u_1^+(x_1))^2\equiv C_2.$$

In terms of the Rankine-Hugoniot conditions (1.8) we obtain

$$\rho^+(x_1)u_1^+(x_1)\equiv \tilde{\rho}_0\tilde{q}_0, \qquad P(\rho^+(x_1))+\rho^+(x_1)(u_1^+(x_1))^2\equiv P(\tilde{\rho}_0)+\tilde{\rho}_0\tilde{q}_0^2. \tag{4.12}$$

Since $(\tilde{\rho}_0,\tilde{q}_0)\neq(\rho_0,q_0)$, then combining with the entropy condition (1.9) yields $\rho^+(x_1)\neq\rho_+$. This is contradictory with the boundary condition (1.10). Therefore, we complete the proof on Theorem 1.2.

Next, we turn to the proof of Theorem 1.3.

Proof of Theorem 1.3 Introducing the polar coordinates

$$x_1-x_1^0=r\cos\theta, \qquad x_2-x_2^0=r\sin\theta.$$

Assume that $u_1^+(x)=U^+(r,\theta)\cos\theta, u_2^+(x)=U^+(r,\theta)\sin\theta$ and $\rho^+(x)=\rho^+(r,\theta)$, and the shock is denoted by $r=\zeta(\theta)$, then (1.1) can be rewritten as

$$\begin{cases} \partial_r(\rho^+U^+)+\frac{\rho^+U^+}{r}=0, \\ U^+\partial_rU^++\frac{c^2(\rho^+)}{\rho^+}\partial_r\rho^+=0. \end{cases} \tag{4.13}$$

Analogous to the proof in Theorem 1.2, under the assumptions in Theorem 1.3, if the supersonic flow is symmetric in $x_1\geq-\frac{1}{4}$, then one can show that

$$U^+(r,\theta) \quad \text{and} \quad \rho^+(r,\theta) \quad \text{are independent of} \quad \theta, \quad \text{and} \quad \zeta(\theta)=r_0$$

with $r_0=(x_1^1-x_1^0)\sec\alpha_0$.

If the supersonic coming flow is $(\rho_0,q_0\cos\theta,q_0\sin\theta)$ in $r\in[(-\frac{1}{4}-x_1^0)\sec\alpha_0,r_0]$, then the conditions (1.8) imply that

$$\rho^+(r_0)=\rho_+, \qquad U^+(r_0)=q_+. \tag{4.14}$$

Furthermore, we derive from (4.13) that

$$\partial_r\begin{pmatrix} U^+ \\ \rho^+ \end{pmatrix}=\begin{pmatrix} -\frac{U^+c^2(\rho^+)}{r(c^2(\rho^+)-(U^+)^2)} \\ \frac{\rho^+(U^+)^2}{r(c^2(\rho^+)-(U^+)^2)} \end{pmatrix} \tag{4.15}$$

Hence the end density $\rho^+(r)|_{r=(1-x_1^0)\sec\alpha_0}$ is completely determined by (4.14) and (4.15). Thus, Theorem 1.3 is proved.

Remark. 4.1. From (4.15), we know that the boundary condition on ρ^+ at the exit $r = 1$ should satisfy

$$(c^2(\rho^+) - (U^+)^2)\partial_r \rho^+ - \frac{(U^+)^2}{r}\rho^+ = 0. \tag{4.16}$$

Noting that $c^2(\rho^+) - (U^+)^2 > 0$ holds for the subsonic flow, thus the boundary condition (4.16) is not an appropriate one for the second order elliptic equation (2.8). In addition, it is obvious from (4.16) that the transonic problem (1.1) is also ill-posed when the pressure is arbitrarily given at the exit.

§5. **Further Results**

In this final section, we will generalize some of the results discussed in the previous sections. The first main result is that the ill-posedness of Theorem 1.2 can be generalized to the non-isentropic flows, and the second result is that the Courant-Friedrich conjecture holds true for some special nozzle flows.

First, we claim that for nonisentropic Euler system, the transonic shock problem as formulated in §1 is still ill-posed when the pressure at the exit is arbitrarily given. Indeed, we consider the steady full Euler system:

$$\begin{cases} \partial_1(\rho u_1) + \partial_2(\rho u_2) = 0, \\ \partial_1(\rho u_1^2 + P) + \partial_2(\rho u_1 u_2) = 0, \\ \partial_1(\rho u_1 u_2) + \partial_2(\rho u_2^2 + P) = 0, \\ \partial_1((\rho e + \frac{1}{2}\rho|u|^2 + P)u_1) + \partial_2((\rho e + \frac{1}{2}\rho|u|^2 + P)u_2) = 0, \end{cases} \tag{5.1}$$

which is equivalent to

$$\begin{cases} \partial_1(\rho u_1) + \partial_2(\rho u_2) = 0, \\ u_1\partial_1 u_1 + u_2\partial_2 u_1 + \frac{\partial_1 P}{\rho} = 0, \\ u_1\partial_1 u_2 + u_2\partial_2 u_2 + \frac{\partial_2 P}{\rho} = 0, \\ u_1\partial_1 S + u_2\partial_2 S = 0, \end{cases} \tag{5.2}$$

for any C^1-smooth solution (ρ, u_1, u_2, S), where S is the specific entropy. Then we have the following generalization of Theorem 1.2:

Theorem 5.1 (Ill-Posedness) *Assume that the walls of the nozzle are flat, i.e., $f_1(x_1) = (-1)^i$, $(i = 1, 2)$, and the constant supersonic incoming flow $(\tilde{\rho}_0, \tilde{q}_0, 0, S_0)$ and the end pressure $\tilde{\rho}_+(x_2) = \rho_+$ satisfy $(\tilde{\rho}_0, \tilde{q}_0, S_0) \neq (\rho_0, q_0, S_0)$, where $(\rho_0, q_0, 0, S_0)$ and $(\rho_+, q_+, 0, S_+)$ form a normal transonic shock for (5.1). Then the problem (5.1) with boundary conditions (1.7) - (1.11) has no transonic shock solution $(\rho(x), u_1(x), u_2(x), S(x); \xi(x_2))$ such that*

$$(\rho^+(x), u_1^+(x), u_2^+(x), S^+(x); \xi(x_2)) \equiv (\rho, u_1(x), u_2(x), S(x); \xi(x_2))|_{\bar{\Omega}^+},$$

where $\Omega^+ = \Omega \cap \{x_1 > \xi(x_2)\}$, has the following properties:

(i). $\xi(x_2) \in C^3[-1, 1]$, moreover, $\|\xi(x_2) - x_1^1\|_{C^3[-1,1]} \leq C\varepsilon$.

(ii). $(\rho^+, u_1^+, u_2^+, S^+)(x) \in C^2(\bar{\Omega}_+)$ and satisfies

$$\|\rho^+ - \rho_+\|_{C^2} + \|u_1^+ - q_+\|_{C^2} + \|u_2^+\|_{C^2} + \|S^+ - S_+\|_{C^2} \leq C\varepsilon.$$

Sketch of Proof First, in an analogous way as in the proof of Theorem 1.1, one can establish a similar uniqueness result with the corresponding regularity assumptions (see Remark 1.5). Then, based on the uniqueness result, the ill-posedness in Theorem 5.1 can be proved in the same way as for Theorem 1.2 provided one can show that

$$\partial_2\rho^+(x_1,\pm 1)=\partial_2 u_1^+(x_1,\pm 1)=\partial_2 S^+(x_1,\pm 1)=\partial_2^2 u_2^+(x_1,\pm 1)=\xi^{(3)}(\pm 1)=0 \tag{5.3}$$

in order to ensure the required regularity of extended solutions.

We now prove (5.3). On the shock $x_1=\xi(x_2)$, the Rankine-Hugoniot conditions are

$$\begin{cases} [\rho u_1]-\xi'(x_2)[\rho u_2]=0, \\ [P(\rho,S)+\rho u_1^2]-\xi'(x_2)[\rho u_1 u_2]=0, \\ [\rho u_1 u_2]-\xi'(x_2)[P(\rho,S)+\rho u_2^2]=0, \\ [(\rho e(\rho,S)+\frac{1}{2}\rho(u_1^2+u_2^2)+P(\rho,S))u_1]-\xi'(x_2)[(\rho e(\rho,S)+\frac{1}{2}\rho(u_1^2+u_2^2)+P(\rho,S))u_2]=0. \end{cases} \tag{5.4}$$

As in Lemma 3.2, we can derive

$$\xi'(\pm 1)=0. \tag{5.5}$$

Analogous to (3.11), it follows from (5.5) and $u_2^+(x_1,\pm 1)=0$ that

$$\begin{cases} \partial_2[\rho u_1](\xi(\pm 1),\pm 1)=0, \\ \partial_2[P(\rho,S)+\rho u_1^2](\xi(\pm 1),\pm 1)=0, \\ \partial_2[(\rho e(\rho,S)+\frac{1}{2}\rho(u_1^2+u_2^2)+P(\rho,S))u_1](\xi(\pm 1),\pm 1)=0 \end{cases} \tag{5.6}$$

This implies at the points $(\xi(\pm 1),\pm 1)$ that

$$\begin{cases} \rho^+\partial_2 u_1^+ + u_1^+\partial_2\rho^+=0, \\ 2\rho^+u_1^+\partial_2 u_1^+ + (c^2(\rho^+,S^+)+(u_1^+)^2)\partial_2\rho^+ + \partial_S P(\rho^+,S^+)\partial_2 S^+=0, \\ \Big(\rho^+e(\rho^+,S^+)+\frac{3}{2}\rho^+(u_1^+)^2+P(\rho^+,S^+)\Big)\partial_2 u_1^+ + u_1^+\Big(e(\rho^+,S^+)+\rho^+\partial_\rho e(\rho^+,S^+)+\frac{1}{2}(u_1^+)^2 \\ \qquad +\partial_\rho P(\rho^+,S^+)\Big)\partial_2\rho^+ + u_1^+\Big(\rho^+\partial_S e(\rho^+,S^+)+\partial_S P(\rho^+,S^+)\Big)\partial_2 S^+=0. \end{cases} \tag{5.7}$$

For the polytropic gas, the determinant of the coefficient matrix of (5.7) equals to

$$(\rho^+)^2 u_1^+\partial_S e(\rho^+,S^+)(c^2(\rho^+)-(u_1^+)^2)\neq 0,$$

so (5.7) yields:

$$\partial_2 u_1^+(\xi(\pm 1),\pm 1)=\partial_2\rho^+(\xi(\pm 1),\pm 1)=\partial_2 S^+(\xi(\pm 1),\pm 1)=0. \tag{5.8}$$

Next, we prove that $\partial_2\rho^+(x_1,\pm 1)=\partial_2 u_1^+(x_1,\pm 1)=\partial_2 S^+(x_1,\pm 1)=\partial_2^2 u_2^+(x_1,\pm 1)=\xi^{(3)}(\pm 1)=0$ hold.

Due to $u_2^+(x_1,\pm 1)=0$ and the fourth equation in (5.2), we get

$$\partial_1 S^+(x_1,\pm 1)=0.$$

This leads to

$$S^+(x_1,\pm1)\equiv S^+(\xi(\pm1),\pm1). \tag{5.9}$$

Furthermore, it follows from the fourth equation in (5.2) and (5.9) that

$$u_1^+(x_1,\pm1)\partial_1(\partial_2 S^+(x_1,\pm1))+\partial_2 u_2^+(x_1,\pm1)\partial_2 S^+(x_1,\pm1)=0. \tag{5.10}$$

This, together with (5.8), shows

$$\partial_2 S^+(x_1,\pm1)\equiv 0. \tag{5.11}$$

Note that the third equation in (5.2) yields:

$$\left(\frac{c^2(\rho^+,S^+)}{\rho^+}\right)(x_1,\pm1)\partial_2\rho^+(x_1,\pm1)+\left(\frac{\partial_S P(\rho^+,S^+)}{\rho^+}\right)(x_1,\pm1)\partial_2 S^+(x_1,\pm1)=0. \tag{5.12}$$

As a consequence of (5.11) and (5.12), one gets

$$\partial_2\rho^+(x_1,\pm1)\equiv 0. \tag{5.13}$$

Now making use of (5.9), (5.11) and (5.13), and proceeding in the same way as in the proof of Theorem 1.2, we can arrive at

$$\partial_2 u_1^+(x_1,\pm1)=\partial_2^2 u_1^+(x_1,\pm1)=\xi^{(3)}(\pm1)=0.$$

Consequently, the proof of Theorem 5.1 is considered as completed.

Finally, we would like to emphasize that despite the ill-posedness results in Theorem 1.2 - 1.3 and Theorem 5.1, yet there are large class of nozzles and flows such that the Courant-Friedrich's conjecture for transonic shock phenomena in nozzles holds true. Indeed, we consider a two-dimensional nozzle whose divergent part is symmetric for $r\equiv\sqrt{x_1^2+x_2^2}\in(r_1,r_2)$ for some positive constants $r_1<r_2$. Furthermore, we assume that the supersonic incoming is symmetric in r near $r=r_1$, i.e.,

$$(\rho^-,u_1^-,u_2^-,S^-)(x_1,x_2)=(\rho_0(r),U_0(r)\cos\theta,U_0(r)\sin\theta,S_0)\quad\text{for}\quad r\ge r_1, \tag{5.14}$$

where (r,θ) is the plan coordinates in $\mathbb{R}^2$, and S_0 is constant.

For definiteness, we assume that the flow is polytropic. Then we have the following well-posedness result:

Theorem 5.2 *There exist two positive constants P_1 and P_2 with $P_1<P_2$, which are determined by the incoming flow and the nozzle, such that if the constant end pressure P_e satisfies $P_e\in(P_1,P_2)$, then the full Euler system (5.1) has a unique symmetric (i.e. depending only on r) solution. Moreover, the strength and location of the transonic shock is determined uniquely by the end pressure P_e.*

Proof Across the shock $r=r_0$, $r_1\le r_0\le r_2$, the specific entropy S may change. However, we are looking for smooth solution before and after the shock, so the entropies S^- and S^+ are constants respectively. Then, due to the symmetry properties, the full Euler system is equivalent to the following equations:

$$\begin{cases}\dfrac{d}{dr}(r\rho^\pm U^\pm)=0,\\ \dfrac{d}{dr}\left(\dfrac{1}{2}(U^\pm)^2+h(\rho^\pm,S^\pm)\right)=0.\end{cases} \tag{5.15}$$

The Rankine-Hugoniot conditions across the shock at $r = r_0$ are

$$\begin{cases} [\rho U] = 0, \\ [\rho U^2 + P] = 0, \\ \left[\rho\left(\frac{1}{2}U^2 + e\right)U + PU\right] = 0. \end{cases} \tag{5.16}$$

It follows from (5.15) and (5.16) that

$$r\rho^+(r)U^+(r) \equiv m_0 \quad \text{for} \quad r \geq r_0, \quad \text{and} \quad r\rho^-(r)U^-(r) \equiv m_0 \quad \text{for} \quad r \leq r_0 \tag{5.17}$$

with $m_0 = r_1\rho^-(r_1)U^-(r_1)$.

In addition, we derive from (5.15) that

$$\begin{cases} (c^2(\rho^+) - \dfrac{m_0^2}{r^2(\rho^+(r))^2})\dfrac{d\rho^+(r)}{dr} = \dfrac{m_0^2}{r^3\rho^+(r)} \quad \text{for} \quad r \geq r_0, \\ (c^2(\rho^-) - \dfrac{m_0^2}{r^2(\rho^-(r))^2})\dfrac{d\rho^-(r)}{dr} = \dfrac{m_0^2}{r^3\rho^-(r)} \quad \text{for} \quad r \leq r_0. \end{cases} \tag{5.18}$$

and

$$\begin{cases} \dfrac{dU^\pm}{dr} = \dfrac{m_0c^2(\rho^\pm)}{r^2\rho^\pm((U^\pm)^2 - c^2(\rho^\pm))}, \\ \dfrac{d((U^\pm)^2 - c^2(\rho^\pm))}{dr} = \dfrac{\big(2\partial_\rho P(\rho^\pm, S^\pm) + \rho^\pm\partial_\rho^2 P(\rho^\pm, S^\pm)\big)(U^\pm)^2}{r((U^\pm)^2 - c^2(\rho^\pm))}. \end{cases} \tag{5.19}$$

For the polytropic gas, $2\partial_\rho P(\rho^\pm, S^\pm) + \rho^\pm\partial_\rho^2 P(\rho^\pm, S^\pm) > 0$ (see [8]).

Thus, for the subsonic flow in $r \geq r_0$ it holds that $\frac{d\rho^+(r)}{dr} > 0$ and $\frac{dU^+}{dr} < 0$. This implies that $\rho^+(r)$ is an increasing function and $U^+(r)$ is a decreasing function for $r \geq r_0$. Similarly, $\rho^-(r)$ is an decreasing function and $U^-(r)$ is an increasing function for $r \leq r_0$. Moreover, it follows from a direct computation that (5.16) has a unique subsonic state $(\rho^+(r_0), U^+(r_0), S^+(r_0))$ for $r = r_0$. From these properties, one can derive that the supersonic flow and the subsonic flow exist uniquely in $[r_1, r_0]$ and $[r_0, r_2]$ respectively, in particular, $S^+(r) \equiv S^+(r_0)$ holds for $r \geq r_0$.

To show Theorem 5.2, we only need to show that for P_e in an appropriate range, there exists a unique $r_0 \in [r_1, r_2]$ so that the end pressure of the subsonic state (ρ^+, U^+, S^+) is P_e.

In fact, it follows from (5.15) and (5.16) that

$$\begin{cases} r\rho^+(r)U^+(r) \equiv m_0, \\ \frac{1}{2}(U^+(r))^2 + h(\rho^+(r), S^+(r_0)) \equiv B, \end{cases} \tag{5.20}$$

with $h(\rho, S) = e(\rho, S) + \frac{P(\rho,S)}{\rho}$.

In particular,

$$\begin{cases} r_2\rho^+(r_2)U^+(r_2) \equiv m_0, \\ \frac{1}{2}(U^+(r_2))^2 + h(\rho^+(r_2), S^+(r_0)) \equiv B, \end{cases} \tag{5.21}$$

Now we derive a relation between the shock position r_0 and the end pressure $P^+(r_2)$.

Below we should keep in mind that $\rho^+(r_0), U^+(r_0), S^+(r_0), r_0, \rho^+(r_2)$ and $U^+(r_2)$ will be a smooth function of $P^+(r_2)$.

Due to the second law of thermodynamics $de = TdS - Pd(\frac{1}{\rho})$ (here T is the absolute temperature), we derive from the first and the second equations in (5.20) that

$$\begin{cases} d(\rho^+(r_0)U^+(r_0)) = -\rho^+(r_0)U^+(r_0)\frac{dr_0}{r_0}, \\ \rho^+(r_0)U^+(r_0)dU^+(r_0) = -\rho^+(r_0)T^+(r_0)dS^+(r_0) - dP^+(r_2). \end{cases} \tag{5.22}$$

In addition, it follows from the second equation in (5.16) and (5.22) that

$$[\rho U^2](r_0)\frac{dr_0}{r_0} = -\rho^+(r_0)T^+(r_0)dS^+(r_0). \tag{5.23}$$

By the second equation in (5.21) and the state equations of polytropic gas we have

$$dS^+(r_0) = \frac{((U^+(r_2))^2 - c^2(\rho^+(r_2)))}{c^2(\rho^+(r_2))\Big(\rho^+(r_2)T^+(r_2) - (U^+(r_2))^2(\partial_S\rho^+)(r_2)\Big)}dP^+(r_2). \tag{5.24}$$

Thus it follows from (5.23) and (5.24) that

$$[\rho U^2](r_0)\frac{dr_0}{r_0} = \frac{\rho^+(r_0)T^+(r_0)(c^2(r_2) - (U^+(r_2))^2)}{c^2(r_2)\big(\rho^+(r_2)T^+(r_2) - (U^+(r_2))^2(\partial_S\rho^+)(r_2)\big)}dP^+(r_2) \tag{5.25}$$

with $\rho^+(r_2) = \rho(P(r_2), S^+(r_0))$.

Since $\rho^+(r_2)T^+(r_2) - (U^+(r_2))^2\partial_S\rho^+(r_2) > \rho^+(r_2)T^+(r_2) - c^2(r_2)(\partial_S\rho^+)(r_2) = \rho^+(r_2)T^+(r_2) + (\partial_S P^+)(r_2) > 0$ for the polytropic gas, and $[\rho U^2](r_0) < 0$ holds due to $[\rho U^2 + P](r_0) = 0$ and $[P](r_0) > 0$, then we conclude that r_0 is a strictly decreasing function of the end pressure $P^+(r_2)$. When $r_0 = r_1$ or $r_0 = r_2$, it follows from (5.16), (5.18), (5.19) and the analysis above that we can obtain two different pressures P_1 and P_2 with $P_1 < P_2$ (P_1 and P_2 correspond to the end pressures for $r_0 = r_2$ and $r_0 = r_1$ respectively). Therefore, by the monotonicity and continuity of r_0 with respect to the end pressure $P^+(r_2)$, one can obtain the unique symmetric transonic shock for $P^+(r_2) \in (P_1, P_2)$. Namely, Theorem 5.2 is proved.

Remark The uniqueness of the transonic shock in Theorem 5.2 in two space dimensions will be given in our forthcoming paper [24].

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