Recent mathematical results and open problems about Shallow Water equations

Didier Bresch [*], Benoît Desjardins [†], Guy Métivier [‡]

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Abstract

The purpose of this work is to present recent mathematical results about the shallow water model. We will also mention related open problems of high mathematical interest.

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1 Introduction

The shallow water equations are the simplest form of equation of motion that can be used to describe the horizontal structure of the atmosphere. They describe the evolution of an incompressible fluid in response to gravitational and rotational accelerations. The solutions of the shallow water equations represent many types of motion, including Rossby waves and inertia-gravity waves. The aim of this paper is to discuss several problems related to the

[*]LMC–IMAG, (CNRS–INPG–UJF) 38051 Grenoble cedex, France and The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, NT, Hong Kong, email: didier.bresch@imag.fr

[†]CEA/DIF, B.P. 12, 91680 Bruyères le Châtel, France, and E.N.S. Ulm, D.M.A., 45 rue d'Ulm, 75230 Paris cedex 05, France, email: Benoit.Desjardins@cea.fr

[‡]MAB Université Bordeaux 1 351, cours de la Libération F-33405 Talence cedex, email: metivier@math.u-bordeaux.fr

general form of shallow water equations written as follows, on a two dimensional domain Ω:

$$\mathrm{St}\,\partial_t h + \mathrm{div}(hu) = 0, \tag{1}$$

$$\mathrm{St}\,\partial_t(hu) + \mathrm{div}(hu\otimes u) = -h\frac{\nabla(h-b)}{\mathrm{Fr}^2} - h\frac{fu^\perp}{\mathrm{Ro}} + \frac{2}{\mathrm{Re}}\mathrm{div}(hD(u)) + \frac{2}{\mathrm{Re}}\nabla(h\mathrm{div}u) - \frac{1}{\mathrm{We}}h\nabla\Delta h + \mathcal{D}, \tag{2}$$

where $h\in\mathbf{R}$ denotes the height of the free surface, u is the mean horizontal velocity and f a function depending on the latitude y in order to describe the variability of the Coriolis force. In addition, b is a given function and describes the topography of the bottom level of the fluid, $b>0$ on Ω and b can vanish on $\partial\Omega$; moreover, $b\le h$. In the second equation, $\perp$ denotes the direct rotation of angle $\frac{\pi}{2}$, namely $G^\perp = (-G_2, G_1)$ when $G=(G_1,G_2)$. The numbers St, Ro, Re, Fr and We respectively denote the Strouhal number, the Rossby number, the Reynolds number, the Froude number and the Weber number. Some damping terms $\mathcal{D}$ coming from friction may be added or not. We will discuss this point in Section 3. Remark that in 1871, Saint–Venant wrote in a note [1] a system which describes the flow of a river and corresponds to the inviscid shallow water model written here.

Equations (6) and (7), respectively express the conservation of height, momentum energy. System (6)–(7) is supplemented with initial conditions

$$h|_{t=0} = h_0, \qquad (hu)|_{t=0} = q_0, \tag{3}$$

The functions h_0, q_0, are assumed to satisfy

$$h_0 \ge 0 \quad \text{a.e. on} \quad \Omega, \quad \text{and} \quad \frac{|q_0|^2}{h_0} = 0 \quad \text{a.e. on} \quad \{x\in\Omega\,/\,h_0(x)=0\}. \tag{4}$$

The formal derivation of such system from the Navier-Stokes equations with free boundary may be found in [37]. Validity of such approximation will be discussed in the last section.

2 Conservation of potential vorticity

The inviscid case. For the two-dimensional inviscid shallow water equations, the vorticity ω is a scalar, defined as

$$\omega = \partial_x u_2 - \partial_y u_1$$

[1] de Saint Venant, *Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et è l'introduction des marées dans leur lit*, C.R.Ac. Sc. Paris, LXXIII, 1871, 147-154

To derive an evolution equation for ω , we take the curl of the momentum equation divided by h, and use the identity

$$\mathrm{curl}\Big(\mathrm{St}\,\partial_t u+u\cdot\nabla u+f\frac{u^\perp}{\mathrm{Ro}}\Big)=\mathrm{St}\,\partial_t\omega+u\cdot\nabla\omega+\Big(\omega+\frac{f}{\mathrm{Ro}}\Big)\mathrm{div}u+u_2\frac{\partial_y f}{\mathrm{Ro}}.$$

We can eliminate the term involving $\mathrm{div}u$, multiplying the vorticity equation by h and the conservation of mass equation by $\omega+f/\mathrm{Ro}$ and write the difference of the two obtained identities. This yields

$$h\,(\mathrm{St}\,\partial_t+u\cdot\nabla)\Big(\omega+\frac{f}{\mathrm{Ro}}\Big)-\Big(\omega+\frac{f}{\mathrm{Ro}}\Big)(\mathrm{St}\,\partial_t h+u\cdot\nabla h)=0.$$

Therefore :

$$\Big(\mathrm{St}\,\partial_t+u\cdot\nabla\Big)\frac{\omega_R}{h}=0,\quad \omega_R:=\omega+\frac{f}{\mathrm{Ro}}. \tag{5}$$

The quantity ω_R is called the *relative vorticity*. The equation means that the ratio of ω_R and the effective depth h is conserved along the particle trajectories of the flow. This constraint is called the *potential vorticity*. It provides a powerful constraint in large scale motions of the atmosphere. If $\omega+f/\mathrm{Ro}$ is constant initially, the only way that $\omega+f/\mathrm{Ro}$ remains constant at a latter time is if h itself is constant. In the general, the conservation of potential vorticity tells us that if h increases then $\omega+f/\mathrm{Ro}$ must increase, and conversely, if h decreases, then $\omega+f/\mathrm{Ro}$ must decrease.

The viscous case. If h is assumed to be constant equal to 1, $\mathcal{D}=0$ and $f(y)=\beta y$ with β a constant, a system on the relative vorticity $\omega+f$ and stream function is easily written. It reads

$$\mathrm{St}\,\partial_t(\omega+f)+u\cdot\nabla(\omega+f)-\frac{1}{\mathrm{Re}}\Delta(\omega+f)=0,\quad -\Delta\Psi=\omega,\quad u=\nabla^\perp\Psi.$$

When h is not constant, even when it does not depend on time, there are no simple equation for ω_R nor ω_P, analogous to (5). The term

$$\mathrm{curl}\Big(\frac{1}{h}(2\nu\mathrm{div}(hD(u))+\nabla(2h\mathrm{div}u))\Big)$$

generates cross terms between derivatives of h and u. As a consequence, even in the case where $h=b(x)$ is a given function depending only on x, we are not able to get global existence and uniqueness of a strong solution for this viscous model if we allow b to vanish on the shore. The lack of equation for the vorticity also induces problems in the proof of convergence from the viscous case to the inviscid case when the Reynolds number tends to infinity. This is as an interesting open problem. Other interesting questions are to analyze the domain of validity of viscous shallow-water equations in bounded domain, and to determine the relevant boundary conditions when h vanishes on the shore.

3 Leray solutions.

3.1 A new mathematical entropy.

We introduce in this section a new mathematical entropy that has been recently discovered in [11] for more general compressible flows namely Korteweg system. The domain Ω which is considered is either the periodic domain or the whole space. In [11] the capillarity coefficient σ is taken equal to 0, more precisely the Weber number is equal to ∞, and the considered system may be written in the following form

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{6}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) = -\nabla p(\rho) + \mathrm{div}(2\mu(\rho)D(u)) + \nabla(\lambda(\rho)\mathrm{div} u). \tag{7}$$

The energy identity for such system reads

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho|u|^2 + 2\pi(\rho)) + \int_\Omega \Big(2\mu(\rho)|D(u)|^2 + \lambda(\rho)|\mathrm{div} u|^2\Big) = 0,$$

where π denotes the internal energy per unit volume given by

$$\pi(\rho) = \rho \int_{\overline{\rho}}^{\rho} \frac{p(s)}{s^2} ds$$

for some constant reference density $\overline{\rho}$.

In [11], a new mathematical entropy has been discovered that helps to get a great variety of mathematical results about compressible flows with density dependent viscosities. Namely if $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$, then the following equality holds

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(\rho|u + 2\nabla\varphi(\rho)|^2 + 2\pi(\rho)\Big) + \int_\Omega \frac{p'(\rho)\mu'(\rho)}{\rho}|\nabla\rho|^2 + \int_\Omega 2\mu(\rho)|A(u)|^2 = 0$$

where $A(u) = (\nabla u - {}^t\nabla u)/2$ and $\rho\varphi'(\rho) = \mu'(\rho)$. For the reader's convenience, we recall here the alternate proof given in [9].

Proof of the new mathematical entropy identity. Using the mass equation we know that for all smooth function $\xi(\cdot)$

$$\partial_t \nabla \xi(\rho) + (u \cdot \nabla)\nabla \xi(\rho) + \sum_i \nabla u_i \partial_i \xi(\rho) + \nabla(\rho \xi'(\rho) \mathrm{div} u) = 0.$$

Thus, using once more the mass equation, we see that $v = \nabla \xi(\rho)$ satisfies:

$$\partial_t(\rho v) + \mathrm{div}(\rho u \otimes v) + \rho \sum_i \nabla u_i \partial_i \xi(\rho) + \rho \nabla(\rho \xi'(\rho)\mathrm{div} u) = 0$$

which gives, using the momentum equation on u,

$$\begin{aligned}\partial_t(\rho(u+v)) + \mathrm{div}(\rho u \otimes (u+v)) - \mathrm{div}(2\mu(\rho)D(u)) - \nabla(\lambda(\rho)\mathrm{div} u) \\ + \nabla p(\rho) + \rho \sum_i \nabla u_i \partial_i \xi(\rho) + \rho \nabla(\rho \xi'(\rho)\mathrm{div} u) = 0\end{aligned}$$

Next, we write the diffusion term as follows:

$$-\mathrm{div}(2\mu(\rho)D(u)) = -\mathrm{div}(2\mu A(u)) - 2\sum_i \nabla u_i \partial_i \mu - 2\nabla(\mu \mathrm{div} u) + 2\mathrm{divu}\nabla \mu$$

where $A(u) = (\nabla u - {}^t\nabla u)/2$. Therefore, the equation for $u+v$ reads

$$\begin{aligned}\partial_t(\rho(u+v)) + \mathrm{div}(\rho u \otimes (u+v)) - 2\mathrm{div}(\mu(\rho)A(u)) - 2\mu'(\rho)\sum_i \nabla u_i \partial_i \rho \\ -2\nabla(\mu(\rho)\mathrm{div} u) + 2\mu'(\rho)\nabla\rho \mathrm{div} u + \nabla p(\rho) - \nabla(\lambda(\rho)\mathrm{div} u) + \rho\xi'(\rho)\sum_i \nabla u_i \partial_i \rho \\ + \nabla(\rho^2 \xi'(\rho)\mathrm{div} u) - \rho \xi'(\rho)\nabla\rho \mathrm{div} u = 0.\end{aligned}$$

This equation can be simplified under the form

$$\begin{aligned}\partial_t(\rho(u+v)) + \mathrm{div}(\rho u \otimes (u+v)) - 2\mathrm{div}(\mu(\rho)A(u)) + \nabla p(\rho) \\ + \nabla((\rho^2\xi'(\rho) - 2\mu - \lambda)\mathrm{div} u) + (\rho\xi'(\rho) - 2\mu'(\rho))\sum_i \nabla u_i \partial_i \rho \\ + (2\mu'(\rho) - \rho\xi'(\rho))\nabla\rho\, \mathrm{div} u = 0 \end{aligned} \tag{8}$$

If we choose ξ such that $2\mu'(\rho) = \xi'(\rho)\rho$, then $\lambda = \xi'(\rho)\rho^2 - 2\mu$ and the last three terms cancel, which implies:

$$\partial_t(\rho(u+v)) + div(\rho u \otimes (u+v)) - 2\mathrm{div}(\mu(\rho)A(u)) + \nabla p(\rho) = 0.$$

Multiplying this equation by $(u+v)$ and the mass equation by $|u+v|^2/2$ and adding we easily get the new mathematical entropy equality. We just have to observe that

$$\int_\Omega \operatorname{div}(\mu(\rho)A(u))\cdot v = 0,$$

since v is a gradient. The term involving $\nabla p(\rho)$ gives

$$\begin{aligned}\int_\Omega \nabla p(\rho)\cdot(u+v) &= \int_\Omega \rho\nabla\pi'(\rho)\cdot u + \int_\Omega \frac{p'(\rho)\mu'(\rho)}{\rho}|\nabla\rho|^2 \\ &= \frac{d}{dt}\int_\Omega \pi(\rho) + + \int_\Omega \frac{p'(\rho)\mu'(\rho)}{\rho}|\nabla\rho|^2 \end{aligned} \tag{9}$$

where $\pi(\rho) = \rho\int_{\overline{\rho}}^{\rho} p(s)/s^2 ds$ with $\overline{\rho}$ a constant reference density.

We remark that the mathematical entropy estimate gives an extra information on ρ, namely

$$\mu'(\rho)\nabla\rho/\sqrt{\rho} \in L^\infty(0,T;L^2(\Omega))$$

assuming $\mu'(\rho_0)\nabla\rho_0/\sqrt{\rho}_0 \in L^2(\Omega)$ initially.

Such an information is crucial in the analysis of viscous compressible flows with density dependent viscosities and it should help in various cases. Recent applications have been given. For instance, in [38], A. Mellet and A. Vasseur study the stability of isentropic compressible Navier-Stokes equations with barotropic pressure law $p(\rho) = a\rho^\gamma$ with $\gamma > 1$ in dimension $d = 1, 2$ and 3. The diffusive term is assumed under the form $-\operatorname{div}(2\mu(\rho)\nabla u) - \nabla(\lambda(\rho)\operatorname{div} u)$.

An other interesting result concerns the full compressible Navier-Stokes equations. Existence of global weak solutions has been obtained in [10] assuming perfect gas law except close to vacuum where cold pressure is used to get compactness on the temperature. This completes the result recently obtained by E. Feireisl in [22] where the temperature satisfies an inequality instead of an equality in the sense of distributions and where the assumptions on the equation of state prevent from considering perfect polytropic gas laws even away from vacuum. Note that the two above results do not involve the same kind of viscosity laws. Something has therefore to be understood.

We remark that the relation between λ and μ

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)) \tag{10}$$

and the conditions

$$\mu(\rho) \geq c, \qquad \lambda(\rho) + 2\mu(\rho)/d \geq 0$$

can not be fulfilled simultaneously. Indeed the viscosity μ has to vanish in vacuum. In [27], D. Hoff and D. Serre prove that only viscosities vanishing with the density may prevent failure of continuous dependence on initial data for the Navier–Stokes equations of compressible flow. Our relation (10) between λ and μ push to consider such degenerate viscosities. We refer the reader to [38] for interesting mathematical comments on the relation imposed between λ and μ.

Remark on the viscous shallow water equations. We stress that, in the equation written at the beginning of the paper, the viscous term does not satisfy the conditions imposed above. Indeed, we have $\mu = \rho$ but $\lambda \not\equiv 0$. As a consequence, the usual viscous shallow water equations are far from being solved for weak solutions except in 1D where $-2\mathrm{div}(hD(u)) - 2\nabla(h\mathrm{div}u) = -4\partial_x(h\partial_x u)$

Equations with capillarity. Capillarity can be taken into account, as it is observed in [11], introducing a term of the form $-\rho\nabla(G'(\rho)\Delta G(\rho))/\mathrm{We}$ at the right hand side of the momentum equation. It turns out that it suffices to choose a viscosity equal to $\mu(\rho) = G(\rho)$. This gives the mathematical entropy

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(\rho|u + 2\nabla\varphi(\rho)|^2 + 2\pi(\rho) + \frac{1}{\mathrm{We}}|\nabla\mu(\rho)|^2\Big) + \int_\Omega \frac{p'(\rho)\mu'(\rho)}{\rho}|\nabla\rho|^2$$
$$+\frac{1}{\mathrm{We}}\sigma\int_\Omega \mu'(\rho)|\Delta\mu(\rho)|^2 + \int_\Omega 2\mu(\rho)|A(u)|^2 = 0 \qquad (11)$$

where $A(u) = (\nabla u - {}^t\nabla u)/2$ and $\rho\varphi'(\rho) = \mu'(\rho)$. Applications of such mathematical entropy will be provided in [14], to approximations of hydrodynamics. It may also be used to construct suitably smooth sequences of approximate solutions to non capillary models corresponding to the limit of infinite Weber number.

The temperature dependent case. For the full compressible Navier-Stokes equation, the mathematical energy equality reads

$$\frac{d}{dt}\int_\Omega (\frac{1}{2}\rho|u|^2 + \pi(\rho)) + \int_\Omega 2\mu(\rho)|D(u)|^2 + \int_\Omega \lambda(\rho)|\mathrm{div}u|^2 = -\int_\Omega \nabla p\cdot u,$$

and the new mathematical entropy equality reads

$$\frac{d}{dt}\int_\Omega (\frac{1}{2}\rho|u+v|^2 + \pi(\rho)) + \int_\Omega 2\mu(\rho)|A(u)|^2 = -\int_\Omega \nabla p\cdot(u+v)$$

where $A(u) = (\nabla u - {}^t\nabla u)/2$. To close the estimate, it is sufficient to prove that the extra terms $\int_\Omega \nabla p \cdot u$ and $\int_\Omega \nabla p \cdot (u+v)$ can be controlled by the left hand side. This has been done in [10], in which the global existence of weak solutions is proved for the full compressible Navier–Stokes equations with perfect polytropic pressure laws modified by a cold pressure component (at zero temperature) to control density close to vacuum. The proof uses the new mathematical entropy to control the density far from vacuum.

3.2 Weak solutions with drag terms.

When drag terms such as $\mathcal{D} = r_0 u + r_1 h|u|u$ are present in the Saint Venant model, with diffusion term equal to $-2\mathrm{div}(hD(u))$, the existence of global weak solution is proved in [12] without capillarity term ($r_1 > 0$ and $r_0 > 0$). The authors mention that in 1D, r_1 may be taken equal to 0.

Of course such drag terms is helpful from a mathematical view point since it gives the extra information we need on u to prove stability results. Anyway, we will see in the last section that the derivation of the shallow water equations is far from being understood. We will give an example where drag forces appear from the underlying 3D incompressible free boundary Navier–Stokes equations with Dirichlet conditions at the bottom and an example where no drag terms appear when other boundary conditions are considered, namely no slip conditions.

3.3 Dropping drag terms

A. Mellet and A. Vasseur, in [38], show how to ignore the drag terms for the shallow water equations without capillary terms assuming the diffusive term under the form $-\mathrm{div}(h\nabla u)$. This is useful for the mathematical analysis. Indeed, controlling $\nabla\sqrt{\rho}$ implies further information on u assuming this regularity initially. More precisely, multiplying the momentum equation by $|u|^\delta u$ with δ small enough and assuming that $\sqrt{\rho}_0 u_0 \in L^{2+\delta}(\Omega)$, one proves that $\sqrt{\rho}u \in L^\infty(0,T; L^{2+\delta}(\Omega))$ using the estimates given by the new mathematical entropy. This new piece of information is sufficient to pass to the limit in the nonlinear term without the help of drag terms. Note that such technic hold for general μ and λ satisfying the relation needed for the new mathematical entropy.

Important remark. Note that the new mathematical entropy has been used to get stability results for approximate solutions for various models: compressible barotropic Navier–Stokes equations, compressible Navier-Stokes equations with thermal conductivity, Korteweg equations...... In [12], [10] for instance, the authors claim that such approximate solutions may be built to get global existence of weak solutions, i.e. to built actual sequences of suitably smooth approximate solutions. The details will be given in [8] for readers convenience.

4 Strong solutions.

There are several results about the local existence of strong solutions for the shallow water equations written as follows

$$\partial_t h + \mathrm{div}(hu) = 0, \tag{12}$$

$$\partial_t(hu) + \mathrm{div}(hu \otimes u) - \nu \mathrm{div}(h\nabla u) + h\nabla h = 0. \tag{13}$$

Following the energy method of A. Matsumura and T. Nishida, it is natural to show the global (in time) existence of classical solutions to the dissipative shallow water equations on a different domain. The external force field and the initial data being assumed to be small in a suitable space. Such result has been proved in [30], [17]. In [45], a global existence and uniqueness theorem of strong solutions for the initial-value problem for the viscous shallow water equations is established for small initial data and no forcing. Polynomial L^2 and decay rates are established and the solution is shown to be classical for $t > 0$.

More recently, in [47], is studied the Cauchy problem for viscous shallow water equations. The authors work in the Sobolev spaces of index $s > 2$ to obtain local solutions for any initial data, and global solutions for small initial data. The proof is based on Littlewood-Paley decomposition of solutions. The result reads

Theorem 4.1 *Let $s > 0$, u_0, $h_0 - \overline{h_0} \in H^{s+2}(R^2)$, $\|h_0 - \overline{h_0}\|_{H^{s+2}} \ll \overline{h_0}$. Then there exists a positive time T, a unique solution (u, h) of Cauchy problem (12)–(13) such that*

$$u, h - \overline{h_0} \in L^\infty(0, T; H^{2+s}(R^2)), \qquad \nabla u \in L^2(0, T; H^{2+s}(R^2)).$$

Furthermore, there exists a constant c such that if $\|h_0 - \overline{h_0}\|_{H^{s+2}(R^2)} + \|u_0\|_{H^{s+2}(R^2)} \le c$ then we can choose $T = +\infty$.

Nothing has been done so far, using for instance the new mathematical entropy, to obtain better results such as local existence of strong solution with initial data including vacuum. Remark that such a situation is important from a physical point of view: the dam break situation.

5 Other viscous terms in the literature.

In [34], different diffusive terms are proposed, namely $-2\nu\mathrm{div}(hD(u))$, or $-\nu h\Delta u$ or else $-\nu\Delta(hu)$. The reader is referred to [3] for the study of such diffusive terms in the low Reynolds approximation. Let us give some comments around the last two propositions.

Diffusive term equal to $-\nu h\Delta$.

It is shown by P. Gent in [23], that this form, which is frequently used for the viscous adiabatic shallow-water equations, is energetically inconsistent compared to the primitive equations. An energetically consistent form of the shallow-water equations is then given and justified in terms of isopycnal coordinates. This energetical form is exactly the form considered in the present paper. Examples are given of the energetically inconsistent shallow-water equations used in low-order dynamical systems and simplified coupled models of tropical airsea interaction and the E1 Nino-Southern Oscillation phenomena.

From a mathematical point of view, this inconsistency can be easily identified. Let us multiply, formally, the momentum equation by u/h. We get

$$\frac{d}{dt}\int_\Omega |u|^2 + \nu\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u\cdot\nabla|u|^2 + \int_\Omega u\cdot\nabla h = 0.$$

The last term may be written

$$\int_\Omega u\cdot\nabla h = -\int_\Omega \log h\,\mathrm{div}(hu) = \int_\Omega \log h\,\partial_t h = \frac{d}{dt}\int_\Omega (h\log h - h).$$

The third term may be estimated as follows

$$\Big|\int_\Omega |u|^2\mathrm{div}u\Big| \le \|u\|^2_{L^4(\Omega)}\|u\|_{H^1(\Omega)} \le c\|u\|^2_{H^1(\Omega)}\|u\|_{L^2(\Omega)}.$$

Thus if we want to get dissipation, we have look at solutions such that

$$\|u\|_{L^2(\Omega)} < \nu/c,$$

that means sufficiently small velocity solutions. Such an analysis has been performed in [43], looking at solutions of the above system such that u bounded in $L^2(0,T;H^1(\Omega))$, $h\in L^\infty(0,T;L^1(\Omega))$, $h\log h\in L^\infty(0,T;L^1(\Omega))$. The mathematical difficulty is to prove the convergence of sequences of approximate solutions $(\rho_n u_n)_{n\in N}$ in the stability proof. For this, the author uses strongly the fact that h_n and $h_n\log h_n$ are uniformly bounded in

$L^\infty(0,T;L^1(\Omega))$. Using Dunford-Pettis theorem and Trudinger-Moser inequality, he can conclude by a compactness argument.

Remark that using this diffusive term, several papers have been devoted to the low Reynolds approximation namely the following system

$$\partial_t h + \mathrm{div}(hu) = 0, \quad \partial_t u - \Delta u + \nabla h^\alpha = f,$$

with $\alpha \geq 1$. The most recent one, see [33], deals with blow up phenomena, if the initial density contains vacuum, using an uniform bounds with respect to time of the L^∞ norm on the density. It would be interesting to understand what happens without simplification by h in the momentum equation allowing the height to vanish. Remark that such simplification has been also done in [36] to study high rotating and low Froude number limit of inviscid shallow water equations.

Diffusive term equal to $-\nu\Delta(hu)$. Using such diffusion term also gives energetical inconsistency. Only results about the existence of global weak solutions for small data have been obtained.

6 The quasi-geostrophic model.

The well-known quasigeostrophic system for zero Rossby and Froude number flows has been used extensively in oceanography and meteorology for modelling and forecasting mid-latitude oceanic and atmospheric circulation. Deriving this system requires a (singular) perturbation expansion. The quasigeostrophic equation expresses conservation of the zero-order potential vorticity of the flow. In 2D, that means neglecting the stratification, such a model can be derived from the shallow water equations. We just have to choose $\mathrm{Fr} = \mathrm{Ro} = O(\varepsilon)$ and $b = 1 + \varepsilon\eta_b$ with $\eta_b = O(1)$ and let ε go to 0. It yields the following two dimensional system

$$\mathrm{St}\,\partial_t u + \mathrm{div}(u \otimes (u + \eta_b)) = -\mathcal{D} - \nabla p + \frac{1}{\mathrm{Re}}\Delta u - \mathrm{St}\partial_t \Delta^{-1} u, \quad (14)$$

$$\mathrm{div} u = 0, \quad (15)$$

We also note the presence of the new term $\partial_t \Delta^{-1} u$ coming from the free surface, which cannot be derived from the standard rotating Navier–Stokes equations in a fixed domain. To the knowledge of the authors, there exists only one mathematical paper concerning the derivation of such models from the viscous Shallow water equations. It concerns global weak solutions, see [12].

7 The lake equations.

The so-called lake equations arise as the shallow water limit of the rigid lid equations - three dimensional Euler or Navier–Stokes equations with a rigid lid upper boundary condition - in a horizontal basin with bottom topography. It could also be seen as a low Froude approximation of the shallow water equation assuming $b = O(1)$. Neglecting the Coriolis force, it gives

$$\begin{aligned} &\mathrm{St}\,\partial_t(bu) + \mathrm{div}(bu \otimes u) \\ &\qquad\qquad = -b\nabla p + \frac{2}{\mathrm{Re}}\mathrm{div}(bD(u)) + \frac{2}{\mathrm{Re}}\nabla(b\mathrm{div}u), \qquad (16) \\ &\mathrm{div}(bu) = 0, \qquad (17) \end{aligned}$$

7.1 The viscous case.

This model has been formally derived and studied by D. Levermore and B. Samartino, see [32]. In this paper, assuming that the depth b is positive and smooth up to the boundary of Ω, they prove that the system is globally well posed. Note that such model has been used to simulate the currents in Lake Erie. Concerning the boundary conditions, they consider the no-slip boundary conditions

$$u \cdot n = 0, \quad \tau \cdot ((\nabla u + {}^t\nabla u)n) = -\beta u \cdot \tau.$$

where n and τ are the outward unit normal and unit tangent to the boundary. They assume $\beta \geq \kappa$ where κ is the curvature of the boundary.

7.2 The inviscid case.

Let us assume formally that $\mathrm{Re} \to \infty$ to model an inviscid flow and consider a two-dimensional bounded domain Ω. We get the following system

$$\mathrm{St}\,\partial_t(bu) + \mathrm{div}(bu \otimes u) + b\nabla p = 0, \qquad (18)$$

$$\mathrm{div}(bu) = 0, \quad u \cdot n|_{\partial\Omega} = 0. \qquad (19)$$

That means a generalization of the standard two dimensional incompressible Euler equation obtained if $b \equiv 1$.

A strictly positive bottom function. Yudovich's method may be applied using the fact that the relative vorticity ω/b is transported by the flow.

More precisely, the inviscid lake equation may be written using a stream-relative vorticity formulation under the following form

$$\mathrm{St}\,\partial_t\big(\frac{\omega}{b}\big) + u\cdot\nabla\big(\frac{\omega}{b}\big) = 0, \tag{20}$$

$$-\mathrm{div}(\nabla\Psi/b) = \omega, \quad \omega = \mathrm{curl}, \quad \Psi|_{\partial\Omega} = 0. \tag{21}$$

Assuming $b \geq c > 0$ smooth enough, L^p regularity on the stream function Ψ remains true, that is

$$\|\Psi|_{W^{2,p}(\Omega)} \leq Cp\|\omega\|_{L^p(\Omega)}$$

where C does not depend on p. Such elliptic estimates with non-degenerate coefficients comes from [1]-[2]. This result allows D. Levermore, M. Oliver, E. Titi, in [31], to conclude to global existence and uniqueness of strong solutions.

A degenerate bottom function. This is the case when b vanishes on the boundary (the shore). Suppose that φ is a function equivalent to the distance to the boundary, that is $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$, $\Omega = \{\varphi > 0\}$ and $\nabla\varphi \neq 0$ on $\partial\Omega$. Assuming that $b = \varphi^a$ where $a > 0$ and the problem on the stream function may be written under the form

$$-\varphi\Delta\Psi + a\nabla\varphi\cdot\nabla\Psi = \varphi^{a+1}\omega \text{ in } \Omega, \qquad \Psi|_{\partial\Omega} = 0 \tag{22}$$

The case $a = 1$, that is $b = \varphi$, is physically the most natural. This equation belongs to a well known class of degenerate elliptic equations (see [25, 5]).

In [15], the authors prove that, for such degenerate equation, the L^p regularity estimate remains true. The analysis is based on Schauder's estimates solutions of (22) and on a careful analysis of the associated Green function which depends on the degenerate function b.

Using such estimate, the authors are able to follow the lines of the proof given by Yudovitch to get the existence and uniqueness of a global strong solution. Moreover as a corollary, they prove that the boundary condition $u\cdot n|_{\partial\Omega} = 0$ holds on the velocity.

Remark. To the authors' knowledge, there exists only one paper dealing with the derivation of the lake equations from the Euler equations, locally in time, see [42].

Remark. It would be very interesting to investigate the influence of b on properties which are known for the two–dimensional Euler equations, see [4].

An interesting open problem: Open sea boundary conditions. Let us mention here an open problem which have received a lot of attention from applied mathematicians, especially A. KAZHIKHOV. Consider the Euler equations, or more generally the inviscid lake equations, formulated in a two-dimensional bounded domain. When the boundary is of inflow type, all the velocity components are prescribed. Along an impervious boundary, flux vanishes everywhere. This is known as a slip condition. Along a boundary of outflow type, the flux normal to the boundary surface is prescribed for all points of the boundary.

A local existence and uniqueness theorem is proved in a class of smooth solutions by A. KAZHIKHOV in [28]. A global existence theorem is also proved under the assumption that the flow is almost uniform and initial data are small. But the question of global existence and uniqueness in the spirit of V. YUDOVITCH's results remains open, see [48]. In this paper, the author shows how the boundary conditions can be augmented in this more general case to obtain a properly posed problem. Under the additional condition $\mathrm{curl}\, v|_{S^-} = \boldsymbol{\pi}(x,t)$, where $\boldsymbol{\pi}(x,t)$ is—modulo some necessary restrictions—arbitrary, the author shows the existence, in the two-dimensional case, of a unique solution for all time. The method is constructive, being based on successive approximations, and it brings out clearly the physical basis for the additional condition. To understand it in a better way, open sea boundary conditions could be helpful for shallow-water equations for instance with an application to strait of Gibraltar modelling.

8 Multi-level models.

We now give an example where it could be important to write multi-level shallow water equations: it concerns the modelling of the dynamics of water in the Alboran sea and the strait of Gibraltar. In this sea, two layers of water can be distinguished: the surface Atlantic water penetrating into the Mediterranean through the strait of Gibraltar and the deeper, denser Mediterranean water flowing into the Atlantic. This observation shows that, if we want to use two dimensional models to simulate such phenomena, we have to consider at least two layers models. The model which is usually used to study this phenomena considers sea water as composed of two immiscible layers of different densities. In this model, waves appear not only at the surface but also at the interface. It is assumed that for the phenomena under consideration, the wavelength is sufficiently large to make accurate the shallow water approximation in each layer. Therefore the resulting equations form a coupled system of shallow water equations. Concerning viscous bi-layer shallow water equations, to the authors knowledge, only papers with the diffusion $-h\Delta u$ in each layer has been studied, that is the energetically inconsistent one, see [40]. Nothing has been done with the structure studied in the mono-layer case in [12]-[38]. We also note that there are very few mathematical studies concerning the propagation of waves in multi-levels geophysical models.

Shallow water flow with non-constant density. Shallow water equations taking into account non-constant density of the material are subject to investigation. There exists only few results in this direction. They may be seen as perturbations of the known results for the standard shallow water equations. Let us for instance comment on the model studied recently in [26]. This model reads

$$\partial_t(\rho h) + \partial_x(\rho h u) - 0, \tag{23}$$

$$\partial_t(\rho h u) + \partial_x(\rho h u^2 + \tfrac{1}{2}\beta(x)\rho h^2) = \rho h g. \tag{24}$$

where $\rho = h^\alpha$, $\alpha \geq 0$ being a constant, $\beta = \beta(x)$ and $g = g(v, x)$ are given functions. Here h stands for height, ρ for density and v for velocity, and the whole models an avalanche down an inclined slope.

Moreover, the author considers the following particular choices for β and g: $\beta(x) = k\cos(\gamma(x))$, $g(v,x) = \sin(\gamma(x)) - \text{sign}(v)\cos(\gamma(x))\tan(\delta_F(x))$, where δ_F and γ are given functions. Here sign is the sign function, with $\text{sign}(0) = [-1, 1]$. After a change of unknown, this system can be rewritten in the form

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}F(u) \in \tilde{G}(u, x),$$

with essentially the same structure as the system of isentropic gas dynamics in one dimension of space (see for instance [35]), except for the fact that there is an inclusion instead of an equality. A precise (natural) definition of entropy solutions of such systems and a long-time existence theorem of such solutions, under the assumption that the initial height is bounded below by a positive constant, which corresponds to avoiding vacuum may be performed using well-known tools, without additional difficulties.

To propose and study better shallow water models taking into account the density variability of the material could be an interesting research area.

Remark. Using the mathematical entropy estimate, perhaps it could be possible to find a physical viscous and capillary approximation for the 1D

Euler equation replacing the viscous mathematical approximation used in [35], namely

$$\partial_t \rho + \partial_x(\rho u) = \varepsilon \partial_x^2 \rho, \tag{25}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \nabla_x p(\rho) = \varepsilon \partial_x^2(\rho u). \tag{26}$$

9 Derivation of shallow water equations.

Recently D. Coutand, S. Shkoller, see [20]–[21], have written two papers dedicated to the free surface incompressible Euler and Navier-Stokes equations with or without surface tension. We also mention two recent papers dedicated to the formal derivation of viscous shallow water equations from the Navier-Stokes equations with free surface, see [24] for 1D shallow water equations and see [37] for 2D shallow water equations.

It would be interesting to prove mathematically such formal derivations. We make here a few remarks concerning the hypothesis which have been used to derive formally these viscous shallow water equations with damping terms.

First hypothesis. The viscosity is of order ε, meaning that the viscosity is of the same order than the depth, and the asymptotic analysis is performed at order 1.

Second hypothesis. The boundary condition at the bottom for the Navier-Stokes equations is taken using wall laws. Namely the boundary conditions are of the form $(\sigma n)_{\rm tang} = r_0 u$ on the bottom with $u \cdot n = 0$. These boundary conditions can lead to a drag term. (there is no such drag term if $r_0 = 0$).

Dirichlet boundary condition. Suppose that one starts with standard Dirichlet boundary conditions instead of a wall law which by itself is a modelled view of boundary layers near to the bottom. Then we get a linear drag term due to the parabolic profile of the velocity, see [46] and the quadratic term $\partial_x(hu^2)$ is replaced by $6\partial_x(hu^2)/5$.

To conclude, we stress that the rigorous derivation of applicable shallow water equations is far from being understood and a deep mathematical analysis is needed in this direction. Let us mention for the reader convenience some recent works made in that direction by [6], [7] in order to try to propose some generalization of shallow water equations which take into account order one variation in the slope of the bottom. We also mention recent works by J.-P. Vila, see [46], where a precise asymptotic is performed in the description of the velocity profile. Looking at the asymptotic with adherence condition on the bottom and free surface conditions on the surface, he proves that depending on the Ansatz for the horizontal velocity and for the viscosity coefficient, we can formally get various asymptotic inviscid models at the main order.

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References

[1] S. Agmon, A. Douglis, L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations I, *Comm. Pures and Appl. Math.,* 12, (1959), 623-727.

[2] S. Agmon, A. Douglis, L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations II, *Comm. Pures and Appl. Math.,* 17, (1964), 35–92.

[3] C. Bernardi, O. Pironneau. On the shallow water equations at low Reynolds number. *Commun. Partial Diff. Eqs.* 16, 59–104 (1991).

[4] A.L. Bertozzi, A.J. Majda. *Vorticity and incompressible flows.* Cambridge University Press, (2001).

[5] P. Bolley, J. Camus, G. Métivier, Estimations de Schauder et régularité Hölderienne pour une classe de problèmes aux limites singuliers, *Comm. Partial Diff. Equ.* 11 (1986), 1135–1203.

[6] F. Bouchut, A. Mangeney-Castelnau, B. Perthame, J.P. Vilotte. A new model of Saint Venant and Savage-Hutter type for gravity driven shallow water flows. *C.R. Acad. Sci. Paris,* série I, 336(6):531–536, (2003).

[7] F. Bouchut, M. Westdickenberg. Gravity driven shallow water models for arbitrary topography. *Comm. in Math. Sci.,* 2(3):359–389, (2004).

[8] D. Bresch, B. Desjardins. Numerical approximation of compressible fluid models with density dependent viscosity. In preparation (2005).

[9] D. Bresch, B. Desjardins, J.–M. Ghidaglia. On bi-fluid compressible models. In preparation (2005).

[10] D. Bresch, B. Desjardins. Existence globale de solutions pour les équations de Navier–Stokes compressibles complètes avec conduction thermique. *C. R. Acad. Sci.,* Paris, Section mathématiques. Submitted (2005).

[11] D. Bresch, B. Desjardins. Some diffusive capillary models of Korteweg type. *C. R. Acad. Sciences,* Paris, Section Mécanique. Vol **332** no 11 (2004), p 881–886.

[12] D. Bresch, B. Desjardins. Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. *Commun. Math. Phys.,* **238**, 1-2, (2003), p. 211–223.

[13] D. Bresch, M. Gisclon, C.K. Lin. An example of low Mach (Froude) number effects for compressible flows with nonconstant density (height) limit. *M2AN,* Vol. 39, N3, pp. 477-486, (2005).

[14] D. Bresch, A. Juengel, H.-L. Li, Z.P. Xin. Effective viscosity and dispersion (capillarity) approximations to hydrodynamics. Forthcoming paper, (2005).

[15] D. Bresch, G. Métivier. Global existence and uniqueness for the lake equations with vanishing topography : elliptic estimates for degenerate equations. To appear in *Nonlinearity,* (2005).

[16] D. Bresch, B. Desjardins, C.K. Lin. On some compressible fluid models: Korteweg, lubrication and shallow water systems. *Comm. Partial Differential Equations,* **28**, 3–4, (2003), p. 1009–1037.

[17] A.T. Bui. Existence and uniqueness of a classical solution of an initial boundary value problem of the theory of shallow waters. *SIAM J. Math. Anal.* 12 (1981) 229-241.

[18] J.F. Chatelon, P. Orenga. Some smoothness and uniqueness for a shallow water problem. *Adv. Diff. Eqs,* 3, 1 (1998), 155-176.

[19] C. Cheverry. Propagation of oscillations in Real Vanishing Viscosity Limit, *Commun. Math. Phys.* 247, 655-695 (2004).

[20] D. Coutand, S. Shkoller. Unique solvability of the free-boundary Navier-Stokes equations with surface tension, *Arch. Rat. Mech. Anal.* (2005).

[21] D. Coutand, S. Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension , Submitted (2005).

[22] E. Feireisl. *Dynamics of viscous compressible fluids.* Oxford Science Publication, Oxford, (2004).

[23] P.R. Gent. The energetically consistent shallow water equations. *J. Atmos. Sci.,* 50, 1323-1325, (1993).

[24] J.F. Gerbeau, B. Perthame. Derivation of Viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation, Discrete and Continuous Dynamical Systems, Ser. B, Vol. 1, Num. 1, 89–102, (2001).

[25] C. Goulaouic, N. Shimakura, Régularité Höldérienne de certains problèmes aux limites dégénérés, *Ann. Scuola Norm. Sup. Pisa,* 10, (1983), 79–108.

[26] P. Gwiazda. An existence result for a model of granular material with non-constant density. *Asymptotic Analysis,* **30**,

[27] D. Hoff, D. Serre. The failure of continuous dependence on initial data for the Navier-Stokes equations of compressible flow. *SIAM J. Appl. Math.,* 51(4):887898, (1991).

[28] A.V. Kazhikhov, Initial-boundary value problems for the Euler equations of an ideal incompressible fluid. *Moscow Univ. Math. Bull.* 46 (1991), no. 5, 10–14.

[29] A.V. Kazhikhov, A. Veigant. Global solutions of equations of potential fluids for small Reynolds number. *Diff. Eqs,* 30, (1994), 935–947.

[30] P.E. Kloeden, Global existence of classical solutions in the dissipative shallow water equations. *SIAM J. Math. Anal.* 16 (1985), 301–315.

[31] D. Levermore, M. Oliver, E.S. Titi, Global well-posedness for models of shallow water in a basin with a varying bottom, *Indiana Univ. Math. J.* 45 (1996), 479-510.

[32] D. Levermore, B. Sammartino. A shallow water model in a basin with varying bottom topography and eddy viscosity, *Nonlinearity,* Vol.14, n.6, 1493-1515 (2001).

[33] J. Li, Z.P. Xin. Some Uniform Estimates and Blowup Behavior of Global Strong Solutions to the Stokes Approximation Equations for Two-Dimensional Compressible Flows. To appear in *J. Diff. Eqs.* (2005)

[34] P.–L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models.* Oxford Science Publication, Oxford, (1998).

[35] P.-L. Lions, B. Perthame, P. E. Souganidis, Existence of entropy solutions to isentropic gas dynamics System. *Comm. Pure Appl. Math.* 49 (1996), no. 6, 599–638.

[36] A. Majda. *Introduction to PDEs and waves for the atmospher and ocean.* Courant lecture notes in Mathematics, (2003).

[37] F. Marche. Derivation of a new two-dimensional shallow water model with varying topography, bottom friction and capillary effects. Submitted (2005).

[38] A. Mellet, A. Vasseur. On the isentropic compressible Navier-Stokes equation. Submitted (2005).

[39] L. Min, A. Kazhikhov, S. Ukai. Global solutions to the Cauchy problem of the Stokes approximation equations for two-dimensional compressible flows. *Comm. Partial Diff. Eqs,* 23, 5-6, (1998), 985–1006.

[40] M.L. Muoz-Ruiz, F.–J. Chatelon, P. Orenga. On a bi-layer shallow-water problem. *Nonlinear Anal. Real World Appl.* 4 (2003), no. 1, 139–171.

[41] A. Novotny, I. Straskraba. *Introduction to the mathematical theory of compressible flow.* Oxford lecture series in Mathematics and its applications, (2004).

[42] M. Oliver, Justification of the shallow water limit for a rigid lid flow with bottom topography, *Theoretical and Computational Fluid Dynamics* 9 (1997), 311-324.

[43] P. Orenga. Un théorème d'existence de solutions d'un problème de shallow water, *Arch. Rational Mech. Anal.* 130 (1995) 183-204

[44] L. Sundbye. Existence for the Cauchy Problem for the Viscous Shallow Water Equations. *Rocky Mountain Journal of Mathematics,* 1998, 28 (3), 1135-1152.

[45] L. Sundbye. Global existence for Dirichlet problem for the viscous shallow water equations. *J. Math. Anal. Appl.* 202 (1996), 236–258.

[46] J.–P. Vila. Shallow water equations for laminar flows of newtonian fluids. Paper in preparation and private communication, (2005).

[47] W. Wang, C.-J. Xu. The Cauchy problem for viscous shallow water equations *Rev. Mat. Iberoamericana* 21, no. 1 (2005), 1-24.

[48] V.I. Yudovich. The flow of a perfect, incompressible liquid through a given region. *Soviet Physics Dokl.* 7 (1962) 789–791.