Periodic Bifurcation and Soliton deflexion for Kadomtsev-Petviashvili Equation

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Abstract

The exact two-soliton solution to KPI and a class of periodic soliton solutions to KPII are obtained by using bilinear form and variable translation. At the same time, making a temporal and spatial transformation induces the doubly periodic solution to KPI and another class of periodic soliton solution to KPII. It is also investigated that the equilibrium solution \( u_0 = -\frac{1}{6} \) is an unique bifurcation which is periodic bifurcation for KPI and deflexion of soliton for KPII.

Key Words
periodic bifurcation, soliton deflexion, Kadomtsev-Petviashvili Equation

1. Introduction

It is well established now that the Kadomtsv-Petviashvili(KP) equation is the key ingredient in a number of remarkable nonlinear problems, both in physics and mathematics [1,2]. It was derived initially to examine small effects in a direction perpendicular to the propagation direction would have on a KdV soliton in a plasma[3]. It is only weakly dependent on the transverse coordinate and has been frequently stated to apply to one and one-half dimensions.

The solutions of KP equation have been extensively studied since they were first found. Various methods had been tried and many special solutions were given[4-9]. A prominent feature of the KP equation is that it admits an interaction and resonance of several individual solitons resulting in a drastic increase of the amplitude if the solitons in the interaction region[10-12].

In this paper, several types of exact solutions to KP equation were constructed by bilinear form and variable translation. It is explicitly analyzed that the feature of the solutions is different on the both sides of equilibrium solution \( u_0 = -\frac{1}{6} \), which is a unique periodic bifurcation point for KPI and deflexion point of soliton for KPII. As for KPI, when the equilibrium \( u_0 \) varies from one side of \( -\frac{1}{6} \) to another side, two-soliton solution changes into doubly periodic solution. Whereas, the \( y \)-periodic soliton changes into \( x - t \)-periodic soliton for KPII.

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2. The two-solitons to KPI and periodic solutions to KPIII

The Kadomtsev-Petviashvili equation in normalised variable \((u, x, y, t)\) reads

\[
u_{xt} - u_{xxxx} - 3(u^2)_{xx} - p^2 u_{yy} = 0
\]

(1)

where \(u : \mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_t \rightarrow \mathbb{R}\). Eq(1) is called KPI when \(p^2 = 1\), and KPII when \(p^2 = -1\). It is easily to note that \(u = u_0\) is an equilibrium solution of KP equation, where \(u_0\) is an arbitrary constant.

Now consider KPI equation

\[
u_{xt} - u_{xxxx} - 3(u^2)_{xx} - u_{yy} = 0
\]

(2)

Setting \(\xi = i(x - t)\) in eq(2) gives

\[
u_{yy} - u_{\xi\xi} - 3(u^2)_{\xi\xi} + u_{\xi\xi\xi\xi} = 0
\]

(3)

Let \(u = u_0 + 2v\), \(v = -(\ln F)\xi\), then (3) can be reduced into the following bilinear form

\[
[D_y^2 - (1 + 6u_0)D_x^2 + D_t^4 - A] F \cdot F = 0
\]

(4)

where \(A\) is an integration constant and the Hirota bilinear operator \(D_x^m D_t^n\) are defined by ref.[4]

\[
D_x^m D_t^n a \cdot b = (\frac{\partial}{\partial x})^m (\frac{\partial}{\partial t})^n a(x, t)b(x, t)|_{x'=x, t'=t}
\]

With regard to (4), we are going to seek the solution of the form

\[
F = 1 + b_1 (e^{ip\xi} + e^{-ip\xi}) e^{\Omega y + \gamma} + b_2 e^{2\Omega y + 2\gamma}
\]

(5)

where \(A, p, \Omega, \gamma, b_1\) and \(b_2\) are all real.

Substituting (5) into (4) yields the exact solution of eq(3) of the form

\[
u = u_0 + \frac{2p^2[4b_1^2 + b_1(e^{ip\xi} + e^{-ip\xi})(b_2 e^{\Omega y + \gamma} + e^{-\gamma})]}{[b_1(e^{ip\xi} + e^{-ip\xi}) + (b_2 e^{\Omega y + \gamma} + e^{-\gamma})]^2}
\]

(6)

where coefficients satisfy

\[
A = 0 \quad \Omega^2 = -p^4 - (1 + 6u_0)p^2 \quad b_1^2 = \frac{\Omega^2 b_2}{\Omega^2 - 3p^4}
\]

(7)

It is obviously that \(u_0 < -\frac{1}{6}\) is required so that the conditions \(\Omega^2 > 0\), \(b_1^2 > 0\) and \(p^2 < -\frac{1+6u_0}{3}\) can be satisfied in eq(7), in this case \(u_0\) is considered as a parameter.

Taking \(\xi = i(x - t)\) into eq(6), the exact solution to KPI equation is expressed by

\[
u_1(x, y, t) = u_0 + \frac{2p^2[4b_1^2 + b_1(e^{ip(x-t)} + e^{-ip(x-t)})(b_2 e^{\Omega y + \gamma} + e^{-\gamma})]}{[b_1(e^{ip(x-t)} + e^{-ip(x-t)}) + (b_2 e^{\Omega y + \gamma} + e^{-\gamma})]^2}
\]

(8)

where

\[
\begin{cases}
u_0 < -\frac{1}{6} \\ p^2 < -\frac{1+6u_0}{4} \\ \Omega^2 = -p^4 - (1 + 6u_0)p^2 \\ b_1^2 = \frac{\Omega^2 b_2}{\Omega^2 - 3p^4}
\end{cases}
\]

(9)
According to expression eq(8) of $u_1$, the two-soliton solution to KPI is obtained. (ref Fig.1)

The KPII equation is given by

$$u_{xt} - u_{xxxx} - 3(u^2)_{xx} + u_{yy} = 0$$

(10)

Making a variable transformation $\xi = x - t$, eq(10) can be transformed into the following form

$$u_{yy} - u_{\xi\xi} - 3(u^2)_{\xi\xi} + u_{\xi\xi\xi\xi} = 0$$

(11)

Letting $u = u_0 + 2v, \ v = (\ln F)_{\xi\xi}$, being similar to the way of dealing with KPI, we take

$$F = 1 + b_1(e^{ip\xi} + e^{-ip\xi})e^{\Omega y + \gamma} + b_2 e^{2\Omega y + 2\gamma}$$

By computing, the exact solution of (11) is given by

$$u = u_0 - \frac{2p^2[4b_1^2 + b_1(e^{ip\xi} + e^{-ip\xi})(b_2e^{\Omega y + \gamma} + e^{-\Omega y - \gamma})]}{[b_1(e^{ip\xi} + e^{-ip\xi}) + (b_2e^{\Omega y + \gamma} + e^{-\Omega y - \gamma})]^2}$$

(12)

where coefficients satisfy

$$\Omega^2 = p^4 - (1 + 6u_0)p^2 \quad b_1^2 = \frac{\Omega^2 b_2}{\Omega^2 + 3p^4}$$

(13)

It is easily to see that $u_0 \geq -\frac{1}{6}$ is available as long as $p^2 \geq 1 + 6u_0$.

Taking $\xi = x - t$ into eq(12), the exact solution to KPII is expressed

$$u_2(x, y, t) = u_0 - \frac{2p^2[4b_1^2 + b_1(e^{ip(x-t)} + e^{-ip(x-t)})(b_2e^{\Omega y + \gamma} + e^{-\Omega y - \gamma})]}{[b_1(e^{ip(x-t)} + e^{-ip(x-t)}) + (b_2e^{\Omega y + \gamma} + e^{-\Omega y - \gamma})]^2}$$

(14)

where

$$\begin{cases} 
  u_0 \geq -\frac{1}{6} \\
  p^2 \geq 1 + 6u_0 \\
  \Omega^2 = p^4 - (1 + 6u_0)p^2 \\
  b_1^2 = \frac{\Omega^2 b_2}{\Omega^2 + 3p^4} 
\end{cases}$$

(15)
Obviously, the $\cos p(x-t)$ is periodic, so the solution given by (14) is a periodic soliton solution with $x-t$-direction. (ref. fig.2)

Figure 2: The $x-t$-periodic soliton solution for KPI equation as $u_0 = -\frac{1}{8}$

3. The doubly periodic solution to KPI and $y$-periodic soliton solution to KPII

Comparing (3) with (11), it is easily to find that the eq(3) may be changed into eq(11) and vice versa by using the temporal and spatial transformation $(\xi, y) \rightarrow (i\xi, iy)$. Because the solutions of KP equation are real functions, it is naturally to take specially $b_2 = 1$, $\gamma = 0$. And, making variable transformation $\xi \rightarrow i\xi$, $y \rightarrow iy$ in eq(6) and eq(12) yields

$$u = u_0 \pm \frac{2p^2[4b_1^2 + b_1(e^{ip(x-t)} + e^{-ip(x-t)})(e^{i\Omega y} + e^{-i\Omega y})]}{[b_1(e^{ip(x-t)} + e^{-ip(x-t)}) + (e^{i\Omega y} + e^{-i\Omega y})]^2}(16)$$

Hence, the doubly periodic solution to KPI equation is obtained readily

$$u_3(x, y, t) = u_0 - \frac{2p^2[4b_1^2 + b_1(e^{ip(x-t)} + e^{-ip(x-t)})(e^{i\Omega y} + e^{-i\Omega y})]}{[b_1(e^{ip(x-t)} + e^{-ip(x-t)}) + (e^{i\Omega y} + e^{-i\Omega y})]^2}(17)$$

i.e.

$$u_3(x, y, t) = u_0 - \frac{2p^2[b_1^2 + b_1 \cos(p(x-t)) \cos(\Omega y)]}{[b_1 \cos(p(x-t)) + \cos(\Omega y)]^2}(18)$$

where

$$\begin{align*}
u_0 &\geq -\frac{1}{6} \\
p^2 &\geq 1 + 6u_0 \\
\Omega^2 &\equiv p^4 - (1 + 6u_0)p^2 \\
b_1^2 &\equiv \frac{\Omega^2 - \Omega^2 y^2}{1 + 3p^2}
\end{align*}(19)$$

It is noted that the solution given by eq(18) is a singular periodic solution to KPI equation. In order to avoid the singularity, we set $\cos(p(x-t)) > 0$ and $\cos(\Omega y) > 0$. (ref. fig.3)

Besides, the $y$-periodic soliton solution to KPII is also given by

$$u_4(x, y, t) = u_0 + \frac{2p^2[4b_1^2 + b_1(e^{ip(x-t)} + e^{-ip(x-t)})(e^{i\Omega y} + e^{-i\Omega y})]}{[b_1(e^{ip(x-t)} + e^{-ip(x-t)}) + (e^{i\Omega y} + e^{-i\Omega y})]^2}(20)$$
Figure 3: (a) The doubly periodic solution for KPI equation with $x - t$ direction as $u_0 = -\frac{1}{6}$
(b) The doubly periodic solution for KPI equation with $y$ direction as $u_0 = -\frac{1}{8}$

where

$$
\begin{align*}
\left\{ 
\begin{array}{l}
u_0 < -\frac{1}{6} \\p^2 < -\frac{1 + 6u_0}{4} \\\Omega^2 = -p^4 - (1 + 6u_0)p^2 \\
b_1^2 = \frac{\Omega^2 p^2}{1 - 3p^2}
\end{array}
\right.
\end{align*}
$$

(21)

The solution given by (20) represents periodic soliton with $y$-direction. (ref.fig.4)

Figure 4: The $y$-periodic soliton solution for KPII equation as $u_0 = -\frac{1}{4}$

According to above discuss, we draw a conclusion that $u_0 = -\frac{1}{6}$ is a unique periodic bifurcation point for KPI and deflexion of soliton for KPII. Around the both sides at $u_0$, the property of solutions to KPI and KPII is all changed. As for KPI, when the equilibrium $u_0$ varies from one side of $-\frac{1}{6}$ to another side, two-soliton solution changes into doubly periodic solution. Whereas, the $y$-periodic soliton changes into $x$-periodic soliton for KPII. The two-soliton waves and doubly periodic soliton waves of KPI, periodic soliton waves on different spatial variable of KPII are interchanged around $u_0$. (ref.fig.5, fig.6)
Figure 5: (a) (b)
(a) The doubly periodic solution for KPI equation as $u_0 = -\frac{1}{6}$
(b) The $x - t$-periodic soliton solution for KPII equation as $u_0 = -\frac{1}{6}$

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Reference