# Global Subsonic and Subsonic-Sonic Flows through Infinitely Long Nozzles*

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**Abstract:** In this paper, we study global subsonic and subsonic-sonic flows through a general infinitely long nozzle. First, it is proved that there exists a critical value for the incoming mass flux so that a global uniformly subsonic flow exists in the nozzle as long as the incoming mass flux is less than the critical value. More importantly, we establish some uniform estimates for the deflection angles of the subsonic flows and the monotonicity of the maximum of the flow speed with respect to the incoming mass flux by combining hodograph transformation, partial hodograph transformation, and the comparison principle for elliptic equations. With the help of these properties and a compensated compactness framework, we get the existence of a global subsonic-sonic flow solution in the case of the critical incoming mass flux.

**Key Words:** subsonic flow, boundary gradient estimate, Hölder gradient estimate, comparison principle, partial hodograph transformation, hodograph transformation, compensated compactness, subsonic-sonic flow.

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# 1 Introduction and Main Results

Mathematical theory of multidimensional compressible fluid flows give rise to many outstanding challenging problems. There are a lot of experiments and numerical simulations involved in this field. For the global potential subsonic flow, one of the most significant progress was due to Bers[2], who showed that for two dimensional flow past a profile, if the Mach number of the freestream is small enough, then the whole flow field will be subsonic outside the profile; furthermore, as the freestream Mach number increases, the maximum of flow speed will tend to the sound speed. Later on, Finn and Gilbarg[8] showed uniqueness of subsonic flow past a profile by maximum principles and asymptotic behavior of flows at far field. For the three dimensional flows, it was studied initially by Finn and Gilbarg[9], and then by Dong[6], the final results are quite similar to the two dimensional case, that a subsonic flow exists globally if the freestream Mach number is suitably small, moreover, the maximum of the flow speed will tend to the sound speed if the freestream Mach number increases to some critical value.

We note that Bers' result does not apply to the flow with the critical freestream Mach number. In fact, by the maximum principle, Gilbarg and Shiffman[10] asserts that the sonic point should occur on the profile, which presupposed the existence of the smooth critical flows. In this regards, Gilbarg and Shiffman[10] remarked in footnote 8: "The actual existence of critical flows past finite profiles of bounded curvature has been proved by M. Shiffman (unpublished)". Bers also mentioned this unpublished result in [3]. However, so far, there are no detailed and precise rigorous proof to refer to.

On the other hand, for flows through an infinitely long nozzle, so far it does not have a complete theory as what had been obtained by Bers, et al, for flow past an obstacle. In his famous survey, Bers[3] proposed the following problem, for the given nozzle, show that there is a global subsonic flow through the nozzle for an appropriately given incoming mass flux. Although it seems that this problem is quite similar to the airfoil problem physically, however, it does not seem to be true mathematically. As Bers said in his book[3], "No proof, however, has yet been carried out along these lines". One of the aims of this paper is to give a positive answer to this problem. Moreover, we would like to show that there exists a critical value such that a global uniform subsonic flow exists uniquely in a general nozzle as long as the incoming mass flux is less than the critical value. More importantly, we would like to investigate the properties of these uniform subsonic flows, in particular, the dependence of the flow speed on the incoming mass flux, so that we can obtain a class of subsonic-sonic flows corresponding to the critical incoming mass flux as the limits of uniform subsonic flows associated with the incoming mass fluxes which increase to the critical value.

It should be noted there have been some studies related to subsonic flow problems since 1980's for nozzles of finite length, (see [7] and references therein), whose physical significance, however, is not clear.

To describe the problem mathematically, let us consider two dimensional steady, isentropic, compressible Euler equations

$$\begin{cases} (\rho u)_{x_1} + (\rho v)_{x_2} = 0, \\ (\rho u^2)_{x_1} + (\rho u v)_{x_2} + p_{x_1} = 0, \\ (\rho u v)_{x_1} + (\rho v^2)_{x_2} + p_{x_2} = 0, \end{cases} \tag{1}$$

where $\rho$ is the density, $(u, v)$ is the velocity, and $p = p(\rho)$ denotes the pressure. In general, we assume that $p'(\rho) > 0$ for $\rho > 0$ and $p''(\rho) \geq 0$, where $c(\rho) = \sqrt{p'(\rho)}$ is called sound speed. The most important examples include polytropic gases and isothermal gases, for polytropic gases, $p = A\rho^\gamma$ where $A$ is a constant and $\gamma$ is the adiabatic constant with $\gamma > 1$; and for isothermal gases, $p = c^2\rho$ with constant sound speed $c$.

Suppose that the flow is also irrotational, i.e.

$$u_{x_2} = v_{x_1}. \tag{2}$$

Then it is easy to deduce the following Bernoulli's law[5],

$$\frac{q^2}{2} + h = \frac{\hat{q}^2}{2}, \tag{3}$$

where $h = h(\rho)$ is the enthalpy satisfying $h'(\rho) = c^2(\rho)/\rho$, $q = \sqrt{u^2 + v^2}$ is the flow speed, and $\hat{q}$ is a constant. With the aid of Bernoulli's law (3), we would like to point out some useful and important facts for the flow[5]. First, $\rho$ is a decreasing function of $q$, attains its maximum at $q = 0$. Second, there is a critical speed $q_c$ such that $q < c$(subsonic) if and only if $q < q_c$. Finally, $\rho q$ is a nonnegative function of $q$, for $q \geq 0$, which is increasing for $q \in (0, q_c)$ and decreasing for $q \geq q_c$, and vanishes at $q = 0$. so $\rho q$ attains its maximum at $q = q_c$, therefore, that the flow is subsonic is equivalent to $\rho q < \rho_c q_c$ and $\rho > \rho_c$.

Using the critical speed, one can introduce the nondimensionalized velocity, density and pressure as

$$\tilde{u} = \frac{u}{q_c},\ \tilde{v} = \frac{v}{q_c},\ \tilde{\rho} = \frac{\rho}{\rho_c},\ \tilde{q} = \frac{q}{q_c} \text{ and } \tilde{p}(\tilde{\rho}) = \frac{p(\tilde{\rho}\rho_c)}{\rho_c q_c^2}.$$

With an abuse of notation, we will take $\tilde{}$ away, and just regard $\rho$, $u$ , $v$, $q$ and $p$ as the nondimensionalized quantities. Then Euler equations (1) become

$$\begin{cases} (\rho u)_{x_1} + (\rho v)_{x_2} = 0, \\ (\rho u^2)_{x_1} + (\rho uv)_{x_2} + p_{x_1} = 0, \\ (\rho uv)_{x_1} + (\rho v^2)_{x_2} + p_{x_2} = 0. \end{cases} \tag{4}$$

Note that the nondimensionalized pressure also satisfies that $p'(\rho) > 0$ for $\rho > 0$, $p'(1) = 1$, and $p''(\rho) \geq 0$ for $\rho \geq 0$. For example, one has $p = \rho^\gamma/\gamma$ for polytropic gases and $p = \rho$ for isothermal gases. At the same time, Bernoulli's law (3) reduces to

$$\frac{q^2}{2} - \frac{1}{2} + \int_1^\rho \frac{p'(s)}{s} ds = 0. \tag{5}$$

By this Bernoulli's law, one can represent $\rho = \rho(q^2)$ by the implicit function theorem. With this nondimensionalization, it is easy to see that $\rho q \leq 1$ for $q \geq 0$ and that subsonic flow means $q < 1$ or $\rho > 1$. For example, for polytropic gases, (5) is nothing but

$$\frac{q^2}{2} + \frac{\rho^{\gamma-1}}{\gamma - 1} = \frac{\gamma + 1}{2(\gamma - 1)}, \tag{6}$$

which yields

$$\rho = \rho(q^2) = (\frac{\gamma + 1 - (\gamma - 1)q^2}{2})^{\frac{1}{\gamma-1}}; \tag{7}$$

in the case of isothermal gases, instead of (7), one has

$$\rho = \rho(q^2) = \exp(\frac{1 - q^2}{2}). \tag{8}$$

Based on the continuity equation, the stream function $\psi$ can be introduced such that

$$\psi_{x_1} = -\rho v, \qquad \psi_{x_2} = \rho u. \tag{9}$$

Obviously, $|\nabla\psi| = \rho q$, therefore, $\rho$ is a two-valued function of $|\nabla\psi|^2$. Subsonic flow is corresponding to the branch where $\rho > 1$ if $|\nabla\psi|^2 \in [0,1)$. Set $\rho = H(|\nabla\psi|^2)$ such that $\rho > 1$ if $|\nabla\psi|^2 \in [0,1)$, therefore, $H$ is a positive decreasing function defined on $[0,1]$, twice differentiable on $[0,1)$, and satisfies $H(1) = 1$. For example, for polytropic gases, $\rho$ can be solved from the following equation,

$$\frac{\rho^{\gamma-1}}{\gamma-1} + \frac{|\nabla\psi|^2}{2\rho^2} = \frac{\gamma+1}{2(\gamma-1)}. \tag{10}$$

Now, the irrotationality (2) reduces to a single equation

$$\mathrm{div}\left(\frac{\nabla\psi}{H(|\nabla\psi|^2)}\right) = 0. \tag{11}$$

For flows passing through a nozzle, when the nozzle walls are impermeable solid walls, the boundary conditions are given by

$$(u,v)\cdot\overrightarrow{n} = 0, \tag{12}$$

where $\overrightarrow{n}$ is the inner normal of the domain. By definition of stream function, (12) implies that the nozzle walls are streamlines, that is, $\psi =$constant on each nozzle wall. Without loss of generality, we assume $\psi = 0$ on one of the walls.

Let two nozzle walls be $S_i = \{(x_1,x_2) | x_2 = f_i(x_1), -\infty < x_1 < \infty\}$, $(i = 1,2)$. Suppose that $f_2(x_1) > f_1(x_1)$ for $x_1 \in (-\infty,\infty)$ satisfying the following conditions:

$$f_1(x_1) \to 0, \quad f_2(x_1) \to 1, \qquad \text{as } x_1 \to -\infty, \tag{13}$$

$$f_1(x_1) \to a, \quad f_2(x_1) \to b > a, \qquad \text{as } x_1 \to +\infty, \tag{14}$$

and

$$f_i'(x_1), \quad f_i''(x_1) \to 0, \qquad \text{as } |x_1| \to \infty. \tag{15}$$

Moreover, $f_i \in C^{2,\alpha}_{loc}(\mathbb{R})$ for some $\alpha > 0$. So the domain of the flow is given by $\Omega = \{(x_1,x_2) | f_1(x_1) < x_2 < f_2(x_1), -\infty < x_1 < \infty\}$. Under above conditions on $f_i (i = 1,2)$, it follows that $\Omega$ satisfies the uniform exterior sphere condition with some uniform radius $r > 0$.

For the given nozzle satisfying all the above conditions, we have the following theorem on the existence of uniform subsonic flows in the nozzle.

**Theorem 1** *(1). There exists a constant $\bar{m} > 0$ which depends only on $S_1$ and $S_2$ such that if $0 \le m < \bar{m}$, then the problem*

$$\begin{cases} div(\frac{\nabla\psi}{H(|\nabla\psi|^2)}) = 0, & in\ \Omega, \\ \psi = 0, & on\ S_1, \\ \psi = m, & on\ S_2, \end{cases} \tag{16}$$

*has a unique global uniformly subsonic solution.*

*(2). Moreover, at the far fields, the flow approximates to uniform flows, i.e.*

$$\nabla\psi(x_1, x_2) \to \begin{cases} (0, m), & x_1 \to -\infty, \\ (0, \frac{m}{b-a}), & x_1 \to +\infty. \end{cases}$$

If the incoming mass flux is increased, then the following sharp result holds

**Theorem 2** *For the given nozzle, there exists a constant $\hat{m}$ such that if $0 \le m < \hat{m}$, (16) has a unique uniformly subsonic solution satisfying*

$$M(m) = \sup_{(x_1,x_2)\in\Omega} |\nabla\psi| < 1; \tag{17}$$

*moreover, $M(m)$ ranges over $[0, 1)$ as $m$ varies in $[0, \hat{m})$. Furthermore, if $0 < m < \hat{m}$, the horizontal velocity is always positive in $\bar{\Omega}$, i.e.*

$$\partial_{x_2}\psi > 0, \tag{18}$$

*and, the deflection angle of the flow, $\theta = \arctan\frac{-\partial_{x_1}\psi}{\partial_{x_2}\psi}$, satisfies*

$$\underline{\theta} \le \theta \le \bar{\theta}, \tag{19}$$

*where*

$$\underline{\theta} = \inf_{x_1} \min\{\arctan f_1'(x_1), \arctan f_2'(x_1)\}, \tag{20}$$

$$\bar{\theta} = \sup_{x_1} \max\{\arctan f_1'(x_1), \arctan f_2'(x_1)\}. \tag{21}$$

*Moreover, if* $0 \le m_1 < m_2 < \hat{m}$*, then*

$$M(m_1) < M(m_2), \tag{22}$$

*therefore, as* $m \uparrow \hat{m}$*,* $M(m) \uparrow 1$*.*

In fact, as $m \uparrow \hat{m}$, the corresponding flow fields tend a limit which yields a subsonic-sonic flow in the nozzle.

**Theorem 3** *Let* $\{m_n\}$ *be any monotone sequence such that* $m_n \to \hat{m}$*. Denote by* $(u_n, v_n)$ *the global uniformly subsonic flow corresponding to* $m_n$ *as guaranteed by Theorem 2. Then there exists a subsequence, still labelled by* $\{(u_n, v_n)\}$ *associated with* $\{m_n\}$ *such that*

$$u_n \to u, \quad v_n \to v, \tag{23}$$

$$\rho(q_n^2)u_n \to \rho(q^2)u, \quad \rho(q_n^2)v_n \to \rho(q^2)v, \tag{24}$$

*where* $q_n^2 = u_n^2 + v_n^2$*,* $q^2 = u^2 + v^2$*, and* $\rho(q^2)$ *is the function defined by Bernoulli's law (5), all the above convergence are weak-*$*$ *convergence in* $L^\infty(\Omega)$*. Hence* $(u, v)$ *satisfies*

$$\begin{cases} (\rho(q^2)u)_{x_1} + (\rho(q^2)v)_{x_2} = 0, \\ u_{x_2} - v_{x_1} = 0, \end{cases} \tag{25}$$

*in the sense of distribution. Furthermore, the limit velocity* $(u, v)$ *satisfies the boundary condition (12), as the normal trace of the divergence field* $(\rho(q^2)u, \rho(q^2)v)$ *on the boundary.*

There are a few remarks in order.

**Remark 1:** Though it seems that the nozzle flow may be simpler than the airfoil flow. Yet, there are some difficulties both physically and mathematically. Physically, a symmetric nozzle can be regarded as two pieces of bumps, then all flow patterns appear in flow past a profile may appear in flows through the nozzle. Mathematically, it is an exterior problem for partial differential equations for flows past a profile. So one can use some techniques, for example, Kelvin transformation, etc, to reduce the problem into a boundary value problem in bounded domain, which seems essential for the estimates by Bers[2] and Dong[6], et al, for flows past an obstacle. However, it seems not easy to use Kelvin transformation to transform the domain for nozzle flows into a bounded domain. Moreover, there is another major difference between nozzle flow and and the flow past an obstacle that flow at far fields may not be same, which does not occur for airfoil flows.

**Remark 2:** It is well-known that the existence of subsonic potential flows is equivalent to the existence of quasiconformal mapping between the physical space $(x_1, x_2)$ and the space $(\varphi, \psi)$, where $\varphi$ is the velocity potential and $\psi$ is the stream function for a given flow, [3]. There are some important and general results for the existence of quasiconformal mappings for the domain we considered here, see [12], [13]. The key assumption in [12], [13] is the uniform ellipticity of the equation, however, for our problem, apriorily, we do not have the uniform ellipticity. It can be seen that Theorem 1 will be obtained easily for a more general class of nozzles if the uniform ellipticity is known a priorily.

**Remark 3:** Note also that in the formulation of the problem about subsonic flows past a profile, it is required a priorily that the flow field is uniformly subsonic. However, for a general nozzle, one can not require that the flow approximates to uniform flow at far fields a priorily, otherwise, mathematically, the problem for elliptic equations is overdetermined. To establish this uniform ellipticity, we exploit the relationship between the incoming mass flux and the nozzle boundaries (see Lemma 5), and study the flow at far field by our key estimate, Lemma 7, which is the main reason for the conditions (13)-(15) on the nozzle boundaries. In other words, we give a sufficient condition to ensure the uniform ellipticity mentioned in Remark 2.

**Remark 4:** The significance of Theorem 1 lies in that we can give an explicit form of $\bar{m}$ for a given nozzle, see Section 3. Moreover, it can be seen in Section 3 that $\bar{m}$ does not depend on the equation of states under our nondimensionalization. In Theorem 2, we assert only the existence of $\hat{m}$ for the given nozzle, but we don't know how large it is. Of course, $\bar{m}$ can be regarded as a lower bound of $\hat{m}$.

**Remark 5:** It is an open problem whether the maximum of flow speed for the whole flow field is monotonously increasing with respect to the freestream Mach number for general profiles[2] and obstacles[6]. Here we obtain this property for general nozzles.

**Remark 6:** The estimate for deflection angle is very important for the limiting procedure in Theorem 3. In fact, this is the one of crucial assumptions in Morawetz's compensated compactness framework[14].

**Remark 7:** In Theorem 3, although the strong convergence of the velocity fields is not known, yet we can obtain some commutate relations of weak convergence with those nonlinear functions which ensure the existence of weak solutions to (25) in the sense of distribution.

**Remark 8:** All the above theorems hold true for general equation of states which satisfies $p'(\rho) > 0$ for $\rho > 0$ and $p''(\rho) \geq 0$.

The rest of this paper is organized as follows: in Section 2, we will prove the existence of solutions to the elliptic problem in the unbounded domain with a subsonic truncation. In Section 3, we will use the Hölder gradient estimate for elliptic equations of two variables to show that the flow approximate to uniform flows at far fields, combining this with some boundary gradient estimates, we can show that the flow is actually globally subsonic; uniqueness will be proved subsequently by the maximum principle for uniformly subsonic flow. In Section 4, we first use Bers' idea[2] to show the existence of a critical incoming mass flux. Next, with the help of a comparison principle, we prove the positivity of the horizontal velocity. Then, the estimate for deflection angle becomes a direct consequence of the maximum principle for the equation in the hodograph plane. At last, we will combine the comparison principle and certain partial hodograph transformation to prove the monotonicity of the maximum of flow speed with respect to the incoming mass flux. In the last section, Section 5, Theorem 3 will be shown by the theory of compensated compactness ([14], [4]) based on Theorem 2. The appendix contains an elementary explicit construction of truncated domains used in Section 2.

## 2 Subsonic Truncated Problem

We now study the boundary value problem (16). Note that the function $H = H(|\nabla\psi|^2)$ is defined only on $[0,1]$, and $H'(|\nabla\psi|^2)$ goes to infinity as $|\nabla\psi| \to 1$. Thus, the ellipticity of the equation in (16) depends crucially on the upper bound for $|\nabla\psi|$. Hence, instead of the problem (16), we first consider the following truncated problem. Define

$$\tilde{H}(s) = \begin{cases} H(s), & \text{if } 0 \le s < \tilde{m}^2, \\ H((\frac{\tilde{m}+1}{2})^2), & \text{if } s \ge (\frac{\tilde{m}+1}{2})^2, \end{cases} \tag{26}$$

where $\tilde{m}(< 1)$ is a positive constant to be determined, moreover, $\tilde{H}$ is a smooth decreasing function. Finally, it will be shown that our solution satisfies $|\nabla\psi| < \tilde{m}$ if $m < \bar{m}$ so that the subsonic truncation can be taken away.

We first solve the problem

$$\begin{cases} \operatorname{div}(\frac{\nabla\psi}{\tilde{H}(|\nabla\psi|^2)}) = 0, & \text{in } \Omega, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)} m, & \text{on } \partial\Omega. \end{cases} \tag{27}$$

Note that the problem (27) is a boundary value problem for a uniformly elliptic equation in an unbounded domain. Although the existence of this problem is a corollary of general results in [13], we would like to present a sketch of the proof of the existence and give some important estimates which will be used later.

To solve problem (27), we will truncate the domain first, and use a series of boundary value problems in bounded domains to approximate the problem (27). Thus, consider first the problem

$$\begin{cases} \operatorname{div}(\frac{\nabla\psi}{\tilde{H}(|\nabla\psi|^2)}) = 0, & \text{in } \Omega_L, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)} m, & \text{on } \partial\Omega_L, \end{cases} \tag{28}$$

where $\Omega_L$ satisfies $\{(x_1,x_2)|(x_1,x_2) \in \Omega, -L < x_1 < L\} \subset \Omega_L \subset \{(x_1,x_2)|(x_1,x_2) \in \Omega, -4L < x_1 < 4L\}$ for $\forall L \in \mathbb{N}$. Furthermore, one may choose $\Omega_L$ so that $\Omega_L \in C^{2,\alpha_1}(0 < \alpha_1 \le \alpha)$ satisfies the uniform exterior sphere condition with uniform radius $r_0$, $0 < r_0 < r$, sufficiently small for all $L > L_0$ with some $L_0$ sufficiently large. See the appendix for the construction of $\Omega_L$.

The problem (28), a Dirichlet boundary value problem for a quasilinear elliptic equation, can be solved by standard fixed point arguments. Indeed, applying Theorem 11.4 in [11], one needs only to show that all $C^{2,\alpha_1}$ solutions to the problem

$$\begin{cases} \operatorname{div}(\frac{\nabla\psi}{\tilde{H}(|\nabla\psi|^2)}) = 0, & \text{in } \Omega_L, \\ \psi = \sigma\frac{x_2 - f_1(x_1)}{f_2(x_1)-f_1(x_1)}m, & \text{on } \partial\Omega_L. \end{cases} \tag{29}$$

with $\sigma \in [0,1]$ satisfy

$$\|\psi\|_{C^{1,\beta}(\Omega_L)} \leq C. \tag{30}$$

for some fixed constants $C$ and $\beta \in (0,1)$, which do not depend on $\sigma$ and $\psi$.

To obtain the desired estimate (30), we proceed as that described in §11.3 in [11] to divide the estimate into four steps. First, by the maximum principle, one has

$$0 \leq \psi \leq \sigma m \leq m. \tag{31}$$

Moreover, the Berstein estimate holds for this equation, i.e.

$$\max_{\overline{\Omega}_L} |\nabla\psi|^2 \leq \max_{\partial\Omega_L} |\nabla\psi|^2, \tag{32}$$

by checking the conditions in Theorem 15.1 in [11]. Furthermore, due to Theorem 13.2 in [11], one can obtain

$$[D\psi]_{\beta,\Omega_L} \leq C(|\psi|_{1;\Omega}), \tag{33}$$

for some $\beta \in (0,1)$. Thus, if one can show the following boundary gradient estimate

$$\max_{\partial\Omega_L} |\nabla\psi|^2 \leq C, \tag{34}$$

then the desired estimate (30) follows. This, in turn, shows the existence of a solution to the problem (28). In fact we have a stronger estimate for the boundary gradient in the following lemma, which is important to obtain the existence of solutions in the case of general nozzles.

**Lemma 4** *There exists a constant $C$ depending only on $r_0$, $\tilde{m}$, and $C^2$ norm of $f_i (i = 1,2)$, but not on $L$, such that (34) holds for all $L$.*

**Proof:** This lemma can be proved by a barrier argument. Decompose $\psi = g(x_1, x_2) + \phi$, with $g = \sigma \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)}$. Then

$$Q\phi = \tilde{H}(|\nabla(\phi+g)|^2)\Delta(\phi+g) - 2\tilde{H}'(|\nabla(\phi+g)|^2)\partial_i(\phi+g)\partial_j(\phi+g)\partial_{ij}(\phi+g) = 0.$$

For $x^0 \in \partial\Omega_L$, there exists $y \in \overline{\Omega_L}^c$, such that $\overline{B(y, r_1)} \cap \overline{\Omega_L} = \{x^0\}$ with some $0 < r_1 \le r_0$ to be determined. Set $d = |x - y|$. We will construct a barrier function of the form $w = h(d)$. Then it follows that

$$\begin{aligned} Qw &= \tilde{H}(|\nabla(w+g)|^2)\Delta w - 2\tilde{H}'(|\nabla(w+g)|^2)\partial_i(w+g)\partial_j(w+g)\partial_{ij}w \\ &\quad + \tilde{H}(|\nabla(w+g)|^2)\Delta g - 2\tilde{H}'(|\nabla(w+g)|^2)\partial_i(w+g)\partial_j(w+g)\partial_{ij}g \\ &= I_1 + I_2. \end{aligned}$$

Direct calculation yields

$$\begin{aligned} I_1 &= \big(\tilde{H}(|\nabla(w+g)|^2) - 2\tilde{H}'(|\nabla(w+g)|^2)\frac{|\nabla(w+g)\cdot(x-y)|^2}{|x-y|^2}\big)(h''(d) + \frac{h'(d)}{d}) \\ &\quad + 4\tilde{H}'(|\nabla(w+g)|^2)\frac{|\nabla(w+g)\cdot(x-y)|^2}{|x-y|^2}\frac{h'(d)}{d} \\ &\quad - 2\tilde{H}'(|\nabla(w+g)|^2)|\nabla(w+g)|^2\frac{h'(d)}{d}. \end{aligned}$$

Note that $\tilde{H}' \le 0$. So

$$\begin{aligned} I_1 &\le \big(\tilde{H}(|\nabla(w+g)|^2) - 2\tilde{H}'(|\nabla(w+g)|^2)\frac{|\nabla(w+g)\cdot(x-y)|^2}{|x-y|^2}\big)(h''(d) + \frac{h'(d)}{d}) \\ &\quad - 2\tilde{H}'(|\nabla(w+g)|^2)|\nabla(w+g)|^2\frac{h'(d)}{d}, \end{aligned}$$

provided

$$h'(d) \ge 0. \tag{35}$$

It follows from (26) and direct computations that

$$0 < C_1 \le \tilde{H}(|\nabla(w+g)|^2) - 2\tilde{H}'(|\nabla(w+g)|^2)\tfrac{|\nabla(w+g)\cdot(x-y)|^2}{|x-y|^2} \le C_2,$$

$$0 \le C_3 \le -2\tilde{H}'(|\nabla(w+g)|^2)|\nabla(w+g)|^2 \le C_4,$$

$$C_5 < I_2 = \tilde{H}(|\nabla(w+g)|^2)\Delta g - 2\tilde{H}'(|\nabla(w+g)|^2)\partial_i(w+g)\partial_j(w+g)\partial_{ij}g \le C_6,$$

for some uniform constants $C_i(i=1,\cdots,6)$, which depend only on $\tilde{m}$, and $C^2$ norm of $f_j$ $(j=1,2)$, moreover, except for $C_3$ and $C_5$, all others are positive. Therefore,

$$\begin{aligned} Qw &\le C_1(h''(d)+\frac{h'(d)}{d})+C_4\frac{h'(d)}{d}+C_6 \\ &\le C_1(h''(d)+(1+\frac{C_4}{C_1})\frac{h'(d)}{d}+\frac{C_6}{C_1}) \\ &= C_1(h''(d)+(1+C_7)\frac{h'(d)}{d}+C_8), \end{aligned}$$

if

$$h''(d)+\frac{h'(d)}{d}\le 0, \tag{36}$$

and (35) holds. Thus, set

$$h(d)=-\frac{C_8}{2(2+C_7)}d^2+\frac{C_9}{C_7}d^{-C_7}+C_{10},$$

with constants $C_9$ and $C_{10}$ to be determined. Then $Qw\le 0$. Choose $C_{10}$ such that $h(r_1)=0$, i.e. $C_{10}=\frac{C_8}{2(2+C_7)}r_1^2-\frac{C_9}{C_7}(r_1)^{-C_7}$. Then clearly one can select $C_9<0$ small enough such that

$$h'(d)=-C_9d^{-C_7-1}-\frac{C_8}{2+C_7}d>0, \quad \text{for } \forall d\in(r_1,2r_1), \tag{37}$$

and

$$h(2r_1)=-\frac{C_8}{2(2+C_7)}(2r_1)^2+\frac{C_9}{C_7}(2r_1)^{-C_7}+\frac{C_8}{2(2+C_7)}r_1^2-\frac{C_9}{C_7}(r_1)^{-C_7}>2m, \tag{38}$$

hold simultaneously when $r_1>0$ is sufficiently small. Clearly, (37) implies that (35) holds. Since $h(d)$ satisfies the differential equation

$$h''(d)+(1+C_7)\frac{h'(d)}{d}+C_8=0,$$

it is trivial that $h(d)$ satisfies (36) for $h'(d)\ge 0$ and $C_7$, $C_8>0$.

Without loss of generality, one may assume that $\Omega_L\setminus B(y,2r_1)$ is connected, then $w\ge\phi$ on $\partial\mathcal{N}$ with $\mathcal{N}=B(y,2r_1)\cap\Omega_L$. It follows from the comparison principle that $\phi\le w$ in $\mathcal{N}$. Therefore,

$$\frac{\partial\phi}{\partial n}\le h'(r_1)=C,$$

for some uniform positive constant $C$, which depends only on $r_1$, $\tilde{m}$, and $f_j (j = 1, 2)$, where $n$ is the inner normal of $\Omega_L$ at $x_0$. Similarly, one can prove that

$$\frac{\partial \phi}{\partial n} \geq -C.$$

Hence,

$$|\nabla \psi| \leq C$$

on the boundary, for some constant $C$ which depends only on $r_0$, $\tilde{m}$, and $f_j (j = 1, 2)$. $\square$

For each $L$, the problem (28) has a solution, which is denoted by $\psi_L$. In addition, the maximum principle, Lemma 4, and the Bernstein estimate imply that

$$|\psi_L|_{1;\Omega_L} \leq C, \tag{39}$$

with $C$ depending only on $r_0$, $\tilde{m}$, and $f_j (j = 1, 2)$. Moreover, by the standard interior Schauder estimate and Schauder estimate on a boundary portion[11], one has

$$|\psi_n|_{C^{2,\alpha}(\Omega_L)} \leq C, \tag{40}$$

for any $n > 2L+1$, where the constant $C$ may depend on $L$ and the restriction of $f_i$ on $\Omega_{2L}$, but not on $n$. Therefore, it follows from the Arzela-Ascoli lemma and a diagonal procedure that there exists a subsequence $\{\chi_{\Omega_{n_k}} \psi_{n_k}\}$, where $\chi_{\Omega_{n_k}}$ is the characteristic function of $\Omega_{n_k}$, which converges to a function $\psi$ in $C^{2,\delta}(\Omega_L)$ for $\forall L > 0$ and $\forall \delta < \alpha_1$. Obviously, $\psi$ solves the problem (27).

## 3 Subsonic Estimates

In this section, we will show that in fact the solution $\psi$ to the problem (27) obtained in Section 2, is a subsonic solution if $m < \bar{m}$ . Moreover, $|\nabla \psi| < \tilde{m}$ if $m < \bar{m}$, thus the subsonic truncation can be taken away. Due to the Berstein estimate, $|\nabla \psi|^2$ can only attain its maximum on the boundaries, thus, we need only to estimate $|\nabla \psi|^2$ on the solid boundaries and at the far fields.

First, let us estimate the gradient of $\psi$ on the boundaries of the nozzle. Notice that the solution attains its minimum and maximum on $S_1$ and $S_2$ repectively. Moreover, $\psi = 0$ on $S_1$ and $\psi = m$ on $S_2$, both of them are constant. Thus it is the same as the boundary gradient estimate for the homogeneous boundary value problems, since the equation in (27) has no lower order terms.

Let $R_i(x_1)$ be the largest radius of disks, whose closure intersects with $\bar{\Omega}$ only at $x^i = (x_1, f_i(x_1))$. Suppose the center of the disk is the point $y = (y_1, y_2)$, define $d_i(x_1) = \text{dist}(y, S_j) - R_i(x_1)$ if $R_i(x_1) < \infty$; $d_i(x_1) = \text{dist}(x^i, S_j)$, if $R_i(x_1) = \infty$, where $j \neq i$ $(i, j = 1, 2)$. Define

$$D = \inf_{x_1 \in \mathbb{R}} \min_{1 \le i \le 2} g_i(x_1), \tag{41}$$

where $g_i(x_1)$ is defined as follows

$$g_i(x_1) = \begin{cases} R_i(x_1) \ln(1 + \frac{d_i(x_1)}{R_i(x_1)}), & \text{if} \quad R_i(x_1) < \infty, \\ d_i(x_1), & \text{if} \quad R_i(x_1) = \infty. \end{cases} \tag{42}$$

It follows from our assumptions on the boundaries that both $f_i(x_1)$ and $f_i''(x_1)$ tend to zero as $|x_1| \to \infty$, and $f_i \in C^{2,\alpha}_{loc}(\mathbb{R})$, therefore, $D > 0$. Furthermore, clearly, from (42), $D \le \min\{1, b - a\}$ because $R_i \to \infty$ as $|x_1| \to \infty$.

Now we estimate the flow speed on the boundary.

**Lemma 5** *The solution $\psi$ to (27) satisfies $|\nabla\psi| \le m/D$ on $\partial\Omega$.*

**Proof:** The proof is again by a barrier argument. Let $x^0 \in S_1$. Without loss of generality, assume that $R = R_1(x^0) < \infty$, and $\overline{B_R(y)} \cap S_1 = \{x^0\}$. Set $d = |x - y|$, $w = h(d)$ with $h' \ge 0$. Then direct computation yields

$$\begin{aligned} Qw &= \tilde{H}(|\nabla w|^2)\Delta w - 2\tilde{H}'(|\nabla w|^2)\partial_i w \partial_j w \partial_{ij} w \\ &= \big(\tilde{H}(|h'(d)|^2) - 2\tilde{H}'(|h'(d)|^2)|h'(d)|^2\big)\big(h''(d) + \frac{h'(d)}{d}\big) + 2\tilde{H}'(|h'(d)|^2)\frac{(h'(d))^3}{d} \\ &\le (h''(d) + h'(d)\frac{1}{d})\big(\tilde{H}(|h'(d)|^2) - 2\tilde{H}'(|h'(d)|^2)|h'(d)|^2\big), \end{aligned}$$

since $\tilde{H}' \le 0$. Therefore, $Qw \le 0$ provided

$$h''(d) + h'(d)\frac{1}{d} \le 0. \tag{43}$$

It is equivalent to

$$(dh'(d))' \le 0,$$

which is satisfied by the following class of functions:

$$h(d) = C_1 \ln \frac{d}{R} + C_2,$$

with constants $C_1$ and $C_2$ to be determined. Let $C_2 = 0$. Then $h(R) = 0$. Moreover, choose $C_1$ so that $h(d_1 + R) = m$, i.e.,

$$w = h(d) = \frac{m}{\ln \frac{d_1+R}{R}} \ln \frac{d}{R}.$$

Now it is easy to see that in the domain $\mathcal{N} = \Omega \cap B(y, d_1 + R)$, $Qw \le 0$, and $w \ge \psi$ on $\partial\mathcal{N}$. Thus by the comparison principle, $w \ge \psi$ in $\mathcal{N}$, which yields,

$$\frac{\partial \psi}{\partial n}|_{x^0} \le h'(R) \le m/D,$$

where $n$ is the inner normal direction of $\Omega$. On the other hand, $\psi$ is constant on $S_2$, moreover, it is the maximum of $\psi$ on $\Omega$, therefore, in a similar way as that on $S_1$, one can show

$$\frac{\partial \psi}{\partial n} \ge -m/D.$$

Consequently, we have finished the proof of the lemma. □

To estimate the flow speed at the far fields, we first consider a special case in which there exists a sufficient large number $L_0$ such that

$$f_1(x_1) = 0, \quad f_2(x_1) = 1, \quad \text{for } x_1 < -L_0. \tag{44}$$

**Lemma 6** *In addition to the conditions in Theorem 1, it is assumed that $f_1$ and $f_2$ satisfy (44). Then there exists a $\mu$, $0 < \mu < 1$, depending on $\tilde{m}$, such that the solution $\psi$ to (27) satisfies*

$$|\nabla\psi - (0, m)| \le \frac{Cm}{|x_1|^{1+\mu}} \quad \textit{for } x_1 < -2L_0, \tag{45}$$

*where $C$ depends only on $\tilde{m}$.*

**Proof:** Set $\phi = \psi - mx_2$ and $E = \{(x_1, x_2)|(x_1, x_2) \in \Omega, x_1 < -L_0\}$. Then, it is easy to see that

$$\begin{cases} a_{ij}(x_1, x_2)\partial_{ij}\phi = 0, & \text{in } E, \\ \phi(x_1, 0) = 0, & x_1 < -L_0, \\ \phi(x_1, 1) = 0, & x_1 < -L_0, \end{cases} \tag{46}$$

where

$$a_{ij}(x_1, x_2) = \tilde{H}(|\nabla\psi|^2)\delta_{ij} - 2\tilde{H}'(|\nabla\psi|^2)\partial_i\psi\partial_j\psi. \tag{47}$$

In the case of the truncation we used, the elliptic equation in (46) has uniform elliptic ration $\nu$ depending only on $\tilde{m}$ such that $\frac{\tilde{H}(|\nabla\psi|^2)}{\tilde{H}(|\nabla\psi|^2)-2\tilde{H}'(|\nabla\psi|^2)|\nabla\psi|^2} \geq \nu > 0$.

For a fixed $k$, consider the domain $E_{l,k} = \{(x_1, x_2)| - l - k < x_1 < -l + k, f_1(x_1) < x_2 < f_2(x_1)\}$, for $l$ large enough to ensure $-l + k < -L_0$. $\phi$ is extended as follows

$$\tilde{\phi}(x_1, x_2) = -\phi(x_1, -x_2), \quad -1 < x_2 < 0,$$

and then $\tilde{\phi}$ is extended periodically with respect to $x_2$ with period 2, i.e.,

$$\tilde{\phi}(x_1, x_2) = \phi(x_1, x_2 + 2n), \quad \text{if } -1 < x_2 + 2n < 1 \text{ for some } n \in \mathbb{Z}.$$

It follows from above extensions that

$$\partial_{ii}\tilde{\phi}(x_1, x_2) = -\partial_{ii}\tilde{\phi}(x_1, -x_2),\ i = 1, 2;\ \partial_{12}\tilde{\phi}(x_1, x_2) = \partial_{12}\tilde{\phi}(x_1, -x_2),\ \text{for } -1 < x_2 < 0.$$

Therefore, one may extend $a_{ij}$ as

$$\tilde{a}_{ii}(x_1, x_2) = a_{ii}(x_1, -x_2),\ i = 1, 2;\ \tilde{a}_{12}(x_1, x_2) = -a_{12}(x_1, -x_2) \text{ for } -1 < x_2 < 0,$$

and then periodically as

$$\tilde{a}_{ij}(x_1, x_2) = a_{ij}(x_1, x_2 + 2n),\ \text{if } -1 < x_2 + 2n < 1 \text{ for some } n \in \mathbb{Z}.$$

Then $\tilde{\phi} \in W^{2,\infty}(\tilde{E})$, and $\tilde{\phi}$ is a strong solution of the equation

$$\tilde{a}_{ij}\partial_{ij}\tilde{\phi} = 0, \quad \text{in } \tilde{E}, \tag{48}$$

where $\tilde{E} = \{(x_1, x_2) | -l-k < x_1 < -l+k, x_2 \in \mathbb{R}\}$. With an abuse of notation, we remove $\tilde{}$ of $\tilde{a}_{ij}$ and $\tilde{\phi}$ away, and regard the functions $a_{ij}$ and $\phi$ as the above extensions defined in $\tilde{E}$.

Before estimating the solution to the elliptic equation (48), we introduce the weighted Hölder norms as in [11]. Let $d_x = \mathrm{dist}(x, \partial\tilde{E})$ and $d_{x,y} = \min(d_x, d_y)$. Then

$$[\phi]^*_{1,\mu} = \sup_{x,y\in\tilde{E}} d^{1+\mu}_{x,y} \frac{|D\phi(x) - Du(y)|}{|x-y|^{\mu}}, \text{ and } |f|^{(2)}_0 = \sup_{x\in\tilde{E}} d^2_x |f(x)|. \tag{49}$$

By Theorem 12.4 in [11], there exists a $0 < \mu < 1$, depending on $\nu$, such that

$$[\phi]^*_{1,\mu} \le C|\phi|_0, \tag{50}$$

holds, where $C$ depends only on $\nu$. Although it is required in Theorem 12.4 in [11] that $u \in C^2$, as remarked in §12.1 in [11], Theorem 12.4 in [11] is also valid for $u \in W^{2,\infty}$. Note that

$$d(x) = \mathrm{dist}(x, \partial\tilde{E}) = \min\{|x_1 - (-l-k)|, |x_1 - (-l+k)|\}. \tag{51}$$

Therefore, the estimate (50) implies that

$$k^{1+\mu} \frac{|D\phi(-l, x_2) - D\phi(-l, y_2)|}{|x_2 - y_2|^{\mu}} \le Cm.$$

So,

$$|D\phi(-l, x_2) - D\phi(-l, y_2)| \le \frac{Cm}{k^{1+\mu}} |x_2 - y_2|^{\mu}.$$

In particular,

$$|D\phi(-l, x_2) - D\phi(-l, 0)| \le \frac{Cm}{k^{1+\mu}} |x_2|^{\mu}, \quad \text{for any } x_2 \in (0, 1).$$

On the other hand, thanks to the boundary conditions for $\phi$, i.e., $\phi(x_1, 0) = \phi(x_1, 1) = 0$ for $x_1 < -L_0$, we get

$$\begin{aligned} 0 &= \int_0^1 \partial_2\phi(-l, x_2)dx_2 \\ &= \int_0^1 \partial_2\phi(-l, 0)dx_2 + \int_0^1 (\partial_2\phi(-l, x_2) - \partial_2\phi(-l, 0))dx_2 \\ &= \partial_2\phi(-l, 0) + \int_0^1 (\partial_2\phi(-l, x_2) - \partial_2\phi(-l, 0))dx_2. \end{aligned}$$

Therefore,

$$\begin{aligned} |\partial_2\phi(-l,0)| &\le \int_0^1 |\partial_2\phi(-l,x_2)-\partial_2\phi(-l,0)|dx_2 \\ &\le \frac{Cm}{k^{1+\mu}}. \end{aligned}$$

Hence,

$$|\partial_2\phi(-l,x_2)|\le 2\frac{Cm}{k^{1+\mu}},\quad |\partial_1\phi(-l,x_2)|\le \frac{Cm}{k^{1+\mu}},\quad \text{for } x_2\in(0,1). \tag{52}$$

Therefore, choosing $l>2L_0$ and $k=l/2$, for $x_1<-2L_0$, one obtains from (52) that

$$|\nabla\phi(x_1,x_2)|\le \frac{Cm}{|x_1|^{1+\mu}}, \tag{53}$$

where $C$ depends only on $\nu$. Thus

$$|\nabla\psi-(0,m)|\le \frac{Cm}{|x_1|^{1+\mu}}\quad \text{for } x_1<-2L_0,$$

with some positive constant $C$ depending only on $\tilde{m}$. This finishes the proof of the lemma. □

Similarly, if two walls are straight near positive infinity, then

$$|\nabla(\psi-\frac{m}{b-a}x_2)|\le \frac{Cm}{|x_1|^{1+\mu}}\quad \text{for } x_1>2L_1 \text{ with some } L_1>0.$$

Next, we discuss the asymptotic behavior at far fields for general nozzles, we will show that the flows tends to uniform flows at far fields as follows

**Lemma 7** *For any nozzle which satisfies the conditions in Theorem 1, the solution $\psi$ to (27) satisfies*

$$|\nabla\psi-(0,m)|\to 0\quad \textit{as } x_1\to-\infty, \tag{54}$$

$$|\nabla\psi-(0,\frac{m}{b-a})|\to 0\quad \textit{as } x_1\to\infty. \tag{55}$$

**Proof:** If the two boundaries are not straight lines at infinity, we introduce the following transformation,

$$\begin{cases} X_1=x_1, \\ X_2=\frac{x_2-f_1(x_1)}{f_2(x_1)-f_1(x_1)}. \end{cases} \tag{56}$$

Set $\bar{\psi}(X_1, X_2) = \psi(x_1, x_2)$, then

$$\partial_{x_1}\psi = \partial_{X_1}\bar{\psi} + (\partial_{X_2}\bar{\psi})g_1, \quad \partial_{x_2}\psi = (\partial_{X_2}\bar{\psi})g_2, \tag{57}$$

with

$$g_1 = \frac{f_1'(x_1)(x_2 - f_2(x_1)) - f_2'(x_1)(x_2 - f_1(x_1))}{(f_2(x_1) - f_1(x_1))^2}, \text{ and } g_2 = \frac{1}{f_2(x_1) - f_1(x_1)}. \tag{58}$$

Then $\bar{\psi}$ satisfies

$$A_{ij}(X_1, X_2)\partial_{X_iX_j}\bar{\psi} = f(X_1, X_2) \tag{59}$$

and the boundary conditions $\bar{\psi}(X_1, 0) = 0$ and $\bar{\psi}(X_1, 1) = m$ for all $X_1 \in \mathbb{R}$. Moreover, $A_{ij}\partial_{ij}$ is a uniformly elliptic operator with two eigenvalues $\lambda_1$ and $\lambda_2$ satisfying $0 < \lambda < \lambda_1 < \lambda_2$ for some constant $\lambda$, moreover, $\frac{\lambda_1}{\lambda_2} \geq \bar{\nu} > 0$ for some constant $\bar{\nu}$. Direct calculation shows that

$$f(X_1, X_2) = -\Big(a_{11}(\nabla\psi)\frac{\partial\bar{\psi}}{\partial X_2}\frac{\partial^2 X_2}{\partial x_1^2} + 2a_{12}(\nabla\psi)\frac{\partial\bar{\psi}}{\partial X_2}\frac{\partial^2 X_2}{\partial x_1\partial x_2}\Big), \tag{60}$$

where $a_{ij}$ is defined by (47), and

$$\begin{aligned} \frac{\partial^2 X_2}{\partial x_1^2} &= \frac{(f_1(x_1) - x_2)f_2''(x_1) + (x_2 - f_2(x_1))f_1''(x_1)}{(f_2 - f_1)^2} \\ &\quad + \frac{2(x_2 - f_1(x_1))(f_2'(x_1) - f_1'(x_1))^2}{(f_2 - f_1)^3} + \frac{2f_1'(f_2' - f_1')}{(f_2 - f_1)^2}, \end{aligned} \tag{61}$$

$$\frac{\partial^2 X_2}{\partial x_1\partial x_2} = -\frac{f_2' - f_1'}{(f_2 - f_1)^2}. \tag{62}$$

By (39), we have

$$|\nabla\psi|_{1;\Omega} \leq C, \tag{63}$$

with some positive constant $C$ which depends only on $\tilde{m}$ and $f_i (i = 1, 2)$. Therefore, it follows from (57), (58) and (63) that

$$|f(X_1, X_2)| \to 0 \quad \text{uniformly with respect to } X_2 \text{ as } |X_1| \to \infty, \tag{64}$$

under our assumptions on $f_i (i = 1, 2)$.

Set $\bar{\phi} = \bar{\psi} - mX_2$, then $\bar{\phi}$ satisfies

$$\begin{cases} A_{ij}(X_1, X_2)\partial_{X_iX_j}\bar{\phi} = f(X_1, X_2), & \text{in } \{(X_1, X_2)|X_2 \in (0,1), X_1 \in \mathbb{R}\}, \\ \bar{\phi}(X_1, 0) = 0, \ \bar{\phi}(X_1, 1) = 0, & X_1 \in \mathbb{R}. \end{cases} \tag{65}$$

The only difference between (46) and (65) is that equation in (65) has the source term $f(X_1, X_2)$. Thus, one may extend the coefficients $A_{ij}$ and the function $\bar{\phi}$ as before

$$A_{ii}(X_1, X_2) = A_{ii}(X_1, -X_2), \ i = 1, 2; \ A_{12}(X_1, X_2) = -A_{12}(X_1, -X_2), \ \text{for } -1 < X_2 < 0,$$

$$\bar{\phi}(X_1, X_2) = -\bar{\phi}(X_1, -X_2) \quad \text{for } -1 < X_2 < 0,$$

and extend $f$ as follows,

$$f(X_1, X_2) = -f(X_1, -X_2), \ -1 < X_2 < 0.$$

Then all these functions can be extended periodically with period 2 with respect to $X_2$, that is

$$\begin{cases} A_{ij}(X_1, X_2) = A_{ij}(X_1, X_2 + 2n), \ i, j \in \{1, 2\}, \\ \bar{\phi}(X_1, X_2) = \bar{\phi}(X_1, X_2 + 2n), & \text{if } X_2 + 2n \in (-1, 1) \text{ for some } n \in \mathbb{Z}. \\ f(X_1, X_2) = f(X_1, X_2 + 2n). \end{cases}$$

Then $\bar{\phi} \in W^{2,\infty}(\tilde{E}_{l,k})$, with $\tilde{E}_{l,k} = \{(X_1, X_2)| - l - k < X_1 < -l + k, X_2 \in \mathbb{R}\}$, such that

$$A_{ij}(X_1, X_2)\partial_{X_iX_j}\bar{\phi} = f(X_1, X_2) \quad \text{in } \tilde{E}_{l,k}.$$

Using Theorem 12.4 in [11] in the case of $W^{2,\infty}$ solutions again, one can conclude that there exists $0 < \mu < 1$ depending on $\bar{\nu}$ such that

$$[\bar{\phi}]^*_{1,\mu} \leq C(|\bar{\phi}|_0 + |\frac{f}{\lambda_1}|^{(2)}_0)$$

holds for some constant $C$ depending only on $\bar{\nu}$. In particular, we have

$$\begin{aligned} |D\bar{\phi}(-l, X_2) - D\bar{\phi}(-l, 0)| &\leq C(\frac{m}{k^{1+\mu}} + \frac{C}{k^{1+\mu}} \sup_{-l-k \leq Z_1 \leq -l+k} d_Z^2 |f(Z_1, Z_2)|) \\ &\leq C(\frac{m}{k^{1+\mu}} + \frac{C}{k^{1+\mu}} \sup_{-l-k \leq Z_1 \leq -l+k} k^2 |f(Z_1, Z_2)|) \\ &\leq C(\frac{m}{k^{1+\mu}} + Ck^{1-\mu} \sup_{-l-k \leq Z_1 \leq -l+k} |f(Z_1, Z_2)|), \ \text{for } X_2 \in (0, 1). \end{aligned}$$

Since $f(X_1,X_2)$ tends to 0 as $|X_1|\to\infty$, then $\forall\varepsilon>0$, there exists $L_0$ such that

$$|f(X_1,X_2)|<\varepsilon,\quad \text{for } |X_1|>L_0.$$

Now choosing $k=\frac{1}{\varepsilon^{(1-\mu)/2}}$, if $l>L_0+k$, then

$$\begin{aligned}|D\bar{\phi}(-l,X_2)-D\bar{\phi}(-l,0)| &\le C(m\varepsilon^{\frac{1-\mu^2}{2}}+C\varepsilon^{\frac{1+2\mu-\mu^2}{2}})\\ &\le C\varepsilon^{(1-\mu^2)/2}.\end{aligned}$$

Therefore, as in the proof of Lemma 6, one has

$$|D\bar{\psi}(X_1,X_2)-(0,m)|\to 0,\quad \text{as } X_1\to-\infty.$$

Then using the transformation (57), one gets

$$|D\psi(x_1,x_2)-(0,m)|\to 0,\quad \text{as } x_1\to-\infty.$$

Similarly, one can prove (55). This completes the proof of the lemma. □

**Remark 8:** It follows from Lemma 6 that if both nozzle walls are straight at far fields, then we get not only the convergence to uniform flows at far fields, but also a convergence rate, (45); however, for the general case, the rate of convergence is not clear.

Combining Lemma 5, Lemma 7 and the Bernstein estimate, we have

$$\sup_{\Omega}|\nabla\psi|\le\max\{\frac{m}{D},\, m,\, \frac{m}{b-a}\}.$$

Since $D\le\min\{1,b-a\}$, therefore,

$$\sup_{\Omega}|\nabla\psi|\le\frac{m}{D}.$$

Thus, set $\bar{m}=D$. So that if $m<\bar{m}$, then $m/D<1$. We now choose $\tilde{m}=(\frac{m}{D}+1)/2$, then $\sup_{\Omega}|\nabla\psi|\le\frac{m}{D}<\tilde{m}$, so $\psi$ solves (16). Hence we obtain the existence of a solution to (16).

**Remark 9:** If the nozzle walls change very slowly, then $R_i(x_1)\to\infty$, then $D$ approximates to the smallest distance $d$ between two nozzle walls. Therefore, $m<D$ is equivalent to $m\cdot 1<1\cdot d$, this is nothing but a necessary and sufficient condition for global existence of subsonic flows for quasi-one-dimensional nozzle.

Now let us prove the uniqueness of uniformly subsonic solutions.

**Lemma 8** *The uniformly subsonic solution to (16) is unique.*

**Proof:** Let two solutions $\psi_1$ and $\psi_2$ be both uniformly subsonic. Then there are two positive constants $c_1$ and $c_2 < 1$ such that $|\nabla\psi_1| < c_1$, and $|\nabla\psi_2| < c_2$. It is easy to see that Lemma 6 and Lemma 7 are both valid for uniformly subsonic flows, therefore, both flows approximate to uniform flows at far fields. Thus $\nabla\psi_i \to (0, m)$, as $x_1 \to -\infty$ and $\nabla\psi_i \to (0, \frac{m}{b-a})$ as $x_1 \to \infty (i = 1, 2)$. Let $\psi = \psi_2 - \psi_1$. Then

$$\begin{cases} a_{ij}(\nabla\psi_2)\partial_{ij}\psi + (a_{ij}(\nabla\psi_2) - a_{ij}(\nabla\psi_1))\partial_{ij}\psi_1 = 0 & \text{in } \Omega \\ \psi = 0 & \text{on } S_1 \cup S_2 \\ \nabla\psi \to 0 & \text{as } |x_1| \to +\infty. \end{cases}$$

By the maximum principle, we have

$$\psi = 0.$$

This finishes the proof of the lemma. □

**Remark 10:** From the proof of Lemma 8, it is easy to see that uniqueness holds for the flows which are subsonic in the whole domain, and only uniformly subsonic at far field. However, so far, we have not been able to prove the uniqueness for subsonic solutions as what had been obtained by Finn and Glibarg[8] for flow past a profile. For the flow past a profile, the velocity potential satisfies an elliptic equation in an exterior domain. Finn and Gilbarg[8] showed that, for the solution $\varphi$ to an elliptic equation of two variables, if the gradient $D\varphi$ satisfies $|D\varphi| = O\left((\ln r)^{1-\delta}\right)$ as $r \to \infty$ with $\delta > 0$, then $D\varphi$ tends Hölder continuously to a certain limit. Consequently, for the flow past a profile, the flow must be uniformly subsonic if it is subsonic. However, for flows through the nozzle, it seems difficult to show that the flow is actually uniformly subsonic if it is subsonic.

Collecting all the results obtained so far, we have finished the proof of Theorem 1.

# 4 Properties of Subsonic Flows

In this section, we will prove Theorem 2. The main idea of the proof for the first part of the theorem, existence of $\hat{m}$, stems from that of Bers[2]. Positivity of the horizontal velocity is proved by a comparison principle. We estimate the deflection angle in hodograph plane, where the deflection angle satisfies an elliptic equation. For the last part of the theorem, i.e., the monotonicity of the maximum of flow speed with respect to the incoming mass flux, is a consequence of certain partial hodograph transformation and a comparison principle for elliptic equations.

Let $\{r_i\}_{i=1}^{\infty}$ be a strictly increasing sequence satisfying $\lim_{i\to\infty} r_i = 1$. For $\forall t > 0$, we first solve the problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla\psi}{H_n(|\nabla\psi|^2)}\right) = 0, & \text{in } \Omega, \\ \psi = 0, & \text{on } S_1, \\ \psi = t, & \text{on } S_2, \end{cases} \tag{66}$$

where

$$H_n(s) = \begin{cases} H(s), & \text{if } 0 \le s \le r_n^2, \\ H\left(\left(\frac{r_n+1}{2}\right)^2\right), & \text{if } s > \left(\frac{r_n+1}{2}\right)^2, \end{cases}$$

is a smooth decreasing function and satisfies $H_n(s) - 2H_n'(s)s < \Gamma_n$ with some $\Gamma_n > 0$, for all $s \ge 0$. Denote the solution to the problem (66) by $\psi_n(\cdot;t)$, and set $M_n(t) = \sup_\Omega |\nabla\psi_n(\cdot;t)|$. Then clearly,

$$M_n(t) \ge t.$$

Moreover, we have the following lemma,

**Lemma 9** *$M_n(t)$ is a continuous function of $t$.*

Assume that the lemma holds first. Then there exists the largest $R_n > 0$ such that

$$M_n(t) < r_n, \quad \forall t \in (0, R_n),$$

and, furthermore, there exists the smallest $S_n \in (0, R_n)$ such that

$$M_n(t) > r_{n-1}, \quad \forall t \in (S_n, R_n).$$

Moreover, obviously, $R_{n+1} \geq R_n$. Set

$$\hat{m} = \lim_{n\to\infty} R_n.$$

Then it is clear that $\forall m < \hat{m}$, there exists $R_n$ such that $m < R_n$, thus $M(m) = M_n(m) < r_n < 1$. Moreover, $\forall M \in (0,1)$, there exists $n$, such that $M \in (0, r_n)$, therefore, there is $m \in (0, R_n)$ by Lemma 9, such that $M(m) = M_n(m) = M$. This finishes the proof for the first part of Theorem 2.

It remains to prove Lemma 9.

**Proof of the Lemma 9:** Let $t_j \to t$. Without loss of generality, we assume that $\sup_{j\geq 1} t_j < T$.

In a same way as what we have done in Section 2, it is easy to see that the solution, $\psi_n(\cdot; t_j)$, to the problem (66) satisfies the estimates

$$|\psi_n(\cdot; t_j)|_{1;\Omega} \leq C(T) \tag{67}$$

and

$$|\psi_n(\cdot; t_j)|_{C^{2,\alpha}(\Omega_L)} \leq C(T, L). \tag{68}$$

Thanks to the estimates (67) and (68), there exists a subsequence of $\{\psi_n(\cdot; t_j)\}$ by Arzela-Ascoli lemma and a diagonal procedure such that

$$\psi_n(\cdot; t_{k_j}) \to \Psi(\cdot) \text{ in } C^{2,\beta}(\Omega_L) \quad \text{as } t_{k_j} \to t,$$

for each $L$ and $0 < \beta < \alpha$. Clearly, $\Psi$ solves the boundary value problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla\Psi}{H_n(|\nabla\Psi|^2)}\right) = 0, & \text{in } \Omega, \\ \Psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)} t, & \text{on } \partial\Omega. \end{cases}$$

Moreover,

$$|\Psi|_{1;\Omega} \leq C(T)$$

by (67). Therefore, the flow associated with $\Psi$ approximates to uniform flows at far fields by Lemma 6 and Lemma 7. On the other hand, as in Section 2, there exists a $\psi_n(\cdot; t)$ which solves

$$\begin{cases} \operatorname{div}(\frac{\nabla\psi}{H_n(|\nabla\psi|^2)}) = 0, & \text{in } \Omega, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)} t, & \text{on } \partial\Omega, \end{cases}$$

and $\psi_n(\cdot;t)$ approximates to uniform flows at far fields by Lemma 6 and Lemma 7. It follows from the proof of lemma 8 that

$$\Psi(\cdot) = \psi_n(\cdot;t).$$

Using the uniqueness lemma again, we have

$$\psi_n(\cdot;t_j) \to \psi_n(\cdot;t) \quad \text{in } C^{2,\beta}(\Omega_L) \text{ for } \forall L.$$

Therefore,

$$M_n(t_j) \to M_n(t).$$

This completes the proof of the Lemma.

Now let us show that the horizontal velocity is always positive, which is important for the subsequent applications.

**Lemma 10** *Suppose* $0 < m < \hat{m}$*, then the solution* $\psi$ *to (16) satisfies* $\psi_{x_2} > 0$ *in* $\bar{\Omega}$*.*

**Proof:** We rewrite the equation in (16) as

$$H(|\nabla\psi|^2)\Delta\psi - 2H'(|\nabla\psi|^2)\partial_i\psi\partial_j\psi\partial_{ij}\psi = 0.$$

It follows that the function $w = \partial_2\psi$ satisfies the equation

$$\bar{a}_{ij}\partial_{ij}w + \bar{b}_i\partial_i w = 0, \tag{69}$$

where

$$\begin{aligned} \bar{a}_{ij} &= H(|\nabla\psi|^2)\delta_{ij} - 2H'(|\nabla\psi|^2)\partial_i\psi\partial_j\psi, \\ \bar{b}_i &= 2H'(|\nabla\psi|^2)\Delta\psi\partial_i\psi - 4H'(|\nabla\psi|^2)\partial_j\psi\partial_{ij}\psi - 4H''(|\nabla\psi|^2)\partial_l\psi\partial_j\psi\partial_{lj}\psi\partial_i\psi. \end{aligned}$$

So the equation for $w$ satisfies the maximum principle. Note that along $S_1$, $\psi = 0$, so,

$$\partial_1\psi(x_1, f_1(x_1)) + \partial_2\psi(x_1, f_1(x_1))f_1'(x_1) = 0.$$

Therefore, the inner normal derivative satisfies

$$\frac{\partial\psi}{\partial n}(x_1, f_1(x_1)) = \partial_2\psi(1+(f_1')^2).$$

On the other hand, $\psi$ attains its minimum of at $S_1$, by the Hopf lemma, one has $\frac{\partial\psi}{\partial n} > 0$. Thus $\partial_2\psi > 0$ on $S_1$. Similarly, one can prove that $\partial_2\psi > 0$ on $S_2$. Therefore, $\partial_2\psi > 0$ on the solid boundaries. Since the flow approximates the uniform flows at the far fields, thus $\partial_2\psi$ tends to some uniform positive constants at the far fields, hence $\partial_2\psi > 0$ by the maximum principle for equation (69). □

With the help of the positivity of the horizontal velocity, we can obtain an estimate for the deflection angle of the flow.

**Lemma 11** *If $0 < m < \hat{m}$, then for any uniformly subsonic solution $\psi$ to (16), the estimate*

$$\underline{\theta} \le \theta \le \bar{\theta}, \tag{70}$$

*holds for the deflection angle of the fluid, $\theta = \arctan\frac{-\partial_{x_1}\psi}{\partial_{x_2}\psi}$, where $\underline{\theta}$ and $\bar{\theta}$ are given in (20) and (21) respectively.*

**Proof:** To prove the lemma, we first introduce the hodograph transformation. For a smooth solution to the equation

$$\mathrm{div}\left(\frac{\nabla\psi}{H(|\nabla\psi|^2)}\right) = 0,$$

denote

$$u = \frac{\partial_{x_2}\psi}{H(|\nabla\psi|^2)},\ v = -\frac{\partial_{x_1}\psi}{H(|\nabla\psi|^2)},\ \text{and } \rho = H(|\nabla\psi|^2).$$

Then, the velocity potential can be defined as

$$\varphi(x_1, x_2) = \int_{(0,f_1(0))}^{(x_1,x_2)} u dx_1 + v dx_2.$$

It follows from equation (16) that $\varphi$ is well-defined in $\Omega$, and $\varphi_{x_1} = u$, $\varphi_{x_2} = v$. Moreover, the Jocabian

$$J = \det\frac{\partial(\varphi,\psi)}{\partial(x_1,x_2)} = \frac{|\nabla\psi|^2}{H(|\nabla\psi|^2)} > 0, \tag{71}$$

by Lemma 10. Thus by the inverse function theorem, one can represent $(x_1, x_2)$ in terms of $(\varphi, \psi)$ as

$$\begin{cases} x_1 = h_1(\varphi, \psi), \\ x_2 = h_2(\varphi, \psi). \end{cases} \tag{72}$$

Although the representation (72) is usually valid only locally under the condition (71), we would like to show this representation holds globally in the case of uniformly subsonic flows. Suppose that there exist $x = (x_1, x_2)$ and $y = (y_1, y_2)$ such that $\varphi(x_1, x_2) = \varphi(y_1, y_2)$ and $\psi(x_1, x_2) = \psi(y_1, y_2)$. Then it follows that $x$ and $y$ are on the same streamline. Define the streamline as follows

$$\begin{cases} \frac{dz_1}{ds} = u(z_1, z_2), \\ \frac{dz_2}{ds} = v(z_1, z_2), \\ z_1(0) = x_1, \; z_2(0) = x_2. \end{cases}$$

Then there exists $s_0$ such that $z_1(s_0) = y_1$ and $z_2(s_0) = y_2$. Without loss of generality, we assume that $s_0 \geq 0$. By the definition of $\varphi$, one has

$$\varphi(y_1, y_2) = \varphi(x_1, x_2) + \int_0^{s_0} (u^2(z(s)) + v^2(z(s)))ds.$$

Therefore, $\varphi(y_1, y_2) = \varphi(x_1, x_2)$ holds if and only if $s_0 = 0$ because of positivity of $u$. Thus the map $(x_1, x_2) \mapsto (\varphi, \psi)$ is a globally one-to-one map. Therefore, the representation (72) is valid globally.

Using the polar coordinates, $u = q\cos\theta$, $v = q\sin\theta$, after direct calculations, one has

$$\begin{aligned} u_{x_2} - v_{x_1} &= -q^2\theta_\varphi + \rho q q_\psi, \\ (\rho u)_{x_1} + (\rho v)_{x_2} &= \frac{d}{dq}(\rho q) q q_\varphi + \rho^2 q^2 \theta_\psi. \end{aligned}$$

With the help of Bernoulli's law (5), one obtains

$$q_\psi = \frac{q}{\rho}\theta_\varphi, \quad q_\varphi = \frac{\rho q}{M^2 - 1}\theta_\psi.$$

where $M^2 = q^2/c^2$ is Mach number. Thus the deflection angle $\theta$ satisfies the equation

$$(\frac{q}{\rho}\theta_\varphi)_\varphi + (\frac{\rho q}{1 - M^2}\theta_\psi)_\psi = 0. \tag{73}$$

Note that the flow is subsonic, which implies $M^2 < 1$, therefore, the equation (73) is elliptic. Now the domain $\Omega$ becomes $\tilde{\Omega} = \{(\varphi,\psi)|0 < \psi < m, \varphi \in \mathbb{R}\}$, and the boundary conditions in (16) become

$$\theta = \arctan f_1'(x_1),\ \psi = 0; \quad \theta = \arctan f_2'(x_1),\ \psi = m.$$

Since at far fields, the flow approximates to uniform flows whose vertical components of the velocity fields tend to zero, therefore,

$$\theta \sim 0 \text{ when } |\varphi| \text{ is sufficiently large.}$$

Thus by the maximum principle, we have

$$\underline{\theta} \leq \theta \leq \bar{\theta}.$$

This finishes the proof of the lemma □

With all these properties of subsonic flows at hand, we are ready to prove the monotonicity of the maximum of the flow speed with respect to the incoming mass flux.

**Lemma 12** *Let $0 < m_1 < m_2 < \hat{m}$. Suppose that $\psi_i$ are solutions to (16) associated with the incoming mass flux $m_i(i = 1, 2)$. Then*

$$|\nabla\psi_1(x)| < |\nabla\psi_2(x)|, \quad \forall x \in \partial\Omega. \tag{74}$$

**Proof:** Let $\psi$ be the solution to (16). Then set

$$\begin{cases} Y_1 = x_1, \\ Y_2 = \psi(x_1, x_2). \end{cases} \tag{75}$$

By Lemma 10, $\psi_{x_2} > 0$, one may conclude that the transformation $(x_1, x_2) \mapsto (Y_1, Y_2)$ by (75) is invertible so that one can represent $x_2$ as a function of $Y_1$ and $Y_2$. Set $x_2 = \Phi(Y_1, Y_2)$. Then $\partial_{Y_2}\Phi > 0$ and $\Phi$ satisfies the following problem

$$\begin{cases} B_{ij}(\nabla\Phi)\partial_{ij}\Phi = 0, \\ \Phi = f_1(Y_1), \quad \text{on } Y_2 = 0, \\ \Phi = f_2(Y_1), \quad \text{on } Y_2 = m, \end{cases}$$

where

$$B_{11}=\frac{H(\frac{1+\Phi_{Y_1}^2}{\Phi_{Y_2}^2})-2H'(\frac{1+\Phi_{Y_1}^2}{\Phi_{Y_2}^2})\frac{\Phi_{Y_1}^2}{\Phi_{Y_2}^2}}{H(\frac{1+\Phi_{Y_1}^2}{\Phi_{Y_2}^2})-2H'(\frac{1+\Phi_{Y_1}^2}{\Phi_{Y_2}^2})\frac{1+\Phi_{Y_1}^2}{\Phi_{Y_2}^2}}\Phi_{Y_2}^2,\ B_{12}=B_{21}=-\Phi_{Y_1}\Phi_{Y_2},\ B_{22}=1+\Phi_{Y_1}^2.$$

Moreover,

$$\frac{\partial\Phi}{\partial Y_1}=-\frac{\partial\psi}{\partial x_1}\Big/\frac{\partial\psi}{\partial x_2},\quad \frac{\partial\Phi}{\partial Y_2}=1\Big/\frac{\partial\psi}{\partial x_2}. \tag{76}$$

Define $\Phi_i$ as the transformation corresponding to $\psi_i (i=1,2)$. Then for $l=1,2$,

$$\begin{cases} B_{ij}(\nabla\Phi_l)\partial_{ij}\Phi_l=0, & \text{in } G_l,\\ \Phi_l=f_1(Y_1), & \text{on } Y_2=0,\\ \Phi_l=f_2(Y_1), & \text{on } Y_2=m_l, \end{cases}$$

where $G_l=\{(Y_1,Y_2)|Y_2\in(0,m_l),Y_1\in\mathbb{R}\}$. Since $\partial_{Y_2}\Phi_2>0$, $\Phi_2(Y_1,m_1)<\Phi_2(Y_1,m_2)$. Furthermore, since the flows approximate to uniform flows at far fields, therefore, it follows from (76) that, for any $\varepsilon$ sufficiently small satisfying $0<\varepsilon<\min\{\frac{m_2-m_1}{2m_1m_2},\frac{(b-a)(m_2-m_1)}{2m_1m_2}\}$, there exists a positive number $L_0$ sufficiently large, such that if $Y_1<-L_0$,

$$\Phi_1(Y_1,Y_2)=f_1(Y_1)+\int_0^{Y_2}\frac{\partial\Phi_1}{\partial Y_2}(Y_1,t)dt\geq f_1(Y_1)+(\frac{1}{m_1}-\varepsilon)Y_2,$$

and

$$\Phi_2(Y_1,Y_2)=f_1(Y_1)+\int_0^{Y_2}\frac{\partial\Phi_2}{\partial Y_2}(Y_1,t)dt\leq f_1(Y_1)+(\frac{1}{m_2}+\varepsilon)Y_2,$$

Similarly, for $Y_1\geq L_0$, it holds that

$$\Phi_1(Y_1,Y_2)=f_1(Y_1)+\int_0^{Y_2}\frac{\partial\Phi_1}{\partial Y_2}(Y_1,t)dt\geq f_1(Y_1)+(\frac{b-a}{m_1}-\varepsilon)Y_2, \tag{77}$$

$$\Phi_2(Y_1,Y_2)=f_1(Y_1)+\int_0^{Y_2}\frac{\partial\Phi_2}{\partial Y_2}(Y_1,t)dt\leq f_1(Y_1)+(\frac{b-a}{m_2}+\varepsilon)Y_2. \tag{78}$$

Thus in the domain $G_{1,L}=\{(Y_1,Y_2)|Y_2\in(0,m_1),|Y_1|<L\}$, when $L\geq L_0$ is large enough, the function $\bar\Phi=\Phi_2-\Phi_1$ satisfies

$$\begin{cases} B_{ij}(\nabla\Phi_2)\partial_{ij}\bar\Phi+(B_{ij}(\nabla\Phi_2)-B_{ij}(\nabla\Phi_1))\partial_{ij}\Phi_1=0, & \text{in } G_{1L},\\ \bar\Phi=0, & Y_2=0,\ |Y_1|<L,\\ \bar\Phi<0, & Y_2=m_1,\ |Y_1|<L,\\ \bar\Phi<0, & |Y_1|=L,\ Y_2\in(0,m_1). \end{cases}$$

Therefore, $\bar{\Phi}$ attains its maximum on $Y_2 = 0$, thus it follows from the Hopf Lemma that

$$\frac{\partial \bar{\Phi}}{\partial n} < 0,$$

where $n$ is the inner normal, i.e. $\frac{\partial \Phi_2}{\partial n} < \frac{\partial \Phi_1}{\partial n}$. Note that $\Phi_i$ $(i = 1, 2)$, attain their minimum at $Y_2 = 0$. Thus

$$\frac{\partial \Phi_1}{\partial n} > \frac{\partial \Phi_2}{\partial n} > 0.$$

Note that $|\nabla \psi|^2 = \frac{1+|\partial_1 \Phi|^2}{|\partial_2 \Phi|^2}$ by (76), therefore

$$\begin{aligned} |\nabla \psi_1(x_1, f_1(x_1))|^2 &= \frac{1+|\partial_1 \Phi_1|^2}{|\partial_2 \Phi_1|^2}(Y_1, 0) = \frac{1+|f_1'(Y_1)|^2}{|\partial_2 \Phi_1|^2(Y_1, 0)} \\ < \frac{1+|f_1'(Y_1)|^2}{|\partial_2 \Phi_2|^2(Y_1, 0)} &= \frac{1+|\partial_1 \Phi_2|^2}{|\partial_2 \Phi_2|^2}(Y_1, 0) = |\nabla \psi_2(x_1, f_1(x_1))|^2. \end{aligned}$$

It is the same to prove that (74) holds on $S_2$ by studying $\psi_i - m_i$ instead of $\psi_i$. This completes the proof of the Lemma. □

Since the flow approximates to uniform flows at far fields, therefore, the flow tends to its supermum only on the solid boundaries, this is the consequence of the Berstein estimate of the governing equation in (16) for potential flows. Thus if $m_1 < m_2$, by Lemma 12, one has that

$$M(m_1) < M(m_2).$$

Therefore, as $m \uparrow \hat{m}$, $M(m) \uparrow 1$. This finishes the proof of Theorem 2.

## 5 Subsonic-Sonic Flows

In this section, we will employ the theory of compensated compactness developed by Morawetz[14] (see also [4]) to obtain a global subsonic-sonic flow. It will be shown that the existence of a weak solution for subsonic-sonic flow is a direct consequence of the properties obtained in Section 4 and a compensated compactness framework.

Let a sequence of functions $w^\varepsilon(x_1, x_2) = (q^\varepsilon, \theta^\varepsilon)(x_1, x_2)$ be defined in an open set $\Omega \subset \mathbb{R}^2$, and satisfy the following conditions (C):

(C.1) $0 \le q^\varepsilon(x_1, x_2) \le 1$ a.e. in $\Omega$.

(C.2) $|\theta^\varepsilon(x_1, x_2)| \le \hat{\theta} < \frac{\pi}{2}$, for some constant $\hat{\theta}$ independent of $\varepsilon$.

(C.3) $\partial_{x_1}\eta_\pm(w^\varepsilon)+\partial_{x_2}\Lambda_\pm(w^\varepsilon)$ are confined in a compact set in $H^{-1}_{loc}(\Omega)$ for the momentum entropy-entropy flux pair

$$(\eta_+, \Lambda_+) = (\rho q^2 \cos^2\theta + p(\rho), \rho q^2 \sin\theta\cos\theta), \quad (\eta_-, \Lambda_-) = (\rho q^2 \sin\theta\cos\theta, \rho q^2 \sin^2\theta + p(\rho)),$$

where $p = p(\rho)$, and $\rho = \rho(q^2)$ is determined by Bernoulli's law (5). For example, for polytropic gases, $p = \frac{\rho^\gamma}{\gamma}$, and $\rho = \rho(q^2)$ is determined by (7).

**Remark 11:** (C.2) appeared as an assumption in Morawetz's theory[14], while (C.1) and (C.3) are conditions in [4]. Here we will use $(q, \theta)$ instead of $(u, v)$ which was used in [4].

By the Young measure representation theorem and the div-curl lemma [15][16], under the conditions (C), one has the following identity

$$\begin{aligned}&\langle \nu(w), \eta_+(w)\Lambda_-(w) - \eta_-(w)\Lambda_+(w)\rangle \\ &= \langle \nu(w), \eta_+(w)\rangle\langle \nu(w), \Lambda_-(w)\rangle - \langle \nu(w), \eta_-(w)\rangle\langle \nu(w), \Lambda_+(w)\rangle, \end{aligned} \tag{79}$$

where $\nu = \nu_{(x_1,x_2)}(w)$ is the associated Young measure for the sequence $w^\varepsilon(x_1, x_2) = (q^\varepsilon, \theta^\varepsilon)(x_1, x_2)$. The main point for the compensated compactness framework is to obtain commutation of certain nonlinear compositions and the weak-$*$ limits by studying the properties of the Young measures.

First of all, let us prove the following compensated compactness framework which is based on some observations in[4].

**Theorem 13** *Let a sequence of function $w^\varepsilon(x_1, x_2) = (q^\varepsilon, \theta^\varepsilon)(x_1, x_2)$ satisfy the Conditions (C). Then there exists a subsequence $\{w^{\varepsilon_k}\}$ of $\{w^\varepsilon\}$ and $w(x_1, x_2) = (q, \theta)(x_1, x_2)$ such that*

$$(q^{\varepsilon_k}, \theta^{\varepsilon_k}) \to (q, \theta), \tag{80}$$

$$q^{\varepsilon_k}\cos\theta^{\varepsilon_k} \to q\cos\theta, \quad q^{\varepsilon_k}\sin\theta^{\varepsilon_k} \to q\sin\theta, \tag{81}$$

$$\rho((q^{\varepsilon_k})^2)q^{\varepsilon_k}\cos\theta^{\varepsilon_k} \to \rho(q^2)q\cos\theta, \quad \rho((q^{\varepsilon_k})^2)q^{\varepsilon_k}\sin\theta^{\varepsilon_k} \to \rho(q^2)q\sin\theta, \tag{82}$$

*where all the convergence in (80)-(82) are weak-$*$ convergence in $L^\infty(\Omega)$, and $w = (q, \theta)$ satisfies*

$$
\begin{aligned}
0 \le q(x_1, x_2) &\le 1, \\
|\theta(x_1, x_2)| &\le \hat{\theta}.
\end{aligned}
$$

**Proof:** It is easy to check that the identity (79) is equivalent to

$$\langle \nu(w_1) \otimes \nu(w_2), I(w_1, w_2) \rangle = 0, \tag{83}$$

where

$$I(w_1, w_2) = (\eta_+(w_1) - \eta_+(w_2))(\Lambda_-(w_1) - \Lambda_-(w_2)) - (\eta_-(w_1) - \eta_-(w_2))(\Lambda_+(w_1) - \Lambda_+(w_2)).$$

By direct calculations as in [4], one has

$$I(w_1, w_2) = -\rho_1\rho_2 q_1^2 q_2^2 \sin^2(\theta_2 - \theta_1) + (p_1 - p_2)^2 + (p_1 - p_2)(\rho_1 q_1^2 - \rho_2 q_2^2).$$

It follows from Bernoulli's law (5) that

$$(p_1 - p_2)^2 + (p_1 - p_2)(\rho_1 q_1^2 - \rho_2 q_2^2) = (p(\rho_1) - p(\rho_2))(J(\rho_1) - J(\rho_2)),$$

where

$$J(s) = p(s) + s - 2s \int_1^s \frac{p'(t)}{t} dt.$$

It is easy to calculate that $J''(s) < 0$, and $J'(1) = 0$. Therefore, $J'(s) < 0$ if $s \in (1, \rho_{max})$. Since $p$ is an increasing function of $\rho$, thus $(p(\rho_1) - p(\rho_2))(J(\rho_1) - J(\rho_2)) \le 0$, for $\rho_1,\ \rho_2 \ge 1$. So, $I(w_1, w_2) \le 0$ and, moreover,

$$I(w_1, w_2) = 0 \text{ if and only if } q_1 = q_2 = 0, \text{ or } q_1 = q_2 \neq 0,\ \theta_1 = \theta_2. \tag{84}$$

Now we have the following claim:

Claim: supp$\nu \subset \{(0, \theta) | \theta \in [-\hat{\theta}, \hat{\theta}]\}$ or $\nu$ is a Dirac measure supported on some point $(q, \theta)$ with $q \neq 0$.

Proof of the claim: Trivially, one has supp$\nu \subset E_1 \cup E_2$, where $E_1 = \{(0, \theta) | \theta \in [-\hat{\theta}, \hat{\theta}]\}$, and $E_2 = \{(q, \theta) | 0 < q \le 1, \theta \in [-\hat{\theta}, \hat{\theta}]\}$. If both $\nu(E_1)$ and $\nu(E_2)$ are not equal to zero, then $(\nu\otimes\nu)(E_1\times E_2)\neq 0$, which contradicts with (83) and (84). Therefore, the support of $\nu$ is included in either $E_1$ or $E_2$. If $\nu(E_1)=0$, suppose $\nu$ is not a Dirac measure, then there exists $E_3\subset E_2$ such that $\nu(E_3)>0$ and $\nu(E_2\setminus E_3)>0$, therefore,

$$(\nu\otimes\nu)(E_3\times(E_2\setminus E_3))>0.$$

This also contradicts to (83) and (84). Thus $\nu$ is a Dirac measure. This finishes the proof of the claim.

Since $\{w^\varepsilon\}$ is uniformly bounded, without loss of generality, one assumes that $\{w^{\varepsilon_k}\}$ satisfies

$$(q^{\varepsilon_k},\theta^{\varepsilon_k})\to(q,\theta),\quad \text{weak}-*\text{ in } L^\infty(\Omega).$$

On the other hand, it follows from the Young measure representation theorem that

$$w^{\varepsilon_k}=(q^{\varepsilon_k},\theta^{\varepsilon_k})\to\int(q,\theta)d\nu_x((q,\theta))=\begin{cases}(q_0,\theta_0) & \text{if supp}\nu_x=\{(q_0,\theta_0)\}\subset E_2,\\ (0,\theta) & \text{if supp}\nu_x\subset E_1.\end{cases}$$

Therefore

$$\begin{aligned}\rho((q^{\varepsilon_k})^2)q^{\varepsilon_k}\cos\theta^{\varepsilon_k}(x) &\to \int\rho(q^2)q\cos\theta d\nu_x((q,\theta))\\ &= \begin{cases}\rho(q_0^2)q_0\cos\theta_0 & \text{if supp}\nu_x=\{(q_0,\theta_0)\}\\ 0 & \text{if supp}\nu_x\subset E_1\end{cases}\\ &= \rho(q^2)q\cos\theta.\end{aligned}$$

Similarly, we can obtain the convergence (82). Hence the proof of the theorem is completed.

□

**Remark 12:** Theorem 13 holds for general equation of states, $p'(\rho)>0$ for $\rho>0$ and $p''(\rho)\geq 0$, without assuming that the gas is polytropic or isothermal.

As an application of this compensated compactness framework and the properties for uniform subsonic flows, we can take limit for $m\uparrow\hat{m}$ in Theorem 2.

For the sequence $m^\varepsilon \uparrow \hat{m}$, let $\{\psi^\varepsilon(x_1,x_2)\}$ be solutions associated with $m^\varepsilon$ given in Theorem 2. Define

$$q^\varepsilon(x_1,x_2) = \frac{|\nabla\psi^\varepsilon(x_1,x_2)|}{H(|\nabla\psi^\varepsilon(x_1,x_2)|^2)}, \text{ and } \theta^\varepsilon(x_1,x_2) = \arctan\frac{-\partial_{x_1}\psi^\varepsilon(x_1,x_2)}{\partial_{x_2}\psi^\varepsilon(x_1,x_2)}.$$

By Lemma 10, both of them are well-defined. Moreover, when $m^\varepsilon < \hat{m}$, the flow is subsonic, so $q^\varepsilon < 1$. It follows from (70) and the conditions (13)-(15) that $\underline{\theta} \le \theta^\varepsilon \le \bar{\theta}$ and $\underline{\theta} > -\frac{\pi}{2}$ and $\bar{\theta} < \frac{\pi}{2}$. Thus, there exists a constant $0 < \hat{\theta} < \frac{\pi}{2}$ such that $w^\varepsilon = (q^\varepsilon, \theta^\varepsilon)$ satisfies (C.1) and (C.2). With the help of Bernoulli's law and smoothness of the solutions $\{\psi^\varepsilon\}$, it is easy to conclude that

$$\partial_{x_1}\eta_\pm(w^\varepsilon) + \partial_{x_2}\Lambda_\pm(w^\varepsilon) = 0,$$

and thus compact in $H^{-1}_{loc}(\Omega)$. Since the equation in (16) is equivalent to

$$\begin{cases} \partial_{x_1}(\rho((q^\varepsilon)^2)q^\varepsilon\cos\theta^\varepsilon) + \partial_{x_2}(\rho((q^\varepsilon)^2)q^\varepsilon\sin\theta^\varepsilon) = 0 \\ \partial_{x_1}(q^\varepsilon\sin\theta^\varepsilon) - \partial_{x_2}(q^\varepsilon\cos\theta^\varepsilon) = 0 \end{cases}$$

Applying Theorem 13, we obtain that (25) is satisfied in the sense of distribution with $u = q\cos\theta$ and $v = q\sin\theta$.

The fact that the boundary condition (12) is satisfied for $(u,v)$ in the sense of distribution is standard by multiplying the system by a test function and applying divergence theorem and the fact that the sequence of subsonic solutions does satisfy the boundary condition (12), which implies $(u,v)$ satisfies the boundary condition (12) actually as the normal trace of divergence measure field $(\rho u, \rho v)$ on the boundary in the sense of Anzellotti[1]. So we finish the proof of Theorem 3.

Further characterizations of the subsonic-sonic flow we obtained are left for future.

## Appendix

In this appendix, the truncated domain $\Omega_L$ in Section 2 is constructed explicitly. It follows from (13)-(15) that there exists $L_0 > 0$ sufficiently large such that

$$f_1(x_1) < \tfrac{1}{16} \text{ and } f_2(x_1) > \tfrac{15}{16} \quad \text{for } x_1 < -L_0,$$
$$f_1(x_1) < \tfrac{b+15a}{16} \text{ and } f_2(x_1) > \tfrac{15b+a}{16} \quad \text{for } x_1 > L_0,$$

and

$$|f_1'(x_1)| < C, \quad |f_2'(x_1)| < C \quad \text{for any } x_1 \in \mathbb{R}.$$

with some positive constant $C > 1$. We concentrate on constructing $\Omega_L$ in the half plane $\{(x_1, x_2)|x_1 \le 0\}$. The construction in the half plane $\{(x_1, x_2)|x_1 \ge 0\}$ is the same. Define

$$f_{1,L}^l(x_1) = \begin{cases} f_1(x_1), & x_1 > -\frac{3L}{2}, \\ f_1(x_1) + \frac{16C}{\eta^3}(x_1 + \frac{3L}{2})^4, & x_1 \le -\frac{3L}{2}, \end{cases}$$

and

$$f_{2,L}^l(x_1) = \begin{cases} f_2(x_1), & x_1 > -\frac{3L}{2}, \\ f_2(x_1) - \frac{16C}{\eta^3}(x_1 + \frac{3L}{2})^4, & x_1 \le -\frac{3L}{2}. \end{cases}$$

where $\eta > 0$ is a constant to be determined. Then direct calculations show that

$$(f_{1,L}^l)'(x_1) < -C \text{ and } (f_{2,L}^l)'(x_1) > C \text{ for } x_1 \le -\frac{3L}{2} - \frac{\eta}{2}.$$

By choosing $\eta = \frac{1}{256C}$, it follows that

$$f_{1,L}^l(x_1) \le \frac{1}{8} \text{ and } f_{2,L}^l(x_1) \ge \frac{7}{8} \quad \text{for } -\frac{3L}{2} - \eta \le x_1 \le -\frac{3L}{2}.$$

Let $x_1 = g_{i,L}^l(x_2)$ be the inverse functions of $x_2 = f_{i,L}^l(x_1)$ with $x_1 \le -\frac{3L}{2} - \frac{\eta}{2}$ $(i = 1, 2)$. Choose smooth nonnegative functions $x_1 = \zeta_1(x_2)$, $x_1 = \zeta_2(x_2)$ and $x_1 = \zeta(x_2)$ satisfying

$$\zeta_1(x_2) = \begin{cases} 0, & \text{if } x_2 > \frac{3}{8}, \\ 1, & \text{if } x_2 < \frac{1}{4}, \end{cases}$$

$$\zeta_2(x_2) = \begin{cases} 1, & \text{if } x_2 > \frac{3}{4}, \\ 0, & \text{if } x_2 < \frac{5}{8}, \end{cases}$$

and

$$\zeta(x_2) = \begin{cases} 0, & \text{if } x_2 > \frac{7}{8} \text{ or } x_2 < \frac{1}{8}, \\ 1, & \text{if } \frac{1}{4} < x_2 < \frac{3}{4}. \end{cases}$$

Now, define

$$g(x_2) = \zeta_1(x_2) g^l_{1,L}(x_2) + \zeta_1(x_2) g^l_{1,L}(x_2) - \frac{15L}{8}\zeta(x_2)$$

Then $\partial\Omega_L \cap \{(x_1, x_2)|x_1 \le 0\} = \{(x_1, f^l_{1,L}(x_1))| - \frac{3L}{2} - \frac{\eta}{2} < x_1 \le 0\} \cup \{(x_1, f^l_{2,L}(x_1))| - \frac{3L}{2} - \frac{\eta}{2} < x_1 \le 0\} \cup \{(g(x_2), x_2)|f^l_{1,L}(-\frac{3L}{2} - \frac{\eta}{2}) \le x_2 \le f^l_{2,L}(-\frac{3L}{2} - \frac{\eta}{2})\}$. Similarly, one can construct the domain $\Omega_L$ in the half plane $\{(x_1, x_2)|x_1 \ge 0\}$. Thus the construction of truncated domain $\Omega_L$ is completed.

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