Weak and Strong Solutions for the Incompressible Navier-Stokes Equations with Damping Term

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Abstract: In this paper we intend to understand the influences of the damping term $|u|^{\beta-1}u$ on the well-posedness of the classical incompressible Navier-Stokes equations. Our results show that the Cauchy problem of the damped Navier-Stokes equations will have global weak solutions for any $\beta \geq 1$, global strong solutions for any $\beta \geq 7/2$ and will have unique strong solution for any $7/2 \leq \beta \leq 5$. Note that when $1 \leq \beta \leq 4$, the solutions of the damped Navier-Stokes equations do not belong to the Serrin's class, which is the regularity class of the classical Navier-Stokes equations.

Key Words: Navier-Stokes equations, damping term, weak solutions, strong solutions

AMS(1991)Subject Classification: 35Q35, 76B03

1 Introduction

We consider the following incompressible Navier–Stokes equations with damping term

$$\begin{cases} u_t - \mu\triangle u + u\cdot\nabla u + \alpha|u|^{\beta-1}u + \nabla p = 0, & (x,t)\in R^3\times(0,T), \\ \mathrm{div} u = 0, & (x,t)\in R^3\times[0,T), \\ u\mid_{t=0} = u_0, & x\in R^3, \\ |u|\to 0, & as\ \ |x|\to\infty. \end{cases} \tag{1.1}$$

The unknown functions here are $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p(x,t)$, which stand for the velocity fields and the pressure of the flow, respectively. $\alpha|u|^{\beta-1}u$ is a damping term with $\beta \geq 1, \alpha > 0$ two constants. The given

*The research is partially supported by the Key Project of NSFC (No.10431060), Beijing NSF (No.1042003) and Key Project of Beijing NSF and Beijing Eduction Committee.

functions $u_0 = u_0(x)$ is the initial velocity. The constant $\mu > 0$ represents the viscosity coefficient of the flow.

Although many mathematical studies have been made for the well-posedness of the 3-D classical incompressible Navier-Stokes equation (see [7],[9], [5] and references therein), the uniqueness of weak solutions and the global existence (on time) of strong solution remain completely open. Introducing the class $L^s(0,T;L^q)$, Serrin showed that if u is a weak solution in such a class with $2/s + 3/q < 1$, then u is smooth. Since Serrin's criterion, many efforts have been made to obtain a larger class of weak solution in which uniqueness and regularity hold. Generally speaking, up to now, the obtained results show that if the weak solutions $u(x,t)$ of the classical Navier-Stokes equations belongs to $L^s(0,T;L^q)$ with $2/s + 3/q \leq 1$ satisfying $2 \leq s \leq \infty, 3 \leq q \leq \infty$, then the weak solution is regular and unique(see [3],[2], [4], [6], [8], [1], [7] and references therein). The class $L^s(0,T;L^q)$ is also called Serrin's class.

In this paper, we consider the Cauchy problem (1.1) of the classical Navier-Stokes equations with damping term $|u|^{\beta-1}u$. The damping term is from the resistance to the motion of the flow. It describes various physical situations such as porous media flow and so on. Roughly speaking, the damping term will make the solutions of the classical incompressible Navier-Stokes equations "better". In this paper we intend to understand the influences of the damping term $|u|^{\beta-1}u$ on the well-posedness of the classical incompressible Navier-Stokes equations.

The usual a priori energy estimates to the damped Navier-Stokes equations leads directly to $u \in L^{\beta+1}(0,T;L^{\beta+1}(R^3))$. We will show that there exist global weak solutions of (1.1) for any $\beta \geq 1$ and there exists a global strong solution for any $\beta \geq 7/2$. However, the damping term causes new difficulties in the proof of the uniqueness of the strong solution. We will prove that for any $7/2 \leq \beta \leq 5$, the global strong solution is unique. But it is still not available for the case $\beta > 5$. Recalling the Serrin's class to the classical Navier-Stokes equations, the solutions of (1.1) will belong to Serrin's class if and only if $\beta \geq 4$. Therefore, our result shows that the damped Navier-Stokes equations have special features of its own especially in the aspect of regularity of the solutions.

We apply the Galerkin method to construct the approximate solutions and make more delicate a prior estimates to proceed compactness arguments. We are happy to find that new more a priori estimates guarantee that the obtained solution u belongs to $L^\infty(0,T;W^{1,2}_{0,\sigma}(R^3)) \cap L^\infty(0,T;L^{\beta+1}(R^3)) \cap L^2(0,T;H^2(R^3))$ for $\beta \geq \frac{7}{2}$ and the strong solution is unique when $\frac{7}{2} \leq \beta \leq 5$.

Before ending this section, we introduce some notations of function space which will be used later.

The space $L^p(R^3)$, $1 \le p \le \infty$, represents the usual Lebesgue spaces of scalar functions as well as that of vector-valued functions with norm denoted by $\|\cdot\|_p$. Let $C^\infty_{0,\sigma}(R^3)$ denote the set of all C^∞ real vector-valued functions $u = (u_1, u_2, u_3)$ with compact support in R^3 such that $\mathrm{div} u = 0$. Then the function space $L^p_\sigma(R^3)$, $1 < p < \infty$, is defined as the closure of $C^\infty_{0,\sigma}(R^3)$ in $L^p(R^3)$ endowed with norm $\|\cdot\|_p$. We define $W^{k,p}(R^3)$ the usual Sobolev space with the norm $\|\cdot\|_{k,p}$ and $W^{k,p}_{0,\sigma}(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ with respect to $\|\cdot\|_{k,p}$. When $p = 2$, we denote $W^{k,2}(R^3)$ by $H^k(R^3)$. Given a Banach space X with norm $\|\cdot\|_X$, we denote by $L^p(0, T; X)$, $1 \le p \le \infty$, the set of function $f(t)$ defined on $(0, T)$ with values in X such that $\int_0^T \|f(t)\|^p_X \, dt < \infty$. In this paper, we use C to express an absolute constant which may change from line to line.

The rest of the paper is organized as follows. In Section 2, we prove the global weak solutions of (1.1) for any $\beta \ge 1$. In Section 3, we prove global existence of strong solution for any $\beta \ge \frac{7}{2}$ and existence and uniqueness of strong solution for $\frac{7}{2} \le \beta \le 5$ for the Cauchy problem (1.1).

2 Existence of weak solutions

In this section, we prove the global existence of weak solutions for the problem (1.1). The definition of weak solutions is given as usual way as follows.

Definition 1 The functions pair $(u(x, t), p(x, t))$ is called a weak solution of the problem (1.1) if for any $T > 0$, the following conditions are satisfied:

1) $u \in L^\infty(0, T; L^2_\sigma(R^3)) \cap L^2(0, T; W^{1,2}_{0,\sigma}(R^3)) \cap L^{\beta+1}(0, T; L^{\beta+1}(R^3))$,

2) For any $\Phi \in C^\infty_{0,\sigma}([0, T] \times R^3)$ with $\Phi(\cdot, T) = 0$, we have

$$\begin{aligned} &-\int_0^T (u, \Phi_t)\, dt + \mu \int_0^T \int_{R^3} \nabla u : \nabla \Phi \, dxdt - \int_0^T \int_{R^3} (u \cdot \nabla) u \Phi \, dxdt \\ &+\alpha \int_0^T \int_{R^3} |u|^{\beta-1} u \Phi \, dxdt = (u_0, \Phi_0), \end{aligned} \tag{2.1}$$

3) div $u(x, t) = 0$ for *a.e.* $(x, t) \in R^3 \times [0, T)$.

In (2.1), ∇u denotes matrix $(\partial_i u_j)_{3\times 3}$ and for two matrixes $A = (a_{ij})$ and $B = (b_{ij})$, the matrix $A : B = \Sigma^3_{i,j=1} a_{ij} b_{ij}$. Here (,) means the inner product in $L^2(R^3)$.

The following Lemma is a compactness result of which proof is referred to [9].

Lemma 2.1 Let X_0, X be Hilbert spaces satisfying a compact imbedding

$X_0 \hookrightarrow\hookrightarrow X$. Let $0 < \gamma \leq 1$ and $(v_j)_{j=1}^{\infty}$ be a sequence in $L^2(R; X_0)$ satisfying

$$\sup_j \left(\int_{-\infty}^{\infty} \|v_j\|_{X_0}^2 \, dt\right) < \infty, \quad \sup_j \left(\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\hat{v}_j\|_X^2 \, d\tau\right) < \infty$$

where

$$\hat{v}(\tau) = \int_{-\infty}^{+\infty} v(t) \exp^{-2\pi i \tau t} dt$$

is the Fourier transformation of $v(t)$ on the time variable. Then there exists a subsequence of $(v_j)_{j=1}^{\infty}$ which converges strongly in $L^2(R; X)$ to some $v \in L^2(R; X)$.

Our main result of this section reads as

Theorem 1 Suppose that $\beta \geq 1$ **and** $u_0 \in L^2_\sigma(R^3)$. **Then for any given** $T > 0$, **there exists a weak solution** $(u(x,t), p(x,t))$ **to the problem (1.1) such that**

$$u \in L^\infty(0, T; L^2_\sigma(R^3)) \cap L^2(0, T; W^{1,2}_{0,\sigma}(R^3)) \cap L^{\beta+1}(0, T; L^{\beta+1}(R^3)).$$

Moreover,

$$\sup_{0 \leq t \leq T} \|u\|_{L^2}^2 + 2\mu \int_0^T \|\nabla u(t)\|_{L^2}^2 \, dt + 2\alpha \int_0^T \|u(t)\|_{L^{\beta+1}}^{\beta+1} \, dt \leq \|u_0\|_{L^2}^2 \tag{2.2}$$

Proof: We employ the Galerkin approximations to prove the theorem. The approach is similar to that of ([9]) for the classical Navier-Stokes equations.

Since $W^{1,2}_{0,\sigma}$ is separable and $C^\infty_{0,\sigma}$ is dense in $W^{1,2}_{0,\sigma}$, there exists a sequence $\omega_1, \omega_2, \cdots, \omega_m$ of elements of $C^\infty_{0,\sigma}$, which is free and total in $W^{1,2}_{0,\sigma}$. For each m we define an approximate solution u_m as follows:

$$u_m = \Sigma_{i=1}^m g_{im}(t) \omega_i(x)$$

and

$$\begin{aligned} &(u_m'(t), \omega_j) + \mu(\nabla u_m(t), \nabla \omega_j) + (u_m(t) \cdot \nabla u_m(t), \omega_j) \\ &\qquad + (\alpha |u_m|^{\beta-1} u_m(t), \omega_j) = 0, \\ &\qquad\qquad\qquad\qquad t \in [0, T], j = 1, 2, \cdots, m. \end{aligned} \tag{2.3}$$

and $u_{0m} \to u_0$ in L^2_σ, as $m \to \infty$.

We have a priori estimates on the approximate solutions u_m as follows.

Lemma 2.2 Suppose that $u_0 \in L^2_\sigma$. **Then for any given** $T > 0$ **and any** $\beta \geq 1$, **we have**

$$\sup\nolimits_{0 \leq t \leq T} \|u_m\|_{L^2_\sigma} + \|u_m\|_{L^2(0,T; W^{1,2}_{0,\sigma})} + \| \, u_m\|_{L^{\beta+1}(0,T; L^{\beta+1})} \leq C,$$

where C is a constant independent of T and m.

Proof: Multiplying on both sides of (2.3) by $g_{jm}(t)$ and sum over $j = 1, \cdots, m$, we have

$$\frac{1}{2}\frac{d}{dt}\|u_m\|_{L^2}^2 + \mu\|\nabla u_m\|_{L^2}^2 + \alpha\|u_m\|_{L^{\beta+1}}^{\beta+1} \le 0$$

where we have used the fact that $((u\cdot\nabla)v, v) = 0$ for $u \in W_{0,\sigma}^{1,2}$ for $v \in W^{1,2}$.

Integrating over $(0,T)$ on time t, we obtain

$$\sup_{0\le t\le T}\|u_m\|_{L^2}^2(t) + 2\mu\int_0^T \|\nabla u_m\|_{L^2}^2\ dt + 2\alpha\int_0^T \|u_m\|_{L^{\beta+1}}^{\beta+1}\ dt \le \|u_0\|_{L^2}^2 \qquad (2.4)$$

(2.2) is obtained by (2.4). The proof of Lemma 2.2 is finished.

By a standard procedure, applying Lemma 2.2, we obtain the global existence of the approximate solutions $u_m \in L^\infty(0,T;L^2_\sigma(R^3)) \cap L^2(0,T;W_{0,\sigma}^{1,2}(R^3)) \cap L^{\beta+1}(0,T;L^{\beta+1}(R^3))$. Next, we will use Lemma 2.1 to prove the strong convergence of u_m (or its subsequence) in $L^2 \cap L^\beta([0,T]\times R^3)$. To this end, we denote $\tilde{u}_m$ the function from R into $W_{0,\sigma}^{1,2}$, which is equal to u_m on $[0,T]$ and to 0 on the complement of this interval. Similarly, we prolong $g_{im}(t)$ to R by defining $\tilde{g}_{im}(t) = 0$ for $t \in R\backslash[0,T]$. The Fourier transform on time variable of $\tilde{u}_m$ and $\tilde{g}_{im}$ is denoted by $\hat{\tilde{u}}_m$ and $\hat{\tilde{g}}_{im}$ respectively.

Note that the approximate solutions $\tilde{u}_m$ satisfy

$$\begin{aligned}\frac{d}{dt}(\tilde{u}_m, \omega_j) &= \mu(\nabla\tilde{u}_m(t), \nabla\omega_j) + (\tilde{u}_m(t)\cdot\nabla\tilde{u}_m(t), \omega_j)\\ &\quad +(\alpha|\tilde{u}_m|^{\beta-1}\tilde{u}_m(t), \omega_j)\\ &\equiv (\tilde{f}, \omega_j) + (\alpha|\tilde{u}_m|^{\beta-1}\tilde{u}_m(t), \omega_j)\\ &\qquad j = 1, 2, \cdots, m.\end{aligned} \qquad (2.5)$$

where $(\tilde{f}_m, \omega_j) = \mu(\nabla\tilde{u}_m(t), \nabla\omega_j) + (\tilde{u}_m(t)\cdot\nabla\tilde{u}_m(t), \omega_j)$.

Taking the Fourier transform about the time variable, (2.5) gives

$$\begin{aligned}2\pi i\tau(\hat{\tilde{u}}_m, \omega_j) &= (\hat{\tilde{f}}_m, \omega_j) + \alpha(\widehat{|\tilde{u}_m|^{\beta-1}\tilde{u}_m}(t), \omega_j)\\ &\quad +(u_{0m}, \omega_j) - (u_m(T), \omega_j)exp(-2\pi iT\tau),\end{aligned} \qquad (2.6)$$

where $\hat{\tilde{f}}_m$ denote the Fourier transforms of $\tilde{f}_m$.

Multiply (2.6) by $\hat{\tilde{g}}_{jm}(\tau)$ and add the resulting equations for $j = 1, \cdots, m$ to get:

$$\begin{aligned}2\pi i\tau\|\hat{\tilde{u}}_m(\tau)\|_2^2 &= (\hat{\tilde{f}}_m(\tau), \hat{\tilde{u}}_m) + \alpha(\widehat{|\tilde{u}_m|^{\beta-1}\tilde{u}_m}(\tau), \hat{\tilde{u}}_m)\\ &\quad +(u_{0m}, \hat{\tilde{u}}_m) - (u_m(T), \hat{\tilde{u}}_m)exp(-2\pi iT\tau).\end{aligned} \qquad (2.7)$$

For any $v \in L^2((0,T);H_0^1) \cap L^{\beta+1}(0,T;L^{\beta+1})$, we have

$$\begin{aligned}(f_m(t),v) &= (\nabla u_m, \nabla v) + (u_m \cdot \nabla u_m, v)\\ &\le C(\|\nabla u_m\|_2^2 + \|\nabla u_m\|_2)\|v\|_{H^1}.\end{aligned}$$

It follows that for any given $T>0$

$$\int_0^T \|f_m(t)\|_{H^{-1}}\ dt \le \int_0^T C(\|\nabla u_m(t)\|_2^2 + \|\nabla u_m(t)\|_2)\ dt \le C,$$

and hence

$$\sup_{\tau\in R} \|\hat{\tilde{f}}_m(\tau)\|_{H^{-1}} \le \int_0^T \|f_m(t)\|_{H^{-1}}\ dt \le C. \tag{2.8}$$

Moreover, it follows from Lemma 2.2 that

$$\int_0^T \||u_m|^{\beta-1}u_m\|_{\frac{\beta+1}{\beta}}\ dt \le \int_0^T \|u_m\|_{\beta+1}^{\beta}\ dt \le C$$

which implies that

$$\sup_{\tau\in R} \|\widehat{|u_m|^{\beta-1}}u\|_{\frac{\beta+1}{\beta}}(\tau) \le C. \tag{2.9}$$

From Lemma 2.2, we have

$$\|u_m(0)\| \le C, \|u_m(T)\| \le C. \tag{2.10}$$

We deduce from (2.7)-(2.10) that

$$|\tau|\|\hat{\tilde{u}}_m(\tau)(\tau)\|_2^2 \le C(\|\hat{\tilde{u}}_m(\tau)\|_{H^1} + \|\hat{\tilde{u}}_m(\tau)\|_{\beta+1}).$$

For any γ fixed $0<\gamma<\frac{1}{4}$, we observe that

$$|\tau|^{2\gamma} \le C\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}, \forall \tau \in R.$$

Thus

$$\begin{aligned}\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\|\hat{\tilde{u}}_m(\tau)\|_2^2\ d\tau &\le C\int_{-\infty}^{+\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}\|\hat{\tilde{u}}_m(\tau)\|_2^2\ d\tau\\ &\le C\int_{-\infty}^{+\infty}\|\hat{\tilde{u}}_m(\tau)\|_2^2\ d\tau + C\int_{-\infty}^{+\infty}\frac{\|\hat{\tilde{u}}_m(\tau)\|_{H^1}}{1+|\tau|^{1-2\gamma}}\ d\tau \qquad (2.11)\\ &+C\int_{-\infty}^{+\infty}\frac{\|\hat{\tilde{u}}_m(\tau)\|_{\beta+1}}{1+|\tau|^{1-2\gamma}}\ d\tau.\end{aligned}$$

Thanks to the Parseval equality and Lemma 2.2, the first integral on the right hand side of (2.11) is bounded uniformly on m.

By the Schwarz inequality, the Parseval equality and Lemma 2.2, we have

$$\int_{-\infty}^{+\infty}\frac{\|\hat{\tilde{u}}_m(\tau)\|_{H^1}}{1+|\tau|^{1-2\gamma}}\ d\tau \le \Big(\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{1-2\gamma})^2}\ d\tau\Big)^{\frac{1}{2}}\Big(\int_0^T\|u_m(t)\|_{H^1}^2\ dt\Big)^{\frac{1}{2}} \le C \tag{2.12}$$

for $0<\gamma<\frac{1}{4}$.

Similarly, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\|\hat{\tilde{u}}(\tau)\|_{\beta+1}}{1+|\tau|^{1-2\gamma}}\,d\tau &\le (\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{1-2\gamma})^{\frac{\beta+1}{\beta}}})^{\frac{\beta}{\beta+1}}(\int_{-\infty}^{+\infty}\|\hat{\tilde{u}}_m(\tau)\|_{\beta+1}^{\beta+1}(\tau)\;d\tau)^{\frac{1}{\beta+1}}\\ &\le C\int_{-\infty}^{+\infty}\|\tilde{u}_m(\tau)\|_{\beta+1}^{\frac{\beta+1}{\beta}}(\tau)\;d\tau)^{\frac{\beta}{\beta+1}}\\ &\le CT^{\frac{\beta-1}{(\beta+1)}}(\int_0^T\|u_m\|_{\beta+1}^{\beta+1}(t)\;dt)^{\frac{1}{\beta}}\end{aligned} \tag{2.13}$$

It follows from (2.11) that

$$\int_{-\infty}^{+\infty}|\tau|^{2\gamma}\|\hat{\tilde{u}}_m(\tau)\|_2^2\;d\tau\le C \tag{2.14}$$

Taking $X_0=W_{0,\sigma}^{1,2}, X=L^2$ in Lemma 2.1, in view of Lemma 2.2 and (2.14), we obtain that there exists a subsequence of u_m, still denoted by itself, such that $u_m\to u$ strongly in $L^2(0,T;L^2)$ and $\nabla u_m\rightharpoonup\nabla u$ weakly in $L^2(0,T;L^2)$. Noting that $\int_0^T\int_{R^3}|u|^{\beta+1}\,dxdt\le C$, we obtain that $u_m\to u$ strongly in $L^p(0,T;L^p)$ for $2\le p<\beta+1$. These convergences guarantee that $u(x,t)$ is a weak solution of (1.1). The details is referred to [9] and we omit it here.

The proof of Theorem 1 is proved.

3 Existence and uniqueness of strong solution

We call the function pair $(u(x,t),p(x,t))$ the strong solution of the problem (1.1) if it is a weak solution of (1.1) satisfying that

$$u\in L^\infty(0,T;W_{0,\sigma}^{1,2}(R^3))\cap L^2(0,T;H^2(R^3))\cap L^\infty(0,T;L^{\beta+1}(R^3)).$$

We remark that as what has done in the classical Navier-Stikes equations, if $(u(x,t),p(x,t))$ is a strong solution of (1.1), then the pressure function $p(x,t)$ can be determined uniquely from the velocity field $u(x,t)$.

As a preliminary, we recall the known Gagliardo-Nirenberg inequality as follows.

Lemma 3.1 (Gagliardo-Nirenberg inequality) Assume that q and r satisfy $1\le q,r\le\infty$, and j,m are arbitrary integers satisfying $0\le j<m$. Assume $u\in C_0^\infty(R^n)$, then

$$\|D^ju\|_{L^p}\le C\|D^mu\|_{L^r}^a\|u\|_{L^q}^{1-a}, \tag{3.1}$$

where $\frac{1}{p}=\frac{j}{n}+a(\frac{1}{r}-\frac{m}{n})+(1-a)\frac{1}{q}$, $\frac{j}{m}\le a\le 1$, and the constant C only depends on n,m,j,q,r,a. If $m-j-\frac{n}{r}$ is a nonnegative integers, the above inequality holds for $\frac{j}{m}\le a<1$.

Our main results of this section is stated as

Theorem 2 Suppose that $\beta\ge\frac{7}{2}$ and $u_0\in W^{1,2}_{0,\sigma}\cap L^{\beta+1}$, Then there exists a strong solution $(u(x,t),\ p(x,t))$ to the problem (1.1) satisfying

$$u\in L^\infty(0,T;W^{1,2}_{0,\sigma}(R^3))\cap L^\infty(0,T;L^{\beta+1}(R^3))\cap L^2(0,T;H^2(R^3)),$$

$$\nabla u|u|^{\frac{\beta-1}{2}}\in L^2(0,T;L^2(R^3));u_t\in L^2(0,T;L^2(R^3)).$$

Moreover when $\frac{7}{2}\le\beta\le 5$, the strong solution is unique.

Proof: The existence of strong solution is based on the following a priori estimates.

Lemma 3.2 Suppose that $((u(x,t),p(x,t)))$ is a smooth solution of the problem (1.1). Then for any $\beta\ge\frac{7}{2}$, we have

$$\begin{aligned}&\sup_{0\le t\le T}(\|\nabla u\|_2^2+\|u\|_{\beta+1}^{\beta+1})+\|u_t\|_{2,2;T}+\|\Delta u\|_{2,2;T}+\||\nabla u||u|^{\frac{\beta-1}{2}}\|_{2,2;T}\\&+\frac{\alpha(\beta-1)}{2}\int_{R^3}|u|^{\beta-3}|\nabla|u|^2|^2\,dx\le C.\end{aligned}\tag{3.2}$$

Proof of Lemma 3.2: Multiply the first equation of (1.1) by u_t, $-\Delta u$ and integrate the resulting equation on R^3 respectively to obtain

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\int_{R^3}|\nabla u|^2\,dx+\frac{\alpha}{\beta+1}\frac{d}{dt}\int_{R^3}|u|^{\beta+1}\,dx+\int_{R^3}|u_t|^2\,dx\\&=-\int_{R^3}u_tu\cdot\nabla u\,dx,\end{aligned}\tag{3.3}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{R^3}|\nabla u|^2\,dx+\mu\int_{R^3}|\Delta u|^2\,dx+\alpha\int_{R^3}|u|^{\beta-1}|\nabla u|^2\,dx\\&+\frac{\alpha(\beta-1)}{4}\int_{R^3}|u|^{\beta-3}|\nabla|u|^2|^2\,dx=\int_{R^3}(u\cdot\nabla u)\Delta u\,dx.\end{aligned}\tag{3.4}$$

Adding (3.3),(3.4) and using Hölder inequality, Young inequality yield

$$\begin{aligned}&\frac{\mu+1}{2}\frac{d}{dt}\int_{R^3}|\nabla u|^2\,dx+\frac{\alpha}{\beta+1}\frac{d}{dt}\int_{R^3}|u|^{\beta+1}\,dx+\frac{3\mu}{4}\int_{R^3}|\Delta u|^2\,dx\\&+\frac{1}{2}\int_{R^3}|u_t|^2\,dx+\alpha\int_{R^3}|u|^{\beta-1}|\nabla u|^2\,dx\\&+\frac{\alpha(\beta-1)}{4}\int_{R^3}|u|^{\beta-3}|\nabla|u|^2|^2\,dx\le C\int_{R^3}|u\cdot\nabla u|^2\,dx\equiv J.\end{aligned}\tag{3.5}$$

The estimates of J are divided into the following two cases.

Case I: Using Gagliardo-Nirenberg inequality (3.1), we have

$$\|\nabla u\|_{\frac{2(\beta+1)}{\beta-1}} \leq C\|\triangle u\|_2^a \|u\|_{\beta+1}^{1-a} \tag{3.6}$$

where β satisfies

$$\frac{1}{2} \leq a = \frac{11-\beta}{\beta+7} \leq 1, \tag{3.7}$$

that is,

$$2 \leq \beta \leq 5. \tag{3.8}$$

Using Hölder inequality, (3.6) and Young inequality, we have

$$\begin{aligned} J &\leq C\|u\|_{\beta+1}^2 \|\nabla u\|_{\frac{2(\beta+1)}{\beta-1}}^2 \\ &\leq C\|u\|_{\beta+1}^2 \|\triangle u\|_2^{\frac{2(11-\beta)}{\beta+7}} \|u\|_{\beta+1}^{\frac{4(\beta-2)}{\beta+7}} \\ &\leq C\|\triangle u\|_2^{\frac{2(11-\beta)}{\beta+7}} \|u\|_{\beta+1}^{\frac{6(\beta+1)}{\beta+7}} \\ &\leq \tfrac{\mu}{4}\|\triangle u\|_2^2 + C\|u\|_{\beta+1}^{\frac{3(\beta+1)}{\beta-2}}. \end{aligned} \tag{3.9}$$

If $\frac{3(\beta+1)}{\beta-2} \geq \beta+1$, that is, $2 < \beta \leq 5$, it directly follows that

$$J \leq \frac{\mu}{4}\|\triangle u\|_2^2 + C\|u\|_{\beta+1}^{\beta+1}\|u\|_{\beta+1}^{\frac{4\beta-\beta^2+5}{\beta-2}} \tag{3.10}$$

In (3.10) we demand that

$$\begin{cases} 4\beta-\beta^2+5 \geq 0 \Rightarrow -1 \leq \beta \leq 5, \\ 4\beta-\beta^2+5 \leq (\beta-2)(\beta+1) \Rightarrow \beta \geq \frac{7}{2}\ . \end{cases} \tag{3.11}$$

Combing (3.8) with (3.11), we obtain the restrictions of β:

$$\frac{7}{2} \leq \beta \leq 5. \tag{3.12}$$

Substituting (3.10) into (3.5), we have

$$\begin{aligned} &(\mu+1)\sup_{0\leq t\leq T}\|\nabla u(t)\|_2^2 + \tfrac{2\alpha}{\beta+1}\sup_{0\leq t\leq T}\|u(t)\|_{\beta+1}^{\beta+1} + \mu\|\Delta u\|_{2,2;T}^2 \\ &+\|u_t\|_{2,2;T}^2 + 2\alpha\beta\||u|^{\frac{\beta-1}{2}}|\nabla u|\|_{2,2;T}^2 + \frac{\alpha(\beta-1)}{2}\int_{R^3}|u|^{\beta-3}|\nabla|u|^2|^2\,dx \\ &\leq C\exp(\|u\|_{\beta+1,\beta+1;T}^{\frac{4\beta-\beta^2+5}{\beta-2}} T^{\frac{2\beta-7}{\beta-2}}) \times (\|\nabla u_0\|_2^2 + \|u_0\|_{\beta+1}^{\beta+1}), \end{aligned} \tag{3.13}$$

where β satisfies (3.12).

Case II: Using Gagliardo-Nirenberg inequality, we have

$$\|\nabla u\|_{\frac{2(\beta+1)}{\beta-1}} \leq C\|\triangle u\|_2^a \|u\|_2^{1-a} \tag{3.14}$$

$$\frac{1}{2} \leq a = \frac{\beta+4}{2(\beta+1)} \leq 1, \tag{3.15}$$

that is,

$$\beta \geq 2. \tag{3.16}$$

Using Hölder inequality, (3.14) and Young inequality, we obtain

$$\begin{aligned} J &\leq C\|u\|_{\beta+1}^2 \|\nabla u\|_{\frac{2(\beta+1)}{\beta-1}}^2 \\ &\leq C\|u\|_{\beta+1}^2 \|\triangle u\|_2^{\frac{\beta+4}{\beta+1}} \|u\|_2^{\frac{\beta-2}{\beta+1}} \\ &\leq \tfrac{\mu}{4}\|\triangle u\|_2^2 + C\|u\|_{\beta+1}^{\frac{4(\beta+1)}{\beta-2}} \|u\|_2^2. \end{aligned} \tag{3.17}$$

If $\frac{4(\beta+1)}{\beta-2} \leq \beta+1$, that is,

$$\beta \geq 6 \tag{3.18}$$

Substituting (3.17) into (3.5), we have

$$\begin{aligned} &(\mu+1)\sup_{0\leq t\leq T}\|\nabla u(t)\|_2^2 + \tfrac{2\alpha}{\beta+1}\sup_{0\leq t\leq T}\|u(t)\|_{\beta+1}^{\beta+1} + \mu\|\Delta u\|_{2,2;T}^2 \\ &+\|u_t\|_{2,2;T}^2 + 2\alpha\beta\||u|^{\frac{\beta-1}{2}}|\nabla u|\|_{2,2;T}^2 + \frac{\alpha(\beta-1)}{2}\int_{R^3}|u|^{\beta-3}|\nabla|u|^2|^2\,dx \\ &\leq C(\|u\|_{2,\infty;T}^2\|u\|_{\beta+1,\beta+1;T}^{\frac{4(\beta+1)}{\beta-2}}T^{\frac{\beta-6}{\beta-2}}) + (\|\nabla u_0\|_2^2 + \|u_0\|_{\beta+1}^{\beta+1}), \end{aligned} \tag{3.19}$$

where β satisfies (3.18).

If $\frac{4(\beta+1)}{\beta-2} \geq \beta+1$, that is,

$$\beta \leq 6 \tag{3.20}$$

It directly follows that

$$J \leq \frac{\mu}{4}\|\triangle u\|_2^2 + C\|u\|_{\beta+1}^{\beta+1}\|u\|_{\beta+1}^{\frac{5\beta-\beta^2+6}{\beta-2}}\|u\|_2^2 \tag{3.21}$$

In (3.21) we demand that

$$\begin{cases} 5\beta-\beta^2+6 \geq 0 \Rightarrow -1 \leq \beta \leq 6, \\ 5\beta-\beta^2+6 \leq (\beta-2)(\beta+1) \Rightarrow \beta \geq 4. \end{cases} \tag{3.22}$$

Combing (3.20) and (3.22), we obtain

$$4 \leq \beta \leq 6. \tag{3.23}$$

Substituting (3.17) into (3.5), we have

$$\begin{aligned}
&(\mu+1)\sup_{0\le t\le T}\|\nabla u(t)\|_2^2+\tfrac{2\alpha}{\beta+1}\sup_{0\le t\le T}\|u(t)\|_{\beta+1}^{\beta+1}+\mu\|\Delta u\|_{2,2;T}^2\\
&+\|u_t\|_{2,2;T}^2+2\alpha\beta\||u|^{\frac{\beta-1}{2}}|\nabla u|\|_{2,2;T}^2+\frac{\alpha(\beta-1)}{2}\int_{R^3}|u|^{\beta-3}|\nabla|u|^2|^2\,dx\\
&\le C\exp(\|u\|_{2,\infty;T}^2\|u\|_{\beta+1,\beta+1;T}^{\frac{(\beta+1)(6-\beta)}{\beta-2}}T^{\frac{2\beta-8}{\beta-2}})\times(\|\nabla u_0\|_2^2+\|u_0\|_{\beta+1}^{\beta+1})
\end{aligned}\tag{3.24}$$

where β satisfies (3.23).

Combing (3.5), (3.13),(3.19) and (3.24), we obtain that, for any $\beta\ge\frac{7}{2}$,

$$\begin{aligned}
&\sup_{0\le t\le T}(\|\nabla u\|_2^2+\|u\|_{\beta+1}^{\beta+1})+\|u_t\|_{2,2;T}+\|\Delta u\|_{2,2;T}+\||\nabla u||u|^{\frac{\beta-1}{2}}\|_{2,2;T}\\
&+\frac{\alpha(\beta-1)}{2}\int_{R^3}|u|^{\beta-3}|\nabla|u|^2|^2\,dx\le C.
\end{aligned}\tag{3.25}$$

The proof of Lemma 3.2 is proved.

Now we proceed to prove the uniqueness of the strong solutions of Theorem 2. Assume that under the same initial data, there exist two strong solutions (u,p),$(\bar u,p)$ of the equations of (1.1) satisfying

$$\begin{aligned}
&-\int_0^T(u,\Phi_t)\,dt+\mu\int_0^T\int_{R^3}\nabla u:\nabla\Phi\,dxdt-\int_0^T\int_{R^3}(u\cdot\nabla)u\Phi\,dxdt\\
&+\alpha\int_0^T\int_{R^3}|u|^{\beta-1}u\Phi\,dxdt=(u_0,\Phi_0),
\end{aligned}\tag{3.26}$$

$$\begin{aligned}
&-\int_0^T(\bar u,\Phi_t)\,dt+\mu\int_0^T\int_{R^3}\nabla\bar u:\nabla\Phi\,dxdt-\int_0^T\int_{R^3}(\bar u\cdot\nabla)\bar u\Phi\,dxdt\\
&+\alpha\int_0^T\int_{R^3}|\bar u|^{\beta-1}\bar u\Phi\,dxdt=(\bar u_0,\Phi_0)
\end{aligned}\tag{3.27}$$

for $\Phi\in C_{0,\sigma}^\infty([0,T]\times R^3)$ with $\Phi(\cdot,T)=0$ and by the density argument ([7]) and ([9]) hold actually for $\Phi\in L^2(0,T;H^1)$.

Subtracting (3.26) from (3.27) and taking $\Phi=u-\bar u$ in the resulting equations, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u-\bar u\|_2^2+\mu\|\nabla(u-\bar u)\|_2^2+\alpha\||u|^{\frac{\beta-1}{2}}|u-\bar u|\|_2^2\\
&\quad\le\int_{R^3}|u-\bar u|^2|\nabla\bar u|\,dx+\alpha\int_{R^3}|u-\bar u||\bar u||\,|u|^{\beta-1}-|\bar u|^{\beta-1}|\,dx\\
&\quad\equiv I_1+I_2.
\end{aligned}\tag{3.28}$$

where we have used the fact that $((u\cdot\nabla)v,v)=0,u\in W_{0,\sigma}^{1,2},v\in W^{1,2}$.

Applying Hölder and Sobolev inequalities to yield

$$\begin{aligned} I_1 &\le \|u-\bar u\|_4^2\|\nabla \bar u\|_2 \\ &\le C(\|\nabla(u-\bar u)\|_2^{\frac{3}{4}}\|u-\bar u\|_2^{\frac{1}{4}})^2\|\nabla u\|_2 \\ &\le C\|\nabla(u-\bar u)\|_2^{\frac{3}{2}}\|u-\bar u\|_2^{\frac{1}{2}}\|\nabla \bar u\|_2 \\ &\le \varepsilon\|\nabla(u-\bar u)\|_2^2 + C\|u-\bar u\|_2^2\|\nabla\bar u\|_2^4, \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} I_2 &\le \alpha\int_{R^3}|u-\bar u||\bar u|||u|^{\beta-1}-|\bar u|^{\beta-1}|\,dx \\ &\le C(\beta-1)\int_{R^3}|u-\bar u|||u|^{\beta-2}+|\bar u|^{\beta-2}||u-\bar u||\bar u|\,dx \\ &\le C\|u-\bar u\|_4^2\|\bar u\|_6\||u|^{\beta-2}+|\bar u|^{\beta-2}\|_3 \\ &\le C(\|\nabla(u-\bar u)\|_2^{\frac{3}{4}}\|u-\bar u\|_2^{\frac{1}{4}})^2\|\bar u\|_6\||u|+|\bar u|\|_{3(\beta-2)}^{\beta-2} \\ &\le C\|\nabla(u-\bar u)\|_2^{\frac{3}{2}}\|u-\bar u\|_2^{\frac{1}{2}}\|\bar u\|_6\||u|+|\bar u|\|_{3(\beta-2)}^{\beta-2} \\ &\le \varepsilon\|\nabla(u-\bar u)\|_2^2 + C\|u-\bar u\|_2^2\|\bar u\|_6^4\||u|+|\bar u|\|_{3(\beta-2)}^{4(\beta-2)}. \end{aligned} \tag{3.30}$$

In the second inequality of I_2, we used the fact that

$$|x^p-y^p|\le Cp(|x|^{p-1}+|y|^{p-1})|x-y|$$

for any $x,y\ge 0$, where C is an absolute constant.

Substituting the estimates of I_1, I_2 into inequality (3.28), choosing $\varepsilon=\frac{\mu}{4}$, we obtain

$$\begin{aligned} &\frac{d}{dt}\|u-\bar u\|_{L^2}^2+\mu\|\nabla(u-\bar u)\|_{L^2}^2+2\alpha\||u|^{\frac{\beta-1}{2}}|u-\bar u|\|_2^2 \\ &\le C\|u-\bar u\|_{L^2}^2(\|\nabla\bar u\|_2^4+\|\bar u\|_6^4[\|u\|_{3(\beta-2)}^{4(\beta-2)}+\|\bar u\|_{3(\beta-2)}^{4(\beta-2)}]. \end{aligned} \tag{3.31}$$

Note that

$$\int_0^T \||u|\|_{3(\beta-2)}^{4(\beta-2)} \le \int_0^T \|u\|_{\beta+1}^{\frac{4(\beta^2+\beta)}{\beta+7}}\|\triangle u\|_2^{\frac{8(2\beta-7)}{\beta+7}} \le \sup_{0\le t\le T}\|u\|_{\beta+1}^{\frac{4(\beta^2+\beta)}{\beta+7}}\|\Delta u\|_{2,2;T}^{\frac{8(2\beta-7)}{\beta+7}}T^{\frac{35-7\beta}{\beta+7}}, \tag{3.32}$$

and similar estimate hold true for $\bar u$ instead of u in (3.32). In (3.32), we have a restriction of β as $0\le\frac{8(2\beta-7)}{\beta+7}\le 2$, that is,

$$\frac{7}{2}\le\beta\le 5. \tag{3.33}$$

Substituting (3.32) into (3.31) and applying the Gronwall inequality, we obtain that $u=\bar u$ for a.e. $(x,t)\in R^3\times[0,T]$ under (3.33). This completes the proof of Theorem 2.

Acknowledgements: The paper is started when the authors visited the IMS of the Chinese University of Hong Kong. The authors thank Professor Zhouping Xin for his valuable suggestions, discussions and constant encouragements.

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