Vanishing viscous limits for the 2D lake equations with Navier boundary conditions

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Abstract

The vanishing viscosity limit is considered for the viscous lake equations with Navier friction boundary conditions. We prove that the inviscid limit satisfies the inviscid lake equations, and the results include flows generated by L^p initial vorticity with $1 < p \leq \infty$.

Key Words: viscous lake equations, inviscid limit, Navier boundary conditions

1 Introduction

The viscous lake equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu b^{-1} div(2bD(u) + bdivuI) + \nabla p = f, \\ div(bu) = 0, \end{cases} \tag{1}$$

for $(x,t) \in \Omega \times (0,T)$ with $\Omega \subset R^2$, a bounded and smooth domain. Here, $u(x,t) = (u_1(x,t), u_2(x,t))$ stands for the two-dimensional velocity fields and $D(u) = \frac{\nabla u + \nabla u^t}{2}$ is the deformation tensor. The positive number μ represents the eddy viscosity coefficient and the matrix I is the 2×2 identity one. Moreover, the bottom function $b(x)$ is a given function, which is assumed to be in $C^2(\bar{\Omega})$ and non-degenerate, i.e. there exists two positive constants b_1, b_2 such that

$$0 < b_1 \leq b(x) \leq b_2, \quad x \in \bar{\Omega}. \tag{2}$$

For viscous lake equations (1), we impose the Navier boundary conditions as

$$u \cdot n = 0, \quad 2D(u)n \cdot \tau + \alpha u \cdot \tau = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3}$$

and initial data as

$$u(x,t)\mid_{t=0} = u_0, \quad x \in \Omega. \tag{4}$$

In (3), n, τ means the unit normal vector and tangential vector respectively. $\alpha(x)$ is a nonnegative bounded turbulent boundary drag coefficient defined on $\partial\Omega$. Here we assume that $\alpha(x) \geq \kappa(x)$, where $\kappa(x)$ is the curvature of $\partial\Omega$.

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The Navier boundary conditions, which were firstly used by Navier in 1827, say that there is a stagnant layer of fluid close to the wall allowing a fluid to slip, and the slip velocity is proportional to the shear stress. Such boundary conditions can be induced by effects of free capillary boundaries [2] or a rough boundary [1], or a perforated boundary [9], and so on. A special case of (3) with $\alpha(x) = \kappa(x)$ is called the free Navier boundary condition (see [14], [15]), which is

$$u \cdot n = 0, \qquad curl u = 0, \qquad \text{on } \partial\Omega. \tag{5}$$

Viscous lake equations (1) have been asymptotically derived by D. Levermore and B. Samartino in [12], as the shallow water limit of the 3D Navier-Stokes equations with a rigid lid upper boundary condition in a horizontal basin with bottom topography. Also, in [12], the authors obtained the global existence and uniqueness of strong solution to the 2D viscous lake equations. Obviously, if $b \equiv const$, then (1) becomes the classical 2D incompressible Navier-Stokes equations. Formally, when the viscosity coefficient $\mu = 0$, the viscous lake equations becomes the inviscid lake equations, which is

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = f, \\ div(bu) = 0, \end{cases} \tag{6}$$

for $(x, t) \in \Omega \times (0, T)$.

The investigation of vanishing viscosity limit of solutions of (1) is a big issue both in mathematical study and physical applications, just as the case of the classical Navier-Stokes equations. In particular, for the non-slip boundary conditions $u = 0$ on $\partial\Omega$ instead of Navier boundary conditions (3), there will appear strong boundary layer in general. However, for Navier boundary conditions (1), the corresponding convergence is possible. Some progresses have been made for the Navier-Stokes equations with Navier boundary conditions recently. In [7] T. Clopeau, A, Mikelić, and R. Robert proved the convergence from the 2D Navier Stokes equations to the Euler equations as the viscosity tends to zero under the assumption that the initial vorticity belongs to $L^\infty(\Omega)$. M. C. Lopes Filho, H. J. Nussenzveig Lopes and G. Planas ([16]) improved the results of [7] under the assumption that the initial vorticity belongs to $L^p(\Omega), p > 2$. For the 3D Navier-Stokes equations, Yuelong Xiao and Zhouping Xin [17] established the convergence of strong solution as the viscosity vanishes under the slip boundary condition,

$$u \cdot n = 0, \qquad curl u \cdot \tau = 0, \qquad \text{on } \partial\Omega,$$

The global existence of solutions to (6) has been studied by D. Levermore, M. Oliver and E. Titi in [11] under the assumption that the initial vorticity belongs to $L^p(\Omega)$ with $2 \le p \le \infty$. The solutions in [11] were constructed as the inviscid limit of solutions of a system with an artificial viscosity. And free Navier boundary conditions (5) were used in [11].

In this paper we intend to prove rigorously the convergence from viscous lake equations (1) to inviscid lake equations (3) under Navier type boundary condition (3). More precisely, we prove that the solutions of viscous lake equations (1), denoted by u^μ will tend to the solutions of inviscid lake equations (3) as the viscosity $\mu \to 0$, for both smooth initial data and non-smooth initial data. Due to the appearance of the bottom function $b(x)$ in the equations (1) and (6), the weighted Sobolev spaces instead of the usual Sobolev spaces will be used. Since the vorticity equations of viscous lake equations (1) becomes much complicated, a key step of our analysis is to establish the L^p $(1 < p \leq \infty)$ estimates of the vorticity which makes it available to get the convergence of the viscous lake equations (1) to the inviscid ones. Our results differ from the ones of [11] in three respects. First, our solutions are obtained by taking limit of the solutions of viscous lake equations (1) to inviscid equations (6) as the viscosity coefficient vanishes. The second difference is that our boundary conditions are the general Navier boundary condition. The third respect is that we improve the results of [11] to the case that the initial vorticity belongs to $L^p(\Omega)$ with $1 < p \leq 2$, using free boundary condition (5).

The global existence and uniqueness of the inviscid lake equations with degenerate bottom topography instead of non-degenerate one (2) are proved in [6] very recently. The study of the convergence of the viscous lake equations to the inviscid lake equations with the generate bottom topography will be very interesting and will be investigated in our subsequent work.

The paper is organized as follow. In section 2 we give some mathematical preliminaries and the existence and uniqueness of (1). In section 3, we give two a priori estimates of (1) and the corresponding vorticity equations. Moreover, the convergence and the existence of (6) is also given. Finally, in section 4, we generalize the result to the case of non-smooth initial data.

2 Some mathematical preliminary and the solvability of (1)

For the mathematical setting of (1), we introduce the Sobolev spaces with weight b. For example, we endow $H^m(\Omega)$ for $m \in \mathbb{N}$ with the scalar product $(\phi, \theta)_{H^m} = \sum_{0\leq|\alpha|\leq n} \int_\Omega D^\alpha \phi(x) D^\alpha \theta(x) b(x) dx$. The weighted integral over the domain is abbreviated by $< \cdot >$, i.e.

$$< \phi >= \int_\Omega \phi(x) b(x) dx.$$

The scalar product between u, v is denoted by $(u, v)_{L^2} =< uv >$. We say that u is divergence free if $\int_\Omega bu \cdot \nabla \phi dx = 0$ for $\phi \in C^\infty$. We introduce the space of infinitely differentiable and compactly supported functions which satisfy our weighted incompressible condition

$$\mathbb{D} = \{u \in C_0^\infty(\Omega) : div(bu) = 0 \text{ in } \Omega\}.$$

Moreover, we define the Hilbert spaces

$$\begin{aligned} H &= \{u : \ u \in L^2(\Omega),\ div(bu) = 0,\ u \cdot n = 0,\ x \in \Omega\} \\ V &= \{u : \ u \in H^1(\Omega),\ div(bu) = 0,\ u \cdot n = 0,\ x \in \Omega\} \\ W &= \{u : \ u \in H^2(\Omega) \cap V,\ 2D(u)n \cdot \tau + \alpha u \cdot \tau = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

It is noted that the nondegeneracy of b guarantees that that the Sobolev norm with weight function $b(x)$ is equivalent to the standard Sobolev norm.

The Navier friction condition can be formulated in terms of vorticity, which is stated as

Lemma 1 Suppose $v \in H^2(\Omega)^2, v \cdot n = 0$ on $\partial\Omega$, then we have

$$D(v)n \cdot \tau - \frac{1}{2} curl v + \kappa(v \cdot \tau) = 0 \quad \text{on} \quad \partial\Omega, \tag{7}$$

where $curl v = \partial_1 v_2 - \partial_2 v_1$ and κ is the curvature of $\partial\Omega$.

The proof of Lemma 1 is referred to [7] and we omit it here. It follows from Lemma 1 that the Navier boundary conditions can be written as

$$\omega = (2\kappa - \alpha)b^{-1} u \cdot \tau \quad (x, t) \in \partial\Omega \times (0, T).$$

A main difficulty in our approach is to deal with the vorticity equations which will become complicated due to the presence of the bottom function $b(x)$. Suppose that u^μ is the smooth solution of the viscous lake equations (1) with Navier boundary conditions (3) and the initial data (4). We introduce the potential initial vorticity $\omega_0 = b^{-1}\nabla \times u_0$ and the time dependent vorticity $\omega^\mu = b^{-1}\nabla \times u^\mu$ associated to the solution u^μ. An evolution equation for the potential vorticity is obtained by taking the curl on both sides of the equation (1). The nonlinear term in (1) becomes

$$b^{-1}\nabla \times (u \cdot \nabla u) = (u \cdot \nabla)\omega,$$

where the divergence free condition $div(bu) = 0$ has been used and $\omega = b^{-1}\nabla \times u$. The viscous term becomes more complicated, which is

$$\begin{aligned} & b^{-1}\nabla \times (b^{-1} div(2bD(u) + b div u I)) \\ & = \Delta\omega + 3b^{-1} \sum_{i=1}^{2} \partial_i b \partial_i \omega + G(u, \nabla u) \\ & \equiv A\omega. \end{aligned} \tag{8}$$

where $G(u, \nabla u)$ is the linear combination of u and ∇u. Moreover, we have

$$\|G(u, \nabla u)\|_{L^p(\Omega)} \leq C(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}), \quad p > 1, \tag{9}$$

The precise presentation and the properties of the term $G(u, \nabla u)$ and the operator A will be given in Appendix.

Using the divergence free condition $div(bu) = 0$, we introduce the stream function ϕ satisfying

$$\begin{aligned} & u^\mu = b^{-1}\nabla^\perp \phi, \\ & \omega^\mu = b^{-1}\nabla \times (b^{-1}\nabla^\perp \phi), \\ & \quad\;\; = b^{-2}\Delta\phi - b^{-3}\partial_i b \partial_i \phi, \\ & \phi \mid_{\partial\Omega} = 0, \end{aligned} \tag{10}$$

where $\nabla^\perp = (\partial_2, -\partial_1)$. Then for each fixed time, the velocity u^μ can be recovered from vorticity by means of the Biot-Savart law, which is denoted by

$$u^\mu = K_\Omega(\omega^\mu).$$

The following lemma, which is proved in [10], is useful later.

Lemma 2 For $b(x) \in C^2(\bar{\Omega})$ and every $\omega \in H^{-1}(\Omega)$, there exists a unique function $u = K_\Omega \omega \in H$. Moreover, K is continuous as a mapping among the spaces $H^{-1}(\Omega) \to H, L^2(\Omega) \to V$ and $H^1(\Omega) \to V \cap H^2(\Omega)$ and for some $p_0 > 1$, satisfies the estimate

$$\|K_\Omega(\omega)\|_{W^{1,p}} \le Cp\|\omega\|_{L^p},$$

for all $p \ge p_0$, where the constant C depends only on p_0, Ω.

Now the equations for the potential vorticity read as

$$\begin{cases} \partial_t \omega^\mu + u^\mu \cdot \nabla \omega^\mu - \mu A \omega^\mu = b^{-1} curl f, \quad (x,t) \in \Omega \times (0,T), \\ u^\mu = K_\Omega \omega^\mu, \quad (x,t) \in \Omega \times (0,T), \\ \omega^\mu = (2\kappa - \alpha) b^{-1} u^\mu \cdot \tau \qquad (x,t) \in \partial\Omega \times (0,T), \\ \omega^\mu(\cdot, 0) = \omega_0, \quad x \in \Omega, \end{cases} \tag{11}$$

where $A\omega$ is defined as in (8).

The global existence and uniqueness of the viscous lake equations (1) is stated as follows, similar to the proof in [12],

Theorem 1 Let $b(x), \alpha, \kappa$ are defined as above. For $u_0 \in W,\ f \in H^1(0,T; L^2(\Omega)), curl f \in L^\infty(0,T; L^p(\Omega)), \omega_0 = b^{-1} curl u_0 \in L^p(\Omega), 2 < p \le \infty$, there exists a unique solution $u^\mu \in L^\infty(0,T; H^2), \partial_t u^\mu \in L^2(0,T; V) \cap C(0,T; H)$, which satisfies the variational form of (1) , i.e.

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega \phi u^\mu b dx + 2\mu \int_\Omega Du^\mu : D\phi\ b dx + \mu \int_\Omega div u^\mu div \phi\ b dx \\ \qquad + \displaystyle\int_\Omega u^\mu \cdot \nabla u^\mu \cdot \phi\ b dx + \mu \int_{\partial\Omega} \alpha (u \cdot \tau)(\phi \cdot \tau) b dS = \int_\Omega f \cdot u^\mu b dx. \\ u^\mu(x, t=0) = u_0, \end{cases} \tag{12}$$

for $\phi \in V$. Moreover, $\omega^\mu = b^{-1} curl u^\mu$ is well-defined, and satisfies (11) in the distribution sense with the properties $\omega^\mu \in C([0,T]; H^1(\Omega))$. Finally, there exists a unique pressure field $p \in C([0,T]; H^1(\Omega))$ such that (1) holds a.e. on $\Omega \times (0,T)$.

Proof. The key point of the proof is the global existence of the solutions u^μ of (1).

Just as in [12], the proof is divided into the following steps:

Step1. Weak solutions for elliptic Stokes problem;

Step 2. Regularity for the elliptic Stokes problem;

Step 3. Galerkin approximations.

In the proof in [12], there is a little gap on the regularity for the elliptic Stokes problem and hence we only give the proof of Step 1, 2 to make it clear. The leftover is completely similar to [12] and we omit it here. We write $u^\mu,\ \omega^\mu$ by u, ω respectively for simplicity.

Step 1. Weak solutions for elliptic Stokes problem

Consider the elliptic problem

$$\begin{cases} -b^{-1}div[2bD(u)+bdivuI]+\nabla p=f, & x\in\Omega \\ div(bu)=0, & x\in\Omega \\ u\cdot n=0, & x\in\partial\Omega \\ 2D(u)n\cdot\tau+\alpha u\cdot\tau=0, & x\in\partial\Omega, \end{cases} \tag{13}$$

where $\alpha\geq\kappa, f\in L^2(\Omega)$. We intend to prove that there exists a unique $u\in V$ satisfying (13) in weak sense. To this end, we define the bound bilinear operator E by

$$(Eu,v)=2\int_\Omega Du:Dvbdx+\int_\Omega divudivvbdx+\int_{\partial\Omega}\alpha u\cdot vbds,$$

for $u,v\in V$, and we prove the coercivity of E as follows.

$$\begin{aligned} (Eu,u) &= 2\int_\Omega Du:Dubdx+\int_\Omega(divu)^2\ bdx+\int_{\partial\Omega}\alpha|u|^2bds \\ &\geq b_1(2\int_\Omega Du:Dudx+\|divu\|_{L^2}^2+\int_{\partial\Omega}\alpha|u|^2ds) \\ &\geq b_1\int_\Omega[2(\partial_xu_1)^2+2(\partial_yu_2)^2+(\partial_xu_2+\partial_yu_1)^2]dx+b_1\|divu\|_{L^2}^2+b_1\int_{\partial\Omega}\alpha|u|^2ds \\ &\geq b_1\int_\Omega[(\partial_xu_1-\partial_yu_2)^2+(\partial_xu_2+\partial_yu_1)^2]dx+b_1\|divu\|_{L^2}^2+b_1\int_{\partial\Omega}\alpha|u|^2ds \\ &\geq b_1\|\nabla u\|_{L^2}^2+b_1\|divu\|_{L^2}^2+b_1\int_{\partial\Omega}(\alpha-\kappa)|u|^2ds \\ &\geq O(1)\|u\|_{H^1}. \end{aligned}$$

In above estimates, we have used the identity in [12] for $u\in C^\infty(\Omega)$, satisfying $u\cdot n=0$,

$$\begin{aligned} &2\int_\Omega[\partial_xu_1\partial_yu_2-\partial_yu_1\partial_xu_2]dx \\ &=\int_\Omega div(u_1\partial_yu_2-u_2\partial_yu_1,u_2\partial_xu_1-u_1\partial_xu_2)dx \\ &=\int_{\partial\Omega}(u_1\tau\cdot\nabla u_2-u_2\tau\cdot\nabla u_1)ds \\ &=-\int_{\partial\Omega}(\tau\cdot\nabla u\cdot n)u\cdot\tau ds=\int_{\partial\Omega}\kappa|u|^2ds. \end{aligned}$$

Applying Lax-Milgram theorem, we obtain that there exists a unique $u\in V$ which is weak solution of (13).

Step 2. Regularity for the elliptic Stokes problem

Due to the Navier friction boundary condition, we cannot use the standard elliptic theory directly. Given $u\in V$, the weak solution of (13), we solve the following system

$$\begin{cases} -\Delta\psi+3\partial_ib\partial_i\psi+\beta\psi=-G(u,\nabla u)+\beta\omega+b^{-1}curlf\equiv g(x) & x\in\Omega, \\ \psi=(2\kappa-\alpha)b^{-1}u\cdot\tau\equiv h(x) & x\in\partial\Omega. \end{cases} \tag{14}$$

Note that $g(x) \in H^{-1}(\Omega),\ h(x) \in H^{\frac{1}{2}}(\partial\Omega)$. We have that there exists a unique solution $\psi \in H^1(\Omega)$ of (14) for $\beta > 0$, which is large enough.

Then it follows from the standard elliptic theory that $\bar{\phi} \in H^3(\Omega)$ is the unique solution of

$$\begin{cases} b^2\Delta\bar{\phi} + b^{-3}\partial_i b\partial_i\bar{\phi} = \psi, & x \in \Omega \\ \bar{\phi} = 0, & x \in \partial\Omega. \end{cases}$$

Set $\bar{u} = b^{-1}\nabla^{\perp}\bar{\phi}$. Then $\bar{u} \in H^2(\Omega)$. Moreover, it concludes that $(\bar{u}, p)$ satisfies the system

$$\begin{cases} -b^{-1}div[2bD(\bar{u}) + bdiv\bar{u}I] + \beta\bar{u} + \nabla p = bcurl^{-1}[G(\bar{u}, \nabla\bar{u}) - G(u, \nabla u)] + \beta u + f, & x \in \Omega \\ div(b\bar{u}) = 0, & x \in \Omega \\ \bar{u} \cdot n = 0, & x \in \partial\Omega \\ 2D(\bar{u})n \cdot \tau + 2\kappa\bar{u} \cdot \tau = (2\kappa - \alpha)u \cdot \tau, & x \in \partial\Omega. \end{cases} \tag{15}$$

We remark that $\bar{u}$ satisfies the above system (15) but not the original one (13), which is ignored in (13). Applying similar approach in (13), we obtain that there exists a unique weak solution of (15) under the assumption that β is sufficient large. It is noted that $u \in V$ is a weak solution of (13) and hence u is also a weak solution of (15). On the other hand, $\bar{u} \in H^2(\Omega)$ is also the weak solution of (15). Thus $u = \bar{u} \in V \cap H^2(\Omega)$ due to the uniqueness of (15).

Then for $u^\mu \in L^\infty(0, T; H^2(\Omega))$, we know that $\omega^\mu = \phi \in L^\infty(0, T; H^1(\Omega))$ and satisfies (**??**)a9) in the distribution sense.

3 Estimates and convergence for smooth initial data

In this section, we first obtain the estimates for the solutions of the viscous equations (1) under assumptions of smooth enough initial data, which is presented in Theorem 1. Our result reads

Theorem 2 Under assumptions of Theorem 1, we have

$$\|u^\mu\|_{L^\infty((0,T);L^2(\Omega))} + \sqrt{\mu}\|u^\mu\|_{L^2((0,T);V)} \leq O(1)(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^\infty(0,T;L^2(\Omega))}). \tag{16}$$

$$\|\omega^\mu\|_{L^\infty((0,T);L^p(\Omega))} \leq O(1)(\|\omega_0\|_{L^p(\Omega)} + \|u_0\|_{L^2(\Omega)} + \|f\|_{L^\infty(0,T;L^2(\Omega))} + \|curlf\|_{L^\infty(0,T;L^p(\Omega))}), \tag{17}$$

where u^μ, ω^μ are same as in Theorem 1 and $O(1)$ are the positive constants depending on Ω, b, T and the bound of κ and α.

Proof Multiplying the first equation of (1) by u^μ and integrating by parts, we have

$$\frac{d}{dt}\|u^\mu\|^2_{L^2(\Omega)} + \mu\|Du^\mu\|^2_{L^2(\Omega)} + \mu\int_\Omega(\alpha - \kappa)|u^\mu|^2 bds \leq \|f\|_{L^2(\Omega)}\|u^\mu\|_{L^2(\Omega)},$$

which yields

$$\|u^\mu\|_{L^\infty((0,T);L^2(\Omega))} + \mu\|u^\mu\|_{L^2((0,T);V)} \leq \|u_0\|_{L^2(\Omega)} + \|f\|_{L^\infty(0,T;L^2(\Omega))}.$$

The estimate (16) is proved.

Now we prove (17). Let $K = \|(2\kappa - \alpha)u^\mu \cdot \tau\|_{L^\infty(\partial\Omega\times(0,T))}$. Consider the Dirichlet problem for the linear parabolic equations as follows

$$\begin{cases} \tilde{\omega}_t + u^\mu \cdot \nabla\tilde{\omega} - \mu(\Delta\tilde{\omega} + 3b^{-1}\partial_i b \partial_i \tilde{\omega} + G(u^\mu, \nabla u^\mu)) = b^{-1} curl f, \quad x \in \Omega, \\ \tilde{\omega}(t = 0, \cdot) = |\omega_0|, \quad x \in \Omega, \\ \tilde{\omega}(t, x) = K, \quad x \in \partial\Omega, \end{cases} \tag{18}$$

where $u^\mu \in C([0, T]; H^2(\Omega))$ is given by Theorem 1 and $G(u^\mu, \nabla u^\mu)$ are same as in (8) satisfying (9).

By the hypothesis, we have the properties: $u^\mu \in C([0, T]; H^2(\Omega)), \omega_0 \in L^p(\Omega),\ curl f \in L^\infty(0, T; L^p(\Omega)), p > 2$. It follows from the standard parabolic theory that the problem (18) has a unique weak solution $\tilde{\omega} \in L^2(0, T; H^1(\Omega))$. Setting $\bar{\omega} = \omega^\mu - \tilde{\omega}$ (or $-\omega^\mu - \tilde{\omega}$). Then $\bar{\omega}$ satisfies

$$\begin{cases} \bar{\omega}_t + u^\mu \cdot \nabla\bar{\omega} - \mu(\Delta\bar{\omega} + 3b^{-1}\partial_i b \partial_i \bar{\omega}) = 0 \quad x \in \Omega, \\ \bar{\omega}(t = 0, x) = \omega_0 - |\omega_0| \quad x \in \Omega, \\ \bar{\omega}(t, x) = (2\kappa - \alpha)u^\mu \cdot \tau - K \quad x \in \partial\Omega. \end{cases} \tag{19}$$

According to the comparison theorem of the parabolic equations, we obtain

$$|\omega^\mu| \le \tilde{\omega}, \quad \text{a.e. in} \quad \Omega \times [0, T). \tag{20}$$

Thus, we only need to prove (17) with ω^μ replaced by $\tilde{\omega}$. To this end, we set $\hat{\omega} = \tilde{\omega} - K$. Then we have

$$\begin{cases} \hat{\omega}_t + u^\mu \cdot \nabla\hat{\omega} - \mu(\Delta\hat{\omega} + 3b^{-1}\partial_i b \partial_i \hat{\omega}) = \mu G(u^\mu, \nabla u^\mu) + b^{-1} curl f, \quad x \in \Omega, \\ \hat{\omega}(t = 0, x) = |\omega_0| - K, \quad x \in \Omega, \\ \hat{\omega}(t, x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{21}$$

Multiplying $|\hat{\omega}|^{p-2}\hat{\omega}$ to the first equation of (21) and integrating with respect to x, we have

$$\begin{aligned} &\frac{d}{dt}\|\hat{\omega}\|^p_{L^p} + (p-1)\mu\int_\Omega (|\nabla\hat{\omega}||\hat{\omega}^{\frac{p-2}{2}}|)^2 dx - 3\mu\int_\Omega \partial_i b \partial_i \hat{\omega}|\hat{\omega}|^{p-2}\hat{\omega} dx \\ &= \mu\int_\Omega G(u^\mu, \nabla u^\mu)|\hat{\omega}|^{p-2}\hat{\omega} b dx + \int_\Omega b^{-1} curl f \cdot |\hat{\omega}|^{p-2} \cdot \hat{\omega} b dx + \mu\int_\Omega |\hat{\omega}|^{p-2}\nabla\hat{\omega} \cdot \hat{\omega} \cdot \nabla b dx \\ &\le \mu c_1\|\hat{\omega}\|^p_{L^p} + c_2\|curl f\|_{L^p(\Omega)}\|\hat{\omega}\|^{p-1}_{L^p(\Omega)} + \frac{p}{2}\mu\int_\Omega (|\nabla\hat{\omega}||\hat{\omega}^{\frac{p-2}{2}}|)^2 dx \end{aligned}$$

Applying Lemma 2 and Hölder inequality, and the fact

$$-3\mu\int_\Omega \partial_i b \partial_i \hat{\omega}|\hat{\omega}|^{p-2}\hat{\omega} dx = 3\mu\int_\Omega \Delta b|\hat{\omega}|^p dx,$$

we get

$$\frac{d}{dt}\|\hat{\omega}\|^p_{L^p} + (p-1)\mu\int_\Omega (|\nabla\hat{\omega}||\hat{\omega}^{\frac{p-2}{2}}|)^2 dx \le c_3\mu p\|\hat{\omega}\|^p_{L^p} + c_4\|curl f\|_{L^p(\Omega)}\|\hat{\omega}\|^{p-1}_{L^p(\Omega)}.$$

It follows that

$$\|\hat{\omega}\|_{L^\infty(0,T;L^p(\Omega))} \leq c_5 e^{\mu T}(\|\omega_0\|_{L^p(\Omega)} + \|curl f\|_{L^p(\Omega)}) \leq O(1)(\|\omega_0\|_{L^p(\Omega)} + \|curl f\|_{L^p(\Omega)}). \tag{22}$$

The last step is due to the fact that μ is small positive constant and will tend to zero. Obviously, $c_1, \cdots, c_5,\ O(1)$ are positive constants independent of μ, p. Thus, combining (20) with (22), we have that $\|\omega^\mu\|_{L^\infty(0,T;L^p(\Omega))} \leq O(1)(\|\omega_0\|_{L^p(\Omega)} + \|curl f\|_{L^p(\Omega)}) + K$.

Next we need to estimate K. Since

$$\begin{aligned}
\|u^\mu \cdot \tau\|_{L^\infty(\partial\Omega\times(0,T))} &\leq O(1)\|u^\mu\|^\theta_{L^\infty(0,T;L^2(\Omega)}\|u^\mu\|^{1-\theta}_{L^\infty(0,T;W^{1,p}(\Omega))} \\
&\leq O(1)\|u^\mu\|^\theta_{L^\infty(0,T;L^2(\Omega))}\|\omega^\mu\|^{1-\theta}_{L^\infty(0,T;L^p(\Omega))},
\end{aligned}$$

where $\theta = \dfrac{p-2}{2p-2},\ p > 2$, then we have

$$K \leq \varepsilon\|\omega^\mu\|_{L^\infty(0,T;L^p(\Omega))} + C_\varepsilon\|u^\mu\|_{L^\infty(0,T;L^2(\Omega))}.$$

We take ε small enough to obtain that

$$\|\omega^\mu\|_{L^\infty(0,T;L^p(\Omega))} \leq O(1)(\|\omega_0\|_{L^p(\Omega)} + \|curl f\|_{L^p(\Omega)}) + \|u^\mu\|_{L^\infty(0,T;L^2(\Omega))}).$$

Using the obtained estimate (16), we have

$$\|\omega^\mu\|_{L^\infty(0,T;L^p(\Omega))} \leq O(1)(\|\omega_0\|_{L^p(\Omega)} + \|u_0\|_{L^2(\Omega)} + \|f\|_{L^\infty(0,T;L^2(\Omega))} + \|curl f\|_{L^\infty(0,T;L^p(\Omega))}),$$

which is (17) for $p < \infty$. It also holds for $p = \infty$ due to $O(1)$ independent of p. Then the proof is finished.

Based on the estimates of Theorem 2, we have the following convergence result.

Theorem 3 Under assumptions of Theorem 1, we have that there exists a subsequence of $\{u^\mu\}$ denoted by u^{μ_k} such that

$$u^{\mu_k} \longrightarrow u$$

strongly in $C([0,T];L^2(\Omega))$ as $k \longrightarrow \infty$. The limit function u is the weak solution to the inviscid lake equations (6), which is,

$$\int_\Omega u\phi b dx + \int_0^T \int_\Omega u(u\cdot\nabla)\phi b dx dt = \int_\Omega u_0\phi(\cdot,0) b dx + \int_0^T \int_\Omega f \cdot u b dx dt. \tag{23}$$

for any test functions $\phi \in C([0,T);V)$, which is divergence free and tangent to the boundary.

Remark 1. When $p = \infty$, the weak solution of the inviscid lake equations (6) is unique, similar to Yudovich's theorem to the 2D Euler equations.

Remark 2. When $1 < p \leq 2$, using the free Navier boundary condition (5), we can obtain the similar estimates as (16), (17) and the results of Theorem 3 hold in this case.

Proof We note that the following estimates hold for u^μ

$$\|u^\mu\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq O(1).$$

and

$$\|\partial_t u^\mu\|_{L^2(0,T;H^{-1}(\Omega))} \leq O(1).$$

Where $O(1) > 0$ depends only on the initial velocity u_0 ,the initial vorticity ω_0 and f, independent of viscosity μ. Then we can extract a subsequence u^{μ_k} which converges strongly in $C([0,T];L^2(\Omega))$, and weakly in $L^2((0,T);H^1(\Omega))$. Obviously, the convergence is sufficient to pass to the limit in each term of (12) and guarantee that the limit function u satisfies the 2D incompressible inviscid lake equation in weak sense. This completes the proof.

4 The case of non-smooth data

In this section, we intend to relax the initial conditions of Theorem 3 to non-smooth initial data. To this aim, we firstly approximate the initial data as follows (see [16]).

Lemma 3 Let $\omega = b^{-1}curlu \in L^p(\Omega), 1 < p \leq +\infty$. Then there exists a sequence $\omega^n = b^{-1}curlu^n \in H^1(\Omega) \cap L^\infty(\Omega)$ satisfy

$$\omega^n \longrightarrow \omega \quad \text{in} \quad L^p(\Omega).$$

Using Lemma 3, we can obtain the following theorem similar to Theorem 3.

Theorem 4 Let $u_0 \in V$ and $\omega_0 = b^{-1}curlu_0 \in L^p(\Omega), f \in H^1(0,T;L^2(\Omega)), curlf \in L^\infty(0,T;L^p(\Omega)), 2 < p \leq \infty$, then there exists a unique solution $u^\mu \in C(0,T;H), \partial_t u \in L^2(0,T;V'), \omega^\mu \in L^\infty(0,T;L^p(\Omega))$ for the weak solution of (1). Moreover, u^μ, ω^μ satisfy the estimates (16) and (17).

Our main result of this section reads as

Theorem 5 Let $u_0 \in V, \omega_0 \in L^p(\Omega), f \in H^1(0,T;L^2(\Omega)), curlf \in L^\infty(0,T;L^p(\Omega)), 2 < p \leq \infty$. Let u^μ, ω^μ be the corresponding solution of (1) and (11) presented in Lemma 4. Then we have

$$u^\mu \longrightarrow u \ \text{ in } \ L^q(0,T;W^{\alpha,q'}(\Omega)),$$

where $1 < q < \infty$, $\frac{1}{q'} < \frac{1}{p} - \frac{1-\alpha}{2}$, $\alpha \in (0,1)$. Moreover, u is the weak solution to the 2D incompressible inviscid lake system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, & \text{in } \ \Omega \times (0,T) \\ div(bu) = 0, & \text{in } \ \Omega \times (0,T) \\ \omega = b^{-1}curlu, & \text{in } \ \Omega \times (0,T) \\ u \cdot \tau = 0, & \text{on } \ \partial\Omega \times (0,T) \\ u(t=0,\cdot) = u_0 & \text{in } \ \Omega. \end{cases}$$

Proof From Lemma 4, we know that there exist the solutions u^μ, ω^μ of (1) such that

$$u^\mu \in C(0,T;H) \cap L^2(0,T;H^1(\Omega))$$

and

$$\begin{cases} \displaystyle\int_\Omega \phi u^\mu b dx + 2\mu \int_0^T \int_\Omega Du^\mu : D\phi b dx + \mu \int_0^T \int_\Omega div u^\mu div\phi b dx \\ \qquad + \displaystyle\int_0^T \int_\Omega u^\mu \cdot \nabla u^\mu \cdot \phi b dx + \mu \int_0^T \int_{\partial\Omega} \alpha(u\cdot\tau)(\phi\cdot\tau) b dS \\ \qquad = \displaystyle\int_\Omega u_0 \phi(0,\cdot) b dx + \int_0^T \int_\Omega f \cdot u^\mu b dx, \\ u^\mu = K_\Omega(\omega^\mu), \end{cases} \tag{24}$$

for every test function $\phi \in C([0,T);V)$. From the proof of Theorem 2, we know that it is also true for non-smooth data. Then we know that

$$\begin{aligned} &u^\mu \in L^\infty(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,p}(\Omega)), \\ &\omega^\mu \in L^\infty(0,T;L^p(\Omega)). \end{aligned}$$

Then we can take a subsequence, denoted by u^{μ_k}, such that

$$\begin{aligned} &u^{\mu_k} \rightharpoonup u \quad \text{ in } \; w* - L^\infty(0,T;W^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \\ &\omega^{\mu_k} \rightharpoonup \omega, \quad \text{ in } \; w* - L^\infty(0,T;L^p(\Omega)), \end{aligned}$$

as $k \longrightarrow \infty$.

And

$$u^{\mu_k} \longrightarrow u, \quad \text{ in } \; L^q(0,T;W^{\alpha,q'}(\Omega)) \quad \text{ as } \; k \longrightarrow \infty,$$

where $1 < q < \infty$, $\frac{1}{q'} \le \frac{1}{p} - \frac{1-\alpha}{2}$, $\alpha \in (0,1)$.

Then the limit functions u satisfies the weak form of the inviscid lake equations, that is,

$$\begin{aligned} &\int_\Omega \phi u b dx + \int_0^T \int_\Omega u \cdot \nabla u \cdot \phi b dx = \int_\Omega u_0 \phi(0,\cdot) b dx + \int_0^T \int_\Omega f \cdot u b dx, \\ &u = K_\Omega(\omega), \end{aligned}$$

This completes the proof.

5 Appendix

Here we give the explicit form of $A\omega$, appearring in (8), where $\omega = b^{-1} curl u$ is the potential vorticity of the velocity u.

Using the divergence free condition $div(bu) = 0$, we obtain that

$$\begin{aligned} &b^{-1} div(2bD(u) + b div u I) \\ &\quad = b^{-1} \sum_{i=1}^{2} \partial_i (b \partial_i u_j + b \partial_j u_i + b div u \delta_{ij}) \\ &\quad = b^{-1} \sum_{i=1}^{2} (\partial_i(\partial_i(bu_j) - \partial_i b u_j + \partial_j(bu_i) - \partial_j b u_i)) + \partial_j(b div u) \\ &\quad = \Delta u + b^{-1} \nabla b \cdot \nabla u - b^{-1} u \cdot \nabla(\nabla b) + \nabla(div u), \end{aligned}$$

then we have
$$\begin{aligned}
&b^{-1}\nabla\times(b^{-1}div(2bD(u)+bdivuI))\\
&=b^{-1}\nabla\times(\Delta u+b^{-1}\nabla b\cdot\nabla u-b^{-1}u\cdot\nabla(\nabla b)+\nabla(divu))\\
&=\Delta\omega+3b^{-1}\partial_i b\partial_i\omega+G(u,\nabla u)\\
&\equiv A\omega,
\end{aligned}$$
where
$$\begin{aligned}
G(u,\nabla u)&=b^{-1}\Delta b\omega+b^{-1}\nabla\times(\nabla logb\cdot\nabla u)-b^{-1}\nabla\times(b^{-1}u\cdot\nabla(\nabla b))\\
&=b^{-1}\Delta b\omega+\sum_{i=1}^{2}(\partial_{1i}^2 logb\partial_i u_2-\partial_{2i}^2 logb\partial_i u_1\\
&\qquad -u_i(\partial_1 b^{-1}\partial_{i2}^2 b-\partial_2 b^{-1}\partial_{i1}^2 b)-b^{-1}(\partial_1 u_i\partial_{i2}^2 b-\partial_2 u_i\partial_{i1}^2 b).
\end{aligned}$$
It is noted that $G(u,\nabla u)$ is the linear combination of $u,\nabla u$, satisfying
$$\|G(u,\nabla u)\|_{L^p(\Omega)}\le C(\|u\|_{L^p(\Omega)}+\|\nabla u\|_{L^p(\Omega)}),\quad p>1,$$
where C depending on the bound norm of $b,\nabla b,D_{ij}b$.

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