Spherically Symmetric Isentropic Compressible Flows with Density-Dependent Viscosity Coefficients*

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Abstract

We construct global weak solutions to the compressible Navier-Stokes equations with density-dependent viscosity coefficients when the initial data is large, discontinuous, and spherically symmetric. We focus on the case where those coefficients vanish on vacuum. The solutions are obtained as limits of solutions in annular regions between two balls, and the equations hold in the sense of distribution in the entire space-time domain. In particular, we prove the existence of spherically symmetric solutions to the Saint-Venant model for shallow water.

1 Introduction

The compressible Navier-Stokes equations with density-dependent viscosity coefficients can be written as

$$\rho_t + \mathrm{div}(\rho \mathbf{U}) = 0, \tag{1.1}$$

$$(\rho \mathbf{U})_t + \mathrm{div}(\rho \mathbf{U} \otimes \mathbf{U}) - \mathrm{div}(h(\rho) D(\mathbf{U})) - \nabla(g(\rho)\mathrm{div}\mathbf{U}) + \nabla P(\rho) = 0, \tag{1.2}$$

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where $t \in (0, +\infty)$ is the time and $\mathbf{x} \in \mathbb{R}^N, N = 2, 3$ is the spatial coordinate, while $\rho(\mathbf{x}, t), \mathbf{U}(\mathbf{x}, t)$ and $P(\rho) = \rho^\gamma (\gamma > 1)$ stand for the fluid density, velocity and pressure, respectively. And

$$D(\mathbf{U}) = \frac{\nabla \mathbf{U} +^t \nabla \mathbf{U}}{2}$$

is the strain tensor and $h(\rho), g(\rho)$ are the Lamé viscosity coefficients satisfying

$$h(\rho) > 0, h(\rho) + N g(\rho) \geq 0. \tag{1.3}$$

In the last several decades, significant progress on the system (1.1)-(1.2) with a positive constant viscosity coefficients has been achieved by many authors. Concerning the global existence and the large-time behavior of solutions for sufficiently small data, the system (1.1)-(1.2) (as well as the full compressible Navier-Stokes equations including the conservation law of energy) is well-understood in the sense that if the data are small perturbations of an uniform non-vacuum state, then there exists a (smooth or weak) solution which is time-asymptotically stable (see[22]-[24],[3]). The situation, however, becomes more complex when the data are large, and a number of important questions, for example the existence of global solutions in the case of heat-conducting gases and the uniqueness of weak solutions, still remain open. The first general result on weak solutions was obtained by Lions in [20], in which he used the method of weak convergence to obtain global weak solutions provided that the specific heat ratio γ is appropriately large, for example $\gamma \geq 3N/(N+2)$, $N = 2, 3$. Feireisl, Novotný and Petzeltová [7, 8] extended Lions' existence result to the case $\gamma > N/2$ $(N = 2, 3)$. Jiang and Zhang [15, 16] showed the global existence of weak solutions for any $\gamma > 1$ to the Cauchy problem with spherically symmetric data. More recently, the global existence of axisymmetric and helically symmetric weak solutions for any $\gamma \geq 1$ was studied in [17, 28].

It is noted that in dealing with large amplitude solutions, one has to face the possible appearance of vacuum state in general. However, as observed in [11, 32, 21], the compressible Navier-Stokes equations with constant viscosity coefficients behave singularly in the presence of vacuum. By some physical considerations, Liu, Xin and Yang in [21] introduced the modified compressible Navier-Stokes equations with density-dependent viscosity coefficients for isentropic fluids. In fact, as presented in [21], while deriving the compressible Navier-Stokes equations from the Boltzmann equations by the Chapman-Enskog expansions, the viscosity depends on the temperature, and correspondingly depends on the density for isentropic cases. Meanwhile, in geophysical flows, many mathematical models correspond to (1.1)-(1.2) (see [1, 2, 20]). In particular, the viscous Saint-Venant system for shallow water is expressed exactly as (1.1)-(1.2) with $N = 2, h(\rho) = \rho, g(\rho) = 0$ and $\gamma = 2$. Shallow water equations are to describe vertically averaged flows in three-dimensional shallow domains in term of the mean velocity $\mathbf{U}$ and the variation of the depth ρ due to the free surface (see [20], [2]), which is widely used in geophysical flows. Local smooth solutions or global smooth solutions for data close to equilibrium were established in [29] and related topics have been extensively studied in [1], [2] and references therein. Nevertheless, the global existence of weak solutions for large data to the shallow water equations or more generally to the multi-dimensional compressible Navier-Stokes equations (1.1)-(1.2) $(N = 2, 3)$ is still open. This is mainly due to the facts that for these more physical models new mathematical challenges are encountered. First, the vacuum states may appear for the solutions of (1.1) and (1.2) even if the initial data are far from the vacuum. Second, when dealing with vanishing viscosity coefficients on vacuum, the velocity cannot even be defined when the density vanishes and hence we will have no uniform estimates for the velocity. Finally, the system (1.1)-(1.2) is highly degenerate at vacuum.

For one-dimensional compressible Navier-Stokes equations (1.1) and (1.2) with $h(\rho) = \rho^\alpha, g(\rho) = 0 (\alpha \in (0, 1))$, there are many literatures on the well-posedness theory of the solutions (see[13],[14],[21],[26],[31], [33],[34],[35] and references therein). In particular, initial-boundary-value problems for one-dimensional (1.1)-(1.2) with $h(\rho) = \rho^\alpha(\alpha > 1/2)$ and $P = \rho^\gamma(\gamma \geq 1)$ was studied by Li, Li and Xin recently in [19] and interesting phenomena of vacuum vanishing and blow-up of solutions were found there. However, few results are available for multi-dimensional problems. The first multi-dimensional result is due to Bresch, Desjardins and Lin [2], where they showed the L^1 stability of weak solutions for the Korteweg's system (with the Korteweg stress tensor $k\rho\nabla\triangle\rho$) and their result was later improved in [1] to include the case of vanishing capillarity ($k = 0$), but with an additional quadratic friction term $r\rho|\mathbf{U}|\mathbf{U}$. An interesting new entropy estimate is established in [2] and [1] in a priori way, which provides some high regularity for the density. Recently, Mellet and Vasseur [25] proved the L^1 stability of weak solutions of the system of (1.1)-(1.2) based on the new entropy estimate, extending the corresponding L^1 stability results of [2] and [1] to the case $r = k = 0$. However, although L^1 stability is considered as one of the main steps to prove existence of weak solutions, the global existence of weak solutions of Korteweg's system (see [2]) and the compressible Navier-Stokes equations with density-dependent viscosity (1.1)-(1.2) remains open in the multi-dimensional cases. The key issue now is how to construct approximate solutions satisfying the a priori estimates required in the L^1 stability analysis, among which the lower bound of the density should be crucial and addressed. It seems highly non-trivial to do so due to the degeneracy of the viscosities near vacuum and the additional entropy inequality to be hold in the construction of approximate solutions.

In our paper, we will construct a class of approximate solutions and furthermore prove the global existence of weak solutions for spherically symmetric solutions of the compressible Navier-Stokes equations with the viscosity coefficients depending on the density. For simplicity of the presentation, in this paper we will only give the proof of the global existence of the three-dimensional spherically symmetric solutions of (1.1)-(1.2) with $h(\rho) = \rho, g(\rho) = 0$. Our result holds true for general $h(\rho) = \rho^\alpha, g(\rho) = (\alpha-1)\rho^\alpha$ for some $\alpha > \frac{N-1}{N} (N = 2, 3)$. More general $h(\rho)$ and $g(\rho)$ satisfying $g(\rho) = \rho h'(\rho) - h(\rho)$ and other restrictions same as in [25] can be handled in a similar way. It should be noted that the shallow water equations corresponding to the case of $N = 2, \alpha = 1, \gamma = 2$ in (1.1)-(1.2) are covered and therefore we obtain the global spherically symmetric solutions of the shallow water equations.

It seems to be difficult to adapt the analysis in [7, 20] due to the degeneracy of the viscosities near vacuum which may appear. Thus we construct the approximate solutions by solving the approximate systems of (1.1)-(1.2) with $h^\varepsilon(\rho) = h(\rho) + \varepsilon\rho^\beta, g^\varepsilon(\rho) = g(\rho) + \varepsilon(\beta - 1)\rho^\beta$ for some fixed $0 < \beta < 1$ ($\beta = 3/4$ for example) instead of $h(\rho), g(\rho)$ in (1.1)-(1.2). This is motivated by the approach of Jiang, Xin, and Zhang [14], in which one-dimensional case is considered and $h(\rho)$ can be regarded as ρ^α, and $g(\rho) = (\alpha - 1)\rho^\alpha$ for $0 < \alpha < 1$. However, compared with the one-dimensional equations, there are some new difficulties encountered for radial symmetric 3-dimensional N-S systems. In particular, the three-dimensional spherically symmetric equations become singular at $r = 0$ and more new source terms appear in both Eulerian and Lagrangian radial symmetric equations (see (2.6)-(2.7) in Section 2 and (3.12) in Section 3 respectively), which lead to some difficulties to obtain the lower bound of the density. Therefore we will use the radial symmetric system only on the annular domain $\Omega_\varepsilon = \Omega \setminus \bar{B}_\varepsilon(0)$, where $B_\varepsilon(0)$ is a ball with radius ε and center 0, to exclude the singularity at the origin when we construct approximate solutions, and rewrite the Lagrangian equation as a new form (see (3.23) in section 3) which makes it possible to obtain the lower bounds of the approximate solutions.

By the approach mentioned above, we can obtain a class of approximate solutions with the required a priori uniform estimates such as energy estimates and entropy estimates. However, it should be noted such approximate solutions are defined and estimated on the annular domain $\Omega_\varepsilon = \Omega \setminus \bar{B}_\varepsilon(0)$, and the L^1-stability analysis as in [25] can provide the convergence of the terms in the equations (1.1)-(1.2) for the approximate solutions away from $r = 0$ (in particulars, the strong convergence of $\sqrt{\rho^j}\mathbf{U}^j$ locally in $r > 0$). Thus, to take the limit of the approximate solutions to obtain weak solutions which are defined on the entire domain Ω, we need to define the approximate solutions on $B_\varepsilon(0)$. Note that the usual zero extensions as in [10, 12] are not suitable here since such extension would yield that $\nabla\sqrt{\rho}$ belongs to $L^\infty(0, T; L^2_{loc}(\Omega \setminus \{0\})$ only so that it is difficult to make sense of the nonlinear diffusion terms in the definition of weak solutions. An appropriate extension is presented in this paper, one of whose advantages is that it preserves the uniform $L^\infty(0, T; H^1(\Omega))$ estimate of $\sqrt{\rho^\varepsilon}$ such that we can obtain the convergence of the pressure term $(\rho^\varepsilon)^\gamma$ and the diffusion terms which are difficult to handle due to the density-dependent viscosity coefficients. Also, though it seems difficult to obtain some uniform estimates for $\mathbf{U}^j$ separately because of the possible appearance of the vacuum, an extra estimate for $\mathrm{esssup}_{0\le t\le T}\int_\Omega \rho^j|\mathbf{U}^j|^{2+\eta}dx$ with some small $\eta \in (0, 1)$, which was observed by Mellet and Vasseur ([25], guarantees the convergence of the nonlinear convection terms.

The plan of this paper is as follows. In Section 2 we give the main results of this paper. In Section 3 we give the entropy estimates and the pointwise bounds of the density, which are the starting point for the derivation of smooth approximate solutions and their convergence. In Section 4, we construct approximate solutions and take the limits to obtain the global existence of weak solutions of the original system.

2 Notations and main results

Set $h(\rho) = \rho$ and $g(\rho) = 0$ in (1.1)-(1.2). The isentropic compressible Navier-Stokes equations become

$$\rho_t + \mathrm{div}(\rho\mathbf{U}) = 0, \tag{2.1}$$

$$(\rho\mathbf{U})_t + \mathrm{div}(\rho\mathbf{U} \otimes \mathbf{U}) - \mathrm{div}(\rho D(\mathbf{U})) + \nabla P(\rho) = 0 \tag{2.2}$$

for $t \in (0, +\infty)$ and $\mathbf{x} \in \mathbb{R}^3$. Here $\rho(\mathbf{x}, t), \mathbf{U}(\mathbf{x}, t)$ and $P(\rho) = \rho^\gamma (\gamma > 1)$ are the same as in (1.1)-(1.2). The initial and boundary conditions of (2.1)-(2.2) are imposed as:

$$(\rho, \rho\mathbf{U})|_{t=0} = (\rho_0, \mathbf{m}_0) \tag{2.3}$$

$$\mathbf{m} = \rho\mathbf{U} = 0 \text{ on } \partial\Omega, . \tag{2.4}$$

We are concerned with the spherically symmetric solutions of the system (2.1)-(2.2) in a ball Ω of radius R centered at the origin in $\mathbb{R}^3$. To this end, we denote

$$|\mathbf{x}| = r, \rho(\mathbf{x}, t) = \rho(r, t), \mathbf{U}(\mathbf{x}, t) = u(r, t)\frac{\mathbf{x}}{r}. \tag{2.5}$$

And for simplicity, we will take $D(\mathbf{U}) = \nabla\mathbf{U}$ in (2.2), though the full strain tensor could be considered without any additional difficulty. This leads to the following system of equations for $r > 0$,

$$\rho_t + (\rho u)_r + \frac{2\rho u}{r} = 0, \tag{2.6}$$

$$(\rho u)_t + (\rho u^2 + \rho^\gamma)_r + \frac{2\rho u^2}{r} - (\rho u_r)_r - \rho(\frac{2u}{r})_r = 0, \tag{2.7}$$

with the initial condition

$$(\rho, \rho u)|_{t=0} = (\rho_0, m_0), \tag{2.8}$$

and the boundary conditions

$$\rho u(0, t) = 0, \quad \rho u(R, t) = 0. \tag{2.9}$$

It is easy to get the following usual a priori energy estimate for smooth solutions to (2.6), (2.7) and (2.9):

$$\frac{d}{dt}\int_0^R (\frac{1}{2}\rho u^2 + \frac{1}{\gamma - 1}\rho^\gamma)r^2 dr + \int_0^R \rho(u_r^2 r^2 + 2u^2)dr \leq 0. \tag{2.10}$$

However, the system (2.1)-(2.2) admits an additional a priori estimate, as observed by Bresch, Desjardins and Lin [2], which reads in general case as follows

Lemma 2.1. *(see [25]) Assume that $h(\rho)$ and $g(\rho)$ are two C^2 functions such that*

$$g(\rho) = \rho h'(\rho) - h(\rho)$$

holds true. Then, the following inequality holds for smooth solutions of (1.1)-(1.2) *with $\rho > 0$:*

$$\frac{d}{dt}\int_\Omega (\frac{1}{2}\rho|\mathbf{U} + \nabla\varphi(\rho)|^2 + \frac{1}{\gamma - 1}\rho^\gamma)dx + \int_\Omega \nabla\varphi(\rho) \cdot \nabla\rho^\gamma dx \leq 0, \tag{2.11}$$

with φ such that

$$\varphi'(\rho) = \frac{h'(\rho)}{\rho}.$$

In particular, for three-dimensional spherically symmetric equations (2.6)-(2.7), one has

Lemma 2.2. *If* (ρ, u) *is a smooth solutions to* (2.6)-(2.9) *with* $\rho > 0$*, then the following inequality holds:*

$$\frac{d}{dt}\int_0^R \frac{1}{2}\rho|u + (\log\rho)_r|^2 r^2 dr + \int_0^R \frac{4}{\gamma}((\rho^{\gamma/2})_r r)^2 dr \leq 0,$$

i.e.,

$$\frac{d}{dt}\int_0^R \{\frac{1}{2}\rho u^2 + \rho_r u + |(\sqrt{\rho})_r|^2\} r^2 dr + \int_0^R \frac{4}{\gamma}((\rho^{\gamma/2})_r r)^2 dr \leq 0. \tag{2.12}$$

Proof. Although Lemma 2.2 is a special case of Lemma 2.1, for completeness, we outline the proof here since it is very simple in the radial symmetric case. Multiplying (2.6) by $\frac{|(\log\rho)_r|^2 r^2}{2}$ on both sides gives

$$\rho_t \frac{|(\log\rho)_r|^2}{2} r^2 + \frac{|(\log\rho)_r|^2}{2}(\rho u r^2)_r = 0. \tag{2.13}$$

It follows from (2.6) that

$$\begin{aligned} &\rho(\frac{|(\log\rho)_r|^2}{2})_t r^2 + (\frac{|(\log\rho)_r|^2}{2})_r \rho u r^2 \\ &= -\rho_r u_{rr} r^2 - \rho_r u_r (\log\rho)_r r^2 - 2\rho_r u_r r + 2\rho_r u. \end{aligned} \tag{2.14}$$

Summing over (2.13) and (2.14), integrating the resulting equation with respect to r from 0 to R, one gets from (2.9) that

$$\begin{aligned} &\frac{d}{dt}\int_0^R \frac{\rho|(\log\rho)_r|^2}{2} r^2 dr \\ &= \int_0^R \{2\rho_r u - \rho_r u_{rr} r^2 - \rho_r u_r (\log\rho)_r r^2 - 2\rho_r u_r r\} dr. \end{aligned} \tag{2.15}$$

Note that

$$\begin{aligned} \frac{d}{dt}\int_0^R (\log\rho)_r \rho u r^2 dr &= \int_0^R (\log\rho)_r \partial_t(\rho u) r^2 dr + \int_0^R \rho u \partial_t((\log\rho)_r) r^2 dr, \\ &= \int_0^R (\log\rho)_r \partial_t(\rho u) r^2 dr + \int_0^R (\rho u r^2)_r \{(\log\rho)_r u + u_r + \frac{2u}{r}\} dr, \end{aligned} \tag{2.16}$$

due to (2.6) and (2.9). While (2.7) gives

$$\begin{aligned} \int_0^R (\log\rho)_r \partial_t(\rho u) r^2 dr &= -\int_0^R \frac{4}{\gamma}((\rho^{\gamma/2})_r r)^2 dr - \int_0^R (\rho u^2 r^2)_r (\log\rho)_r dr \\ &+ \int_0^R \{(\rho u_r + \frac{2\rho u}{r})_r (\log\rho)_r r^2 - 2\rho_r u r (\log\rho)_r\} dr. \end{aligned} \tag{2.17}$$

Putting (2.17) into (2.16) shows that

$$\frac{d}{dt}\int_0^R (\log\rho)_r \rho u r^2 dr + \int_0^R \frac{4}{\gamma}((\rho^{\gamma/2})_r r)^2 dr = \int_0^R \{\rho u_r^2 r^2 + 4\rho u u_r r + 4\rho u^2 + \rho_r u_{rr} r^2 + 2\rho_r u_r r - 2\rho_r u + (\log\rho)_r \rho_r u_r r^2 + 2\rho_r u^2 r\} dr \tag{2.18}$$

It follows from (2.15) and (2.18) that

$$\begin{aligned}
&\frac{d}{dt}\int_0^R \{(\log\rho)_r \rho u + \frac{\rho|(\log\rho)_r|^2}{2}\} r^2 dr + \int_0^R \frac{4}{\gamma}((\rho^{\gamma/2})_r r)^2 dr \\
&= \int_0^R \{\rho u_r^2 r^2 + 4\rho u u_r r + 4\rho u^2 + 2\rho_r u^2 r\} dr \\
&= \int_0^R \rho(u_r^2 r^2 + 2u^2) dr.
\end{aligned} \tag{2.19}$$

Combing (2.19) with the energy inequality (2.10), one obtains the desired estimate (2.12) and the lemma is proved. □

Now, we give a definition of weak solutions to (2.1)-(2.4).

Definition 2.1. *A pair $(\rho, \mathbf{U})$ is said to be a weak solution to (2.1)-(2.2) provided that*

(1) $\rho \geq 0$ a.e., and

$$\rho \in L^\infty(0,T; L^1(\Omega) \cap L^\gamma(\Omega)) \cap C([0,\infty); W^{1,\infty}(\Omega)^*),\ \sqrt{\rho} \in L^\infty(0,T; H^1(\Omega)),$$
$$\sqrt{\rho}\mathbf{U} \in L^\infty(0,T; L^2(\Omega)),\ \sqrt{\rho}\nabla\mathbf{U} \in L^2(0,T; W^{-1,1}(\Omega)),$$

where $W^{1,\infty}(\Omega)^*$ is the dual space of $W^{1,\infty}(\Omega)$;

(2) For any $t_2 \geq t_1 \geq 0$ and any $\psi \in C^1(\bar\Omega \times [t_1, t_2])$, the mass equation (2.1) holds in the following sense:

$$\int_\Omega \rho\psi dx|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_\Omega (\rho\psi_t + \rho\mathbf{U}\cdot\nabla\psi) dxdt; \tag{2.20}$$

(3) For any $\psi = (\psi^1, \psi^2, \psi^3) \in C^2(\bar\Omega \times [0,T])$ satisfying $\psi(\mathbf{x}, t) = 0$ on $\partial\Omega$ and $\psi(\mathbf{x}, T) = 0$, it holds that

$$\begin{aligned}
&\int_\Omega \mathbf{m_0}\cdot\psi(0,\cdot)dx + \int_0^T\int_\Omega [\sqrt{\rho}(\sqrt{\rho}\mathbf{U})\cdot\partial_t\psi + \sqrt{\rho}\mathbf{U}\otimes\sqrt{\rho}\mathbf{U} : \nabla\psi] dxdt \\
&+ \int_0^T\int_\Omega \rho^\gamma \mathrm{div}\psi dxdt - < \rho\nabla\mathbf{U}, \nabla\psi >= 0,
\end{aligned} \tag{2.21}$$

where the diffusion term makes sense as

$$\begin{aligned}
< \rho\nabla\mathbf{U}, \nabla\psi >= &-\int_0^T\int_\Omega \sqrt{\rho}(\sqrt{\rho}\mathbf{U})\cdot\Delta\psi dxdt \\
&-2\int_0^T\int_\Omega (\sqrt{\rho}\mathbf{U})\cdot(\nabla\sqrt{\rho}\cdot\nabla)\psi dxdt.
\end{aligned} \tag{2.22}$$

In this paper, we will construct global three-dimensional spherically symmetric weak solutions of (2.1)-(2.2) with the initial-boundary conditions (2.3)-(2.4). The initial data are assumed to satisfy

$$\rho_0 \geq 0 \text{ a.e. in } \Omega; \mathbf{m}_0 = 0 \text{ a.e. on } \{x \in \Omega | \rho_0(x) = 0\}; \tag{2.23}$$

$$\rho_0 \in W^{1,4}(\Omega); \nabla\sqrt{\rho_0} \in L^2(\Omega); \mathbf{m}_0^4 \in L^1(\Omega); \frac{\mathbf{m}_0^{2+\eta}}{\rho_0^{1+\eta}} \in L^1(\Omega), \tag{2.24}$$

here $\eta \in (0,1)$ is some small constant. It follows from the assumptions of (2.24) that

$$\rho_0 \in L^\infty(\Omega); \rho_0 \mathbf{U}_0^{2+\eta} \in L^1(\Omega); \rho_0 \mathbf{U}_0^2 \in L^1(\Omega). \tag{2.25}$$

The main results of this paper can be stated as

Theorem 2.1. *For $N = 3$ and $1 < \gamma < 3$, if the initial data have the form*

$$\rho_0 = \rho_0(|\mathbf{x}|), \mathbf{U}_0 = u_0(|\mathbf{x}|)\frac{\mathbf{x}}{r}$$

and satisfy (2.23)-(2.24), then the initial-boundary-value problem (2.1)-(2.4) has a global spherically symmetric weak solution

$$\rho = \rho(|\mathbf{x}|, t), \mathbf{U} = u(|\mathbf{x}|, t)\frac{\mathbf{x}}{r}$$

satisfying for all $T > 0$,

$$\rho(\mathbf{x}, t) \in C([0, T]; L^{\frac{3}{2}}(\Omega)), \sqrt{\rho}\mathbf{U} \in L^\infty(0, T; L^2(\Omega)), \tag{2.26}$$

$$\int_\Omega \rho(\mathbf{x}, t)dx = \int_\Omega \rho_0(\mathbf{x})dx. \tag{2.27}$$

Moreover, it holds that

$$\sup_{t\in[0,T]} \int_\Omega (\frac{1}{2}\rho|\mathbf{U}|^2 + \frac{1}{\gamma - 1}\rho^\gamma)dx \leq C, \tag{2.28}$$

$$\sup_{t\in[0,T]} \int_\Omega |\nabla\sqrt{\rho}|^2 dx \leq C, \tag{2.29}$$

where C is a constant.

Remark 2.1. *In fact, our analysis applies to slightly more general viscosity coefficients $h(\rho)$ and $g(\rho)$. For instance, our results hold true for the following situations:*

1) $h(\rho) = \rho^\alpha$ and $g(\rho) = (\alpha - 1)\rho^\alpha$ with $\alpha > \frac{N-1}{N}$, where the restriction of α results from the Lamé viscosity coefficients relation (1.3) and the usual energy estimates.

2) $h(\rho)$ and $g(\rho)$ satisfy the relation

$$g(\rho) = \rho h'(\rho) - h(\rho)$$

and some additional restrictions presented in [25].

Remark 2.2. *It can be checked easily that for $N = 2$, the conclusions in Theorem 2.1 hold true for any $\gamma > 1$. Consequently, we obtain existence of a global spherically symmetric solution to the Saint-Venant model for shallow water, which is a particular case of (2.1)-(2.2) with $N = 2, h(\rho) = \rho, g(\rho) = 0$ and $\gamma = 2$ (see [2, 20]).*

Remark 2.3. *It should be noted that the boundary condition (2.4) is appropriate from the physical point of view since if the vacuum appears on the boundary the velocity itself is meaningless and the momentum can be controllable. On the other hand, if no vacuum appears on the boundary, the boundary condition (2.4) is equivalent to $\mathbf{U}(R, t) = 0$.*

To make sense of the boundary condition (2.4) for weak solutions in Theorem 2.1, we note that $\mathbf{U} = u(r)\frac{\mathbf{x}}{r}$ and ρu satisfies

$$\int_0^R \rho\varphi r^2 dr|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_0^R (\rho\varphi_t + \rho u\varphi_r) r^2 dr dt. \tag{2.30}$$

for functions φ which are C^1 on $[0, R] \times [t_1, t_2]$, see (4.31) in Proposition 4.5 in section 4.

Actually, (2.30) holds for any φ which is Lipschitz continuous. In particular, set $\varphi(r, t) = \varphi_1(t)\varphi_2(r)$, where $\varphi_1(t)$ and $\varphi_2(r)$ are Lipschitz continuous functions satisfying $\varphi_1(t) \equiv 1$ in $[t_1, t_2]$ and

$$\varphi_2(r) = \begin{cases} 1, & x \in [0, R - \delta] \\ 1 - \frac{1}{\delta}(r - (R - \delta)), & r \in [R - \delta, R]. \end{cases}$$

Substituting $\varphi_1(t)$ and $\varphi_2(r)$ into (2.30) gives

$$\begin{aligned}\frac{1}{\delta}\int_{t_1}^{t_2}\int_{R-\delta}^R \rho u r^2 dr dt &= \int_0^R \rho(r, t_2)\varphi(r, t_2) r^2 dr - \int_0^R \rho(r, t_1)\varphi(r, t_1) r^2 dr \\ &= \int_0^R \rho(r, t_2) r^2 dr - \int_0^R \rho(r, t_1) r^2 dr + \int_{R-\delta}^R \rho(r, t_2)(\varphi_2(t_2) - 1) r^2 dr \\ &\quad - \int_{R-\delta}^R \rho(r, t_1)(\varphi_2(t_1) - 1) r^2 dr.\end{aligned}$$

It follows from this and the conservation of mass (2.27) that

$$\frac{1}{\delta}|\int_{t_1}^{t_2}\int_{R-\delta}^R \rho u r^2 dr dt| \le |\int_{R-\delta}^R \rho(r, t_2)(\varphi_2(t_2) - 1) r^2 dr - \int_{R-\delta}^R \rho(r, t_1)(\varphi_2(t_1) - 1) r^2 dr| \to 0$$

as $\delta \to 0$. This implies that $(\rho u)(R, t) = 0$ in the sense of trace.

3 Approximate Solutions and Their Estimates

The key point of the proof of Theorem 2.1 is to construct smooth approximate solutions satisfying the a priori estimates required in the L^1 stability analysis. The crucial issue is to obtain lower and upper bounds of the density, as mentioned in the introduction. To this end, we study the following system as an approximate system of (2.1)-(2.2).

$$\rho_t + \operatorname{div}(\rho\mathbf{U}) = 0, \tag{3.1}$$

$$(\rho\mathbf{U})_t + \operatorname{div}(\rho\mathbf{U}\otimes\mathbf{U}) - \operatorname{div}((\rho+\varepsilon\rho^{\frac{3}{4}})\nabla\mathbf{U}) + \nabla(\frac{\varepsilon}{4}\rho^{\frac{3}{4}}\operatorname{div}\mathbf{U}) \\ +\nabla P(\rho) = 0, \tag{3.2}$$

where $\varepsilon > 0$ is a constant.

When $\rho(\mathbf{x},t) = \rho(r,t), \mathbf{U}(\mathbf{x},t) = u(r,t)\frac{\mathbf{x}}{r}$, the system (3.1)-(3.2) becomes

$$\rho_t + (\rho u)_r + \frac{2\rho u}{r} = 0, \tag{3.3}$$

$$(\rho u)_t + (\rho u^2 + \rho^\gamma)_r + \frac{2\rho u^2}{r} + (\rho+\varepsilon\rho^{\frac{3}{4}})_r\frac{2u}{r} = ((\rho + \frac{3\varepsilon}{4}\rho^{\frac{3}{4}})(u_r + \frac{2u}{r}))_r, \tag{3.4}$$

for $r > 0$. We will first construct the smooth solution of (3.3)-(3.4) in the truncated region $0 < \varepsilon < r < R$ with the following initial condition

$$(\rho, \rho u)(r,0) = (\rho_0 + \varepsilon, m_0),$$

and boundary conditions

$$u(r,t)|_{r=\varepsilon} = 0, \quad u(r,t)|_{r=R} = 0. \tag{3.5}$$

For the approximate solutions which will have lower bound of the density, the boundary conditions of (3.5) is equivalent to $\rho u(r,t)|_{r=\varepsilon} = 0, \quad \rho u(r,t)|_{r=R} = 0$.

We assume that the initial data are smooth and satisfy the bounds (2.23)-(2.24) with constants independent of ε. As discussed in the introduction, we shall eventually take a sequence of inner radii ε_j tending to 0, and the dependence on j will be suppressed if there would be no confusions.

In the following, we will state the energy and entropy estimates which have been proved in the preceding section for these approximate solutions.

Lemma 3.1. *Let $(\rho^\varepsilon, u^\varepsilon)$ be smooth solutions of (3.3)-(3.4) defined on $[\varepsilon, R]\times[0,T]$ with boundary conditions (3.5) such that $\rho^\varepsilon > 0$. Then there exists a constant C independent of ε such that*

$$\int_\varepsilon^R \rho^\varepsilon(r,t)r^2dr \le C, \tag{3.6}$$

$$\int_\varepsilon^R (\frac{1}{2}\rho^\varepsilon(u^\varepsilon)^2 + \frac{1}{\gamma-1}(\rho^\varepsilon)^\gamma)r^2dr + \int_0^T\int_\varepsilon^R(\rho^\varepsilon + \frac{\varepsilon}{4}(\rho^\varepsilon)^{\frac{3}{4}})((u^\varepsilon_r)^2r^2 + (u^\varepsilon)^2)drdt \\ \le C, \tag{3.7}$$

$$\int_\varepsilon^R \frac{1}{2}\rho^\varepsilon|u^\varepsilon + (\log\rho^\varepsilon)_r + \frac{3\varepsilon}{4}(\rho^\varepsilon)^{-\frac{5}{4}}\rho^\varepsilon_r|^2r^2dr \\ + \int_0^T\int_\varepsilon^R(\gamma(\rho^\varepsilon)^{\gamma-2} + \frac{3\varepsilon}{4}\gamma(\rho^\varepsilon)^{\gamma-\frac{9}{4}})|\rho^\varepsilon_r|^2r^2drdt \le C. \tag{3.8}$$

Remark 3.1. *Notes that* $h(\rho) = \rho + \varepsilon\rho^{\frac{3}{4}}$ *and* $g(\rho) = -\frac{\varepsilon}{4}\rho^{\frac{3}{4}}$ *satisfy the relation* $g(\rho) = \rho h'(\rho) - h(\rho)$*. In general, one can choose to approximate the system* (2.1)-(2.2) *by taking*

$$h_\varepsilon(\rho) = \rho + \varepsilon\rho^\alpha, g_\varepsilon(h) = \varepsilon(\alpha - 1)\rho^\alpha$$

which satisfy

$$g_\varepsilon(\rho) = \rho h_\varepsilon'(\rho) - h_\varepsilon(\rho),$$

where $\frac{N-1}{N} < \alpha < 1, N = 2, 3$*. We take* $\alpha = \frac{3}{4}$ *for 3-dimensional case here.*

To make these a priori estimates valid globally, we need to give some detailed estimates on the density. We start with the following pointwise bounds for ρ^ε.

Lemma 3.2. *Given* $\varepsilon > 0$*, there is an absolute constant* C *which is independent of* ε*, such that*

$$0 \le \rho^\varepsilon(r, t) \le \frac{C}{\varepsilon^2} \tag{3.9}$$

for $\varepsilon \le r \le R$ *and* $t \ge 0$.

Proof. To simplify the presentation, we drop the superscript ε.

Let $r(t)$ denote a particle path by

$$\frac{dr(t)}{dt} = u(r(t), t).$$

Then along the particle path, (3.3) can be solved to get

$$\rho(r(t), t)r^2 = \rho_0(r(0))r(0)^2 e^{-\int_0^t u_r(r(s),s)ds},$$

which implies that $\rho \ge 0$ provided that $\rho_0 \ge 0$.

It follows from (3.7) and (3.8) that

$$\int_\varepsilon^R \frac{\rho_r^2}{\rho} r^2 dr \le C \tag{3.10}$$

for some absolute constant C independent of ε.

Then, it follows from (3.6) and (3.8) that for $\varepsilon \le r \le R$,

$$\begin{aligned}
\rho(r, t) &\le \int_\varepsilon^R \rho(r, t)dr + \int_\varepsilon^R |\rho_r(r, t)|dr \\
&\le \frac{1}{\varepsilon^2}\int_\varepsilon^R \rho(r, t)r^2 dr + \frac{1}{\varepsilon^2}\int_\varepsilon^R \sqrt{\rho}\frac{|\rho_r(r, t)|}{\sqrt{\rho}} r^2 dr \\
&\le \frac{C}{\varepsilon^2}
\end{aligned} \tag{3.11}$$

for all $t \ge 0$. The proof of the lemma is finished. □

To derive the a priori estimates about the velocity of the approximate solutions, the crucial step is to obtain lower bounds of the density. To this end, we introduce Lagrangian coordinates for the radial system (3.3)-(3.4) as follows. Let $\varepsilon > 0$ be fixed and define

$$x(r,t) = \int_{\varepsilon}^{r} \rho r^2 dr, \tau = t$$

Set $\int_{\varepsilon}^{1} \rho r^2 dr = 1$ for any fixed $\varepsilon > 0$ without loss of generality. Then,

$$\frac{\partial x}{\partial r} = \rho r^2, \frac{\partial x}{\partial t} = -\rho u r^2, \frac{\partial \tau}{\partial r} = 0, \frac{\partial \tau}{\partial t} = 1.$$

Then the system (3.3)-(3.4) becomes

$$\begin{cases} \rho_\tau + \rho^2 (r^2 u)_x = 0, \\ r^{-2} u_\tau + (\rho^\gamma)_x = [(\rho^2 + \frac{3\varepsilon}{4}\rho^{\frac{7}{4}})(r^2 u)_x]_x - (\rho + \varepsilon \rho^{\frac{3}{4}})_x \frac{2u}{r}, \end{cases} \tag{3.12}$$

for $\tau > 0$ and $0 \le x \le 1$.

The corresponding initial data is

$$(\rho, \rho u)(\cdot, 0) = (\rho_0 + \varepsilon, m_0),$$

and the boundary conditions are

$$u(0,\tau) = 0, \quad u(1,\tau) = 0. \tag{3.13}$$

For this system, the following a priori estimates hold.

Lemma 3.3. *For all $\tau \in [0, T]$, it holds that*

$$\begin{aligned} &\int_0^1 (\frac{u^2(x,\tau)}{2} + \frac{\rho^{\gamma-1}(x,\tau)}{\gamma - 1}) dx + \int_0^\tau \int_0^1 (\frac{2u^2}{r^2} + \rho^2 u_x^2 r^4) dx ds \\ &+ (1 - \frac{\lambda}{2}) \int_0^\tau \int_0^1 \varepsilon \frac{u^2}{\rho^{\frac{1}{4}} r^2} dx ds + (\frac{3}{4} - \frac{1}{2\lambda}) \int_0^\tau \int_0^1 \varepsilon \rho^{\frac{7}{4}} u_x^2 r^4 dx ds \\ &\le \int_0^1 (\frac{u_0^2}{2} + \frac{\rho_0^{\gamma-1}}{\gamma-1}) dx \ , \forall \lambda \in (\frac{2}{3}, 2). \end{aligned} \tag{3.14}$$

$$0 \le \rho(x,\tau) \le C(\varepsilon, T), \tag{3.15}$$

$$\varepsilon \le r(x,\tau) \le R, \tag{3.16}$$

$$\begin{aligned} &\int_0^1 u^4 dx + \int_0^\tau \int_0^1 (\frac{4u^4}{r^2} + 6\rho^2 u^2 u_x^2 r^4 + \frac{2\varepsilon u^4}{\rho^{\frac{1}{4}} r^2} + \varepsilon \rho^{\frac{7}{4}} u^2 u_x^2 r^4) dx ds \\ &\le \int_0^1 u_0^4 dx + C(\varepsilon, T). \end{aligned} \tag{3.17}$$

Proof. Multiplying $(3.12)_2$ by $r^2 u$, using $(3.12)_1$ and integration by parts, one gets

$$\begin{aligned} &\frac{d}{d\tau} \int_0^1 (\frac{u^2}{2} + \frac{\rho^{\gamma-1}}{\gamma-1}) dx + \int_0^1 (\rho^2 + \frac{3}{4}\varepsilon \rho^{\frac{7}{4}})((r^2 u)_x)^2 dx = \int_0^1 (\rho + \varepsilon \rho^{\frac{3}{4}})(2u^2 r)_x dx \\ &= 4 \int_0^1 (\rho + \varepsilon \rho^{\frac{3}{4}}) u u_x r dx + 2 \int_0^1 (1 + \frac{\varepsilon}{\rho^{\frac{1}{4}}}) \frac{u^2}{r^2} dx. \end{aligned} \tag{3.18}$$

Since

$$((r^2u)_x)^2 = (\frac{2u}{\rho r} + r^2u_x)^2 = 4\frac{u^2}{\rho^2 r^2} + 4\frac{uu_x r}{\rho} + u_x^2 r^4,$$

then from (3.18), one has

$$\frac{d}{d\tau}\int_0^1 (\frac{u^2}{2} + \frac{\rho^{\gamma-1}}{\gamma-1})dx + \int_0^1 (\frac{2u^2}{r^2} + \rho^2 u_x^2 r^4)dx + \int_0^1 \{\varepsilon\frac{u^2}{\rho^{\frac14} r^2} + \frac34 \varepsilon\rho^{\frac74} u_x^2 r^4\}dx$$
$$= \varepsilon\int_0^1 \rho^{\frac34} uu_x r dx \le \frac{\lambda}{2}\int_0^1 \frac{\varepsilon u^2}{\rho^{\frac14} r^2}dx + \frac{1}{2\lambda}\int_0^1 \varepsilon\rho^{\frac74} u_x^2 r^4 dx, \quad \forall\lambda\in(\frac23, 2).$$

Thus (3.14) holds.

Next, (3.15) follows from Lemma 3.2 and (3.16) holds trivially.

Now, we prove (3.17). In fact, multiplying $(3.12)_2$ by r^2u^3, using $(3.12)_1$ and integration by parts, we have

$$\frac14\frac{d}{d\tau}\int_0^1 u^4 dx + \int_0^1 (\rho^2 + \varepsilon\frac34\rho^{\frac74})((r^2u)_x)^2 u^2 dx + 2\int_0^1 (\rho^2 + \varepsilon\frac34\rho^{\frac74})u^2u_x^2 r^4$$
$$= -4\int_0^1 (\rho + \varepsilon\frac34\rho^{\frac34})u^3u_x r dx + \int_0^1 \rho^\gamma (u^3r^2)_x dx + \int_0^1 (\rho + \varepsilon\rho^{\frac34})(2u^4 r)_x dx.$$

Thus

$$\frac14\frac{d}{d\tau}\int_0^1 u^4 dx + \int_0^1 (\frac{2u^4}{r^2} + 3\rho^2u^2u_x^2r^4)dx + \int_0^1 (\frac{\varepsilon u^4}{\rho^{\frac14} r^2} + \frac94\varepsilon\rho^{\frac74}u^2u_x^2r^4)dx$$
$$= 2\int_0^1 \varepsilon\rho^{\frac34}u^3u_x r dx + \int_0^1 (\rho^{\gamma-1}\frac{2u^3}{r} + 3\rho^\gamma u^2u_x r^2)dx. \tag{3.19}$$

Using Hölder and Young's inequality and Lemma 3.2, one can estimate each term of the right hand side of (3.19) as follows:

$$2\varepsilon\int_0^1 \rho^{\frac34}u^3u_x r dx \le \frac12\int_0^1 \frac{\varepsilon u^4}{\rho^{\frac14}r^2}dx + 2\int_0^1 \varepsilon\rho^{\frac74}u^2u_x^2r^4 dx;$$
$$2\int_0^1 \rho^{\gamma-1}\frac{u^3}{r}dx \le 2(\int_0^1 \rho^{4(\gamma-1)}r^2 dx)^{\frac14}(\int_0^1 \frac{u^4}{r^2}dx)^{\frac34} \le \frac12\int_0^1 \frac{u^4}{r^2}dx + C,$$

and

$$3\int_0^1 \rho^\gamma u^2u_x r^2 dx \le 3(\int_0^1 \rho^{2\gamma-2}u^2 dx)^{\frac12}(\int_0^1 \rho^2u^2u_x^2r^4dx)^{\frac12}$$
$$\le \frac32\int_0^1 \rho^2u^2u_x^2r^4 dx + \frac32\int_0^1 \rho^{2\gamma-2}u^2 dx$$
$$\le \frac32\int_0^1 \rho^2u^2u_x^2r^4 dx + \frac32(\int_0^1 \rho^{4\gamma-4}r^2 dx)^{\frac12}(\int_0^1 \frac{u^4}{r^2}dx)^{\frac12}$$
$$\le \frac32\int_0^1 \rho^2u^2u_x^2r^4 dx + \frac12\int_0^1 \frac{u^4}{r^2}dx + C.$$

Putting the above three estimates into (3.19) yields

$$\frac14\frac{d}{d\tau}\int_0^1 u^4 dx + \int_0^1 (\frac{u^4}{r^2} + \frac32\rho^2u^2u_x^2r^4)dx + \int_0^1 (\frac{\varepsilon u^4}{2\rho^{\frac14}r^2} + \frac14\varepsilon\rho^{\frac74}u^2u_x^2r^4)dx \le C,$$

i.e.,

$$\int_0^1 u^4 dx + \int_0^\tau \int_0^1 (\frac{4u^4}{r^2} + 6\rho^2 u^2 u_x^2 r^4 + \frac{2\varepsilon u^4}{\rho^{\frac{1}{4}} r^2} + \varepsilon \rho^{\frac{7}{4}} u^2 u_x^2 r^4) dx ds$$
$$\leq \int_0^1 u_0^4 dx + C. \tag{3.20}$$

This proves (3.17). □

Remark 3.2.

$$\int_0^1 u_0^4 dx = \int_\varepsilon^R \frac{m_0^4}{(\rho_0 + \varepsilon)^3} r^2 dr \leq C(\varepsilon) \|\mathbf{m_0}\|_{L^4(\Omega)}.$$

The following estimate can be obtained by modifying the analysis in [14]:

Lemma 3.4. *There is a positive constant* $C = C(\|\rho_0\|_{W^{1,4}(\Omega)}, \|\mathbf{m}_0\|_{L^4(\Omega)}, \varepsilon, T)$, *such that*

$$\int_0^1 ((\rho^{\frac{3}{4}})_x)^4 (x, \tau) dx \leq C, \ \forall \tau \in [0, T]. \tag{3.21}$$

Proof. We rewrite $(3.12)_1$ in the form:

$$(\rho + \varepsilon \rho^{\frac{3}{4}})_{x\tau} = -[(\rho^2 + \frac{3\varepsilon}{4} \rho^{\frac{7}{4}})(r^2 u)_x]_x. \tag{3.22}$$

Thus, substituting (3.22) into $(3.12)_2$ yields

$$r^2 (\rho + \varepsilon \rho^{\frac{3}{4}})_{x\tau} + (\rho + \varepsilon \rho^{\frac{3}{4}})_x 2ur = -u_\tau - (\rho^\gamma)_x r^2.$$

Notes that

$$r^3(x, \tau) = \varepsilon^3 + 3 \int_0^x \frac{1}{\rho(y, \tau)} dy, \quad \frac{\partial r}{\partial x} = \frac{1}{\rho r^2},$$

and so

$$3r^2 \frac{\partial r}{\partial \tau} = 3 \int_0^x (\frac{1}{\rho})_\tau (y, t) dy$$
$$= 3 \int_0^x (r^2 u)_y (y, \tau) dy = 3r^2 u(x, \tau).$$

Thus

$$\frac{\partial r}{\partial \tau} = u.$$

So the above equality can be rewritten as

$$(r^2 (\rho + \varepsilon \rho^{\frac{3}{4}})_x)_\tau = -u_\tau - (\rho^\gamma)_x r^2. \tag{3.23}$$

Integrating it over $[0, t]$ shows

$$u(x, t) - u_0(x) + \int_0^t (\rho^\gamma)_x r^2 (x, s) ds$$
$$= r_0^2 (\frac{4}{3} \rho_0^{\frac{1}{4}} + \varepsilon) \partial_x (\rho_0^{\frac{3}{4}}) - r^2 (\frac{4}{3} \rho^{\frac{1}{4}} + \varepsilon) \partial_x (\rho^{\frac{3}{4}}). \tag{3.24}$$

Multiplying (3.24) by $(\partial_x(\rho^{\frac{3}{4}})r^2)^3$ and integrate over $[0,1]$ with respect to x, one gets

$$\begin{aligned}
&\int_0^1 (\frac{4}{3}\rho^{\frac{1}{4}}+\varepsilon)(\partial_x(\rho^{\frac{3}{4}})r^2)^4 dx = \int_0^1 r_0^2(\frac{4}{3}\rho_0^{\frac{1}{4}}+\varepsilon)\partial_x(\rho_0^{\frac{3}{4}})(\partial_x(\rho^{\frac{3}{4}})r^2)^3 dx\\
&-\int_0^1 \{u-u_0+\int_0^t (\rho^\gamma)_x r^2(x,s)ds\}(\partial_x(\rho^{\frac{3}{4}})r^2)^3 dx\\
&\le C(\int_0^1 (\partial_x(\rho^{\frac{3}{4}})r^2)^4 dx)^{\frac{3}{4}}\{\|u-u_0\|_{L^4}+\|\partial_x(\rho_0^{\frac{3}{4}})\|_{L^4}\\
&+(\int_0^t \|\partial_x\rho^\gamma\|_{L^4}^4 ds)^{\frac{1}{4}}\}.
\end{aligned} \tag{3.25}$$

Using Lemma 3.3, $\varepsilon \le r, r_0 \le R$ and Young's inequality, one gets from (3.25) that there is a positive constant C depending on $\|\rho_0\|_{W^{1,4}[0,1]}, \|u_0\|_{L^4[0,1]}, \varepsilon$ and T, such that

$$\varepsilon\int_0^1 (\partial_x(\rho^{\frac{3}{4}})r^2)^4 dx \le \frac{\varepsilon}{2}\int_0^1 (\partial_x(\rho^{\frac{3}{4}})r^2)^4 dx + C\int_0^t\int_0^1 (\partial_x\rho^\gamma)^4 dxds + C, \tag{3.26}$$

whence,

$$\int_0^1 (\partial_x(\rho^{\frac{3}{4}}))^4 dx \le C + C\int_0^t \max_{[0,1]}(\rho^{4\gamma-3})\int_0^1 (\partial_x(\rho^{\frac{3}{4}}))^4 dxds. \tag{3.27}$$

Applying Gronwall's inequality to (3.27) and making use of Lemma 3.2, we obtain

$$\int_0^1 (\partial_x(\rho^{\frac{3}{4}}))^4 dx \le C. \tag{3.28}$$

This completes the proof. $\square$

Remark 3.3.

$$\begin{aligned}
&\int_0^1 |\partial_x\rho_0^{\frac{3}{4}}|^4 dx = \frac{3}{4}\int_\varepsilon^R \frac{|\partial_r\rho_0|^4}{(\rho_0+\varepsilon)^4 r^8} r^2 dr\\
&\le C(\varepsilon)\int_\varepsilon^R |\partial_r\rho_0|^4 r^2 dr \le C(\varepsilon)\|\rho_0\|_{W^{1,4}(\Omega)}.
\end{aligned}$$

Now we can obtain the lower bound of the density.

Lemma 3.5. *There is a positive constant*

$$C = C(\varepsilon, T, \|\rho_0\|_{W^{1,4}(\Omega)}, \|\mathbf{m}_0\|_{L^4(\Omega)}),$$

such that

$$\rho \ge C, \ \forall x\in[0,1], \tau\in[0,T]. \tag{3.29}$$

Proof. Set $v(x,\tau)=\frac{1}{\rho(x,\tau)}$, and $V(\tau)=\max_{[0,1]\times[0,\tau]}v(x,s)$. The equation $(3.12)_1$ can be written as $v_\tau=(ru)_x$, which implies that $\int_0^1 v(x,\tau)dx=\int_0^1 v(x,0)dx\le C_0$, thanks to the boundary conditions (3.13). Then it follows from Sobolev's embedding $W^{1,1}([0,1])\hookrightarrow L^\infty([0,1])$ that, for any $0<\beta<1$,

$$
\begin{aligned}
v^\beta(x,\tau)&\le\int_0^1 v^\beta(x,\tau)dx+\int_0^x|\partial_x v^\beta|dx\\
&\le(\int_0^1 vdx)^\beta+\beta\int_0^x v^{\beta+\frac34}|\frac{\rho_x}{\rho^{\frac14}}|dx\\
&\le C+C\beta(\int_0^1(v^{\beta+\frac34})^{\frac43}dx)^{\frac34}(\int_0^1((\rho^{\frac34})_x)^4dx)^{\frac14}\\
&\le C+C\beta V^\beta(\int_0^1 vdx)^{\frac34}(\int_0^1((\rho^{\frac34})_x)^4dx)^{\frac14}\\
&\le C+C\beta V^\beta,
\end{aligned}
\tag{3.30}
$$

Thus choosing $\beta>0$ small enough, which may depend on ε and T, we obtain

$$V(T)\le C,$$

where $C=C(\varepsilon,T,\|\rho_0\|_{W^{1,4}(\Omega)},\|\mathbf{m}_0\|_{L^4(\Omega)})$. The proof of the lemma is completed. $\square$

4 Proof of Theorem 2.1

In this section, we will prove Theorem 2.1 by completing the constructions of smooth, approximate solutions, applying the a priori bounds of Section 2 and Section 3, and taking appropriate limits.

4.1 The existence of the approximate solutions

Consider the following approximate system in Lagrangian coordinate

$$
\begin{cases}
\rho_\tau+\rho^2(r^2u)_x=0,\\
r^{-2}u_\tau+(\rho^\gamma)_x=[(\rho^2+\frac{3\varepsilon}{4}\rho^{\frac74})(r^2u)_x]_x-(\rho+\varepsilon\rho^{\frac34})_x\frac{2u}{r}
\end{cases}
\tag{4.1}
$$

for $\tau>0, 0\le x\le 1$, with

$$(\rho,\rho u)(\cdot,0)=(\rho_0+\varepsilon,m_0),$$

and

$$u(0,\tau)=0,\ \ u(1,\tau)=0.$$

First we regularize the initial data as follows. Let J_δ be a standard mollifier (in r) of width δ. Let $(\rho_0+\varepsilon,u_0)$ be the initial data in Eulerian coordinate, where $u_0=\frac{m_0}{\rho_0+\varepsilon}$.

(1) Extend $\rho_0 + \varepsilon$ continuously outside $[\varepsilon, R]$ by taking $\rho_0(\varepsilon) + \varepsilon$ on $[0, \varepsilon]$ and $\rho_0(R) + \varepsilon$ on $[R, \infty)$, mollify with J_δ, restrict it to $[\varepsilon, R]$, and then multiply by a constant to normalize the total mass to be

$$M_0 = \int_0^R (\rho_0 + \varepsilon) r^2 dr.$$

The resulting density function is denoted by $\rho_0^{\varepsilon,\delta}(r)$.

(2) Redefine u_0 to be zero on $[0, \varepsilon + 2\delta]$ and $[R - 2\delta, R]$, then mollify it with J_δ to get the smooth approximate initial velocity denoted by $u_0^{\varepsilon,\delta}(r)$. Note that $u_0^{\varepsilon,\delta}(r)$ is identically zero on a neighborhood of $r = \varepsilon$ and $r = R$.

The resulting data $(\rho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta})$ then satisfy the hypotheses (2.23)-(2.24) with constants which are independent of ε and δ. For any fixed $\varepsilon > 0$, we denote the corresponding initial data in Lagrangian coordinate by $(\rho_0^\delta, u_0^\delta)$. Then $\rho_0^\delta \in C^{1+\beta}[0, 1]$ and $u_0^\delta \in C^{2+\beta}[0, 1]$ for any $0 < \beta < 1$. Moreover,

$$\rho_0^\delta \to \rho_0 + \varepsilon \text{ in } W^{1,4}([0, 1]), u_0^\delta \to u_0 \text{ in } L^4([0, 1]) \tag{4.2}$$

as $\delta \to 0$ and

$$u_0^\delta(0, \tau) = u_0^\delta(1, \tau) = 0.$$

Now, consider the initial boundary value problem (4.1) with the initial data $(\rho_0 + \varepsilon, u_0)$ replaced by $(\rho_0^\delta, u_0^\delta)$. Note, however, that ε is fixed and positive at this stage of the argument, so that there are no singularities in the equations, and the construction of these approximate solutions is essentially an one dimensional problem. For this problem one can apply the standard argument to obtain the existence of a unique local solution (ρ^δ, u^δ) with $\rho^\delta, \rho_x^\delta, \rho_{\tau x}^\delta, u^\delta, u_x^\delta, u_\tau^\delta, u_{xx}^\delta \in C^{\beta,\beta/2}([0, 1] \times [0, T^*])$ for some $T^* > 0$. It follows from Lemma 3.2-Lemma 3.5 and (4.2) that ρ^δ is bounded from below and above, $(u^\delta)^2$ and ρ_x^δ are bounded in $L^\infty([0, T]; L^2)$, and u_x^δ is bounded in $L^2([0, T]; L^2)$ for any $T > 0$ because of $\varepsilon < r < R$. Furthermore, one can differentiate the equations (4.1) and apply the energy method to derive bounds of high-order derivatives of (ρ^δ, u^δ). Then we can apply the Schauder theory for linear parabolic equations to conclude that the $C^{\beta,\beta/2}([0, 1] \times [0, T])$-norms of ρ^δ, ρ_x^δ, $\rho_{\tau x}^\delta$, u^δ, u_x^δ, u_τ^δ and u_{xx}^δ are bounded a priorly. Therefore, we can continue the local solution globally in time and obtain that there exists a unique global solution (ρ^δ, u^δ) of (4.1) with the initial data (ρ_0, u_0) replaced by $(\rho_0^\delta, u_0^\delta)$, such that for any $T > 0$,

$$\rho^\delta, \rho_x^\delta, \rho_{\tau x}^\delta, u^\delta, u_x^\delta, u_\tau^\delta, u_{xx}^\delta \in C^{\beta,\beta/2}([0, 1] \times [0, T])$$

for some $0 < \beta < 1$, and $\rho^\delta > 0$ on $[0, 1] \times [0, T]$. This can be done in a similar way as in [14]. Thus the solutions which can be denoted as $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$, satisfies (4.1). Transforming it into Euler coordinates again by

$$x = \int_\varepsilon^r \rho(r, \tau) r^2 dr, \tau = t,$$

we can obtain the solutions $(\rho^{\varepsilon,\delta}(r, t), u^{\varepsilon,\delta}(r, t))$ to the approximate system (3.3)-(3.4), and consequently Lemma 3.1 holds for these approximate solutions.

4.2 The passage to limit

So far, $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})$ are defined on $\varepsilon \le r \le R$. To take the limit passage as $\{\varepsilon_j, \delta_j\} \to 0$, we extend $\rho^{\varepsilon_j,\delta_j}(r,t), u^{\varepsilon_j,\delta_j}(r,t)$ to the whole domain Ω in the following way,

$$\tilde{\rho}^{\varepsilon_j,\delta_j} = \begin{cases} \rho^{\varepsilon_j,\delta_j}(r,t), \;\; r \in [\varepsilon_j, R], \\ \rho^{\varepsilon_j,\delta_j}(\varepsilon_j,t), \;\; r \in [0, \varepsilon_j], \end{cases} \tag{4.3}$$

$$\tilde{u}^{\varepsilon_j,\delta_j} = \begin{cases} u^{\varepsilon_j,\delta_j}(r,t), \;\; r \in [\varepsilon_j, R], \\ 0, \;\; r \in [0, \varepsilon_j], \end{cases} \tag{4.4}$$

and still denote the so obtained approximate solutions $\{\tilde{\rho}^{\varepsilon_j,\delta_j}, \tilde{u}^{\varepsilon_j,\delta_j}\}$ by $\{\rho^{\varepsilon_j,\delta_j}, u^{\varepsilon_j,\delta_j}\}$. Let $\rho^{\varepsilon_j,\delta_j}(\mathbf{x},t) = \rho^{\varepsilon_j,\delta_j}(r,t), \mathbf{U}^{\varepsilon_j,\delta_j}(\mathbf{x},t) = u^{\varepsilon_j,\delta_j}(r,t)\frac{\mathbf{x}}{r}$. For simplicity, we write $(\rho^j, \mathbf{U}^j)$ instead of $(\rho^{\varepsilon_j,\delta_j}, \mathbf{U}^{\varepsilon_j,\delta_j})$ and denote $\Omega_\varepsilon = \Omega\backslash B_\varepsilon(0)$ for $\epsilon > 0$ and $\Omega_{\frac{1}{n}} = \Omega\backslash B_{\frac{1}{n}}(0)$ for $n \in N$, where N is the set of the positive integers.

It then follows from Lemma 3.1 that

Lemma 4.1. *Let $(\rho^j, \mathbf{U}^j)(\mathbf{x},t)$ be the approximate solutions of (3.1)-(3.2) constructed above. Then there exists a constant C independent of ε such that*

$$\sup_{t\in[0,T]} \int_{\Omega_{\varepsilon_j}} \rho^j(x,t)dx \le C, \tag{4.5}$$

$$\begin{aligned} &\sup_{t\in[0,T]} \int_{\Omega_{\varepsilon_j}} (\frac{1}{2}\rho^j|\mathbf{U}^j|^2 + \frac{1}{\gamma-1}(\rho^j)^\gamma)(x,t)dx + \int_0^T \int_{\Omega_{\varepsilon_j}} \rho^j|\nabla \mathbf{U}^j|^2(x,t)dxdt \\ &+\frac{1}{4}\int_0^T \int_{\Omega_{\varepsilon_j}} \varepsilon(\rho^j)^{\frac{3}{4}}|\nabla \mathbf{U}^j|^2(x,t)dxdt \le C, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\sup_{t\in[0,T]} \int_{\Omega_{\varepsilon_j}} \frac{1}{2}\rho^j|\mathbf{U}^j + \nabla \log \rho^j + \frac{3}{4}\varepsilon(\rho^j)^{-\frac{5}{4}}\nabla\rho^j|^2(x,t)dx \\ &+ \int_0^T \int_{\Omega_{\varepsilon_j}} \frac{4}{\gamma}|\nabla(\rho^j)^{\frac{\gamma}{2}}|^2(x,t)dxdt \\ &+ \int_0^T \int_{\Omega_{\varepsilon_j}} \frac{48\varepsilon\gamma}{(4\gamma-1)^2}|\nabla(\rho^j)^{\frac{4\gamma-1}{8}}|^2(x,t)dxdt \le C. \end{aligned} \tag{4.7}$$

Moreover, the following uniform estimate hold

$$\sup_{t\in[0,T]} \|\sqrt{\rho^j}\|_{H^1(\Omega)} \le C, \tag{4.8}$$

$$\sup_{t\in[0,T]} \int_\Omega \rho^j|\mathbf{U}^j|^2 dx \le C. \tag{4.9}$$

Proof. (4.5)-(4.7) follow directly from Lemma 3.1 and (4.9) can be checked easily. It suffices to prove (4.8).

First, it holds that

$$\sup_{t\in[0,T]} \|\nabla\sqrt{\rho^j}\|_{L^2(\Omega)} \le C, \tag{4.10}$$

where C is a constant independent of ε. Indeed, for any $\phi \in C_0^\infty(\Omega)$, one has

$$\int_\Omega \sqrt{\rho^j}\partial_i\phi dx = [\int_{\Omega_{\varepsilon_j}} + \int_{B_{\varepsilon_j}(0)}]\sqrt{\rho^j}\partial_i\phi dx$$
$$= -[\int_{\Omega_{\varepsilon_j}} + \int_{B_{\varepsilon_j}(0)}]\partial_i\sqrt{\rho^j}\phi dx + \int_{\partial\Omega_{\varepsilon_j}} \sqrt{\rho^j}n_i\phi dS + \int_{\partial B_{\varepsilon_j}(0)} \sqrt{\rho^j}\bar{n}_i\phi dS,$$

where n_i and $\bar{n}_i$ are the unit outer normal vector of $\partial\Omega_\varepsilon$ and $\partial B_\varepsilon(0)$ respectively, and $i = 1, 2, 3$. For any $\phi \in C_0^\infty(\Omega)$, in view of the extension (4.3), we have

$$\int_\Omega \sqrt{\rho^j}\partial_i\phi = -[\int_{\Omega_{\varepsilon_j}} + \int_{B_{\varepsilon_j}(0)}]\partial_i\sqrt{\rho^j}\phi dx$$
$$= -\int_{\Omega_{\varepsilon_j}} \partial_i\sqrt{\rho^j}\phi dx,$$

which implies that for *a.e.* $t \in [0, T]$,

$$\partial_i\sqrt{\rho^j}(x,t) = \begin{cases} \partial_i\sqrt{\rho^j}, & x \in \Omega_{\varepsilon_j}, \\ 0, & x \in \bar{B}_{\varepsilon_j}, \end{cases}$$

for $i = 1, 2, 3$. Consequently, (4.10) follows from (4.6) and (4.7).

Next, we verify that

$$\sup_{t\in[0,T]} \|\sqrt{\rho^j}\|_{L^2(\Omega)} \le C, \tag{4.11}$$

where C is a constant independent of ε.

Thanks to the upper bound estimate of the density (3.9) and (4.5), there exists an absolute constant C independent of ε and T such that

$$\sup_{t\in[0,T]} \int_0^R \rho^j r^2 dr \le \sup_{t\in[0,T]} \int_0^{\varepsilon_j} \rho^j r^2 dr + \sup_{t\in[0,T]} \int_{\varepsilon_j}^R \rho^j r^2 dr$$
$$\le \frac{C}{\varepsilon_j^2} \int_0^{\varepsilon_j} r^2 dr + C \le \frac{C\varepsilon_j}{3} + C \le C$$

for all $0 < \varepsilon_j < R$, which gives (4.11). Combining (4.10) with (4.11) shows (4.8). $\square$

Remark 4.1. *Compared with the usual zero extensions in [10, 12], the extensions (4.3) and (4.4) keep the $L^\infty(0,T;H^1(\Omega))$-norm of $\sqrt{\rho^j}$, which is needed in the following convergence arguments.*

Proposition 4.1. *Let $T > 0$ be fixed. Then there are a sequence $(\varepsilon_j, \delta_j)$, and a limiting function $\rho(x,t)$ such that*

$$\rho^j(\mathbf{x},t) \to \rho(\mathbf{x},t), \ \text{in } C([0,T], L^{3/2}(\Omega)). \tag{4.12}$$

Moreover, $\rho(\mathbf{x},t) = \rho(r,t)$ is a spherically symmetric function.

Proof. It follows from (4.8) that $\sqrt{\rho^j}$ is bounded in $L^\infty(0,T;L^q(\Omega))$ for $q \in [2,6]$. Thus ρ^j is bounded in $L^\infty(0,T;L^3(\Omega))$, and therefore

$$\rho^j \mathbf{U}^j = \sqrt{\rho^j}\sqrt{\rho^j}\mathbf{U}^j$$

is bounded in $L^\infty(0,T;L^{3/2}(\Omega))$ due to (4.9). The continuity equation thus yields $\partial_t \rho^j$ bounded in $L^\infty(0,T;W^{-1,3/2}(\Omega))$. Moreover, since $\nabla\rho^j = 2\sqrt{\rho^j}\nabla\sqrt{\rho^j}$, we also have that $\nabla\rho^j$ is bounded in $L^\infty(0,T;L^{3/2}(\Omega))$, hence the compactness of ρ^j in $C([0,T],L^{3/2}(\Omega))$, i.e., (4.12) is obtained. Moreover, since $\rho^j(\mathbf{x},t) = \rho^j(r,t)$ is spherically symmetric, it is clear to get that $\rho(\mathbf{x},t) = \rho(r,t)$ is a spherically symmetric function.

□

Proposition 4.2. *Suppose that* $1 < \gamma < 3$. *Then* $(\rho^j)^\gamma$ *converges to* ρ^γ *strongly in* $L^1((0,T);L^1(\Omega))$.

Proof. This follows directly from the fact that ρ^j is bounded in $L^\infty(0,T;L^3(\Omega))$ and (4.12).

□

The following proposition will enable us to take the limit in the nonlinear convection term.

Proposition 4.3. *If* $1 < \gamma < 3$, *and*

$$\int_0^R \rho_0|u_0|^{2+\eta}r^2dr \le C, \tag{4.13}$$

then the following estimate is true

$$\begin{aligned}&\frac{d}{dt}\int_{\varepsilon_j}^R \rho^j\frac{|u^j|^{2+\eta}}{2+\eta}r^2dr + \int_{\varepsilon_j}^R (\frac{3}{4}\rho^j + \frac{\varepsilon}{8}(\rho^j)^{\frac{3}{4}})|u^j|^\eta(u^j_r)^2r^2dr\\ &+\int_{\varepsilon_j}^R (\frac{7}{4}\rho^j + \frac{3\varepsilon}{8}(\rho^j)^{\frac{3}{4}})|u^j|^{\eta+2}dr \le C\end{aligned}$$

for some small $\eta \in (0,1)$. *That is*

$$\int_{\Omega_{\varepsilon_j}} \rho^j\frac{|\mathbf{U}^j|^{2+\eta}}{2+\eta}dx \le C, \tag{4.14}$$

where $\Omega_{\varepsilon_j} = \Omega \setminus B_{\varepsilon_j}(0)$ *and* C *is a constant independent of* ε.

To prove Proposition 4.3, we need the following lemma.

Lemma 4.2. *The pressure* $(\rho^j)^\gamma$ *is bounded in* $L^{\frac{5}{3}}((0,T);L^{\frac{5}{3}}(\Omega_{\varepsilon_j}))$.

Proof. It follows from Lemma 4.1 that $(\rho^j)^{\gamma/2} \in L^2(0,T;H^1(\Omega_{\varepsilon_j}))$, and so, $(\rho^j)^\gamma \in L^1(0,T;L^3(\Omega_{\varepsilon_j}))$. Since $(\rho^j)^\gamma$ is bounded in $L^\infty(0,T;L^1(\Omega_{\varepsilon_j}))$ by (4.6), Hölder inequality gives

$$\|(\rho^j)^\gamma\|_{L^{5/3}((0,T)\times\Omega_{\varepsilon_j})} \le \|(\rho^j)^\gamma\|^{2/5}_{L^\infty(0,T;L^1(\Omega_{\varepsilon_j}))}\|(\rho^j)^\gamma\|^{3/5}_{L^1(0,T;L^3(\Omega_{\varepsilon_j}))} \le C,$$

where C is independent of ε_j. This finishes the proof of the lemma. □

Now we can prove Proposition 4.3.

Proof of Proposition 4.3. Let $\eta \in (0,1)$ satisfy $0<\eta<1/2$. Multiplying (3.4) by $r^2 u^j|u^j|^\eta$ and integrating the resulting equation yield

$$\begin{aligned}
&\int_{\varepsilon_j}^R \rho^j\partial_t\frac{|u^j|^{2+\eta}}{2+\eta}r^2dr+\int_{\varepsilon_j}^R \rho^j u^j(\frac{|u^j|^{2+\eta}}{2+\eta})_r r^2dr\\
&+(1+\eta)\int_{\varepsilon_j}^R(\rho^j+\frac{3\varepsilon_j}{4}(\rho^j)^{\frac{3}{4}})|u^j|^\eta(u^j_r)^2r^2dr\\
&+\int_{\varepsilon_j}^R(2\rho^j+\varepsilon_j(\rho^j)^{\frac{3}{4}})|u^j|^{\eta+2}dr+\int_{\varepsilon_j}^R|u^j|^\eta\, u^j\,|((\rho^j)^\gamma)_r|\,r^2dr\\
&\le(\varepsilon_j+\frac{\eta\varepsilon_j}{2})\int_{\varepsilon_j}^R(\rho^j)^{\frac{3}{4}}|u^j|^{\eta+1}|u^j_r|rdr\\
&\le(\frac{\varepsilon_j}{2}+\frac{\eta\varepsilon_j}{4})[\int_{\varepsilon_j}^R(\rho^j)^{\frac{3}{4}}|u^j|^\eta(u^j_r)^2r^2dr+(\rho^j)^{\frac{3}{4}}|u^j|^{\eta+2}dr.
\end{aligned}$$

Since $\eta<\frac{1}{2}$, one deduces that

$$\begin{aligned}
&\int_{\varepsilon_j}^R \rho^j\partial_t\frac{|u^j|^{2+\eta}}{2+\eta}r^2dr+\int_{\varepsilon_j}^R \rho^j u^j(\frac{|u^j|^{2+\eta}}{2+\eta})_r r^2dr+\int_{\varepsilon_j}^R(\rho^j+\frac{\varepsilon_j}{8}(\rho^j)^{\frac{3}{4}})|u^j|^\eta(u^j_r)^2r^2dr\\
&+\int_{\varepsilon_j}^R(2\rho^j+\frac{3\varepsilon_j}{8}(\rho^j)^{\frac{3}{4}})|u^j|^{\eta+2}dr+\int_{\varepsilon_j}^R|u^j|^\eta u^j((\rho^j)^\gamma)_r r^2dr\le 0.
\end{aligned}$$

Moreover, multiplying (3.3) by $\frac{r^2|u^j|^{\eta+2}}{2+\eta}$ and integrating by parts show that

$$\int_{\varepsilon_j}^R\frac{|u^j|^{2+\eta}}{2+\eta}\partial_t\rho^j r^2dr-\int_{\varepsilon_j}^R\rho^j u^j(\frac{|u^j|^{2+\eta}}{2+\eta})_r r^2dr=0.$$

Summing over the last two inequalities, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\varepsilon_j}^R\rho^j\frac{|u^j|^{2+\eta}}{2+\eta}r^2dr+\int_{\varepsilon_j}^R(\rho^j+\frac{\varepsilon_j}{8}(\rho^j)^{\frac{3}{4}})|u^j|^\eta(u^j_r)^2r^2dr\\
&+\int_{\varepsilon_j}^R(2\rho^j+\frac{3\varepsilon_j}{8}(\rho^j)^{\frac{3}{4}})|u^j|^{\eta+2}dr\le|\int_{\varepsilon_j}^R|u^j|^\eta\, u^j\,|((\rho^j)^\gamma)_r|\,r^2dr|. \qquad (4.15)
\end{aligned}$$

It remains to bound the right hand side of (4.15). It follows from Young's inequality that

$$\begin{aligned}
&|\int_{\varepsilon_j}^R|u^j|^\eta\, u^j\,|((\rho^j)^\gamma)_r|\,r^2dr|\le(1+\eta)|\int_{\varepsilon_j}^R|u^j|^\eta|u^j_r|(\rho^j)^\gamma r^2dr|+2\int_{\varepsilon_j}^R|u^j|^{\eta+1}(\rho^j)^\gamma rdr\\
&\le(1+\eta)(\int_{\varepsilon_j}^R\rho^j|u^j|^\eta|u^j_r|^2r^2dr)^{1/2}(\int_{\varepsilon_j}^R(\rho^j)^{2\gamma-1}|u^j|^\eta r^2dr)^{1/2}+2\int_{\varepsilon_j}^R|u^j|^{\eta+1}(\rho^j)^\gamma rdr\\
&\le\frac{1}{4}\int_{\varepsilon_j}^R\rho^j|u^j|^\eta|u^j_r|^2r^2dr+C\int_{\varepsilon_j}^R(\rho^j)^{2\gamma-1}|u^j|^\eta r^2dr+2\int_{\varepsilon_j}^R|u^j|^{\eta+1}(\rho^j)^\gamma rdr. \qquad (4.16)
\end{aligned}$$

The last two terms in (4.16) can be estimated as follows:

$$\int_{\varepsilon_j}^R (\rho^j)^{2\gamma-1}|u^j|^\eta r^2 dr \le (\int_{\varepsilon_j}^R ((\rho^j)^{2\gamma-1-\frac{\eta}{2}})^{\frac{2}{2-\eta}} r^2 dr)^{\frac{2-\eta}{2}} (\int_{\varepsilon_j}^R \rho^j |u^j|^2 r^2 dr)^{\frac{\eta}{2}}$$
$$\le C \int_{\varepsilon_j}^R ((\rho^j)^{2\gamma-1-\frac{\eta}{2}})^{\frac{2}{2-\eta}} r^2 dr + C, \tag{4.17}$$

and

$$\int_{\varepsilon_j}^R |u^j|^{\eta+1}(\rho^j)^\gamma r dr \le (\int_{\varepsilon_j}^R (\rho^j)^{(\gamma-\frac{\eta+1}{\eta+2})(2+\eta)} r^{2+\eta} dr)^{\frac{1}{2+\eta}} (\int_{\varepsilon_j}^R \rho^j |u^j|^{2+\eta} dr)^{\frac{1+\eta}{2+\eta}}$$
$$\le C(R) \int_{\varepsilon_j}^R (\rho^j)^{(\gamma-\frac{\eta+1}{\eta+2})(2+\eta)} r^2 dr + \frac{1}{4}\int_{\varepsilon_j}^R \rho^j |u^j|^{2+\eta} dr. \tag{4.18}$$

Then it follows from (4.15)-(4.18) that

$$\frac{d}{dt}\int_{\varepsilon_j}^R \rho^j \frac{|u^j|^{2+\eta}}{2+\eta} r^2 dr + \int_{\varepsilon_j}^R (\frac{3}{4}\rho^j + \frac{\varepsilon_j}{8}(\rho^j)^{\frac{3}{4}})|u^j|^\eta (u^j_r)^2 r^2 dr$$
$$+ \int_{\varepsilon_j}^R (\frac{7}{4}\rho^j + \frac{3\varepsilon_j}{8}(\rho^j)^{\frac{3}{4}})|u^j|^{\eta+2} dr$$
$$\le C\int_{\varepsilon_j}^R (\rho^j)^{(2\gamma-1-\frac{\eta}{2})\frac{2}{2-\eta}} r^2 dr + C(R)\int_{\varepsilon_j}^R (\rho^j)^{(\gamma-\frac{\eta+1}{\eta+2})(2+\eta)} r^2 dr + C. \tag{4.19}$$

Using Lemma 4.2, one can check easily that the right hand side of (4.19) is bounded for small η under the condition

$$2\gamma - 1 < \frac{5}{3}\gamma,$$

which is satisfied if $1 < \gamma < 3$. This gives Proposition 4.3.

It is noted that the initial data (4.13) will be satisfied if we assume (2.24). Moreover, since we have extended u^j to be zero outside $[\varepsilon_j, R]$, it follows from (4.14) that

$$\int_\Omega \rho^j \frac{|\mathbf{U}^j|^{2+\eta}}{2+\eta} dx \le C. \tag{4.20}$$

Consequently, since

$$\int_\Omega (\rho^j|\mathbf{U}^j|^2)^{1+\zeta} dx \le (\int_\Omega \rho^j |\mathbf{U}^j|^{2+\eta} dx)^{\frac{2+2\zeta}{2+\eta}} (\int_\Omega (\rho^j)^{1+\frac{(2+\eta)\zeta}{\eta-2\zeta}} dx)^{\frac{\eta-2\zeta}{2+\eta}},$$

and as ζ small enough, we deduce that

Corollary 4.1. *If $1 < \gamma < 3$, then $\sqrt{\rho^j}\mathbf{U}^j$ is bounded in $L^\infty(0,T;L^{2+2\zeta}(\Omega))$ for some small $\zeta > 0$.*

Thanks to Proposition 4.1, Proposition 4.3, Corollary 4.1, and Lemmas 4.4 and 4.6 in [25], we have

Proposition 4.4. *1) Up to a subsequence, the momentum* $\mathbf{m}^j = \rho^j \mathbf{U}^j$ *converges strongly in* $L^1((0,T)\times\Omega_{\frac{1}{n}})$ *and* $L^2(0,T;L^{1+\zeta}(\Omega_{\frac{1}{n}}))$ *and almost everywhere to some* $\mathbf{m}(\mathbf{x},t)$*, where* $n \in N$ *is any positive integer.*

2) The quantity $\sqrt{\rho^j}\mathbf{U}^j$ *converges strongly in* $L^2((0,T)\times\Omega_{\frac{1}{n}})$ *to* $\frac{\mathbf{m}}{\sqrt{\rho}}$ *(define to be zero when* $m=0$*) for any* $n \in N$*. In particular, we have* $\mathbf{m}(x,t)=0$ *a.e. on* $\{\rho(\mathbf{x},t)=0\}$ *and there exists a function* $\mathbf{U}(\mathbf{x},t)$ *such that*

$$\mathbf{m}(\mathbf{x},t) = \rho(\mathbf{x},t)\mathbf{U}(\mathbf{x},t).$$

Proof. This proposition can be proved exactly as in [25]. For completeness, we sketch it here.

1) Since

$$\rho^j\mathbf{U}^j = \sqrt{\rho^j}\sqrt{\rho^j}\mathbf{U}^j,$$

where $\sqrt{\rho^j}$ is bounded in $L^\infty(0,T;L^p(\Omega_{\varepsilon_j}))$ for $p\in[2,6]$, and $\sqrt{\rho^j}\mathbf{U}^j$ is bounded in $L^\infty(0,T;L^2(\Omega_{\varepsilon_j}))$, we deduce that $\rho^j\mathbf{U}^j$ is bounded in $L^\infty(0,T;L^q(\Omega_{\varepsilon_j}))$ for all $q\in[1,\frac{3}{2}]$. Next, since

$$\begin{aligned}\partial_i(\rho^j\mathbf{U}^j_k) &= \rho^j\partial_i\mathbf{U}^j_k + \mathbf{U}^j_k\partial_i\rho^j\\ &= \sqrt{\rho^j}\sqrt{\rho^j}\partial_i\mathbf{U}^j_k + 2\sqrt{\rho^j}\mathbf{U}^j_k\partial_i\sqrt{\rho^j},\end{aligned}$$

then it follows from Corollary 4.1 and the energy estimates that the second term on the right hand side above is bounded in $L^\infty(0,T;L^{1+\zeta}(\Omega_{\varepsilon_j}))$ for some small $\zeta>0$, while the first term is bounded in $L^2(0,T;L^p(\Omega_{\varepsilon_j}))$ for all $p\in[1,\frac{3}{2}]$. This means

$$\nabla(\rho^j\mathbf{U}^j)\in L^2(0,T;L^{1+\zeta}(\Omega_{\varepsilon_j})).$$

In particular,

$$\rho^j\mathbf{U}^j\in L^2(0,T;W^{1,1+\zeta}(\Omega_{\varepsilon_j})).$$

On the other hand, it follows from Corollary 4.1 and Lemma 4.2 that

$$\begin{aligned}\operatorname{div}(\sqrt{\rho^j}\mathbf{U}^j\otimes\sqrt{\rho^j}\mathbf{U}^j)&\in L^\infty(0,T;W^{-1,1+\zeta}(\Omega_{\varepsilon_j}))\\ \nabla(\rho^j)^\gamma&\in L^{\frac{5}{3}}(0,T;W^{-1,1+\zeta}(\Omega_{\varepsilon_j})).\end{aligned}$$

Next we check that

$$\nabla((\rho^j+\varepsilon_j(\rho^j)^{\frac{3}{4}})\nabla\mathbf{U}^j),\ \nabla(\varepsilon_j(\rho^j)^{\frac{3}{4}}\operatorname{div}\mathbf{U}^j)$$

are uniformly bounded in $L^\infty(0,T;W^{-2,\frac{3}{2}}(\Omega_{\varepsilon_j}))$. Indeed, note that

$$(\rho^j+\varepsilon_j(\rho^j)^{\frac{3}{4}})\nabla\mathbf{U}^j = \nabla((\rho^j+\varepsilon_j(\rho^j)^{\frac{3}{4}})\mathbf{U}^j) - \mathbf{U}^j\nabla(\rho^j+\varepsilon_j(\rho^j)^{\frac{3}{4}}), \tag{4.21}$$

$$\varepsilon_j(\rho^j)^{\frac{3}{4}}\operatorname{div}\mathbf{U}^j = \operatorname{div}(\varepsilon_j(\rho^j)^{\frac{3}{4}}\mathbf{U}^j) - \mathbf{U}^j\nabla(\varepsilon_j(\rho^j)^{\frac{3}{4}}). \tag{4.22}$$

The second term on the right hand side in (4.21) is

$$\mathbf{U}^j\nabla(\rho^j+\varepsilon_j(\rho^j)^{\frac{3}{4}}) = \sqrt{\rho^j}\mathbf{U}^j(2\nabla\sqrt{\rho^j}) + \varepsilon_j\nabla((\rho^j)^{\frac{3}{4}}))\mathbf{U}^j$$

which is uniformly bounded in $L^\infty(0,T;L^{1+\zeta}(\Omega_{\varepsilon_j}))$ thanks to Lemma 4.1 and Corollary 4.1. The first term on the right hand side in (4.21) can be rewritten as

$$\nabla((\rho^j+\varepsilon_j(\rho^j)^{\frac{3}{4}})\mathbf{U}^j) = \nabla((\sqrt{\rho^j}+\varepsilon_j(\rho^j)^{\frac{1}{4}})\sqrt{\rho^j}\mathbf{U}^j)$$

which is uniformly bounded in $L^\infty(0,T;W^{-1,\frac{3}{2}}(\Omega_{\varepsilon_j}))$ because $\sqrt{\rho^j}$ and $\varepsilon_j(\rho^j)^{\frac{1}{4}}$ are uniformly bounded in $L^\infty(0,T;L^6(\Omega_{\varepsilon_j}))$ due to the entropy estimates in Lemma 4.1.

Similarly, one can show that $\varepsilon_j(\rho^j)^{\frac{3}{4}}\mathrm{div}\mathbf{U}^j$ is uniformly bounded in $L^\infty(0,T;W^{-1,\frac{3}{2}}(\Omega_{\varepsilon_j}))$. Thus, noting that $L^{1+\zeta}(\Omega_{\varepsilon_j}) \hookrightarrow W^{-1,\frac{3}{2}}(\Omega_{\varepsilon_j})$, we obtain that

$$(\rho^j+\varepsilon_j(\rho^j)^{\frac{3}{4}})\nabla\mathbf{U}^j, \varepsilon_j(\rho^j)^{\frac{3}{4}}\mathrm{div}\mathbf{U}^j$$

are uniformly bounded in $L^\infty(0,T;W^{-1,\frac{3}{2}}(\Omega_{\varepsilon_j}))$ and hence that

$$\nabla((\rho^j+\varepsilon_j(\rho^j)^{\frac{3}{4}})\nabla\mathbf{U}^j),\ \nabla(\varepsilon_j(\rho^j)^{\frac{3}{4}}\mathrm{div}\mathbf{U}^j)$$

are uniformly bounded in $L^\infty(0,T;W^{-2,\frac{3}{2}}(\Omega_{\varepsilon_j}))$. Moreover, it follows from (3.2) that

$$\partial_t(\rho^j\mathbf{U}^j) \text{ is uniformly bounded in } L^{\frac{5}{3}}(0,T;W^{-2,\frac{3}{2}}(\Omega_{\varepsilon_j})). \tag{4.23}$$

In fact, since $W_0^{1,3}(\Omega_{\varepsilon_j}) \hookrightarrow L^{1+\frac{1}{\zeta}}(\Omega_{\varepsilon_j})$ for small ζ, therefore

$$L^{1+\zeta}(\Omega_{\varepsilon_j}) \hookrightarrow W^{-1,\frac{3}{2}}(\Omega_{\varepsilon_j}) \hookrightarrow W^{-2,\frac{3}{2}}(\Omega_{\varepsilon_j}).$$

Thus (4.23), together with Aubin's Lemma and the diagonal principle, yields the compactness of $\mathbf{m}^j=\rho^j\mathbf{U}^j$ in $L^2(0,T;L^{1+\zeta}(\Omega_{\frac{1}{n}}))$ for all $n\in N$.

2) From the proof of 1), if we define $\frac{\mathbf{m}^2}{\rho}$ to be zero when $\mathbf{m}=0$, we have $\mathbf{m}(x,t)=0$ a.e. in $\{\rho(x,t)=0\}$. Since $\frac{\mathbf{m}^j}{\sqrt{\rho^j}}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega_{\varepsilon_j}))$ and hence in $L^\infty(0,T;L^2(\Omega_{\frac{1}{n}}))$ for any $n\in N$ satisfying $\varepsilon_j\le\frac{1}{n}$, then by Fatou's lemma, we have

$$\int_{\Omega_{\frac{1}{n}}}\frac{\mathbf{m}^2}{\rho}dx\le C.$$

Because $\sqrt{\rho^j}|\mathbf{U}^j|$ is uniformly bounded in $L^\infty(0,T;L^{2+2\zeta}(\Omega_{\varepsilon_j}))$ for small $\zeta>0$. It is thus enough to prove the convergence of $\sqrt{\rho^j}|\mathbf{U}^j|$ in $L^1((0,T)\times\Omega_{\frac{1}{n}})$ for all $n\in N$.

To this end, we fix $n\in N$ first and denote the set of vacuum by

$$\mathcal{F}=\{x\in\Omega_{\frac{1}{n}}|\rho(x,t)=0\}.$$

Notes that $\sqrt{\rho^j}\mathbf{U}^j$ converges almost everywhere to $\frac{\mathbf{m}}{\sqrt{\rho}}$ in the region $\mathcal{F}^c$. To control $\sqrt{\rho^j}\mathbf{U}^j$ on the vacuum set, one sets

$$\mathcal{V}^j=\{x\in\Omega_{\frac{1}{n}}|(\rho^j)^{\frac{1}{2+\eta}}|\mathbf{U}^j|\ge \mathrm{M}\}$$

for $\mathrm{M}>0$ and small $\eta>0$ to be specified later. Consider

$$\begin{aligned}\int_0^T\int_{\Omega_{\frac{1}{n}}}|\sqrt{\rho^j}\mathbf{U}^j-\frac{\mathbf{m}}{\sqrt{\rho}}|dxdt&=\int_0^T\int_{(\mathcal{V}^j)^c\setminus\mathcal{F}}|\sqrt{\rho^j}\mathbf{U}^j-\frac{\mathbf{m}}{\sqrt{\rho}}|dxdt\\&+\int_0^T\int_{(\mathcal{V}^j)^c\cap\mathcal{F}}|\sqrt{\rho^j}\mathbf{U}^j-\frac{\mathbf{m}}{\sqrt{\rho}}|dxdt+\int_0^T\int_{\mathcal{V}^j}|\sqrt{\rho^j}\mathbf{U}^j-\frac{\mathbf{m}}{\sqrt{\rho}}|dxdt.\end{aligned} \tag{4.24}$$

For the first term, the $L^\infty(0,T;L^2(\Omega_{\varepsilon_j}))$ bound and the fact that $\sqrt{\rho^j}\mathbf{U}^j$ converges almost everywhere to $\frac{\mathbf{m}}{\sqrt{\rho}}$ in the region $\mathcal{F}^c$ gives

$$\int_0^T\int_{(\mathcal{V}^j)^c\setminus\mathcal{F}}|\sqrt{\rho^j}\mathbf{U}^j-\frac{\mathbf{m}}{\sqrt{\rho}}|dxdt\to 0,\ \text{as } j\to\infty.$$

Moreover, since $\sqrt{\rho^j}\mathbf{U}^j$ is uniformly bounded in $L^\infty(0,T;L^{2+2\zeta}(\Omega_{\varepsilon_j}))$ and Tchebychev's inequality yields

$$|\mathcal{V}^j|\le\frac{C}{\mathrm{M}^2},$$

and so

$$\int_0^T\int_{\mathcal{V}^j}|\sqrt{\rho^j}\mathbf{U}^j-\frac{\mathbf{m}}{\sqrt{\rho}}|dxdt\le\sqrt{|\mathcal{V}^j|}(\int_{\Omega_j}(\rho^j|\mathbf{U}^j|^2+\frac{\mathbf{m}^2}{\rho})dx)^{\frac12}\le\frac{C}{\mathrm{M}},$$

this means that the third term of (4.24) also goes to zero as M tends to ∞.

It remains to treat the second term on the right hand side of (4.24). Notes that, on the region $(\mathcal{V}^j)^c\cap\mathcal{F}$, we have

$$|\sqrt{\rho^j}\mathbf{U}^j|\le\mathrm{M}(\rho^j)^{\frac12-\frac{1}{2+\eta}}\to 0,$$

since $\rho^j\to 0$ as $j\to\infty$ and $\frac12-\frac{1}{2+\eta}>0$ for all small $\eta>0$. So $1_{(\mathcal{V}^j)^c\cap\mathcal{F}}|\sqrt{\rho^j}\mathbf{U}^j|$ converges almost everywhere to zero. In particular, the $L^\infty(0,T;L^2(\Omega_{\varepsilon_j}))$ bound of $\sqrt{\rho^j}\mathbf{U}^j$ gives

$$\int_0^T\int_{(\mathcal{V}^j)^c\cap\mathcal{F}}|\sqrt{\rho^j}\mathbf{U}^j|dxdt\to 0,\ \text{as } j\to\infty.$$

Since we defined $\frac{\mathbf{m}}{\sqrt{\rho}}$ to be zero on $\mathcal{F}$, we also have

$$1_{(\mathcal{V}^j)^c\cap\mathcal{F}}\frac{\mathbf{m}}{\sqrt{\rho}}(x,t)=0,a.e.,\forall j,$$

hence we also can conclude that the second term of (4.24) goes to zero as $j\to\infty$.

Combining all the arguments above, using the diagonal principle, we obtain $\sqrt{\rho^j}|\mathbf{U}^j|$ converges to $\frac{\mathbf{m}}{\sqrt{\rho}}$ in $L^1((0,T)\times\Omega_{\frac1n})$ strongly for any $n\in N$. The lemma follows. $\square$

It follows from Propositions 4.1 and 4.4 that

Corollary 4.2. *Let $m^j(r,t)=\rho^j u^j(r,t)$. Then*

1) there exists a function $m(r,t)$ such that $\mathbf{m}(\mathbf{x},t)=m(r,t)\frac{\mathbf{x}}{r}$ and $m^j(r,t)=\rho^j u^j(r,t)$ converges strongly in $L^2(0,T;L^{1+\zeta}_{loc}((0,R);r^2dr))$ and almost everywhere to $m(r,t)$;

2) there exists a function $u(r,t)$ such that $\mathbf{U}(\mathbf{x},t)=u(r,t)\frac{\mathbf{x}}{r}$ and the quantity $\sqrt{\rho^j}u^j$ converges strongly in $L^2((0,T);L^2_{loc}((0,R);r^2dr))$ to $\frac{m}{\sqrt{\rho}}$ (define to be zero when $m=0$)

Proof. Since $\mathbf{m}^j(\mathbf{x},t) = m^j(r,t)\frac{\mathbf{x}}{r}$, we have $m^j(r,t) = |\mathbf{m}^j(x,t)|$ converges almost everywhere to $m(r,t) = |\mathbf{m}(\mathbf{x},t)|$ due to the fact that $\mathbf{m}^j(\mathbf{x},t)$ converges almost everywhere to $\mathbf{m}(x,t)$ by the first part of Proposition 4.4. Therefore $\mathbf{m}(\mathbf{x},t) = m(r,t)\frac{\mathbf{x}}{r}$. Moreover, noting that $\rho(\mathbf{x},t) = \rho(r,t)$ by Proposition 4.1 and $\mathbf{m}(\mathbf{x},t) = \rho(\mathbf{x},t)\mathbf{U}(\mathbf{x},t)$ by Proposition 4.4, we obtain

$$m(r,t)\frac{\mathbf{x}}{r} = \rho(r,t)\mathbf{U}(\mathbf{x},t).$$

Therefore there exists a spherically function $u(r,t)$ such that $m(r,t) = \rho u(r,t)$.

The rest parts of the Corollary follow directly from Proposition 4.4 and the proof of the corollary is finished. □

Now we show that $(\rho, \mathbf{U})$ obtained in Proposition 4.1-4.4 satisfy the weak form of the mass equation (2.1), i.e., (2.20) holds.

Proposition 4.5. *Let $(\rho, \mathbf{U})$ be the limit described as in Proposition 4.1-4.4. Then the weak form of the mass equation, (2.20), holds for C^1 test function $\psi : \bar{\Omega} \times [t_1, t_2] \to \mathbb{R}$. Moreover, $\rho \in C([0,\infty); W^{1,\infty}(\Omega)^*)$, where $W^{1,\infty}(\Omega)^*$ is the dual space of $W^{1,\infty}(\Omega)$.*

Proof. We first derive the weak form of the one-dimensional equation (2.6). Let $\varphi(r,t)$ be a C^1 function on $[0,R] \times [t_1,t_2]$. Then

$$\int_{\varepsilon_j}^{R} \rho^j \varphi r^2 dr|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\varepsilon_j}^{R} (\rho^j \varphi_t + \rho^j u^j \varphi_r) r^2 dr dt. \tag{4.25}$$

That is

$$\begin{aligned}
&\int_0^R \rho^j \varphi r^2 dr|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_0^R (\rho^j \varphi_t + \rho^j u^j \varphi_r) r^2 dr dt \\
&= \int_0^{\varepsilon_j} \rho^j \varphi r^2 dr|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_0^{\varepsilon_j} \rho^j \varphi_t r^2 dr dt,
\end{aligned} \tag{4.26}$$

because of the extension (4.3)-(4.4). Then Proposition 4.1 shows

$$\int_0^R \rho^j \varphi r^2 dr \to \int_0^R \rho \varphi r^2 dr \tag{4.27}$$

and

$$\int_{t_1}^{t_2}\int_0^R \rho^j \varphi_t r^2 dr dt \to \int_{t_1}^{t_2}\int_0^R \rho \varphi_t r^2 dr dt, \tag{4.28}$$

as $j \to \infty$.

It follows from (4.8) that $\sqrt{\rho^j}$ is bounded in $L^\infty(0,T;L^q(\Omega))$ for $q \in [2,6]$. Thus we get that $\sqrt{\rho^j}$ (or its subsequence) converges strongly in $L^2(0,T;L^2(\Omega))$ to $\sqrt{\rho}$ due to Proposition 4.1. Moreover, Corollary 4.1 yields that $\sqrt{\rho^j}u^j$ is bounded in

$L^\infty(0,T;L^{2+2\zeta}(\Omega))$ and Corollary 4.2 yields that $\sqrt{\rho^j}u^j$ converges almost everywhere to $\sqrt{\rho}u$. Thus $\sqrt{\rho^j}u^j$ converges strongly to $\sqrt{\rho}u$ in $L^2(0,T;L^2(\Omega))$. Hence,

$$\int_{t_1}^{t_2}\int_0^R \rho^j u^j \varphi_r r^2 drdt = \int_{t_1}^{t_2}\int_0^R \sqrt{\rho^j}(\sqrt{\rho^j}u^j)\varphi_r r^2 drdt \to$$
$$\int_{t_1}^{t_2}\int_0^R \sqrt{\rho}(\sqrt{\rho}u)\varphi_r r^2 drdt = \int_{t_1}^{t_2}\int_0^R \rho u\varphi_r r^2 drdt, \tag{4.29}$$

as $j\to\infty$.

By (4.27)-(4.29), it is clear that the terms on the left hand side of the equation (4.26) converge respectively to the corresponding ones without superscript j. It remains to prove that the terms on the right hand side of (4.26) vanish as $j\to\infty$.

For the first term on the right hand side of (4.26), it follows from the proof of Proposition 4.1 that

$$|\max_{t\in[0,T]}\int_0^{\varepsilon_j}\rho^j\varphi r^2 dr| \le C\max_{t\in[0,T]}(\int_0^R(\rho^j)^{\frac{3}{2}}r^2dr)^{\frac{2}{3}}(\varepsilon_j)^{\frac{1}{3}}$$
$$\le C(\varepsilon_j)^{\frac{1}{3}}\to 0 \tag{4.30}$$

as $j\to\infty$.

The second term on the right hand side of (4.26) can be treated similarly. Therefore, taking limit $j\to\infty$ in (4.26), we obtain that, for functions φ which are C^1 on $[0,R]\times[t_1,t_2]$,

$$\int_0^R \rho\varphi r^2 dr|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_0^R(\rho\varphi_t+\rho u\varphi_r)r^2drdt. \tag{4.31}$$

Now let $\psi:\bar\Omega\times[t_1,t_2]\to\mathbb{R}$ be any C^1 function. Define

$$\varphi(r,t) := \int_S \psi(ry,t)dS_y,$$

where the integral is over the unit sphere $S=S^2$ in $\mathbb{R}^3$. Equation (4.31) then holds for φ, and it is easy to get that, for $t=t_1$ or t_2,

$$\int_0^R \rho(r,t)\varphi(r,t)r^2dr = \int_\Omega \rho(x,t)\psi(x,t)dx.$$

We note that the second term on the right hand side of (4.31) may be rewritten as

$$\int_{t_1}^{t_2}\int_0^R \rho(r,t)u(r,t)\varphi_r(r,t)r^2drdt = \int_{t_1}^{t_2}\int_0^R \rho(r,t)u(r,t)\nabla\psi(ry,t)\cdot yr^2dS_ydrdt$$
$$= \int_{t_1}^{t_2}\int_\Omega \rho(r,t)u(r,t)\frac{x}{r}\nabla\psi(x,t)dxdt = \int_{t_1}^{t_2}\int_\Omega \rho(x,t)\mathbf{U}(x,t)\nabla\psi(x,t)dxdt.$$

The first term on the right hand side of (4.31) is treated in a similar way. This establishes the weak form of the mass equation, that is

$$\int_\Omega \rho\psi(x,\cdot)dx|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_\Omega\{\rho\psi_t+\rho(x,t)\mathbf{U}\cdot\nabla\psi\}(x,t)dxdt$$

for C^1 functions $\psi : \bar{\Omega} \times [t_1, t_2] \to \mathbb{R}$.

Now we prove that $\rho \in C([0,\infty); W^{1,\infty}(\Omega)^*)$. If ϕ is a C^1 function of x, then by the continuity equation, we have

$$\begin{aligned}
|\int_\Omega \rho\phi dx|_{t_1}^{t_2}| &= |\int_{t_1}^{t_2}\int_\Omega \rho(x,t)\mathbf{U}\cdot\nabla\phi dxdt| \\
&\le \|\nabla\phi\|_{L^\infty}\int_{t_1}^{t_2}(\int_\Omega \rho dx)^{1/2}(\int_\Omega \rho|\mathbf{U}|^2dx)^{1/2}dt \\
&\le C(T)\|\nabla\phi\|_{L^\infty}|t_2-t_1|.
\end{aligned}$$

A straightforward argument enables us to extend this to functions $\phi \in W^{1,\infty}(\Omega)$, so that

$$\|\rho(\cdot,t_2)-\rho(\cdot,t_1)\|_{W^{1,\infty}(\Omega)^*} \le C(T)|t_2-t_1|,$$

for $t_1, t_2 \in [0,T]$. The proof of the proposition is complete. $\square$

Finally, we prove that $(\rho, \mathbf{U})$ satisfies the weak form of the momentum equation, (2.21), in the sequel.

Proposition 4.6. *The weak form of the momentum equation, (2.21), holds as stated in Definition 2.1.*

Proof. Let ϕ be a C^2-function on $[0,R]\times[0,T]$ with $\phi(0,t) = \phi(R,t) = 0$ for all $t \in [0,T]$. Then it follows from (3.4) that

$$\begin{aligned}
&\int_{\varepsilon_j}^R \rho_0^j u_0^j \phi(r,0)r^2dr + \int_0^T\int_{\varepsilon_j}^R (\rho^j u^j\phi_t + \rho^j(u^j)^2\phi_r + (\rho^j)^\gamma(\phi_r + \frac{2\phi}{r}))r^2drdt \\
&- \int_0^T\int_{\varepsilon_j}^R \rho^j(u_r^j\phi_r + \frac{2u^j\phi}{r^2})r^2drdt = \int_0^T\int_{\varepsilon_j}^R \frac{3}{4}\varepsilon_j(\rho^j)^{\frac{3}{4}}(u_r^j + \frac{2u^j}{r})(\phi_r + \frac{2}{r}\phi)r^2drdt \\
&- \int_0^T\int_{\varepsilon_j}^R \varepsilon_j(\rho^j)^{\frac{3}{4}}(\frac{2u_r^j\phi}{r} + \frac{2u^j\phi_r}{r} + \frac{2}{r^2}u^j\phi)r^2drdt + \varepsilon_b^j.
\end{aligned} \tag{4.32}$$

where

$$\varepsilon_b^j = \int_0^T \{[\rho^j + \frac{3}{4}\varepsilon_j(\rho^j)^{\frac{3}{4}}u_r^j](\varepsilon_j,t)\varepsilon_j^2\phi(\varepsilon_j,t) - \varepsilon_j^2(\rho^j)^\gamma(\varepsilon_j,t)\phi(\varepsilon_j,t)\}dt. \tag{4.33}$$

We claim that

$$\lim_{\varepsilon_j\to 0^+}\varepsilon_b^j = 0. \tag{4.34}$$

To check this, we drop the subscript j for convenience. First, we show that

$$\lim_{\varepsilon\to 0^+}\varepsilon^2\int_0^T \rho^\gamma(\varepsilon,t)\phi(\varepsilon,t)dt = 0. \tag{4.35}$$

Indeed, note that

$$\begin{aligned}
&\left|\varepsilon^2\int_0^T \rho^\gamma(\varepsilon,t)\phi(\varepsilon,t)dt\right| \le \max_{0\le t\le T}|\phi(\varepsilon,t)|\int_0^T \varepsilon^2\rho^\gamma(\varepsilon,t)dt \\
&\le \max_{0\le t\le T}|\phi(\varepsilon,t)|\left[\int_0^T\int_\varepsilon^R \rho^\gamma(r,t)r^2drdt + \int_0^T\int_\varepsilon^R |\partial_r(\rho^\gamma)(r,t)|r^2drdt\right].
\end{aligned}$$

Since

$$\int_0^T\int_\varepsilon^R \rho^\gamma(r,t)r^2drdt \le C_0,$$

and

$$\begin{aligned}\int_0^T\int_\varepsilon^R |\partial_r(\rho^\gamma)|r^2drdt = & \ 2\int_0^T\int_\varepsilon^R |\rho^{\frac{\gamma}{2}}||\partial_r(\rho^{\frac{\gamma}{2}})|r^2drdt \\ \le & \int_0^T\int_\varepsilon^R \rho^\gamma r^2drdt + \int_0^T\int_\varepsilon^R |\partial_r(\rho^{\frac{\gamma}{2}})|^2r^2drdt \le C_0\end{aligned}$$

due to (4.6) and (4.7), so (4.36) follows from the fact that $\lim\limits_{\varepsilon\to 0^+}\max\limits_{0\le t\le T}|\phi(\varepsilon,t)| = 0$ since $\phi(0,t)\equiv 0$ and $\phi\in C^2$. Next, we show that

$$\lim_{\varepsilon\to 0^+}\int_0^T (\rho u_r)(\varepsilon,t)\phi(\varepsilon,t)\varepsilon^2dt = 0. \tag{4.36}$$

Thanks to (3.3) and the boundary condition that $u(\varepsilon,t)=0$, one has

$$\rho_t(\varepsilon,t)+\rho(\varepsilon,t)\partial_r u(\varepsilon,t) = 0.$$

Thus,

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\int_0^T (\rho u_r)(\varepsilon,t)\phi(\varepsilon,t)\varepsilon^2dt = \lim_{\varepsilon\to 0^+}\left(-\varepsilon^2\int_0^T \partial_t\rho(\varepsilon,t)\phi(\varepsilon,t)dt\right) \\ = & \lim_{\varepsilon\to 0^+}\left[\varepsilon^2\rho_0(\varepsilon)\phi(\varepsilon,0)+\varepsilon^2\int_0^T \rho(\varepsilon,t)\partial_t\phi(\varepsilon,t)dt\right] = \lim_{\varepsilon\to 0^+}\left[\varepsilon^2\int_0^T \rho(\varepsilon,t)\partial_t\phi(\varepsilon,t)dt\right].\end{aligned}$$

On the other hand, it is easy to get

$$\begin{aligned}\varepsilon^2\left|\int_0^T \rho(\varepsilon,t)\partial_t\phi(\varepsilon,t)dt\right| &\le \varepsilon^{2-\frac{2}{\gamma}}\left(\varepsilon^2\int_0^T \rho^\gamma(\varepsilon,t)dt\right)^{\frac{1}{\gamma}}||\partial_t\phi(\varepsilon,\cdot)||_{L^{\frac{\gamma}{\gamma-1}}} \\ &\le C_0\,\varepsilon^{2-\frac{2}{\gamma}}\to 0 \qquad \text{as} \qquad \varepsilon\to 0^+.\end{aligned}$$

Hence (4.37) holds. Similarly, one can show that

$$\lim_{\varepsilon\to 0^+}\frac{3}{4}\varepsilon\int_0^T \varepsilon^2\left(\rho^{\frac{3}{4}}u_r\right)(\varepsilon,t)\phi(\varepsilon,t) = 0. \tag{4.37}$$

Indeed, it follows from (3.3) and $u(\varepsilon,t)=0$ that

$$\frac{3}{4}\int_0^T \varepsilon^3\left(\rho^{\frac{3}{4}}u_r\right)(\varepsilon,t)\phi(\varepsilon,t)dt = \varepsilon^3\rho_0^{\frac{3}{4}}(\varepsilon)\phi(\varepsilon,0)+\int_0^T \varepsilon^3\rho^{\frac{3}{4}}(\varepsilon,t)\partial_t\phi(\varepsilon,t)dt.$$

Since

$$\varepsilon^3\left|\int_0^T \rho^{\frac{3}{4}}(\varepsilon,t)\partial_t\phi(\varepsilon,t)dt\right| \le \varepsilon^{3(\frac{2\gamma-1}{2\gamma})}\left(\varepsilon^2\int_0^T \rho^\gamma(\varepsilon,t)dt\right)^{\frac{3}{4\gamma}}||\partial_t\phi(\varepsilon,\cdot)||_{L^{\frac{4\gamma}{4\gamma-3}}},$$

so (4.38) follows. Now (4.35) is a consequence of (4.36) - (4.38).

Now, for any $\psi = (\psi^1, \psi^2, \psi^3) \in C^2(\bar{\Omega} \times [0,T])$ satisfying $\psi(x,t) = 0$ for all $x \in \partial\Omega$ and $\psi(x,T) = 0$, we set

$$\phi(r,t) = \int_S \psi(ry,t) \cdot y\, dS_y \tag{4.38}$$

with $S = S^2$ the unit sphere n $\mathbb{R}^3$, and transform the terms of (4.33) into integrals in Cartesian coordinates. The treatments of the first two integrals on the left hand side of (4.33) are similar to those in the proof of Proposition 4.5. The next integral can be taken care of by direction calculations. Indeed, note that

$$(r^2\phi)_r = \partial_r \int_{|x|\le r} \operatorname{div}\psi(x,t)dx = r^2 \int_S (\psi^i)_{x_i}(ry,t)dS_y.$$

Thus,

$$\begin{aligned}
&-\int_0^T\int_{\varepsilon_j}^R \rho^j \left(u_r^j\phi_r + \frac{2u^j\phi}{r^2}\right) r^2 drdt \\
= &-\int_0^T\int_{\varepsilon_j}^R \left\{\rho^j\left[\left(\frac{u^j}{r}\right)_r r\phi_r + \frac{u^j}{r}r^{-2}(r^2\phi)_r\right]\right\} r^2drdt \\
= &-\int_0^T\int_{\varepsilon_j}^R \rho^j \left\{\left(\frac{u^j}{r}\right)_r r\left(\int_S y_iy_k(\psi^i)_{x_k}(ry,t)dS_y\right) + \frac{u^j}{r}\left(\int_S \operatorname{div}\psi(ry,t)dS_y\right)\right\} r^2drdt \\
= &-\int_0^T\int_{\Omega_{\varepsilon_j}} \rho^j\left\{\left(\frac{u^j}{r}\right)_r \frac{x_ix_k}{r} + \frac{u^j}{r}\delta_{ik}\right\}\psi^i_{x_k}\, dxdt \\
= &-\int_0^T\int_{\Omega_{\varepsilon_j}} \rho^j \frac{\partial}{\partial x_k}\left(\frac{u^j}{r}x_i\right)(\psi^i)_{x_k}dxdt \\
= &-\int_0^T\int_{\Omega_{\varepsilon_j}} \rho^j\nabla(\mathbf{U}^j)^i : \nabla\psi^i dxdt.
\end{aligned}$$

Similarly, one has

$$\int_0^T\int_{\varepsilon_j}^R \frac{3}{4}\varepsilon_j(\rho^j)^{\frac{3}{4}}\left(u_r^j + \frac{2u^j}{r}\right)\left(\phi_r + \frac{2}{r}\phi\right) r^2drdt = \int_0^T\int_{\Omega_{\varepsilon_j}} \frac{1}{4}\varepsilon_j(\rho^j)^{\frac{3}{4}} \operatorname{div}\mathbf{U}^j \operatorname{div}\psi dxdt,$$

and

$$-\int_0^T\int_{\varepsilon_j}^R \varepsilon_j(\rho^j)^{\frac{3}{4}}\left(2\frac{u_r^j\phi}{r} + 2\frac{u^j\phi_r}{r} + \frac{2}{r^2}u^j\phi\right) r^2drdt = -\int_0^T\int_{\Omega_{\varepsilon_j}} \varepsilon_j(\rho^j)^{\frac{3}{4}}\nabla\mathbf{U}^j : \nabla\psi dxdt$$

Thus, we have shown that

$$\begin{aligned}
&\int_{\Omega_{\varepsilon_j}} \rho_0^j\mathbf{U}_0^j\cdot\psi(0,x)dx + \int_0^T\int_{\Omega_{\varepsilon_j}} \left\{\sqrt{\rho^j}(\sqrt{\rho^j}\mathbf{U}^j)\cdot\partial_t\psi + \sqrt{\rho^j}\mathbf{U}^j\otimes\sqrt{\rho^j}\mathbf{U}^j : \nabla\psi\right\} dxdt \\
&+\int_0^T\int_{\Omega_{\varepsilon_j}} (\rho^j)^\gamma \operatorname{div}\psi dxdt - \int_0^T\int_{\Omega_{\varepsilon_j}} \rho^j\nabla\mathbf{U}^j : \nabla\psi dxdt \\
= &\frac{1}{4}\varepsilon_j\int_0^T\int_{\Omega_{\varepsilon_j}} (\rho^j)^{\frac{3}{4}} \operatorname{div}\mathbf{U}^j \operatorname{div}\psi dxdt - \varepsilon_j\int_0^T\int_{\Omega_{\varepsilon_j}} (\rho^j)^{\frac{3}{4}}\nabla\mathbf{U}^j : \nabla\psi dxdt + \varepsilon_b^j.
\end{aligned}$$

It follows from this and (4.4) that

$$
\begin{aligned}
&\int_\Omega \rho_0^j \mathbf{U}_0^j \cdot \psi(0,\cdot)dx + \int_0^T\int_\Omega \Big\{\sqrt{\rho^j}(\sqrt{\rho^j}\mathbf{U}^j)\cdot\partial_t\psi + \sqrt{\rho^j}\mathbf{U}^j \otimes \sqrt{\rho^j}\mathbf{U}^j : \nabla\psi\Big\}\,dxdt \\
&+\int_0^T\int_\Omega (\rho^j)^\gamma \mathrm{div}\psi dxdt - \int_0^T\int_\Omega \rho^j \nabla\mathbf{U}^j : \nabla\psi dxdt \\
= &\int_0^T\int_{B_{\varepsilon_j}} (\rho^j)^\gamma \mathrm{div}\psi dxdt + \frac{\varepsilon_j}{4}\int_0^T\int_{\Omega_{\varepsilon_j}} (\rho^j)^{\frac{3}{4}}\mathrm{div}\mathbf{U}^j\,\mathrm{div}\psi dxdt \\
&-\varepsilon_j\int_0^T\int_{\Omega_{\varepsilon_j}} (\rho^j)^{\frac{3}{4}}\nabla\mathbf{U}^j : \nabla\psi dxdt + \varepsilon_b^j.
\end{aligned}
\tag{4.39}
$$

We proceed to show that each term on the left hand side of (4.40) converges to corresponding term in (2.21) and each term on the right hand side of (4.40) vanishes as $j\to\infty$.

First, the convergence of the term $\rho^j\mathbf{U}^j\phi_t$ can be established just as what has been done for the term $\rho^j u^j\phi_r$ in the proof of Proposition 4.5.

Next,

$$
\begin{aligned}
&|\int_0^T\int_\Omega [\sqrt{\rho^j}\mathbf{U}^j\otimes\sqrt{\rho^j}\mathbf{U}^j - \sqrt{\rho}\mathbf{U}\otimes\sqrt{\rho}\mathbf{U}] : \nabla\psi dxdt| \\
\le\ &\|\nabla\psi\|_{L^\infty}\int_0^T\int_{B_{\frac{1}{n}}} (|\sqrt{\rho^j}\mathbf{U}^j|^2 + \sqrt{\rho}|\mathbf{U}|^2)dxdt \\
&+|\int_0^T\int_{\Omega_{\frac{1}{n}}} [\sqrt{\rho^j}\mathbf{U}^j\otimes\sqrt{\rho^j}\mathbf{U}^j - (\sqrt{\rho}\mathbf{U}\otimes\sqrt{\rho}\mathbf{U}) : \nabla\psi]dxdt|,
\end{aligned}
\tag{4.40}
$$

for all $n\in N$.

By virtue of Proposition 4.3, one has

$$
\begin{aligned}
\int_0^T\int_{B_{\frac{1}{n}}} |\sqrt{\rho^j}\mathbf{U}^j|^2 dxdt &\le (\int_0^T\int_{B_{\frac{1}{n}}} \rho^j dxdt)^{\frac{\eta}{2+\eta}}(\int_0^T\int_{B_{\frac{1}{n}}} \rho^j|\mathbf{U}^j|^{2+\eta}dxdt)^{\frac{2}{2+\eta}} \\
&\le C(\int_0^T\int_{B_{\frac{1}{n}}} \rho^j dxdt)^{\frac{\eta}{2+\eta}}.
\end{aligned}
$$

As proved in Proposition 4.5, the following convergence holds

$$
\begin{aligned}
\int_0^T\int_{B_{\frac{1}{n}}} \rho^j dxdt &\le C(T)(\int_0^T\int_{B_{\frac{1}{n}}} (\rho^j)^3 dxdt)^{\frac{1}{3}}|B_{\frac{1}{n}}|^{\frac{2}{3}} \\
&\le C(T)|B_{\frac{1}{n}}|^{\frac{2}{3}} \to 0
\end{aligned}
\tag{4.41}
$$

as $n\to\infty$, where (4.8) has been used. Consequently, it holds that

$$
\int_0^T\int_{B_{\frac{1}{n}}} |\sqrt{\rho^j}\mathbf{U}^j|^2 dxdt \to 0,
$$

uniformly on j, as $n \to \infty$. Also,

$$\int_0^T \int_{B_{\frac{1}{n}}} |\sqrt{\rho}\mathbf{U}|^2 dxdt \leq \liminf_{j\to\infty} \int_0^T \int_{B_{\frac{1}{n}}} |\sqrt{\rho^j}\mathbf{U}^j|^2 dxdt \to 0,$$

as $n \to \infty$. It follows from (4.36) and Proposition 4.4 that

$$\int_0^T \int_\Omega \sqrt{\rho^j}\mathbf{U}^j \otimes \sqrt{\rho^j}\mathbf{U}^j : \nabla\psi dxdt \to \int_0^T \int_\Omega \sqrt{\rho}\mathbf{U} \otimes \sqrt{\rho}\mathbf{U} : \nabla\psi dxdt, \quad (4.42)$$

as $j \to \infty$. For the pressure term, Proposition 4.2 implies that

$$\int_0^T \int_\Omega (\rho^j)^\gamma \operatorname{div}\psi\, dxdt \to \int_0^T \int_\Omega \rho^\gamma \operatorname{div}\psi\, dxdt, \qquad j \to \infty. \quad (4.43)$$

Concerning the diffusion terms on the left hand side of (4.39), it follows from (4.4) and integration by parts that

$$\begin{aligned} \int_0^T \int_\Omega \rho^j \nabla \mathbf{U}^j : \nabla\psi dxdt &= -\int_0^T \int_\Omega \sqrt{\rho^j}(\sqrt{\rho^j}\mathbf{U}^j) \cdot \Delta\psi dxdt \\ -2\int_0^T \int_\Omega (\sqrt{\rho^j}\mathbf{U}^j) \cdot (\nabla\sqrt{\rho^j} \cdot \nabla)\psi dxdt. \end{aligned} \quad (4.44)$$

Using Proposition 4.2- Proposition 4.4, one can prove the convergence for the first term on the right hand side of (4.44) as follows,

$$\int_0^T \int_\Omega \sqrt{\rho^j}(\sqrt{\rho^j}\mathbf{U}^j)\Delta\phi dxdt \to \int_0^T \int_\Omega \sqrt{\rho}(\sqrt{\rho}\mathbf{U})\Delta\phi dxdt, \quad (4.45)$$

as $j \to \infty$, in a similar way as in the proof of (4.29).

Due to Lemma 4.1, it holds that

$$\|\nabla\sqrt{\rho^j}\|_{L^\infty(0,T;L^2(\Omega))} \leq C,$$

and hence there exists a function $g \in L^(0, T; L^2(\Omega))$ such that

$$\nabla\sqrt{\rho^j} \rightharpoonup g \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)).$$

Meanwhile, by Proposition 4.1, up to a subsequence, $\sqrt{\rho^j}$ converges almost everywhere to $\sqrt{\rho}$. Combining the fact that $\sqrt{\rho^j}$ is uniformly bounded in $L^\infty(0,T;L^6(\Omega)$, one has

$$\sqrt{\rho^j} \rightharpoonup \sqrt{\rho} \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)),$$

and hence $g = \sqrt{\rho}$. Consequently, it yields

$$\nabla\sqrt{\rho^j} \rightharpoonup \nabla\sqrt{\rho} \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)).$$

Due to Proposition 4.3 and Proposition 4.4, we finally obtain

$$\begin{aligned} &-2\int_0^T \int_\Omega (\sqrt{\rho^j}\mathbf{U}^j) \cdot (\nabla\sqrt{\rho^j} \cdot \nabla)\psi dxdt \to \\ &-2\int_0^T \int_\Omega (\sqrt{\rho}\mathbf{U}) \cdot (\nabla\sqrt{\rho} \cdot \nabla)\psi dxdt, \end{aligned} \quad (4.46)$$

similar to the proof of (4.42). Substituting (4.45) and (4.46) into (4.44) yields

$$\begin{aligned}&\int_0^T\int_\Omega \rho^j\nabla\mathbf{U}^j:\nabla\psi dxdt \to <\rho\nabla\mathbf{U},\nabla\psi>\\ \equiv& -\int_0^T\int_\Omega\sqrt{\rho}(\sqrt{\rho}\mathbf{U})\cdot\Delta\psi dxdt-2\int_0^T\int_\Omega(\sqrt{\rho}\mathbf{U})\cdot(\nabla\sqrt{\rho}\cdot\nabla)\psi dxdt. \quad (4.47)\end{aligned}$$

Up to now, we have proved that the terms on the left hand side of (4.39) converge to corresponding ones in (2.21) as $j\to\infty$. In the following, we prove that each term on the right hand side of (4.39) vanishes as $j\to\infty$.

First, since $\sqrt{\rho^j}$ is uniformly bounded in $L^\infty(0,T;L^6(\Omega))$ due to (4.8), it holds that

$$|\int_0^T\int_{B_{\varepsilon_j}}(\rho^j)^\gamma \mathrm{div}\psi dxdt|\le C(\int_0^T\int_{B_{\varepsilon_j}}(\rho^j)^3dxdt)^{\frac{\gamma}{3}}|B_{\varepsilon_j}|^{\frac{3-\gamma}{3}}\le C|B_{\varepsilon_j}|^{\frac{3-\gamma}{3}} \quad (4.48)$$

for $1<\gamma<3$, which tends to zero as $\varepsilon_j\to 0$.

Next, with the help of Lemma 4.1 again, one has

$$\begin{aligned}&|\frac{\varepsilon_j}{4}\int_0^T\int_{\Omega_{\varepsilon_j}}(\rho^j)^{\frac34}\mathrm{div}\mathbf{U}^j\mathrm{div}\psi dxdt|\\ \le& C\sqrt{\varepsilon_j}(\varepsilon_j\int_0^T\int_{\Omega_{\varepsilon_j}}(\rho^j)^{\frac34}|\nabla\mathbf{U}^j|^2dxdt)^{\frac12}(\int_0^T\int_{\Omega_{\varepsilon_j}}(\rho^j)^{\frac34}dxdt)^{\frac12}\\ \le& C\sqrt{\varepsilon_j}. \quad (4.49)\end{aligned}$$

Finally, the integral $\varepsilon_j\int_0^T\int_{\Omega_j}(\rho^j)^{\frac34}\nabla\mathbf{U}^j:\nabla\psi dxdt$ admits same bound as in (4.49).

It follows from this, (4.34), and (4.48)-(4.49) that each term on the right hand side of (4.39) converges to 0 as $j\to\infty$.

Taking the limit $j\to\infty$ in (4.39), we finish the proof of the proposition. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. The weak forms of the mass conservation and momentum equations follow from Proposition 4.5 and 4.6 respectively. The first part in the definition of the weak solutions (see Definition 2.1) follows from Lemma 3.2, Lemma 4.1, Proposition 4.5, and the proof of (4.47) in Proposition 4.6 which shows that $\rho^j\nabla\mathbf{U}^j\rightharpoonup\rho\nabla\mathbf{U}$ in the sense of distribution and $\rho\nabla\mathbf{U}\in L^2(0,T;W^{-1,1}(\Omega))$. Moreover, $\rho\in C([0,T];L^{\frac32}(\Omega))$ and the equation of mass conservation (2.27) are obtained by Propositions 4.1 and 4.5. The energy estimate (2.28) and entropy estimate (2.29) are due to Lemma 4.1. Finally, the radial symmetry of the weak solutions is a consequence of Corollary 4.2. The proof of Theorem 2.1 is thus finished.

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