Energy Scattering for the Klein-Gordon Equation with a Cubic Convolution Nonlinearity

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Abstract

In this paper, we study the theory of scattering in the energy space for the Klein-Gordon equation with a cubic convolution in space dimension $n \geq 3$. By means of the strategy of frequency decomposition to distinguish the dispersive effects and the flexibility of the Strichartz estimates for the Klein-Gordon equation, along with the method of Morawetz-Strauss [18] and Ginibre-Velo [10], we prove the asymptotic completeness for the radial nonnegative, nonincreasing potentials satisfying suitable regularity properties at the origin and suitable decay properties at infinity. The results cover in particular the case of the potential $|x|^{-\nu}$ for $2 < \nu < \min(n, 4)$.

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1 Introduction

This paper is devoted to the theory of scattering for the Klein-Gordon equation with a cubic convolution

$$\ddot{u} - \Delta u + u + f(u) = 0, \qquad t \in \mathbb{R}, \ \ x \in \mathbb{R}^n, \ \ n \geq 3 \tag{1.1}$$

where $f(u) = (V * |u|^2)u$. Here u is a complex valued function defined in space time $\mathbb{R}^{n+1}$, the dot denotes the time derivative, Δ is the Laplacian in $\mathbb{R}^n$, V is a real valued radial function defined in $\mathbb{R}^n$, hereafter called the potential, and $*$ denotes the convolution in $\mathbb{R}^n$. For simplicity, we have taken the mass equal to 1 in the equation (1.1). It is known that for the local nonlinearity (e.g. $f(u) = |u|^{p-1}u$), the scattering theory has been obtained, see [3], [4], [5], [6], [8], [19] and [21] for details. For the equation (1.1) with the small data, Mochizuki [17] took use of the ideas of Strauss[23], [24] and

Pecher [22], and shown that if $n \geq 3$, $2 \leq \nu < \min(n, 4)$, then the scattering operator $S : B(\delta; H^1 \times L^2) \longrightarrow H^1 \times L^2$ is well defined for some small $\delta > 0$, where $B(\delta; H^1 \times L^2)$ denote the set $\{(f, g) \in H^1 \times L^2, \|(f, g)\|_{H^1 \times L^2} \leq \delta\}$. That is, the scattering operator S is well defined in the low energy space. In this paper, we develop a complete theory of scattering for the equation (1.1) in the energy space, which turns out to be the Sobolev space $H^1 \times L^2$, under suitable assumptions on V, the results cover in particular the case of the potential $|x|^{-\nu}$ for $2 < \nu < \min(n, 4)$.

The scattering theory in the energy space for the Hartree equation

$$i\dot{u} = -\Delta u + (V * |u|^2)u$$

has been studied by many authors. For subcritical cases(see Remark 1.2), Ginibre and Velo [10] derived the Morawetz inequality and extracted an useful estimate of the solution in Birman-Solomjak norm to obtain the asymptotic completeness. Nakanishi [20] improved the results by a new Morawetz estimate which depending not on nonlinearity. For critical cases, Miao, Xu and Zhao [14] took advantage of the local Morawetz estimate to rule out the possibility of energy concentration and establish the scattering theory for the radial data in dimension $n \geq 5$. Please see [15] for the general data.

In this paper, we devote attention to study the Klein-Gordon equation (1.1) with same potential V as the Hartree equation in [10]. But if we directly proceed with the approach of Ginibre-Velo [10] in conjunction with the Strichartz estimates in [16], the asymptotic completeness be only obtained for the potential $|x|^{-\nu}$ with $2 < \nu < \min(n, 4 - \frac{1}{n})$, which is not natural. The reason for this lies in that the dispersive estimates for the Klein-Gordon equation is different from those of the Schrödinger equation; one can get sufficient time decay rate by compensating the condition of the high regularity, which conflicts with the solution belong to the energy space H^1. The major innovation of the paper is to conquer the difficulty by exploiting sufficiently the dispersive effects. Indeed, the dispersive natures of high and low frequencies of the solution for the Klein-Gordon equation are different; the solution on the low frequency behaves as that of the nonrelativistic equation where the condition of regularity can be dropped by the Bernstein inequality, while the high frequency part with bad dispersive effect can be controlled by the energy of solution. Therefore we take advantage of the strategy of frequency decomposition to gain the time decay estimates and establish the scattering theory. It is valuable to mention that the high frequency decay depending only on frequency not on time, thus we merely carry iteration on low frequency region, which is also a key point in the proof of the time decay. A more detailed description is provided in Section 4.

In all this paper, we assume that the potential V satisfies the following assumption, which ensure the local existence of the energy solution.

(H1) V is a real radial function and $V \in L^{p_1} + L^{p_2}$ for some p_1, p_2 satisfying

$$1 \vee \frac{n}{4} \leq p_2 \leq p_1 \leq \infty.$$

Furthermore, to exploit the Morawetz estimate and demonstrate the theory of asymptotic completeness, we still need an additional assumption on V.

(H2) V is radial and nonincreasing, namely $V(x) = v(r)$ where v is nonincreasing in $\mathbb{R}+$. Furthmore, for some $\alpha \geq 2$, v satisfies the following condition:

(A_α): There exists $a > 0$ and $A_\alpha > 0$ such that

$$v(r_1) - v(r_2) \geq \frac{A_\alpha}{\alpha}(r_2^\alpha - r_1^\alpha) \text{ for } 0 < r_1 < r_2 \leq a.$$

One easily verifies that as soon as $V(x) \in L^p$ for some $p < \infty$, (H2) implies that V is nonnegative and tends to zero at infinity.

Remark 1.1. *Different with the Hartree equation[20], the assumption (A_α) is still needed for the equation* (1.1). *In fact, it seems difficult to prove that the corresponding term*

$$2Re\big((V * |u|^2)u, m\big) = -\big(\nabla V * |u|^2, \frac{x}{\lambda}|u|^2\big) - \big(V * |u|^2, \frac{|x|^2}{\lambda^3}|u|^2\big) - \big(V * |u|^2, \frac{t}{\lambda}\frac{d}{dt}|u|^2\big),$$

is nonnegative or controlled by energy, where

$$\mathcal{D} = \big(-\partial_t, \nabla\big), \quad \lambda = |(t,x)|, \quad m = \frac{(t,x)}{\lambda} \cdot \mathcal{D}u + \Big(\frac{n-1}{2\lambda} + \frac{t^2 - |x|^2}{2\lambda^3}\Big)u.$$

We believe that how to remove the condition (A_α) is still an interesting problem.

The main theorem of this paper is the following.

Theorem 1.1. *Assume that V satisfies (H1) with $2 < \frac{n}{p_1} \leq \frac{n}{p_2} < \min\big(n, 4\big)$ and (H2). Then, the wave operators and the scattering operator for* (1.1) *are homeomorphisms in $H^1 \times L^2$. Precisely, for any global solution u of* (1.1) *with $(u(0), \dot{u}(0)) \in H^1 \times L^2$, there exists a solution v of the free Klein-Gordon equation*

$$\ddot{v} - \Delta v + v = 0 \tag{1.2}$$

with $v(0) \in H^1, \dot{v}(0) \in L^2$ such that

$$\big\|\big(u(t), \dot{u}(t)\big) - \big(v(t), \dot{v}(t)\big)\big\|_{H^1 \times L^2} \longrightarrow 0, \quad as \quad t \longrightarrow \infty. \tag{1.3}$$

Moreover, the corresponding $(u(0), \dot{u}(0)) \longrightarrow (v(0), \dot{v}(0))$ defines a homeomorphism in $H^1 \times L^2$.

Remark 1.2. *If we take the special potential $V(x) = |x|^{-\nu}$, then the restriction on ν is just $2 < \nu < \min(4, n)$. From the analysis of scaling $s_c = \frac{\nu}{2} - 1 = \frac{n}{2p} - 1$ for V belonging to the single exponent L^p, we know that the case of $\nu = 2$ is corresponding to L^2-critical case; while if $n \geq 5$, the case of $\nu = 4$ is corresponding to H^1-critical case.*

The paper is organized as follows. In Section 2, we deal with the Cauchy problem at finite time for the equation (1.1). We prove the local wellposedness in $H^1 \times L^2$ and then prove the global wellposedness in $H^1 \times L^2$ by deriving the conservation law of the energy. In Section 3, we prove the existence of the wave operators. we solve the local Cauchy problem in a neighborhood of of infinity in time to illustrate the existence and some properties of asymptotic states for the solutions. Finally in Section 4, we prove the main result of this paper, namely asymptotic completeness in $H^1 \times L^2$. We derive the finiteness of the propagation speed and the Morawetz-type estimates of the solution in suitable norms. In conjunction with frequency decomposition we prove the time decay of the arbitrary finite energy solutions, hence prove the asymptotic completeness.

We conclude this introduction by giving some notations which will be used freely throughout this paper. We always assume the space dimension $n \geq 3$ and let $2^* = \frac{2n}{n-2}$ in this paper. For any $r, 1 \leq r \leq \infty$, we denote by $\|\cdot\|_r$ the norm in $L^r = L^r(\mathbb{R}^n)$ and by r' the conjugate exponent defined by $\frac{1}{r} + \frac{1}{r'} = 1$. For any $s \in \mathbb{R}$, we denote by $H^s \equiv H^s(\mathbb{R}^n)$ the usual Sobolev spaces. For $s \in \mathbb{R}$ and $1 \leq r, m \leq \infty$, denote by $B^s_{r,m}(\mathbb{R}^n)$ the Besov space defined as the space of distributions u such that $\{2^{js}\|\varphi_j * u\|_r\}_{j=0}^{\infty} \in \ell^m$, where $*$ stands for the convolution and $\{\varphi_j\}$ is a dyadic decomposition on $\mathbb{R}^n$, and by $\dot{B}^s_{r,m}(\mathbb{R}^n)$ the homogeneous Besov space defined as the space of distributions u modulo polynomials such that $\{2^{js}\|\psi_j * u\|_r\}_{j=-\infty}^{\infty} \in \ell^m$, where $\{\psi_j\}$ is a dyadic decomposition on $\mathbb{R}^n \backslash \{0\}$. For the detailed definitions of the above function spaces and the associated inequalities, see [1]. We shall omit $\mathbb{R}^n$ from spaces and norms. For any interval $I \in \mathbb{R}$ and any Banach space X we denote by $\mathcal{C}(I; X)$ the space of strongly continuous functions from I to X and by $L^q(I; X)$ the space of strongly measurable functions from I to X with $\|u(\cdot); X\| \in L^q(I)$. Given n, we define, for $2 \leq r \leq \infty$,

$$\delta(r) = n\Big(\frac{1}{2} - \frac{1}{r}\Big).$$

Sometimes abbreviate $\delta(r)$, $\delta(r_i)$ to δ, δ_i separately. We denote by $< \cdot, \cdot >$ the scalar product in L^2. Finally for any real number a and b, we let $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$, $a_+ = a \vee 0$ and $a_- = (-a)_+$.

2 The Cauchy problem at finite times

In this section, we consider the Cauchy problem for the equation (1.1)

$$\begin{cases} \ddot{u} - \Delta u + u + f(u) = 0, \\ u(0) = u_0, \ \dot{u}(0) = u_1. \end{cases} \tag{2.1}$$

where

$$f(u) = (V * |u|^2)u. \tag{2.2}$$

The form of the integral equation for the Cauchy Problem (2.1) can be read as

$$u(t) = \dot{K}(t)u_0 + K(t)u_1 - \int_0^t K(t-s)f(u(s))ds, \tag{2.3}$$

or

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = V_0(t)\begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix} - \int_0^t V_0(t-s)\begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds, \tag{2.4}$$

where $K(t)$ denotes

$$K(t) = \frac{\sin(t\omega)}{\omega}, \quad V_0(t) = \begin{pmatrix} \dot{K}(t), K(t) \\ \ddot{K}(t), \dot{K}(t) \end{pmatrix}, \quad \omega = \big(1-\Delta\big)^{1/2}.$$

Let $U(t) = e^{it\omega}$, then

$$\dot{K}(t) = \frac{U(t)+U(-t)}{2}, \qquad K(t) = \frac{U(t)-U(-t)}{2i\omega}.$$

We first give the following dispersive estimates due to Brenner [5] and Ginibre, Velo [8] for the operator $U(t) = e^{it\omega}$. The proof is omitted.

Lemma 2.1. *Let $2 \le r \le \infty$ and $0 \le \theta \le 1$. Then*

$$\big\|e^{i\omega t}u\big\|_{B_{r,2}^{-(n+1+\theta)(\frac{1}{2}-\frac{1}{r})/2}} \le \mu(t)\big\|u\big\|_{B_{r',2}^{(n+1+\theta)(\frac{1}{2}-\frac{1}{r})/2}},$$

where

$$\mu(t) = C\min\Big(|t|^{-(n-1-\theta)(\frac{1}{2}-\frac{1}{r})_+}, |t|^{-(n-1+\theta)(\frac{1}{2}-\frac{1}{r})}\Big).$$

According to the above lemma, the abstract duality and interpolation argument [9], [11] , it is well known that $U(t)$ satisfies the following Strichartz estimates [16].

Lemma 2.2. *Let $0 \le \theta_i \le 1$, $\rho_i \in \mathbb{R}$, $2 \le q_i, r_i \le \infty$, $(\theta_i, n, q_i, r_i) \ne (0,3,2,\infty)$, $i = 1,2$ satisfy following admissible conditions*

$$\begin{cases} 0 \le \dfrac{2}{q_i} \le \min\Big\{(n-1+\theta_i)(\dfrac{1}{2}-\dfrac{1}{r_i}), 1\Big\}, \\ \rho_i + (n+\theta_i)\Big(\dfrac{1}{2}-\dfrac{1}{r_i}\Big) - \dfrac{1}{q_i} = 0; \end{cases}$$

and let $\mathscr{A}$ denote the set of all $(q_i, r_i, \theta_i, \rho_i)$ satisfying the above conditions.

1. *For $u \in L^2$, then*
$$\big\|U(\cdot)u\big\|_{L^{q_1}\big(\mathbb{R};B_{r_1,2}^{\rho_1}\big)} \le C\|u\|_2; \tag{2.5}$$

2. *For $I \subseteq \mathbb{R}$, then*
$$\big\|U * f\big\|_{L^{q_1}\big(I;B_{r_1,2}^{\rho_1}\big)} \le C\big\|f\big\|_{L^{q_2'}\big(I;B_{r_2',2}^{-\rho_2}\big)}; \tag{2.6}$$

3. *For $I = [0,T) \subset \mathbb{R}$, then*
$$\big\|U_R * f\big\|_{L^{q_1}\big(I;B_{r_1,2}^{\rho_1}\big)} \le C\big\|f\big\|_{L^{q_2'}\big(I;B_{r_2',2}^{-\rho_2}\big)}. \tag{2.7}$$

   where the subscript R stands for retarded.

Remark 2.1. *One easily checks that* (2.6) *and* (2.7) *hold for any* $(q_i, r_i, \theta_i, \rho_i) \in \mathscr{A},\ i = 1, 2$, *thus the choices of exponents (especially in* θ*) are very flexible which is significant in the estimate of nonlinearity. In fact, for any* $(q_i, r_i, \theta_i, \rho_i) \in \mathscr{A},\ i = 1, 2$, *we let*

$$B := L^{q_1}\big(I; B^{\rho_1}_{r_1,2}\big) \cap L^{q_2}\big(I; B^{\rho_2}_{r_2,2}\big),$$

then the dual space of B *is*

$$B^* := L^{q_1'}\big(I; B^{-\rho_1}_{r_1',2}\big) \oplus L^{q_2'}\big(I; B^{-\rho_2}_{r_2',2}\big).$$

It follows from (2.5) *and the abstract* TT^* *method that*

$$\big\|U * f\big\|_{L^{q_1}\left(I; B^{\rho_1}_{r_1,2}\right)} \leq \big\|U * f\big\|_B \leq C\big\|f\big\|_{B^*} \leq C\big\|f\big\|_{L^{q_2'}\left(I; B^{-\rho_2}_{r_2',2}\right)}.$$

The above lemma suggests that we study the Cauchy problem for the equation (2.3) in spaces of the following type. Let I be an interval, $\sigma_n = 1$ if $n \geq 4$ and $\sigma_n < 1$ can be any constant close to 1 if $n = 3$. Fix $\rho = -\frac{1}{2}$, we define Banach spaces $X^1_\theta(I)$ and $X^1(I)$ by

$$\begin{aligned}
X_\theta(I) = \Big\{ & u : u \in \big(\mathcal{C} \cap L^\infty\big)\big(I, L^2\big) \cap L^q\big(I, B^\rho_{r,2}\big), \\
& 0 \leq \frac{2}{q} \leq (n - 1 + \theta)(\frac{1}{2} - \frac{1}{r}) \leq \sigma_n,\ \rho + (n + \theta)(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q} = 0 \Big\}; \\
& X^1_\theta(I) = \big\{ u : u, \dot{u} \text{ and } \nabla u \in X_\theta(I) \big\}; \\
& X^1(I) := \bigcap_{0 \leq \theta \leq 1} X^1_\theta(I) \triangleq X^1_0 \cap X^1_1.
\end{aligned} \tag{2.8}$$

For noncompact I, we define the space $X^1_{loc}(I)$ in a similar way as L^q_{loc}.

For future reference we state additional time integrability properties of functions in $X^1(I)$ which are immediately obtained by means of Lemma 2.2 and Sobolev embedding theorem.

Lemma 2.3. *Let* I *be an interval, possibly unbounded. Let* (q, r) *satisfy*

$$0 \leq \frac{2}{q} \leq \sigma_n, \quad \frac{2}{q} \leq \delta(r) \leq 1 + \frac{1}{q}$$

Then

$$\big\|u\big\|_{L^q(I;L^r)} \leq C\big\|u\big\|_{X^1(I)},$$

where C *is independent of* I.

We can now state the main result on the local Cauchy problem for the equation (1.1) with $H^1 \times L^2$ initial data.

Proposition 2.1. *Let* V *satisfy (H1), and* $u_0 \in H^1, u_1 \in L^2$. *Then*

1. *There exists a maximal interval $(-T_-, T_+)$ with $T_\pm > 0$ such that the equation* (2.3) *has a unique solution $u \in X^1_{loc}(-T_-, T_+)$;*

2. *For any interval I containing* 0*, the equation* (2.3) *has at most one solution in $X^1(I)$;*

3. *For $-T_- < T_1 \le T_2 < T_+$, the map $(u_0, u_1) \to (u, u_t)$ is continuous from $H^1 \times L^2$ to $X^1([T_1, T_2])$;*

4. *Let in addition $p_2 > \frac{n}{4}$. Then if $T_+ < \infty$ (resp. $T_- < \infty$), $\|\dot{u}\|_2 + \|u\|_{H^1} \to \infty$ when t increases to T_+ (resp. decreases to $-T_-$).*

Proof: We shall apply the Banach fixed point argument to prove this proposition in the Banach space $X^1(I)$. The main technique point consists in proving that the operator defined by the RHS of (2.3) is a contraction on suitable bounded sets of $X^1(I)$ for $I = [-T, T]$ and T sufficiently small. From Lemma 2.2, by applying the fractional Leibnitz rule and the Hölder and Young inequalities in space and followed by the Hölder inequality in time, the key estimate consists in

$$\|f(u)\|_{L^{q'}\left(I;B^{1/2}_{r',2}\right)} \le \|V\|_p \|u\|_{L^q\left(I;B^{1/2}_{r,2}\right)} \|u\|^2_{L^k\left(I;L^s\right)} T^\vartheta, \tag{2.9}$$

where we have assumed for simplicity that $V \in L^p$, and where the exponents satisfy

$$\begin{cases} \dfrac{n}{p} = 2\delta(r) + 2\delta(s), \\ \dfrac{2}{q} + \dfrac{2}{k} = 1 - \vartheta, \\ 0 \le \vartheta \le 1 \end{cases} \tag{2.10}$$

and the exponents (θ, q, r, k, s) possibly depending on p.

If $\frac{n}{p} \le 2$, (2.9) can be replaced by

$$\|f(u)\|_{L^{q'}\left(I;B^{0}_{r',2}\right)} \le \|V\|_p \|u\|_{L^q\left(I;B^{0}_{r,2}\right)} \|u\|^2_{L^k\left(I;L^s\right)} T^\vartheta, \tag{2.11}$$

provided that one choose $r = 2$, $\delta(s) \le 1$ and $k = q = \infty$. This implies that $\vartheta = 1$.

If $2 \le \frac{n}{p} \le 3$, one can choose $\theta = 0$, $k = \infty$, $0 \le \delta(s) \le 1$ and $q = r = \frac{2(n+1)}{n-1}$ so that $0 < \vartheta = 1 - \frac{2}{q} < 1$.

If $\frac{n}{p} \ge 3$, one can choose $\theta = 0$, $q = r = \frac{2(n+1)}{n-1}$, $\delta(s) = 1 + \frac{1}{k}$ so that $\frac{n}{p} = 4 - \vartheta$ which yields $\vartheta \ge 0$ for $\frac{n}{p} \le 4$.

The H^1-critical case $p = \frac{n}{4}$ yields $\vartheta = 0$ and requires a slightly more refined treatment than the subcritical case $p > \frac{n}{4}$. For general V satisfying (H1), the contribution of the components in L^{p_1} and L^{p_2} are treated separately. Therefore, we obtain the desired results.

It is well known that the equation (1.1) formally satisfies the conservation of the energy

$$E(u)(t) = \frac{1}{2}\|\dot{u}\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{1}{4}\int dxdy\big|u(t,x)\big|^2 V(x-y)\big|u(t,y)\big|^2 = E(u)(0). \tag{2.12}$$

Actually it turns out that the X^1 regularity of the solutions constructed in Proposition 2.1 is sufficient to ensure this conservation law.

Proposition 2.2. *Let V satisfy (H1). Let I be an interval and let $u \in X^1(I)$ be a solution of the equation* (1.1). *Then u satisfies*

$$E(u)(t_1) = E(u)(t_2)$$

for all $t_1, t_2 \in I$.

Proof: Let $\varphi \in C_0^\infty$ be a smooth approximation of the Dirac distribution δ in $\mathbb{R}^n$. By an elementary computation which is allowed by the available regularity, we can obtain

$$\begin{aligned}\frac{d}{dt}E(\varphi * u)(t) = \quad & \mathrm{Re} < \varphi * \dot{u},\ \varphi * \ddot{u} > + \mathrm{Re} < \varphi * \nabla\dot{u},\ \varphi * \nabla u > \\ & + \mathrm{Re} < \varphi * \dot{u},\ \varphi * u > + \mathrm{Re} < \varphi * \dot{u},\ f(\varphi * u) > \\ = & \mathrm{Re} < \varphi * \dot{u},\ \varphi * \ddot{u} - \varphi * \Delta u + \varphi * u + f(\varphi * u) > \\ = & \mathrm{Re} < \varphi * \dot{u},\ f(\varphi * u) - \varphi * f(u) >\end{aligned}$$

Integrating over the time interval (t_1, t_2), we obtain

$$E(\varphi * u)(t_2) - E(\varphi * u)(t_1) = \mathrm{Re}\int_{t_1}^{t_2} < \varphi * \dot{u},\ f(\varphi * u) - \varphi * f(u) > (t)\ dt. \tag{2.13}$$

We can let φ tend to δ, using the fact that convolution with φ tends strongly to the unit operator in L^r for $1 \leq r < \infty$. The LHS of (2.13) tends to $E\big(u(t_2)\big) - E\big(u(t_1)\big)$ and the RHS is shown to tend to zero by the Lebesgue dominated convergence theorem applied to the time integration. For that purpose we need an estimate of the integrand which is uniform in φ and integrable in time. That estimate essentially boils down to

$$\big| < \dot{u}, f(u) > \big| \leq C\big\|V\big\|_p \big\|\dot{u}\big\|_{B^{-1/2}_{r,2}} \big\|u\big\|_{B^{1/2}_{r,2}} \big\|u\big\|_s^2 \tag{2.14}$$

We choose the same values of r, s as in the proof of Proposition 2.1, so that the RHS of (2.14) belongs to L^∞ in time for $\frac{n}{p} \leq 2$ and to L^1 for $\frac{n}{p} \geq 2$.

We now turn to the global Cauchy problem for the equation (1.1). For that purpose we need to ensure that the conservation of the energy provides an a priori estimate of the norm $\big\|\dot{u}\big\|_2 + \big\|u\big\|_{H^1}$ of the solution. This is the case if the potential V satisfies the nonnegative assumption.

Together this inequality with the local well-posedness (Proposition 2.1) and the conservation of the energy (Proposition 2.2), we can now state the main result on the global Cauchy problem for the equation (1.1)

Proposition 2.3. *Let V be nonnegative and satisfy (H1) with $p_2 > \frac{n}{4}$. Let $(u_0, u_1) \in H^1 \times L^2$ and let u be a solution of the equation* (2.3) *constructed in Proposition* 2.1. *Then $T_+ = T_- = \infty$ and $u \in X^1_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R}, H^1)$.*

Note that the result is stated only for the H^1 subcritical case $p_2 > \frac{n}{4}$.

3 Scattering Theory I: Existence of the wave operators

In this section we begin the study of the theory of scattering for the equation (1.1) by addressing the first question, namely the existence of the wave operators. We restrict our attention to positive time. We consider an asymptotic state $(u_{0+}, u_{1+}) \in H^1 \times L^2$ and we look for a solution u of the equation (1.1) which is asymptotic to the solution

$$\begin{pmatrix} v(t) \\ \dot{v}(t) \end{pmatrix} = V_0(t) \begin{pmatrix} u_{0+} \\ u_{1+} \end{pmatrix}$$

of the free equation. For that purpose, we introduce the solution $u_{t_0}(t)$ of the equation (1.1) satisfying the initial condition

$$\begin{pmatrix} u_{t_0}(t_0) \\ \dot{u}_{t_0}(t_0) \end{pmatrix} = \begin{pmatrix} v(t_0) \\ \dot{v}(t_0) \end{pmatrix} = V_0(t_0) \begin{pmatrix} u_{0+} \\ u_{1+} \end{pmatrix}.$$

We then let t_0 tend to ∞. In favorable circumstances, we expect $u_{t_0}(t)$ to converge to a solution $u(t)$ of the equation (1.1) which is asymptotic to $v(t)$. The previous procedure is easily formulated in terms of integral equations. The Cauchy problem with initial data $(u(t_0), \dot{u}(t_0))$ at time t_0 is equivalent to the equation

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = V_0(t - t_0) \begin{pmatrix} u(t_0) \\ \dot{u}(t_0) \end{pmatrix} - \int_{t_0}^{t} V_0(t-s) \begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds. \tag{3.1}$$

The solution $(u_{t_0}(t), \dot{u}_{t_0}(t))$ with initial data $V_0(t_0)\binom{u_{0+}}{u_{1+}}$ at time t_0 should therefore be a solution of the equation

$$\begin{aligned} \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} &= V_0(t-t_0)V_0(t_0) \begin{pmatrix} u_{0+} \\ u_{1+} \end{pmatrix} - \int_{t_0}^{t} V_0(t-s) \begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds \\ &= V_0(t) \begin{pmatrix} u_{0+} \\ u_{1+} \end{pmatrix} - \int_{t_0}^{t} V_0(t-s) \begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds, \end{aligned} \tag{3.2}$$

where we have used the trigonometric identity.

The limiting solution $(u, \dot{u})$ is then expected to satisfy the equation

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = V_0(t) \begin{pmatrix} u_{0+} \\ u_{1+} \end{pmatrix} - \int_{+\infty}^{t} V_0(t-s) \begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds. \tag{3.3}$$

The problem of existence of the wave operators is therefore the Cauchy problem with infinite initial time. We first solve it locally in a neighborhood of infinity by a contraction method. We then extend the solutions thereby obtained to all times by using the available results on the Cauchy problem at finite times. In order to solve the local Cauchy problem at infinity, we need to use function spaces including some time decay in their definition, so that at the very least the integral in (3.3) converges at infinity. Furthermore the free solution $V_0(t)\binom{u_{0+}}{u_{1+}}$ should belong to those spaces. In view of Lemma 2.2, natural candidates are the spaces $X^1(I)$ for some $I = [T, \infty)$, where the time decay is expressed by the L^q integrability at infinity, and we shall therefore study that problem in those spaces. We shall also need the fact that the time decay of u implies sufficient time decay of $f(u)$. This will show up through additional assumptions on V in the form of an upper bound on p_1, namely $p_1 \le \frac{n}{2}$.

We shall use freely the notation $\binom{\widetilde{u}(t)}{\tilde{\dot{u}}(t)} = V_0(-t)\binom{u(t)}{\dot{u}(t)}$ for u a suitably regular function of space time. We also recall the notation $\overline{\mathbb{R}}$ for $\mathbb{R} \cup \{\pm\infty\}$ and $\overline{I}$ for the closure of an interval I in $\overline{\mathbb{R}}$ equipped with the obvious topology.

We can now state the main result on the local Cauchy problem in a neighborhood of infinity.

Proposition 3.1. *Let V satisfy (H1) with $p_1 \le \frac{n}{2}$. Let $(u_{0+}, u_{1+}) \in H^1 \times L^2$. Then*

1. *There exists $T < \infty$ such that for any $t_0 \in \overline{I}$ where $I \in [T, \infty)$, the equation* (3.2) *has a unique solution u in $X^1(I)$.*
2. *For any $T' > T$, the solution u is strongly continuous from $(u_{0+}, u_{1+}) \in H^1 \times L^2$ and $t_0 \in \overline{I'}$ to $X^1(I')$, where $I' = [T', \infty)$.*

Proof: The proof proceeds by a contraction argument in $X^1(I)$. The main technical point consists in proving that the operator defined by the RHS of (3.2) is contraction in suitable bounded sets of $X^1(I)$ for T sufficiently large. Let (q, r) is admissible pair, the basic estimate is again

$$\left\| f(u) \right\|_{L^{q'}\left(I; B^{1/2}_{r',2}\right)} \le \left\| V \right\|_p \left\| u \right\|_{L^q\left(I; B^{1/2}_{r,2}\right)} \left\| u \right\|^2_{L^k\left(I; L^s\right)}, \tag{3.4}$$

where we have assumed for simplicity that $V \in L^p$, and where the exponents satisfy

$$\begin{cases} \dfrac{n}{p} = 2\delta(r) + 2\delta(s), \\ \dfrac{2}{q} + \dfrac{2}{k} = 1. \end{cases} \tag{3.5}$$

The fact that we use spaces where the time decay appears in the form of an L^q integrability condition in time forces the condition $\vartheta = 0$, so that we are in a critical situation, as was the case for the local Cauchy problem at finite times in the H^1 critical case $p_2 = \frac{n}{4}$.

Since $L^p \subset L^{p_1} + L^{p_2}$, $(1 \le p_2 \le p \le p_1)$, it suffices to consider the two endpoint cases $\frac{n}{p} = 2$ and $\frac{n}{p} = \min(4, n)$.

On the one hand, we can take $\theta = 1$, $\delta(s) = \frac{2}{k}$ and admissible pair $q = r = \frac{2(n+2)}{n}$ (then $\delta(r) = \frac{n}{n+2}, k = n+2$) such that

$$\frac{n}{2p} = \delta(r) + \delta(s) = \frac{n}{n+2} + \frac{2}{k} = 1.$$

On the other hand, for $n \geq 4$, we can take $\theta = 0$, $\delta(s) = 1 + \frac{1}{k}$ and admissible pair $q = r = \frac{2(n+1)}{n-1}$ (then $\delta(r) = \frac{n}{n+1}, k = n+1$) such that

$$\frac{n}{2p} = \delta(r) + \delta(s) = \frac{n}{n+1} + 1 + \frac{1}{k} = 2.$$

For $n = 3$, we can take $\theta = 0$, $k = q = r = 4$, and $\delta(r) = \delta(s) = \frac{3}{4}$ such that $\frac{n}{2p} = \frac{3}{2}$.

The smallness condition which ensures the contraction takes the form

$$\left\| \dot{K}(t)u_{0+} + K(t)u_{1+} \right\|_{L^q(I;B^{1/2}_{r,2})} \leq R_0 \tag{3.6}$$

for some absolute small constant R_0. (3.6) can be ensured by Lemma 2.2. In particular the time T of local resolution cannot be expressed in terms of the $H^1 \times L^2$ norm of (u_{0+}, u_{1+}) alone, as is typical of a critical situation.

The continuity in t_0 up to and including infinity follows from an additional application of the same estimates.

An immediate consequence of the estimates in the proof of Proposition 3.1 is the existence of asymptotic states for solutions of the equation (1.1) in $X^1([T,\infty))$ for some T. Furthermore the conservation law of the energy is easily extended to infinite time for such solutions.

Proposition 3.2. *Let V satisfy (H1) with $p_1 \leq \frac{n}{2}$. Let $T \in \mathbb{R}$, $I = [T,\infty)$ and let $u \in X^1(I)$ be a solution of the equation* (1.1)*. Then*

1. *$(\widetilde{u}, \widetilde{u}_1) \in \mathcal{C}_b\big(I, H^1 \times L^2\big)$. In particular the following limit exists*

$$\big(\widetilde{u}_0(\infty), \widetilde{u}_1(\infty)\big) = \lim_{t\to\infty} \big(\widetilde{u}(t), \widetilde{\dot{u}}(t)\big)$$

   as a strong limit in $H^1 \times L^2$.

2. *u satisfies the equation* (3.3) *with $(u_{0+}, u_{1+}) = \big(\widetilde{u}_0(\infty), \widetilde{u}_1(\infty)\big)$.*

3. *u satisfies the conservation law*

$$E(u) = \frac{1}{2}\left\|\widetilde{u}_1(\infty)\right\|_2^2 + \frac{1}{2}\left\|\nabla\widetilde{u}_0(\infty)\right\|_2^2 + \frac{1}{2}\left\|\widetilde{u}_0(\infty)\right\|_2^2.$$

Proof: Part (1) We estimate for $T \le t_1 \le t_2$

$$\begin{aligned} &\big\|\widetilde{u}(t_2) - \widetilde{u}(t_1)\big\|_{H^1} + \big\|\widetilde{\dot{u}}(t_2) - \widetilde{\dot{u}}(t_1)\big\|_{L^2} \\ \le&\Big\| \int_{t_1}^{t_2} K(t_2 - s) f(u(s)) ds\Big\|_{H^1} \\ \le&\big\|K * f\big\|_{X^1([t_1,t_2])} \end{aligned}$$

and it reduces to the estimate of term $\big\|f(u)\big\|_{L^{q'}\left([t_1,t_2];B^{1/2}_{r',2}\right)}$ with the same choice of exponents as in the proof of Proposition 3.1.

Part (2) follows from Part (1) and from Proposition 3.1, especially part (2).

Part (3) From the conservation law at finite time and from Part (1), it follows that the following limits exist

$$\begin{aligned} \lim_{t\to\infty} F(u) =&E(u) - \frac{1}{2}\lim_{t\to\infty}\big\|\widetilde{\dot{u}}\big\|_2^2 - \frac{1}{2}\lim_{t\to\infty}\big\|\widetilde{\nabla u}\big\|_2^2 - \frac{1}{2}\lim_{t\to\infty}\big\|\widetilde{u}\big\|_2^2 \\ =&E(u) - \frac{1}{2}\big\|\widetilde{u}_1(\infty)\big\|_2^2 - \frac{1}{2}\big\|\nabla\widetilde{u}_0(\infty)\big\|_2^2 - \frac{1}{2}\big\|\widetilde{u}_0(\infty)\big\|_2^2, \end{aligned} \tag{3.7}$$

where

$$F(u) = \frac{1}{4}\int dxdy\big|u(t,x)\big|^2 V(x-y)\big|u(t,y)\big|^2 \in \mathcal{C}\big(\mathbb{R}\big).$$

On the other hand,

$$\big|F(u)\big| \le C\big\|V\big\|_p\big\|u\big\|_r^4 \in L^1_t \tag{3.8}$$

by the Hölder and Young inequalities and by Lemma 2.3 with $q = 4$, $\frac{1}{2} \le \delta(r) \le 1 + \frac{1}{4}$ for $\frac{n}{p} = 4\delta(r)$. It then follows from (3.8) that the limit in (3.7) is zero.

The existence and the properties of the wave operators now follow from the previous local result at infinity and from the global result of Section 2.

Proposition 3.3. *Let V be nonnegative and satisfy (H1) with $p_1 \le \frac{n}{2}$ and $p_2 > \frac{n}{4}$. Then*

1. *For any $\big(u_{0+}, u_{1+}\big) \in H^1 \times L^2$, the equation (3.3) has a unique solution u in $X^1_{loc}(\mathbb{R}) \cap X^1(\mathbb{R}^+)$, such that $\widetilde{u}(t) \in \mathcal{C}\big(\mathbb{R}\cup\{+\infty\}; H^1\big)$. Furthermore u satisfies the conservation law*

$$E(u) = \frac{1}{2}\big\|u_{1+}\big\|_2^2 + \frac{1}{2}\big\|\nabla u_{0+}\big\|_2^2 + \frac{1}{2}\big\|u_{0+}\big\|_2^2,$$

   for all $t \in \mathbb{R}$.

2. *The wave operator $\Omega_+ : (u_{0+}, u_{1+}) \to (u(0), \dot{u}(0))$ is well-defined in $H^1 \times L^2$, and is continuous and bounded in the $H^1 \times L^2$ norm.*

Proof: Part (1) follows immediately from Proposition 2.2, 2.3, 3.1 and 3.2. In Part (2), boundedness of Ω_+ follows from the conservation law of the energy, while the continuity follows from the corresponding statements in Proposition 2.1 and 3.1.

4 Scattering Theory II: Asymptotic completeness

In this section, we continue the study of the theory of scattering for the equation (1.1) by addressing the second question, namely the asymptotic completeness holds in the energy space $H^1 \times L^2$ for radial and suitably repulsive potentials. In view of the result of Section 3, especially Proposition 3.2, it will turn out that the crux of the argument consists in showing that the global solutions of the equation (1.1) in $X^1_{loc}(\mathbb{R})$ constructed in Proposition 2.3 actually belong to $X^1(\mathbb{R})$, namely exhibit the time decay properties contained in the definition of that space. To this end we have to use the strategy of frequency decomposition and the method in [18],[10] which base on two basic facts. The first one is the finiteness of the propagation speed. The second fact follows from the Morawetz-type estimate, which is closely related to the approximate dilation invariance of the equation. Space time is split into an internal and an external region where $|x|$ is small or large respectively as compared with $|t|$. For radial repulsive potentials according to the assumption (H2), the Morawetz-type estimate implies an a priori estimate for a suitable norm of the internal part of u. One use that estimate in the internal and the propagation estimate in the external region. Plugging those estimates into the integral equation for the solution u, we prove successively that a suitable norm of u is small in large intervals and tends to zero at infinity and that u belongs to $X^1(\mathbb{R})$ for some $\theta \in [0, 1]$.

We continue to restrict our attention to positive time. We first state an elementary property of $H^1 \times L^2$ solutions of the free Klein-Gordon equation.

Lemma 4.1. *Let $(u_0, u_1) \in H^1 \times L^2$ and $2 < r \leq 2^*$. Then $\dot{K}(t)u_0 + K(t)u_1$ tends to zero in L^r norm when $|t| \to \infty$.*

Proof: It suffices to prove for $f \in H^1$

$$\big\|e^{it\omega} f\big\|_r \longrightarrow 0, \qquad \text{as} \quad |t| \longrightarrow \infty.$$

We approximate f in H^1 norm by $g \in B^{(n+2)(\frac{1}{2}-\frac{1}{r})}_{r',2} \cap H^1$. By Lemma 2.1 and the unit property in H^1, we estimate

$$\begin{aligned}\big\|e^{it\omega} f\big\|_r \leq & \big\|e^{it\omega} g\big\|_r + C\big\|f-g\big\|_2^{1-\delta(r)}\big\|\nabla(f-g)\big\|_2^{\delta(r)} \\ \leq & C|t|^{-\delta(r)}\big\|g\big\|_{B^{(n+2)(\frac{1}{2}-\frac{1}{r})}_{r',2}} + C\big\|f-g\big\|_{H^1},\end{aligned}$$

from which we easily obtain the result as $|t| \longrightarrow \infty$.

We are now in a position to prove the finiteness of the propagation speed in the form of local energy conservation. For any open ball $\Omega = B(x, R)$ of center x and radius R in $\mathbb{R}^n$, for any $t \in \mathbb{R}$, we define $\Omega_{\pm}(t) = B(x, R \pm |t|)$, with the convention that $B(x, R)$ is empty if $R \leq 0$. For any measurable set $\Omega \subset \mathbb{R}^n$, for any $(u_0, u_1) \in H^1 \times L^2$, we define

$$E(u_0, u_1; \Omega) = \frac{1}{2}\int_\Omega dx\big(|u_1|^2 + |\nabla u_0|^2 + |u_0|^2\big) + \frac{1}{4}\int_\Omega dx \int dy\big|u_0(x)\big|^2 V(x-y)\big|u_0(y)\big|^2.$$

Lemma 4.2. *Let V satisfy (H1). Let u be a finite energy solution of the equation* (1.1). *Then for any open ball $\Omega \subset \mathbb{R}^n$, for any $t \in \mathbb{R}$, the following inequalities hold:*

$$E(u(t), \dot{u}(t); \Omega_-(t)) \leq E(u(0), \dot{u}(0); \Omega), \tag{4.1}$$

and

$$E(u(t), \dot{u}(t); \Omega_+^c(t)) \leq E(u(0), \dot{u}(0); \Omega^c), \tag{4.2}$$

where the subscrip c denotes the complement in $\mathbb{R}^n$.

Proof: It suffices to prove the estimate for C^2 solution by approximation.

Without loss of generality, we can assume that $\Omega = B(0, R)$ and that t is positive. The formal proof of (4.1) proceeds as follows. Define

$$\begin{aligned} l(u) =& \frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + \frac{1}{4}\int dy \big|u(t,x)\big|^2 V(x-y)\big|u(t,y)\big|^2; \\ \overrightarrow{M}(u) =& -\ \operatorname{Re}(\dot{\bar{u}}\nabla u). \end{aligned}$$

Then

$$\dot{l}(u) + \nabla \cdot \overrightarrow{M}(u) = 0.$$

Integrating this equality over the region

$$Q(\Omega, t) = \big\{(t', x') \in \mathbb{R}^{n+1} : 0 \leq t' < t \text{ and } x' \in \Omega_-(t')\big\},$$

Applying Gauss's theorem and taking into account the fact that the vector (l, M) in $\mathbb{R}^{n+1}$ is time-like and outgoing on the side surface of $Q(\Omega, t)$, we obtain (4.1). For more detail, please refer to [12] or [13].

The inequality (4.2) follows immediately from (4.1), from the conservation of the energy and the reversibility in time of the equation.

For any function u of space time and for $t \geq 1$, we define

$$u_{\gtrless}(t, x) = u(t, x)\chi(|x| \gtrless 2t), \tag{4.3}$$

so that $u = u_< + u_>$. As a consequence of the above lemma, we have

Corollary 4.1. *Let V satisfy (H1). Let u be a finite energy solution of the equation* (1.1). *Then*

$$\big\|u_>(t)\big\|_2 \to 0, \quad \text{as } t \to \infty.$$

Proof: In fact, It follows from Lemma 4.2 that for each t, we have

$$\begin{aligned} \big\|u_>(t)\big\|_2^2 = \int_{|x|\geq 2t} \big|u(t,x)\big|^2\, dx &\leq E(u(t), \dot{u}(t), B^c(0, 2t)) \\ &\leq E(u(0), \dot{u}(0), B^c(0, t)), \end{aligned}$$

from which, the conservation of the energy and the Lebesgue dominated convergence theorem we obtain the result.

The second main ingredient of the proof is the Morawetz inequality, which for the equation (1.1) can be written as following

Proposition 4.1. *Let V satisfy (H1). Let u be a finite energy solution of the equation* (1.1). *Then for any s and t in $\mathbb{R}$, $s \le t$, u satisfies the inequality*

$$-\int_s^t \int |u(\tau,x)|^2 \frac{x}{|x|} \cdot (V * \nabla |u|^2) \; dx d\tau \le 4 \|\dot{u}\|_{L^\infty L^2} \|\nabla u\|_{L^\infty L^2}. \tag{4.4}$$

Proof: For the sake of completeness, we give a complete proof to the above Morawetz estimate even if it is similar to the case of the Hartree equation. We first need derive the corresponding result at the available level of regularity, we introduce the same regularization as in the proof of energy conservation in Proposition 2.2, and we then let φ tend to the Dirac distribution δ. Using the fact that $u_\varphi = \varphi * u \in \mathcal{C}^1\big(\mathbb{R}, H^k\big)$, and let $Mu_\varphi = (h \cdot \nabla + \frac{\nabla \cdot h}{2}) u_\varphi$, we obtain

Proof: Let $Mu_\varphi = h \cdot \nabla u_\varphi + \frac{1}{2}(\nabla \cdot h) u_\varphi$, we obtain

$$\begin{aligned}
&\frac{d}{dt} \operatorname{Re} < \dot{u}_\varphi, Mu_\varphi > \\
&= \operatorname{Re} < \ddot{u}_\varphi, Mu_\varphi > + \operatorname{Re} < \dot{u}_\varphi, M\dot{u}_\varphi > \\
&= \operatorname{Re} < \Delta u_\varphi, Mu_\varphi > - \operatorname{Re} < u_\varphi, Mu_\varphi > + \operatorname{Re} < \varphi * f(u), Mu_\varphi > + \operatorname{Re} < \dot{u}_\varphi, M\dot{u}_\varphi > \\
&= \operatorname{Re} < \Delta u_\varphi, Mu_\varphi > - \operatorname{Re} < u_\varphi, Mu_\varphi > + \operatorname{Re} < f(u_\varphi), Mu_\varphi > + \operatorname{Re} < \dot{u}_\varphi, M\dot{u}_\varphi > \\
&\quad + \operatorname{Re} < \varphi * f(u) - f(u_\varphi, Mu_\varphi > \\
&\quad =: I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned}$$

Let $\rho_\varphi = |u_\varphi|^2$, we consider the first four terms, respectively.

$$\begin{aligned}
I_2 &= \operatorname{Re} \int u_\varphi \overline{Mu}_\varphi dx = \operatorname{Re} \int u_\varphi h \cdot (\nabla \bar{u}_\varphi) dx + \frac{1}{2} \int (\nabla \cdot h) |u_\varphi|^2 dx \\
&= \frac{1}{2} \int \Big[h \cdot \nabla |u_\varphi|^2 + (\nabla \cdot h)|u_\varphi|^2 \Big] dx = \frac{1}{2} \int \Big[-(\nabla \cdot h)|u_\varphi|^2 + (\nabla \cdot h)|u_\varphi|^2 \Big] dx = 0
\end{aligned}$$

Arguing similarly in deriving I_2, we have $I_4 = 0$.

Making use of the following equalities

$$\begin{aligned}
\operatorname{Re} \int (h \cdot \nabla \bar{u}_\varphi)(V * \rho_\varphi) u_\varphi dx &= \frac{1}{2} \int (h \cdot \nabla |u_\varphi|^2)(V * \rho_\varphi) \\
&= -\frac{1}{2} \int (\nabla \cdot h)(V * \rho_\varphi) \rho_\varphi dx - \frac{1}{2} \int h \cdot (V * \nabla \rho_\varphi) \rho_\varphi dx,
\end{aligned}$$

it follows that

$$I_3 = -\frac{1}{2}\int \rho_\varphi h \cdot (V * \nabla \rho_\varphi) dx.$$

Now we are in position to consider I_1. Since

$$\begin{aligned}
\mathrm{Re}\int \Delta u_\varphi h \cdot \nabla \bar{u}_\varphi dx &= -\mathrm{Re}\int \nabla u_\varphi \cdot \nabla (h \cdot \nabla \bar{u}_\varphi) dx \\
&= -\mathrm{Re}\int \partial_i u_\varphi \cdot \partial_i (h_j \cdot \partial_j \bar{u}_\varphi) dx \\
&= -\mathrm{Re}\int \partial_i u_\varphi \cdot \partial_i h_j \cdot \partial_j \bar{u}_\varphi dx - \mathrm{Re}\int \partial_i u_\varphi \cdot h_j \cdot \partial_i \partial_j \bar{u}_\varphi \\
&= -\mathrm{Re}\int \partial_i u_\varphi \cdot \partial_i h_j \cdot \partial_j \bar{u}_\varphi dx - \frac{1}{2}\int h_j \cdot \partial_j (|\nabla u_\varphi|^2) \\
&= -\mathrm{Re}\int \partial_i u_\varphi \cdot \partial_i h_j \cdot \partial_j \bar{u}_\varphi dx + \frac{1}{2}\int (\nabla \cdot h) |\nabla u_\varphi|^2
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Re}\int \Delta u_\varphi \frac{1}{2}(\nabla \cdot h) \bar{u}_\varphi dx &= -\frac{1}{2}\mathrm{Re}\int \nabla u_\varphi (\nabla \cdot h) \nabla \bar{u}_\varphi dx - \frac{1}{2}\mathrm{Re}\int \nabla u_\varphi \cdot \nabla (\nabla \cdot h) \bar{u}_\varphi dx \\
&= -\frac{1}{2}\int (\nabla \cdot h) |\nabla u_\varphi|^2 dx - \frac{1}{4}\int \nabla |u_\varphi|^2 \nabla (\nabla \cdot h) dx \\
&= -\frac{1}{2}\int (\nabla \cdot h) |\nabla u_\varphi|^2 dx + \frac{1}{4}\int |u_\varphi|^2 \Delta (\nabla \cdot h) dx.
\end{aligned}$$

we derive that

$$I_1 = -\mathrm{Re}\int \partial_i u_\varphi \cdot \partial_i h_j \cdot \partial_j \bar{u}_\varphi dx + \frac{1}{4}\int |u_\varphi|^2 \Delta (\nabla \cdot h) dx.$$

We next choose $a(x) = \big(|x|^2 + |\sigma|^2\big)^{1/2}$ for some $\sigma > 0$, and verify by simple computation

$$h(x) = \nabla a = \frac{x}{\big(|x|^2 + |\sigma|^2\big)^{1/2}}, \quad h_j(x) = \frac{x_j}{\big(|x|^2 + |\sigma|^2\big)^{1/2}}.$$

Further,

$$\partial_i h_j = \nabla_{ij}^2 a = a^{-1}\big(\delta_{ij} - a^{-2} x_i x_j\big)$$

is a positive matrix, and

$$\Delta (\nabla \cdot h) = \Delta^2 a = -(n-1)(n-3) a^{-3} - 6(n-3)\sigma^2 a^{-5} - 15\sigma^4 a^{-7}$$

is negative. Hence, we have by taking integration with respect to time t

$$\mathrm{Re} < \dot{u}_\varphi, M u_\varphi > \big|_s^t \geq -\frac{1}{2}\int_s^t \int \rho_\varphi h \cdot (V * \nabla \rho_\varphi) dx d\tau + \int_s^t I_5 d\tau \tag{4.5}$$

In addition, for I_5 it boils down to

$$\big| < h \cdot \nabla u, f(u) > \big| \leq C \big\|h\big\|_\infty \big\|V\big\|_p \big\|\nabla u\big\|_2 \big\|u\big\|_s^3, \tag{4.6}$$

with $\delta(s) = \frac{n}{3p} \leq \frac{4}{3}$, so that the RHS of (4.6) belongs to L^1_{loc} in time and I_5 will converge to 0 when φ to $\delta(x)$ by the Lebesgue dominated theorem and (4.6).

Now we take the harmless limit $\sigma \to 0$ and φ to $\delta(x)$ by the Lebesgue dominated theorem in (4.5) to obtain

$$-\int_s^t \int |u(\tau, x)|^2 \frac{x}{|x|} \cdot (V * \nabla |u|^2) \; dx d\tau \leq 2\text{Re} < \dot{u}, Mu > |_s^t.$$

This shows (4.4).

In the same way as in [10], the estimate of Proposition 4.1 will be used through its following consequence. The assumption on V made so far are not stronger than those made in Section 2. In order to proceed further, we need to exploit the fact the LHS of (4.4) controls some suitable norm of u. For that purpose we need the repulsive condition $(H2)$ on V (see Introduction).

In order to exploit the Morawetz inequality (4.4), we shall need the following spaces. Let $\sigma > 0$ and let Q_i be the cube with edge σ centered at $i\sigma$ where $i \in \mathbb{Z}^n$ so that $R^n = \cup_i Q_i$. Let $1 \leq r, m \leq \infty$. We define

$$l^m(L^r) = \big\{ u \in L^r_{loc} : \big\|u\big\|_{l^m(L^r)} = \big\| \|u\|_{L^r(Q_i)} \big\|_{l^m} \big\}.$$

The spaces $l^m(L^r)$ do not depend on σ, and different values of σ yield equivalent norms. The previous spaces have been introduced by Birman and Solomjak [2]. They allow for an independent characterization of local regularity and of decay at infinity in terms of integrability properties. The Hölder and Young inequalities hold in those spaces, with the exponents m and r treated independently.

Proposition 4.2. *Let V satisfy (H1) with $\frac{n}{4} < p_2 \leq p_1 < \infty$ and (H2). Let $u \in X^1_{loc}(\mathbb{R})$ be a finite energy solution of the equation* (1.1). *Then, for any $t_1, t_2 \in \mathbb{R}$ with $1 \leq t_1 \leq t_2$, the following estimate holds*

$$\int_{t_1}^{t_2} \frac{1}{2t + a} \big\|u_<(t)\big\|^{\alpha+4}_{l^{\alpha+4}(L^2)} \; dt \leq C \frac{E^{1+\frac{\alpha}{2}}}{A_\alpha}, \tag{4.7}$$

where $u_<$ is defined by (4.3) *and C depends only on n, α and a.*

Proof: We only need to note that the definition of $u_<(t)$ and replacing $\|u\|_2$ by $\|\dot{u}\|_2$. From this, the proof is the same as that of the corresponding result for the Hartree equation (see Proposition 4.2 in [10]).

The basic estimate (4.7) is not convenient for the direct application to the integral equation, now we give a more usable consequence.

Corollary 4.2. *Let V satisfy (H1) with $\frac{n}{4} < p_2 \le p_1 < \infty$ and (H2). Let $u \in X^1_{loc}(\mathbb{R})$ be a finite energy solution of the equation* (1.1)*. Then for any $t_1 \ge 1$, any $\varepsilon > 0$ and any $l \ge a$, there exists $t_2 \ge t_1 + l$ such that*

$$\int_{t_2 - l}^{t_2} \left\| u_<(t) \right\|_{l^{\alpha+4}(L^2)}^{\alpha+4} \, dt \le \varepsilon.$$

We can find such a t_2 satisfying

$$t_2 \le e^{\frac{(2+a)Ml}{\varepsilon}} (t_1 + l + 1) - 1, \tag{4.8}$$

where M is the RHS of (4.7)*, namely*

$$M = C \frac{E^{1+\frac{\alpha}{2}}}{A_\alpha}.$$

Proof: The proof is the same as that of Lemma 4.4 in [10]. Let N be a positive integer. From (4.7) we obtain

$$M \ge \sum_{j=1}^{N} \frac{K_j}{2(t_1 + jl) + a},$$

where

$$K_j = \int_{t_1+(j-1)l}^{t_1+jl} \left\| u_<(t) \right\|_{l^{\alpha+4}(L^2)}^{\alpha+4} \, dt.$$

If $K_j \ge \varepsilon$ for $1 \le j \le N$, then

$$\begin{aligned} M \ge & \varepsilon \sum_{j=1}^{N} \frac{1}{2(t_1 + jl) + a} \\ \ge & \frac{\varepsilon}{l} \int_{t_1+l}^{t_1+(N+1)l} \frac{1}{2t + a} \, dt \\ \ge & \frac{\varepsilon}{(2+a)l} \log \frac{t_1 + (N+1)l + 1}{t_1 + l + 1}, \end{aligned}$$

which is an upper bound on N, namely

$$t_1 + (N+1)l \le e^{\frac{(2+a)Ml}{\varepsilon}} (t_1 + l + 1) - 1.$$

For the first N not satisfying that estimate, there is a j with $K_j \le \varepsilon$ and we can take $t_2 = t_1 + jl$ for that j. That t_2 is easily seen to satisfy (4.8).

We turn to exploit the estimates of Corollary 4.1 and Corollary 4.2 together with the integral equation for u to prove that for $2 < r < 2^*$, the L^r norm of u is small in large intervals. The idea of the proof partially follows the version given in [7] for the NLS equation, in [8] for the NLKS and NLS equations, in [10] for the Hartree equation.

Now we devote attention to deal with the difficulty which is related with the dispersive estimates for the Klein-Gordon equation. It is well known that the dispersive estimates play an crucial role in the proof of time decay property, besides the Morawetz inequality and the finite propagation speed. Comparing with the Schrödinger or wave equation, the dispersive $L^p - L^q$ estimates for the Klein-Gordon equation are more complicated, which turn out to be mixture of the dispersive properties of the relativistic and non-relativistic equations. We can exploit the high and low frequency decomposition to distinguish those to derive desired results.

Firstly, let us fix a real-valued radially symmetric bump function $\varphi(\xi) \in C_0^\infty(\mathbb{R}^n)$ adapted to the ball $\{\xi \in \mathbb{R}^n : \ |\xi| \leq 2\}$ which equals 1 on the ball $\{\xi \in \mathbb{R}^n : \ |\xi| \leq 1\}$. For each $N > 1$, we define the Fourier multiplier

$$\widehat{P_{\leq N} f}(\xi) := \varphi(\xi/N)\hat{f}(\xi), \qquad \widehat{P_{>N} f}(\xi) := (1-\varphi(\xi/N))\hat{f}(\xi).$$

The projection operators $P_{\leq N},\ P_{>N}$ can commute with the groups $K(t),\ \dot{K}(t)$. Then, Lemma 2.1 can be restated as follows.

Lemma 4.3. *Let $\frac{n}{n+2} \leq \delta(r) \leq \frac{n}{2}$. Then we have the following estimates*

(1) Dispersive estimates in low frequency

$$\big\|K(t)P_{\leq N} f\big\|_r \leq C\mu(t)N^s\big\|f\big\|_{r'},\ t > 0, \tag{4.9}$$

where

$$\mu(t) = \min\big(|t|^{-(n-2)(\frac{1}{2}-\frac{1}{r})},\ |t|^{-\delta(r)}\big),\ s = (n+2)(\frac{1}{2}-\frac{1}{r}) - 1 \geq 0;$$

(2) Dispersive estimates in high frequency

$$\big\|K(t)P_{>N} f\big\|_r \leq Ct^{-(n-1)(\frac{1}{2}-\frac{1}{r})}\big\|P_{>N} f\big\|_{W^{(n+1)(\frac{1}{2}-\frac{1}{r})-1,r'}}\ , t > 1. \tag{4.10}$$

Remark 4.1. *Recall $K(t) = \frac{U(t)-U(-t)}{2i\omega}$, one easily verifies that Lemma 4.3 is just a direct consequence of the Bernstein inequality and Lemma 2.1 with $\theta = 1, 0$. The condition $\frac{n}{n+2} \leq \delta(r)$ ensures the order of derivative $s \geq 0$. This lemma reflects the fact that for long time the solution of the Klein-Gordon equation disperses as the Schrödinger equation in low frequency (non-relativistic) regions and disperses as the wave equation in high frequency (relativistic) regions, and for short time it has better time decay than the Schrödinger equation in low frequency regions.*

For the sake of convenience, we define

$$f_{\gtrless}(u) := \big(V * |u_{\gtrless}|^2\big)u, \quad u_2^{\gtrless}(t) := -\int_{t-l_2}^{t} K(t-s)f_{\gtrless}(u(s))ds,$$

where $u_{\gtrless}$ is the same as (4.3). One easily shows that

$$|f(u)| \leq 2\big(|f_>(u)| + |f_<(u)|\big).$$

For convenient reference, we need one useful nonlinear estimate due to Ginibre and Velo (cf (4.59) in [10]), the proof is omitted here.

Lemma 4.4. *[10] Let V satisfy (H1) with $2 < \frac{n}{p_1} \le \frac{n}{p_2} < \min\big(n, 4\big)$ and (H2), there exist r with $\frac{n}{n+1} \le \delta(r) < 1$ such that the internal estimate*

$$\|f_< u(t)\|_{r'} \le M \|u_<(t)\|^{m/\beta}_{l^m(L^2)}, \quad m = \alpha + 4 \tag{4.11}$$

holds for some β with $[1 - (n-2)(\frac{1}{2} - \frac{1}{r})]\beta > 1$ and for all $t \ge 1$.

Proposition 4.3. *Let V satisfy (H1) with $1 < \frac{n}{p_1} \le \frac{n}{p_2} < \min\big(n, 4\big)$ and (H2). Let $u \in X^1_{loc}(\mathbb{R})$ be a finite energy solution of the equation (1.1). Let $2 < r < 2^*$, then for any $\varepsilon > 0$, and any $l_1 > 0$, there exists $t_2 \ge l_1$ such that*

$$\big\|u\big\|_{L^\infty\big([t_2 - l_1, t_2], L^r\big)} < \varepsilon. \tag{4.12}$$

Proof: Since $u \in L^\infty(\mathbb{R}, H^1)$, it is sufficient to derive the result for one value of r with $2 < r < 2^*$, $\delta(r)$ sufficiently close to 1. The result for general r will then follow by interpolation with uniform boundedness in H^1.

For further reference, we note that for any $s_1, s_2, t \in \mathbb{R}$,

$$\begin{aligned}\mathcal{N}(s_1, s_2, t) :=& \int_{s_1}^{s_2} K(t-s) f(u(s)) ds \\ =& \dot{K}(t - s_2)u(s_2) + K(t-s_2)\dot{u}(s_2) - \dot{K}(t-s_1)u(s_1) - K(t-s_1)\dot{u}(s_1),\end{aligned}$$

so that

$$\big\|\mathcal{N}(s_1, s_2, t)\big\|_{H^1} \le 2\big(\big\|u\big\|_{H^1} + \|\dot{u}\|_2\big) \le 2\sqrt{E}. \tag{4.13}$$

Decay of the high frequency part.

Since

$$\big\| <\xi> \widehat{P_{>N}u(t)}(\xi)\big\|_{L^2} = \|P_{>N}u(t)\|_{H^1} \le \|u(t)\|_{H^1} \le \sqrt{E},$$

we have

$$\big\|P_{>N}u(t)\|_{L^2} \le \sqrt{E}N^{-1}.$$

By interpolation, one has for each $\varepsilon > 0$, there exists N_0 sufficiently large such that

$$\|P_{>N_0}u(t)\|_{L^r} \le \|P_{>N_0}u(t)\|^{1-\delta}_{L^2} \|P_{>N_0}u(t)\|^{\delta}_{L^{2^*}} \le C\sqrt{E}N_0^{\delta - 1} < \varepsilon/4. \tag{4.14}$$

for some $0 \le \delta = \delta(r) < 1$ and all $t > 0$.

Decay of the low frequency part.

For technical reasons, we also introduce an r_1 with $\delta(r_1) > 1$, which will have to satisfy various compatible conditions.

Now ε be given as above, and fix N_0. Now we deal with the low frequency regions . For each l_1, we introduce $l_2 \ge 1, t_1 > 0$ and $t_2 \ge t_1 + l$ where $l = l_1 + l_2$, to be chosen later; l_2 and t_1 will have to be sufficiently large, depending on ε but not on l_1 for given u. We split the integral equation for $P_{\le N_0}u(t)$ with $t \in [t_2 - l_1, t_2]$ as follows

$$\begin{aligned} P_{\le N_0}u(t) =& \dot{K}(t)P_{\le N_0}u(0) + K(t)P_{\le N_0}\dot{u}(0) \\ & - \Big(\int_0^{t-l_2} + \int_{t-l_2}^{t} \Big) K(t-s)P_{\le N_0} f(u(s))ds \\ =:& \, u^{(0)}(t) + u_1(t) + u_2(t). \end{aligned}$$

We estimate the various terms in L^r successively. In all proof, M denotes various constants, depending only on r, r_1 and $E(u)$, possibly varying from one estimate to the next.

Estimate of $u^{(0)}(t)$

It follows from Lemma 4.1 that

$$\big\|\dot{K}(t)u_0 + K(t)u_1\big\|_{L^r} \to 0$$

when $t \to \infty$, so that for $t > t_2 - l_1 \ge t_1 + l_2 > l_2$

$$\big\|u^{(0)}(t)\big\|_{L^r} \le \big\|\dot{K}(t)u_0 + K(t)u_1\big\|_{L^r} < \frac{\varepsilon}{4}, \tag{4.15}$$

for l_2 sufficiently large depending on ε.

Estimate of $u_1(t)$ We estimate by the Hölder inequality

$$\big\|u_1\big\|_r \le \big\|u_1\big\|_2^{1-\frac{\delta}{\delta_1}} \big\|u_1\big\|_{r_1}^{\frac{\delta}{\delta_1}}, \tag{4.16}$$

where $\delta = \delta(r)$, $\delta_1 = \delta(r_1)$. The L^2 norm of u_1 is estimated by (4.13). Let $C_{N_0,r_1} := CN_0^{(n+2)(\frac{1}{2}-\frac{1}{r_1})-1}$, the L^{r_1} norm is estimated by the use of Lemma 4.3 as

$$\begin{aligned} \big\|u_1(t)\big\|_{r_1} \le& C_{N_0,r_1} \int_0^{t-l_2} (t-s)^{-\delta(r_1)} \big\|f(u(s))\big\|_{r_1'} \, ds \\ \le& C_{N_0,r_1} \frac{l_2^{1-\delta(r_1)}}{\delta(r_1)-1} \big\|f(u)\big\|_{L^\infty\left(\mathbb{R},L^{r_1'}\right)} \\ \le& C_{N_0,r_1} M \frac{l_2^{1-\delta(r_1)}}{\delta(r_1)-1}. \end{aligned} \tag{4.17}$$

Here we used the fact

$$\big\|f(u)\big\|_{L^\infty\left(\mathbb{R},L^{r_1'}\right)} \le M \tag{4.18}$$

Indeed, by Hölder and Young's inequalities, one has

$$\big\|f(u)\big\|_{L^{r_1'}} \le \|V\|_p \|u\|_{L^{r_2}}^3 \le \|V\|_p \|u\|_{H^1}^3$$

with $\delta(r_1) + 3\delta(r_2) = \frac{n}{p}$. Thus for any $1 < \frac{n}{p} < \infty$, we can take admissible r_1, r_2 such that $0 \le \delta(r_2) \le 1 < \delta(r_1)$.

Furthermore, by (4.16), (4.17), we can ensure that for $l_2 < t_2 - l_1 \le t \le t_2$

$$\|u_1\|_r \le C_{N_0,r_1} M l_2^{\frac{\delta}{\delta_1}(1-\delta(r_1))} < \frac{\varepsilon}{4} \tag{4.19}$$

for l_2 sufficiently large depending on ε. We now choose $l_2 = l_2(\varepsilon)$ so as to ensure both (4.15) and (4.19).

Fix l_2, we now turn to the estimate of $u_2(t)$. Here we need to consider the contributions of internal and external regions separately.

Contribution of the external region of u_2

By the similar computation as in (4.17), note that better dispersive effect in short time we have

$$\begin{aligned}\|u_2^>(t)\|_r \le & C N_0^{(n+2)(\frac{1}{2}-\frac{1}{r})-1} \Big\{ \int_{t-l_2}^{t-1} (t-s)^{-\delta(r)} + \int_{t-1}^{t} (t-s)^{-(n-2)(\frac{1}{2}-\frac{1}{r})} \Big\} \|f_>(u(s))\|_{L^{r'}}\, ds \\ \le & C_{N_0,l_2,r} \|f_>(u(s))\|_{L^\infty\left([t-l_2,t],L^{r'}\right)}.\end{aligned}$$

holds for $\frac{n}{n+2} \le \delta(r) < 1$, where $C_{N_0,l_2,r}$ is a finite constant depending on N_0, l_2, r since $(n-2)(\frac{1}{2}-\frac{1}{r}) < 1$.

By the Hölder and Young inequalities again, we get

$$\|f_>(u(s))\|_{L^{r'}} \le \|V\|_p \|u\|_{r_2} \|u_>\|_{r_3}^2 \le \|V\|_p \|u\|_{r_2} \|u_>\|_2^\sigma \|u\|_{2^*}^{2-\sigma}, \tag{4.20}$$

where $\delta(r) + \delta(r_2) + 2\delta(r_3) = \frac{n}{p}$.

For each p with $1 < \frac{n}{p} < 4$, we can take suitable r, r_2, r_3 such that $\frac{n}{n+2} \le \delta(r) \le 1, 0 \le \delta(r_2) \le 1$, $0 \le \delta(r_3) < 1$ and $0 < \sigma \le 2$.

By the finiteness of the propagation speed, that is, Corollary 4.1, we can ensure that the contribution of the external region to $\|u_2(t)\|_r$ satisfies

$$\|u_2^>(t)\|_r \le C_{N_0,l_2,r} M \|u_>\|_{L^\infty([t_2-l,t_2],L^2)}^\sigma < \frac{\varepsilon}{4}, \tag{4.21}$$

for all $t \in [t_2 - l_1, t_2]$ by taking t_1 sufficiently large depending on ε, since we have imposed $t_2 \ge t_1 + l = t_1 + l_1 + l_2$. We now choose $t_1 = t_1(\varepsilon)$ such that (4.21) holds.

Contribution of the internal region of u_2

By (4.9), Hölder inequality and (4.11), we get

$$\begin{aligned}\|u_2^<(t)\|_r \le & C_{N_0,r} \Big(\int_{t-l_2}^{t-1} (t-s)^{-\delta(r)} + \int_{t-1}^{t} (t-s)^{-(n-2)(\frac{1}{2}-\frac{1}{r})} \Big) \|f_<(u(s))\|_{L^{r'}}\, ds \\ \le & C_{N_0,l_2,r} M \left(\int_{t-l_2}^{t} \|u_<(s)\|_{l^m(L^2)}^m\, ds \right)^{1/\beta},\end{aligned}$$

where $C_{N_0,l_2,r}$ is a finite constant since $(n-2)(\frac{1}{2}-\frac{1}{r})\beta' < 1$.

For l_2, t_1 given above, applying Corollary 4.2 to conclude that there exists $t_2 \geq t_1 + l = t_1 + l_1 + l_2$ such that the contribution of the external region to $\left\|u_2(t)\right\|_r$ satisfies

$$\left\|u_2^{<}(t)\right\|_r \leq C_{N_0,l_2,r} M\Big(\int_{t_2-l}^{t_2} \left\|u_<(s)\right\|_{l^m(L^2)}^m \, ds\Big)^{\frac{1}{\beta}} < \frac{\varepsilon}{4}, \tag{4.22}$$

for all $t \in [t_2 - l_1, t_2]$.

Collecting (4.15), (4.19), (4.21) and (4.22) yields (4.12).

Indeed, the arguments above have showed the more delicate consequence.

Corollary 4.3. *Under the conditions in Proposition 4.3, then for any $\rho > 0$ and any $\varepsilon > 0$ there exists $N_\varepsilon > 1$ sufficient large such that*

$$\left\|P_{>N_\varepsilon} u(t)\right\|_{L^r} < M N_\varepsilon^{\delta-1} =: \varepsilon^\rho, \qquad t > 0 \tag{4.23}$$

and for any $l_1 > 0$ there exists $t_2 \geq l_1$ such that

$$\left\|P_{\leq N_\varepsilon} u\right\|_{L^\infty\left([t_2-l_1,t_2],L^r\right)} < \varepsilon \tag{4.24}$$

hold for $2 < r < 2^*$.

We observe an useful fact that the estimates on the high frequency holds uniformly in $t > 0$, hence the iteration scheme will only be carried on the low frequency part which is key ingredient of the following proof.

Proposition 4.4. *Let V satisfy (H1) with $2 < \frac{n}{p_1} \leq \frac{n}{p_2} < \min(4, n)$ and (H2). Let $u \in X^1_{loc}(\mathbb{R})$ be a finite energy solution of the equation (1.1). Let $2 < r < 2^*$. Then $\left\|u(t)\right\|_{L^r}$ tends to zero when $t \to \infty$.*

Proof: As (4.23) and (4.24) hold, it suffices to show that for given ε and N_ε above, and for some l_1 sufficiently large (depending on u, ε and N_ε), then there exists $t_2 > l_1$ such that

$$\left\|P_{\leq N_\varepsilon} u\right\|_{L^\infty\left([t_2-l_1,\ \infty),L^r\right)} < \varepsilon. \tag{4.25}$$

Here we use the bootstrap argument.

Since (4.24) holds and the map $t \mapsto \|u(t)\|_{L^r}$ is continuous, we can assume that there exists t_0 with $t_2 \leq t_0 < \infty$ such that

$$\|P_{\leq N_\varepsilon} u(t)\|_{L^\infty\left([t_2-l_1,t_0),L^r\right)} < \varepsilon, \tag{4.26}$$

and

$$\|P_{\leq N_\varepsilon} u(t_0)\|_{L^r} = \varepsilon. \tag{4.27}$$

We write the integral equation for $P_{\le N_\varepsilon}u(t_0)$ as follows

$$\begin{aligned}P_{\le N_\varepsilon}u(t_0) =&\{\dot K(t_0)P_{\le N_\varepsilon}u(0)+K(t_0)P_{\le N_\varepsilon}u_t(0)\}\\ &\qquad -\Big(\int_0^{t_0-l_1}+\int_{t_0-l_1}^{t_0-1}+\int_{t_0-1}^{t_0}\Big)K(t_0-s)P_{\le N_\varepsilon}f(u(s))ds\\ &=: u^{(0)}+u_1+u_2+u_3.\end{aligned}$$

In the same way as in Proposition 4.3, we can ensure that

$$\big\|u^{(0)}\big\|_{L^r}<\frac{\varepsilon}{4}, \tag{4.28}$$

and

$$\big\|u_1\big\|_r\le MC_{N_\varepsilon}l_1^{\frac{\delta}{\delta_1}(1-\delta_1)}<\frac{\varepsilon}{4}, \tag{4.29}$$

for $t_0\ge t_2>l_1$, l_1 sufficiently large depending on u, ε and N_ε.

By the interpolation, one has

$$\big\|u_2\big\|_r\le\big\|u_2\big\|_2^{1-\frac{\delta}{\delta_1}}\big\|u_2\big\|_{r_1}^{\frac{\delta}{\delta_1}}. \tag{4.30}$$

For $\frac{n}{n+1}<\delta(r)<1$ and $\delta_1>1$, by the similar argument as driving (4.17), we estimate

$$\begin{aligned}\big\|u_2\big\|_r\le&\frac{CN_\varepsilon^{(n+2)(\frac12-\frac{1}{r_1})-1}}{\delta(r_1)-1}\big\|f(u(s))\big\|_{L^\infty\big([t_0-l_1,t_0],L^{r_1'}\big)},\\ \big\|u_3\big\|_r\le&\frac{CN_\varepsilon^{(n+2)(\frac12-\frac{1}{r})-1}}{1-\delta(r)}\big\|f(u(s))\big\|_{L^\infty\big([t_0-l_1,t_0],L^{r'}\big)}.\end{aligned}$$

Here, we need the following inequalities

$$\|f(u)\|_{r_1{}'}^{\delta/\delta_1}\le M\big\|u\big\|_{L^r}^{1+\nu}, \tag{4.31}$$

$$\|f(u)\|_{r'}\le M\big\|u\big\|_{L^r}^{1+\nu} \tag{4.32}$$

valid for some $\nu>0$, which is the same as in [10], so we sketch the proof.

In fact, we can take $2<2\delta_1<\frac{n}{p}<4\delta<4$ and $\delta,\ \delta_1$ both close to 1 such that

$$\|f(u)\|_{r'_{(1)}}\le\|V\|_p\|u\|_{r_2}^3\le\|V\|_p\|u\|_2^{3(1-\delta_2/\delta)}\|u\|_r^{3\delta_2/\delta},$$

where $\delta_{(1)}+3\delta_2=\frac{n}{p}$. Hence

$$\max\big\{\|f(u)\|_{r_1{}'}^{\delta/\delta_1},\|f(u)\|_{r'}\big\}\le M\|u\|^{1+\nu},$$

where $1+\nu=3\delta_2/\delta_{(1)}=\big(\frac{n}{p}-\delta_{(1)}\big)/\delta_{(1)}>1$ since $\frac{n}{p}>2\delta_{(1)}$.

Then applying (4.31), (4.32) and Proposition 4.3 to conclude that for $\nu > 0$ we can take $\rho > 0$ small enough depending only on r, r_1, n such that

$$\begin{aligned}\big\|u_2\big\|_r + \big\|u_3\big\|_r \leq & MN_\varepsilon^{(n+2)(\frac{1}{2}-\frac{1}{r_1})-1}\big\|u\big\|^{1+\nu}_{L^\infty\left([t_0-l_1,t_0],L^r\right)} \\ \leq & MN_\varepsilon^{(n+2)(\frac{1}{2}-\frac{1}{r_1})-1}\varepsilon^{1+\nu} \\ \leq & M\varepsilon^{\rho(\delta-1)^{-1}((n+2)(\frac{1}{2}-\frac{1}{r_1})-1)}\varepsilon^{1+\nu} \\ < & \frac{\varepsilon}{2},\end{aligned} \tag{4.33}$$

holds for sufficient small ε depending only on $\|u\|_{H^1}$.

Collecting (4.28) – (4.33), we have

$$\begin{aligned}\varepsilon = & \|P_{\leq N_\varepsilon}u(t_0)\|_{L^r} \\ & \leq \|u^{(0)}\|_{L^r} + \|u_1\|_{L^r} + \|u_2\|_{L^r} + \|u_3\|_{L^r} \\ & < \varepsilon.\end{aligned}$$

This is a contradiction. That is to say

$$\big\|P_{\leq N_\varepsilon}u\big\|_{L^\infty\left([t_2-l_1,\infty),L^r\right)} \leq \varepsilon. \tag{4.34}$$

Combining (4.23) with (4.34), one gets

$$\|u\|_{L^\infty[t_2-l_1,\infty),L^r)} \leq 2\varepsilon^\rho.$$

Since ε is arbitrarily small, it yields the desired result.

We can now state the global space-time integrability of the solution.

Proposition 4.5. *Assume that V satisfies (H1) with $2 < \frac{n}{p_1} \leq \frac{n}{p_2} < \min(n,4)$ and (H2). Let $u \in X^1_{loc}(\mathbb{R})$ be a finite energy solution of the equation* (1.1). *Then $u \in X^1(\mathbb{R})$.*

Proof: We give the proof in the special case where $V \in L^p$. The general case of V satisfying (H1) with $p_2 < p_1$ can be treated by a straightforward extension of the proof based on Lemma 2.2. Let (q,r) be the admissible pair satisfying

$$\begin{cases} \dfrac{2}{q} \leq (n-1+\theta)(\dfrac{1}{2}-\dfrac{1}{r}), \\ (n+\theta)(\dfrac{1}{2}-\dfrac{1}{r}) - \dfrac{1}{q} = \dfrac{1}{2}. \end{cases} \tag{4.35}$$

Let $0 < t_1 < t_2$, the key estimate consists in again

$$\big\|f(u)\big\|_{L^{q'}\left([t_1,t_2],B^{1/2}_{r',2}\right)} \leq \big\|V\big\|_p\big\|u\big\|_{L^q\left([t_1,t_2],B^{1/2}_{r,2}\right)}\big\|u\big\|^2_{L^k\left([t_1,t_2],L^s\right)} \tag{4.36}$$

with

$$\begin{cases} \dfrac{1}{k} + \dfrac{1}{q} = \dfrac{1}{2}, \\ 2\delta(s) + 2\delta(r) = \dfrac{n}{p}. \end{cases} \tag{4.37}$$

By interpolation and Lemma 2.3, let $c < k < \infty$ we have

$$\begin{aligned} \|u\|_{L^k\left([t_1,t_2],L^s\right)} &\leq \|u\|^{1-\lambda}_{L^c\left([t_1,t_2],L^b\right)} \|u\|^{\lambda}_{L^\infty\left([t_1,t_2],L^a\right)} \\ &\leq C\|u\|^{1-\lambda}_{X^1\left([t_1,t_2]\right)} \|u\|^{\lambda}_{L^\infty\left([t_1,t_2],L^a\right)} \end{aligned}$$

with

$$\begin{cases} \dfrac{1}{k} = \dfrac{1-\lambda}{c}, \\ \delta(s) = (1-\lambda)\delta(b) + \lambda\delta(a). \end{cases} \tag{4.38}$$

and $\frac{2}{c} \leq \delta(b) \leq 1 + \frac{1}{c}$.

Similar as the arguments of Proposition 3.1, we only consider the two endpoint cases.

On the one hand, one can choose $\theta = 1,\ \delta(b) = \frac{2}{c}$ and $q = r = c = \frac{2(n+2)}{n}$ (then $\delta(r) = \frac{n}{n+2},\ k = n+2,\ \lambda = \frac{n-2}{n}$) such that

$$\begin{aligned} \frac{n}{2p} &= (1-\lambda)\delta(b) + \delta(r) + \lambda\delta(a) \\ &= (1-\lambda)\frac{2}{c} + \delta(r) + \lambda\delta(a) \\ &= \frac{2}{k} + \delta(r) + \lambda\delta(a) \\ &= 1 + \lambda\delta(a) > 1 \end{aligned}$$

closes to 1 if $\delta(a) > 0$ closes to 0.

On the other hand, for $n \geq 4$, one can choose $\theta = 0,\ \delta(b) = 1 + \frac{1}{c}$ and $q = r = c = \frac{2(n+1)}{n-1}$ (then $\delta(r) = \frac{n}{n+1},\ k = n+1,\ \lambda = \frac{n-3}{n-1}$) such that

$$\begin{aligned} \frac{n}{2p} &= (1-\lambda)(1 + \frac{1}{c}) + \delta(r) + \lambda\delta(a) \\ &= (1-\lambda) + \frac{1}{k} + \delta(r) + \lambda\delta(a) \\ &= 2 - \lambda + \lambda\delta(a) < 2 \end{aligned}$$

closes to 2 if $\delta(a) < 1$ closes to 1; For $n = 3$, one can choose $q = r = c = b = a = \frac{2(4+\theta)}{2+\theta}$ (then $\delta(a) = \delta(b) = \delta(r) = \frac{3}{4+\theta},\ k = 4+\theta,\ \lambda = \frac{\theta}{2+\theta}$) such that

$$\frac{n}{2p} = (1-\lambda)\delta(b) + \delta(r) + \lambda\delta(a) = \frac{6}{4+\theta} < \frac{3}{2}$$

closes to $\frac{3}{2}$ if $\theta > 0$ closes to 0.

Therefore, applying the estimates to the integral equation for u with initial time t_1, we get that for any $2 < \frac{n}{p} < \min(4, n)$, there exist $\theta \in [0, 1]$ and $\delta(a) \in (0, 1)$ such that

$$\begin{aligned} y :=& \|u\|_{X^1([t_1,t_2])} \\ \leq & C\big\|u_0\big\|_{H^1} + C\big\|u_1\big\|_{L^2} + C\big\|f(u)\big\|_{L^{q'}\left([t_1,t_2],B^{1/2}_{r',2}\right)} \\ \leq & M + C\big\|V\big\|_p \|u\|^{3-2\lambda}_{X^1\left([t_1,t_2]\right)} \|u\|^{2\lambda}_{L^\infty\left([t_1,t_2],L^a\right)} \\ \leq & M + C\big\|V\big\|_p \|u\|^{2\lambda}_{L^\infty\left([t_1,t_2],L^a\right)} y^{3-2\lambda}. \end{aligned} \tag{4.39}$$

By Proposition 4.4, $\|u\|_{L^\infty\left([t_1,t_2],L^a\right)}$ can be made arbitrarily small by taking t_1 sufficiently large, uniformly with respect to t_2. Furthermore for fixed t_1, y is a continuous function of t_2, starting from zero for $t_2 = t_1$. It then follows from (4.39) that for t_1 sufficiently large, y is bounded uniformly in t_2, namely that $u \in X^1\big([t_1, \infty)\big)$. Plugging that result again into the integral equation yields that $u \in X^1(\mathbb{R}^+)$. The same argument holds for negative times.

As a direct consequence of the global space-time integrability, one easy derives the scattering result as following:

Theorem 4.1. *Assume that V satisfies (H1) with $2 < \frac{n}{p_1} \leq \frac{n}{p_2} < \min\big(4, n\big)$ and (H2). Then there exist homeomorphisms $\Omega_\pm$ on $H^1 \times L^2$ with the following property. For any $(\varphi, \psi) \in H^1 \times L^2$, let v be the solution to*

$$\begin{cases} \Box v + v = 0, \\ (v(0), \dot{v}(0)) = (\varphi, \psi), \end{cases}$$

and let $u_\pm$ be the global solution to

$$\begin{cases} \Box u_\pm + u_\pm + (V * |u_\pm|^2)u_\pm = 0, \\ (u_\pm(0), \dot{u}_\pm(0)) = \Omega_\pm(\varphi, \psi). \end{cases}$$

Then we have

$$\lim_{t\to\pm\infty} \big\|(v(t), \dot{v}(t)) - (u_\pm(t), \dot{u}_\pm(t))\big\|_{H^1\times L^2} = 0.$$

Moreover, this property uniquely determines $\Omega_\pm$. Thus the scattering operator $S = \Omega_+^{-1}\Omega_-$ is also a homeomorphism on $H^1 \times L^2$.

This completes the proof of Theorem 1.1.

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