The regularity of weak solutions to magneto-micropolar fluid equations

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Abstract

In this paper, we study the regularity of weak solutions and the blow-up criteria of smooth solutions to the magneto-micropolar fluid equations in $\mathbb{R}^3$. We obtain the classical blow-up criteria for smooth solutions (u, ω, b), ie. $u \in L^q(0, T; L^p(\mathbb{R}^3))$ for $\frac{2}{q} + \frac{3}{p} \leq 1$ with $3 < p \leq \infty$, $u \in C([0, T); L^3(\mathbb{R}^3))$ or $\nabla u \in L^q(0, T; L^p)$ for $\frac{3}{2} < p \leq \infty$ satisfying $\frac{2}{q} + \frac{3}{p} \leq 2$. Moreover, our results indicate that the regularity of weak solutions is dominated by the velocity u of fluid. In the end-point case $p = \infty$, the blow-up criteria can be extended to more general spaces $(u, \omega, b) \in L^2(0, T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^3))$ or $\nabla(u, \omega, b) \in L^1(0, T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^3))$.

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1 Introduction

This paper concerns about the regularity of weak solutions and blow-up criteria of smooth solutions to the magneto-micropolar fluid equations in 3 dimensions

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\mu + \chi)\Delta u + u \cdot \nabla u - b \cdot \nabla b + \nabla(p + b^2) - \chi \nabla \times \omega = 0, \\ \dfrac{\partial \omega}{\partial t} - \gamma \Delta \omega - \kappa \nabla div\omega + 2\chi\omega + u \cdot \nabla \omega - \chi \nabla \times u = 0, \\ \dfrac{\partial b}{\partial t} - \nu \Delta b + u \cdot \nabla b - b \cdot \nabla u = 0, \\ divu = divb = 0 \\ u(x, 0) = u_0(x), \ \omega(x, 0) = \omega_0(x), \ b(x, 0) = b_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1(x, t), u_2(x, t), u_3(x, t))$ denotes the velocity of the fluid at a point $x \in \mathbb{R}^3$, $t \in [0, T)$, $\omega = (\omega_1(x, t), \omega_2(x, t), \omega_3(x, t))$, $b = (b_1(x, t), b_2(x, t), b_3(x, t))$ and $p = p(x, t)$ denote, respectively, the micro-rotational velocity, the magnetic field and the hydrostatic pressure. u_0, ω_0 and b_0 are the prescribed initial data for the velocity and angular velocity and magnetic field with properties $div u_0 = 0$ and $div b_0 = 0$. μ is the kinematic viscosity, χ is the vortex viscosity, κ and γ are spin viscosities, and $\frac{1}{\nu}$ is the magnetic Reynold. If the magnetic field $b = 0$, (1.1) reduces to the micropolar fluid system. Theory of micropolar fluid was first proposed by Eringen [6] in 1966, which enable us to consider some physical phenomena that cannot be treated by the classical Navier-Stokes equations for the viscous incompressible fluid, for example, the motion of animal blood, liquid crystals and dilute aqueous polymer solutions etc. The existences of weak solutions and strong solution were treated by Galdi and Rionero [7] for weak solutions, Yamaguchi [20] for strong solution. If, further, the vortex viscosity $\chi = 0$, the velocity u does not depend on the micro-rotation field ω, and the first equation reduces to the classical Navier-Stokes equation which has been greatly analyzed, see, for example, the classical books by Ladyzhenskaya [11], Lions [13] or Lemarié-Rieusset [12]. If we ignore the micro-rotation of particles, it reduces to the viscous incompressible magneto-hydrodynamic equations, which has also been studied extensively [18, 5, 2, 8, 4]. It is worthy to note that He and Xin [8] proved the regularity criterion of weak solutions to the magneto-hydrodynamic equations, which only need the velocity u or its gradient ∇u or the vorticity $\nabla \times u$ satisfy some conditions. This indicates that the velocity field u plays a more dominate role than the magnetic field b does on the regularity of solutions to the magneto-hydrodynamic equations.

The magneto-micropolar fluid system (1.1) was studied by Galdi and Rionero in [7]. Rojas-Medar [16] studied it and established the local in time existence and uniqueness of strong solutions by the spectral Galerkin method, Ortega-Torres and Rojas-Medar [15] proved global in time existence of strong solution for small initial data. Rojas-Medar and Boldrini [17] proved the existence of weak solutions by the Galerkin method, and in 2D case, also proved the uniqueness of the weak solutions.

The purpose of this paper is to study the regularity of weak solutions and the break-down criteria of smooth solution to the magneto-micropolar fluid system (1.1). The classical blow-up criteria of smooth solution to Navier-Stokes equation also holds for the magneto-micropolar fluid equations. As demonstrated in paper [8], we also prove that to guarantee the regularity of weak solutions to (1.1), one only need impose conditions on the velocity field of fluid. This also demonstrates that in the regularity of weak solutions the micro-rotational velocity ω of particles and the magnetic field b play less important role than the velocity u does, and the regularity of weak solutions to (1.1) is dominated by the velocity u of fluid. In a more general case, i.e. Besov space $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$, we need all the velocity field u, micro-rotational velocity ω and magnetic field b to control the blow-up of fluids. This indicates that, in the weaker topology, only use of velocity u is not sufficient to guarantee the regularity of solutions.

For the convenient of following discussion, we introduce some function spaces and notations. Let $C^\infty_{0,\sigma}(\mathbb{R}^3)$ denote the set of all C^∞ vector functions $f(x) = (f_1(x), f_2(x), f_3(x))$ with compact support such that $div f(x) = 0$. $L^r_\sigma(\mathbb{R}^3)$ is the closure of $C^\infty_{0,\sigma}(\mathbb{R}^3)$-function with respect to the L^r-norm $\|\cdot\|_r$ for $1 \le r \le \infty$. $H^s_\sigma(\mathbb{R}^3)$ denotes the closure of $C^\infty_{0,\sigma}(\mathbb{R}^3)$ with respect to the H^s-norm $\|f\|_{H^s} = \|(1-\Delta)^{\frac{s}{2}} f\|_2$, for $s \ge 0$.

In the following arguments the letters C and C_i denote inessential constants which may vary from line to line, but do not depend on particular solutions or functions.

Now we state our main results.

Theorem 1.1. *Let* $(u_0, b_0) \in H^1_\sigma(\mathbb{R}^3)$ *and* $w_0 \in H^1(\mathbb{R}^3)$. *Assume that* $(u, b) \in C[0, T; H^1_\sigma(\mathbb{R}^3)) \cap C(0, T; H^2_\sigma(\mathbb{R}^3))$ *and* $\omega \in C[0, T; H^1(\mathbb{R}^3)) \cap C(0, T; H^2(\mathbb{R}^3))$ *is a smooth solution to equations (1.1). If* (u, ω, b) *satisfies*

(1) $u \in L^q(0, T; L^p(\mathbb{R}^3))$ *for* $\frac{2}{q} + \frac{3}{p} \leq 1$ *with* $3 < p \leq \infty$.

(2) $u \in C([0, T); L^3(\mathbb{R}^3))$.

Then the solution (u, ω, b) *can be extended beyond* $t = T$.

Theorem 1.2. *Let* $(u_0, b_0) \in H^1_\sigma(\mathbb{R}^3)$ *and* $\omega_0 \in H^1(\mathbb{R}^3)$. *Suppose that* $(u, b) \in C[0, T; H^1_\sigma(\mathbb{R}^3)) \cap C(0, T; H^2_\sigma(\mathbb{R}^3))$ *and* $\omega \in C[0, T; H^1(\mathbb{R}^3)) \cap C(0, T; H^2(\mathbb{R}^3))$ *is a smooth solution to equations (1.1). If* u *satisfies*

$$\int_0^T \|\nabla u(t)\|_p^q dx < \infty, \tag{1.2}$$

for $\frac{3}{2} < p \leq \infty$ *satisfying* $\frac{2}{q} + \frac{3}{p} \leq 2$. *Then the solution* (u, ω, b) *can be extended to* $(0, T')$ *for some* $T' > T$.

In the endpoint case $p = \infty$, we can extend the blow-up criteria to more general space $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$.

Theorem 1.3. *Let* $(u_0, b_0) \in H^1_\sigma(\mathbb{R}^3)$ *and* $\omega_0 \in H^1(\mathbb{R}^3)$. *Suppose that* $(u, b) \in C[0, T; H^1_\sigma(\mathbb{R}^3)) \cap C(0, T; H^2_\sigma(\mathbb{R}^3))$ *and* $\omega \in C[0, T; H^1(\mathbb{R}^3)) \cap C(0, T; H^2(\mathbb{R}^3))$ *is a smooth solution to equations (1.1). If* (u, ω, b) *satisfies one of the conditions*

$$\int_0^T \|(u(t), \omega(t), b(t))\|^2_{\dot{B}^0_{\infty,\infty}} dt < \infty, \tag{1.3}$$

or

$$\int_0^T \|\nabla(u(t), \omega(t), b(t))\|_{\dot{B}^0_{\infty,\infty}} dt < \infty. \tag{1.4}$$

then the solution (u, ω, b) *can be extended beyond* $t = T$.

We next consider the criteria on regularity of weak solutions to the magneto-micropolar equations (1.1), thus we introduce the definition of a weak solution.

Definition 1.1. *Let* $(u_0(x), b_0(x)) \in L^2_\sigma(\mathbb{R}^3)$ *and* $\omega_0(x) \in L^2(\mathbb{R}^3)$. *A measurable function* $(u(x, t), \omega(x, t), b(x, t))$ *is called a weak solution to the magneto-micropolar equations (1.1) on* $(0, T)$ *if*
(a)

$$(u(x, t), b(x, t)) \in L^\infty(0, T; L^2_\sigma(\mathbb{R}^3)) \cap L^2(0, T; H^1_\sigma(\mathbb{R}^3)),$$

and

$$\omega \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3));$$

(b)

$$\int_0^T \{-(u,\partial_\tau\varphi)+(\mu+\chi)(\nabla u,\nabla\varphi)+(u\cdot\nabla u,\varphi)-(b\cdot\nabla b,\varphi)\}-\chi(\nabla\times\omega,\varphi)d\tau$$
$$=-(u_0,\varphi(0)),$$

$$\int_0^T \{-(\omega,\partial_\tau\phi)+\gamma(\nabla\omega,\nabla\phi)+\kappa(div\omega,div\phi)+2\chi(\omega,\phi)+(u\cdot\nabla\omega,\phi)-\chi(\nabla\times u,\phi)d\tau$$
$$=-(\omega_0,\phi(0)),$$

and

$$\int_0^T \{-(b,\partial_\tau\psi)+\nu(\nabla b,\nabla\psi)+(u\cdot\nabla b,\psi)-(b\cdot\nabla u,\psi)\}\,d\tau=-(b_0,\psi(0)).$$

for any $(\varphi(x,t),\psi(x,t))\in H^1((0,T);H^1_\sigma(\mathbb{R}^3)$ *and* $\phi(x,t)\in H^1((0,T);H^1(\mathbb{R}^3)$*with* $\varphi(T)=0$*,* $\phi(T)=0$ *and* $\psi(T)=0$*.*

In the reference [17], Rojas-Medar and Boldrini proved the global existence of weak solutions to the equations (1.1) of the magneto-micropolar fluid motion by the Galerkin method. The weak solutions also satisfy the energy inequality

$$\|(u,\omega,b)\|_2^2+2\mu\int_0^t\|\nabla u\|_2^2\mathrm{d}s+2\gamma\int_0^t\|\nabla\omega\|_2^2\mathrm{d}s+2\nu\int_0^t\|\nabla b\|_2^2\mathrm{d}s \tag{1.5}$$
$$+2\kappa\int_0^t\|div\omega\|_2^2\mathrm{d}s+2\chi\int_0^t\|\omega\|_2^2\mathrm{d}s\le\|(u_0,\omega_0,b_0)\|_2^2.$$

Theorem 1.4. *Let* $(u_0,\omega_0,b_0)\in L^2(\mathbb{R}^3)$ *with* $divu_0=divb_0=0$*. Assume that there exists a weak solution* (u,ω,b) *satisfying energy inequality (1.5) of strong form. If one of the following conditions hold:*

(1) $u\in L^q(0,T;L^p(\mathbb{R}^3))$ *for* $\frac{2}{q}+\frac{3}{p}\le 1$ *with* $3<p\le\infty$*.*

(2) $u\in C([0,T);L^3(\mathbb{R}^3))$*.*

Then (u,ω,b) *is a unique solution with the initial value* (u_0,ω_0,b_0)*. Moreover the solution* $(u,\omega,b)\in C^\infty((0,T)\times\mathbb{R}^n)$ *for some* $T>0$*.*

Theorem 1.5. *Let* $(u_0,\omega_0,b_0)\in L^2(\mathbb{R}^3)$ *with* $divu_0=divb_0=0$*. Assume that there exists a weak solution* (u,ω,b) *satisfying the energy inequality (1.5) and*

$$\int_0^T\|\nabla u(t)\|_p^q dx<\infty, \tag{1.6}$$

for $\frac{3}{2}<p\le\infty$ *satisfying* $\frac{2}{q}+\frac{3}{p}\le 2$*. Then* (u,ω,b) *is a unique solution with the initial value* (u_0,ω_0,b_0)*. Moreover the solution* $(u,\omega,b)\in C^\infty((0,T)\times\mathbb{R}^n)$ *for some* $T>0$*.*

Theorem 1.6. *Let* $(u_0,\omega_0,b_0)\in L^2(\mathbb{R}^3)$ *with* $divu_0=divb_0=0$*. Assume that there exists a weak solution* (u,ω,b) *satisfying the energy inequality (1.5). If* (u,ω,b) *satisfies one of the conditions*

$$\int_0^T\|(u(t),\omega(t),b(t))\|^2_{\dot{B}^0_{\infty,\infty}}dt<\infty,$$

or

$$\int_0^T \|\nabla(u(t),\omega(t),b(t))\|_{\dot{B}^0_{\infty,\infty}}\,dt<\infty.$$

Then (u,ω,b) is a unique solution with the initial value (u_0,ω_0,b_0). Moreover the solution $(u,\omega,b)\in C^\infty((0,T)\times\mathbb{R}^n)$ for some $T>0$.

Remark 1.1. *In the more general case we require that $(u,\omega,b)\in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3))$ or $\nabla(u,\omega,b)\in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3))$ to guarantee the regularity of weak solutions, moreover that ω is not divergence free brings some difficulty, see the proof of Theorem 1.3 in section 3. We naturally ask QUESTION: in the case of Theorem 1.3, whether the blow-up criterion (1.3) and (1.4) can be replaced by conditions only on the velocity field of fluids?*

The proofs of Theorem 1.4, 1.5 and 1.6 are standard. Let $(u(x,t),\omega(x,t),b(x,t))$ be a weak solution satisfying the strong energy inequality (1.5) for any $0<t_0<t\le T$. Since $(u(t_0),b(t_0))\in H^1_\sigma(\mathbb{R}^3)$ and $\omega(t_0)\in H^1(\mathbb{R}^3)$, it follows from the classical local existence theorem of strong solution that there exist a time $T'>t_0$ and a unique solution $(u',b')\in C((t_0,T');H^1_\sigma(\mathbb{R}^3))$ and $\omega'\in C((t_0,T');H^1(\mathbb{R}^3))$ with $(u'(t_0),\omega'(t_0),b'(t_0))=(u(t_0),\omega(t_0),b(t_0))$. Since (u,ω,b) is a weak solution satisfying the energy inequality (1.5), we conclude that $(u',\omega',b')=(u,\omega,b)$ on $[t_0,T')$. we assert that $T'=T$. If not, let $T'<T$, without loss of generality, we may assume that T' is the maximal existent time for (u',ω',b'). Since $(u'(t),\omega'(t),b'(t))=(u(t),\omega(t),b(t))$ on $[t_0,T')$, one has that $(u'(t),\omega'(t),b'(t))$ satisfies the conditions of Theorem 1.4, 1.5 or 1.6 on $[t_0,T')$. By virtue of Theorem 1.1, 1.2 and 1.3, it follows that $(u'(t),\omega'(t),b'(t))$ can be extended to interval $(0,T_1)$ for some $T_1>T'$, which is contradictory to the maximality of T'. Thus we prove the Theorem 1.3 and 1.4.

2 Proofs of Theorem 1.1 and Theorem 1.2

In this section we prove the Theorems 1.1 and 1.2 by the most fundamental tools.

Proof of Theorem 1.1: We differentiate the equations (1.1) with respect to x_i, then multiply the resulting equations by $\partial_{x_i}u,\ \partial_{x_i}\omega,\ \partial_{x_i}b$, respectively, integrate with respect to x and sum them up, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|(\partial_{x_i}u,\partial_{x_i}\omega,\partial_{x_i}b)\|_2^2)+\sum_{j=1}^3\Big((\mu+\chi)\|\partial^2_{x_ix_j}u\|_2^2+\gamma\|\partial^2_{x_ix_j}\omega\|_2^2+\nu\|\partial^2_{x_ix_j}b\|_2^2\Big)\\
&\quad+\kappa\|div\partial_{x_i}\omega\|_2^2+2\chi\|\partial_{x_i}\omega\|_2^2 \qquad (2.1)\\
\le&\quad |(\partial_{x_i}u\cdot\nabla u,\partial_{x_i}u)|+|(\partial_{x_i}b\cdot\nabla b,\partial_{x_i}u)|+|(\partial_{x_i}u\cdot\nabla b,\partial_{x_i}b)|\\
&\quad+|(\partial_{x_i}b\cdot\nabla u,\partial_{x_i}b)|+|(\partial_{x_i}u\cdot\nabla\omega,\partial_{x_i}\omega)|+2\chi|(\nabla\times\partial_{x_i}u,\partial_{x_i}\omega)|\\
=&\quad I_1+I_2+I_3+I_4+I_5+I_6,
\end{aligned}$$

where use has been made of the facts that

$$(\nabla\times\partial_{x_i}u,\partial_{x_i}\omega)=(\nabla\times\partial_{x_i}\omega,\partial_{x_i}u),\ \ (b\cdot\nabla\partial_{x_i}b,\partial_{x_i}u)+(b\cdot\nabla\partial_{x_i}u,\partial_{x_i}b)=0$$

and

$$(u\cdot\nabla\partial_{x_i}u,\partial_{x_i}u)=(u\cdot\nabla\partial_{x_i}b,\partial_{x_i}b)=0,$$

where $(\cdot,\cdot)$ denotes the L^2 inner product on $\mathbb{R}^3$. For conciseness, the short notation

$$\|(A,B,C)\|_2^2=\|A\|_2^2+\|B\|_2^2+\|C\|_2^2.$$

has been used and will be used in the following part.

(1) We estimate the terms $I_j,\ j=1,2,\cdots,6$. First of all, by Hölder, Gagliardo-Nirenberge and Young inequalities, one has

$$\begin{aligned} I_1 &\le |\int_{\mathbb{R}^3}\partial_{x_i}u\cdot\nabla\partial_{x_i}u\cdot u(x)\mathrm{d}x|+|\int_{\mathbb{R}^3}\partial_{x_i}\partial_{x_i}u\cdot\nabla u\cdot u(x)\mathrm{d}x| \\ &\le C\|u\|_p\|\nabla u\|_{\frac{2p}{p-2}}\|D^2u\|_2 \\ &\le C\|u\|_p\|\nabla u\|_2^{1-3/p}\|D^2u\|_2^{1+3/p} \\ &\le \frac{\chi}{12}\|D^2u\|_2^2+C\|u\|_p^{2p/(p-3)}\|\nabla u\|_2^2. \end{aligned}\tag{2.2}$$

Similarly, for I_2 one can deduce

$$\begin{aligned} I_2 &\le |\int_{\mathbb{R}^3}\partial_{x_i}b\cdot\nabla\partial_{x_i}b\cdot u(x)\mathrm{d}x|+|\int_{\mathbb{R}^3}\partial_{x_i}\partial_{x_i}b\cdot\nabla b\cdot u(x)\mathrm{d}x \\ &\le \frac{\nu}{18}\|D^2b\|_2^2+C\|u\|_p^{2p/(p-3)}\|\nabla b\|_2^2. \end{aligned}\tag{2.3}$$

In the same way, for I_3, I_4 and I_5, we have

$$I_3,\ I_4\le\frac{\nu}{18}\|D^2b\|_2^2+C\|u\|_p^{2p/(p-3)}\|\nabla b\|_2^2,\tag{2.4}$$

and

$$I_5\le\frac{\gamma}{6}\|D^2\omega\|_2^2+C\|u\|_p^{2p/(p-3)}\|\nabla\omega\|_2^2.\tag{2.5}$$

Finally, we deal with the term I_6. Applying Hölder and Young inequalities, one has

$$I_6\le\frac{\chi}{2}\|\nabla\times\partial_{x_i}u\|_2^2+2\chi\|\nabla\omega\|_2^2.\tag{2.6}$$

Inserting the estimates (2.2)-(2.6) of I_1-I_6 into the inequality (2.1) and summing up i from 1 to 3 to arrive at

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|(\nabla u,\nabla\omega,\nabla b)\|_2^2)+(2\mu+\frac{1}{2}\chi)\|D^2u\|_2^2+\gamma\|D^2\omega\|_2^2+\nu\|D^2b\|_2^2+2\kappa\|\nabla div\omega\|_2^2 \\ \le\ & C\|u\|_p^{2p/(p-3)}\|(\nabla u,\nabla\omega,\nabla b)\|_2^2. \end{aligned}$$

Gronwall inequality implies the a priori estimate

$$\|(\nabla u,\nabla\omega,\nabla b)\|_2^2\le\|(\nabla u_0,\nabla\omega_0,\nabla b_0)\|_2^2\exp\left\{C\int_0^t\|u(s)\|_p^{2p/(p-3)}\mathrm{d}s\right\}.\tag{2.7}$$

The above estimates are also valid for $p=\infty$ provided we modify them accordingly.

Combining the a priori estimate (2.7) with the energy inequality (1.5) and by standard arguments of continuation of local solutions, we conclude that the solutions $(u(x,t),\omega(x,t),b(x,t))$ can be extended beyond $t=T$ provided that $u\in L^q(0,T;L^p(\mathbb{R}^3))$ for $\frac{2}{q}+\frac{3}{p}\le 1$ with $3<p\le\infty$.

(2) In the case $u\in C([0,T);L^3(\mathbb{R}^3))$, one can decompose $u=u^1+u^2$ with $\|u^1;C([0,T];L^3)\|\le\varepsilon$ and $\|u^2;L^\infty([0,T)\times\mathbb{R}^3\|\le C(\varepsilon,\|u;C([0,T];L^3)\|)$ for any $\varepsilon>0$.

First we estimate I_1.

$$\begin{aligned} I_1 &\le C\int_{\mathbb{R}^3}|u||D^2u||\nabla u|\mathrm{d}x\\ &\le C\|u^1\|_3\|\nabla u\|_6\|D^2u\|_2+C\|u^2\|_\infty\|\nabla u\|_2\|D^2u\|_2\\ &\le C\varepsilon\|D^2u\|_2^2+C\|u^2\|_\infty^2\|\nabla u\|_2^2\\ &\le \frac{\chi}{12}\|D^2u\|_2^2+C\|u^2\|_\infty^2\|\nabla u\|_2^2.\end{aligned}$$

Similarly, for I_2-I_4, one has

$$\begin{aligned} I_2,\ I_3,\ I_4 &\le C\int_{\mathbb{R}^3}|u||D^2b||\nabla b|\mathrm{d}x\\ &\le C\|u^1\|_3\|\nabla b\|_6\|D^2b\|_2+C\|u^2\|_\infty\|\nabla b\|_2\|D^2b\|_2\\ &\le C\varepsilon\|D^2b\|_2^2+C\|u^2\|_\infty^2\|\nabla b\|_2^2\\ &\le \frac{\nu}{18}\|D^2b\|_2^2+C\|u^2\|_\infty^2\|\nabla b\|_2^2,\end{aligned}$$

and

$$I_5\le\frac{\gamma}{6}\|D^2\omega\|_2^2+C\|u^2\|_\infty^2\|\nabla\omega\|_2^2.$$

Collecting the above estimates of I_1-I_5 and the estimate (2.6) of I_6 and summing up i from 1 to 3, then applying Gronwall inequality, we have

$$\begin{aligned} &\|(\nabla u,\nabla\omega,\nabla b)\|_2^2+C_1\int_0^t(\|(D^2u,D^2\omega,D^2b)\|_2^2)+2\kappa\|\nabla div\omega\|_2^2\mathrm{d}s \qquad (2.8)\\ \le\ &\|(\nabla u_0,\nabla\omega_0,\nabla b_0)\|_2^2\exp\left\{C\int_0^t\|u^2(s)\|_\infty^2\mathrm{d}s\right\}<\infty.\end{aligned}$$

We thus obtain the proof of Theorem 1.1. □

Proof of Theorem 1.2: Similarly as in the proof of Theorem 1.1, we differentiate the equations (1.1) with respect to x_i then multiply the resulting equations by $\partial_{x_i}u,\ \partial_{x_i}\omega,\ \partial_{x_i}b$, respectively, and sum them, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|(\partial_{x_i}u,\partial_{x_i}\omega,\partial_{x_i}b)\|_2^2)+\sum_{j=1}^3\Big((\mu+\chi)\|\partial^2_{x_ix_j}u\|_2^2+\gamma\|\partial^2_{x_ix_j}\omega\|_2^2+\nu\|\partial^2_{x_ix_j}b\|_2^2\Big)\\ &+\kappa\|div\partial_{x_i}\omega\|_2^2+2\chi\|\partial_{x_i}\omega\|_2^2 \qquad (2.9)\\ \le\ &|(\partial_{x_i}u\cdot\nabla u,\partial_{x_i}u)|+|(\partial_{x_i}b\cdot\nabla b,\partial_{x_i}u)|+|(\partial_{x_i}u\cdot\nabla b,\partial_{x_i}b)|\\ &+|(\partial_{x_i}b\cdot\nabla u,\partial_{x_i}b)|+|(\partial_{x_i}u\cdot\nabla\omega,\partial_{x_i}\omega)|+2\chi|(\nabla\times\partial_{x_i}u,\partial_{x_i}\omega)|\\ =\ &I_1+I_2+I_3+I_4+I_5+I_6,\end{aligned}$$

This time we estimate I_j, $j=1,2,\cdots,5$ in another way. By means of Hölder,Gagliardo-Nirenberg and Young inequalities one has

$$\begin{aligned} I_1 &\leq \|\nabla u\|_p \|\nabla u\|_{2p/(p-1)}^2 \\ &\leq C\|\nabla u\|_p \|\nabla u\|_2^{2-3/p} \|D^2 u\|_2^{3/p} \\ &\leq \frac{\chi}{12}\|D^2 u\|_2^2 + C\|\nabla u\|_p^{2p/(2p-3)} \|\nabla u\|_2^2. \end{aligned} \tag{2.10}$$

Similarly, for $I_2 - I_5$ one has

$$I_2,\ I_3,\ I_4 \leq \frac{\nu}{18}\|D^2 b\|_2^2 + C\|\nabla u\|_p^{2p/(2p-3)}\|\nabla b\|_2^2, \tag{2.11}$$

and

$$I_5 \leq \frac{\gamma}{6}\|D^2\omega\|_2^2 + C\|\nabla u\|_p^{2p/(2p-3)}\|\nabla \omega\|_2^2. \tag{2.12}$$

Inserting the estimates (2.10)-(2.12) and (2.6) into (2.9) and summing up i from 1 to 3 to arrive at

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|(\nabla u, \nabla\omega, \nabla b)\|_2^2) + (2\mu + \frac{1}{2}\chi)\|D^2 u\|_2^2 + \gamma\|D^2\omega\|_2^2 + \nu\|D^2 b\|_2^2 + 2\kappa\|\nabla div\omega\|_2^2 \\ \leq\ & C\|\nabla u\|_p^{2p/(2p-3)}\|(\nabla u, \nabla\omega, \nabla b)\|_2^2. \end{aligned}$$

We get the a priori estimate

$$\begin{aligned} &\|(\nabla u, \nabla\omega, \nabla b)\|_2^2 + C_1\int_0^t (\|(D^2 u, D^2\omega, D^2 b)\|_2^2 + 2\kappa\|\nabla div\omega\|_2^2)\mathrm{d}s \\ \leq\ & \|(\nabla u_0, \nabla\omega_0, \nabla b_0)\|_2^2 \exp\left\{C\int_0^t \|\nabla u\|_p^{2p/(2p-3)}\mathrm{d}s\right\}, \end{aligned} \tag{2.13}$$

by Gronwall inequality.

The above proof is also valid for $p=\infty$ provided we modify it accordingly. Combining the a priori estimate (2.13) with the energy inequality (1.5) and by standard arguments of continuation of local solutions, we conclude that the solutions $(u(x,t), \omega(x,t), b(x,t))$ can be extended to $(0, T')$ for some $T' > T$, provided that u satisfies $\int_0^T \|\nabla u(t)\|_p^q \mathrm{d}x < \infty$ for $\frac{3}{2} < p \leq \infty$ with $\frac{2}{q} + \frac{3}{p} \leq 2$. We thus complete the proof of Theorem 1.2. □

3 Proofs of Theorem 1.3

In this section we use two different methods to prove the blow-up criteria of smooth solutions for conditions (1.3) and (1.4), respectively. To condition (1.3) the Littlewood-Paley decomposition and Bony's para-product decomposition have been used; To condition (1.4) we use the estimate obtained in Theorem 1.2 and a logarithmic Sobolev inequality to give a simple proof. To do this we first introduce the Littlewood-Paley decomposition and Bony's para-product decomposition.

Choose a nonnegative radial function $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \chi(\xi) \leq 1$ and

$$\chi(\xi) = \begin{cases} 1, & \text{for } |\xi| \leq \frac{3}{4}, \\ 0, & \text{for } |\xi| > \frac{4}{3}, \end{cases}$$

and let $\hat{\varphi}(\xi) = \chi(\xi/2) - \chi(\xi)$, $\chi_j(\xi) = \chi(\frac{\xi}{2^j})$ and $\hat{\varphi}_j(\xi) = \hat{\varphi}(\frac{\xi}{2^j})$ for $j \in \mathbb{Z}$. Write

$$\begin{aligned} h(x) &= \mathcal{F}^{-1}\chi(\xi), \ h_j(x) = 2^{nj}h(2^j x); \\ \varphi_j(x) &= 2^{nj}\varphi(2^j x), \end{aligned}$$

where $\hat{f}(\xi)$ and $\mathcal{F}^{-1}f(\xi)$ denote the Fourier transform and inverse transform, respectively. Define the Littlewood-Paley projection operators S_j and $\triangle_j$, respectively, as

$$\begin{aligned} S_j u(x) &= h_j * u(x), \quad \text{for } j \in \mathbb{Z}, \\ \triangle_j u(x) &= \varphi_j * u(x) = S_{j+1}u(x) - S_j u(x), \quad \text{for } j \in \mathbb{Z}. \end{aligned}$$

Formally $\triangle_j$ is a frequency projection to the annulus $|\xi| \sim 2^j$, while S_j is a frequency projection to the ball $|\xi| \lesssim 2^j$ for $j \in \mathbb{Z}$. For any $u(x) \in L^2(\mathbb{R}^n)$ we have the Littlewood-Paley decomposition

$$\begin{aligned} u(x) &= h * u(x) + \sum_{j \geq 0} \varphi_j * u(x) \\ u(x) &= \sum_{j=-\infty}^{\infty} \varphi * u(x), \end{aligned}$$

where the series is convergent in the sense of L^2 norm. Clearly,

$$\begin{aligned} \text{supp}\chi(\xi) \cap \text{supp}\hat{\varphi}_j(\xi) &= \emptyset \text{ for } j \geq 1, \\ \text{supp}\hat{\varphi}_j(\xi) \cap \text{supp}\hat{\varphi}_{j'}(\xi) &= \emptyset, \text{ for } |j - j'| \geq 2. \end{aligned}$$

Next, we recall the definition of Besov spaces. Let $s \in \mathbb{R}$ and $1 \leq p, \ q \leq \infty$, the Besov space $B^s_{p,q}(\mathbb{R}^n)$, abbreviated as $B^s_{p,q}$, is defined by

$$B^s_{p,q} = \{f(x) \in \mathcal{S}(\mathbb{R}^n); \|f\|_{B^s_{p,q}} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}} = (\|h * f\|_p^q + \sum_{j \geq 0} 2^{jsq}\|\varphi_j * f\|_p^q)^{1/q}$$

is the Besov norm. The homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by the dyadic decomposition as

$$\dot{B}^s_{p,q} = \{f(x) \in \mathcal{Z}'(\mathbb{R}^n); \ \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = (\sum_{j=-\infty}^{\infty} 2^{jsq}\|\varphi_j * f\|_p^q)^{1/q}$$

is the homogeneous Besov norm, and $\mathcal{Z}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{Z}(\mathbb{R}^n) = \{f(x) \in \mathcal{S}(\mathbb{R}^n);\ D^\alpha \hat{f}(0) = 0,\ \text{for any} \alpha \in \mathbb{N}^n \text{ multi-index}\}$ and can be identified by the quotient space $\mathcal{S}'/\mathcal{P}$ with the polynomial functional set $\mathcal{P}$. In particular, if $p = q = 2$ and $s = m$ is positive integer, $\dot{B}^s_{2,2}(\mathbb{R}^n)$ and $B^s_{2,2}(\mathbb{R}^n)$ are equivalent to the Sobolev spaces $\dot{H}^m(\mathbb{R}^n)$ and $H^m(\mathbb{R}^n)$. For details, please refer to [1], [14] and [19].

For convenience, we recall the definition of Bony's para-product formula which gives the decomposition of the product $f \cdot g$ of two functions $f(x)$ and $g(x)$.

Definition 3.1. *The para-product of two functions f and g is defined by*

$$T_g f = \sum_{i \le j-2} \triangle_i g \triangle_j f = \sum_{j \in \mathbb{Z}} S_{j-1} g \triangle_j f.$$

The remainder of the para-product is defined by

$$R(f, g) = \sum_{|i-j| \le 1} \triangle_i g \triangle_j f.$$

Then Bony's para-product formula reads

$$f \cdot g = T_g f + T_f g + R(f, g). \tag{3.1}$$

Below we recall the Bernstein's lemma that will be used in proofs of our results.

Proposition 3.1. *(Bernstein's inequality)*

(a) Let $g(x) \in L^p(\mathbb{R}^n) \bigcap L^{p_1}(\mathbb{R}^n)$, and supp$\hat{g} \subset \{|\xi| \le r\}$. Then there exists a constant C such that

$$\|g\|_{p_1} \le C r^{n(\frac{1}{p} - \frac{1}{p_1})} \|g\|_p,$$

for $1 \le p \le p_1 \le \infty$.

(b) Assume that $f(x) \in L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$ and supp$\hat{f} \subset \{2^{j-1} \le |\xi| \le 2^{j+1}\}$ for $j \in \mathbb{Z}$, there exists a constant C_k so that the following inequality holds:

$$C_k^{-1} 2^{jk} \|f\|_p \le \|D^k f\|_p \le C_k 2^{jk} \|f\|_p. \tag{3.2}$$

The proof is an immediate consequence of Young's inequality, please refer to [3] for details.

Proof of Theorem 1.3 First of all, we prove the theorem under condition (1.4). Similarly as in the proof of Theorem 1.2, we differentiate the equations (1.1) with respect to x_i and x_j then multiply the resulting equations by $\partial^2_{ij} u,\ \partial^2_{ij} \omega,\ \partial^2_{ij} b$, respectively, and sum them up, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|(\partial_{ij}^2 u, \partial_{ij}^2\omega, \partial_{ij}^2 b)\|_2^2) + \sum_{k=1}^{3}\Big((\mu+\chi)\|\partial_{ijk}^3 u\|_2^2 + \gamma\|\partial_{ijk}^3\omega\|_2^2 + \nu\|\partial_{ijk}^3 b\|_2^2\Big)\\
&+\kappa\|div\partial_{ij}^2\omega\|_2^2 + 2\chi\|\partial_{ij}^2\omega\|_2^2\\
\leq\ & |(\partial_{ij}^2 u\cdot\nabla u, \partial_{ij}^2 u)| + 2|(\partial_i u\cdot\partial_j u, \partial_{ij}^2 u)| + \cdots + |(\partial_{ij}^2 u\cdot\nabla\omega, \partial_{ij}^2\omega)|\\
&+2|(\partial_i u\cdot\partial_j\omega, \partial_{ij}^2\omega)| + 2\chi|(\nabla\times\partial_{ij}^2 u, \partial_{ij}^2\omega)|,
\end{aligned}
$$

where $\partial_i f$, $\partial_{ij}^2 f$ and $\partial_{ijk}^3 f$ are short forms of $\partial_{x_i} f$, $\partial_{x_i x_j}^2 f$ and $\partial_{x_i x_j x_k}^3 f$.

Arguing similarly as the deriving of estimate (2.13) and noting the energy inequality (1.5), it is not difficult to deduce that

$$
\begin{aligned}
&\|(u,\omega,b)\|_{H^2}^2 + C_1\int_{t_0}^{t}(\|(u,\omega,b)\|_{\dot H^3}^2 + 2\kappa\|\mathrm{div}\omega\|_{\dot H^2}^2)\mathrm{d}s\\
\leq\ & \|(u(t_0),\omega(t_0),b(t_0))\|_{H^2}^2\exp\Big\{\int_{t_0)}^{t}\|(u(s),\omega(s),b(s)\|_\infty)\mathrm{d}s\Big\}.
\end{aligned}
$$

Applying a logarithmic Sobolev inequality (See [21] or [10])

$$
\|f(x)\|_\infty \leq C(1+\|f\|_{\dot B^0_{\infty,\infty}}\log(\mathrm{e}+\|f\|_{W^{s,p}})),
$$

for any $s>\frac{n}{p}$, where n is the space dimension and C is a constant independent of $f(x)$ to arrive at

$$
\begin{aligned}
\|(u,\omega,b)\|_{H^2}^2 \leq\ & \|(u(t_0),\omega(t_0),b(t_0))\|_{H^2}^2\\
&\exp\Big\{\int_{t_0}^{t}C(1+\|(u,\omega,b)(s)\|_{\dot B^0_{\infty,\infty}}\log(\mathrm{e}+\|(u,\omega,b)(s)\|_{H^2}^2))\mathrm{d}s\Big\},
\end{aligned}
$$

for some $t_0>0$. Write $Z(t)=\log(\mathrm{e}+\|(u,\omega,b)(s)\|_{H^2}^2)$, applying Gronwall inequality, we have

$$
Z(t)\leq Z(t_0)\exp\Big\{\int_{t_0}^{t}C(1+\|(u,\omega,b)(s)\|_{\dot B^0_{\infty,\infty}})\mathrm{d}s\Big\}<\infty. \tag{3.3}
$$

Next we apply the Littlewood-Paley projection operator $\triangle_j$ ($j\in\mathbb{Z}$) on both sides of equations (1.1), multiply the resulting equations by $2^{2j}\triangle_j u$, $2^{2j}\triangle_j\omega$ and $2^{2j}\triangle_j b$, respectively, then integrate them on $\mathbb{R}^3$, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}2^{2j}\|\triangle_j(u,\omega,b)\|_2^2 + 2^{2j}\sum_{i=1}^{3}((\mu+\chi)\|\triangle_j\partial_i u\|_2^2 + \gamma\|\triangle_j\partial_i\omega\|_2^2 + \nu\|\triangle_j\partial_i b\|_2^2)\\
&+\kappa 2^{2j}\|\triangle_j\mathrm{div}\omega\|_2^2 + 2\chi 2^{2j}\|\triangle_j\omega\|_2^2 \qquad\qquad (3.4)\\
\leq\ & 2^{2j}\int_{\mathbb{R}^3}(|\triangle_j(u\cdot\nabla u)\cdot\triangle_j u| + |\triangle_j(b\cdot\nabla b)\cdot\triangle_j u| + |\triangle_j(u\cdot\nabla b)\cdot\triangle_j b|\\
&+|\triangle_j(b\cdot\nabla u)\cdot\triangle_j b| + |\triangle_j(u\cdot\nabla\omega)\cdot\triangle_j\omega| + 2\chi|\nabla\times\triangle_j u\cdot\triangle_j\omega|)\mathrm{d}x\\
=\ & I_1+I_2+\cdots+I_6.
\end{aligned}
$$

We estimate I_j $(j=1,\cdots 6)$ in the following by means of Bony's para-product decomposition (3.1). It can be handled by the same method for I_j $(j=1,\cdots 4)$, so we only give an estimate for I_3. Applying the para-product decomposition (3.1) to I_3 one has

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^3} 2^{2j}\triangle_j(u\cdot\nabla b)\cdot\triangle_j b\mathrm{d}x\\
=&\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^3}\Big(\sum_{k=j-3}^{j+5}2^{2j}\triangle_j(\triangle_k u\cdot\nabla S_{k-1}b)\cdot\triangle_j b+\sum_{k=j-3}^{j+5}2^{2j}\triangle_j(\triangle_k b\cdot\nabla S_{k-1}u)\cdot\triangle_j b\\
&+\sum_{k\geq j-4}\sum_{l=k-1}^{k+1}2^{2j}\triangle_j(\triangle_k u\cdot\nabla\triangle_l b)\cdot\triangle_j b\Big)\mathrm{d}x\\
=&\ A_1+A_2+A_3.
\end{aligned}$$

Here use has been made of the fact that $S_{k-1}u\cdot\nabla\triangle_j b=\nabla\cdot(S_{k-1}\otimes\triangle_k b)=\triangle_k b\cdot\nabla S_{k-1}u$, because of the divergence free property of magnetic b. For A_1, applying Hölder inequality, Bernstein inequality (3.2) and discrete Young inequality one has

$$\begin{aligned}
A_1 &= \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^3}\sum_{l=-3}^{5}2^{2j}\triangle_j(\sum_{k\leq l+j-1}\triangle_{l+j}u\cdot\nabla\triangle_k b)\cdot\triangle_j b\mathrm{d}x\\
&\leq \sum_{j\in\mathbb{Z}}\sum_{l=-3}^{5}2^{2j}\|\triangle_{l+j}u\|_2\sum_{k\leq l+j-1}2^k\|\triangle_k b\|_\infty\|\triangle_j b\|_2\\
&\leq \sum_{l=-3}^{5}\sum_{j\in\mathbb{Z}}2^j\|\triangle_j u\|_2\sum_{k\leq j-1}\|\triangle_k b\|_\infty 2^{2(j-l)}\|\triangle_{j-l}b\|_2 2^{k-j}\\
&\leq C\|u\|_{\dot{B}^1_{2,2}}\|b\|_{\dot{B}^0_{\infty,\infty}}\|b\|_{\dot{B}^2_{2,2}}.
\end{aligned}$$

By means of the Young inequality with ε it follows

$$A_1\leq\frac{\gamma}{48}\|b\|^2_{\dot{H}^2}+C\|b\|^2_{\dot{B}^0_{\infty,\infty}}\|u\|^2_{\dot{H}^1}. \tag{3.5}$$

Arguing similarly as deriving A_1, for A_2 one also has

$$A_2\leq\frac{\gamma}{48}\|b\|^2_{\dot{H}^2}+C\|u\|^2_{\dot{B}^0_{\infty,\infty}}\|b\|^2_{\dot{H}^1}. \tag{3.6}$$

For A_3, applying Hölder inequality, Bernstein inequality (3.2) and discrete Young inequality, it can be deduced that

$$\begin{aligned}
A_3 &= \sum_{j\in\mathbb{Z}}\sum_{k\geq j-4}\sum_{m=-1}^{1}2^{2j}\triangle_j(\triangle_k u\cdot\nabla\triangle_{k+m}b)\cdot\triangle_j b\\
&\leq \sum_{m=-1}^{1}\sum_{j\in\mathbb{Z}}\sum_{k\geq j-4}\|\triangle_k u\|_\infty 2^{2(k+m)}\|\triangle_{k+m}b\|_2 2^j\|\triangle_j b\|_2 2^{j-k}2^{-m}\\
&\leq C\|u\|_{\dot{B}^0_{\infty,\infty}}\|b\|_{\dot{H}^2}\|b\|_{\dot{H}^1}.
\end{aligned}$$

According to the Young inequality with ε it follows

$$A_3 \le \frac{\gamma}{48}\|b\|^2_{\dot{H}^2} + C\|u\|^2_{\dot{B}^0_{\infty,\infty}}\|b\|^2_{\dot{H}^1}. \tag{3.7}$$

Collecting the estimates (3.5)-(3.7), one arrives at the a priori estimate of I_3 as

$$\begin{aligned}\sum_{j\in\mathbb{Z}} I_3 &= \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^3} 2^{2j}\triangle_j(u\cdot\nabla b)\cdot\triangle_j b\mathrm{d}x \\ &\le \frac{\gamma}{12}\|b\|^2_{\dot{H}^2} + C(\|u\|^2_{\dot{B}^0_{\infty,\infty}} + \|b\|^2_{\dot{B}^0_{\infty,\infty}})(\|u\|^2_{\dot{H}^1} + \|b\|^2_{\dot{H}^1}). \end{aligned}\tag{3.8}$$

It is a little complicate to estimated I_5, because ω is not divergence free. We first write I_5 in the form of a commutator, then estimate it by a cancel property of the commutator.

$$\begin{aligned}&\sum_{j\in\mathbb{Z}} 2^{2j}\int_{\mathbb{R}^3}\triangle_j(u\cdot\nabla\omega)\cdot\triangle_j\omega\mathrm{d}x\\
= &-\sum_{j\in\mathbb{Z}} 2^{2j}\int_{\mathbb{R}^3}[u,\triangle_j]\cdot\nabla\omega\cdot\triangle_j\omega\mathrm{d}x\Big(\triangleq\sum_{j\in\mathbb{Z}} 2^{2j}\int_{\mathbb{R}^3}(\triangle_j(u\cdot\nabla\omega)-u\cdot\nabla\triangle_j\omega)\cdot\triangle_j\omega\mathrm{d}x\Big)\\
= &\sum_{j\in\mathbb{Z}} 2^{2j}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\varphi_j(x-y)(u(y)-u(x))\cdot\nabla\omega(y)\mathrm{d}y\cdot\triangle_j\omega(x)\mathrm{d}x\\
= &-\sum_{j\in\mathbb{Z}} 2^{2j}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^1\varphi_j(x-y)(x-y)\cdot\nabla u(x-\tau(x-y))\cdot\nabla\omega(y)\mathrm{d}\tau\mathrm{d}y\cdot\triangle_j\omega(x)\mathrm{d}x\\
= &-\sum_{j\in\mathbb{Z}} 2^{j}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^1\varphi(z)z\cdot\nabla u(x-\tau 2^{-j}z)\cdot\nabla\omega(x-2^{-j}z)\mathrm{d}\tau\mathrm{d}z\cdot\triangle_j\omega(x)\mathrm{d}x.\end{aligned}$$

Applying the Bony's para-product decomposition (3.1) to $\nabla u(x-\tau 2^{-j}z)\cdot\nabla\omega(x-2^{-j}z)$, I_5 can be decomposed as

$$\begin{aligned}&\sum_{j\in\mathbb{Z}} 2^{2j}\int_{\mathbb{R}^3}|\triangle_j(u\cdot\nabla\omega)\cdot\triangle_j\omega|\mathrm{d}x\\
\le\ & C\sum_{j\in\mathbb{Z}} 2^{j}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_0^1\varphi(z)\Big|\Big(z\cdot\nabla S_{j-1}u(x-\tau 2^{-j}z)\cdot\nabla\triangle_j\omega(x-2^{-jz})\\
&+z\cdot\nabla\triangle_j u(x-\tau 2^{-j}z)\cdot\nabla S_{j-1}\omega(x-2^{-j}z)\\
&+\sum_{k\ge j-4} z\cdot\nabla\triangle_k u(x-\tau 2^{-j}z)\cdot\nabla\triangle_k\omega(x-2^{-j}z)\Big)\mathrm{d}z\mathrm{d}\tau\cdot\triangle_j\omega(x)\Big|\mathrm{d}x\\
=\ & B_1+B_2+B_3.\end{aligned}$$

Here we regard $k\approx j$ in the para-product term and $k\approx l$ in the remainder term for the sake of concise exposition.

$$\begin{aligned}
& B_1 \\
\leq\ & C\sum_{j\in\mathbb{Z}}\int_0^1\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} 2^j\varphi(z)|z\cdot\sum_{k\leq j-1}\nabla\triangle_k u(x-\tau 2^{-j}z) \\
& \cdot\nabla\triangle_j\omega(x-2^{-j}z)\mathrm{d}z\cdot\triangle_j\omega(x)|\mathrm{d}x\mathrm{d}\tau \\
\leq\ & C\sum_{j\in\mathbb{Z}}\int_0^1\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} 2^j\varphi(z)|z|\sum_{k\leq j-1}2^k\|\triangle_k u\|_\infty|\nabla\triangle_j\omega(x-2^{-j}z)||\triangle_j\omega(x)|\mathrm{d}x\mathrm{d}z\mathrm{d}\tau \\
\leq\ & C\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^3}|z|\varphi(z)\mathrm{d}z\sum_{k\leq j-1}\|\triangle_k u\|_\infty 2^{2j}\|\triangle_j\omega\|_2 2^j\|\triangle_j\omega\|_2 2^{k-j}
\end{aligned}$$

Thus the Young inequalities yield

$$\begin{aligned}
B_1 &\leq C\|u\|_{\dot{B}^0_{\infty,\infty}}\|\omega\|_{\dot{H}^2}\|\omega\|_{\dot{H}^1} \\
&\leq \frac{\gamma}{48}\|\omega\|^2_{\dot{H}^2}+C\|u\|^2_{\dot{B}^0_{\infty,\infty}}\|\omega\|^2_{\dot{H}^1}. \qquad (3.9)
\end{aligned}$$

Arguing similarly as the above, for B_2, one has

$$B_2\leq\frac{\gamma}{48}\|u\|^2_{\dot{H}^2}+C\|\omega\|^2_{\dot{B}^0_{\infty,\infty}}\|\omega\|^2_{\dot{H}^1}. \qquad (3.10)$$

Similarly, for B_3 it follows that

$$\begin{aligned}
B_3 &\leq C\sum_{j\in\mathbb{Z}}\sum_{k\geq j-4}2^k\|\triangle_k u\|_2 2^{2k}\|\triangle_k\omega\|_2\|\triangle_j\omega\|_\infty 2^{j-k} \\
&\leq \frac{\gamma}{48}\|\omega\|^2_{\dot{H}^2}+C\|\omega\|^2_{\dot{B}^0_{\infty,\infty}}\|u\|^2_{\dot{H}^1}. \qquad (3.11)
\end{aligned}$$

Collecting the estimates (3.9)-(3.11) to arrive at

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}I_5 &= \sum_{j\in\mathbb{Z}}2^{2j}\int_{\mathbb{R}^3}|\triangle_j(u\cdot\nabla\omega)\cdot\triangle_j\omega|\mathrm{d}x \\
&\leq \frac{\gamma}{48}(\|(u,\omega)\|^2_{\dot{H}^2}+C\|(u,\omega)\|^2_{\dot{B}^0_{\infty,\infty}}\|(u,\omega)\|^2_{\dot{H}^1}. \qquad (3.12)
\end{aligned}$$

Concerning I_6, by means of the Young inequality with ε one has

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}I_6 &= 2\chi\sum_{j\in\mathbb{Z}}2^{2j}\int_{\mathbb{R}^3}|\nabla\times\triangle_j u\cdot\triangle_j\omega|\mathrm{d}x \\
&\leq 2\chi\sum_{j\in\mathbb{Z}}(\frac{1}{4}2^{2j}\|\triangle_j\nabla u\|_2^2+2^{2j}\|\triangle_j\omega\|_2^2) \\
&\leq \frac{\chi}{2}\|\nabla u\|_{\dot{H}^1}+2\chi\|\omega\|_{\dot{H}^1}. \qquad (3.13)
\end{aligned}$$

Inserting the estimates (3.12), (3.13) of I_5 and I_6 and similar estimates to (3.8) for I_1-I_4 into (3.4) it follows

$$\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}(\|(u,\omega,b)\|^2_{\dot{H}^1})+(2\mu+\frac{1}{2}\chi)\|\nabla u\|^2_{\dot{H}^1}+\gamma\|\nabla\omega\|^2_{\dot{H}^1}+\nu\|\nabla b\|^2_{\dot{H}^1}+2\kappa\|div\omega\|^2_{\dot{H}^1} \\
\leq\ & C(\|u\|^2_{\dot{B}^0_{\infty,\infty}}+\|\omega\|^2_{\dot{B}^0_{\infty,\infty}}+\|b\|^2_{\dot{B}^0_{\infty,\infty}})(\|u\|^2_{\dot{H}^1}+\|\omega\|^2_{\dot{H}^1}+\|b\|^2_{\dot{H}^1}).
\end{aligned}$$

Gronwall inequality implies

$$\begin{aligned} & \|(u,\omega,b)\|_{\dot{H}^1}^2 \\ \leq \quad & \|(u_0,\omega_0,b_0)\|_{\dot{H}^1}^2 \exp\left\{C\int_0^t \|(u(s),\omega(s),b(s))\|_{\dot{B}^0_{\infty,\infty}}^2 \mathrm{d}s\right\} < \infty. \end{aligned} \tag{3.14}$$

Combining the energy inequality (1.5) with estimates (3.14) and (3.3), by standard arguments of continuation of local solutions, the proof of Theorem 1.3 is thus complete. □

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