Stability of Contact Discontinuity for Jin-Xin Relaxation System

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Abstract: In this paper, a contact wave for 1-dimensional Jin-Xin relaxation system [10], which is a relaxation version of contact discontinuity of the corresponding hyperbolic system, is shown to be nonlinearly stable. The time-decay rate is also obtained. The proof is given by a weighted energy estimate.

1 Introduction

The relaxation phenomena often arises in many physical situations, such as the kinetic theory, non-equilibrium gas dynamics, elasticity with memory, flood flow with friction and magnetohydrodynamics etc. Mathematically, the investigation of the behavior of the solutions to the relaxation system is an important subject.

In this paper, we consider the initial value problem of 1-dimensional Jin-Xin relaxation system [10] which reads

$$\begin{cases} u_t + v_x = 0, \\ v_t + a^2 u_x = \dfrac{1}{\varepsilon}(f(u) - v), \quad x \in \mathbb{R}^1, t \geq 0, \\ (v, u)(x, t = 0) = (v_0, u_0)(x), \quad x \in \mathbb{R}^1, \end{cases} \tag{1.1}$$

where $u = u(x,t), v = v(x,t)$ are vector-valued functions in $\mathbb{R}^n$, $f(u)$ is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$, $a > 0$ is a given constant satisfying the sub-characteristic condition (1.5) below, and $\varepsilon > 0$ represents the relaxation coefficient.

Assume that the initial data satisfies

$$(v_0(x), u_0(x)) \to (v_\pm, u_\pm), \qquad \text{as } x \to \pm\infty, \tag{1.2}$$

where $v_\pm, u_\pm$ are given constants satisfying $v_\pm = f(u_\pm)$.

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When the relaxation coefficient $\varepsilon \to 0$, formally, the first order approximation of the system (1.1) is the following conservation laws:

$$u_t + f(u)_x = 0. \tag{1.3}$$

The relaxation system (1.1) is designed by Jin-Xin in [10] to approximate the conservation laws (1.3) by the numerical scheme. The main advantage of this scheme is its generality and simplicity since the relaxation system (1.1) is semilinear.

We assume that the system (1.3) is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate, i.e., the Jacobian matrix $Df(u)$ of the flux $f(u)$ has real and distinct eigenvalues $\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$ with corresponding left and right eigenvectors $l_j(u), r_j(u)$ $(j = 1, 2, \cdots, n)$ satisfying

$$\begin{array}{l} L(u)Df(u)R(u) = diag(\lambda_1(u), \lambda_2(u), \cdots, \lambda_n(u)) \equiv \Lambda(u), \\ L(u)R(u) = Id., \end{array} \tag{1.4}$$

where $L(u) = (l_1(u), \cdots, l_n(u))^t$, $R(u) = (r_1(u), \cdots, r_n(u))$, $Id.$ = Identity matrix; and each $i-$field is either genuinely nonlinear, i.e., $\nabla\lambda_i(u) \cdot r_i(u) \neq 0$, or linearly degenerate, namely, $\nabla\lambda_i(u) \cdot r_i(u) \equiv 0$.

Under the above assumptions, it is well-known that the hyperbolic conservation laws (1.3) has rich wave phenomenon. In the genuinely nonlinear field, the nonlinear waves, i.e., shock waves or rarefaction waves, may appear, and contact discontinuities, which are the linear wave, may occur in the linearly degenerate field.

To ensure the dissipative nature of the system (1.1), it is important (may necessary) to require a sub-characteristic condition, [10], [11], i.e.,

$$-a < \lambda_i(u) < a, \qquad \forall\, u, \forall\, i = 1, 2, \cdots, n. \tag{1.5}$$

Due to the effect of the relaxation term, the system (1.1) is dissipative under the sub-characteristic condition (1.5). The elementary hyperbolic waves, i.e., shock waves, rarefaction waves and contact discontinuities, become smooth in the system (1.1). It is interesting to investigate the asymptotic stability of the relaxation versions of the hyperbolic waves in the relaxation system.

Liu [11] first considered a general 2×2 1-dimensional relaxation system and gave the stability criteria for the shock waves, rarefaction waves and also diffusion waves. Since then, many authors have stuided the stability of the shock waves and rarefaction waves to the relaxation system in 1-dimension or several space dimension under some small conditions, see [2], [3], [14], [15], [18], [19], [20] etc. However, there is no result corresponding contact discontinuities for the relaxation system (1.1) as far as we know. The investigation of the asymptotic stability of contact discontinuity for the viscous conservation laws begins with Xin [17] in 1996, which concerned with the Euler system with uniform artificial viscosity. It was first discovered in [17] that the inviscid contact discontinuity can not be an asymptotic state for the viscous system, but a viscous contact wave which approximates the contact discontinuity on any finite time interval as the viscosity tends to zero, is nonlinear stable. This is so called meta-stability [17].

In this paper, we study the meta-stability of contact discontinuities for the relaxation system (1.1) under the assumptions (1.4) and (1.5). That is, we construct a contact wave, which approximates the contact discontinuity of the corresponding hyperbolic system (1.3) on any finite time interval as the relaxation coefficient tends to zero, and prove that the contact wave is nonlinear stable. Our idea is following: it is observed that the relaxation system (1.1) is equivalent to the perturbed conservation laws with uniform artificial viscosity, see (1.9) below. We treat the perturbation term u_{tt} as a higher-order term in (1.9) due to the sub-characteristic condition (1.5) and expect that the long time behavior of the solutions to (1.1) is similar to that for the viscous conservation laws

$$u_t + f(u)_x = a^2 \varepsilon u_{xx} - \varepsilon u_{tt}. \tag{1.6}$$

For (1.6), Liu-Xin [13] in 1997 showed that the inviscid contact discontinuity is meta-stable by the pointwise estimates. Liu-Xin's analysis is based on approximated Green's function, which is very difficult to construct in many physical systems whose viscosity matrix is only semi-definite, such as compressible Navier-Stokes and Boltzmann equation. Thus it is difficult to apply Liu-Xin's approach to some physical systems. Recently Huang-Matsumura-Xin [5] and Huang-Xin-Yang [7] develop a new energy method to treat the stability of the contact discontinuity for the compressible Navier-Stokes equations and Boltzmann equation. Such approach admits that the energy estimate involving the lower order grows at the rate $(1+t)^{\frac{1}{2}}$. But it can be compensated by the decay in the energy estimate for derivatives which is of the order of $(1+t)^{-\frac{1}{2}}$ due to the underlying properties of the viscous contact wave. Thus, these reciprocal order of decay rates for the time evolution can close the priori estimate containing the uniform bounds of the L^∞ norm on the lower order estimate. This method can be widely applied to many physical systems, see [8] and [6]. In this paper, we shall apply the ideas of [5] and [7] to investigate the stability of the contact discontinuity for the relaxation system (1.1).

Assume that $p-$field of system (1.3) is linearly degenerate, i.e. $\exists p : 1 \leq p \leq n$, s.t. $\nabla \lambda_p(u) \cdot r_p(u) \equiv 0$. Consider the hyperbolic system (1.3) with the following Reimann initial data

$$u(x,0) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases}$$

Then (1.3), (1.6) admit a $p-$contact discontinuity solution

$$\hat{U}(x,t) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \end{cases} \tag{1.7}$$

provided that

$$f(u_+) - f(u_-) = s(u_+ - u_-), \quad s = \lambda_p(u_+) = \lambda_p(u_-). \tag{1.8}$$

Without loss of generality, we assume that $s \equiv 0$ in (1.8).

From (1.1), we obtain a system for $u(x,t)$ by eliminating $v(x,t)$

$$\begin{cases} u_t + f(u)_x = a^2 \varepsilon u_{xx} - \varepsilon u_{tt} \\ u(x,0) = u_0(x), \\ u_t(x,0) = -v_{0x}(x). \end{cases} \tag{1.9}$$

Now we construct the viscous $p-$contact wave for (1.1) motivated by [13]. Firstly choosing non-singular parameter ρ, we define the $p-$contact wave curve by

$$C_p(u_-) = \{u | u = u(\rho), \frac{du}{d\rho} = r_p(u(\rho)), u(\rho_-) = u_-\}. \tag{1.10}$$

Along the curve $C_p(u_-)$,

$$\frac{d\lambda_p(u(\rho))}{d\rho} = \nabla \lambda_p(u(\rho)) \cdot \frac{du(\rho)}{d\rho} = \nabla \lambda_p \cdot r_p \equiv 0.$$

So we have

$$\lambda_p(u(\rho)) = \lambda_p(u_+) = \lambda_p(u_-) \equiv 0. \tag{1.11}$$

This means that the $p-$eigenvalue $\lambda_p(u)$ is zero along the curve $C_p(u_-)$.

To define the viscous $p-$contact wave, we choose the non-singular parameter ρ in (1.10) satisfying

$$u(\rho_-) = u_-, \ u(\rho_+) = u_+,$$

and

$$\begin{cases} \rho_t - a^2 \varepsilon \rho_{xx} = 0, \qquad x \in \mathbb{R}^1, t \geq -1, \\ \rho(x, t = -1) = \begin{cases} \rho_-, & x < 0, \\ \rho_+, & x > 0. \end{cases} \end{cases} \tag{1.12}$$

Without loss of generality, we assume that $0 < \rho_- < \rho_+$. Now we define the viscous $p-$contact wave $\bar{U}(x,t)$ by

$$\bar{U}(x,t) \in C_p(u_-), \qquad \bar{U}(x,t) \equiv u(\rho(x,t)), \tag{1.13}$$

where the parameter $\rho(x,t)$ is defined in (1.12). From the construction of $\bar{U}(x,t)$, we have

$$\begin{aligned} &\bar{U}_t(x,t) = r_p(u(\rho))\rho_t, \qquad \bar{U}_x(x,t) = r_p(u(\rho))\rho_x, \\ &\bar{U}_{xx}(x,t) = r_p(u(\rho))\rho_{xx} + \nabla r_p(u(\rho)) \cdot r_p(u(\rho))(\rho_x)^2. \end{aligned} \tag{1.14}$$

Now we impose the following structure condition to the system (1.1) or (1.3)

$$\nabla r_p(u(\rho)) \cdot r_p(u(\rho)) \equiv 0, \qquad \forall \ u \in C_p(u_-), \tag{1.15}$$

in order that, on one hand, $\bar{U}(x,t)$ satisfies the equation in the conservative form (1.6) so that we can introduce the anti-derivative variable in the proof; on the other hand, the error term $\nabla r_p(u(\rho)) \cdot r_p(u(\rho))(\rho_x)^2$ vanishes, which is not good enough with the decay rate $(1+t)^{-1}$.

It is remarked that we only require here that the $p-$right eigenvector $r_p(u(\rho))$ is constant along the curve $C_p(u_-)$ in the structure condition (1.15), while in [13], the left eigenvector $l_p(u(\rho))$ is also required to be constant along the curve $C_p(u_-)$.

Under the structure condition (1.15), the viscous contact wave $\bar{U}(x,t)$ defined in (1.12) satisfies the system

$$\bar{U}_t + f(\bar{U})_x - a^2\varepsilon \bar{U}_{xx} = 0. \tag{1.16}$$

The parameter $\rho(x,t)$ in (1.12) has the following properties as $x \to \pm\infty$:

$$\begin{aligned}
&|\rho - \rho_\pm| = O(1)(\rho_+ - \rho_-)e^{-\frac{x^2}{8a^2\varepsilon(1+t)}},\\
&|\rho_x| = O(1)(\rho_+ - \rho_-)[\varepsilon(1+t)]^{-\frac{1}{2}}e^{-\frac{x^2}{8a^2\varepsilon(1+t)}},\\
&|\rho_t, \varepsilon\rho_{xx}| = O(1)(\rho_+ - \rho_-)(1+t)^{-1}e^{-\frac{x^2}{8a^2\varepsilon(1+t)}}.
\end{aligned} \tag{1.17}$$

Consequently the contact wave $\bar{U}(x,t)$ satisfies the properties:

$$\begin{aligned}
&|\bar{U} - u_\pm| = O(\delta)e^{-\frac{x^2}{8a^2\varepsilon(1+t)}},\\
&|\bar{U}_x| = O(\delta)[\varepsilon(1+t)]^{-\frac{1}{2}}e^{-\frac{x^2}{8a^2\varepsilon(1+t)}},\\
&|\bar{U}_t, \varepsilon\bar{U}_{xx}| = O(\delta)(1+t)^{-1}e^{-\frac{x^2}{8a^2\varepsilon(1+t)}},
\end{aligned} \tag{1.18}$$

as $x \to \pm\infty$, where $\delta = |u_+ - u_-| = O(1)(\rho_+ - \rho_-)$.

It is straightforward to compute that

$$\|\bar{U} - \hat{U}\|_{L^q(\mathbb{R}^1)} = O(1)\varepsilon^{\frac{1}{2q}}(1+t)^{\frac{1}{2q}}, \qquad q \geq 1.$$

where $\hat{U}$ is the inviscid contact discontinuity defined in (1.6). The above property means that the viscous contact wave $\bar{U}(x,t)$ for (1.1) approximate the inviscid contact discontinuity $\hat{U}(x,t)$ to the system (1.3) in L^q norm, $q \geq 1$, on any finite time interval as the relaxation coefficients $\varepsilon \to 0$.

In the following, we only consider the asymptotic behavior of the solutions of the system (1.1) for fixed relaxation constant ε. Without loss of generality, we fix $\varepsilon = 1$.

Usually the integral

$$\int_{-\infty}^{+\infty} (u(x,0) - \bar{U}(x,0))dx$$

does not be equal to zero. We shall introduce some linear diffusion waves to remove the excessive initial mass. We remark that the nonlinear diffusion waves is first introduced by [12] in studying the nonlinear stability of the viscous shock wave to remove the the excessive initial mass. But in our case, as in [5], it is sufficient to use the linear diffusion waves due to the different stability analysis from [12].

For weak contact discontinuity, i.e. $\delta \ll 1$, the vectors $r_1(u_-), \cdots, r_{p-1}(u_-)$, $u_+ - u_-, r_{p+1}(u_+), \cdots, r_n(u_+)$ form a basis in $\mathbb{R}^n$. We thus decompose the excessive initial mass as

$$\int_{-\infty}^{+\infty} (u(x,0) - \bar{U}(x,0))dx = \sum_{i \neq p} \alpha_i r_i(\hat{u}_i) + x_0(u_+ - u_-), \tag{1.19}$$

with the uniquely determined constants α_i $(i \neq p), x_0$, where and in the sequel, we use the notation

$$\hat{u}_i = \begin{cases} u_-, & i < p, \\ u_+, & i > p. \end{cases}$$

Define the linear diffusion waves by

$$\begin{cases} \theta_{it} + \lambda_i(\hat{u}_i)\theta_{ix} = a^2 \theta_{ixx}, & x \in \mathbb{R}^1, t \geq -1, \quad i \neq p, \\ \theta_i(x, t = -1) = \alpha_i \delta(x), \end{cases}$$

where $\delta(x)$ is the Dirac function satisfying $\int_{-\infty}^{+\infty} \delta(x)dx = 1$.

Then we have

$$\theta_i(x,t) = \frac{\alpha_i}{\sqrt{4\pi a^2(1+t)}} e^{-\frac{|x-\lambda_i(\hat{u}_i)(1+t)|^2}{4a^2(1+t)}}, \qquad \int_{-\infty}^{+\infty} \theta_i(x,t)dx = \alpha_i. \tag{1.20}$$

Now we define the ansantz $\tilde{U}(x,t)$ by

$$\tilde{U}(x,t) = \bar{U}(x + x_0, t) + \theta(x,t) \tag{1.21}$$

with $\theta(x,t) = \sum_{i \neq p} \theta_i(x,t) r_i(\hat{u}_i)$.

Thus we have

$$\int_{-\infty}^{+\infty} (u(x,0) - \tilde{U}(x,0))dx = 0. \tag{1.22}$$

A direct computations gives

$$\tilde{U}_t + \tilde{U}_{tt} + f(\tilde{U})_x - a^2 \tilde{U}_{xx} = R_x, \tag{1.23}$$

with the error term

$$\begin{aligned} R(x,t) &= [f(\tilde{U}) - f(\bar{U}) - \sum_{i \neq p} \lambda_i(\hat{u}_i)\theta_i r_i(\hat{u}_i)] \\ &\quad + [-f(\bar{U})_t + a^2 \tilde{U}_{xt} - \sum_{i \neq p} \lambda_i(\hat{u}_i)\theta_{it} r_i(\hat{u}_i)] \\ &= O(\bar{\delta})(1+t)^{-1} \sum_{i=1}^{n} e^{-\frac{|x-\lambda_i(\hat{u}_i)(1+t)|^2}{8a^2(1+t)}}, \end{aligned} \tag{1.24}$$

where we have used the fact

$$
\begin{aligned}
& f(\tilde{U}) - f(\bar{U}) - \sum_{i\neq p} \lambda_i(\hat{u}_i)\theta_i r_i(\hat{u}_i) \\
&= Df(\bar{U})\theta - \sum_{i\neq p} \lambda_i(\hat{u}_i)\theta_i r_i(\hat{u}_i) + O(1)|\theta|^2 \\
&= \sum_{i\neq p} [Df(\bar{U}) - Df(\hat{u}_i)]\theta_i r_i(\hat{u}_i) + O(1)|\theta|^2 \\
&= O(1)(\delta|\alpha| + |\alpha|^2)(1+t)^{-1} \sum_{i=1}^{n} e^{-\frac{|x-\lambda_i(\hat{u}_i)(1+t)|^2}{8a^2(1+t)}} \\
&= O(\bar{\delta})(1+t)^{-1} \sum_{i=1}^{n} e^{-\frac{|x-\lambda_i(\hat{u}_i)(1+t)|^2}{8a^2(1+t)}},
\end{aligned}
$$

and the diffusion wave strength $|\alpha| = \sum_{i\neq p} |\alpha_i|$ and $\bar{\delta} = \delta + |\alpha|$.

Without loss of generality, we assume that $x_0 = 0$ from now on. In view of the equation (1.9) for $u(x,t)$, (1.23) for $\tilde{U}(x,t)$, we have

$$
\frac{d}{dt}H(t) + \frac{d^2}{dt^2}H(t) = 0,
$$

with

$$
H(t) = \int_{-\infty}^{+\infty} (u(x,t) - \tilde{U}(x,t))dx.
$$

Thus we have for all $t \geq 0$,

$$
H(t) = \int_{-\infty}^{+\infty} (u(x,t) - \tilde{U}(x,t))dx = 0, \tag{1.25}
$$

due to the initial excessive mass H(0)=0 (see (1.22)) and

$$
\begin{aligned}
H'(0) &= \int_{-\infty}^{+\infty} (u_t(x,0) - \tilde{U}_t(x,0))dx \\
&= \int_{-\infty}^{+\infty} (-v_x(x,0) - \bar{U}_t(x,0) - \sum_{i\neq p} \theta_{it}(x,0) r_i(\hat{u}_i))dx \\
&= -(v_+ - v_-) + (f(u_+) - f(u_-)) = 0.
\end{aligned}
$$

Set the perturbation by

$$
\phi(x,t) = u(x,t) - \tilde{U}(x,t)
$$

and introduce the anti-derivative variable

$$
\Phi(x,t) = \int_{-\infty}^{x} \phi(y,t)dy.
$$

The equation (1.25) ensures that the anti-derivative variable $\Phi(x,t)$ is well-defined in some Soblev spaces like $L^2(\mathbb{R}^1), H^1(\mathbb{R}^1)$ etc.

Now we construct the ansatz $\tilde{V}(x,t)$ for $v(x,t)$. From the first equation in (1.1), we set

$$\tilde{V}(x,t) = f(\tilde{U}) - a^2\tilde{U}_x + \int_{-\infty}^{x} \tilde{U}_{tt} dx - R, \tag{1.26}$$

then we have

$$\tilde{U}_t + \tilde{V}_x = 0. \tag{1.27}$$

Set

$$\psi(x,t) = v(x,t) - \tilde{V}(x,t).$$

From $(1.1)_1$ and (1.27), we have

$$\phi_t + \psi_x = 0$$

and

$$\Phi_t = -\psi, \qquad \Phi_x = \phi. \tag{1.28}$$

Our main result is

Theorem 1.1 *Consider the relaxation problem (1.1)-(1.2) under the sub-characteristic condition (1.5) with ε being fixed to be 1. Suppose that the corresponding conservation system (1.3) satisfies the condition (1.4) and the structure condition (1.15) and $p-$character field is linearly degenerate $(1 \leq p \leq n)$. Let $\tilde{U}(x,t)$ be the ansatz in (1.21) superposed by the viscous $p-$contact wave $\bar{U}(x,t)$ and the linear diffusion wave $\theta(x,t)$ in the transversal families. Then there exists a small positive constant δ_0 such that if the wave strength $\bar{\delta}$ and the initial values $(v_0(x), u_0(x))$ satisfy*

$$\bar{\delta} + \|\Phi_0\|_{H^3}^2 + \|\psi_0\|_{H^2}^2 \leq \delta_0^2,$$

then the problem (1.1) admits a unique global solution $(v(x,t), u(x,t))$ satisfying

$$u(x,t) \in C([0,+\infty); H^2) \cap L^2(0,+\infty; H^3),$$

$$v(x,t) \in C([0,+\infty); H^1) \cap L^2(0,+\infty; H^2),$$

and

$$\|(u - \tilde{U}, v - \tilde{V})\|_{L^\infty} \leq C\delta_0(1+t)^{-\frac{1}{4}}, \tag{1.29}$$

where C is a positive constant independent of t.

The rest of the paper will be arranged as follows. In the next section, we will give the desired a priori energy estimates. Theorem 1.1 will be given in Sections 3.

2 Energy Estimate

From (1.9) and (1.23), we obtain a system for $\phi(x,t)$

$$\phi_t + \phi_{tt} + (f(u) - f(\tilde{U}))_x - a^2\phi_{xx} = -R_x, \tag{2.1}$$

Integrating the system (2.1) over $(-\infty, x)$ yields

$$\Phi_t + \Phi_{tt} + (f(u) - f(\tilde{U})) - a^2\Phi_{xx} = R. \tag{2.2}$$

Linearizing the above system (2.2) gives

$$\begin{aligned}
&\Phi_t + \Phi_{tt} + Df(\bar{U})\Phi_x - a^2\Phi_{xx} \\
&= -[f(u) - f(\bar{U}) - Df(\bar{U})(u - \bar{U})] + [f(\tilde{U}) - f(\bar{U}) - Df(\bar{U})\theta] + R \\
&=: R_1,
\end{aligned} \tag{2.3}$$

with

$$R_1 = O(1)(|\Phi_x|^2 + |\theta|^2 + R). \tag{2.4}$$

To diagonalize the system (2.3), we introduce the new variable

$$W(x,t) = L(\bar{U})\Phi(x,t), \Phi(x,t) = R(\bar{U})W(x,t), \tag{2.5}$$

where $L(\bar{U})$ and $R(\bar{U})$ are defined in (1.4). Multiplying the system (2.3) by $L(\bar{U})$ in the left, we get

$$\begin{aligned}
W_t + W_{tt} + \Lambda(\bar{U})W_x - a^2W_{xx} &= L(\bar{U})_tR(\bar{U})W + 2L(\bar{U})_t(R(\bar{U})W)_t \\
&+L(\bar{U})_{tt}R(\bar{U})W + \Lambda(\bar{U})L(\bar{U})_xR(\bar{U})W - a^2L(\bar{U})_{xx}R(\bar{U})W \\
&-2a^2L(\bar{U})_x(R(\bar{U})W)_x + L(\bar{U})R_1,
\end{aligned} \tag{2.6}$$

Let

$$W = (W_1, W_2, \cdots, W_{p-1}, W_p, W_{p+1}, \cdots, W_n)^t, \tag{2.7}$$

where and in the sequel the notation $()^t$ represents the transpose of a vector or matrix ().

Introduce a weight function

$$\eta(x,t) = \frac{\rho(x,t)}{\rho_+}, \tag{2.8}$$

where $\rho(x,t), \rho_+$ is the parameter defined in (1.12). If $\delta \ll 1$, then $|\eta(x,t) - 1| \ll 1$. Note that $0 < \rho_- < \rho_+$, thus $\rho_x > 0$.

Denote that

$$\bar{W} = (\eta^N W_1, \eta^N W_2, \cdots, \eta^N W_{p-1}, W_p, \eta^{-N} W_{p+1}, \cdots, \eta^{-N} W_n)^t,$$

where N is a large positive constant to be determined later. Also we have if N is large enough, η^N and η^{-N} is very close to 1.

Since the local existence of the solution of (2.3) is well-known, we omit the proof for brevity. To prove Theorem 1.1, it is sufficient to prove the following a priori assumption by using the continuum process,

$$N(T) = \sup_{t\in[0,T]} (\|\Phi\|_{L_\infty} + \|\phi\|_{H^2} + \|\Phi_t\|_{H^1} + (1+t)^{\frac{1}{4}}\|\phi\|_{L^2}) \le \varepsilon_0, \tag{2.9}$$

where the small positive constant ε_0 is only depending on the initial values and the wave strength $\bar{\delta}$.

Multiplying the system (2.6) by $\bar{W}$, we have

$$\begin{aligned}
&(\frac{\eta^N}{2}\sum_{i=1}^{p-1} W_i^2 + \frac{1}{2}W_p^2 + \frac{\eta^{-N}}{2}\sum_{i=p+1}^{n} W_i^2)_t - (\frac{\eta^N}{2})_t\sum_{i=1}^{p-1} W_i^2 - (\frac{\eta^{-N}}{2})_t\sum_{i=p+1}^{n} W_i^2\\
&+(\eta^N\sum_{i=1}^{p-1} W_iW_{it} + W_pW_{pt} + \eta^{-N}\sum_{i=p+1}^{n} W_iW_{it})_t - (\eta^N)_t\sum_{i=1}^{p-1} W_iW_{it}\\
&-(\eta^{-N})_t\sum_{i=p+1}^{n} W_iW_{it} - \eta^N\sum_{i=1}^{p-1} W_{it}^2 - W_{pt}^2 - \eta^{-N}\sum_{i=p+1}^{n} W_{it}^2\\
&+(\eta^N\sum_{i=1}^{p-1}\lambda_i(\bar{U})\frac{W_i^2}{2} + \eta^{-N}\sum_{i=p+1}^{n}\lambda_i(\bar{U})\frac{W_i^2}{2})_x\\
&-\eta^{N-1}\sum_{i=1}^{p-1}(N\eta_x\lambda_i(\bar{U}) + \eta\lambda_{ix}(\bar{U}))\frac{W_i^2}{2} + \eta^{-N-1}\sum_{i=p+1}^{n}(N\eta_x\lambda_i(\bar{U}) - \eta\lambda_{ix}(\bar{U}))\frac{W_i^2}{2}\\
&-a^2(\eta^N\sum_{i=1}^{p-1} W_iW_{ix} + W_pW_{px} + \eta^{-N}\sum_{i=p+1}^{n} W_iW_{ix})_x + a^2N\eta^{N-1}\eta_x\sum_{i=1}^{p-1} W_iW_{ix}\\
&+a^2(-N\eta^{-N-1}\eta_x)\sum_{i=p+1}^{n} W_iW_{ix} + a^2\eta^N\sum_{i=1}^{p-1} W_{ix}^2 + a^2W_{px}^2 + a^2\eta^{-N}\sum_{i=p+1}^{n} W_{ix}^2\\
&= \bar{W}\cdot[L(\bar{U})_tR(\bar{U})W + 2L(\bar{U})_t(R(\bar{U})W)_t + L(\bar{U})_{tt}R(\bar{U})W\\
&+\Lambda(\bar{U})L(\bar{U})_xR(\bar{U})W - a^2L(\bar{U})_{xx}R(\bar{U})W - 2a^2L(\bar{U})_x(R(\bar{U})W)_x + L(\bar{U})R_1]
\end{aligned} \tag{2.10}$$

Note that $|\lambda_{ix}(\bar{U})| \le C\rho_x$, we choose N is large enough such that

$$\begin{aligned}
&-\eta^{N-1}\sum_{i=1}^{p-1}(N\eta_x\lambda_i(\bar{U}) + \eta\lambda_{ix}(\bar{U}))\frac{W_i^2}{2}\\
&\quad+\eta^{-N-1}\sum_{i=p+1}^{n}(N\eta_x\lambda_i(\bar{U}) - \eta\lambda_{ix}(\bar{U}))\frac{W_i^2}{2}\\
&\quad-\bar{W}\cdot\Lambda(\bar{U})L(\bar{U})_xR(\bar{U})W \ge C^{-1}\rho_x\sum_{i\neq p} W_i^2
\end{aligned} \tag{2.11}$$

with a positive constant C, where we have used the construction condition (1.15) such that $r_p(\bar{U})_x = \nabla r_p(\bar{U})\cdot r_p(\bar{U})\rho_x = 0$ and

$$|\bar{W}\cdot\Lambda(\bar{U})L(\bar{U})_xR(\bar{U})W| \le C\rho_x\sum_{i\neq p} W_i^2.$$

Integrating (2.10) and using the a priori assumption (2.9) as well as (2.4) give

$$
\begin{aligned}
&[\int \eta^N \sum_{i=1}^{p-1}(\frac{W_i^2}{2}+W_iW_{it})+(\frac{W_p^2}{2}+W_pW_{pt})+\eta^{-N}\sum_{i=p+1}^{n}(\frac{W_i^2}{2}+W_iW_{it})dx]_t\\
&+a^2\|W_x\|^2-(1+C\bar{\delta})\|W_t\|^2\\
&\le C\bar{\delta}(1+t)^{-1}\|W\|^2+C(\bar{\delta}+\varepsilon_0)\|W_x\|^2dx+C\bar{\delta}(1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{2.12}
$$

Multiplying the system (2.6) by $\bar{C}W_t$ with the positive constant $\bar{C}$ determined later, we obtain

$$
\begin{aligned}
&\bar{C}\sum_{i=1}^{n}W_{it}^2+\bar{C}(\sum_{i=1}^{n}\frac{W_{it}^2}{2})_t+\bar{C}\sum_{i=1}^{n}\lambda_i(\bar{U})W_{ix}W_{it}-a^2\bar{C}(\sum_{i=1}^{n}W_{ix}W_{it})_x\\
&\quad+a^2\bar{C}(\sum_{i=1}^{n}\frac{W_{ix}^2}{2})_t=\bar{C}W_t\cdot[L(\bar{U})_tR(\bar{U})W+2L(\bar{U})_t(R(\bar{U})W)_t\\
&\quad+L(\bar{U})_{tt}R(\bar{U})W+\Lambda(\bar{U})L(\bar{U})_xR(\bar{U})W-a^2L(\bar{U})_{xx}R(\bar{U})W\\
&\quad-2a^2L(\bar{U})_x(R(\bar{U})W)_x+L(\bar{U})R_1].
\end{aligned} \tag{2.13}
$$

Integrating (2.13) yields

$$
\begin{aligned}
&\bar{C}(\int\sum_{i=1}^{n}(\frac{W_{it}}{2}+\frac{a^2W_{ix}^2}{2})dx)_t+\bar{C}\int\sum_{i=1}^{n}\lambda_i(\bar{U})W_{ix}W_{it}dx+\bar{C}\|W_t\|^2\\
&\le C\bar{\delta}(1+t)^{-1}\|W\|^2+C(\bar{\delta}+\varepsilon_0)(\|W_x\|^2+\|W_t\|^2)+C\bar{\delta}(1+t)^{-\frac{3}{2}}.
\end{aligned} \tag{2.14}
$$

Combining (2.12) and (2.14), we have

$$
\begin{aligned}
&[\int\sum_{i=1}^{p-1}(\frac{\eta^N}{2}W_i^2+\eta^NW_iW_{it}+\frac{\bar{C}}{2}W_{it}^2)+(\frac{W_p^2}{2}+W_pW_{pt}+\frac{\bar{C}}{2}W_{pt}^2)\\
&+\sum_{i=p+1}^{n}(\frac{\eta^{-N}}{2}W_i^2+\eta^{-N}W_iW_{it}+\frac{\bar{C}}{2}W_{it}^2)+\frac{a^2\bar{C}}{2}|W_x|^2dx]_t\\
&+\int\sum_{i=1}^{n}[a^2W_{ix}^2+\lambda_i(\bar{U})W_{ix}W_{it}+(\bar{C}-1)W_{it}^2]dx\\
&\le C\bar{\delta}(1+t)^{-1}\|W\|^2+C(\bar{\delta}+\varepsilon_0)(\|W_x\|^2+\|W_t\|^2)+C\bar{\delta}(1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{2.15}
$$

Choosing N large enough, $\bar{C}$ suitably, s.t. the discriminant of each quadratic in the left side of (2.15) is strictly negative, i.e.

$$
(\eta^N)^2-4\times\frac{\eta^N}{2}\times\frac{\bar{C}}{2}<0, 1^2-4\times\frac{1}{2}\times\frac{\bar{C}}{2}<0,
$$

$$
(\eta^{-N})^2-4\times\frac{\eta^{-N}}{2}\times\frac{\bar{C}}{2}<0, (\bar{C}\lambda_i(\bar{U}))^2-4a^2(\bar{C}-1)<0,
$$

Set $\bar{C}=2-\beta$, where $\beta\ll 1$, then all the above inequalities hold when N is large enough due to the sub-characteristic condition (1.5) and $\eta^{\pm N}\sim 1$. The sub-characteristic condition (1.5) plays a crucial role in our proof. With $\bar{C}$ being chosen above, then $\exists C_1 > 0$, s.t.

$$\begin{aligned}
&C_1^{-1}(\|W\|^2 + \|W_x\|^2 + \|W_t\|^2) \le E_1 \le C_1(\|W\|^2 + \|W_x\|^2 + \|W_t\|^2), \\
&\int \sum_{i=1}^{n} [a^2 W_{ix}^2 + \lambda_i(\bar{U}) W_{ix} W_{it} + (\bar{C} - 1) W_{it}^2] dx \ge C_1^{-1}(\|W_x\|^2 + \|W_t\|^2) =: K_1
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
E_1 \equiv &\int \sum_{i=1}^{p-1} (\frac{\eta^N}{2} W_i^2 + \eta^N W_i W_{it} + \frac{\bar{C}}{2} W_{it}^2) + (\frac{W_p^2}{2} + W_p W_{pt} + \frac{\bar{C}}{2} W_{pt}^2) + \\
&+ \sum_{i=p+1}^{n} (\frac{\eta^{-N}}{2} W_i^2 + \eta^{-N} W_i W_{it} + \frac{\bar{C}}{2} W_{it}^2) dx + \frac{1}{2} a^2 \bar{C} \|W_x\|^2.
\end{aligned} \tag{2.17}$$

Furthermore, choosing $\bar{\delta}, \varepsilon_0$ small enough in (2.15), we have the lower estimate

Lemma 2.1. It follows that

$$E_{1t} + \frac{1}{2} K_1 \le C\bar{\delta}(1+t)^{-1} E_1 + C\bar{\delta}(1+t)^{-\frac{1}{2}}, \tag{2.18}$$

where E_1 and K_1 are defined in (2.16) and (2.17).

Now we estimate the higher order estimate of $\Phi_x = \phi$. Let

$$Z(x,t) = L(\bar{U})\phi(x,t),$$

then

$$\phi(x,t) = R(\bar{U}) Z(x,t).$$

Applying ∂_x to the system (2.3), we have the system for $\phi(x,t)$

$$\phi_t + \phi_{tt} + (Df(\bar{U})\phi)_x - a^2 \phi_{xx} = R_{1x}. \tag{2.19}$$

Multiplying (2.19) by $L(\bar{U})$ in the left, we get the system for $Z(x,t)$,

$$\begin{aligned}
&Z_t + Z_{tt} + (\Lambda(\bar{U})Z)_x - a^2 Z_{xx} = L(\bar{U})_x \Lambda(\bar{U}) R(\bar{U}) Z + L(\bar{U})_t R(\bar{U}) Z \\
&+ L(\bar{U})_{tt} R(\bar{U}) Z + 2L(\bar{U})_t (R(\bar{U})Z)_t - a^2 L(\bar{U})_{xx} R(\bar{U}) Z \\
&- 2a^2 L(\bar{U})_x (R(\bar{U})Z)_x + L(\bar{U}) R_{1x}.
\end{aligned} \tag{2.20}$$

Let

$$\begin{aligned}
Z &= (Z_1, \cdots, Z_{p-1}, Z_p, Z_{p+1}, \cdots, Z_n)^t, \\
\bar{Z} &= (\eta^N Z_1, \cdots, \eta^N Z_{p-1}, Z_p, \eta^{-N} Z_{p+1}, \cdots, \eta^{-N} Z_n)^t,
\end{aligned}$$

with the weight function $\eta(x,t)$ defined in (2.8) and N being large constant to be determined later. Here N may be different from the previous one, and we use the same notation without confusion.

Multiplying $\bar{Z}$ to the system (2.20), we obtain

$$
\begin{aligned}
&(\frac{\eta^N}{2}\sum_{i=1}^{p-1}Z_i^2+\frac{1}{2}Z_p^2+\frac{\eta^{-N}}{2}\sum_{i=p+1}^{n}Z_i^2)_t-(\frac{\eta^N}{2})_t\sum_{i=1}^{p-1}Z_i^2-(\frac{\eta^{-N}}{2})_t\sum_{i=p+1}^{n}Z_i^2\\
&+(\eta^N\sum_{i=1}^{p-1}Z_iZ_{it}+Z_pZ_{pt}+\eta^{-N}\sum_{i=p+1}^{n}Z_iZ_{it})_t-(\eta^N)_t\sum_{i=1}^{p-1}Z_iZ_{it}\\
&-(\eta^{-N})_t\sum_{i=p+1}^{n}Z_iZ_{it}-(\eta^N\sum_{i=1}^{p-1}Z_{it}^2+Z_{pt}^2+\eta^{-N}\sum_{i=p+1}^{n}Z_{it}^2)\\
&+(\eta^N\sum_{i=1}^{p-1}\lambda_i(\bar{U})\frac{Z_i^2}{2}+\eta^{-N}\sum_{i=p+1}^{n}\lambda_i(\bar{U})\frac{Z_i^2}{2})_x\\
&-\eta^{N-1}\sum_{i=1}^{p-1}(N\eta_x\lambda_i(\bar{U})-\eta\lambda_{ix}(\bar{U}))\frac{Z_i^2}{2}+\eta^{-N-1}\sum_{i=p+1}^{n}(N\eta_x\lambda_i(\bar{U})+\eta\lambda_{ix}(\bar{U}))\frac{Z_i^2}{2}\\
&-a^2(\eta^N\sum_{i=1}^{p-1}Z_iZ_{ix}+Z_pZ_{px}+\eta^{-N}\sum_{i=p+1}^{n}Z_iZ_{ix})_x+a^2N\eta^{N-1}\eta_x\sum_{i=1}^{p-1}Z_iZ_{ix}\\
&+a^2(-N\eta^{-N-1}\eta_x)\sum_{i=p+1}^{n}Z_iZ_{ix}+a^2\eta^N\sum_{i=1}^{p-1}Z_{ix^2}+a^2Z_{px}^2+a^2\eta^{-N}\sum_{i=p+1}^{n}Z_{ix}^2\\
&=\bar{Z}\cdot\{L(\bar{U})_x\Lambda(\bar{U})R(\bar{U})Z+L(\bar{U})_tR(\bar{U})Z+L(\bar{U})_{tt}R(\bar{U})Z+2L(\bar{U})_t(R(\bar{U})Z)_t\\
&-a^2L(\bar{U})_{xx}R(\bar{U})Z-2a^2L(\bar{U})_x(R(\bar{U})Z)_x+L(\bar{U})R_{1x}\}.
\end{aligned}
\tag{2.21}
$$

Note that $R_1=-[f(u)-f(\bar{U})-Df(\bar{U})(u-\bar{U})]+[f(\tilde{U})-f(\bar{U})-Df(\bar{U})\theta]+R$, we calculate $\int\bar{Z}\cdot L(\bar{U})R_{1x}dx$ term by term. We have

$$
\begin{aligned}
&\int\bar{Z}\cdot L(\bar{U})R_xdx=\int[-\bar{Z}_x\cdot L(\bar{U})R-\bar{Z}\cdot L(\bar{U})_xR]dx\\
&\le C\bar{\delta}(1+t)^{-1}\|Z\|^2+C\bar{\delta}\|Z_x\|^2+C\bar{\delta}(1+t)^{-\frac{3}{2}},
\end{aligned}
\tag{2.22}
$$

$$
\begin{aligned}
&\int\bar{Z}\cdot L(\bar{U})[f(u)-f(\bar{U})-Df(\bar{U})(R(\bar{U})Z+\theta)]_xdx\\
&=\int\bar{Z}\cdot L(\bar{U})\{[Df(u)-Df(\bar{U})](R(\bar{U})Z+\theta)_x\\
&\quad+[Df(u)-Df(\bar{U})]\bar{U}_x-\nabla^2f(\bar{U})(\bar{U}_x,R(\bar{U})Z+\theta)\}dx\\
&\le C\int|Z|[(|Z|+|\theta|)(|Z_x|+|\theta_x|+\rho_x|Z|)+\rho_x(|Z|^2+|\theta|^2)]dx\\
&\le\varepsilon_1\|Z_x\|^2+C_{\varepsilon_1}\|Z\|^6+C\bar{\delta}(1+t)^{-1}\|Z\|^2+C\bar{\delta}(1+t)^{-\frac{3}{2}},
\end{aligned}
\tag{2.23}
$$

and

$$
\begin{aligned}
&\int\bar{Z}\cdot L(\bar{U})[f(\tilde{U})-f(\bar{U})-Df(\bar{U})\theta]_xdx\\
&\le C\bar{\delta}\|Z_x\|^2+C\bar{\delta}(1+t)^{-1}\|Z\|^2+C\bar{\delta}(1+t)^{-\frac{3}{2}}.
\end{aligned}
\tag{2.24}
$$

Choosing N large enough and integrating (2.21) over $\mathbb{R}$, we obtain

$$\begin{aligned}
&[\int \eta^N \sum_{i=1}^{p-1}(\frac{Z_i^2}{2}+Z_iZ_{it})+(\frac{Z_p^2}{2}+Z_pZ_{pt})+\eta^{-N}\sum_{i=p+1}^{n}(\frac{Z_i^2}{2}+Z_iZ_{it})dx]_t\\
&+a^2\|Z_x\|^2-\|Z_t\|^2\\
&\le C\bar{\delta}(1+t)^{-1}\|Z\|^2+C(\bar{\delta}+\varepsilon_0)(\|Z_x\|^2+\|Z_t\|^2)+C\bar{\delta}(1+t)^{-\frac{3}{2}}
\end{aligned} \tag{2.25}$$

Multiplying (2.20) by $\bar{C}Z_t$ with $\bar{C}=2-\beta,\beta\ll 1$, gives

$$\begin{aligned}
&\bar{C}\sum_{i=1}^{n}Z_{it}^2+\bar{C}(\sum_{i=1}^{n}\frac{Z_{it}^2}{2})_t+\bar{C}\sum_{i=1}^{n}\lambda_i(\bar{U})Z_{ix}Z_{it}+\bar{C}\sum_{i=1}^{n}\lambda_{ix}(\bar{U})Z_iZ_{it}\\
&-a^2\bar{C}(\sum_{i=1}^{n}Z_{ix}Z_{it})_x+a^2\bar{C}(\sum_{i=1}^{n}\frac{Z_{ix}^2}{2})_t=\bar{C}Z_t\cdot[L(\bar{U})_x\Lambda(\bar{U})R(\bar{U})Z\\
&+L(\bar{U})_tR(\bar{U})Z+2L(\bar{U})_t(R(\bar{U})Z)_t+L(\bar{U})_{tt}R(\bar{U})Z-a^2L(\bar{U})_{xx}R(\bar{U})Z\\
&-2a^2L(\bar{U})_x(R(\bar{U})Z)_x+L(\bar{U})R_{1x}]
\end{aligned} \tag{2.26}$$

Integrating (2.26) yields

$$\begin{aligned}
&\bar{C}(\int\sum_{i=1}^{n}(\frac{Z_{it}}{2}+\frac{a^2Z_{ix}^2}{2})dx)_t+\bar{C}\int\sum_{i=1}^{n}\lambda_i(\bar{U})Z_{ix}Z_{it}dx+\bar{C}\|Z_t\|^2\\
&\le C\bar{\delta}(1+t)^{-1}\|Z\|^2+C(\bar{\delta}+\varepsilon_0)(\|Z_x\|^2+\|Z_t\|^2)+C\bar{\delta}(1+t)^{-\frac{3}{2}},
\end{aligned} \tag{2.27}$$

Combining (2.25) and (2.27), we have

$$\begin{aligned}
&[\int\sum_{i=1}^{p-1}(\frac{\eta^N}{2}Z_i^2+\eta^NZ_iZ_{it}+\frac{\bar{C}}{2}Z_{it}^2)+(\frac{Z_p^2}{2}+Z_pZ_{pt}+\frac{\bar{C}}{2}Z_{pt}^2)\\
&+\sum_{i=p+1}^{n}(\frac{\eta^{-N}}{2}Z_i^2+\eta^{-N}Z_iZ_{it}+\frac{\bar{C}}{2}Z_{it}^2)+\frac{a^2\bar{C}}{2}\sum_{i=1}^{n}Z_{ix}^2dx]_t\\
&+\int\sum_{i=1}^{n}[a^2Z_{ix}^2+\lambda_i(\bar{U})Z_{ix}Z_{it}+(\bar{C}-1)Z_{it}^2]dx\\
&\le C\bar{\delta}(1+t)^{-1}\|Z\|^2+C(\bar{\delta}+\varepsilon_0)(\|Z_x\|^2+\|Z_t\|^2)+C\bar{\delta}(1+t)^{-\frac{3}{2}}.
\end{aligned} \tag{2.28}$$

Choosing N large enough, $\bar{\delta},\varepsilon_0$ small enough as in (2.18), there $\exists C_2>0$, s.t.

$$\begin{aligned}
&C_2^{-1}(\|Z\|^2+\|Z_x\|^2+\|Z_t\|^2)\le E_2\le C_1(\|Z\|^2+\|Z_x\|^2+\|Z_t\|^2),\\
&\int\sum_{i=1}^{n}[a^2Z_{ix}^2+\lambda_i(\bar{U})Z_{ix}Z_{it}+(\bar{C}-1)Z_{it}^2]dx\ge C_2^{-1}(\|Z_x\|^2+\|Z_t\|^2)=:K_2,
\end{aligned} \tag{2.29}$$

where

$$
\begin{aligned}
E_2 \equiv \int \sum_{i=1}^{p-1}(\frac{\eta^N}{2}Z_i^2 + \eta^N Z_i Z_{it} + \frac{\bar{C}}{2}Z_{it}^2) + (\frac{Z_p^2}{2} + Z_p Z_{pt} + \frac{\bar{C}}{2}Z_{pt}^2) + \\
+ \sum_{i=p+1}^{n}(\frac{\eta^{-N}}{2}Z_i^2 + \eta^{-N} Z_i Z_{it} + \frac{\bar{C}}{2}Z_{it}^2)dx + \frac{1}{2}a^2\bar{C}\|Z_x\|^2.
\end{aligned}
\tag{2.30}
$$

Thus we have the higher order estimate

Lemma 2.2. It follows that

$$
E_{2t} + \frac{1}{2}K_2 \leq C\bar{\delta}(1+t)^{-1}\|Z\|^2 + C\bar{\delta}(1+t)^{-\frac{3}{2}}, \tag{2.31}
$$

where E_2 and K_2 are defined in (2.29) and (2.30).

Then we estimate the term $Z_{xx}(x,t)$ and $Z_{xt}(x,t)$. Multiplying the system (2.20) by $-Z_{xx}$, we obtain

$$
\begin{aligned}
& -(Z_t \cdot Z_x)_x + (\sum_{i=1}^{n}\frac{Z_{ix}^2}{2})_t - (Z_{tt} \cdot Z_x)_x + (Z_x \cdot Z_{xt})_t - |Z_{xt}|^2 \\
& - \sum_{i \neq p}\lambda_{ix}(\bar{U})Z_i Z_{ixx} - \sum_{i \neq p}\lambda_{ix}(\bar{U})Z_{ix}Z_{ixx} + a^2\sum_{i=1}^{n}Z_{ixx}^2 \\
& = -Z_{xx} \cdot \{L(\bar{U})_x\Lambda(\bar{U})R(\bar{U})Z + L(\bar{U})_t R(\bar{U})Z \\
& + L(\bar{U})_{tt}R(\bar{U})Z + 2L(\bar{U})_t(R(\bar{U})Z)_t - a^2 L(\bar{U})_{xx}R(\bar{U})Z \\
& - 2a^2 L(\bar{U})_x(R(\bar{U})Z)_x + L(\bar{U})R_{1x}\}.
\end{aligned}
\tag{2.32}
$$

Integrating the above system yields

$$
\begin{aligned}
& [\int \sum_{i=1}^{n}(\frac{Z_{ix}^2}{2} + Z_{ix}Z_{ixt})dx]_t + a^2\|Z_{xx}\|^2 - \|Z_{xt}\|^2 \\
& - \int \sum_{i \neq p}\lambda_i(\bar{U})Z_{ix}Z_{ixx}dx \leq C\bar{\delta}(1+t)^{-1}\|Z\|^2 \\
& + C(\bar{\delta} + \varepsilon_0)(\|Z_{xx}\|^2 + \|Z_t\|^2 + \|Z_x\|^2) + C\bar{\delta}(1+t)^{-\frac{3}{2}}.
\end{aligned}
\tag{2.33}
$$

Applying ∂_x to the system (2.20), we get

$$
\begin{aligned}
& Z_{xt} + Z_{xtt} + (\Lambda(\bar{U})Z)_{xx} - a^2 Z_{xxx} = \{L(\bar{U})_x\Lambda(\bar{U})R(\bar{U})Z + L(\bar{U})_t R(\bar{U})Z \\
& + L(\bar{U})_{tt}R(\bar{U})Z + 2L(\bar{U})_t(R(\bar{U})Z)_t - a^2 L(\bar{U})_{xx}R(\bar{U})Z \\
& - 2a^2 L(\bar{U})_x(R(\bar{U})Z)_x + L(\bar{U})R_{1x}\}_x.
\end{aligned}
\tag{2.34}
$$

Multiplying (2.34) by $\bar{C}Z_{xt}$ with $\bar{C}=2-\beta, \beta \ll 1$, we get

$$\begin{aligned}
&\bar{C}Z_{xt}^2+\bar{C}(\frac{Z_{xt}^2}{2})_t+\bar{C}\sum_{i=1}^n\lambda_i(\bar{U})Z_{ixx}Z_{ixt}+2\bar{C}\sum_{i=1}^n\lambda_{ix}(\bar{U})Z_{ix}Z_{ixt}\\
&+\bar{C}\sum_{i=1}^n\lambda_{ixx}(\bar{U})Z_iZ_{ixt}-a^2\bar{C}(Zxx\cdot Z_{xt})_x+a^2\bar{C}(\frac{Z_{xx}^2}{2})_t\\
&=\bar{C}Z_{xt}\cdot\{L(\bar{U})_x\Lambda(\bar{U})R(\bar{U})Z+L(\bar{U})_tR(\bar{U})Z\\
&+L(\bar{U})_{tt}R(\bar{U})Z+2L(\bar{U})_t(R(\bar{U})Z)_t-a^2L(\bar{U})_{xx}R(\bar{U})Z\\
&-2a^2L(\bar{U})_x(R(\bar{U})Z)_x+L(\bar{U})R_{1x}\}_x.
\end{aligned}\tag{2.35}$$

Integrating (2.35) implies

$$\begin{aligned}
&\bar{C}[\int\sum_{i=1}^n(\frac{Z_{ixt}^2}{2}+\frac{a^2Z_{ixx}^2}{2})dx]_t+\bar{C}\|Z_{xt}\|^2+\bar{C}\int\sum_{i=1}^n\lambda_i(\bar{U})Z_{ixx}Z_{ixt}dx\\
&\le C\bar{\delta}(1+t)^{-1}\|Z\|^2+C(\bar{\delta}+\varepsilon_0)(\|Z_x\|^2+\|Z_t\|^2+\|Z_{xx}\|^2+\|Z_{xt}\|^2)\\
&+C\bar{\delta}(1+t)^{-\frac{3}{2}}.
\end{aligned}\tag{2.36}$$

Combining (2.33) and (2.36), we have

$$\begin{aligned}
&[\int\sum_{i=1}^n(\frac{Z_{ix}^2}{2}+Z_{ix}Z_{ixt}+\frac{\bar{C}Z_{ixt}^2}{2})+\frac{a^2\bar{C}}{2}\sum_{i=1}^nZ_{ixx}^2dx]_t\\
&+\int\sum_{i=1}^n[a^2Z_{ixx}^2+\bar{C}\lambda_i(\bar{U})Z_{ixx}Z_{ixt}+(\bar{C}-1)Z_{ixt}^2]dx\\
&\le\int\sum_{i\neq p}\lambda_i(\bar{U})Z_{ix}Z_{ixx}dx+C\bar{\delta}(1+t)^{-1}\|Z\|^2\\
&+C(\bar{\delta}+\varepsilon_0)(\|Z_x\|^2+\|Z_t\|^2+\|Z_{xx}\|^2+\|Z_{xt}\|^2)+C\bar{\delta}(1+t)^{-\frac{3}{2}}\\
&\le C\bar{\delta}(1+t)^{-1}\|Z\|^2+C(\bar{\delta}+\varepsilon_0+\varepsilon_2)\|Z_{xx}\|^2\\
&+C_{\varepsilon_2}\|Z_x\|^2+C(\bar{\delta}+\varepsilon_0)(\|Z_t\|^2+\|Z_{xt}\|^2)+C\bar{\delta}(1+t)^{-\frac{3}{2}},
\end{aligned}\tag{2.37}$$

where we have used Young inequality in the second inequality and ε_2 is a small positive constant.

Thus $\exists C_3, C_4>0$, s.t.

$$\begin{aligned}
&[\int\sum_{i=1}^n(\frac{Z_{ix}^2}{2}+Z_{ix}Z_{ixt}+Z_{ixt}^2)+a^2\sum_{i=1}^nZ_{ixx}^2dx]_t\\
&+C_3^{-1}(\|Z_{xx}\|^2+\|Z_{xt}\|^2)\\
&\le C_4\bar{\delta}(1+t)^{-1}\|Z\|^2dx+C_4\|Z_x\|^2\\
&+C_4(\bar{\delta}+\varepsilon_0)\|Z_t\|^2+C_4\bar{\delta}(1+t)^{-\frac{3}{2}},
\end{aligned}\tag{2.38}$$

if we take $\bar{\delta}, \varepsilon_0, \varepsilon_2$ small enough in (2.37).

Multiplying (2.31) a large constant $\hat{C}>1$ s.t.

$$\frac{\hat{C}}{2}K_2-C_4\|Z_x\|^2\ge\frac{\hat{C}}{4}K_2,$$

then combining to (2.38), we get

Lemma 2.3. It follows that

$$E_{3t} + K_3 \leq C\bar{\delta}(1+t)^{-1}\|Z\|^2 + C\bar{\delta}(1+t)^{-\frac{3}{2}}, \tag{2.39}$$

where

$$E_3 = \hat{C}E_2 + \int \sum_{i=1}^{n} (\frac{Z_{ix}^2}{2} + Z_{ix}Z_{ixt} + Z_{ixt}^2)dx + a^2\|Z_{xx}\|^2,$$

$$K_3 = \frac{\hat{C}}{4}K_2 + C_3^{-1}(\|Z_{xx}\|^2 + \|Z_{xt}\|^2).$$

In order to get the decay rate of $\psi = -\Phi_t$, finally we estimate the term Φ_{tt}. We apply ∂_t to the system (2.3),

$$\Phi_{tt} + \Phi_{ttt} + \nabla^2 f(\bar{U})(\bar{U}_t, \Phi_x) + Df(\bar{U})\Phi_{xt} - a^2\Phi_{xxt} = R_{1t}. \tag{2.40}$$

Multiplying (2.40) by Φ_{tt}, we have

$$\begin{aligned} &\Phi_{tt}^2 + (\frac{\Phi_{tt}^2}{2})_t + \Phi_{tt} \cdot \nabla^2 f(\bar{U})(\bar{U}_t, \Phi_x) + Df(\bar{U})(\Phi_{xt}, \Phi_{tt}) \\ &-a^2(\Phi_{xt} \cdot \Phi_{tt})_x + (\frac{a^2\Phi_{xt}^2}{2})_t = \Phi_{tt} \cdot R_{1t}. \end{aligned} \tag{2.41}$$

Integrating (2.41) and using Young inequality yield

$$\begin{aligned} &(\int \frac{\Phi_{tt}^2}{2} + \frac{a^2\Phi_{xt}^2}{2}dx)_t + \int \Phi_{tt}^2 dx \leq C\bar{\delta}(1+t)^{-1}\|\Phi_x\|^2 \\ &+C(\bar{\delta} + \varepsilon_3)\|\Phi_{tt}\|^2 + C_{\varepsilon_3}\{\|\Phi_{xt}\|^2 + \int |R_{1t}|^2 dx\}. \end{aligned} \tag{2.41}$$

Note that

$$|[f(u) - f(\bar{U}) - Df(\bar{U})(u - \bar{U})]_t| = O(1)[(|\phi| + |\theta|)(|\phi_t| + |\theta_t| + |\bar{U}_t|)],$$

and

$$|[f(\tilde{U}) - f(\bar{U}) - Df(\bar{U})\theta]_t| = O(1)|\theta|(|\theta_t| + |\bar{U}_t|).$$

Thus (2.3)-(2.4) and (2.41) give

$$\begin{aligned} &(\int \frac{\Phi_{tt}^2}{2} + \frac{a^2\Phi_{xt}^2}{2}dx)_t + \frac{1}{2}\|\Phi_{tt}\|^2 \\ &\leq C\bar{\delta}(1+t)^{-1}\|\phi\|^2 + C\|\phi_t\|^2 + C\bar{\delta}(1+t)^{-\frac{3}{2}} \\ &\leq C_5\bar{\delta}(1+t)^{-1}\|Z\|^2 + C_5\|Z_t\|^2 + C_5\bar{\delta}(1+t)^{-\frac{3}{2}}, \end{aligned} \tag{2.42}$$

if we choose $\bar{\delta}, \varepsilon_3$ small enough.

Multiplying (2.39) by a large constant $\tilde{C} > 1$ such that

$$\tilde{C}K_3 - C_5\|Z_t\|^2 \geq \frac{\tilde{C}}{2}K_3,$$

we get our desired higher order estimate

$$\left\{\begin{array}{l} E_{4t}+K_4\le C\bar{\delta}(1+t)^{-1}\|Z\|^2+C\|Z\|^6+C\bar{\delta}(1+t)^{-\frac{3}{2}}, \\ E_4=\tilde{C}E_3+\displaystyle\int\frac{\Phi_{tt}^2}{2}+\frac{a^2\Phi_{xt}^2}{2}dx, \qquad K_4=\frac{\tilde{C}}{2}K_3+\frac{1}{2}\int\Phi_{tt}^2dx. \end{array}\right. \tag{2.43}$$

Note that

$$\begin{aligned}\int|Z|^2dx&=\int|L(\bar{U})\Phi_x|^2dx=\int|L(\bar{U})(R(\bar{U})W)_x|^2dx\\ &\le C\bar{\delta}(1+t)^{-1}\int|W|^2dx+C\int|W_x|^2dx\\ &\le C\bar{\delta}(1+t)^{-1}E_1+CK_1.\end{aligned}$$

Thus from (2.43) and the a priori assumption $(1+t)^{\frac{1}{4}}\|\phi\|\le\varepsilon_0$, we have

Lemma 2.4. It follows that

$$E_{4t}+K_4\le C\bar{\delta}(1+t)^{-2}E_1+C(\bar{\delta}+\varepsilon_0^4)(1+t)^{-1}K_1+C\bar{\delta}(1+t)^{-\frac{3}{2}}, \tag{2.44}$$

where E_4 and K_4 are defined in (2.43).

3 Time decay rate

In view of the lower estimate (2.18), we have

$$E_1\le C(E_1(0)+\bar{\delta})(1+t)^{\frac{1}{2}}. \tag{3.1}$$

Set

$$E_5=E_1+E_4, \qquad K_5=\frac{1}{2}K_1+K_4$$

From (2.18), (2.44), (3.1), we have

$$E_{5t}+K_5\le C\bar{\delta}(1+t)^{-1}E_5+C\bar{\delta}(1+t)^{-\frac{1}{2}},$$

which gives

$$E_5\le C(E_5(0)+\bar{\delta})(1+t)^{\frac{1}{2}}, \qquad \int_0^t K_5d\tau\le C(E_5(0)+\bar{\delta})(1+t)^{\frac{1}{2}}. \tag{3.2}$$

In terms of (3.1) and (3.2), (2.44) implies

$$\begin{aligned}[(1+t)E_4]_t\ &=E_4+(1+t)E_{4t}\\ &\le E_4+C\bar{\delta}(1+t)^{-1}E_1+C(\bar{\delta}+\varepsilon_0^4)K_1+C\bar{\delta}(1+t)^{-\frac{1}{2}}\\ &\le CK_5+C\bar{\delta}(1+t)^{-\frac{1}{2}}.\end{aligned}$$

Integrating the above inequality yields

$$E_4\le C(E_5(0)+\bar{\delta})(1+t)^{-\frac{1}{2}}. \tag{3.3}$$

From (3.1), (3.3), we verify the a priori assumption (2.9)

$$\|\Phi\|_{L_\infty}^2 \le C\|\Phi\|_{L_2}\|\phi\|_{L_2} \le C\|W\|_{L_2}\|Z\|_{L_2} \le CE_1^{\frac{1}{2}}E_4^{\frac{1}{2}} \le C(E_5(0)+\bar{\delta}),$$

$$(1+t)^{\frac{1}{2}}\|\phi\|^2 \le C(1+t)^{\frac{1}{2}}E_4 \le C(E_5(0)+\bar{\delta}),$$

and

$$\|\phi\|_{H^2}^2 \le CE_4 \le C(E_5(0)+\bar{\delta})(1+t)^{-\frac{1}{2}}.$$

Also we can get the desired decay rate of ϕ, ϕ_x in the L_∞ norm

$$\|(\phi,\phi_x)\|_{L_\infty} \le C\|\phi\|_{H^2} \le C(E_5(0)+\bar{\delta})^{\frac{1}{2}}(1+t)^{-\frac{1}{4}}.$$

Now we verify the decay rate of $\psi = -\Phi_t$. From the system (2.3), we get

$$\Phi_t = -\Phi_{tt} - Df(\bar{U})\Phi_x + a^2\Phi_{xx} + R.$$

Thus

$$\begin{aligned}\|\Phi_t\|_{L_2} &\le C(\|\Phi_{tt}\|_{L_2} + \|\Phi_x\|_{L_2} + \|\Phi_{xx}\|_{L_2} + \|R\|_{L_2})\\ &\le CE_4^{\frac{1}{2}} + C\bar{\delta}(1+t)^{-\frac{1}{4}}\\ &\le C(E_5(0)+\bar{\delta})^{\frac{1}{2}}(1+t)^{-\frac{1}{4}}.\end{aligned}$$

Finally,

$$\|\psi\|_{L_\infty} = \|\Phi_t\|_{L_\infty} \le C\|\Phi_t\|_{L_2}^{\frac{1}{2}}\|\Phi_{xt}\|_{L_2}^{\frac{1}{2}} \le C(E_5(0)+\bar{\delta})^{\frac{1}{2}}(1+t)^{-\frac{1}{4}}.$$

Note that

$$E_5(0) \approx \|\Phi_0\|_{H^3}^2 + \|\psi_0\|_{H^1}^2.$$

Thus Theorem 1.1 is proved.

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