# Transonic Shock Wave in an Infinite Nozzle with Decay Cross-Sections

**Feng Xie**[†,‡], **Chunpeng Wang**[§,‡]

[†] LCP, Institute of Applied Physics and Computational Mathematics,
P.O. Box 8009, Beijing 100088, China

[‡] The Institute of Mathematical Sciences, CUHK Shatin N.T. HongKong

[§] Department of Mathematics, Jilin University Changchun,
Jilin, 130012, P.R.China

E-mails: tzxief@yahoo.com, matwcp@email.jlu.edu.cn

**Abstract**

We construct a single transonic shock wave pattern in an infinite curved nozzle with decay cross-section, which is close to a unform transonic shock wave. In other words, suppose there is a uniform transonic shock wave in a non-curved nozzle which can be constructed easily, if we perturbed the supersonic incoming flow and the infinite nozzle a little bit, we can obtain a transonic wave near the uniform one. As a consequence, we can show that the uniform transonic wave is stable with respect to the perturbation of the incoming flow and nozzle wall. Based on the theory of [5], the crucial parts of this paper is to derive the uniform Schauder estimates of the linear elliptic equation for the infinite curved nozzle with decay cross-section.

**Keywords:** transonic shock wave, infinite nozzle, decay cross-sections.

**AMS subject classifications:** 35M10, 35J65, 35R35, 76H05, 76L05, 35B45.

## 1 Introduction

In this paper we study the existence and stability of multi-dimensional transonic shocks to the steady flow through an infinite curved nozzle with decay cross-section. Such problems naturally arise in the physical experiments and the engineering designs. (For detail applications, see the chapter V in Courant-Friedriches [8] and references cited therein). Since the length of the nozzle is always much longer than its cross-section in the practical application, the problem can be always formulated mathematically into an infinite nozzle problem. In this paper we mainly consider the following question. For an infinite curved nozzle, given a suitable supersonic incoming flow at the entrance and uniform subsonic flow condition at the infinite exit, can we construct a transonic shock wave pattern in such a nozzle? Such a question may also be expressed in the other words. Suppose there exists an uniform transonic shock in an infinite non-curved nozzle which can be constructed easily, such an uniform transonic shock is called the background solution. If we perturb the nozzle and the incoming flow small enough in some sense, is there still a transonic shock wave pattern which is close enough to the background one. If so, as a consequence, we can can derive an important by-product that the uniform transonic shock wave is stable with respect to the perturbation of incoming flow and the nozzle wall. Such two questions are elementary but important in the aerodynamics.

The steady flow is assumed to be isentropic and irrotational. It is governed by the potential flow equation for a velocity potential $\varphi : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, which can be deduced from the conservation of mass and the Bernoulli law for the velocity, and is a second order nonlinear equation of mixed elliptic-hyperbolic type for the transonic problems (see [6], [8]),

$$\operatorname{div}(\rho(|D\varphi|^2)D\varphi) = 0, \quad x \in \Omega, \tag{1.1}$$

where the density $\rho(|D\varphi|^2)$ is

$$\rho(|D\varphi|^2) = (1 - \theta|D\varphi|^2)^{1/(2\theta)}$$

and $\theta = (\gamma - 1)/2$ with the adiabatic exponent $\gamma > 1$. It is easy to verify that the nonlinear equation (1.1) is elliptic if

$$\rho(|D\varphi|^2) + 2|D\varphi|^2\rho'(|D\varphi|^2) > 0,$$

which corresponds to a subsonic flow, while is hyperbolic if

$$\rho(|D\varphi|^2) + 2|D\varphi|^2\rho'(|D\varphi|^2) < 0,$$

which corresponds to a supersonic flow. As is well-known that transonic flows and transonic flows with shocks are fundamental subjects in fluid dynamics, especially in gas dynamics, and various models have been put forward and studied extensively in the literature ([1, 8, 19, 20, 21, 22, 23, 24, 25]). Profound understanding has been achieved both physically and mathematically by Morawetz ([20, 21, 22, 24]) and others ([1, 8, 16]) on smooth transonic flows. While for transonic flows with shocks, most previous studies involve either experimental and numerically simulations or special wave patterns ([1, 8, 13] ), except the rigorous results on the existence and stability of the quasi one-dimensional transonic shocks, see ([10, 19]). Recently, some important wave patterns involving truly multi-dimensional transonic have established for various models and geometries, especially for the transonic wave pattern in a nozzle, see([2, 3, 4, 5, 27, 28]).

To outstand the background and motivation of this paper, we would like to discuss some of the recent notable studies on multi-dimensional transonic shocks for the potential flow in a nozzle, see([3, 4, 5, 27, 28]). Roughly speaking, there are mainly two kinds of nozzle problems. The first one is the study for the flat nozzle in ([3, 4]), where Chen and Feldman proved the existence and stability of a steady multi-dimensional transonic shock in a finite flat nozzle $\widetilde{\Omega} = (0, 1)^n$ with the Dirichlet boundary condition for the potential at the exit of the nozzle. Due to the wall of the nozzle is straight, and so the domain $\widetilde{\Omega}$ can be extended periodically and the solution may be considered periodic. Consequently, the influences of corners of $\widetilde{\Omega}$ are avoided. Moreover, thanks for the Dirichlet boundary condition at the exit, Chen and Feldman can apply the maximal principle directly to establish some crucial estimates for the existence and use the technique of sifting the boundary to achieve the uniqueness. Subsequently, by establishing the uniform Schauder estimates, Chen and Feldman ([4]) proved the same results for the infinite flat nozzle case with the uniform flow condition at the infinite exit. We should note that in ([3, 4]) Chen and Feldman developed an iteration scheme which is an effective tool to deal with some kinds of transonic shock problems. The second one is the study for the curved nozzle ([27, 28, 5]). Xin and Yin ([27, 28]) established the existence and stability of a steady multi-dimensional transonic shock in a finite general curved nozzle which is a small perturbation of the flat one. There the boundary condition at the exit is described in terms of the suitable pressure. The authors got the a priori $H^2$ estimate by looking for suitable multipliers, and then used the Sobolev embedding theorem to establish the $L^\infty$ estimate which plays a key and crucial role in doing Schauder estimates. On the other hand, the extension technique is invalid since the nozzle is not flat but curved. They considered the corner singularities by direct and complicated analysis in their papers.

A natural question is that if the multi-dimensional transonic shock is still existent and stable for the potential flow in a infinite curved nozzle. By the iteration scheme developed in ([3, 4]), Chen and Feldman ([5]) established the existence and stability for the multi-dimensional transonic shock in an infinite nozzle with finite curved part. That is to say, the nozzle is flat beyond a finite part. Although they believe this restriction is not essential, their proof depends on this assumption, for example the proof of lemma 6.4 in ([5]), in the non-local curved case the oblique differential boundary are no longer homogenous and thus the maximal principle is invalid. In this paper, the infinite general curved nozzle case is studied. More precisely, we consider the multi-dimensional transonic shock in a infinite general curved nozzle with decay cross-section. Motivated by Chen and Feldman ([5]), we transform the transonic flow problem to a free boundary problem for an uniform elliptic equation and use the similar iteration scheme ([3, 4]) to seek the solution. By the Schauder fixed point theorem and doing the elaborate estimates, we prove that the multi-dimensional transonic shock exists and is also stable for the potential flow in such a infinite curved nozzle. Moreover, we also investigate the asymptotic behavior of the transonic flow and give some decay rate. And the uniqueness is also proved by a special partial hodograph transform which is the same one as that in ([5]). Here, since the nozzle may be curved everywhere, we have to overcome some technical difficulties and do much more complicated asymptotic estimates. In particular, in order to establish the $L^\infty$ estimate which plays a key and crucial role in Schauder estimates, the mean integral estimate and the $L^\infty$ estimate of the gradient are needed.

The paper is arranged as follows. In §2, we first set up the problem, then by the classical nonlinear hyperbolic theory and the cut-off function technique, we reformulate it into a free boundary problem for an uniform elliptic equation. And the main theorem of this paper is presented in the end of this section. To solve the free boundary problem, we introduce a linear iteration scheme and a modified linear problem in §3. Since the modified linear problem is in an unbounded domain, we first solve the approximating problem in the bounded nozzle with the Dirichlet condition on the artificial boundary and establish the uniform estimates in §4. In the last section, the modified problem is solved, and by a fixed point method, we prove that the transonic flow exists and the transonic shock wave pattern is stable. And the uniqueness is also proved in this section.

## 2 Formulation of the Problem and the Main Results

In this section, we set up the transonic shock problem in the similar procedure as that in ([3, 4, 5]), and present the main theorem of our paper. Let first recall some basic definitions.

A function $\varphi \in C^1(\overline{\Omega})$ is said to be a weak solution of the equation (1.1) in a domain $\Omega \subset \mathbb{R}^n$, if

$$|D\varphi| \leq c^* = 1/\sqrt{\theta} \quad \text{in } \Omega$$

and

$$\int_\Omega \rho(|D\varphi|^2) D\varphi \cdot D\zeta dx = 0$$

for any $\zeta \in C_0^\infty(\Omega)$.

Let $\Omega^+$ and $\Omega^-$, separated by a $(n-1)-$dimensional smooth surface $S$, be two open subsets of $\Omega$, satisfying

$$\Omega^+ \cap \Omega^- = \emptyset, \quad \overline{\Omega^+} \cup \overline{\Omega^-} = \overline{\Omega}, \quad S = \partial\Omega^+ \cap \Omega.$$

If $\varphi$ is a weak solution of the equation (1.1) in the whole $\Omega$ and $\varphi \in C^2(\Omega^\pm) \bigcap C^1(\overline{\Omega^\pm})$ satisfies (1.1) in $\Omega^\pm$ respectively, and the following equalities

$$\varphi^+ = \varphi^- \quad \text{on } S$$

and

$$\rho(|D\varphi^+|^2) D\varphi^+ \cdot \nu = \rho(|D\varphi^-|^2) D\varphi^- \cdot \nu \quad \text{on } S \tag{2.1}$$

hold with $\varphi^\pm = \varphi|_{\overline{\Omega^\pm}}$ and $\nu$ being the unit normal to $S$ from $\Omega^-$ to $\Omega^+$, then $\varphi$ is called a shock solution with the shock $S$ of the equation (1.1).

If $(\varphi, S)$ is a shock solution with the shock $S$ of the equation (1.1) satisfying

$$|D\varphi| < c_* \quad \text{in } \Omega^+, \quad |D\varphi| > c_* \quad \text{in } \Omega^-, \quad D\varphi^\pm \cdot \nu\Big|_S > 0,$$

with $c_* = \sqrt{1/(\theta+1)} = \sqrt{2/(\gamma+1)}$ being the sonic speed, then $\varphi$ is said to be a transonic shock solution with the transonic shock $S$ of the equation (1.1). Moreover, if $(\varphi, S)$ satisfies the physical entropy condition (see Courant-Friedrichs [8])

$$\rho(|D\varphi^-|^2) < \rho(|D\varphi^+|^2) \quad \text{along } S,$$

then it is called a physically reasonable transonic shock solution with the transonic shock $S$ of the equation (1.1).

Note that the equation (1.1) is elliptic in the subsonic region and hyperbolic in the supersonic region. Consider the flat nozzle $\Omega_0 = \Lambda \times (-\infty, +\infty)$ and let

$$\varphi_0^-(x) = q_0^- x_n, \quad \varphi_0^+(x) = q_0^+ x_n, \quad x \in \mathbb{R}^n,$$

where $q_0^-$ and $q_0^+$ satisfy

$$\rho((q_0^-)^2) q_0^- = \rho((q_0^+)^2) q_0^+, \quad q_0^- \in (c_*, 1/\sqrt{\theta}), \quad q_0^+ \in (0, c_*).$$

Such a pair $(q_0^-, q_0^+)$ must exist since the function

$$\Phi(s) = \rho(s^2)s, \quad s \in \mathbb{R}$$

under consideration satisfies

$$\Phi(0) = 0, \quad \Phi(c_*) > 0, \quad \Phi(c^*) = 0$$

and

$$\Phi'(s) > 0 \ \text{ for } \ s \in (0, c_*), \quad \Phi'(s) < 0 \ \text{ for } \ s \in (0, c^*).$$

Then the function

$$\varphi_0(x) = \begin{cases} \varphi_0^-(x), & x \in \Omega_0^- = \Omega_0 \cap \{x_n < 0\}, \\ \varphi_0^+(x), & x \in \Omega_0^+ = \Omega_0 \cap \{x_n > 0\} \end{cases}$$

is a planar transonic shock solution in the non-curved nozzle $\Omega_0$, with $\Lambda \times (-\infty, 0)$ and $\Lambda \times (0, +\infty)$ being its supersonic and subsonic regions respectively, and $S = \Lambda \times \{x_n = 0\}$ being the transonic shock. Obviously,

$$\varphi_0(x) = \min\{\varphi_0^+(x), \varphi_0^-(x)\}, \quad x \in \Omega_0. \tag{2.2}$$

We call the pair $(\varphi_0(x), S)$ a background transonic shock solution. In this paper, we will construct a transonic shock solution to the equation (1.1), which is a small perturbations of the background solution, with a general curved nozzle and a general supersonic incoming flow. We should note that the general curved nozzle and the general supersonic incoming flow refer to the small perturbation of the flat nozzle and the uniform supersonic incoming flow. As a consequence, it is shown that the background transonic shock solution is stable with respect to the small perturbations of the nozzle and the supersonic incoming flow.

In this paper we consider the following infinite general curved nozzle with decay cross-sections

$$\Omega = \Psi(\Lambda \times (-\infty, +\infty)) \cap \{x = (x', x_n) : x_n \geq -1\}, \tag{2.3}$$

where $\Lambda \subset \mathbb{R}^{n-1}$ is a bounded domain with a smooth boundary and is diffeomorphic to a $n-1$ dimensional ball, and $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map which is close to the identity map. We assume that

$$\Psi_n(x) = x_n, \quad x = (x', x_n) \in \mathbb{R}^n \tag{2.4}$$

and

$$\partial\Lambda \in C^{[n/2]+3,\alpha}, \quad \|\Psi - Id\|_{[n/2]+3,\alpha;\Lambda\times(-1,2)} \leq \sigma, \quad \|\Psi - Id\|_{1,\alpha;\Lambda\times(2,+\infty)} \leq \sigma, \tag{2.5}$$

where $\Psi_n$ is the $n-$th component of $\Psi$, $Id$ is the identical map in $\mathbb{R}^n$, $\alpha \in (0,1)$ and $\sigma > 0$. Additionally, we assume $\Psi$ satisfies the following decay condition

$$\|(\Psi - Id)(x', x_n)\|^{(m)}_{1,\alpha;\Lambda\times(2,+\infty)} \leq \sigma \tag{2.6}$$

with $m > 1$, where $\|\cdot\|^{(m)}_{2,\alpha}$ is a weighted Hölder normal defined as follows. Let $E \subset \mathbb{R}^n$ and $\beta \in (0,1]$. For any $k \in \mathbb{R}$ and $l \in \mathbb{N} \cup \{0\}$, define

$$[[u]]^{(k)}_{l,0;E} = \sum_{|\theta|=l} \sup_{x\in E}(\delta_x^{l+k}|D^\theta u(x)|),$$

$$[[u]]^{(k)}_{l,\beta;E} = \sum_{|\theta|=l} \sup_{x,y\in E, x\neq y} \left( \delta^{l+k+\beta}_{x,y} \frac{|D^\theta u(x) - D^\theta u(y)|}{|x-y|^\beta} \right),$$

$$|u|^{(k)}_{l,0;E} = \sum_{j=1}^{l} [[u]]^{(k)}_{j,0;E},$$

$$\|u\|^{(k)}_{l,\beta;E} = |u|^{(k)}_{l,0;E} + [[u]]^{(k)}_{l,\beta;E},$$

where

$$\delta_x = |x_n| + 1, \quad \delta_{x,y} = \min(\delta_x, \delta_y) \quad \text{for } x, y \in E,$$

$D^\theta = \partial^{\theta_1}_{x_1}\partial^{\theta_2}_{x_2}\cdots\partial^{\theta_n}_{x_n}$, $\theta = (\theta_1, \theta_2\cdots, \theta_n)$ is a multi-index with $\theta_i \in \mathbb{N}\cup\{0\}\,(i = 1, 2, \cdots, n)$ and $|\theta| = \theta_1 + \theta_2 + \cdots + \theta_n$. Thus we can define the weighted Hölder space $C^{l,\beta}_{(k)}(\overline{E})$,

$$C^{l,\beta}_{(k)}(\overline{E}) = \{u \in C^{l,\beta}(\overline{E}) : \|u\|^{(k)}_{l,\beta;E} < \infty\}.$$

If $k = 0$, this space is just the standard Hölder space.

Denote

$$\partial_l\Omega = \partial\Omega \cap \{(x', x_n) \in \overline{\Omega} : x_n > -1\}, \quad \partial_o\Omega = \partial\Omega \cap \{(x', x_n) \in \overline{\Omega} : x_n = -1\}.$$

The transonic flow in the nozzle satisfies the physical slip boundary condition on the nozzle boundary $\partial_l\Omega$, i.e.

$$D\varphi \cdot \nu = 0 \quad \text{on } \partial_l\Omega \tag{2.7}$$

with $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ the inward unit normal to $\partial_l\Omega$. Finally, the general supersonic incoming flow at the entrance $\partial_o\Omega$ considered in the paper is given by

$$\varphi = \varphi_e^-, \quad \varphi_{x_n} = \psi_e^- \quad \text{on } \partial_o\Omega, \tag{2.8}$$

which is the small perturbation of $\varphi_0^-(x) = q_0^- x_n$ in the $H^{[n/2]+3}$ sense i.e.

$$\|\varphi_e^- - q_0^- x_n\|_{H^{[n/2]+3}(\partial_o\Omega)} + \|\psi_e^- - q_0^-\|_{H^{[n/2]+2}(\partial_o\Omega)} \le \sigma, \tag{2.9}$$

and

$$(\varphi_e^-, \psi_e^-) \text{ satisfies the compatibility conditions up to the } ([n/2] + 3) - \text{th order.} \tag{2.10}$$

Then our transonic nozzle problem can be formulated into the following form,

*Problem: Given an infinite general curved nozzle* $\Omega$ *by* (2.3) *with* (2.4)–(2.6) *and a general supersonic incoming flow* $(\varphi_e^-, \psi_e^-)$ *by* (2.8) *with* (2.9) *and* (2.10)*, find a multi-dimensional transonic flow* $\varphi$ *with the transonic shock* $S$ *of the equation (1.1) in* $\Omega$ *satisfying the initial condition* (2.8)*, the Rankine-Hugoniot condition* (2.1)*, the physical slip boundary condition (2.7) and the uniform subsonic flow condition at the infinite exit* $x_n = +\infty$*, which is written as*

$$\|\varphi(x) - \omega x_n\|_{0,1;(\Omega\cap\{x_n > R\})} \to 0, \quad \text{as } R \to \infty, \text{ for some } \omega \in (0, c_*). \tag{2.11}$$

Since we focus on the pattern with only one transonic shock wave, we can transfer such a problem into a free boundary problem by the following procedure. We first solve the nonlinear hyperbolic equation (1.1) in

$$\Omega_2 = \Omega \cap \{-1 < x_n < 2\}$$

satisfying the boundary condition (2.7), from the data on the nozzle entrance (2.8) with $\sigma > 0$ sufficiently small, by using the standard results on initial-boundary value problems for quasilinear wave equations and the Sobolev embedding theorem. And the solution $\varphi^-$ belongs to $C^{1,\alpha}(\overline{\Omega_2})$ due to (2.5), (2.9) and (2.10). Moreover, the solutions is also close enough to the initial state $\varphi_0^-$ provided $\sigma > 0$ is sufficiently small, namely

$$\|\varphi^- - \varphi_0^-\|_{1,\alpha;\Omega_2} \leq C_1\sigma. \tag{2.12}$$

For details, see [11, 15, 26]. Thus we may assume that the $C^{1,\alpha}$ supersonic solution $\varphi^-$ is given in $\Omega_2$ beforehand. On the other hand, we expect to find a small perturbation solution of background solution, so the perturbed transonic shock surface should be around $\{x_n = 0\}$. In this way we can reformulate the transonic nozzle problem as the following one-phase free boundary problem.

*Given an infinite general curved nozzle* $\Omega$ *by* (2.3) *with* (2.4)–(2.6) *and a supersonic upstream flow* $\varphi^-$, *a weak solution of* (1.1) *in* $\Omega_2$ *satisfying* (2.8), (2.7) *and* (2.12), *find a multidimensional subsonic flow* $\varphi$ *of* (1.1) *satisfying* (2.7) *and* (2.11) *and identify a free boundary* $x_n = f(x') \subset \{-1 < x_n < 1\}$ *dividing the subsonic flow* $\varphi^+$ *from the given supersonic flow* $\varphi^-$ *so that the function*

$$\varphi(x) = \begin{cases} \varphi^+(x), & x_n > f(x'), \\ \varphi^-(x), & x_n < f(x') \end{cases}$$

*is a transonic shock solution with the transonic shock* $S = \{(x', f(x')) : x' \in \mathbb{R}^{n-1}\} \cap \Omega$.

Therefore, we only need to solve a free boundary problem of an elliptic equation in

$$\Omega^+(\varphi) = \{x_n > f(x')\}$$

with the free boundary $\{x_n = f(x')\}$. Since $\varphi^-$ is a local $C^{1,\alpha}$ supersonic solution satisfying (2.12) in the domain $\Omega_2$ of the initial-boundary value problem (1.1), (2.8), (2.7), by the standard extension argument (see [3], [4], [5]), we can extend $\varphi^-$ to the whole infinite nozzle such that

$$\|\varphi^- - \varphi_0^-\|_{1,\alpha;\Omega} \leq 2C_1\sigma, \quad D\varphi^- \cdot \nu\Big|_{\partial_l\Omega} = 0. \tag{2.13}$$

Noticing that $\varphi^-$ is a small $C^{1,\alpha}$ perturbation of $\varphi_0^-$ with (2.12) and $q_0^- > q_0^+$, we may expect that $\varphi^+$ is close to $\varphi_0^+$ in $C^{1,\alpha}(\overline{\Omega^+(\varphi)})$, i.e.

$$\|\varphi^+ - \varphi_0^+\|_{1,\alpha;\Omega_+(\varphi)} \leq C_2\sigma, \tag{2.14}$$

and that the subsonic region $\Omega^+(\varphi)$ can also be defined as

$$\Omega^+(\varphi) = \{x \in \Omega : \varphi(x) < \varphi^-(x)\}.$$

Then we modify the equation (1.1) to make it be uniformly elliptic and to make it coincide with the original equation in the range of $|D\varphi^+|^2$ in the subsonic region $\Omega^+(\varphi)$ for $\varphi^+ \in C^{1,\alpha}(\overline{\Omega^+(\varphi)})$ with (2.14). The truncation procedure is the same as that introduced in §4.2 of [3]. Let $\varepsilon = (c_* - q_0^+)/2$. There exists $\tilde{\rho} \in C^{1,1}([0,\infty))$ and $c_j > 0\,(j = 0, 1, 2)$ depending only on $q_0^+$ and $\gamma$ such that

$$\tilde{\rho}(q^2) = \begin{cases} \rho(q^2), & \text{if } 0 \le q \le c_* - \varepsilon, \\ c_0 + c_1/q, & \text{if } q > c_* - \varepsilon \end{cases}$$

and

$$0 < c_0 \le (\tilde{\rho}(q^2)q)' \le c_2 \quad \text{for } q \in (0, \infty).$$

Then the equation

$$\operatorname{div}(\tilde{\rho}(|D\varphi|^2)D\varphi) = 0 \tag{2.15}$$

is uniformly elliptic, whose ellipticity constants depend only on $q_0^+$ and $\gamma$. And it coincides with the original equation (1.1) in the subsonic region $\Omega^+(\varphi)$ for each $\varphi^+ \in C^{1,\alpha}(\overline{\Omega^+(\varphi)})$ satisfying (2.14) with sufficiently small $\sigma > 0$. We also perform the corresponding truncation of the free boundary condition (2.1) by

$$\tilde{\rho}(|D\varphi|^2)D\varphi \cdot \nu = \rho(|D\varphi^-|^2)D\varphi^- \cdot \nu \quad \text{on } S. \tag{2.16}$$

On the right-hand side of (2.16), we use the original function $\rho$ owing to $\rho \neq \tilde{\rho}$ on the range of $|D\varphi^-|^2$. Note that (2.16), with the right-hand side considered as a known function, is the conormal boundary condition for the uniformly elliptic equation (2.15).

Thus, if we solve the truncated free boundary problem for the uniform elliptic equation (2.15) with the uniform conormal boundary condition (2.16) on the free boundary, slip boundary condition (2.7) on the nozzle wall and the uniform subsonic flow condition (2.11) at the infinite exit, then by the uniform estimates of the solution to this problem, this solution is indeed the solution to the original free boundary problem as discussed above.

Our main results in this paper are presented as follows.

**Theorem 2.1** *There exist $\sigma_0 > 0$ and $C > 0$ depending only on the data $n$, $m$, $\alpha$, $\gamma$, $q_0^+$ and $\Lambda$ such that for any infinite general curved nozzle $\Omega$ by (2.3) satisfying (2.4)–(2.6) and any general supersonic incoming flow $(\varphi_e^-, \psi_e^-)$ by (2.8) satisfying (2.9) and (2.10) with any $\sigma \in (0, \sigma_0)$, then there exists a unique solution $\varphi \in C^1(\overline{\Omega}) \cap C^\infty(\Omega^+)$ with the transonic shock $S = \{(x', f(x')) : x' \in \mathbb{R}^{n-1}\} \cap \Omega$ of the problem (1.1), (2.8), (2.1), (2.7) and (2.11) such that*

$$\|\varphi^- - \varphi_0^-\|_{1,\alpha;\Omega_2} + \|D\varphi - q_0^+ e_n\|_{L^\infty(\Omega^+)} \le C\sigma. \tag{2.17}$$

*Moreover, the solution satisfies the following properties*

*(i) The constant $\omega$ in (2.11) must be $q$, where $q$ is the unique root in the interval $(0, c_*)$ of the equation*

$$\rho(q^2)q = \frac{1}{meas(\Lambda)} \int_{\partial_0 \Omega} \rho(|D\varphi^-|^2)D\varphi^- \cdot \nu d\tau. \tag{2.18}$$

*Thus $\varphi$ satisfies*

$$\|\varphi - qx_n\|_{0,1;\Omega \cap \{x_n > R\}} \to 0 \quad \text{as } R \to \infty,$$

*and $q$ satisfies*

$$|q - q_0^+| \leq C\sigma;$$

*(ii) The function $f(x')$ belongs to $C^{1,\alpha}(\overline{\mathbb{R}^{n-1}})$ and satisfies*

$$\|f\|_{1,\alpha;\mathbb{R}^{n-1}} \leq C\sigma,$$

*and the surface $S = \{(x', f(x')) : x' \in \mathbb{R}^{n-1}\} \cap \Omega$ is orthogonal to $\partial_l \Omega$ at every intersection point;*

*(iii) The function $\varphi$ belongs to $C^{1,\alpha}(\overline{\Omega^+})$ and satisfies*

$$\|\varphi - qx_n\|_{1,\alpha;\Omega^+} \leq C\sigma;$$

*(iv) Furthermore, $\varphi - qx_n$ satisfies the following decay properties*

$$\|\varphi(x', x_n) - qx_n\| \leq C\sigma x_n^{-(m-1)}, \quad (x', x_n) \in \Omega^+, \quad x_n > 2,$$

*and*

$$\|D\varphi(x', x_n) - q\| \leq C\sigma x_n^{-(m-1)}, \quad (x', x_n) \in \Omega^+, \quad x_n > 2.$$

**Remark 2.1** *The relation* (2.17) *shows that the transonic flow $\varphi$ is close to $\varphi_0$, namely the background one, if the infinite general curved nozzle $\Omega$ and the general supersonic incoming flow $(\varphi_e^-, \psi_e^-)$ is the small perturbation of the original ones. Therefore, the transonic shock wave pattern is stable.*

# 3 Linear Iteration Scheme and Modified Linear Problem

We prove the existence of solutions to the truncated free boundary problem (2.15), (2.16) (2.7) and (2.11) by the following iteration procedure which is introduced in [3, 4, 5], for completeness we sketch here:

(i) Choose a function $\psi(x)$ to define an approximate boundary to the free boundary and to linearize the nonlinear second order equation around the function $\psi(x)$;

(ii) Solve the linearized second order equation with this fixed boundary on which the data satisfy the Rankine-Hugoniot condition, and then extend this solution to the whole nozzle;

(iii) Update the boundary by this extension function, which gives rise to a new approximation boundary.

It suffices to make uniform estimates and to show that this iterative procedure has a fixed point, which is just a solution to the truncated free boundary problem (2.15), (2.16), (2.7) and (2.11), by the Schauder fixed point theorem. Furthermore, by the uniform estimates, we may verify that this solution is indeed a solution to the problem (1.1), (2.8), (2.1), (2.7) and (2.11) with the free boundary as the transonic shock, as mentioned in the end of §2.

We begin with a function $\psi(x)$, which can be used to define an approximate boundary. Here $\psi(x)$ belongs to a compact subset of the Banach space $C^{1,\alpha}(\overline{\Omega})$. Let $M \geq 1$ and define

$$\mathcal{K}_M^k = \Big\{\psi \in C^{1,\alpha}(\overline{\Omega}) : \|\psi - qx_n\|_{1,\alpha;\Omega}^{(k)} \leq M\sigma\Big\}, \tag{3.1}$$

where $q$ is defined by (2.18) and $0 < k \le m-1$ is fixed. It is easy to verify that the set $\mathcal{K}_M^k$ is compact and convex in $C^{1,\alpha}(\overline{\Omega})$. We construct the iteration scheme as follows. Let $\psi \in \mathcal{K}_M^k$. Owing to $q_0^- > q_0^+$, it follows that, if

$$\sigma \le \frac{q_0^- - q_0^+}{4C_1 M},$$

then (2.12) implies

$$(\varphi^- - \psi)_{x_n}(x) \ge \frac{q_0^- - q_0^+}{2} > 0 \quad \text{in } \Omega.$$

Thus, by the mean value theorem and the implicit function theorem, there exists a surface $x_n = f(x')$ on which $\psi(x) = \varphi^-(x)$. Therefore, we can define the set

$$\Omega^+(\psi) = \{x \in \Omega : x_n > f(x')\},$$

where $f \in C^{1,\alpha}(\overline{\mathbb{R}^{n-1}})$ satisfying

$$\|f\|_{1,\alpha;\mathbb{R}^{n-1}} \le CM\sigma$$

with $C > 0$ depending only on $q_0^- - q_0^+$. The inward unit normal to $S_\psi = \{x \in \Omega : x_n = f(x')\}$ of $\Omega^+$ is

$$\nu^{(\psi)}(x) = \frac{D\varphi^-(x) - D\psi(x)}{|D\varphi^-(x) - D\psi(x)|} \quad \text{for } x \in S_\psi.$$

Obviously, this formula also defines $\nu^{(\psi)}(x)$ on $\Omega_2$ and

$$\|\nu^{(\psi)}(x) - e_n\|_{0,\alpha;\Omega_2} \le CM\sigma$$

with $C > 0$ depending only on $q_0^-$ and $q_0^+$. Motivated by (2.16), we define the function

$$G_\psi(x) = \rho(|D\varphi^-(x)|^2) D\varphi^-(x) \cdot \nu^{(\psi)}(x) \quad \text{on } \Omega_2$$

and consider the following elliptic problem in the domain $\Omega^+(\psi)$

$$\operatorname{div}(\tilde{\rho}(|D\varphi|^2) D\varphi) = 0 \qquad \text{in } \Omega^+(\psi), \tag{3.2}$$

$$\tilde{\rho}(|D\varphi|^2) D\varphi \cdot \nu^{(\psi)} = G_\psi \qquad \text{on } S_\psi = \{x_n = f(x')\}, \tag{3.3}$$

$$D\varphi \cdot \nu = 0 \qquad \text{on } \partial_l \Omega^+(\psi) = \partial\Omega^+(\psi) \cap \partial_l \Omega, \tag{3.4}$$

$$\lim_{R\to\infty} \|\varphi - qx_n\|_{L^\infty(\Omega^+(\psi) \cap \{x_n > R\})} = 0. \tag{3.5}$$

We will approximate the above problem to a linear elliptic problem. First rewrite the problem (3.2)–(3.5) in terms of the function

$$u(x) = \varphi(x) - qx_n, \quad x \in \Omega^+(\psi)$$

to yield

$$\operatorname{div} A(Du) = 0 \qquad \text{in } \Omega^+(\psi), \tag{3.6}$$

$$A(Du) \cdot \nu^{(\psi)} = g^{\psi(x)} \qquad \text{on } S_\psi, \tag{3.7}$$

$$A(Du)\cdot\nu = -\,\tilde\rho(q^2)q\nu\cdot e_n \qquad \text{on } \partial_l\Omega^+(\psi), \tag{3.8}$$

$$\lim_{R\to\infty}\|u\|_{L^\infty(\Omega^+(\psi)\,\cap\,\{x_n>R\})} = 0, \tag{3.9}$$

where

$$A(P)=\tilde\rho(|P+qe_n|^2)(P+qe_n)-\tilde\rho(q^2)qe_n \quad \text{for } P\in\mathbb{R}^n$$

and

$$g^\psi(x)=G_\psi(x)-\rho(q^2)q\nu^{(\psi)}\cdot e_n \quad \text{on } S_\psi.$$

Clearly, (3.6) satisfies the uniformly elliptic equation with the same ellipticity constants as in (2.15), i.e.

$$\lambda|\xi|^2\le\sum_{i,j=1}^n A^i_{P_j}(P)\xi_i\xi_j\le\lambda^{-1}|\xi|^2 \quad \text{for any } P,\xi\in\mathbb{R}^n$$

with $\lambda>0$ depending only on $q_0^+$ and $\gamma$. Additionally, $A(P)$ satisfies

$$A(0)=0, \quad (1+|P|)|D_PA^i_{P_j}(P)|\le C \quad \text{for any } P\in\mathbb{R}^n \tag{3.10}$$

with $C>0$ depending only on $q_0^+$ and $\gamma$. Now we state a linear problem corresponding to the problem (3.6)–(3.9) and thus to the problem (3.2)–(3.5). Namely, we use (3.10) to find that, for $i=1,\cdots,n$,

$$A^i(Du(x))=\sum_{j=1}^n\tilde a_{ij}(x)u_{x_j}(x), \quad \tilde a_{ij}(x)=\int_0^1 A^i_{p_j}(tDu(x))dt.$$

We replace $u(x)=\varphi(x)-qx_n$ in the definition of the coefficients $\tilde a_{ij}$ by $\psi(x)-qx_n$ for $\psi\in\mathcal{K}^k_M$ to define

$$a^{(\psi)}_{ij}(x)=\int_0^1 A^i_{p_j}(t(D\psi(x)-qe_n))dt \quad \text{for } x\in\Omega,\ i,j=1,\cdots,n.$$

It is easy to verify

$$\begin{aligned} a^{(\psi)}_{ij}(x)=\int_0^1\Big\{&\tilde\rho(|tD\psi(x)+(1-t)qe_n|^2)\delta_i^j+2\tilde\rho'(|tD\psi(x)+(1-t)qe_n|^2)\\ &\cdot(t\psi_{x_i}(x)+(1-t)q\delta_i^n)(t\psi_{x_j}(x)+(1-t)q\delta_j^n)\Big\}dt,\\ &\text{for } x\in\Omega,\ i,j=1,\cdots,n \end{aligned} \tag{3.11}$$

with

$$\delta_i^j=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\ne j.\end{cases}$$

In particular, $a^{(\psi)}_{ij}(x)=a^{(\psi)}_{ji}(x)$. We note that, for $\check\psi_0(x)=qx_n\in\mathcal{K}^k_M$, the corresponding coefficients $\check a_{ij}$ defined by (3.11) are constants and satisfy

$$\check a_{ij}=\kappa_i\delta_i^j \quad \text{for } i,j=1,\cdots,n$$

with

$$\kappa_i = \begin{cases} \tilde{\rho}(q^2) & \text{if } i = 1, \cdots, n-1, \\ (\tilde{\rho}(q^2)q)' & \text{if } i = n. \end{cases}$$

We have

$$\lambda \le \kappa_i \le \lambda^{-1} \quad \text{for } i = 1, \cdots, n,$$

$$\lambda|\xi|^2 \le \sum_{i,j=1}^n a_{ij}^{(\psi)}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2 \quad \text{for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^n$$

and

$$\|a_{ij}^{(\psi)}(x) - \check{a}_{ij}\|_{0,\alpha;\Omega}^{(k+1)} \le CM\sigma, \quad \text{for } \psi \in \mathcal{K}_M^k,$$

where $\lambda > 0$ and $C > 0$ depending only on $q_0^+$ and $\gamma$, but independent of $M$.

Thus we formulate the following conormal fixed boundary elliptic problem

$$\sum_{i,j=1}^n (a_{ij}^{(\psi)} u_{x_j})_{x_i} = 0 \qquad \text{in } \Omega^+(\psi), \tag{3.12}$$

$$\sum_{i,j=1}^n a_{ij}^{(\psi)} u_{x_j} \nu_i^{(\psi)} = g^{(\psi)} \qquad \text{on } S_\psi, \tag{3.13}$$

$$\sum_{i,j=1}^n a_{ij}^{(\psi)} u_{x_j} \nu_i = -\tilde{\rho}(q^2)q\nu \cdot e_n \qquad \text{on } \partial_l\Omega^+(\psi), \tag{3.14}$$

$$\lim_{R\to\infty} \|u\|_{L^\infty(\Omega^+(\psi)\,\cap\,\{x_n > R\})} = 0. \tag{3.15}$$

Since the coefficients are only $C^\alpha$, we can expect to find only a weak solution $u \in C^{1,\alpha}$ in the following sense.

**Definition 3.1** *A function $u \in C^{1,\alpha}(\overline{\Omega^+(\psi)})$ is called a weak solution of the problem* (3.12)–(3.15)*, if $u(x)$ satisfies* (3.15) *and for any $v \in C_0^1(\mathbb{R}^n)$,*

$$\int_{\Omega^+(\psi)} \sum_{i,j=1}^n a_{ij}^{(\psi)} u_{x_j} v_{x_i} dx + \int_{S_\psi} g^{(\psi)} v d\tau - \int_{\partial_l\Omega^+(\psi)} \tilde{\rho}(q^2)q\nu \cdot e_n v d\tau = 0. \tag{3.16}$$

We will define the iteration map

$$J(\psi) = \varphi$$

by solving the problem (3.12)–(3.15) for $u$ and then extending $u$ from $\Omega^+(\psi)$ to $\Omega$ so that $u(x) + qx_n \in \mathcal{K}_M^k$, and defining $\varphi(x) = u(x) + qx_n$. A fixed point of this map is obviously a solution of the truncated free boundary problem (2.15), (2.16), (2.7) and (2.11), and thus a solution of the problem (1.1), (2.8), (2.1), (2.7) and (2.11) with the free boundary as the transonic shock. To guarantee such an operator is well-defined, we should prove the problem (3.12)–(3.15) admits a unique weak solution with some suitable estimates, which will be proved in §5. In particular, to achieve the existence and the suitable estimates, noting the domain $\Omega^+(\psi)$ is unbounded, we will use the standard method for second order elliptic problems in unbounded domain. In other words, we first solve the approximating problem in the bounded nozzle with the Dirichlet condition on the artificial boundary and establish the uniform estimates, which will be considered in the next section, and finally complete a limit process in §5.

## 4 Fixed Boundary Problems in finite Nozzles

As mentioned in the end of §3, in order to find a solution of (3.12)–(3.15) in the unbounded domain $\Omega^+(\psi)$, in this section we solve the corresponding problem in the bounded domain

$$\Omega_R^+(\psi) = \Omega^+(\psi) \cap \{x_n < R\}, \quad R > 4,$$

and then in the next section pass to the limit as $R \to +\infty$, which is assured by the uniform estimates. Precisely, we consider the following problem

$$\sum_{i,j=1}^n (a_{ij}^{(\psi,R)} u_{x_j})_{x_i} = 0 \qquad \text{in } \Omega_R^+(\psi), \tag{4.1}$$

$$\sum_{i,j=1}^n a_{ij}^{(\psi,R)} u_{x_j} \nu_i^{(\psi)} = g_R^{(\psi)} \qquad \text{on } S_\psi, \tag{4.2}$$

$$\sum_{i,j=1}^n a_{ij}^{(\psi,R)} u_{x_j} \nu_i = -\tilde{\rho}(q_R^2) q_R \nu \cdot e_n \qquad \text{on } \partial_l \Omega_R^+(\psi) = \partial_l \Omega^+(\psi) \cap \partial\Omega_R^+(\psi), \tag{4.3}$$

$$u = 0 \qquad \text{on } \partial\Omega^+(\psi) \cap \{x_n = R\}, \tag{4.4}$$

where

$$a_{ij}^{(\psi,R)}(x) = \int_0^1 \Big\{ \tilde{\rho}(|tD\psi(x) + (1-t)q_R e_n|^2)\delta_i^j$$
$$+ 2\tilde{\rho}'(|tD\psi(x) + (1-t)q_R e_n|^2)(t\psi_{x_i}(x) + (1-t)q_R\delta_i^n)(t\psi_{x_j}(x) + (1-t)q_R\delta_j^n)\Big\} dt$$
$$\text{for } x \in \Omega_R^+(\psi),\ i,j = 1,\cdots,n,$$

$$g_R^{(\psi)}(x) = G_\psi(x) - \rho(q_R^2) q_R \nu^{(\psi)} \cdot e_n \quad \text{on } S_\psi$$

and $q_R$ is the unique root in the interval $(0, c_*)$ of the equation

$$\rho(q_R^2) q_R = \frac{1}{\operatorname{meas}(\Omega^+(\psi) \cap \{x_n = R\})} \int_{\partial_0\Omega} \rho(|D\varphi^-|^2) D\varphi^- \cdot \nu d\tau.$$

By (2.5) and (2.6),

$$\|\nu \cdot e_n\|_{0,\alpha;\partial_l\Omega_R^+(\psi)}^{(m+1)} \le C\sigma, \tag{4.5}$$

$$|q_R - q| \le C\sigma R^{-m}, \quad R > 4, \tag{4.6}$$

and

$$\|a_{ij}^{(\psi,R)} - a_{ij}^{(\psi)}\|_{L^\infty(\Omega_R^+(\psi))} \le C\sigma R^{-m}, \quad \|g_R^{(\psi)} - g^{(\psi)}\|_{L^\infty(S_\psi)} \le C\sigma R^{-m}, \quad R > 4 \tag{4.7}$$

with $C > 0$ depending only on $n$, $\gamma$, $q_0^+$ and $\Lambda$. Let $\check{a}_{ij}^{(R)} = a_{ij}^{(q_R x_n, R)}$, i.e.

$$\check{a}_{ij}^{(R)} = \kappa_i^{(R)} \delta_i^j \quad \text{for } i,j = 1,\cdots,n$$

with

$$\kappa_i^{(R)} = \begin{cases} \tilde{\rho}(q_R^2) & \text{if } i = 1, \cdots, n-1, \\ (\tilde{\rho}(q_R^2) q_R)' & \text{if } i = n. \end{cases}$$

From the properties of $a_{ij}^{(\psi)}$ and $\Psi$,

$$\bar{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^{(\psi,R)}(x)\xi_i\xi_j \leq \bar{\lambda}^{-1}|\xi|^2 \quad \text{for any } x \in \Omega_R^+(\psi) \text{ and } \xi \in \mathbb{R}^n \tag{4.8}$$

and

$$\bar{\lambda} \leq \kappa_i^{(R)} \leq \bar{\lambda}^{-1}, \quad \|a_{ij}^{(\psi,R)}(x) - \check{a}_{ij}^{(R)}\|_{0,\alpha;\Omega}^{(k+1)} \leq CM\sigma \tag{4.9}$$

with $\bar{\lambda} > 0$ and $C > 0$ depending only on $m$, $\gamma$ and $q_0^+$, but independent of $M$.

From the $L^2$ theory on uniformly elliptic equations (see [5] Section 3 for details), for any sufficiently small $\sigma \in (0, \sigma_0)$, the problem (4.1)–(4.4) admits a unique weak solution in the following sense.

**Definition 4.1** *A function $u \in H^1(\Omega_R^+(\psi))$ is called a weak solution to the problem* (4.1)–(4.4) *if $u = 0$ on $\partial\Omega^+(\psi) \cap \{x_n = R\}$ in the trace sense and*

$$\int_{\Omega_R^+(\psi)} \sum_{i,j=1}^{n} a_{ij}^{(\psi,R)} u_{x_j} v_{x_i} dx + \int_{S_\psi} g_R^{(\psi)} v d\tau - \int_{\partial_l \Omega_R^+(\psi)} \tilde{\rho}(q_R^2) q_R \nu \cdot e_n v d\tau = 0 \tag{4.10}$$

*for any $v \in H^1(\Omega_R^+(\psi))$ satisfying $v = 0$ on $\partial\Omega_R^+(\psi) \cap \{x_n = R\}$ in the trace sense.*

To complete the limit process as $R \to +\infty$, we should establish the uniform estimates on the solution of the problem (4.1)–(4.4). Let us first list some properties of $g_R^{(\psi)}$, which will be used later, whose proof is similar as the corresponding one of [5] Lemmas 6.1 and 6.2.

**Lemma 4.1** *There exists a constant $C > 0$ depending only on the data $n$ ,$m$, $\alpha$, $\gamma$, $q_0^+$ and $\Lambda$, but independent of $M$ and $R$ such that*

$$\|g_R^{(\psi)}\|_{0,\alpha;\Omega_2(\psi)} \leq C\sigma. \tag{4.11}$$

*Moreover,*

$$\int_{S_\psi} g_R^{(\psi)} d\tau = \tilde{\rho}(q_R^2) q_R \int_{\partial_l \Omega_R^+} \nu \cdot e_n d\tau. \tag{4.12}$$

Before deriving the uniform estimates, we prefer to sketch the procedure. It is well-known that to achieve the desired uniform estimates, the crucial step is to obtain the $L^\infty$ estimate of the solution $u$ with some suitable decay. To reach this key estimate, we adapt the following procedure:

**Step (i)** Using an auxiliary function, we first derive the boundedness estimate of the integral of $u$ on some subset of the cross-section, bounded by the $L^2$ estimate of $Du$;

**Step (ii)** Owing to (i), we obtain the $L^2$ estimate of $Du$ by the method of energy estimate;

**Step (iii)** By (ii) and the DeGiorgi-Nash-Moser iteration, we get the interior boundedness estimate of $Du$, which and (i) then lead to the interior $L^\infty$ estimate of $u$. Furthermore, this and (ii) yield the boundary $L^\infty$ estimate of $u$ by the technique of boundary estimate. Thus we get the global $L^\infty$ estimate of $u$;

**Step (iv)** Based on the $L^\infty$ estimate of $u$, we may control the $L^2$ estimate of $Du$ by a decay bound via (ii) and then control the $L^\infty$ estimate of $u$ by a decay bound via (iii). Repeating this procedure, we get the desired decay $L^\infty$ bound of $u$.

### 4.1 Step (i)

We start with the critical estimate (i), the boundedness estimate of the integral of $u$ on some subset, bounded by the $L^2$ estimate of $Du$. Before we go, for technical reason we first extend the solution of the problems (4.1)–(4.4) to a more big domain by the following standard procedure: Suppose there exists a unique solution $u \in H^1(\Omega_R^+(\psi))$ to problem (4.1)–(4.4), then we can define $\widetilde{u}(\Lambda \times (1,R)) = u(\Psi(\Lambda \times (1,R)))$. On the other hand, thanks to the fact that $\Lambda$ is diffeomorphic to a $n-1$ ball with an uniform diffeomorphism constant. Without loss of generality, we can denote this diffeomorphism map by $\Theta : B \times (1,R) \mapsto \Lambda \times (1,R)$. In this way, we have $\overline{u}(B\times(1,R)) = \widetilde{u}(\Theta(B\times(1,R)))$, thus for each ball, we can extend the function $\overline{u}$ by the classical ball surface reflection extension to the domain $B_{pr}\times(1,R), p > 1$ a fixed constant, which satisfies $\Psi(\Lambda \times (1,R)) \subset \Theta^{-1}(B_{pr} \times (1,R))$. we should note here that the extended function in general $H^1$ and continuous function only without high regularity, however it keeps the $L^\infty$ norm of $u$ and the $L^2$ norm of $Du$ with a generic constant which depends only on the nozzle shape, the dimension $n$ and $p$. Which is enough for our following proof. Without causing the confusion, we still denote $\overline{u}$ by $u$. Whit this fact prepared,we can state the following proposition.

**Proposition 4.1** *Let $\sigma \in (0,\sigma_0)$ be sufficiently small, depending only on the data, and $u \in H^1(\Omega_R^+(\psi))$ be the weak solution of the problem (4.1)–(4.4). Then for any $r \in (1,R)$,*

$$\Big|\int_\Lambda u(x',r)dx'\Big| \le C\sigma r^{-(m-1)} + C(M)\sigma^{1/2}r^{-k/2}\Big(\int_{\Omega^+_{(r,R)}(\psi)}|Du|^2dx\Big)^{1/2}. \tag{4.13}$$

*Here and thereafter, we use $C$ to denote a positive constant depending only on the data $n$, $m$, $\alpha$, $\gamma$, $q_0^+$ and $\Lambda$, but independent of $M$, $r$ and $R$, while $C(M)$ a positive constant depending on the data and $M$ but independent of $r$ and $R$. They may take different value at different position.*

*Proof.* To obtain this estimate, we need introduce an auxiliary function. For $1 < r < R$, let

$$w(x) = x_n - r, \quad x \in \Omega^+_{(r,R)}(\psi),$$

which is the unique solution to the following problem with constant coefficients

$$\sum_{i,j=1}^n (\breve{a}_{ij}^{(R)} w_{x_j})_{x_i} = 0 \qquad \text{in} \quad \Lambda \times (r,R), \tag{4.14}$$

$$w = 0 \qquad \text{on} \quad \Lambda \times \{x_n = r\},$$

$$\sum_{i,j=1}^{n} \breve{a}_{ij}^{(R)} w_{x_j}\nu_i = 0 \qquad \text{on} \quad \partial\Lambda\times(r,R),$$

$$w = R - r \qquad \text{on} \quad \Lambda\times\{x_n = R\}.$$

Take

$$v(x) = \begin{cases} -(R-r), & x\in\overline{\Omega_r^+(\psi)},\\ w(x)-(R-r), & x\in\Omega_{(r,R)}^+(\psi)\end{cases}$$

in (4.10) and use (4.12) to get

$$\begin{aligned}
&\int_{\Omega_{(r,R)}^+(\psi)} \sum_{i,j=1}^{n} a_{ij}^{(\psi,R)} u_{x_j} w_{x_i}\,dx\\
=&\int_{S_\psi} g_R^{(\psi)}(R-r)d\tau - \int_{\partial_l\Omega_r^+(\psi)} \tilde{\rho}(q_R^2)q_R\nu\cdot e_n(R-r)d\tau\\
&\qquad+\int_{\partial_l\Omega_{(r,R)}^+(\psi)} \tilde{\rho}(q_R^2)q_R\nu\cdot e_n(w-(R-r))d\tau\\
=&(R-r)\Big(\int_{S_\psi} g_R^{(\psi)}d\tau - \int_{\partial_l\Omega_R^+(\psi)} \tilde{\rho}(q_R^2)q_R\nu\cdot e_n d\tau\Big) + \int_{\partial_l\Omega_{(r,R)}^+(\psi)} \tilde{\rho}(q_R^2)q_R\nu\cdot e_n w d\tau\\
=&\int_{\partial_l\Omega_{(r,R)}^+(\psi)} \tilde{\rho}(q_R^2)q_R\nu\cdot e_n w d\tau.
\end{aligned}\tag{4.15}$$

Here we note the defined function $v\in H^1(\Omega_R^+(\psi))$ satisfying $v=0$ on $\partial\Omega_R^+(\psi)\cap\{x_n=R\}$ in the trace sense. On the other hand, multiplying the (4.14) by $\overline{u}$ (without causing confuse, still denoted by $u$, we should note that the defined domain of such $u$ contains $\Lambda\times(1,R)$) and integrating by parts over $\Lambda\times(r,R)$ lead to

$$\int_{\Lambda\times(r,R)} \sum_{i,j=1}^{n} \breve{a}_{ij}^{(R)} w_{x_j}u_{x_i}dx = -\int_{\Lambda\times\{x_n=r\}} \sum_{j=1}^{n} \breve{a}_{nj}^{(R)} w_{x_j} u dx' = -\int_{\Lambda\times\{x_n=r\}} \breve{a}_{nn}^{(R)} u dx'. \tag{4.16}$$

It follows from (4.15) and (4.16) that

$$\begin{aligned}
&\int_{\Lambda\times\{x_n=r\}} \breve{a}_{nn}^{(R)} u dx' = -\int_{\Lambda\times(r,R)} \sum_{i,j=1}^{n} \breve{a}_{ij}^{(R)} u_{x_i} w_{x_j} dx\\
&+\int_{\Omega_{(r,R)}^+(\psi)} \sum_{i,j=1}^{n} a_{ij}^{(\psi,R)} u_{x_j} w_{x_i} dx - \int_{\partial_l\Omega_{(r,R)}^+(\psi)} a_{ij}^{(\psi,R)}\nu_i w d\tau\\
=&\int_{\Lambda\times(r,R)} \sum_{i,j=1}^{n} (-\breve{a}_{ij}^{(R)} + a_{ij}^{(R)}) u_{x_i} w_{x_j} dx - \int_{\partial_l\Omega_{(r,R)}^+(\psi)} a_{ij}^{(\psi,R)}\nu_i w d\tau\\
+&\int_{\Omega_{(r,R)}^+(\psi)} \sum_{i,j=1}^{n} a_{ij}^{(R)} u_{x_i} w_{x_j} dx - \int_{\Lambda\times(r,R)} \sum_{i,j=1}^{n} a_{ij}^{(R)} u_{x_i} w_{x_j} dx
\end{aligned}$$

$$
\begin{aligned}
\leq & \Big|\int_{\Lambda\times(r,R)}\sum_{i,j=1}^n(-\check{a}_{ij}^{(R)}+a_{ij}^{(R)})u_{x_i}w_{x_j}dx\Big|+\Big|\int_{\partial_l\Omega^+_{(r,R)}(\psi)}a_{ij}^{(\psi,R)}\nu_i w d\tau\Big|\\
+&\Big|\int_{\Omega^+_{(r,R)}(\psi)\setminus\Lambda\times(r,R)}\sum_{i,j=1}^n a_{ij}^{(R)}u_{x_i}w_{x_j}dx\Big|\\
+&\Big|\int_{\Lambda\times(r,R)\setminus\Omega^+_{(r,R)}(\psi)}\sum_{i,j=1}^n a_{ij}^{(R)}u_{x_i}w_{x_j}dx\Big|.
\end{aligned}
$$

We estimate the three terms on the right-hand side respectively. Firstly, by using the Hölder inequality and (4.9),

$$
\begin{aligned}
&\Big|\int_{\Lambda\times(r,R)}\sum_{i,j=1}^n(a_{ij}^{(\psi,R)}-\check{a}_{ij}^{(R)})u_{x_i}w_{x_j}dx\Big|\\
\leq&\int_{\Lambda\times(r,R)}\sum_{i,j=1}^n|a_{ij}^{(\psi,R)}-\check{a}_{ij}^{(R)}||u_{x_i}||w_{x_j}|dx\\
\leq& C(M)\sigma\int_{\Lambda\times(r,R)}x_n^{-(k+1)}|Du||Dw|dx\\
=&C(M)\sigma\int_{\Lambda\times(r,R)}x_n^{-(k+1)}|Du|dx\\
\leq& C(M)\sigma\Big(\int_{\Lambda\times(r,R)}|Du|^2dx\Big)^{1/2}\Big(\int_{\Lambda\times(r,R)}x_n^{-2(k+1)}dx\Big)^{1/2}\\
\leq& C(M)\sigma r^{-(k+1/2)}\Big(\int_{\Omega^+_{(r,R)}(\psi)}|Du|^2dx\Big)^{1/2}.
\end{aligned}
$$

In the last inequality we use the fact that the extended function $u$ keeps $L^2$ norm of $Du$. Secondly, (4.8), the Hölder inequality give

$$
\begin{aligned}
&\Big|\int_{\Omega^+_{(r,R)}(\psi)\setminus(\Lambda\times(r,R))}\sum_{i,j=1}^n a_{ij}^{(\psi,R)}u_{x_j}w_{x_i}dx\Big|\\
\leq& C\Big(\int_{\Omega^+_{(r,R)}(\psi)\setminus(\Lambda\times(r,R))}|Du|^2dx\Big)^{1/2}\Big(\int_{\Omega^+_{(r,R)}(\psi)\setminus(\Lambda\times(r,R))}|Dw|^2dx\Big)^{1/2}\\
\leq& C\Big(\int_{\Omega^+_{(r,R)}(\psi)}|Du|^2dx\Big)^{1/2}\Big(\mathrm{meas}\big(\Omega^+_{(r,R)}(\psi)\setminus(\Lambda\times(r,R))\big)\Big)^{1/2}\\
\leq& C\sigma^{1/2}r^{-(m-1)/2}\Big(\int_{\Omega^+_{(r,R)}(\psi)}|Du|^2dx\Big)^{1/2}.
\end{aligned}
$$

The similar estimates hold for the term $\Big|\int_{(\Lambda\times(r,R))\setminus\Omega^+_{(r,R)}(\psi)}\sum_{i,j=1}^n a_{ij}^{(\psi,R)}u_{x_j}w_{x_i}dx\Big|$. Thirdly, (4.5) leads to

$$
\Big|\int_{\partial_l\Omega^+_{(r,R)}(\psi)}a_{ij}^{(\psi,R)}\nu_i w d\tau\Big|\leq\tilde{\rho}(q_R^2)q_R\int_{\partial_l\Omega^+_{(r,R)}(\psi)}|\nu\cdot e_n||w|d\tau
$$

$$\leq C\sigma \int_r^R x_n^{-(m+1)}(x_n - r)dx_n$$
$$\leq C\sigma r^{-(m-1)}.$$

Therefore,

$$\begin{aligned}\Big|\int_{\Lambda_r} \check{a}_{nn}^{(R)} u dx'\Big| \leq & C(M)\sigma r^{-(k+1/2)}\Big(\int_{\Omega^+_{(r,R)}(\psi)} |Du|^2 dx\Big)^{1/2} \\ & + C\sigma^{1/2} r^{-(m-1)/2}\Big(\int_{\Omega^+_{(r,R)}(\psi)} |Du|^2 dx\Big)^{1/2} + C\sigma r^{-(m-1)} \\ \leq & C\sigma r^{-(m-1)} + C(M)\sigma^{1/2} r^{-k/2}\Big(\int_{\Omega^+_{(r,R)}(\psi)} |Du|^2 dx\Big)^{1/2}\end{aligned}$$

due to $0 < k \leq m-1$ and $r > 1$. From the definition of $\check{a}_{nn}^{(R)}$ and (4.9), (4.13) follows from this estimate directly. The proof is complete.

## 4.2 Step (ii)

To derive the $L^2$ estimate of $Du$, we need the following modified Poincaré inequality.

**Lemma 4.2** *Let $E$ be a bounded domain and $E_0 \subset E$ be a non-empty subset. Then there exists a constant $C > 0$ such that for any $u \in H^1(E)$,*

$$\|u - (u)_{E_0}\|_{L^2(E)} \leq C\|Du\|_{L^2(E)}, \tag{4.17}$$

*where* $(u)_{E_0} = \dfrac{1}{meas(E_0)} \displaystyle\int_{E_0} u(x)dx.$

*Proof.* We prove the lemma by contradiction. Assume that (4.17) were not true, then for each integer $j = 1, 2, ...$, there exists a function $u_j \in H^1(E)$ satisfying

$$\|u_j - (u_j)_{E_0}\|_{L^2(E)} > j\|Du_j\|_{L^2(E)}.$$

Define

$$v_j = \frac{u_j - (u_j)_{E_0}}{\|u_j - (u_j)_{E_0}\|_{L^2(E)}}.$$

Then

$$(v_j)_{E_0} = 0, \quad \|v_j\|_{L^2(E)} = 1, \quad \|Dv_j\|_{L^2(E)} < \frac{1}{j}, \quad j = 1, 2, \cdots. \tag{4.18}$$

In particular, (4.18) implies that $\{v_j\}_{j=1}^\infty$ is bounded in $H^1(E)$. Therefore, there exist a subsequence of $\{v_j\}_{j=1}^\infty$, denoted by itself for convenience, and a function $v \in H^1(E)$ such that

$$v_j \to v \quad \text{strongly in } L^2(E), \quad Dv_j \rightharpoonup Dv \quad \text{weakly in } L^2(E).$$

From (4.18), we get that

$$(v)_{E_0} = 0, \quad \|v\|_{L^2(E)} = 1, \quad |Dv\|_{L^2(E)} \leq \lim_{j\to\infty} |Dv_j\|_{L^2(E)} = 0.$$

However, it is clear that the $v$ with above properties is nonexistent. This complete the proof.

By a similar proof, we may see that

**Remark 4.1** *Let $\{E^j\}_{j\in J}$ be a family of bounded domains and $E_0^j \subset E^j$ $(j \in J)$ be non-empty subsets. If any subsets of $\{E^j\}_{j\in J}$ and $\{E_0^j\}_{j\in J}$ with infinite number both have convergent subsequences, then the constants $C_j$ $(j \in J)$ in (4.17) are uniformly bounded.*

Let us run Step (ii), to do the $L^2$ estimate of $Du$.

**Proposition 4.2** *Let $\sigma \in (0, \sigma_0)$ be sufficiently small, depending only on the data, and $u \in H^1(\Omega_R^+(\psi))$ be the weak solution of (4.1)–(4.4). Then*

$$\|Du\|_{L^2(\Omega_R^+(\psi))} \leq C\sigma(1 + C(M)\sigma^{1/2}). \tag{4.19}$$

*Proof.* Choosing $v = \overline{u}$ in (4.10) (without causing confusion, we still denoted by $u$) gives

$$\int_{\Omega_R^+(\psi)} \sum_{i,j=1}^n a_{ij}^{(\psi,R)} u_{x_j} u_{x_i} dx = -\int_{S_\psi} g_R^{(\psi)} u d\tau + \int_{\partial_l \Omega_R^+(\psi)} \tilde{\rho}(q_R^2) q_R \nu \cdot e_n u d\tau. \tag{4.20}$$

Set

$$Q = \frac{1}{\mathrm{meas}(\{x \in \Omega_{(1,2)(\psi)}^+ : x' \in \Lambda\})} \int_1^2 \int_\Lambda u(x', x_n) dx' dx_n.$$

From (4.13),

$$\begin{aligned}
|Q| \leq & C\Big|\int_1^2 \int_\Lambda u(x', x_n) dx' dx_n\Big| \\
\leq & C\sigma + C(M)\sigma^{1/2}\Big(\int_{\Omega_{(1,R)}^+(\psi)} |Du|^2 dx\Big)^{1/2} \\
\leq & C\sigma + C(M)\sigma^{1/2}\Big(\int_{\Omega_R^+(\psi)} |Du|^2 dx\Big)^{1/2}.
\end{aligned} \tag{4.21}$$

By (4.12) and (4.20), for $R > 4$, we have

$$\begin{aligned}
&\int_{\Omega_R^+(\psi)} \sum_{i,j=1}^n a_{ij}^{(\psi,R)} u_{x_j} u_{x_i} dx \\
=& -\int_{S_\psi} g_R^{(\psi)} (u - Q) d\tau + \int_{\partial_l \Omega_R^+(\psi)} \tilde{\rho}(q_R^2) q_R \nu \cdot e_n (u - Q) d\tau \\
=& \Big(-\int_{S_\psi} g_R^{(\psi)} (u - Q) d\tau + \int_{\partial_l \Omega_2^+(\psi)} \tilde{\rho}(q_R^2) q_R \nu \cdot e_n (u - Q) d\tau\Big)
\end{aligned}$$

$$
\begin{aligned}
&- Q\tilde{\rho}(q_R^2)q_R\int_{\partial_l\Omega^+_{(2,R)}(\psi)}\nu\cdot e_n d\tau+\tilde{\rho}(q_R^2)q_R\int_{\partial_l\Omega^+_{(2,R)}(\psi)}\nu\cdot e_n u d\tau\\
=&I_1+I_2+I_3. 
\end{aligned}\tag{4.22}
$$

where

$$
I_1=-\int_{S_\psi}g_R^{(\psi)}(u-Q)d\tau+\int_{\partial_l\Omega_2^+(\psi)}\tilde{\rho}(q_R^2)q_R\nu\cdot e_n(u-Q)d\tau,
$$

$$
I_2=-Q\tilde{\rho}(q_R^2)q_R\int_{\partial_l\Omega^+_{(2,R)}(\psi)}\nu\cdot e_n d\tau,\quad I_3=\tilde{\rho}(q_R^2)q_R\int_{\partial_l\Omega^+_{(2,R)}(\psi)}\nu\cdot e_n u d\tau.
$$

We estimate these three terms respectively. Firstly, by the Hölder inequality and the trace theorem, and by using (4.11), (4.5) and Lemma 4.2, we get

$$
\begin{aligned}
|I_1|=&\Big|-\int_{S_\psi}g_R^{(\psi)}(u-Q)d\tau+\int_{\partial_l\Omega_2^+(\psi)}\tilde{\rho}(q_R^2)q_R\nu\cdot e_n(u-Q)d\tau\Big|\\
\le&\Big(\int_{S_\psi}(g_R^{(\psi)})^2d\tau\Big)^{1/2}\Big(\int_{S_\psi}(u-Q)^2d\tau\Big)^{1/2}\\
&+\tilde{\rho}(q_R^2)q_R\Big(\int_{\partial_l\Omega_2^+(\psi)}|\nu\cdot e_n|^2d\tau\Big)^{1/2}\Big(\int_{\partial_l\Omega_2^+(\psi)}(u-Q)^2d\tau\Big)^{1/2}\\
\le&C\sigma\Big(\int_{\Omega_2^+(\psi)}\big((u-Q)^2+|Du|^2\big)dx\Big)^{1/2}\\
\le&C\sigma\Big(\int_{\Omega_2^+(\psi)}|Du|^2dx\Big)^{1/2}.
\end{aligned}\tag{4.23}
$$

Here the constant $C>0$ is independent of $\psi\in\mathcal{K}_M^k$ since the constant in Lemma 4.2 is uniformly for sufficiently small $\sigma\in(0,\sigma_0)$, as mentioned in Remark 4.1. In the above calculation, we used the following fact:

$$
\begin{aligned}
\Big(\int_{\Omega_2^+(\psi)}\big((u-Q)^2+|Du|^2\big)dx\Big)^{1/2}\le&\Big(\int_{\Omega_2^+(\psi)\bigcup\{\Lambda\times(1,2)\}}\big((u-Q)^2+|Du|^2\big)dx\Big)^{1/2}\\
\le&C\Big(\int_{\Omega_2^+(\psi)\bigcup\{\Lambda\times(1,2)\}}|Du|^2dx\Big)^{1/2}\\
\le&C\Big(\int_{\Omega_2^+(\psi)}|Du|^2dx\Big)^{1/2}.
\end{aligned}\tag{4.24}
$$

The last inequality is assured by the definition of the extension function $u$, which keeps the $L^2$ norm of $Du$. Secondly, (4.5) and (4.21) lead to

$$
\begin{aligned}
|I_2|=&\Big|Q\tilde{\rho}(q_R^2)q_R\int_{\partial_l\Omega^+_{(2,R)}(\psi)}\nu\cdot e_n d\tau\Big|\\
\le&C\sigma\Big(C\sigma+C(M)\sigma^{1/2}\Big(\int_{\Omega_R^+(\psi)}|Du|^2dx\Big)^{1/2}\Big)
\end{aligned}
$$

$$\leq C\sigma^2 + C(M)\sigma^{3/2}\Big(\int_{\Omega_R^+(\psi)} |Du|^2 dx\Big)^{1/2}. \tag{4.25}$$

Finally, we estimate $I_3$. Define

$$h(x_n) = \frac{1}{|\Psi(\Lambda \times \{x_n\})|} \int_{\Psi(\Lambda\times\{x_n\})} u(x', x_n) dx', \quad r < x_n < R.$$

By the trace theorem and Lemma 4.2,

$$\begin{aligned}
&\int_{\partial\Psi(\Lambda\times\{x_n\})} u^2(x', x_n) d\tau' \\
\leq & C \int_{\Psi(\Lambda\times\{x_n\})} \sum_{i=1}^{n-1} (u_{x_i}(x', x_n))^2 dx' + C\int_{\Psi(\Lambda\times\{x_n\})} u^2(x', x_n) dx' \\
\leq & C \int_{\Psi(\Lambda\times\{x_n\})} \sum_{i=1}^{n-1} (u_{x_i}(x', x_n))^2 dx' + C\int_{\Psi(\Lambda\times\{x_n\})} (u(x', x_n) - h(x_n))^2 dx' + C\int_{\Psi(\Lambda\times\{x_n\})} h^2(x_n) dx' \\
\leq & C \int_{\Psi(\Lambda\times\{x_n\})} \sum_{i=1}^{n-1} (u_{x_i}(x', x_n))^2 dx' + C\int_{\Psi(\Lambda\times\{x_n\})} h^2(x_n) dx'.
\end{aligned}$$

Here the constant $C > 0$ is independent of $x_n$ since the domains $\Psi(\Lambda \times \{x_n\})$ are of the same structure and have limits as $x_n \to +\infty$, as mentioned in Remark 4.1. Multiplying this inequality by $x_n^{-2}$ and then integrating over $(2, R)$ with respect to $x_n$ and using (4.13), we get

$$\begin{aligned}
\int_{\partial_l\Omega^+_{(2,R)}(\psi)} x_n^{-2} u^2(x) d\tau \leq & C\int_{\Omega^+_{(2,R)}(\psi)} x_n^{-2} \sum_{i=1}^{n-1} (u_{x_i}(x', x_n))^2 dx + C\int_{\Omega^+_{(2,R)}(\psi)} x_n^{-2} h^2(x_n) dx \\
\leq & C\int_{\Omega^+_{(2,R)}(\psi)} x_n^{-2} |Du(x)|^2 dx + C\int_2^R r^{-2} h^2(r) dr \\
\leq & C\int_{\Omega^+_{(2,R)}(\psi)} |Du|^2 dx \\
& \quad + C\int_2^R r^{-2}\Big(\sigma^2 r^{-2(m-1)} + C^2(M)\sigma r^{-k}\int_{\Omega^+_{(r,R)}(\psi)} |Du|^2 dx\Big) dr \\
\leq & C\sigma^2 + C(1 + C(M)\sigma^{1/2})^2 \int_{\Omega^+_{(2,R)}(\psi)} |Du|^2 dx
\end{aligned} \tag{4.26}$$

due to $0 < k \leq m - 1$ and $r > 2$. From (4.5), (4.26) and the Hölder inequality, we have

$$\begin{aligned}
|I_3| =& \Big|\tilde{\rho}(q_R^2) q_R \int_{\partial_l\Omega^+_{(2,R)}(\psi)} \nu \cdot e_n u d\tau\Big| \\
\leq & C\int_{\partial_l\Omega^+_{(2,R)}(\psi)} |\nu \cdot e_n||u| d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le C\sigma \int_{\partial_l \Omega^+_{(2,R)}(\psi)} x_n^{-(m+1)} |u(x)| d\tau \\
&\le C\sigma \Big( \int_{\partial_l \Omega^+_{(2,R)}(\psi)} x_n^{-2m} d\tau \Big)^{1/2} \Big( \int_{\partial_l \Omega^+_{(2,R)}(\psi)} x_n^{-2} u^2(x) d\tau \Big)^{1/2} \\
&\le C\sigma \big( \sigma + (1 + C(M)\sigma^{1/2}) \big) \Big( \int_{\Omega^+_{(2,R)}(\psi)} |Du|^2 dx \Big)^{1/2} \\
&\le C\sigma^2 + C\sigma(1 + C(M)\sigma^{1/2}) \Big( \int_{\Omega^+_R(\psi)} |Du|^2 dx \Big)^{1/2} \qquad (4.27)
\end{aligned}
$$

owing to $m > 1$ and $r > 2$. From (4.22)–(4.25), (4.27) and (4.8), by the Young inequality, we get that

$$
\int_{\Omega^+_R(\psi)} |Du|^2 dx \le \frac{1}{\bar\lambda} \int_{\Omega^+_R(\psi)} \sum_{i,j=1}^n a_{ij}^{(\psi,R)} u_{x_j} u_{x_i} dx \le C\sigma^2 (1 + C(M)\sigma^{1/2})^2 + \frac{1}{2} \int_{\Omega^+_R(\psi)} |Du|^2 dx,
$$

which leads to (4.19) directly, and this complete the proof.

## 4.3 Step (iii)

We run Step (iii) and arrive at the following maximum estimate.

**Proposition 4.3** *Let $\sigma \in (0, \sigma_0)$ be sufficiently small, depending only on the data, and $u \in H^1(\Omega^+_R(\psi))$ be the weak solution of (4.1)–(4.4) with $R > 4$. Then*

$$
\|u\|_{L^\infty(\Omega^+_R)} \le C\sigma(1 + C(M)\sigma^{1/2}). \qquad (4.28)
$$

*Proof.* We first prove that for any $x^0 = (x^0_1, ..., x^0_n) \in \overline{\Omega^+_{(2,R-1)}(\psi)}$,

$$
\begin{aligned}
&\|Du\|_{L^\infty(B_{1/4}(x^0) \cap \Omega^+_R(\psi))} \\
\le & C\Big( \|Du\|_{L^2(B_1(x^0) \cap \Omega^+_R(\psi))} + \|\tilde\rho(q_R^2) q_R \nu \cdot e_n\|_{0,\alpha; B_1(x^0) \cap \partial_l \Omega^+_R(\psi)} \Big), \qquad (4.29)
\end{aligned}
$$

where $B_r(x^0)$ is the ball in $\mathbb{R}^n$ centered at $x^0$ with the radius $r$. We use the fact that $u - K$ also satisfies the equation (4.1) and the conormal boundary condition (4.3) for any constant $K$. On the one hand, when $x^0 \in \overline{\Omega^+_{(2,R-1)}(\psi)}$ satisfies $B_1(x^0) \subset \Omega^+_R(\psi)$, applying Theorem 3.13 in [14] to $u - K$ shows that

$$
\|u\|_{1,\alpha; B_{1/4}(x^0) \cap \Omega^+_R(\psi)} \le C \|u - K\|_{L^2(B_1(x^0) \cap \Omega^+_R(\psi))}.
$$

On the other hand, when $x^0 \in \partial_l \Omega^+_{(2,R-1)}(\psi)$, we use Theorem 5.1 in [17] to obtain

$$
\|Du\|_{0,\alpha; B_{1/4}(x^0) \cap \Omega^+_R(\psi)} \le C\Big( \|u - K\|_{L^\infty(B_{1/2}(x^0) \cap \Omega^+_R(\psi))}
$$

$$+ \|\tilde{\rho}(q_R^2) q_R \nu \cdot e_n\|_{0,\alpha;B_{1/2}(x^0)\,\cap\,\partial_l\Omega_R^+(\psi)}\Big).$$

By using the modification of Theorem 6.41 in [18] for the case near the boundary (similar to [5] Proposition 6.1), we may get

$$\|u-K\|_{L^\infty(B_{1/2}(x^0)\,\cap\,\Omega_R^+(\psi))} \le C\|u-K\|_{L^2(B_1(x_0)\cap\Omega_R^+(\psi))}.$$

Therefore,

$$\begin{aligned}\|Du\|_{0,\alpha;B_{1/4}(x^0)\,\cap\,\Omega_R^+(\psi)} \le & C\Big(\|u-K\|_{L^2(B_1(x^0)\,\cap\,\Omega_R^+(\psi))} \\ & + \|\tilde{\rho}(q_R^2) q_R \nu \cdot e_n\|_{0,\alpha;B_1(x^0)\,\cap\,\partial_l\Omega_R^+(\psi)}\Big), \quad x^0 \in \overline{\Omega_{(2,R-1)}^+(\psi)}.\end{aligned}$$

Thus (4.29) follows by choosing

$$K = \frac{1}{\mathrm{meas}(B_1(x_0)\cap\Omega_R^+(\psi))}\int_{B_1(x_0)\,\cap\,\Omega_R^+} u dx$$

and using the standard Poincaré inequality.

From (4.29),

$$\|Du\|_{L^\infty(\Omega_{(2,R-1)}^+(\psi))} \le C\Big(\|Du\|_{L^2(\Omega_R^+(\psi))} + \|\tilde{\rho}(q_R^2) q_R \nu \cdot e_n\|_{0,\alpha;\partial_l\Omega_R^+(\psi)}\Big).$$

This together with Proposition 4.1 and Proposition 4.2 yield

$$\begin{aligned}\|u\|_{L^\infty(\Omega_{(2,R-1)}^+(\psi))} \le & C\sup_{2<r<R-1}\Big|\int_\Lambda u(x',r)dx'\Big| + C\|Du\|_{L^\infty(\Omega_{(2,R-1)}^+(\psi))} \\ \le & C\sup_{2<r<R-1}\Big(C\sigma r^{-(m-1)} + C(M)\sigma^{1/2} r^{-k/2}\Big(\int_{\Omega_{(r,R)}^+(\psi)}|Du|^2dx\Big)^{1/2}\Big) \\ & + C\Big(\|Du\|_{L^2(\Omega_R^+(\psi))} + \|\tilde{\rho}(q_R^2) q_R \nu \cdot e_n\|_{0,\alpha;\partial_l\Omega_R^+(\psi)}\Big) \\ \le & C\sigma(1+C(M)\sigma^{1/2}).\end{aligned} \tag{4.30}$$

To extend this bound to the domains $\Omega_2^+(\psi)$ and $\Omega_{(R-1,R)}^+(\psi)$, we note that these domains are of the fixed size and structure. Thus we can use the standard estimates ([12] Theorem 8.15) for the equations of divergence form (extended to the case when we have the conormal boundary conditions on a part of the boundary, see e.g. [5] Proposition 6.2) to get

$$\|u\|_{L^\infty(\Omega_2^+(\psi))} \le C\|u\|_{L^2(\Omega_4^+(\psi))}, \quad \|u\|_{L^\infty(\Omega_{(R-1,R)}^+(\psi))} \le C\|u\|_{L^2(\Omega_{(R-2,R)}^+(\psi))}.$$

Then, from (4.30) and (4.19), Lemma 4.2 leads to

$$\begin{aligned}\|u\|_{L^\infty(\Omega_2^+(\psi))} \le C\|u\|_{L^2(\Omega_4^+(\psi))} \le & C\|u\|_{L^2(\Omega_{(2,4)}^+(\psi)} + C\|Du\|_{L^2(\Omega_4^+(\psi))} \\ \le & C\sigma(1+C(M)\sigma^{1/2})\end{aligned}$$

and

$$\begin{aligned}\|u\|_{L^\infty(\Omega_{(R-1,R)}^+(\psi))} \le C\|u\|_{L^2(\Omega_{(R-2,R)}^+(\psi))} \le & C\|u\|_{L^2(\Omega_{(R-2,R-1)}^+(\psi)} + C\|Du\|_{L^2(\Omega_{(R-2,R)}^+(\psi))} \\ \le & C\sigma(1+C(M)\sigma^{1/2}).\end{aligned}$$

Thus (4.28) follows from these two estimates and (4.30), and the proof is complete.

### 4.4 Step (iv)

Next we run Step (iv) and first establish the following decay estimate.

**Lemma 4.3** *Let $\sigma \in (0, \sigma_0)$ be sufficiently small, depending only on the data, and $u \in H^1(\Omega_R^+(\psi))$ be the weak solution of the problem (4.1)–(4.4) with $R > 4$. Then for any $2 < r < R$,*

$$\|Du\|_{L^2(\Omega^+_{(r,R)}(\psi))} \leq C\sigma(1 + C(M)\sigma^{1/2})r^{-1/2}. \tag{4.31}$$

*Proof.* Let $\eta(x_n) \in C_0^\infty(0, \infty)$ satisfy $\eta(x_n) = 0$ in $(0, r/2)$, $\eta(x) = 1$ in $(r, R)$ and

$$0 \leq \eta(x_n) \leq 1, \quad |\eta'(x_n)| \leq \frac{C}{r}, \quad x_n \in (0, \infty).$$

Choose $v(x) = \eta^2(x_n)u(x)$ in (4.10) and use the Hölder inequality and the Young inequality to obtain

$$\begin{aligned}
&\int_{\Omega_R^+(\psi)} \sum_{i,j=1}^n \eta^2 a_{ij}^{(\psi,R)} u_{x_j} u_{x_i} dx \\
=&-2\int_{\Omega_R^+(\psi)} \sum_{j=1}^n \eta u a_{nj}^{(\psi,R)} u_{x_j} \eta'(x_n) dx - \int_{\partial_l \Omega_R^+(\psi)} \tilde{\rho}(q_R^2) q_R \nu \cdot e_n \eta^2 u d\tau \\
\leq& \frac{1}{2}\int_{\Omega_R^+(\psi)} \sum_{i,j=1}^n \eta^2 a_{ij}^{(\psi,R)} u_{x_j} u_{x_i} dx + C\int_{\Omega_R^+(\psi)} |\eta'(x_n)|^2 u^2 dx - \int_{\partial_l \Omega_R^+(\psi)} \tilde{\rho}(q_R^2) q_R \nu \cdot e_n \eta^2 u d\tau.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\Omega_R^+(\psi)} \sum_{i,j=1}^n \eta^2 a_{ij}^{(\psi,R)} u_{x_j} u_{x_i} dx \\
\leq& C\int_{\Omega_R^+(\psi)} |\eta'(x_n)|^2 u^2 dx + C\int_{\partial_l \Omega_R^+(\psi)} |\nu \cdot e_n| \eta^2 |u| d\tau \\
\leq& C\|u\|^2_{L^\infty(\Omega^+_{(r/2,r)})} \int_{r/2}^r |\eta'(x_n)|^2 dx \\
&\quad + C\|u\|_{L^\infty(\Omega^+_{(r/2,R)})} \int_{\partial_l \Omega^+_{(r/2,R)}(\psi)} |\nu \cdot e_n| d\tau \\
\leq& C\|u\|^2_{L^\infty(\Omega^+_{(r/2,r)})} r^{-1} + C\sigma \|u\|_{L^\infty(\Omega^+_{(r/2,R)})} \int_{r/2}^R x_n^{-(m+1)} dx_n \\
\leq& C\|u\|_{L^\infty(\Omega^+_{(r/2,R)})} \big(\|u\|_{L^\infty(\Omega^+_{(r/2,R)})} + \sigma r^{-(m-1)}\big) r^{-1}.
\end{aligned}$$

Owing to $m > 1$ and $r > 2$, from (4.28),

$$\int_{\Omega_R^+(\psi)} \sum_{i,j=1}^n \eta^2 a_{ij}^{(\psi,R)} u_{x_j} u_{x_i} dx \leq C\sigma^2(1 + C(M)\sigma^{1/2})^2 r^{-1},$$

which implies (4.31) from (4.8). The proof is complete.

After this lemma, replacing (4.19) by (4.31) in the proof of Proposition 4.3, we may get that

**Lemma 4.4** *Let $\sigma \in (0, \sigma_0)$ be sufficiently small, depending only on the data, and $u \in H^1(\Omega_R^+(\psi))$ be the weak solution of (4.1)–(4.4) with $R > 4$. Then for any $2 < r < R$,*

$$\|u\|_{L^\infty(\Omega^+_{(r,R)}(\psi))} \leq C\sigma(1 + C(M)\sigma^{1/2})r^{-\min\{1/2, m-1\}}. \tag{4.32}$$

*Proof.* Similar as the proof in Proposition 4.3, we may prove that for any $x^0 = (x_1^0, ..., x_n^0) \in \overline{\Omega^+_{(r,R-1)}(\psi)}$,

$$\begin{aligned}
&\|Du\|_{L^\infty(B_{1/4}(x^0) \cap \Omega^+_R(\psi))} \\
\leq& C\Big(\|Du\|_{L^2(B_1(x^0) \cap \Omega^+_{(r/2,R)}(\psi))} + \|\tilde{\rho}(q_R^2)q_R\nu \cdot e_n\|_{0,\alpha;B_1(x^0) \cap \partial_l\Omega^+_{(r/2,R)}(\psi)}\Big).
\end{aligned}$$

This, (4.31) and (4.5) yield

$$\begin{aligned}
&\|Du\|_{L^\infty(\Omega^+_{(r,R-1)}(\psi))} \\
\leq& C\Big(\|Du\|_{L^2(\Omega^+_{(r/2,R)}(\psi))} + \|\tilde{\rho}(q_R^2)q_R\nu \cdot e_n\|_{0,\alpha;B_1(x^0) \cap \partial_l\Omega^+_{(r/2,R)}(\psi)}\Big) \\
\leq& C\sigma(1 + C(M)\sigma^{1/2})r^{-1/2} + C\sigma r^{-(m+1)} \\
\leq& C\sigma(1 + C(M)\sigma^{1/2})r^{-1/2}.
\end{aligned}$$

Therefore, from the above estimate and Proposition 4.1,

$$\begin{aligned}
\|u\|_{L^\infty(\Omega^+_{(r,R-1)}(\psi))} \leq& C \sup_{r<x_n<R-1} \Big|\int_\Lambda u(x', x_n)dx'\Big| + C\|Du\|_{L^\infty(\Omega^+_{(r,R-1)}(\psi))} \\
\leq& C\sigma r^{-(m-1)} + C(M)\sigma^{1/2}r^{-k/2}\Big(\int_{\Omega^+_{(r,R)}(\psi)} |Du|^2dx\Big)^{1/2} \\
&+ C\sigma(1 + C(M)\sigma^{1/2})r^{-1/2} \\
\leq& C\sigma(1 + C(M)\sigma^{1/2})r^{-\min\{1/2, m-1\}}
\end{aligned} \tag{4.33}$$

due to $k > 0$. Similarly, we can extend this bound to the domain $\Omega^+_{(R-1,R)}(\psi)$ to get

$$\|u\|_{L^\infty(\Omega^+_{(R-1,R)}(\psi))} \leq C\|u\|_{L^2(\Omega^+_{(R-2,R)}(\psi))}.$$

Then, from (4.31) and (4.19), Lemma 4.2 leads to

$$\begin{aligned}
\|u\|_{L^\infty(\Omega^+_{(R-1,R)}(\psi))} \leq& C\|u\|_{L^2(\Omega^+_{(R-2,R)}(\psi))} \\
\leq& C\|u\|_{L^2(\Omega^+_{(R-2,R-1)}(\psi)} + C\|Du\|_{L^2(\Omega^+_{(R-2,R)}(\psi))}
\end{aligned}$$

$$\leq C\sigma(1 + C(M)\sigma^{1/2})R^{-\min\{1/2, m-1\}}.$$

Thus (4.32) follows from this estimate and (4.33), and the proof is complete.

After this lemma, replace (4.28) by (4.32) in the proof of Lemma 4.3 to get

$$\begin{aligned}
\|Du\|_{L^2(\Omega^+_{(r,R)}(\psi))} \leq & C\Big(\|u\|_{L^\infty(\Omega^+_{(r/2,R)})}\big(\|u\|_{L^\infty(\Omega^+_{(r/2,R)})} + \sigma r^{-(m-1)}\big)r^{-1}\Big)^{1/2} \\
\leq & C\sigma(1 + C(M)\sigma^{1/2})r^{-\min\{1/2, m-1\} - 1/2} \\
\leq & C\sigma(1 + C(M)\sigma^{1/2})r^{-\min\{1, m-1/2\}}, \quad 2 < r < R. \qquad (4.34)
\end{aligned}$$

If $1 < m \leq 3/2$, then we have gotten that

$$\|u\|_{L^\infty(\Omega^+_{(r,R)}(\psi))} \leq C\sigma(1 + C(M)\sigma^{1/2})r^{-(m-1)}, \quad 2 < r < R$$

and

$$\|Du\|_{L^2(\Omega^+_{(r,R)}(\psi))} \leq C\sigma(1 + C(M)\sigma^{1/2})r^{-(m-1/2)}, \quad 2 < r < R.$$

Otherwise, replacing (4.31) by (4.34) in the proof of Lemma 4.4, we may get that for any $2 < r < R$,

$$\|u\|_{L^\infty(\Omega^+_{(r,R)}(\psi))} \leq C\sigma(1 + C(M)\sigma^{1/2})r^{-\min\{1, m-1\}}.$$

Then replace (4.33) by this estimate in the proof of Lemma 4.3 to get

$$\|Du\|_{L^2(\Omega^+_{(r,R)}(\psi))} \leq C\sigma(1 + C(M)\sigma^{1/2})r^{-\min\{3/2, m-1\}}, \quad 2 < r < R.$$

Repeating this procedure for $[2(m-2)] + 1$ times, we complete Step (iv) and achieve the desired global decay estimate as follows.

**Proposition 4.4** *Let $\sigma \in (0, \sigma_0)$ be sufficiently small, depending only on the data, and $u \in H^1(\Omega^+_R(\psi))$ be the weak solution of the problem* (4.1)–(4.4) *with $R > 4$. Then for any $2 < r < R$,*

$$\|u\|_{L^\infty(\Omega^+_{(r,R)}(\psi))} \leq C\sigma(1 + C(M)\sigma^{1/2})r^{-(m-1)}$$

*and*

$$\|Du\|_{L^2(\Omega^+_{(r,R)}(\psi))} \leq C\sigma(1 + C(M)\sigma^{1/2})r^{-(m-1/2)}.$$

# 5 Fixed Boundary Problem in an Infinite Nozzle and Transonic Shock Problem

In this section, we first prove the problem (3.12)–(3.15) admits a unique weak solution with some suitable estimates. Then by the Schauder fixed point theorem, we obtain a solution of the truncated free boundary problem (2.15), (2.16), (2.7) and (2.11). This solution is just a solution of the transonic nozzle problem (1.1), (2.8), (2.1), (2.7) and (2.11) with the free boundary as the transonic shock, as mentioned in the end of §3. The uniqueness of solution of this transonic nozzle problem is proved by a special partial hodograph transform which is the same as that in ([5]).

Firstly, we have

**Proposition 5.1** *There exists at most one weak solution to the problem* (3.12)–(3.15).

*Proof.* Assume $u_1(x), u_2(x) \in C^{1,\alpha}(\overline{\Omega^+(\psi)})$ are two weak solutions to the problem (3.12)–(3.15). Set

$$u(x) = u_1(x) - u_2(x), \quad x \in \overline{\Omega^+(\psi)}.$$

For $R > 0$, define

$$v(x) = \begin{cases} \big(\eta^{(R)}(x)\big)^2 u(x), & x \in \overline{\Omega^+(\psi)}, \\ 0, & x \in \mathbb{R}^n \setminus \overline{\Omega^+(\psi)}, \end{cases}$$

where

$$\eta^{(R)}(x) = \eta\Big(\frac{|x|}{R}\Big) \quad x \in \mathbb{R}^n$$

and $\eta \in C^\infty(\overline{R_+})$ is a nonnegative function satisfying $\eta = 1$ in $(0,1)$ and $\eta = 0$ in $(2, +\infty)$. Then $v(x) \in C_0^1(\mathbb{R}^n)$ and

$$|D\eta^{(R)}(x)| \le \frac{C}{R}, \quad x \in \mathbb{R}^n.$$

Choosing this $v(x)$ as the test function in (3.16) for $u_1$ and $u_2$ respectively, and then subtracting the two equalities yields

$$\int_{\Omega^+(\psi)} \sum_{i,j=1}^n a_{ij}^{(\psi)} \big(u_{x_j} u_{x_i} (\eta^{(R)})^2 + 2\eta^{(R)} u u_{x_j} \eta_{x_i}^{(R)}\big) dx = 0. \tag{5.1}$$

By the Hölder inequality and the Young inequality,

$$\begin{aligned} \int_{\Omega^+(\psi)} (\eta^{(R)})^2 \sum_{i,j=1}^n a_{ij}^{(\psi)} u_{x_j} u_{x_i} dx =& -2 \int_{\Omega^+(\psi)} \eta^{(R)} u \sum_{i,j=1}^n a_{ij}^{(\psi)} u_{x_j} \eta_{x_i}^{(R)} dx \\ \le& \frac{1}{2} \int_{\Omega^+(\psi)} (\eta^{(R)})^2 \sum_{i,j=1}^n a_{ij}^{(\psi)} u_{x_j} u_{x_i} dx \\ &+ 2 \int_{\Omega^+(\psi)} |u|^2 \sum_{i,j=1}^n a_{ij}^{(\psi)} \eta_{x_j}^{(R)} \eta_{x_i}^{(R)} dx. \end{aligned}$$

Using the ellipticity of the equation (3.12) and the definition of $\eta^{(R)}$ gives

$$\int_{\Omega^+(\psi) \cap \{|x| < R\}} |Du|^2 dx \le \frac{1}{\lambda} \int_{\Omega^+(\psi)} (\eta^{(R)})^2 \sum_{i,j=1}^n a_{ij}^{(\psi)} u_{x_j} u_{x_i} dx$$

$$\le \frac{4}{\lambda}\int_{\Omega^+(\psi)} |u|^2 \sum_{i,j=1}^{n} a_{ij}^{(\psi)} \eta_{x_j}^{(R)} \eta_{x_i}^{(R)} dx$$
$$\le C\|u\|^2_{L^\infty(\Omega^+(\psi))} \int_{\Omega^+(\psi)\,\cap\,\{R<|x|<2R\}} |D\eta^{(R)}|^2 dx$$
$$\le C\|u\|^2_{L^\infty(\Omega^+(\psi))} R^{-1}.$$

Let $R \to +\infty$ to get

$$\int_{\Omega^+(\psi)} |Du|^2 dx = 0,$$

which and (3.15) imply

$$u(x) = u_1(x) - u_2(x) = 0, \quad x \in \overline{\Omega^+(\psi)}.$$

This complete the proof.

Based on the uniform estimates of solutions to the problem (4.1)–(4.4) in §4, we may establish the existence of solutions to the problem (3.12)–(3.15) in the unbounded domain $\Omega^+(\psi)$ and the suitable decay estimates for the solution.

**Proposition 5.2** *Let $\sigma \in (0, \sigma_0)$ be sufficiently small, depending only on the data. There exists a weak solution $u \in C^{1,\alpha}(\overline{\Omega^+(\psi)})$ of the problem (3.12)–(3.15). Furthermore, this solution satisfies*

$$|u(x', x_n)| \le C\sigma(1 + C(M)\sigma^{1/2}) x_n^{-(m-1)}, \quad x = (x', x_n) \in \Omega^+(\psi) \text{ with } x_n > 2, \tag{5.2}$$

$$|Du(x', x_n)| \le C\sigma(1 + C(M)\sigma^{1/2}) x_n^{-(m-1)}, \quad x = (x', x_n) \in \Omega^+(\psi) \text{ with } x_n > 2, \tag{5.3}$$

$$\|Du\|_{L^2(\Omega^+(\psi)\,\cap\,\{r<x_n<+\infty\})} \le C\sigma(1 + C(M)\sigma^{1/2}) r^{-(m-1/2)}, \quad r > 2 \tag{5.4}$$

*and*

$$\|u\|^{(k)}_{1,\alpha;\Omega^+(\psi)} \le C\sigma(1 + C(M)\sigma^{1/2}), \tag{5.5}$$

*where $C > 0$ depending only on the data $n$, $m$, $\alpha$, $\gamma$, $q_0^+$ and $\Lambda$ but independent of $M$, while $C(M) > 0$ depending on the data and $M$.*

*Proof.* Let $u_R$ be the weak solution to the problem (4.1)–(4.4) with $R > 4$. From Propositions 4.2–4.4, we have

$$\|u_R\|_{L^\infty(\Omega^+_R(\psi))} \le C\sigma(1 + C(M)\sigma^{1/2}), \quad \|Du_R\|_{L^2(\Omega^+_R(\psi))} \le C\sigma(1 + C(M)\sigma^{1/2}) \tag{5.6}$$

and

$$\|u_R\|_{L^\infty(\Omega^+_{(r,R)}(\psi)(\psi))} \le C\sigma(1 + C(M)\sigma^{1/2}) r^{-(m-1)}, \qquad 2 < r < R, \tag{5.7}$$

$$\|Du_R\|_{L^2(\Omega^+_{(r,R)}(\psi)(\psi))} \le C\sigma(1 + C(M)\sigma^{1/2}) r^{-(m-1/2)}, \quad 2 < r < R, \tag{5.8}$$

where $C > 0$ depending only on the data but independent of $M$, while $C(M) > 0$ depending on the data and $M$.

Now, by using the classical Schauder estimate, from (5.6)– (5.8), (4.11) and (4.5), we may get that

$$\|u_R\|_{1,\alpha;\Omega_R^+(\psi)\cap\{-1<x_n<2\}} \leq C\sigma(1+C(M)\sigma^{1/2}) \tag{5.9}$$

and

$$\|u_R\|_{1,\alpha;\Omega_R^+(\psi)\cap\{r<x_n<r+1\}} \leq C\sigma(1+C(M)\sigma^{1/2})r^{-(m-1)}, \quad 2<r<r+2<R, \tag{5.10}$$

where $C>0$ and $C(M)>0$ defined as above. Therefore, there exist a subsequence $\{u_{R_j}\}_{j=1}^{\infty}$ with $\{R_j\}_{j=1}^{\infty}\subset(4,+\infty)$ increasing to $+\infty$, and a function $u\in C^{1,\alpha}(\overline{\Omega^+(\psi)})$ such that

$$u_{R_j}\to u, \quad Du_{R_j}\to Du \quad \text{strongly in any compact subset of } \Omega^+(\psi) \text{ as } j\to\infty. \tag{5.11}$$

Furthermore, (5.6)–(5.10) implies

$$\|u\|_{1,\alpha;\Omega^+(\psi)\cap\{-1<x_n<2\}} \leq C\sigma(1+C(M)\sigma^{1/2}), \tag{5.12}$$

$$\|u\|_{1,\alpha;\Omega^+(\psi)\cap\{r<x_n<r+1\}} \leq C\sigma(1+C(M)\sigma^{1/2})r^{-(m-1)}, \quad r>2 \tag{5.13}$$

and

$$\|Du\|_{L^2(\Omega^+(\psi)\cap\{r<x_n<+\infty\})} \leq C\sigma(1+C(M)\sigma^{1/2})r^{-(m-1/2)}, \quad r>2. \tag{5.14}$$

Due to (4.6), (4.7) and (5.11), it is easy to verify that $u\in C^{1,\alpha}(\overline{\Omega^+(\psi)})$ is just a weak solution of the problem (3.12)–(3.15). And the estimates (5.2)–(5.5) just follow from (5.12)–(5.14) due to $0<k\leq m-1$. The proof is complete.

Based on the existence and uniqueness of solutions to the problem (3.12)–(3.15) and the suitable decay estimates in Proposition 5.2, we may prove Theorem 2.1.

*Proof of Theorem 2.1.* We first prove the existence. Let $C>0$ and $C(M)>0$ be the constants defined in Proposition 5.2. Now, first choose $M>0$ such that $M\geq 4C$, then choose $\sigma_0>0$ sufficiently small to satisfy the condition in Proposition 5.2 and satisfy $C(M)\sigma_0^{1/2}\leq 1$. Thus, for any $\psi\in\mathcal{K}_M^k$, we have $u\in C^{1,\alpha}(\overline{\Omega^+(\psi)})$ and

$$\|u\|_{1,\alpha;\Omega^+(\psi)}^{(k)} \leq C\sigma(1+C(M)\sigma^{1/2}) \leq 2C \leq \frac{1}{2}M,$$

where $u$ is the unique solution to the problem (3.12)–(3.15). Define

$$\varphi(x)=u(x)+qx_n, \quad x\in\overline{\Omega^+(\psi)}$$

Similar to Proposition 7.2 in [5], we may define an extension operator $\mathcal{P}_\psi : C_{(k)}^{1,\alpha}(\overline{\Omega^+(\psi)}) \to C_{(k)}^{1,\alpha}(\overline{\Omega})$ such that

$$\mathcal{P}_\psi(\varphi)=\varphi \quad \text{on } \overline{\Omega^+(\psi)}$$

and

$$\|\mathcal{P}_\psi(\varphi)-qx_n\|_{1,\alpha;\Omega}^{(k)} \leq \|u\|_{1,\alpha;\Omega^+(\psi)}^{(k)}+\frac{1}{2}M\leq M,$$

which implies $\mathcal{P}_\psi(\varphi) \in \mathcal{K}_M^k$. Therefore, we define an iteration map $J : \mathcal{K}_M^k \to \mathcal{K}_M^k$ by

$$J(\psi) = \mathcal{P}_\psi(\varphi) \quad \text{for any } \psi \in \mathcal{K}_M^k.$$

Fix $\beta \in (0, \alpha)$. It is easy to verify that the set $\mathcal{K}_M^k$ is a compact and convex and closed subset of $C^{1,\beta}_{(k)}(\overline{\Omega})$, and the map $J$ is continuous in the $\|\cdot\|^{(k)}_{1,\beta;\Omega}$-norm, namely, for any sequence $\{\psi_j\}_{j=1}^\infty \subset \mathcal{K}_M^k$ converging to $\psi \in C^{1,\beta}_{(k)}(\overline{\Omega})$ in the $\|\cdot\|^{(k)}_{1,\beta;\Omega}$-norm, we have that $\psi \in \mathcal{K}_M^k$ and $J(\psi_j)$ converges to $J(\psi)$ in the $\|\cdot\|^{(k)}_{1,\beta;\Omega}$-norm (see more details in [5] Propositions 7.2).

Then, by the Schauder fixed point theorem, $J$ has a fixed point $\varphi \in \mathcal{K}_M^k$. Assume $\tilde{\varphi}$ is such a fixed point and let

$$\hat{\varphi}(x) = \min\{\varphi^-(x), \tilde{\varphi}(x)\}.$$

Obviously, $\hat{\varphi}$ is just a solution of the truncated free boundary problem (2.15), (2.16), (2.7) and (2.11) with the free boundary $S_{\hat{\varphi}} = \{x \in \Omega : \varphi^-(x) = \tilde{\varphi}(x)\}$, and thus a solution of the transonic nozzle problem (1.1), (2.8), (2.1), (2.7) and (2.11) with the transonic shock $S_{\hat{\varphi}}$. From (2.1) and (2.7), the transonic shock $S_{\hat{\varphi}}$ is orthogonal to $\partial_l\Omega$ at every intersection point. Note

$$\hat{\varphi}(x) = \tilde{\varphi}(x) = J(\tilde{\varphi})(x) \quad \text{on } \overline{\Omega^+(\tilde{\varphi})}, \tag{5.15}$$

where $J(\tilde{\varphi}) \in C^{1,\alpha}(\overline{\Omega^+(\tilde{\varphi})})$ is just the unique solution of the problem (3.12)–(3.15) with $\psi = \tilde{\varphi} \in \mathcal{K}_M^k$. The properties in the theorem follows from Proposition 5.2 directly according to (5.15).

Finally, let us turn to the uniqueness, whose proof is mainly based on a special partial hodograph transform. Assume $\varphi \in C^1(\overline{\Omega}) \cap C^\infty(\Omega^+)$ with the transonic shock $S$ is a solution of the problem (1.1), (2.8), (2.1), (2.7) and (2.11) satisfying (2.17) with sufficiently small $\sigma \in (0, \sigma_0)$. Define the partial hodograph transform

$$y = T(x), \quad x \in \overline{\Omega},$$

where

$$\begin{cases} y_i = (T(x))_i = \Psi_i(x) & \text{for } i = 1, 2, \cdots, n-1, \\ y_n = (T(x))_n = \varphi^+(x) - \varphi^-(x) \end{cases}$$

with $\Psi = (\Psi_1, \Psi_2, \cdots, \Psi_n)$ being the inverse function of $\Phi$. Then the function

$$u(y) = \varphi^+(T^{-1}(y)) - \varphi^-(T^{-1}(y)), \quad y \in \overline{\Lambda} \times [0, +\infty)$$

is just a weak solution of some conormal problem to some quasilinear elliptic equation in the cylindric domain $\Lambda \times [0, +\infty)$, where $T^{-1}$ is just the inverse transform of $T$. By a standard step, we may prove the uniqueness of weak solution of this conormal problem in the cylindric domain $\Lambda \times [0, +\infty)$, which deduces the uniqueness for the transonic nozzle problem (1.1), (2.1), (2.9), (2.7) and (2.11). See details in §8 of [5]. The proof of Theorem 2.1 is complete.

**Acknowledgement** This is part of the Ph.D thesis of Xie Feng in IMS CUHK, the first author would like to express the hearty thanks to his supervisor Professor Xin zhouping for his encouragement and help.

## References


[1] Bers, L., *Mathematical aspects of subsonic and transonic gas dynamics.* Surveys in Applied Mathematics, Vol. 3, New York; Chapman and Hall, London, 1958.

[2] Canic, S., Keyfitz, B. L. and Lieberman, G. M., A proof of existence of perturbed steady transonic shocks via a free boundary problem. *Comm. Pure Appl. Math.* **53** (2000), no. 4, 484–511.

[3] Chen, G.-Q. and Feldman, M., Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type. *J. Amer. Math. Soc.* **16** (2003), no. 3, 461–494.

[4] Chen, G.-Q. and Feldman, M., Steady transonic shocks and free boundary problems for the Euler equations in infinite cylinders. *Comm. Pure Appl. Math.* **57** (2004), no. 3, 310–356.

[5] Chen, G.-Q. and Feldman, M., existence and stability of multidimensional transonic flows through an infinite nozzle of arbitrary cross-sections. 2005 *Arch. Ration. Mech. Anal.*, to appear.

[6] Chen, S., Xin, Z.P., Yin, H., Global shock waves for the supersonic flow past a perturbed cone. *Comm. Math. Phys.* **228** (2002), no. 1, 47–84.

[7] Chen, Y. and Wu, L., *Second order elliptic equations and elliptic systems.* Translated from the 1991 Chinese original by Bei Hu. Translations of Mathematical Monographs, 174. American Mathematical Society, Providence, RI, 1998.

[8] Courant, R. and Friedrichs, K.O., *Supersonic Flow and Shock Waves.* Interscience Publishers, Inc., New York, N. Y. 1948.

[9] Dong,G.-C., *Nonlinear partial differential equations of second order.* . Translations of Mathematical Monographs, **95**. American Mathematical Society, Providence, RI, 1991.

[10] Embid, P., Goodman, J., Majda, A., Multiple steady states for 1-D transonic flow. *SIAM J. Sci. Statist. Comput.* **5** (1984), no. 1, 21–41.

[11] John, Fritz, *Nonlinear wave equations, formation of singularities.* University Lecture Series, 2. American Mathematical Society, Providence, RI, 1990.

[12] Gilbarg, D. and Trudinger, N. S., *Elliptic partial differential equations of second order.* Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 224. Springer-Verlag, Berlin, 1983.

[13] Glaz,H. M. and Liu, T.-P., The asymptotic analysis of wave interactions and numerical calculations of transonic nozzle flow. *Adv. in Appl. Math.* **5** (1984), no. 2, 111–146.

[14] Han,Q., Lin, F.H., *Elliptic partial differential equations.* Courant Lecture Notes in Mathematics, 1. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.

[15] Koch, H., Mixed problems for fully nonlinear hyperbolic equations. *Math. Z.* **214** (1993), no. 1, 9–42.

[16] Kru'min, A.G., *Boundary-Value problem for transonic flow, John wiley and Sons, LTD*, (2002).

[17] Lieberman, G.M., Holder continuity of the gradient of solutions of uniformly parabolic equations with conormal boundary conditions. *Ann. Mat. Pura Appl.* (4) **148** (1987), 77–99.

[18] Lieberman, G.M., *Second order parabolic differential equations.* World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[19] Liu, T. P., Nonlinear stability and instability of transonic flows through a nozzle. *Comm. Math. Phys.* **83** (1982), no. 2, 243–260.

[20] Morawetz, C. S., On the non-existence of continuous transonic flows past profiles. I. *Comm. Pure Appl. Math.* **9** (1956), 45–68.

[21] Morawetz, C. S., On the non-existence of continuous transonic flows past profiles. II. *Comm. Pure Appl. Math.* **10** (1957), 107–131.

[22] Morawetz, C. S., On the non-existence of continuous transonic flows past profiles. III. *Comm. Pure Appl. Math.* **11** (1958), 129–144.

[23] Morawetz, C. S., On a weak solution for a transonic flow problem. *Comm. Pure Appl. Math.* **38** (1985), no. 6, 797–817.

[24] Morawetz, C. S., Mathematical problems in transonic flow. *Canad. Math. Bull.* **29** (1986), no. 2, 129–139.

[25] Morawetz, C., S., Potential theory for regular and Mach reflection of a shock at a wedge. *Comm. Pure Appl. Math.* **47** (1994), no. 5, 593–624.

[26] Shibata, Y., Kikuchi, M., On the mixed problem for some quasilinear hyperbolic system with fully nonlinear boundary condition. *J. Differential Equations* **80** (1989), no. 1, 154–197.

[27] Xin, Z. and Yin, H., Transonic shock in a nozzle. I. Two-dimensional case. *Comm. Pure Appl. Math.* **58** (2005), no. 8, 999–1050.

[28] Xin, Z. and Yin, H., Transonic shock in a nozzle. II. Three-dimensional case. Preprint, 2003.