Cross-Constrained Variational Problem for a Davey-Stewartson System*

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Abstract. This paper concerns the sharp threshold of blowup and global existence of the solution as well as the strong instability of standing wave $e^{i\omega t}u(x)$ to the system

$$i\phi_t + \Delta\phi + a|\phi|^{p-1}\phi + bE_1(|\phi|^2)\phi = 0 \qquad (DS)$$

in $\mathbf{R}^N$, where $a > 0$, $b > 0$, $1 \leq p < \frac{N+2}{(N-2)^+}$, $N \in \{2,3\}$ and u is a ground state. First, by constructing a type of cross-constrained variational problem and establishing so-called cross-invariant manifolds of the evolution flow, we derive a sharp threshold for global existence and blowup of the solution to the Cauchy problem for (DS) provided $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$. Secondly, by using the scaling argument, we show that how small the initial data are for the global solutions to exist. Finally, we prove the strong instability of the standing waves with finite time blow up by combining the former results, which partially answer the open problem proposed in [16,Remark 8].

Key Words. Davey-Stewartson system; Cross-invariant manifold; Sharp threshold; Global existence; Blowup; Instability

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1. Introduction

In this paper we study the generalized Davey-Stewartson system:

$$i\phi_t + \Delta\phi + a|\phi|^{p-1}\phi + bE_1(|\phi|^2)\phi = 0, \quad t \geq 0, \quad x \in \mathbf{R}^N, \tag{1.1}$$

where $\phi = \phi(t,x)$ is a complex-valued function of $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^N$, $N \in \{2,3\}$, Δ is the usual Laplacian operator in $\mathbf{R}^N$, a and b are positive constants, $1 \leq p < \frac{N+2}{(N-2)^+}$ (If $N = 2$, then $\frac{N+2}{(N-2)^+} = \infty$ and if $N = 3$, then $(N-2)^+ = N-2$) and E_1 is the singular integral operator with symbol $\sigma_1(\xi) = \xi_1^2/|\xi|^2$, $\xi \in \mathbf{R}^N$ and $E_1(\psi) = \mathcal{F}^{-1}\left(\xi_1^2/|\xi|^2\right)\mathcal{F}\psi$, $\mathcal{F}^{-1}$ and $\mathcal{F}$ are the Fourier inverse transform and Fourier transform on $\mathbf{R}^N$, respectively (see [5,6,10,11]).

The Davey-Stewartson system (1.1) has its origin in fluid mechanics, where for $p = 3$ and $N = 2$, it appears as mathematical models for the evolution of shallow-water waves having one predominant direction of travel (see[4,5,6]). Moreover, (1.1) is the N-dimensional extension of the Davey-Stewartson systems in the elliptic-elliptic case,

$$\begin{cases} i\phi_t + \lambda\phi_{xx} + \mu\phi_{yy} + a|\phi|^{p-1}\phi = b_1\phi\psi_x, \\ \nu\psi_{xx} + \psi_{yy} = -b_2\left(|\phi|^2\right)_x, \end{cases} \tag{1.2}$$

where λ, μ, $\nu > 0$, b_1, b_2 are positive constants and $a \in \mathbf{R}$. In this case, (1.2) describes the time evolution of two-dimensional surface of water waves having one propagation preponderantly in the x-direction (see [4,5,6]). In addition, according to the signs of μ and ν, system (1.2) may be classified as:

$$-elliptic - elliptic: \qquad \mu > 0, \quad \nu > 0, \tag{1.3}$$

$$-elliptic - hyperbolic: \qquad \mu > 0, \quad \nu < 0, \tag{1.4}$$

$$-hyperbolic - elliptic: \qquad \mu < 0, \quad \nu > 0, \tag{1.5}$$

$$-hyperbolic - hyperbolic: \qquad \mu < 0, \quad \nu < 0, \tag{1.6}$$

although the last case does not seem to occur in the context of water waves [see also [4]].

A large amount of work [10,11,12,15,16,17,18,22,23] has been devoted to the study of the Davey-Stewartson systems. The Cauchy problem for the Davey-Stewartson systems in all physical relevant cases ((1.3)-(1.5)) has been studied in [10] by using functional analytical methods, in which Ghidaglia and Saut proved the solvability in the Sobolev spaces $H^1 = H^1(\mathbf{R}^2)$. In the case (1.4), Tsutsumi in [22] obtained the $L^p(\mathbf{R}^2)$-decay estimates of solutions of (1.1) ($2 < p < \infty$). In [18], Ozawa obtained the exact blowup solutions of Cauchy problem for (1.2). Ohta in [15,16,17] discussed the existence and nonexistence of stable standing waves for (1.1) under some conditions. Moreover, Guo and Wang in [11] studied blow-up in a finite time and global existence of the solutions to the Cauchy problem for the generalized Davey-Stewartson system with the case (1.3); Wang and Guo in [23] investigated the initial value problem and scattering of solutions to the generalized Davey-Stewartson systems; Li, Guo and Jian in [12] exploited the global existence and blowup of solutions to a degenerate Davey-Stewartson equations. From the view-point of physics, the following problems are very important. Under what conditions, will the water waves become unstable to collapse (blowup)? And Under what conditions, will the water waves stable for all time (global existence)? Especially the sharp thresholds for blowup and global existence are pursued strongly (see Zhang[25,26,27] and their references).

In the present paper, we investigate the sharp threshold of blowup and global existence of the Cauchy problem to the generalized Davey-Stewartson system (1.1) and the instability of the standing waves for (1.1) provided $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$. Although there are some results [26,27,30] about the sharp condition for global existence of the solutions to the Cauchy problem for nonlinear Schrödinger equations, on the study of sharp threshold of blowup and global existence of the solutions to the Cauchy problem of the generalized Davey-Stewartson system (1.1), there is little work in the literature. For $1 + \frac{4}{N} \leq p \leq 3$, in [28,29], the authors obtained the sharp threshold of global existence to the Cauchy problem for (1.1) by using potential well argument [31] and concavity method [32]. However, the methods in [28,29] can not be used to solve the above problems on (1.1) for $3 \leq p < \frac{N+2}{(N-2)^+}$. Therefore, we must introduce some other method to solve the related problems for $3 \leq p < \frac{N+2}{(N-2)^+}$. In the present paper, motivated by the study of nonlinear Schrödinger equations [1,21,24], we construct a type of cross-constrained variational problem and establish its property, then apply it to (1.1). By defining the corresponding cross-invariant manifolds under the flow generated by the Cauchy problem of the Davey-

Stewartson system (1.1), we establish the sharp threshold for global existence and blowup of the solutions to the Cauchy problem for (1.1). In addition, by using the scaling arguments, we show that how small the initial data are for the global solution of the Cauchy problem for (1.1) to exist. Finally, applying the sharp threshold and the property of the cross constrained variational problem, we can also prove the strong instability of standing waves for (1.1).

As for the standing waves for (1.1), Cipolatti [4] treated the standing waves with the existence of ground state by means of P. L. Lion's concentration-compactness method [13,14]. By standing waves we mean special periodic solutions of the form

$$\phi(t,x) = e^{i\omega t}\varphi(x), \tag{1.7}$$

where $\omega \in \mathbf{R}$ and φ is a ground state of the problem:

$$\begin{cases} -\triangle\psi + \omega\psi - a|\psi|^{p-1}\psi - bE_1(|\psi|^2)\psi = 0, \quad x \in \mathbf{R}^N, \\ \psi \in H^1(\mathbf{R}^N), \quad \psi \not\equiv 0. \end{cases} \tag{1.8}$$

The so-called ground states are standing waves which minimize the action among all nontrivial solutions of the form (1.7). Concerning the problem of stability and instability of standing waves for nonlinear Schrödinger equations has been studied by many authors. Berestycki and Cazenave [1] investigated the instability of ground states; Cazenave and Lions [3] as well as Grillakis, Shatah and Strauss [9] obtained the existence of stable standing waves; Shatah and Strauss [20] established the instability of nonlinear bound states.

Our idea is initiated by the works of Berestycki and Cazenave [1] as well as Weinstein [24]. In [1] and [24], the related variational problems have to be solved, the Schwarz symmetrization and complicated variational computation have to be conducted. But in the present paper, using our new variational argument, we can refrain from solving the attaching variational problems, and directly establish the sharp threshold for global existence and blowup of solutions to the Cauchy problem for system (1.1). Moreover, initiated by the work of Soffer and Weinstein [21], we also discuss the instability of the standing waves.

The major difficulties in analysis of the Davey-Stewartson system (1.1) are the nonlinearities which include the singular integral operator E_1. In order to study the sharp threshold of global existence and blowup of solutions and gain the instability of standing waves, one has to search for proper functionals and manifolds as well as suitable dilation transformations. In the present paper, we present a cross-constrained variational argument to study the sharp threshold for global existence and blowup of solutions to the Cauchy problem for (1.1). The idea is to construct a type of cross-constrained minimization problem, and establish so-called cross-invariant manifolds. Then we can derive the sharp threshold for global existence and blowup of solutions to the Cauchy problem for (1.1) provided $1+\frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$. Moreover, utilizing the scaling discussion, we can also the question: How small are the initial data, the global solutions of the Cauchy problem for (1.1) exist? At last, combining the former conclusions, we can also obtain the instability of the standing waves for (1.1) with finite time blow up under the condition $1+\frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$, which partially answer the open problem proposed in [16, Remark 8].

It should be pointed out system (1.1) has its origin in Fluid Mechanics for $a \in \mathbf{R}$. But in the present paper, we only obtain our results for $a > 0$. It is easily seen that our results in this paper still hold for $a = 0$. As far as the case $a < 0$ is concerned, whether our results hold or not remains open.

At the end of this section, we give a brief outline of the rest of the paper. In Section 2 we collect some elementary facts which are useful for our analysis later. In Section 3 we establish the cross-invariant manifolds. The sharp threshold for global existence and blowup is treated in Section 4. Finally, we investigate the strong instability of the standing waves.

2. Preliminaries

In this section, we will recall some known facts and give some elementary results which will be used and play important roles later. Firstly, we endow (1.1) with the initial data

$$\phi(0,x) = \phi_0, \quad x \in \mathbf{R}^N. \tag{2.1}$$

For system (1.1), Guo and Wang [11] as well as Cazenave [2] established the local well-posedness of the Cauchy problem in energy class $H^1(\mathbf{R}^N)$. Now we state the following result about the local existence of weak solutions to the Cauchy problem (1.1)-(2.1) in the energy space $H^1(\mathbf{R}^N)$ (see also [5],Theorem 2.1).

Proposition 2.1. Let $N \in \{2,3\}$ and $1 \le p < \frac{N+2}{(N-2)^+}$. Then the following holds:

(1) For any $\phi_0 \in H^1(\mathbf{R}^N)$, there exists a unique solution ϕ of the Cauchy problem (1.1)-(2.1) on a maximal time interval $[0,T)$ such that $\phi \in \mathbf{C}([0,T), H^1(\mathbf{R}^N))$ and either $T=\infty$ or else $T<\infty$ and

$$\lim_{t\to T} \|\phi\|_{H^1(\mathbf{R}^N)} = \infty.$$

(2) We have conservation of charge and energy, that is

$$\int |\phi|^2 dx = \int |\phi_0|^2 dx, \tag{2.2}$$

$$\mathcal{E}(\phi(t)) = \mathcal{E}(\phi_0) \tag{2.3}$$

for all $t \in [0,T)$, where

$$\mathcal{E}(\phi) = \frac{1}{2}\int |\nabla\phi|^2 dx - \frac{a}{p+1}\int |\phi|^4 dx - \frac{1}{4}b\int |\phi|^2 E_1(|\phi|^2)dx. \tag{2.4}$$

Here and hereafter, for simplicity, we denote $\int_{\mathbf{R}^N} \cdot dx$ by $\int \cdot dx$.

For more specific results concerning the Cauchy problem (1.1)-(2.1), we refer the reader to [10].

In addition, by a direct calculation (see also Ohta [15,16,17]) we have

Proposition 2.2. Let $\phi_0 \in H^1(\mathbf{R}^N)$ and $\phi(t)$ be a solution of the Cauchy problem (1.1)-(2.1) on $[0,T)$. Put

$$J(t) := \int |x|^2|\phi|^2 dx. \tag{2.5}$$

Then one has

$$J'(t) = -4Im\int x\phi\nabla\bar{\phi}dx, \tag{2.6}$$

and by (2.4)

$$\begin{aligned} J''(t) &= 8\int |\nabla\phi|^2 dx - 4N\frac{p-1}{p+1}a\int |\phi|^{p+1}dx - 2Nb\int |\phi|^2 E_1(|\phi|^2)dx \\ &= 16E(\phi_0) - \frac{4N(p-1)-16}{p+1}a\int |\phi|^{p+1}dx - (2N-4)b\int |\phi|^2 E_1(|\phi|^2)dx. \end{aligned} \tag{2.7}$$

Thus the following result is true.

Corollary 2.1. Let $\phi_0 \in H^1(\mathbf{R}^N)$ and $N \in \{2,3\}$. For $1+\frac{4}{N} \le p < \frac{N+2}{(N-2)^+}$, when $\mathcal{E}(\phi_0) < 0$, the solution ϕ of the Cauchy problem (1.1)-(2.1) blows up in a finite time.

Proof. We prove this proposition by contradiction. Suppose that the maximal existence time T of the solution ϕ to the Cauchy problem (1.1)-(2.1) is infinity. Since $N \in \{2,3\}$, and $1+\frac{4}{N} \le p < \frac{N+2}{(N-2)^+}$, by (2.7) we have

$$J''(t) \le 16\mathcal{E}(\phi_0), \quad 0 \le t < \infty. \tag{2.8}$$

Through a classical analysis, the following identity is true:

$$J(t) = J(0) + J'(0)t + \int_0^t (t-s)J''(s)ds, \quad 0 \le t < \infty. \tag{2.9}$$

From (2.8) it follows that

$$J(t) \le 8\mathcal{E}(\phi_0)t^2 + J'(0)t + J(0), \quad 0 \le t < \infty. \tag{2.10}$$

Noting that $J(t)$ is a nonnegative function, and

$$J(0) = \int |x|^2|\phi_0|^2 dx, \quad J'(0) = -4Im \int x\phi_0 \nabla\bar{\phi}_0 dx, \tag{2.11}$$

by $\mathcal{E}(\phi_0) < 0$ and (2.10) we get that there exists $T^* < \infty$ such that $\lim_{t\to T^*} J(t) = 0$. Namely,

$$\lim_{t\to T^*} \int |x|^2|\phi|^2 dx = 0, \tag{2.12}$$

which together with (2.2) leads us to the contradiction.

Remark 2.1. For system (1.1) without the singular integral operator E_1, by Glassey [33] and (2.2), the following conclusion holds:

Conclusion : Let $\phi_0 \in H^1(\mathbf{R}^N)$. Then for $1 \le p < 1+\frac{4}{N}$, the Cauchy problem (1.1)-(2.1) has a unique bounded global solution. For $1+\frac{4}{N} \le p < \frac{N+2}{(N-2)^+}$, when $\|\phi_0\|_{H^1(\mathbf{R}^N)}$ is sufficiently small, the Cauchy problem (1.1)-(2.1) has a unique bounded global solution. Therefore, for system (1.1) without the singular integral operator E_1, from Corollary 2.1 we see that $p = 1+\frac{4}{N}$ is the critical nonlinearity index for blowup and global existence of the Cauchy problem (1.1)-(2.1). Thus, in that case, we call $p = 1+\frac{4}{N}$ a critical case, $1 \le p < 1+\frac{4}{N}$ a subcritical case, $1+\frac{4}{N} < p < \frac{N+2}{(N-2)^+}$ a supercritical case. But for system

(1.1) with the singular integral operator E_1, when $1 \le p < 1 + \frac{4}{N}$, the Cauchy problem (1.1)-(2.1) has a unique bounded global solution only for $N = 1$ using the method in Glassey [33], and whether the Cauchy problem (1.1)-(2.1) has a unique bounded solution or not for $N \in \{2, 3\}$ remains open. For $1 + \frac{4}{N} \le p < \frac{N+2}{(N-2)^+}$, it is evident that the Cauchy problem (1.1)-(2.1) has either global solutions or blowup solutions. Therefore, it is a natural topic to search for the sharp threshold for global existence and blowup of the solutions to the Cauchy problem (1.1)-(2.1) for $1 + \frac{4}{N} \le p < \frac{N+2}{(N-2)^+}$, which is one of the aims in the present paper.

Remark 2.2. From (2.7), when $N = 2$, for the critical case $p = 3$ one has

$$J(t) = 8\mathcal{E}(\phi_0)t^2 + J'(0)t + J(0),$$

which is a quadratic function about time t. This is coincide with the case of system (1.1) without the singular integral operator E_1.

Now we give some known facts in Cipolatti [4,5].

Lemma 2.1(Cipolatti [4]). Let E_1 be the singular integral operator defined in Fourier variables by

$$\mathcal{F}\{E_1(\psi)\}(\xi) = \sigma_1(\xi)\mathcal{F}\{\psi\}(\xi),$$

where $\sigma_1(\xi) = \xi_1^2/|\xi|^2$, $\xi \in \mathbf{R}^N$ and $\mathcal{F}$ denotes the Fourier transform in $\mathbf{R}^N$:

$$\mathcal{F}\{\psi\}(\xi) = (1/2\pi)^{\frac{N}{2}} \int e^{-i\xi x}\psi(x)dx.$$

For $1 < p < \infty$, E_1 satisfies the following properties:

i) $E_1 \in \mathcal{L}(L^p, L^p)$,

where $\mathcal{L}(L^p, L^p)$ denotes the space of bounded linear operators from L^p to L^p.

ii) If $\psi \in H^s$, then $E_1(\psi) \in H^s$, $s \in \mathbf{R}$,

iii) If $\psi \in W^{m,p}$, then $E_1(\psi) \in W^{m,p}$ and

$$\partial_k E_1(\psi) = E_1(\partial_k \psi), \qquad k = 1, \cdots, N,$$

iv) E_1 preserves the following operations:

-translation: $E_1(\psi(\cdot + y))(x) = E_1(\psi)(x + y)$, $y \in \mathbf{R}^N$,

-dilatation: $E_1(\psi(\lambda\cdot))(x) = E_1(\psi)(\lambda x)$, $\lambda > 0$,

-conjugation: $\overline{E_1(\psi)} = E_1(\overline{\psi})$,

where $\overline{\psi}$ is the complex conjugate of ψ.

Thus Lemma 2.1 yields directly the following properties.

Remark 2.3(Cipolatti [4] and Ohta [16]). Let B_1 be the quadratic functional on L^2 defined by

$$B_1(\psi) = \int \sigma_1(\xi)|\mathcal{F}(\psi)(\xi)|^2 d\xi.$$

It follows from the Parseval identity

$$\int f \cdot \overline{g} dx = \int \mathcal{F}[f]\mathcal{F}[\overline{g}] d\xi, \quad d\xi = (2\pi)^{-N} dx \tag{2.13}$$

that

$$B_1(\psi) = \int E_1(\psi)\overline{\psi} dx,$$

and in particular we have

$$B_1(\psi) \leq \int |\psi|^2 dx, \tag{2.14}$$

$$B_1 \in C^\infty(L^2, R), \ with \ B_1' = 2E_1.$$

Therefore, from the definition of E_1, the Parseval identity (2.13) and (2.14), we have

$$\int |\psi|^2 E_1(|\psi|^2) dx \leq \int |\psi|^4 dx \tag{2.15}$$

and

$$\begin{aligned} \int |\psi|^2 E_1(|\psi|^2) dx &= \int |\psi|^2 \mathcal{F}^{-1} \sigma_1(\xi) F(|\psi|^2) dx \\ &= \int \sigma_1(\xi) |\mathcal{F}(|\psi|^2)|^2 d\xi > 0. \end{aligned}$$

Finally, we state an elementary Lemma.

Lemma 2.2 (Cipolatti [5]). For all $\phi \in \mathcal{S}(R^N, R)$ (the Schwartz space of rapidly decreasing functions), the following identities hold:

$$(i) \qquad \int \phi x \cdot \nabla \phi dx = -\frac{N}{2} \int |\phi|^2 dx.$$

$$(ii) \qquad \int |\phi|^{p-1}\phi x\cdot\nabla\phi dx = -\frac{N}{p+1}\int |\phi|^{p+1}dx.$$

$$(iii) \qquad \int E_1(|\phi|^2)\phi x\cdot\nabla\phi = -\frac{N}{4}\int |\phi|^2 E_1(|\phi|^2)dx.$$

3. The Cross-Constrained Variational Problem and the Cross-Invariant Manifolds

In this section, we first define some functionals and manifolds, then consider the constrained minimization problems. Since system (1.1) include the singular integral operator E_1, our arguments on system (1.1) are more difficult than the related discussions on the system (1.1) without including the singular integral operator E_1.

For $u\in H^1(\mathbf{R}^N)$, $\omega>0$ and $1\le p<\frac{N+2}{(N-2)^+}$, we define the following functionals:

$$I(u) := \frac{1}{2}\int |\nabla u|^2dx + \frac{\omega}{2}\int |u|^2dx - \frac{a}{p+1}\int |u|^{p+1}dx - \frac{b}{4}\int |u|^2E_1(|u|^2)dx, \tag{3.1}$$

$$S(u) := \int |\nabla u|^2dx + \omega\int |u|^2dx - a\int |u|^{p+1}dx - b\int |u|^2E_1(|u|^2)dx, \tag{3.2}$$

$$Q(u) := \int |\nabla u|^2dx - \frac{N(p-1)}{2(p+1)}a\int |u|^{p+1}dx - \frac{N}{4}b\int |u|^2E_1(|u|^2)dx. \tag{3.3}$$

From Remark 2.3 it follows that

$$\int |u|^2E_1(|u|^2)dx \le \int |u|^4dx. \tag{3.4}$$

By (3.4) and the Sobolev's embedding theorem, the above functionals are well-defined. In addition, we define a manifold

$$\mathcal{N} := \left\{u\in H^1(\mathbf{R}^N)\setminus\{0\},\quad S(u)=0\right\}.$$

We first define the constrained variational problem:

$$d_{\mathcal{N}} := \inf_{\mathcal{N}} I(u). \tag{3.5}$$

Thus, we have the following results.

Lemma 3.1 Let $1 < p < \frac{N+2}{(N-2)^+}$, then $d_{\mathcal{N}} > 0$.

Proof. From (3.1),(3.2) and (3.5), on $\mathcal{N}$ we get

$$\begin{aligned} I(u) &= \frac{p-1}{2(p+1)} a \int |u|^{p+1} dx + \frac{b}{4} \int |u|^2 E_1(|u|^2) dx, \\ &= \frac{p-3}{4(p+1)} a \int |u|^{p+1} dx + \frac{b}{4} \int (|\nabla u|^2 + \omega |u|^2) dx, \end{aligned} \tag{3.6}$$

and

$$\int (|\nabla u|^2 + \omega |u|^2) dx = a \int |u|^{p+1} dx + b \int |u|^2 E_1(|u|^2) dx. \tag{3.7}$$

Since $1 < p < \frac{N+2}{(N-2)^+}$,

$$I(u) \geq \frac{1}{2} \min \left\{ \frac{p-1}{p+1}, \frac{1}{2} \right\} \left(a \int |u|^{p+1} dx + b \int |u|^2 E_1(|u|^2) dx \right). \tag{3.8}$$

By the Sobolev embedding theorem and (3.4), one gets

$$\begin{aligned} a \int |u|^{p+1} dx + b \int |u|^2 E_1(|u|^2) dx &\\ &\leq a \int |u|^{p+1} dx + b \int |u|^4 \\ &\leq C_1 \left(\int |\nabla u|^2 dx + \omega \int |u|^2 dx \right)^{\frac{p+1}{2}} + C_2 \left(\int |\nabla u|^2 dx + \omega \int |u|^2 dx \right)^2 \\ &\leq C \left(\int |\nabla u|^2 dx + \omega \int |u|^2 dx \right)^{\frac{\delta}{2}}, \end{aligned} \tag{3.9}$$

where $\delta = \max\{p+1, 4\} \geq 4$ or $\delta = \min\{p+1, 4\} > 2$ since $p > 1$. Here and hereafter, $C > 0$ denotes various positive constants. From (3.7) and (3.9), it follows that

$$\begin{aligned} a \int |u|^{p+1} dx &+ b \int |u|^2 E_1(|u|^2) dx \\ &\leq C \left(a \int |u|^{p+1} dx + b \int |u|^2 E_1(|u|^2) dx \right)^{\frac{\delta}{2}}, \end{aligned}$$

thus

$$\left(a \int |u|^{p+1} dx + b \int |u|^2 E_1(|u|^2) dx \right)^{\frac{\delta}{2} - 1} \geq C > 0. \tag{3.10}$$

Since $\frac{\delta}{2} - 1 > 0$, (3.10) implies that

$$a \int |u|^{p+1} dx + b \int |u|^2 E_1(|u|^2) dx \geq C > 0. \tag{3.11}$$

Therefore (3.8), (3.11) and $1 < p < \frac{N+2}{(N-2)^+}$ imply that $I(u) \geq C > 0$, that is,

$$d_{\mathcal{N}} \geq C > 0. \tag{3.12}$$

Lemma 3.2. There exists $u \in H^1(\mathbf{R}^N) \backslash \{0\}$ such that both $S(u) = 0$ and $Q(u) = 0$.

Proof. From Ohta [16,17], it follows that there exists $u \in H^1(\mathbf{R}^N) \setminus \{0\}$ such that u is a solution of the following Euler-Lagrangian equation

$$-\triangle u + \omega u - a|u|^{p-1}u - bE_1(|u|^2)u = 0. \tag{3.13}$$

Thus $S(u) = 0$, which is obtained by multiplying (3.13) with u and then integrating with respect to x on R^N. Moreover, by (3.13) and Lemma 2.2, we have the Pohozaev identity

$$\frac{N-2}{N} \int |\nabla u|^2 dx + \omega \int |u|^2 dx - \frac{2}{p+1} a \int |u|^{p+1} dx - \frac{1}{2} b \int |u|^2 E_1(|u|^2) dx = 0, \tag{3.14}$$

which is obtained from multiplying (3.13) by $x \cdot \nabla u$. Noting that $S(u) = 0$, we get $Q(u) = 0$.

Now we define a cross-manifold in $H^1(\mathbf{R}^N)$ as follows

$$\mathcal{M} := \left\{ u \in H^1(\mathbf{R}^N), \quad S(u) < 0, \quad Q(u) = 0 \right\}. \tag{3.15}$$

Then the following result is true.

Lemma 3.3. Let $N \in \{2, 3\}$. Then $\mathcal{M}$ is not empty provided $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$.

Proof. We divide the proof into two cases: **i)** $p = 3$ and **ii)** $p \neq 3$.

We first treat the case **i)** $p = 3$. In this case, from Lemma 3.2 it follows that there exists $u \in H^1(\mathbf{R}^N) \setminus \{0\}$ such that both $S(u) = 0$ and $Q(u) = 0$. Let $v_\lambda = \lambda u(\lambda x)$ for $\lambda > 0$, then $v_\lambda \in H^1(\mathbf{R}^N) \setminus \{0\}$. By (3.2) and (3.3), we get

$$\begin{aligned} S(v_\lambda) \quad &= \quad \lambda^{4-N} \int |\nabla u|^2 dx + \lambda^{2-N} \omega \int |u|^2 dx \\ &\quad -\lambda^{4-N} a \int |u|^4 dx - \lambda^{4-N} b \int |u|^2 E_1(|u|^2) dx \end{aligned}$$

$$= \lambda^{4-N}\left(\int |\nabla u|^2dx - a\int |u|^4dx - b\int |u|^2E_1(|u|^2)dx\right)$$

$$+\lambda^{2-N}\omega\int |u|^2dx, \qquad (3.16)$$

$$Q(v_\lambda) = \lambda^{4-N}\left(\int |\nabla u|^2dx - \frac{N}{4}a\int |u|^4dx - \frac{N}{4}b\int |u|^2E_1(|u|^2)dx\right). \qquad (3.17)$$

Thus $S(u)=0$ implies that there exists $\lambda^*>1$ such that $S(v_{\lambda^*})<0$.

On the other hand, from $\lambda^*>1$ and $Q(u)=0$, we still have $Q(v_{\lambda^*})=0$. So $v_{\lambda^*}\in\mathcal{M}$. This proves $\mathcal{M}$ is not empty for i) $p=3$.

Next we treat case **ii)** $p\neq 3$. In this case, by $1+\frac{4}{N}\le p<\frac{N+2}{(N-2)^+}$, we can divide this case into the following three cases:

(**a**) $\quad 3<p<\infty, \qquad when \quad N=2;$

(**b**) $\quad \frac{7}{3}\le p<3, \qquad when \quad N=3;$

(**c**) $\quad 3<p<5, \qquad when \quad N=3.$

From Lemma 3.2, it follows that there exists $u\in H^1(\mathbf{R}^N)\setminus\{0\}$ such that both $S(u)=0$ and $Q(u)=0$. For $\lambda>1$, let $v=\lambda u$, then $v\in H^1(\mathbf{R}^N)\setminus\{0\}$. From (3.2) and (3.3) it follows that

$$S(v) = \int |\nabla v|^2dx + \omega\int |v|^2dx - a\int |v|^{p+1}dx - b\int |v|^2E_1(|v|^2)dx$$

$$= \lambda^2\int |\nabla u|^2dx + \lambda^2\omega\int |u|^2dx$$

$$-\lambda^{p+1}a\int |u|^{p+1}dx - \lambda^4 b\int |u|^2E_1(|u|^2)dx, \qquad (3.18)$$

$$Q(v) = \int |\nabla v|^2dx - \frac{N(p-1)}{2(p+1)}a\int |v|^{p+1}dx - \frac{N}{4}b\int |v|^2E_1(|v|^2)dx$$

$$= \lambda^2 \int |\nabla u|^2 dx - \lambda^{p+1} \frac{N(p-1)}{2(p+1)} a \int |u|^{p+1} dx$$

$$-\lambda^4 \frac{N}{4} b \int |u|^2 E_1(|u|^2) dx. \tag{3.19}$$

By $S(u) = 0$ and $Q(u) = 0$, from $\frac{N(p-1)}{2(p+1)} < 1$ since $1 + \frac{4}{N} \le p < \frac{N+2}{(N-2)^+}$ and $N \in \{2, 3\}$, we can choose $\lambda > 1$ large enough such that $S(v) < 0$, $Q(v) < 0$,

$$\int |\nabla v|^2 dx - \frac{N(p-1)}{2(p+1)} a \int |v|^{p+1} dx > 0, \tag{3.20}$$

$$\int |\nabla v|^2 dx - \frac{N}{4} b \int |v|^2 E_1(|v|^2) dx > 0, \tag{3.21}$$

$$\int |\nabla v|^2 dx - a \int |v|^{p+1} dx < 0, \tag{3.22}$$

and

$$\int |\nabla v|^2 dx - b \int |v|^2 E_1(|v|^2) dx < 0. \tag{3.23}$$

We first prove case (a) $3 < p < \infty$ **and** $N = 2$.

(a) Let $v_\beta = \beta^{\frac{2}{p-1}} v(\beta x)$. Thus we have

$$S(v_\beta) = \beta^{\frac{4}{p-1}} \left(\int |\nabla v|^2 dx - a \int |v|^{p+1} dx \right)$$

$$+\beta^{\frac{6-2p}{p-1}} \omega \int |v|^2 dx - \beta^{\frac{10-2p}{p-1}} b \int |v|^2 E_1(|v|^2) dx, \tag{3.24}$$

$$Q(v_\beta) = \beta^{\frac{4}{p-1}} \left(\int |\nabla v|^2 dx - \frac{p-1}{p+1} a \int |v|^{p+1} dx \right)$$

$$-\frac{1}{2} \beta^{\frac{10-2p}{p-1}} b \int |v|^2 E_1(|v|^2) dx. \tag{3.25}$$

By $3 < p < \infty$, it follows that $\frac{4}{p-1} > \frac{10-2p}{p-1}$ and $\frac{6-2p}{p-1} < 0$. Therefore $Q(v) < 0$, (3.20) and (3.25) imply that there exists $\beta^* > 1$ such that $Q(v_{\beta^*}) = 0$.

On the other hand, from $\beta^* > 1$, $S(v) < 0$, (3.22) and (3.24), it still follows that $S(v_{\beta^*}) < 0$. So $v_{\beta^*} \in \mathcal{M}$.

Next we prove case (b) $\frac{7}{3} \le p < 3$ **and** $N = 3$**.** In this case, $\frac{1}{3} \le p - 2 < 1$.

(b) Let $v_\mu = \mu v(\mu x)$. Thus one has

$$\begin{aligned} S(v_\mu) &= \mu\Big(\int |\nabla v|^2 dx - b\int |v|^2 E_1(|v|^2)dx\Big) \\ &\quad + \mu^{-1}\omega \int |v|^2 dx - \mu^{p-2} a \int |v|^{p+1} dx, \end{aligned} \tag{3.26}$$

$$\begin{aligned} Q(v_\mu) &= \mu\Big(\int |\nabla v|^2 dx - \frac{3}{4} b\int |v|^2 E_1(|v|^2)dx\Big) \\ &\quad - \frac{3(p-1)}{2(p+1)} \mu^{p-2} a \int |v|^{p+1} dx. \end{aligned} \tag{3.27}$$

Therefore by $Q(v) < 0$, from (3.21) and (3.27) it follows that there exists $\mu^* > 1$ such that $Q(v_{\mu^*}) = 0$.

On the other hand, from $\mu^* > 1$, $S(v) < 0$, (3.23) and (3.26), it still follows that $S(v_{\mu^*}) < 0$. So $v_{\mu^*} \in \mathcal{M}$.

Finally, we prove case (c) $3 < p < 5$ **and** $N = 3$**.** In this case, $5 - p > 11 - 3p$ and $7 - 3p < 0$.

(c) Let $v_\xi = \xi^{\frac{2}{p-1}} v(\xi x)$. Thus we have

$$\begin{aligned} S(v_\xi) &= \xi^{\frac{5-p}{p-1}}\Big(\int |\nabla v|^2 dx - a\int |v|^{p+1} dx\Big) \\ &\quad + \xi^{\frac{7-3p}{p-1}}\omega \int |v|^2 dx - \xi^{\frac{11-3p}{p-1}} b \int |v|^2 E_1(|v|^2) dx, \end{aligned} \tag{3.28}$$

$$\begin{aligned} Q(v_\xi) &= \xi^{\frac{5-p}{p-1}}\Big(\int |\nabla v|^2 dx - \frac{3(p-1)}{2(p+1)} a\int |v|^{p+1} dx\Big) \\ &\quad - \frac{3}{4}\xi^{\frac{11-3p}{p-1}} b \int |v|^2 E_1(|v|^2) dx. \end{aligned} \tag{3.29}$$

Therefore, from (3.20), (3.29) and $Q(v) < 0$, it follows that there exists $\xi^* > 1$ such that $Q(v_{\xi^*}) = 0$.

On the other hand, from $\xi^* > 1$, $S(v) < 0$, (3.22) and (3.28), it still follows that $S(v_{\xi^*}) < 0$. So $v_{\xi^*} \in \mathcal{M}$.

From the above argument, we obtain that $\mathcal{M}$ is not empty provided $1+\frac{4}{N}\le p<\frac{N+2}{(N-2)^+}$ and $N\in\{2,3\}$.

So far, we have completed the proof of Lemma 3.3.

Now let us consider the cross-constrained minimization problem

$$d_{\mathcal{M}}:=\inf_{\mathcal{M}} I(u), \tag{3.30}$$

then we have:

Lemma 3.4. Let $N\in\{2,3\}$. Then $d_{\mathcal{M}}>0$ provided $1+\frac{4}{N}\le p<\frac{N+2}{(N-2)^+}$.

Proof. Let $u\in\mathcal{M}$. From $S(u)<0$, it follows that $u\not\equiv 0$. By $Q(u)=0$, we obtain

$$\begin{aligned} I(u) &= \frac{N(p-1)-4}{4(p+1)}a\int |u|^{p+1}dx \\ &+\left(\frac{N}{8}-\frac{1}{4}\right)b\int |u|^2E_1(|u|^2)dx+\frac{\omega}{2}\int |u|^2dx. \end{aligned} \tag{3.31}$$

Since $1+\frac{4}{N}\le p<\frac{N+2}{(N-2)^+}$ and $N\in\{2,3\}$, (3.31) and $u\not\equiv 0$ imply that $d_{\mathcal{M}}\ge 0$.

In the following, we prove $d_{\mathcal{M}}>0$ by dividing the proof into two cases:

1) **the critical case:** $p=1+\frac{4}{N}$;

2) **the supercritical case:** $1+\frac{4}{N}<p<\frac{N+2}{(N-2)^+}$.

We first consider case 1) the critical case: $p=1+\frac{4}{N}$. In this case, we argue by contradiction. If $d_{\mathcal{M}}=0$, then from (3.30) there were a sequence $\{u_n, n\in\mathbf{Z}^+\}\subset\mathcal{M}$ such that $Q(u_n)=0$, $S(u_n)<0$ and $I(u_n)\to 0$ as $n\to\infty$. Since $p=1+\frac{4}{N}$, (3.31) implies that

$$\int |u_n|^2E_1(|u_n|^2)dx\to 0,\quad \int |u_n|^2dx\to 0\quad as\quad n\to\infty. \tag{3.32}$$

From the Gagliardo-Nirenberg inequality

$$\int |v|^{p+1}dx\le C\left(\int |\nabla v|^2dx\right)^{\frac{N(p-1)}{4}}\left(\int |v|^2dx\right)^{\frac{p+1}{2}-\frac{N(p-1)}{4}},\quad v\in H^1(\mathbf{R}^N), \tag{3.33}$$

for $p=1+\frac{4}{N}$ and u_n we have

$$\int |u_n|^{p+1}dx\le C\int |\nabla u_n|^2dx\cdot\left(\int |u_n|^2dx\right)^{\frac{2}{N}}. \tag{3.34}$$

According to $S(u_n) < 0$, $\int |u_n|^2 E_1(|u_n|^2)dx \leq \int |u_n|^4 dx((2.15))$ and (3.34), we get for $p = 1 + \frac{4}{N}$,

$$\begin{aligned} \int |\nabla u_n|^2 dx + \omega \int |u_n|^2 dx &< a \int |u_n|^{p+1} dx + b \int |u_n|^2 E_1(|u_n|^2) dx \\ &\leq C \int |\nabla u_n|^2 dx \cdot \left(\int |u_n|^2 dx \right)^{\frac{2}{N}} \\ &\quad + C \left(\int |\nabla u_n|^2 dx \right)^{\frac{N}{2}} \cdot \left(\int |u_n|^2 dx \right)^{\frac{4-N}{2}}. \end{aligned} \tag{3.35}$$

For C in (3.35), by (3.32), we have when n sufficiently large,

$$\begin{aligned} \int |\nabla u_n|^2 dx &+ \omega \int |u_n|^2 dx \\ &\geq C \int |\nabla u_n|^2 dx \cdot \left(\int |u_n|^2 dx \right)^{\frac{2}{N}} \\ &\quad + C \left(\int |\nabla u_n|^2 dx \right)^{\frac{N}{2}} \cdot \left(\int |u_n|^2 dx \right)^{\frac{4-N}{2}}. \end{aligned} \tag{3.36}$$

It is obvious that (3.35) and (3.36) are contradictory. Since we have showed $d_{\mathcal{M}} \geq 0$, we get $d_{\mathcal{M}} > 0$ for $p = 1 + \frac{4}{N}$.

Next we treat the supercritical case 2) $1 + \frac{4}{N} < p < \frac{N+2}{(N-2)^+}$. In this case, we use the Sobolev embedding inequality

$$\int |u|^{p+1} dx \leq C \left(\int |\nabla u|^2 dx + \omega \int |u|^2 dx \right)^{\frac{p+1}{2}}. \tag{3.37}$$

From $S(u) < 0$, (2.15) and (3.37), it follows that

$$\begin{aligned} \int |\nabla u|^2 dx &+ \omega \int |u|^2 dx < a \int |u|^{p+1} dx + b \int |u|^2 E_1(|u|^2) dx \\ &\leq C \left(\int |\nabla u|^2 dx + \omega \int |u|^2 dx \right)^{\frac{p+1}{2}} + b \int |u|^4 dx \\ &\leq C \left(\int |\nabla u|^2 dx + \omega \int |u|^2 dx \right)^{\frac{p+1}{2}} + C \left(\int |\nabla u|^2 dx + \omega \int |u|^2 dx \right)^2 \end{aligned}$$

$$\leq \begin{cases} C\left(\int |\nabla u|^2dx + \omega \int |u|^2dx\right)^W, \\ \qquad\qquad when \quad \int |\nabla u|^2dx + \omega \int |u|^2dx \geq 1, \\ \\ C\left(\int |\nabla u|^2dx + \omega \int |u|^2dx\right)^m, \\ \qquad\qquad when \quad \int |\nabla u|^2dx + \omega \int |u|^2dx < 1, \end{cases}$$

where $W = \max\{(p+1)/2, 2\} \geq 2$ and $m = \min\{(p+1)/2, 2\}$. Since $1 + \frac{4}{N} < p < \frac{N+2}{(N-2)^+}$ and $N \in \{2, 3\}$, we get $(p+1)/2 \geq 1 + \frac{2}{N} \geq \frac{5}{3} > 1$. Thus we get

$$\int |\nabla u|^2dx + \omega \int |u|^2dx \geq C > 0. \tag{3.38}$$

In order to prove $d_{\mathcal{M}} > 0$, we divide the following proof into two cases:

2-i) $N = 3$, $\frac{7}{3} < p < 5$;

2-ii) $N = 2$, $3 < p < \infty$.

First we consider 2-i) $N = 3$, $\frac{7}{3} < p < 5$. In this case, by (3.31) we get

$$I(u) = \frac{3p-7}{4(p+1)} a \int |u|^{p+1}dx + \frac{1}{8} b \int |u|^2 E_1(|u|^2)dx + \frac{\omega}{2} \int |u|^2dx. \tag{3.39}$$

Let $D = \min\{\frac{3p-7}{4(p+1)}, \frac{1}{8}\}$. Then by $\frac{7}{3} < p < 5$, we get $D = \frac{3p-7}{4(p+1)}$.

Thus by (3.39),

$$I(u) \geq D\left(a \int |u|^{p+1}dx + b \int |u|^2 E_1(|u|^2)dx\right) + \frac{\omega}{2} \int |u|^2dx. \tag{3.40}$$

From $S(u) < 0$ it follows that

$$\int |\nabla u|^2dx + \omega \int |u|^2dx \leq a \int |u|^{p+1}dx + b \int |u|^2 E_1(|u|^2)dx,$$

which together with (3.40) implies that

$$I(u) \geq D\left(\int |\nabla u|^2dx + \omega \int |u|^2dx\right) + \frac{\omega}{2} \int |u|^2dx. \tag{3.41}$$

By (3.38) and (3.41), we get

$$I(u) \geq C > 0. \tag{3.42}$$

Thus when $N = 3$, (3.30) implies that $d_{\mathcal{M}} > 0$ for $\frac{7}{3} < p < 5$.

Next we consider 2-ii) $N = 2$, $3 < p < \infty$. In this case, by (3.31) and (3.32),

$$I(u) = \frac{2p-6}{4(p+1)} a \int |u|^{p+1}dx + \frac{\omega}{2} \int |u|^2dx. \tag{3.43}$$

In the following, we discuss by contradiction. If $d_{\mathcal{M}} = 0$, then from (3.43) there were a sequence $\{u_n, n \in Z^+\} \subset \mathcal{M}$ such that $Q(u_n) = 0$, $S(u_n) < 0$ and $I(u_n) \to 0$ as $n \to \infty$. Since $3 < p < \infty$, (3.43) implies that

$$\int |u_n|^{p+1} dx \to 0, \quad \int |u_n|^2 dx \to 0 \quad as \quad n \to \infty. \tag{3.44}$$

From the Gagliardo-Nirenberg inequality for $N = 2$ and $3 < p < \infty$,

$$\int |v|^{p+1} dx \leq C \left(\int |\nabla v|^2 dx \right)^{\frac{(p-1)}{2}} \left(\int |v|^2 dx \right), \quad v \in H^1(\mathbf{R}^2), \tag{3.45}$$

u_n satisfies

$$\int |u_n|^{p+1} dx \leq C \left(\int |\nabla u_n|^2 dx \right)^{\frac{(p-1)}{2}} \left(\int |u_n|^2 dx \right). \tag{3.46}$$

According to $S(u_n) < 0$, $\int |u_n|^2 E_1(|u_n|^2) dx \leq \int |u_n|^4 dx((2.15))$ and (3.46), we get

$$\begin{aligned} \int |\nabla u_n|^2 dx + \omega \int |u_n|^2 dx \quad &< \quad a \int |u_n|^{p+1} dx + b \int |u_n|^2 E_1(|u_n|^2) dx \\ &\leq \quad C \left(\int |\nabla u_n|^2 dx \right)^{\frac{(p-1)}{2}} \left(\int |u_n|^2 dx \right) \\ &\quad + C \int |\nabla u_n|^2 dx \cdot \int |u_n|^2 dx. \end{aligned} \tag{3.47}$$

For C in (3.47), from (3.44), it follows that when n sufficiently large,

$$\begin{aligned} \int |\nabla u_n|^2 dx \quad &+ \quad \omega \int |u_n|^2 dx \\ &\geq \quad C \left(\int |\nabla u_n|^2 dx \right)^{\frac{(p-1)}{2}} \left(\int |u_n|^2 dx \right) \\ &\quad + C \int |\nabla u_n|^2 dx \cdot \int |u_n|^2 dx. \end{aligned} \tag{3.48}$$

It is obvious that (3.47) and (3.48) are contradictory. Since we have showed $d_{\mathcal{M}} \geq 0$, thus we get $d_{\mathcal{M}} > 0$ for $N = 2$ and $3 < p < \infty$.

So far, we have proved that $d_{\mathcal{M}} > 0$ for the supercritical case $1 + \frac{4}{N} < p < \frac{N+2}{(N-2)^+}$. Therefore from the above arguments of 1) and 2), we obtain $d_{\mathcal{M}} > 0$ for $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$.

This completes the proof of Lemma 3.4.

Now we define

$$d := \min\{d_{\mathcal{N}}, d_{\mathcal{M}}\}. \tag{3.49}$$

Thus by Lemma 3.1 and Lemma 3.4, the following result is true.

Proposition 3.1. Let $N \in \{2,3\}$. Then $d > 0$ provided $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$.

We further define

$$\mathcal{K} := \Big\{\phi \in H^1(\mathbf{R}^N), \quad I(\phi) < d, \quad S(\phi) < 0, \quad Q(\phi) < 0\Big\}. \tag{3.50}$$

Thus we have

Lemma 3.5. Let $N \in \{2,3\}$. Then $\mathcal{K}$ is not empty provided $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$.

Proof. From Lemma 3.2, there exists $u \in H^1(\mathbf{R}^N) \setminus \{0\}$ such that both $S(u) = 0$ and $Q(u) = 0$. Put $u_\lambda(x) = \lambda u(x)$, then by (3.1), (3.2) and (3.3), one has

$$\begin{aligned} S(u_\lambda) &= \lambda^2 \left(\int |\nabla u|^2 dx + \omega \int |u|^2 dx\right) \\ &\quad -\lambda^{p+1} a \int |u|^{p+1} dx - \lambda^4 b \int |u|^2 E_1(|u|^2) dx, \end{aligned} \tag{3.51}$$

$$\begin{aligned} Q(u_\lambda) &= \lambda^2 \int |\nabla u|^2 dx - \frac{N(p-1)}{2(p+1)} \lambda^{p+1} a \int |u|^{p+1} dx \\ &\quad -\frac{N}{4} \lambda^4 b \int |u|^2 E_1(|u|^2) dx, \end{aligned} \tag{3.52}$$

$$\begin{aligned} I(u_\lambda) &= \frac{1}{2}\lambda^2 \left(\int |\nabla u|^2 dx + \omega \int |u|^2 dx\right) \\ &\quad -\frac{a}{p+1} \lambda^{p+1} \int |u|^{p+1} dx - \frac{1}{4}\lambda^4 b \int |u|^2 E_1(|u|^2) dx. \end{aligned} \tag{3.53}$$

Since $d > 0$ and $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$, for $\lambda > 1$ large enough, in view of (3.51), (3.52) and (3.53) as well as $S(u) = 0$ and $Q(u) = 0$, one always has that $S(u_\lambda) < 0$, $Q(u_\lambda) < 0$ and $I(u_\lambda) < d$. Thus $u_\lambda \in \mathcal{K}$.

This completes the proof of Lemma 3.5. Furthermore, we have

Proposition 3.2. Let $1 + \frac{4}{N} < p < \frac{N+2}{(N-2)^+}$ and $N \in \{2,3\}$. Then $\mathcal{K}$ is an invariant manifold of the Cauchy problem (1.1)-(2.1). That is, if $\phi_0(x) \in \mathcal{K}$, then the solution $\phi(t,x)$ of the Cauchy problem (1.1)-(2.1) satisfies $\phi(t,\cdot) \in \mathcal{K}$ for any $t \in [0,T)$.

Proof. Let $\phi_0 \in \mathcal{K}$. According to Proposition 2.1, there exists a unique $\phi(t,\cdot) \in \mathbf{C}([0,T); H^1(\mathbf{R}^N))$ with $T \leq \infty$ such that $\phi(t,x)$ is a solution of the Cauchy problem (1.1)-(2.1). From (2.2), (2.3), (2.4) and (3.1), it follows that

$$I(\phi(t,\cdot)) = I(\phi_0), \qquad t \in [0,T). \tag{3.54}$$

Thus $I(\phi_0) < d$ implies that $I(\phi(t,\cdot)) < d$ for any $t \in [0,T)$.

We first show $S(\phi(t,\cdot)) < 0$ for $t \in [0,T)$ by contradiction. If otherwise, from the continuity, there were a $t^* \in (0,T)$ such that $S(\phi(t^*,\cdot)) = 0$. By (3.54), it has $\phi(t^*,\cdot) \not\equiv 0$. From (3.5) and (3.49), it follows that $I(\phi(t^*,\cdot)) \geq d_{\mathcal{N}} \geq d$. This is contradictory to $I(\phi(t^*,\cdot)) < d$ for $t \in [0,T)$. Therefore $S(\phi(t,\cdot)) < 0$ for all $t \in [0,T)$.

Next we prove $Q(\phi(t,\cdot)) < 0$ for $t \in [0,T)$ by contradiction. If otherwise, from the continuity, there were a $\bar{t} \in (0,T)$ such that $Q(\phi(\bar{t},\cdot)) = 0$. Since we have showed $S(\phi(\bar{t},\cdot)) < 0$, it follows that $\phi(\bar{t},\cdot) \in \mathcal{M}$. Thus (3.30) and (3.49) imply that $I(\phi(\bar{t},\cdot)) \geq d_{\mathcal{M}} \geq d$. This is contradictory to $I(\phi(\bar{t},\cdot)) < d$ for $t \in [0,T)$. Therefore $Q(\phi(t,\cdot)) < 0$ for all $t \in [0,T)$.

From the above argument, we proved that $\phi(t,\cdot) \in \mathcal{K}$ for any $t \in [0,T)$. Thus the proof of Proposition 3.2 is completed.

By the same argument as Proposition 3.2, we get

Proposition 3.3. Let $N \in \{2,3\}$. Define

$$\mathcal{K}_+ := \Big\{\phi \in H^1(\mathbf{R}^N), \quad I(\phi) < d, \quad S(\phi) < 0, \quad Q(\phi) > 0\Big\},$$

$$\mathcal{R}_- := \Big\{\phi \in H^1(\mathbf{R}^N), \quad I(\phi) < d, \quad S(\phi) < 0\Big\},$$

$$\mathcal{R}_+ := \Big\{\phi \in H^1(\mathbf{R}^N), \quad I(\phi) < d, \quad S(\phi) > 0\Big\}.$$

If $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$, then $\mathcal{K}_+$, $\mathcal{R}_-$ and $\mathcal{R}_+$ are all invariant manifolds of the Cauchy problem (1.1)-(2.1).

Remark 3.1. For these manifolds defined in Proposition 3.3, we call $\mathcal{R}_-$ and $\mathcal{R}_+$ are invariant manifolds of the Cauchy problem (1.1)-(2.1). In the course of nature, we call $\mathcal{K}$ and $\mathcal{K}_+$ cross-invariant manifolds of the Cauchy problem (1.1)-(2.1).

By the definitions of $\mathcal{K}$, $\mathcal{K}_+$ and $\mathcal{R}_+$, as well as (3.5), (3.30) and (3.49), the following result holds.

Proposition 3.4. Let $N \in \{2,3\}$ and $1+\frac{4}{N} \le p < \frac{N+2}{(N-2)^+}$. Then

$$\Big\{\phi \in H^1(\mathbf{R}^N)\setminus\{0\}, \quad I(\phi) < d\Big\} = \mathcal{R}_+ \cup \mathcal{K}_+ \cup \mathcal{K}.$$

4. Sharp Threshold for Global Existence and Blowup

In this section, we discuss the sharp sufficient condition for blowup and global existence. Firstly, we give a sufficient condition for blowup of the solutions to the Cauchy problem (1.1)-(2.1).

Theorem 4.1 Let $N \in \{2,3\}$ and $1+\frac{4}{N} \le p < \frac{N+2}{(N-2)^+}$. If $\phi_0(x) \in \mathcal{K}$ and satisfies $|x|\phi_0 \in L^2(\mathbf{R}^N)$, then the solution $\phi(t,x)$ of the Cauchy problem (1.1)-(2.1) blows up in a finite time.

Proof. According to Ginibre and Velo [7,8], from $|x|\phi_0(x) \in L^2(\mathbf{R}^N)$, one has $|x|\phi \in L^2(\mathbf{R}^N)$. By $\phi_0 \in \mathcal{K}$, Proposition 3.2 implies that $\phi(t,.) \in \mathcal{K}$ for $t \in [0,T)$. Thus we have $I(\phi) < d$, $S(\phi) < 0$ and $Q(\phi) < 0$. Now we put

$$J(t) = \int |x|^2|\phi|^2 dx, \tag{4.1}$$

(2.7) and (3.3) imply that

$$J''(t) = 8Q(\phi(t,.)), \quad t \in [0,T). \tag{4.2}$$

Fixed $t \in [0,T)$ and denote $\phi(t,.) = \phi$. Thus ϕ satisfies that $I(\phi) < d$, $Q(\phi) < 0$ and $S(\phi) < 0$.

In the following, we prove Theorem 4.1 through two steps:

Step 1. $p = 3$;

Step 2. $p \neq 3$.

We first consider **Step 1.** $p = 3$.

Let $\phi_\lambda = \lambda^{\frac{N}{2}}\phi(\lambda x)$. Then

$$S(\phi_\lambda) \;=\; \lambda^2 \int |\nabla\phi|^2 dx + \omega \int |\phi|^2 dx$$

$$-a\lambda^N \int |\phi|^4 dx - \lambda^N b \int |\phi|^2 E_1(|\phi|^2) dx, \tag{4.3}$$

$$\begin{aligned} Q(\phi_\lambda) &= \lambda^2 \int |\nabla\phi|^2 dx - \frac{N}{4}\lambda^N a \int |\phi|^4 dx \\ &\quad - \frac{N}{4}\lambda^N b \int |\phi|^2 E_1(|\phi|^2) dx. \end{aligned} \tag{4.4}$$

Since $S(\phi) < 0$, it yields that there exists $0 < \lambda^* < 1$ such that $S(\phi_{\lambda^*}) = 0$, and when $\lambda \in (\lambda^*, 1]$, $S(\phi_\lambda) < 0$. For $\lambda \in [\lambda^*, 1]$ and $Q(\phi) < 0$, $Q(\phi_\lambda)$ has the following three possibilities:

1-i) $Q(\phi_\lambda) < 0$ **for** $\lambda \in [\lambda^*, 1]$**;**

1-ii) $Q(\phi_{\lambda^*}) = 0$**;**

1-iii) **There exist** $\mu \in (\lambda^*, 1)$ **such that** $Q(\phi_\mu) = 0$.

For the case 1-i) and 1-ii), we both have $S(\phi_{\lambda^*}) = 0$ and $Q(\phi_{\lambda^*}) \leq 0$. It follows from (3.5), (3.30) and (3.49) that $I(\phi_{\lambda^*}) \geq d_{\mathcal{N}} \geq d$. Moreover, we have

$$\begin{aligned} I(\phi) - I(\phi_{\lambda^*}) &= \frac{1}{2}(1-\lambda^{*2}) \int |\nabla\phi|^2 dx \\ &\quad - \frac{1}{4}(1-\lambda^{*N}) \left[a \int |\phi|^4 dx + b \int |\phi|^2 E_1(|\phi|^2)) dx \right], \end{aligned} \tag{4.5}$$

$$\begin{aligned} Q(\phi) - Q(\phi_{\lambda^*}) &= (1-\lambda^{*2}) \int |\nabla\phi|^2 dx \\ &\quad - \frac{N}{4}(1-\lambda^{*N}) \left[a \int |\phi|^4 dx + b \int |\phi|^2 E_1(|\phi|^2)) dx \right]. \end{aligned} \tag{4.6}$$

According to $0 < \lambda^* < 1$ and $N \in \{2, 3\}$, (4.5) and (4.6) imply that

$$I(\phi) - I(\phi_{\lambda^*}) \geq \frac{1}{2} Q(\phi) - \frac{1}{2} Q(\phi_{\lambda^*}) \geq \frac{1}{2} Q(\phi). \tag{4.7}$$

For the case 1-iii), we have $Q(\phi_\mu) = 0$ and $S(\phi_\mu) < 0$. Thus $\phi_\mu \in \mathcal{M}$. From (3.30) and (3.49), it follows that $I(\phi_\mu) \geq d_{\mathcal{M}} \geq d$. In addition,

$$I(\phi) - I(\phi_\mu) \geq \frac{1}{2} Q(\phi) - \frac{1}{2} Q(\phi_\mu) \geq \frac{1}{2} Q(\phi). \tag{4.8}$$

Since $I(\phi_{\lambda^*}) \geq d$ and $I(\phi_\mu) \geq d$, in view of (4.7) and (4.8), we get

$$\begin{cases} Q(\phi) \leq 2(I(\phi) - I(\phi_\mu)) \leq 2[I(\phi) - d], \\ \\ Q(\phi) \leq 2(I(\phi) - I(\phi_{\lambda^*})) \leq 2[I(\phi) - d]. \end{cases} \tag{4.9}$$

From (2.2), (2.3), (2.4) and (3.1), it has

$$I(\phi) = I(\phi_0). \tag{4.10}$$

Thus by $\phi_0 \in \mathcal{K}$ and (4.2), we have

$$J''(t) = 8Q(\phi) < 16[I(\phi_0) - d] < 0.$$

Obviously $J(t)$ can not verify the above inequality for all time (see also Glassey [33]). Therefore, from Proposition 2.1, it must be the case that $T < \infty$, which implies

$$\lim_{t \in T} \|\phi(t,.)\|_{H^1(\mathbf{R}^N)} = \infty.$$

Now we consider **Step 2.** $p \neq 3$. In this case, by $N \in \{2,3\}$ and $1+\frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$, we divide the proof into two cases:

2-1) $3 < p < \frac{N+2}{(N-2)^+}$ **and** $N \in \{2,3\}$**;**

2-2) $\frac{7}{3} \leq p < 3$ **and** $N = 3$**.**

Step 2-1) We first consider the case 2-1) $3 < p < \frac{N+2}{(N-2)^+}$ and $N \in \{2,3\}$.

In this case, let $\phi_\beta = \beta^{\frac{N}{p+1}} \phi(\beta x)$. Then

$$\begin{aligned} S(\phi_\beta) &= \beta^{\frac{N+2-p(N-2)}{p+1}} \int |\nabla \phi|^2 dx - a \int |\phi|^{p+1} dx \\ &\quad + \beta^{\frac{(1-p)N}{p+1}} \omega \int |\phi|^2 dx - \beta^{\frac{(3-p)N}{p+1}} b \int |\phi|^2 E_1(|\phi|^2) dx, \end{aligned} \tag{4.11}$$

$$\begin{aligned} Q(\phi_\beta) &= \beta^{\frac{N+2-p(N-2)}{p+1}} \int |\nabla \phi|^2 dx \\ &\quad - \frac{N(p-1)}{4(p+1)} a \int |\phi|^{p+1} dx - \frac{N}{4} \beta^{\frac{(3-p)N}{p+1}} b \int |\phi|^2 E_1(|\phi|^2) dx. \end{aligned} \tag{4.12}$$

Since $3 < p < \frac{N+2}{(N-2)^+}$ and $N \in \{2,3\}$, we have

$$\frac{N+2-p(N-2)}{p+1} > \frac{(3-p)N}{p+1}, \qquad \frac{N+2-p(N-2)}{p+1} > 0, \qquad \frac{(3-p)N}{p+1} < 0.$$

Thus $Q(\phi) < 0$ implies that there exists $\beta^* > 1$ such that $Q(\phi_{\beta^*}) = 0$, and when $\beta \in [1, \beta^*)$, $Q(\phi_\beta) < 0$. For $\beta \in [1, \beta^*]$, because $S(\phi) < 0$, $S(\phi_\beta)$ has the following two possibilities:

2-1-i) $S(\phi_\beta) < 0$ for $\beta \in [1, \beta^*]$;

2-1-ii) There exists $1 < \mu \leq \beta^*$ such that $S(\phi_\mu) = 0$.

For the case 2-1-i), we have $Q(\phi_{\beta^*}) = 0$ and $S(\phi_{\beta^*}) < 0$, that is, $\phi_{\beta^*} \in \mathcal{M}$. From (3.30) and (3.49), it follows that $I(\phi_{\beta^*}) \geq d_{\mathcal{M}} \geq d$. Moreover, we have

$$\begin{aligned} I(\phi) - I(\phi_{\beta^*}) &= \frac{1}{2}\left(1 - \beta^{*\frac{N+2-p(N-2)}{p+1}}\right) \int |\nabla \phi|^2 dx \\ &\quad + \frac{\omega}{2}\left(1 - \beta^{*\frac{(1-p)N}{p+1}}\right) \int |\phi|^2 dx \\ &\quad - \frac{1}{4}\left(1 - \beta^{*\frac{(3-p)N}{p+1}}\right) b \int |\phi|^2 E_1(|\phi|^2) dx, \end{aligned} \tag{4.13}$$

$$\begin{aligned} Q(\phi) - Q(\phi_{\beta^*}) &= \left(1 - \beta^{*\frac{N+2-p(N-2)}{p+1}}\right) \int |\nabla \phi|^2 dx \\ &\quad - \frac{N}{4}\left(1 - \beta^{*\frac{(3-p)N}{p+1}}\right) b \int |\phi|^2 E_1(|\phi|^2) dx. \end{aligned} \tag{4.14}$$

Thus from $\beta^* > 1$, $N \in \{2,3\}$ and $3 < p < \frac{N+2}{(N-2)^+}$, it follows that

$$I(\phi) - I(\phi_{\beta^*}) \geq \frac{1}{2}Q(\phi) - \frac{1}{2}Q(\phi_{\beta^*}) = \frac{1}{2}Q(\phi). \tag{4.15}$$

For the case 2-1-ii), we have $S(\phi_\mu) = 0$ and $Q(\phi_\mu) \leq 0$. Thus (3.5) and (3.49) imply that $I(\phi_\mu) \geq d_{\mathcal{N}} \geq d$. Referring to (4.13) and (4.14), we have

$$I(\phi) - I(\phi_\mu) \geq \frac{1}{2}Q(\phi) - \frac{1}{2}Q(\phi_\mu) \geq \frac{1}{2}Q(\phi). \tag{4.16}$$

For the case 2-1-i) and the case 2-1-ii), since $I(\phi_{\beta^*}) \geq d$, $I(\phi_\mu) \geq d$, from (4.15) and (4.16), it follows that

$$\begin{cases} Q(\phi) \leq 2[I(\phi) - I(\phi_{\beta^*})] \leq 2[I(\phi) - d], \\ \\ Q(\phi) \leq 2[I(\phi) - I(\phi_\mu)] \leq 2[I(\phi) - d]. \end{cases} \tag{4.17}$$

From (2.2), (2.3), (2.4) and (3.1), one has $I(\phi) = I(\phi_0)$. Thus by $\phi_0 \in \mathcal{K}$ and (4.2), we have

$$J''(t) = 8Q(\phi) < 16[I(\phi) - d] = 16[I(\phi_0) - d] < 0. \tag{4.18}$$

Step 2-2) Now we deal with the case 2-2) $\frac{7}{3} \leq p < 3$ and $N = 3$.

In this case, let $\phi_\eta = \eta^{\frac{3}{4}}\phi(\eta x)$. Then

$$\begin{aligned} S(\phi_\eta) &= \eta^{\frac{1}{2}} \int |\nabla\phi|^2 dx - \eta^{\frac{3p-9}{4}} a \int |\phi|^{p+1} dx \\ &\quad + \eta^{-\frac{3}{2}}\omega \int |\phi|^2 dx - b\int |\phi|^2 E_1(|\phi|^2) dx, \end{aligned} \tag{4.19}$$

$$Q(\phi_\eta) = \eta^{\frac{1}{2}} \int |\nabla\phi|^2 dx - \frac{3(p-1)}{4(p+1)}\eta^{\frac{3p-9}{4}} a \int |\phi|^{p+1} dx - \frac{3}{4} b \int |\phi|^2 E_1(|\phi|^2) dx. \tag{4.20}$$

From $\frac{7}{3} \leq p < 3$, it follows that

$$\frac{3p-9}{4} < 0, \qquad \frac{3p-9}{4} \geq -\frac{1}{2} > -\frac{3}{2}.$$

Thus $Q(\phi) < 0$ implies that there exists $\eta^* > 1$ such that $Q(\phi_{\eta^*}) = 0$, and when $\eta \in [1, \eta^*)$, $Q(\phi_\eta) < 0$. For $\eta \in [1, \eta^*]$, by $S(\phi) < 0$, we get $S(\phi_\eta)$ has the following two possibilities:

2-2-a) $S(\phi_\eta) < 0$ for $\eta \in [1, \eta^*]$;

2-2-b) There exists $1 < \lambda \leq \eta^*$ such that $S(\phi_\lambda) = 0$.

For the case 2-2-a), we have $Q(\phi_{\eta^*}) = 0$ and $S(\phi_{\eta^*}) < 0$, that is, $\phi_{\eta^*} \in \mathcal{M}$. From (3.30) and (3.49), it follows that $I(\phi_{\eta^*}) \geq d_{\mathcal{M}} \geq d$. In addition, one has

$$\begin{aligned} I(\phi) - I(\phi_{\eta^*}) &= \frac{1}{2}\left(1 - \eta^{*\frac{1}{2}}\right) \int |\nabla\phi|^2 dx + \frac{\omega}{2}\left(1 - \eta^{*-\frac{3}{2}}\right) \int |\phi|^2 dx \\ &\quad - \frac{1}{4}\left(1 - \eta^{*\frac{(3p-9}{4}}\right) a \int |\phi|^{p+1} dx, \end{aligned} \tag{4.21}$$

$$Q(\phi) - Q(\phi_{\eta^*}) = \left(1 - \eta^{*\frac{1}{2}}\right) \int |\nabla\phi|^2 dx - \frac{3}{4}\left(1 - \eta^{*\frac{3p-9}{4}}\right) a \int |\phi|^{p+1} dx. \qquad (4.22)$$

Thus from $\eta^* > 1$ and $\frac{7}{3} \le p < 3$, it follows that

$$I(\phi) - I(\phi_{\eta^*}) \ge \frac{1}{2}Q(\phi) - \frac{1}{2}Q(\phi_{\eta^*}) = \frac{1}{2}Q(\phi). \qquad (4.23)$$

For the case 2-2-b), we have $S(\phi_\lambda) = 0$ and $Q(\phi_\lambda) \le 0$. Thus (3.5) and (3.49) imply that $I(\phi_\lambda) \ge d_{\mathcal{N}} \ge d$. Referring to (4.21) and (4.22), we have

$$I(\phi) - I(\phi_\lambda) \ge \frac{1}{2}Q(\phi) - \frac{1}{2}Q(\phi_\lambda) \ge \frac{1}{2}Q(\phi). \qquad (4.24)$$

For the case 2-2-a) and the case 2-2-b), since $I(\phi_{\eta^*}) \ge d$, $I(\phi_\lambda) \ge d$, from (4.23) and (4.24), it follows that

$$\begin{cases} Q(\phi) \le 2[I(\phi) - I(\phi_{\eta^*})] \le 2[I(\phi) - d], \\ \\ Q(\phi) \le 2[I(\phi) - I(\phi_\lambda)] \le 2[I(\phi) - d]. \end{cases} \qquad (4.25)$$

From (2.2), (2.3), (2.4) and (3.1), one has $I(\phi) = I(\phi_0)$. Thus by $\phi_0 \in \mathcal{K}$ and (4.2), we have

$$J''(t) = 8Q(\phi) < 16[I(\phi) - d] = 16[I(\phi_0) - d] < 0. \qquad (4.26)$$

Thus, from the arguments of **Step 2-1** and **Step 2-2**, in view of (4.18) and (4.26), obviously $J(t)$ can not verify the above inequality for all time (see also Glassey [33]). Therefore, from Proposition 2.1, it must be the case that $T < \infty$, which implies

$$\lim_{t \in T} \|\phi(t, .)\|_{H^1(\mathbf{R}^N)} = \infty.$$

That is, the solution ϕ of the Cauchy problem (1.1)-(2.1) blows up in a finite time.

From the arguments of **Step 1** and **Step 2**, the proof of Theorem 4.1 is completed.

Remark 4.1. In Theorem 4.1, we developed a new argument to obtain the blowup property of the solutions to the Cauchy problem (1.1)-(2.1). Since $d > 0$, we see that Theorem 4.1 is quite different from Corollary 2.1. If $d \le \frac{1}{2}\int |\phi_0|^2 dx$, then by (2.2), (2.3), (2.4), (3.1) and $I(\phi) < d$, we can obtain $\mathcal{E}(\phi_0) < 0$. Thus in this case, from Theorem 2.1, it can follow the result of Theorem 4.1. On the other hand, if $d > \frac{1}{2}\int |\phi_0|^2 dx$, then from $I(\phi) < d$, it may conclude three cases: (1) $\mathcal{E}(\phi_0) > 0$; (2) $\mathcal{E}(\phi_0) = 0$; (3)

$\mathcal{E}(\phi_0) < 0$. So Theorem 4.1 includes the result that when initial energy is nonnegative, the solution ϕ of the Cauchy problem (1.1)-(2.1) can also blow up in a finite time. This kind of results in Theorem 4.1 generalize the result in Corollary 2.1.

In the following, we give a sufficient condition for global existence of the solutions to the Cauchy problem (1.1)-(2.1).

Theorem 4.2. Let $N \in \{2,3\}$ and $1 + \frac{4}{N} \le p < \frac{N+2}{(N-2)^+}$. If $\phi_0 \in \mathcal{K}_+ \cup \mathcal{R}_+$, then the solution $\phi(t,x)$ of the Cauchy problem (1.1)-(2.1) exists globally in $t \in [0,\infty)$.

Proof. We prove this theorem by two steps.

Step 1. we prove the case $\phi_0 \in \mathcal{K}_+$.

Step 2. we prove the case $\phi_0 \in \mathcal{R}_+$.

Firstly, we consider Step 1. $\phi_0 \in \mathcal{K}_+$.

From $\phi_0 \in K_+$, Proposition 3.3 implies that the solution $\phi(t,x)$ of the Cauchy problem (1.1)-(2.1) satisfies that $\phi(t,\cdot) \in \mathcal{K}_+$ for $t \in [0,T)$. For fixed $t \in [0,T)$, we denote $\phi(t,\cdot) = \phi$. So we have $I(\phi) < d$, $Q(\phi) > 0$ and $S(\phi) < 0$. According to (3.1) and (3.3), we obtain

$$\left(\frac{1}{2} - \frac{2}{N(p-1)}\right) \int |\nabla\phi|^2 dx + \frac{\omega}{2}\int |\phi|^2 dx + \left(\frac{1}{2(p-1)} - \frac{1}{4}\right) b \int |\phi|^2 E_1(|\phi|^2) dx < d. \tag{4.27}$$

In the following, we divide the proof into six cases:

1) $N = 2$ and $p = 3$;

2) $N = 3$ and $p = 3$;

3) $N = 2$ and $3 < p < \infty$;

4) $N = 3$ and $p = \frac{7}{3}$;

5) $N = 3$ and $\frac{7}{3} < p < 3$;

6) $N = 3$ and $3 < p < 5$.

Step 1-1. We first consider the case 1) $N = 2$ and $p = 3$.

By (4.27), we have

$$\frac{\omega}{2}\int |\phi|^2 dx < d. \tag{4.28}$$

In the following, we prove that $\int |\nabla\phi|^2dx$ is bounded. Put $\phi_\lambda = \lambda^{\frac{1}{2}}\phi(\lambda x)$. It follows from (3.3) that

$$Q(\phi_\lambda) = \lambda\int |\nabla\phi|^2dx - \frac{1}{2}a\int |\phi|^4dx - \frac{1}{2}b\int |\phi|^2E_1(|\phi|^2)dx. \tag{4.29}$$

By $Q(\phi) > 0$, it follows that when $\lambda \to 0$, $Q(\phi_\lambda) < 0$ and $\lambda \to 1$, $Q(\phi_\lambda) > 0$. Thus by continuity, there exists a $0 < \lambda^* < 1$ such that $Q(\phi_{\lambda^*}) = 0$. According to (3.1) and (3.3), we have

$$I(\phi_{\lambda^*}) = \frac{\omega}{2}\int |\phi_{\lambda^*}|^2dx = \frac{\omega}{2}\lambda^{*-1}\int |\phi|^2dx. \tag{4.30}$$

From (4.28), it follows that

$$I(\phi_{\lambda^*}) < \lambda^{*-1}d. \tag{4.31}$$

Also, by (3.2) and (3.3), it follows from $Q(\phi_{\lambda^*}) = 0$ that

$$S(\phi_{\lambda^*}) = \omega\int |\phi_{\lambda^*}|^2dx - \int |\nabla\phi_{\lambda^*}|^2dx,$$

which has two possibilities:

1-1-i) $\quad S(\phi_{\lambda^*}) < 0$;

1-1-ii) $\quad S(\phi_{\lambda^*}) \geq 0$.

We first treat **1-1-i)** $\quad S(\phi_{\lambda^*}) < 0$. In this case, noting that $Q(\phi_{\lambda^*}) = 0$, we get $\phi_{\lambda^*} \in \mathcal{M}$. Thus by (3.30) and (3.49), one has

$$I(\phi_{\lambda^*}) \geq d_{\mathcal{M}} \geq d > I(\phi), \tag{4.32}$$

which implies that

$$I(\phi) - I(\phi_{\lambda^*}) < 0. \tag{4.33}$$

That is,

$$\left(\frac{1}{2} - \frac{1}{2}\lambda^*\right)\int |\nabla\phi|^2dx + \left(\frac{1}{2} - \frac{1}{2}\lambda^{*-1}\right)\omega\int |\phi|^2dx < 0. \tag{4.34}$$

By (4.28), it has

$$\int |\nabla\phi|^2dx < \frac{\lambda^{*-1} - 1}{1 - \lambda^*}\omega\int |\phi|^2dx < 2\frac{\lambda^{*-1} - 1}{1 - \lambda^*}d. \tag{4.35}$$

Since $0 < \lambda^* < 1$, one has $\frac{\lambda^{*-1}-1}{1-\lambda^*} > 0$.

Now we consider **1-1-ii)** $\quad S(\phi_{\lambda^*}) \geq 0$. In this case, from (4.31) it follows that

$$I(\phi_{\lambda^*}) - \frac{1}{4}S(\phi_{\lambda^*}) < \lambda^{*-1}d. \tag{4.36}$$

Thus by (3.1) and (3.2), one gets

$$\frac{1}{4}\int|\nabla\phi_{\lambda^*}|^2dx+\frac{1}{4}\omega\int|\phi_{\lambda^*}|^2dx<\lambda^{*-1}d. \tag{4.37}$$

Namely,

$$\lambda^*\int|\nabla\phi|^2dx+\lambda^{*-1}\omega\int|\phi|^2dx<4\lambda^{*-1}d,$$

which implies that

$$\int|\nabla\phi|^2dx<4\lambda^{*-2}d. \tag{4.38}$$

By (4.35) and (4.38), we obtain that $\int|\nabla\phi|^2dx$ is bounded for any $t\in[0,T)$, which together with (4.28) yields that ϕ is bounded in $H^1(\mathbf{R}^N)$. So Proposition 2.1 implies that the solution ϕ of the Cauchy problem (1.1)-(2.1) globally exists on $t\in[0,\infty)$ for $\phi_0\in\mathcal{K}_+$, $p=3$ and $N=2$.

Step 1-2. Secondly, we treat the case 2) $N=3$ and $p=3$.

By (4.27), we get

$$\frac{1}{6}\int|\nabla\phi|^2dx+\frac{\omega}{2}\int|\phi|^2dx<d, \tag{4.39}$$

which implies that the solution ϕ of the Cauchy problem (1.1)-(2.1) is bounded in $H^1(\mathbf{R}^N)$ for any $t\in[0,T)$. So Proposition 2.1 implies that the solution ϕ of the Cauchy problem (1.1)-(2.1) globally exists on $t\in[0,\infty)$ for $\phi_0\in\mathcal{K}_+$, $p=3$ and $N=3$.

Step 1-3. Thirdly, we deal with the case 3) $N=2$ and $3<p<\infty$. In this case, it follows from $Q(\phi)>0$ that

$$\int|\nabla\phi|^2dx>\frac{p-1}{p+1}a\int|\phi|^{p+1}dx+\frac{1}{2}b\int|\phi|^2E_1(|\phi|^2)dx.$$

Thus (3.1) implies that

$$\begin{aligned}
I(\phi) &\geq \frac{p-1}{2(p+1)}a\int|\phi|^{p+1}dx+\frac{1}{4}b\int|\phi|^2E_1(|\phi|^2)dx\\
&+\frac{\omega}{2}\int|\phi|^2dx-\frac{1}{p+1}a\int|\phi|^{p+1}dx-\frac{1}{4}b\int|\phi|^2E_1(|\phi|^2)dx\\
&=\left(\frac{p-1}{2(p+1)}-\frac{1}{p+1}\right)a\int|\phi|^{p+1}dx+\frac{\omega}{2}\int|\phi|^2dx\\
&=\frac{p-3}{2(p+1)}a\int|\phi|^{p+1}dx+\frac{\omega}{2}\int|\phi|^2dx,
\end{aligned}$$

which together with $I(\phi) < d$ yields that

$$\frac{p-3}{2(p+1)}a\int|\phi|^{p+1}dx+\frac{\omega}{2}\int|\phi|^2dx<d.$$

Since $3 < p < \infty$, we obtain

$$\frac{\omega}{2}\int|\phi|^2dx<d. \tag{4.40}$$

In the following, we prove that $\int|\nabla\phi|^2dx$ is bounded.

Put $\phi_\lambda = \lambda^{\frac{2}{p+1}}\phi(\lambda x)$. It follows from (3.3) that

$$Q(\phi_\lambda)=\lambda^{\frac{4}{p+1}}\int|\nabla\phi|^2dx-\frac{p-1}{p+1}a\int|\phi|^{p+1}dx-\frac{1}{2}\lambda^{\frac{6-2p}{p+1}}b\int|\phi|^2E_1(|\phi|^2)dx. \tag{4.41}$$

By $Q(\phi) > 0$, one has when $\lambda \to 0$, $Q(\phi_\lambda) < 0$, and when $\lambda \to 1$, $Q(\phi_\lambda) > 0$. Thus by continuity, there exists $0 < \lambda^* < 1$ such that $Q(\phi_{\lambda^*}) = 0$. According to (3.1) and (3.2), one has

$$\begin{aligned} I(\phi_{\lambda^*}) &= \frac{1}{2}\lambda^{*\frac{4}{p+1}}\int|\nabla\phi|^2dx+\frac{\omega}{2}\lambda^{*\frac{2-2p}{p+1}}\int|\phi|^2dx \\ &\quad -\frac{1}{p+1}a\int|\phi|^{p+1}dx-\frac{1}{4}\lambda^{*\frac{6-2p}{p+1}}b\int|\phi|^2E_1(|\phi|^2)dx, \end{aligned} \tag{4.42}$$

$$\begin{aligned} S(\phi_{\lambda^*}) &= \lambda^{*\frac{4}{p+1}}\int|\nabla\phi|^2dx+\omega\lambda^{*\frac{2-2p}{p+1}}\int|\phi|^2dx \\ &\quad -a\int|\phi|^{p+1}dx-\lambda^{*\frac{6-2p}{p+1}}b\int|\phi|^2E_1(|\phi|^2)dx. \end{aligned} \tag{4.43}$$

From $S(\phi) < 0$, it follows that $S(\phi_{\lambda^*})$ has two possibilities:

1-3-a) $S(\phi_{\lambda^*}) < 0$;

1-3-b) $S(\phi_{\lambda^*}) \geq 0$.

We first consider **1-3- a)** $S(\phi_{\lambda^*}) < 0$. In this case, noting that $Q(\phi_{\lambda^*}) = 0$, one has $\phi_{\lambda^*} \in \mathcal{M}$. Thus by (3.30) and (3.49), one has

$$I(\phi_{\lambda^*}) \geq d_{\mathcal{M}} \geq d > I(\phi). \tag{4.44}$$

(4.44) implies that

$$I(\phi)-I(\phi_{\lambda^*})<0. \tag{4.45}$$

That is,

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{2}\lambda^{*\frac{4}{p+1}}\right) & \int|\nabla\phi|^2dx \\ &+ \left(\frac{1}{2}-\frac{1}{2}\lambda^{*\frac{2-2p}{p+1}}\right)\omega\int|\phi|^2dx \\ &+ \left(\frac{1}{4}\lambda^{*\frac{6-2p}{p+1}}-\frac{1}{4}\right)b\int|\phi|^2E_1(|\phi|^2)dx \\ &< 0. \end{aligned} \tag{4.46}$$

Since $3<p<\infty$ and $0<\lambda^*<1$, one has $\lambda^{*\frac{6-2p}{p+1}}>1$, $\lambda^{*\frac{4}{p+1}}<1$ and $\lambda^{*\frac{2-2p}{p+1}}>1$. So (4.40) and (4.46) imply that

$$\begin{aligned}\int|\nabla\phi|^2dx &< \left(\frac{1}{2}\lambda^{*\frac{2-2p}{p+1}}-\frac{1}{2}\right)/\left(\frac{1}{2}-\frac{1}{2}\lambda^{*\frac{4}{p+1}}\right)\omega\int|\phi|^2dx \\ &< 2\left(\lambda^{*\frac{2-2p}{p+1}}-1\right)/\left(1-\lambda^{*\frac{4}{p+1}}\right)d. \end{aligned} \tag{4.47}$$

Secondly, we deal with **1-3-b)** $\quad S(\phi_{\lambda^*})\geq 0$. In this case, let $\phi_\beta=\beta^{\frac{2}{p+1}}\phi_{\lambda^*}(\beta x)$. By (3.2), we get

$$\begin{aligned}S(\phi_\beta) &= \beta^{\frac{4}{p+1}}\int|\nabla\phi_{\lambda^*}|^2dx+\beta^{\frac{2-2p}{p+1}}\omega\int|\phi_{\lambda^*}|^2dx \\ &- a\int|\phi_{\lambda^*}|^{p+1}dx-\beta^{\frac{6-2p}{p+1}}b\int|\phi_{\lambda^*}|^2E_1(|\phi_{\lambda^*}|^2)dx. \end{aligned} \tag{4.48}$$

From $S(\phi_{\lambda^*})\geq 0$, it has $S(\phi_1)=S(\phi_{\lambda^*})\geq 0$ for $\beta=1$ and $S(\phi_\beta)<0$ for β close to zero. Therefore, there exists $\beta^*\in(0,1]$ such that $S(\phi_{\beta^*})=0$. Thus by (3.5) and (3.49), it has

$$I(\phi_{\beta^*})\geq d_{\mathcal{N}}\geq d>I(\phi). \tag{4.41}$$

Since

$$\begin{aligned}I(\phi_{\beta^*}) &= \frac{1}{2}(\beta^*\lambda^*)^{\frac{4}{p+1}}\int|\nabla\phi|^2dx+\frac{\omega}{2}(\beta^*\lambda^*)^{\frac{2-2p}{p+1}}\int|\phi|^2dx \\ &- \frac{1}{p+1}a\int|\phi|^{p+1}dx-\frac{1}{4}(\beta^*\lambda^*)^{\frac{6-2p}{p+1}}b\int|\phi|^2E_1(|\phi|^2)dx, \end{aligned}$$

which together with (4.49) implies that

$$\begin{aligned}\frac{1}{2}\left(1-(\beta^*\lambda^*)^{\frac{4}{p+1}}\right) & \int|\nabla\phi|^2dx+\frac{\omega}{2}\left(1-(\beta^*\lambda^*)^{\frac{2-2p}{p+1}}\right)\int|\phi|^2dx \\ &+\ \frac{1}{4}\left((\beta^*\lambda^*)^{\frac{6-2p}{p+1}}-1\right)b\int|\phi|^2E_1(|\phi|^2)dx \\ &<\ 0. \end{aligned}\tag{4.50}$$

Since $3<p<\infty$ and $\beta^*\lambda^*\in(0,1)$, (4.40) and (4.50) yield that

$$\int|\nabla\phi|^2dx<2\left((\beta^*\lambda^*)^{\frac{2-2p}{p+1}}-1\right)/\left(1-(\beta^*\lambda^*)^{\frac{4}{p+1}}\right)d. \tag{4.51}$$

By (4.40), (4.47) and (4.51), we get that ϕ is bounded in $H^1(\mathbf{R}^N)$. So Proposition 2.1 implies that the solution ϕ of the Cauchy problem (1.1)-(2.1) globally exists on $t\in[0,\infty)$ for $\phi_0\in\mathcal{K}_+$, $N=2$ and $3<p<\infty$.

Step 1-4. We then consider the case 4) $N=3$ and $p=\frac{7}{3}$. In this case, by (4.27) we have

$$\frac{\omega}{2}\int|\phi|^2dx+\frac{1}{8}b\int|\phi|^2E_1(|\phi|^2)dx<d. \tag{4.52}$$

We put $\phi_\mu=\mu^{\frac{9}{10}}\phi(\mu x)$. Then we get

$$Q(\phi_\mu)=\mu^{\frac{4}{5}}\int|\nabla\phi|^2dx-\frac{3}{5}a\int|\phi|^{\frac{10}{3}}dx-\frac{3}{4}\mu^{\frac{3}{5}}b\int|\phi|^2E_1(|\phi|^2)dx. \tag{4.53}$$

Thus $Q(\phi)>0$ implies that there exists a $\mu^*\in(0,1)$ such that $Q(\phi_{\mu^*})=0$. By (3.1) and (3.3), we have

$$\begin{aligned} I(\phi_{\mu^*}) &= \frac{\omega}{2}\int|\phi_{\mu^*}|^2dx+\frac{1}{8}b\int|\phi_{\mu^*}|^2E_1(|\phi_{\mu^*}|^2)dx \\ &= \frac{\omega}{2}\mu^{*-\frac{6}{5}}\int|\phi|^2dx+\frac{1}{8}\mu^{*\frac{3}{5}}b\int|\phi|^2E_1(|\phi|^2)dx. \end{aligned}\tag{4.54}$$

It follows from (4.53) and $\mu^*\in(0,1)$ that

$$I(\phi_{\mu^*})<\mu^{*-\frac{6}{5}}d. \tag{4.55}$$

Now we consider $S(\phi_{\mu^*})$, which has two possibilities:

1-4-a) $S(\phi_{\mu^*})<0$;

1-4-b) $S(\phi_{\mu^*}) \geq 0$.

At first, we consider **1-4-a)** $S(\phi_{\mu^*}) < 0$. In this case, noting that $Q(\phi_{\mu^*}) = 0$, we have $\phi_{\mu^*} \in \mathcal{M}$, which together with (3.30) and (3.49) implies that

$$I(\phi_{\mu^*}) \geq d_{\mathcal{M}} \geq d > I(\phi). \tag{4.56}$$

So one has

$$I(\phi) - I(\phi_{\mu^*}) < 0, \tag{4.57}$$

namely,

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{2}\mu^{*\frac{4}{5}}\right) & \int |\nabla\phi|^2 dx + \frac{1}{2}\left(1 - \mu^{*-\frac{6}{5}}\right)\omega \int |\phi|^2 dx \\ & -\frac{1}{4}\left(1 - \mu^{*\frac{3}{5}}\right) b \int |\phi|^2 E_1(|\phi|^2) dx \\ < & \quad 0, \end{aligned}$$

which implies that

$$\begin{aligned} \int |\nabla\phi|^2 dx \ < & \ \left(\mu^{*-\frac{6}{5}} - 1\right) / \left(1 - \mu^{*\frac{4}{5}}\right) \omega \int |\phi|^2 dx \\ + & \ \frac{1}{2}\left(1 - \mu^{*\frac{3}{5}}\right) / \left(1 - \mu^{*\frac{4}{5}}\right) b \int |\phi|^2 E_1(|\phi|^2) dx. \end{aligned} \tag{4.58}$$

By (4.52) and $\mu^* \in (0, 1)$, we get

$$\int |\nabla\phi|^2 dx < C. \tag{4.59}$$

Now we deal with **1-4-b)** $S(\phi_{\mu^*}) \geq 0$. In this case, from (4.55), it follows that

$$I(\phi_{\mu^*}) - \frac{3}{10} S(\phi_{\mu^*}) < \mu^{*-\frac{6}{5}} d. \tag{4.60}$$

It follows that

$$\begin{aligned} \frac{1}{5}\mu^{*\frac{4}{5}} \int |\nabla\phi|^2 dx \ & + \ \frac{1}{5}\mu^{*-\frac{6}{5}}\omega \int |\phi|^2 dx \\ & + \ \frac{1}{20}\mu^{*\frac{3}{5}} b \int |\phi|^2 E_1(|\phi|^2) dx \\ & < \ \mu^{*-\frac{6}{5}} d. \end{aligned}$$

Thus

$$\int |\nabla\phi|^2 dx < C. \tag{4.61}$$

Therefore, (4.52), (4.59) and (4.61) show that ϕ is bounded in $H^1(\mathbf{R}^N)$ for any $t \in [0,T)$. Thus by Proposition 2.1, we get that the solution ϕ of the Cauchy problem (1.1)-(2.1) exists globally in $t \in [0,\infty)$ for $\phi_0 \in \mathcal{K}_+$, $N=3$ and $p=\frac{7}{3}$.

Step 1-5. We further treat the case 5) $N=3$ and $\frac{7}{3} < p < 3$. In this case, by (4.27) we get

$$\begin{aligned}\left(\frac{1}{2} - \frac{2}{3(p-1)}\right) & \int |\nabla\phi|^2 dx + \frac{\omega}{2}\int |\phi|^2 dx \\ + & \left(\frac{1}{2(p-1)} - \frac{1}{4}\right) b \int |\phi|^2 E_1(|\phi|^2) dx \\ < & \ d. \end{aligned} \tag{4.62}$$

Since $\frac{7}{3} < p < 3$, it has $\frac{1}{2} - \frac{2}{3(p-1)} > 0$ and $\frac{1}{2(p-1)} - \frac{1}{4} > 0$. Therefore, (4.62) implies that the solution ϕ of the Cauchy problem (1.1)-(2.1) is bounded in $H^1(\mathcal{R}^N)$ for any $t \in [0,T)$. Thus Proposition 2.1 implies that the solution ϕ of the Cauchy problem (1.1)-(2.1) exists globally in $t \in [0,\infty)$ for $\phi_0 \in \mathcal{K}_+$, $N=3$ and $\frac{7}{3} < p < 3$.

Step 1-6. At last, we investigate the case 6) $N=3$ and $3 < p < 5$. In this case, from (3.3) and $Q(\phi) > 0$, it follows that

$$\frac{1}{4} b \int |\phi|^2 E_1(|\phi|^2) dx > -\frac{1}{3}\int |\nabla\phi|^2 dx + \frac{p-1}{2(p+1)} a \int |\phi|^{p+1} dx, \tag{4.63}$$

which together with (3.1) and $I(\phi) < d$ implies that

$$\frac{1}{6}\int |\nabla\phi|^2 dx + \frac{\omega}{2}\int |\phi|^2 dx + \frac{p-3}{2(p+1)} a \int |\phi|^{p+1} dx < d. \tag{4.64}$$

Since $3 < p < 5$, (4.64) yields that

$$\frac{1}{6}\int |\nabla\phi|^2 dx + \frac{\omega}{2}\int |\phi|^2 dx < d. \tag{4.65}$$

(4.65) gives that ϕ is bounded in $H^1(\mathbf{R}^N)$ for any $t \in [0,T)$. Thus Proposition 2.1 implies that the solution ϕ of the Cauchy problem (1.1)-(2.1) exists globally in $t \in [0,\infty)$ for

$\phi_0 \in \mathcal{K}_+$, $N = 3$ and $3 < p < 5$.

Thus from **Step 1-1**-**Step 1-6**, for $\phi_0 \in \mathcal{K}_+$, $N \in \{2, 3\}$ and $1 + \frac{4}{N} \le p < \frac{N+2}{(N-2)^+}$, we have proved that the solution $\phi(t, x)$ of the Cauchy problem (1.1)-(2.1) exists globally in $t \in [0, \infty)$.

Now we deal with **Step 2.** $\phi_0 \in \mathcal{R}_+$.

By $\phi_0 \in \mathcal{R}_+$, Proposition 3.3 implies that the solution $\phi(t, x)$ of the Cauchy problem (1.1)-(2.1) satisfies that $\phi(t, \cdot) \in R_+$ for $t \in [0, T)$. Then we have $I(\phi) < d$ and $S(\phi) > 0$. By $S(\phi) > 0$ and (3.2), one has

$$-a \int |\phi|^{p+1} dx - b \int |\phi|^2 E_1(|\phi|^2) dx > -\left(\int |\nabla\phi|^2 dx + \omega \int |\phi|^2 dx \right). \tag{4.66}$$

From $I(\phi) < d$ and $1 + \frac{4}{N} \le p < \frac{N+2}{(N-2)^+}$, one has the following two results:

(1) For $N \in \{2, 3\}$ and $3 \le p < \infty$, it follows from $I(\phi) < d$ that

$$\begin{aligned} \frac{1}{2} \int |\nabla\phi|^2 dx \ &+ \ \frac{\omega}{2} \int |\phi|^2 dx \\ &- \ \frac{a}{4} \int |\phi|^{p+1} dx - \frac{1}{4} b \int |\phi|^2 E_1(|\phi|^2) dx \\ &\le \ I(\phi) < d. \end{aligned} \tag{4.67}$$

(2) For $N = 3$ and $\frac{7}{3} \le p < 3$, it follows from $I(\phi) < d$ that

$$\begin{aligned} \frac{1}{2} \int |\nabla\phi|^2 dx \ &+ \ \frac{\omega}{2} \int |\phi|^2 dx \\ &- \ \frac{a}{p+1} \int |\phi|^{p+1} dx - \frac{1}{p+1} b \int |\phi|^2 E_1(|\phi|^2) dx \\ &< \ I(\phi) < d. \end{aligned} \tag{4.68}$$

Therefore (4.66), (4.67) and (4.68) imply that

$$\frac{1}{4} \int |\nabla\phi|^2 dx + \frac{\omega}{4} \int |\phi|^2 dx < d, \tag{4.69}$$

or

$$\left(\frac{1}{2} - \frac{1}{p+1} \right) \left(\int |\nabla\phi|^2 dx + \omega \int |\phi|^2 dx \right) < d. \tag{4.70}$$

In view of $1+\frac{4}{N}\leq p<\frac{N+2}{(N-2)^+}$, (4.69) and (4.70) imply that ϕ is bounded in $H^1(\mathbf{R}^N)$ for any $t\in[0,T)$. Thus by Proposition 2.1, we obtain that the solution ϕ of the Cauchy problem (1.1)-(2.1) exists globally in $t\in[0,\infty)$ for $\phi_0\in\mathcal{R}_+$, $1+\frac{4}{N}\leq p<\frac{N+2}{(N-2)^+}$ and $N\in\{2,3\}$.

Through the arguments of **Step 1** and **Step 2**, we complete the proof of Theorem 4.2.

By Theorem 4.1 and 4.2, using Proposition 3.4, we can get a necessary and sufficient condition for blowup of the solution to the Cauchy problem (1.1)-(2.1) for $1+\frac{4}{N}\leq p<\frac{N+2}{(N-2)^+}$ and $N\in\{2,3\}$..

Theorem 4.3. Let $1+\frac{4}{N}\leq p<\frac{N+2}{(N-2)^+}$ and $N\in\{2,3\}$. If ϕ_0 satisfy $I(\phi_0)<d$, then the solution $\phi(t,x)$ of the Cauchy problem (1.1)-(2.1) blows up in a finite time if and only if $\phi_0\in\mathcal{K}$.

In addition, if we limit ω to $0<\omega\leq 1$, then by Theorem 4.2 and using the scaling argument, we can also get another sufficient condition for the global existence of the solution to the Cauchy problem (1.1)-(2.1).

Corollary 4.1. Let $1+\frac{4}{N}\leq p<\frac{N+2}{(N-2)^+}$, $N\in\{2,3\}$ and $0<\omega\leq 1$. If $\phi_0\in H^1(\mathbf{R}^N)$ and satisfies

$$\int|\nabla\phi_0|^2dx+\int|\phi_0|^2dx<2d, \tag{4.71}$$

then the solution $\phi(t,x)$ of the Cauchy problem (1.1)-(2.1) exists globally in $t\in[0,\infty)$.

Proof. According to (3.1), (4.71) and $0<\omega\leq 1$, we can get $I(\phi_0)<d$. In addition, we assert that $S(\phi_0)>0$. If otherwise, from (3.2), there were a $0<\lambda\leq 1$ such that $S(\lambda\phi_0)=0$. Thus by (3.5) and (3.49),

$$I(\lambda\phi_0)\geq d_{\mathcal{N}}\geq d. \tag{4.72}$$

On the other hand,

$$\begin{aligned}\int|\nabla(\lambda\phi_0)|^2dx &+ \int|\lambda\phi_0|^2dx\\ &= \lambda^2\left(\int|\nabla\phi_0|^2dx+\int|\phi_0|^2dx\right)\end{aligned}$$

$$< \ 2\lambda^2 d \le 2d, \tag{4.73}$$

which together with $0 < \omega \le 1$ implies that

$$I(\lambda\phi_0) < d. \tag{4.74}$$

(4.72) and (4.74) are contradictory. So $S(\phi_0) > 0$ holds. From the above argument, we get $\phi_0 \in \mathcal{R}_+$, thus Theorem 4.2 yields the result of Corollary 4.1.

Remark 4.2. Although the condition for global existence in Corollary 4.1 is not sharp, Corollary 4.1 gives an answer to the question: How small are the initial data, the solution of the Cauchy problem (1.1)-(2.1) exists globally?

5. Instability of the Standing Waves

Using the methods in Berestycki and Cazenave [1] as well as Rabinowitz [19], one can easily obtain that the constrained variational problem (3.5) is attained.

Let u be a solution of (3.5), that is we have

$$d_{\mathcal{N}} = \min_{u \in \mathcal{N}} I(u). \tag{5.1}$$

Then $u \in H^1(\mathbf{R}^N) \setminus \{0\}$ is a solution of (3.13). Thus

$$\phi(t, x) = e^{i\omega t} u(x) \tag{5.2}$$

is a standing wave solution of (1.1). From (5.1), it follows that $u(x)$ is a ground state solution of (3.13).

Now we hope to study the instability of the standing wave (5.2). In general, it depends upon the frequency ω and the solvability of the following variational problem

$$d_Q = \inf_{\{u \in H^1(\mathbf{R}^N) \setminus \{0\}, Q(u)=0\}} I(u). \tag{5.3}$$

In the present paper, by Proposition 3.1 and Theorem 4.1, we can refrain from solving the problem (5.3), and show the instability of the standing wave (5.2), which commonly depends upon the frequency ω(see Ohta [16,17], Cipolatti[5]).

In order to obtain the instability result, we first consider two lemmas which are key to our analysis later.

Lemma 5.1. Let $\phi \in H^1(\mathbf{R}^N) \setminus \{0\}$. Then there exists a unique $\mu > 0$ such that $S(\mu\phi) = 0$ and $I(\mu\phi) > I(\lambda\phi)$ for any $\lambda > 0$ and $\lambda \neq \mu$.

Proof. Let $\lambda > 0$. Then we have

$$\begin{aligned} S(\lambda\phi) &= \lambda^2 \left(\int |\nabla\phi|^2 dx + \omega \int |\phi|^2 dx \right) \\ &- \lambda^{p+1} a \int |\phi|^{p+1} dx - \lambda^4 b \int |\phi|^2 E_1(|\phi|^2) dx, \end{aligned} \tag{5.4}$$

$$\begin{aligned} I(\lambda\phi) &= \frac{1}{2}\lambda^2 \left(\int |\nabla\phi|^2 dx + \omega \int |\phi|^2 dx \right) \\ &- \frac{1}{p+1}\lambda^{p+1} a \int |\phi|^{p+1} dx - \frac{1}{4}\lambda^4 b \int |\phi|^2 E_1(|\phi|^2) dx. \end{aligned} \tag{5.5}$$

Noting that (5.4) and (5.5), we obtain

$$\frac{d}{d\lambda} I(\lambda\phi) = \lambda^{-1} S(\lambda\phi). \tag{5.6}$$

Thus by (5.4) and (5.6), the result of Lemma 5.1 is true.

Lemma 5.2 Let u be a minimizer of (5.1). Then $Q(u) = 0$.

Proof. Since u is a minimizer of (5.1), thus u is also a solution of (3.13)(or (1.8)). So multiplying (3.13) by $x \cdot \nabla u$, we get

$$\frac{N-2}{N} \int |\nabla u|^2 dx + \omega \int |u|^2 dx - \frac{2}{p+1} a \int |u|^{p+1} dx - \frac{1}{2} b \int |u|^2 E_1(|u|^2) dx = 0, \tag{5.7}$$

which is called Pohozaev identity. Note that $S(u) = 0$, (5.7) implies that $Q(u) = 0$.

Using Lemma 5.1 and Lemma 5.2, on the standing wave (5.2), we get the following instability theorem.

Theorem 5.1. For $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$ and $N \in \{2, 3\}$, Let ω be such that $d_{\mathcal{M}} \geq d_{\mathcal{N}}$. Then for the minimizer u of (3.5) and any $\varepsilon > 0$, there exists $\phi_0 \in H^1(\mathbf{R}^N)$ with $\|\phi_0 - u\|_{H^1(\mathbf{R}^N)} < \varepsilon$ such that the solution $\phi(t, x)$ of the Cauchy problem (1.1)-(2.1) blows up in a finite time.

Proof. Since $d_{\mathcal{M}} \geq d_{\mathcal{N}}$, one has $d = d_{\mathcal{N}}$ by (3.49). Because u is the minimizer of (3.5), it follows from Lemma 3.2 that there exists $u \in H^1(\mathbf{R}^N) \setminus \{0\}$ such that both $S(u) = 0$ and $Q(u) = 0$. Thus by (3.2) and (3.3), for any $\lambda > 1$ we have

$$S(\lambda u) < 0, \qquad Q(\lambda u) < 0, \qquad \lambda > 1. \tag{5.8}$$

On the other hand, from Lemma 5.1, $S(u) = 0$ implies that $I(\lambda u) < I(u)$ for any $\lambda > 1$. Note that $I(u) = d_{\mathcal{N}} = d$. Thus for any $\lambda > 1$, we have $\lambda u \in \mathcal{K}$. Now we take $\lambda > 1$, and λ is sufficiently close to 1 such that

$$\|\lambda u - u\|_{H^1(\mathbf{R}^N)} = (\lambda - 1)\|u\|_{H^1(\mathbf{R}^N)} < \varepsilon. \tag{5.9}$$

Then take $\phi_0 = \lambda u(x)$. From Theorem 4.1, it follows that the solution $\phi(t, x)$ of the Cauchy problem (1.1)-(2.1) blows up in a finite time.

Remark 5.1. Under the condition $d_{\mathcal{M}} \geq d_{\mathcal{N}}$, Theorem 5.1 gives the strong instability of the ground state standing wave (5.2) for system (1.1) with finite time blow up when $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$ and $N \in \{2, 3\}$. For the related results on the instability of the standing wave (5.2) for system (1.1), there has been a lot of works (see Cipolatti [5] and Ohta [15,16,17]). Cipolatti [5] proved that if $p \geq 3$ and $N \in \{2, 3\}$, then the standing wave (5.2) is unstable for any $\omega \in (0, \infty)$ and that if $N = 2$ and $p = 3$, then the standing wave (5.2) is strongly unstable for any $\omega \in (0, \infty)$. After that, the author [16] proved that if $p \geq 1 + \frac{4}{N}$ and $N \in \{2, 3\}$, then (5.2) is unstable for any $\omega \in (0, \infty)$, and if $N = 3$ and $1 < p < \frac{7}{3}$, then there exists a positive constant $\omega_0 = \omega_0(a, p)$ such that (5.2) is unstable for any $\omega \in (\omega_0, \infty)$. In addition, the author [15] showed that if $N = 3$ and $\frac{7}{3} < p < 5$, then (5.2) is strongly unstable for any $\omega \in (0, \infty)$. On the other hand, when $N = 2$ and $p \leq 3$, the author [15] proved the existence of stable standing waves of (1.1). Further, under the condition $N = 2$ and $p > 3$ or $N = 3$ and $p = \frac{7}{3}$, the author [17] obtained that for any $\omega \in (0, \infty)$, (5.2) is strongly unstable in the sense of Definition 1.1 of [17].

It should be pointed out that in the present paper, we introduce the cross-invariant manifold to discuss the instability of standing waves for system (1.1) with finite time blow up, which partially answer the open problem proposed in [16, Remark 8]. In Theorem 5.1, by limiting frequency ω to satisfy $d_{\mathcal{M}} \geq d_{\mathcal{N}}$, we get the strong instability of (5.2) for $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$ and $N \in \{2, 3\}$. Of course, the condition $d_{\mathcal{M}} \geq d_{\mathcal{N}}$ is still vague.

Further we need to determine for which ω, $d_{\mathcal{M}} \geq d_{\mathcal{N}}$ is true. Moreover, if $d_{\mathcal{M}} < d_{\mathcal{N}}$, is the standing wave (5.2) orbital stable? These problems remain open.

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