On Super-Weakly Compact Sets and Generalized Renormings

Lixin Cheng*†, Qingjin Cheng‡, Wen Zhang§
† ‡ §Department of Mathematics, Xiamen University, Xiamen 361005, China
†The Institute of Mathematical Sciences, The Chinese University of Hong Kong
e-mail: †lxcheng@xmu.edu.cn, ‡qjchen@tom.com, §wenzhang@xmu.edu.cn

Abstract

By extending and developing a series of classical theorems and methods, such as James' finite tree theorem, Enflo's renorming technique, Grothendieck's lemma and the Davis-Figiel-Johnson-Pelzyński Lemma, this paper finally shows that every super-weakly compact convex set is isomorphic to a uniformly convex set of a reflexive relatively uniformly convex Banach space; and that a closed bounded convex set C of a Banach space X is super-weakly compact if and only if there exists a uniformly continuous and uniformly convex function on it, and which is equivalent to that there exists a norm on a reflexive space Y with $C \subset Y \subset X$ such that the norm of Y with respect to the norm of X is uniformly continuous and uniformly convex on C. It proves that every relatively compact set of a Banach space and every bounded set of a super-reflexive space are relatively super-weakly compact. This paper also presents that super-weakly compact sets have very nice properties, for example,every super-weakly compact convex set can be renormed to have normal structure; the image of a super-weakly compact convex set under a uniformly continuous affine mapping is again super-weakly compact; and for any two super-weakly compact convex sets A and B in a Banach space, both $A \times B$ and $A - B$ are still super-weakly compact.

2000 Mathematics Subject Classification: primary 46B20, 46B03, 46B50

Key words and phrases: finite representability, weakly compact set, finite tree property, uniformly convex function, renorming, super-reflexive space, uniformly convex space, Banach space

1 Introduction

It is just like that the notion of weakly compact set can be viewed as a generalization and localization of the notion of reflexive Banach space, the aim of this article is first to generalize and to localize the notion of super-reflexive Banach space to that of super-weakly compact set of Banach spaces in a natural way; then, to show that the behavior of a nonempty super-weakly compact set is much like that of the closed unit ball of a super-reflexive space, such as, a bounded closed convex set is super-weakly compact if

*Partially support by NSFC grant no. 10471114 and by The Institute of Mathematical Sciences at The Chinese University of Hong Kong

and only if it does not have James' finite tree property, and which is also equivalent to that it admits a uniformly continuous and uniformly convex function on it; finally, to present that every super-weakly compact convex set of a Banach space can be affinely embedded into a reflexive relatively uniformly convex space, and that every super-weakly compact convex set admits a uniformly continuous and uniformly convex norm on it. They are done by extending and developing a sequence of classical theorems and methods, such as James' tree theorem for super-reflexive spaces [18], Enflo's renorming technique in showing that every super-reflexive space admits an equivalent uniformly convex norm [10], Grothendieck's lemma for weakly compact sets and the Davis-Figiel-Johnson-Pelzyński Lemma [5].

The study of renorming characterization of various classes of Banach spaces with respect to the convexity and smoothness has continued on for over 70 years since J.A.Clarkson [4] introduced the class of uniformly convex spaces. Now, it is well known that the geometric and topological properties of uniformly convex Banach spaces have played an important part in both linear and non-linear functional analysis, and this fact brought the theoretical research and applications of uniformly convexifiable spaces to mathematicians great attentions (see, for instance, [1], [6-9], [11-12], [14], [20], [23]). For example, an early result of F.E.Browder [2] says that it is just that the uniformly convex spaces to guarantee every non-expansive mapping from a nonempty closed bounded convex subset to itself has a fixed point, and a quite recent conclusion of Kasparrov and Yu [21] explains that the Novikov conjecture holds for any discrete group which admits a uniform embedding into a uniformly convex space. The most spectacular result in this area is certainly the Enflo-Pisier characterization of super-reflexive spaces as those admitting an equivalent uniformly convex norm [10] (or even having power type modulus of uniform convexity [25]). There are also many remarkable results showing characterization of a super-reflexive space or a uniformly convexiflable space, such as the closed unit ball without James' finite tree property [18], and even the girth characterization of super-reflexive spaces [19]. We also know that both uniformly smoothable and uniformly non-square spaces are super-reflexive spaces [16].

Here we should mention that in many cases, the assumption of the uniform convexity on the whole space is not natural, because we need only a localized setting. For example, if T is a non-expansive mapping from a nonempty closed bounded convex set C of a Banach space to itself, the assumptions of weak compactness and of normal structure on C can always guarantee that T has a fixed point [22]. This tells us that we need only to assume that C has some kind of uniform convexity. On the other hand, a Banach space is reflexive if and only if every bounded weakly closed set is weakly compact. Therefore, the notion of weakly compact set can be viewed as a localized setting of reflexive spaces. These facts arise the following natural questions.

Problem 1 How to localize and generalize the notion of the super-reflexivity in a natural way, or more precisely, to introduce a notion of super-weak compactness which is a localization of super-reflexivity?

The Enflo-Pisier deep renorming theorem (see, [10] and [25]) for super-reflexive spaces also reminders us to expect that such a super-weakly compact set can be renormed with some kind of uniform convexity, or equivalently,

Problem 2 Whether every super-weakly compact convex set C of a Banach space admits a norm, which is uniformly convex on it?

The Davis-Figiel-Johnson-Pelzyński Lemma provides an easy way of making reflexive Banach spaces from weakly compact sets of arbitrary Banach spaces. Making a comparison between a weakly compact set and a super-weakly compact set, a further question is arising as follows.

Problem 3 Whether every (super-)weakly compact set can be embedded into a (super-)reflexive space.

This paper, divided into 6 sections, mainly focuses the three questions above. Section 2 generalizes the notion of finite representability between two Banach spaces to the notion of that between two general subsets of Banach spaces, and this is done by substituting simplexes for finite dimensional subspaces.

Section 3, in terms of the generalized finite representability, introduces the notion of super-weakly compact set, that is, a nonempty weakly closed bounded set C in a Banach space is said to be super-weakly compact, if every bounded weakly closed set D which is finitely representable in C is weakly compact, and shows that James' characterization and its consequence for super-reflexive spaces hold again for super-weakly compact sets. Now we state them as follows.

Theorem 1.1 A bounded closed convex set C in a Banach space X is super-weakly compact if and only if it does not have the finite tree property.

Theorem 1.2 A bounded closed convex set C of the space X is not super-weakly compact if and only if there exists $0 < \theta < 1$ such that for every positive integer n, there is a sequence $\{x_i\}_{i=1}^n \subset C$ satisfying

$$\text{dist}(co\{x_1, \cdots, x_k\}, co\{x_{k+1}, \cdots, x_n\}) > \theta$$

for all $1 \leq k < n$, where coA stands for the convex hull of the set A.

In Section 4, through generalizing a sequence of Enflo's Lemmas [10] to general super-weakly compact convex sets, it shows that a bounded closed convex set is super-weakly compact if and only if for every $\varepsilon > 0$, there exists a bounded $\varepsilon-$uniformly convex function on it, and further, the image of a super-weakly compact convex set under a uniformly continuous affine mapping is again super-weakly compact. These, in turn, imply that the product $A \times B$ and the difference $A - B$ of any two super-weakly compact convex sets A and B are again super-weakly compact.

In Section 5, it first verifies that Grothendieck's lemma for weakly compact sets is again valid for super-weakly compact sets, that is

Theorem 1.3 A closed convex subset K in a Banach space is super-weakly compact if and only if for every $\varepsilon > 0$ there exists a super-weakly compact convex set K_ε such that $K \subset K_\varepsilon + \varepsilon B_X$, where B_X denotes the closed unit ball of X.

This section also mentions that there is an important and elegant consequence quite hidden in the original proof of the Davis-Figiel-Johnson-Pelzyński Lemma that we have not discovered for over 30 years, that is, every (super-)weakly compact convex set is linearly isomophic to a (super-)weakly compact convex set of a reflexive space. By employing these results and theorems in the previous sections, it finally shows the following theorem.

Theorem 1.4 Suppose that C is a super-weakly compact set of a Banach space $(X, \|\cdot\|)$. Then there exists a Banach space $(Y, |||\cdot|||)$ such that

i) Y is reflexive and $C \subset B_Y \subset X$;

ii) $\|\cdot\|$ is $|||\cdot|||-$Lipschitz on Y;

iii) $|||\cdot|||$ is $\|\cdot\|-$uniformly continuous and uniformly convex on C;

iv) $(Y, |||\cdot|||)$ is relatively uniformly convex with respect to $\|\cdot\|$, that is, for any two sequences $\{x_n\}$ and $\{y_n\}$ in B_Y, $2(|||x_n|||^2 + |||y_n|||^2) - |||x_n + y_n|||^2 \to 0$ implies $\|x_n - y_n\| \to 0$.

As a consequence, we obtain

Theorem 1.5 A closed bounded convex set of a Banach space is super-weakly compact if and only if there exists a uniformly continuous and uniformly convex function on it.

Motivated by the importance of normal structure of a weakly compact convex set C guaranteeing that every non-expansive mapping ($T: C \to C$) has a fixed point [22] and by the recent renorming theorems of Odell-Schlumprecht [24] and Hájek-Johanis[13], we devote the bulk of Section 6 to some further consideration regarding renormings of (super-)weakly compact convex sets. We begin with discussion of normal structure of super-weakly compact convex sets under renormings. After giving some extensions of Odell-Schlumprecht and Hájek-Johanis' renorming theorems, we finally point out some questions on this topic.

In this paper, the letter X will always be a real Banach space and X^* is its dual. For $x \in X$ and $r > 0$, $B(x, r)$ presents the closed ball centered at x with radius r, and $S(x, r)$, the sphere of $B(x, r)$. We simply denote by B_X, the closed unit ball and by S_X, the sphere of B_X. For a set A in X, ($\overline{co}A$) coA and ($\overline{\text{aff}}A$) aff$A$ stand for the (closed) convex hull and the (closed) affine hull of the set A.

2 Generalized Finite Representability

This section mainly introduces a generalized notion of finite representability between two general sets of Banach spaces, making use of simplexes and affine mappings.

To begin with, we state the classical notion of finite representability introduced by James (see, [17] and [18]), which has played an important rule in studying various kinds of " super-property " of Banach spaces.

Definition 2.1 Suppose that X and Y are two Banach spaces. We say that X is finitely representable in Y, if for every $\varepsilon > 0$ and for every finite dimensional subspace $M \subset X$, there exist a subspace $N \subset Y$ and a linear mapping $T : M \to N$ such that $\|T\| \cdot \|T^{-1}\| \leq 1 + \varepsilon$.

For generalizing the notion above to general sets, a natural way is to substitute simplexes and affine mappings for the linear subspaces and the linear mappings.

Definition 2.2 Suppose that $\{x_i\}_{i=0}^n$ are $n + 1$ vectors in X.

i) $\{x_i\}_{i=0}^n$ are said to be affinely independent if $\{x_i - x_0\}_{i=1}^n$ are linearly independent;

ii) $co\{x_i\}_{i=0}^n$ is called an $n-$simplex of X with vertices at x_i $(i = 0, 1, \cdots n)$ if $\{x_i\}_{i=0}^n$ are affinely independent.

Definition 2.3 Suppose $U \subset X$ and $V \subset Y$ are two affine subspaces, and suppose $A \subset X$ and $B \subset Y$ are two subsets.

i) A mapping $T : U \to V$ is called affine, if

$$T(\alpha u + \beta v) = \alpha Tu + \beta Tv \quad \text{for all } u, v \in U \text{ and } \alpha + \beta = 1$$

ii) In general, a mapping $T : A \to B$ is said to be affine, if T is a restriction of an affine mapping from aff$A$ to aff$B$;

iii) An affine mapping $T : A \to B$ is called a $(1+\varepsilon)-$affine embedding from A to B for some $\varepsilon > 0$, if

$$(1 - \varepsilon)\|x - y\| \leq \|Tx - Ty\| \leq (1 + \varepsilon)\|x - y\|, \quad \forall x, y \in A$$

If such a map T exists, then we also call A can be $(1 + \varepsilon)-$affinely embedded into B, or $(1 + \varepsilon)-$affine embedding into B.

In particular, if $A = X, B = Y$, then a linear map T from X into Y is called a $(1 + \varepsilon)-$linear embedding from X to Y if T satisfies inequality above for all $x, y \in X$. In this case, X is said to be $(1 + \varepsilon)-$linear embedding into Y.

Now we present a generalized notion of finite representability as follows.

Definition 2.4 Suppose that $A \subset X$ and $B \subset Y$ are nonempty subsets. We say that A is finitely representable in B ($A \overset{f.r}{\hookrightarrow} B$) if for every $\varepsilon > 0$ and for each $n-$simplex $S(A)$ with vertices in A there exists an $n-$simplex $S(B)$ with vertices in B such that $S(A)$ is $(1 + \varepsilon)-$affine embedding into $S(B)$.

Clearly, every subset of A is finitely representable in A. Further, it is easy to show the following properties.

Proposition 2.5 Suppose that $A \subset X$, $B \subset Y$ and $C \subset Z$ are three nonempty subsets. Then

i) $A \overset{f.r}{\hookrightarrow} B$ and $B \overset{f.r}{\hookrightarrow} C \Longrightarrow A \overset{f.r}{\hookrightarrow} C$;

ii) For every $x_0 \in A, y_0 \in B, A \overset{f.r}{\hookrightarrow} B \Longleftrightarrow A - x_0 \overset{f.r}{\hookrightarrow} B - y_0$;

iii) $A \overset{f.r}{\hookrightarrow} B \Longleftrightarrow \overline{A} \overset{f.r}{\hookrightarrow} B$;

iv) $A \overset{f.r}{\hookrightarrow} B \Longrightarrow \text{aff}A \overset{f.r}{\hookrightarrow} \text{aff}B$.

The following counter-example shows that the converse of Proposition 2.5 iv) is not true.

Example 2.6 Suppose that X is a separable non-reflexive Banach space and that $\{x_n\}$ is a dense sequence in the unit ball B_X. Let $A = \overline{co}\{\pm\frac{x_n}{n}\}$. Then A is a compact convex set in X. Clearly, $\text{aff}A = \text{span}A$ is a dense subspace of X, and X is not finitely representable in A. But by Proposition 2.5 iii), $X \overset{f.r}{\hookrightarrow} \text{aff}A$. This is a contradiction.

Lemma 2.7 Suppose that $A \subset X$ and $B \subset Y$ are two nonempty subsets, that $E = \text{aff}A, F = \text{aff}B$, and suppose that A has nonempty relative interior (denoted by $\text{int}_E A$). If A is $(1+\varepsilon_0)-$affine embedding in B for some $\varepsilon_0 \geq 0$, then E is again $(1+\varepsilon_0)-$affine embedding in F.

Proof Choose any $a \in \text{int}_E(A)$ and set $A_1 = A - a$ and $E_1 = \text{aff}A_1 (= \text{span}A_1)$. Then $0 \in \text{int}_{E_1}(A_1)$. Let $T : E \to F$ be a $(1+\varepsilon_0)-$affine embedding from A to B. Then $T_1 : E_1 \to F_1$, defined by

$$T_1(x) = T(x+a) - T(a) \quad \text{ for all } x \in E_1$$

is a $(1, \varepsilon_0)-$linear embedding from A_1 to B_1, where $B_1 = B - Ta$ and $F_1 = \text{aff}B_1 = \text{span}B_1$. Therefore, T_1 is also a $(1+\varepsilon_0)-$linear embedding from A_1 to F_1 and which implies that T_1 is again a $(1+\varepsilon_0)-$linear embedding from $E_1 (= \bigcup_{n=1}^{\infty} nA_1)$ to F_1. Thus, T is a $(1+\varepsilon_0)-$affine embedding from $E (= E_1 + a)$ to $F (= F_1 + Ta)$ $\square$

We should mention here that the converse of the lemma above is not true.

By a simple argument of linear homeomorphism, we can present the following result.

Proposition 2.8 Suppose that X and Y are two Banach spaces. Then

$$X \overset{f.r}{\hookrightarrow} Y \Longleftrightarrow B_X \overset{f.r}{\hookrightarrow} B_Y.$$

3 Super-weakly Compact Sets

In this section, we introduce the notion of super-weakly compact set in terms of the generalized finite representability, and discuss their finite tree property (Theorem 1.1), which, along with Theorem 1.2, will be used repeatedly in the sequel. We will see that a super-weakly compact convex set acts just like the closed unit ball of a super-reflexive space. Therefore, this concept can be viewed as a generalized and localized notion of super-reflexivity of Banach space.

First, let us recall a sequence of definitions.

Definition 3.1 A Banach space X is said to be super-reflexive, if every Banach space Y which can be finitely representable in X is reflexive.

Definition 3.2 Suppose that X is a Banach space, $\varepsilon > 0$, and α is a cardinal number less or equal to $\aleph_0$, the cardinal number of $\mathbb{N}$; and suppose, further that $A \subset X$ is defined by

$$A = \Big\{x_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_k} : k \in \mathbb{N} \text{ with } k \leq \alpha, \varepsilon_i = 1,2 \text{ and } i = 1,2,\cdots k\Big\}$$

Then

i) The subset A is called an $\varepsilon-$tree with $n-$branches (or an $(n,\varepsilon)-$tree) if $\alpha = n$ for some $n \in \mathbb{N}$ and it satisfies

$$x_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_k} = \frac{1}{2}\Big(x_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_k,1} + x_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_k,2}\Big) \tag{3.1}$$

and

$$\Big\|x_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_k,1} - x_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_k,2}\Big\| \geq \varepsilon \tag{3.2}$$

for $k = 1,2,\cdots,n-1, \varepsilon_i = 1,2$ and $i = 1,2,\cdots,k$;

ii) The subset A is said to be an infinite $\varepsilon-$tree if $\alpha = \aleph_0$ such that (3.1) and (3.2) hold for all $k \in \mathbb{N}$, $\varepsilon_i = 1,2$ and $i = 1,2,\cdots,k$.

Definition 3.3 Suppose that C is a nonempty convex subset of a Banach space X. Then we say that

i) C has finite tree property if there is $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exists an $(n,\varepsilon)-$tree in C;

ii) C has infinite tree property , if for some $\varepsilon > 0$ there exists an infinite $\varepsilon-$tree in C.

Now we state the definition of super-weakly compact set as follows.

Definition 3.4 A bounded weakly closed set C of X is said to be super-weakly compact, if every bounded weakly closed set D which is finitely representable in C is weakly compact.

We know that Definitions 3.2 and 3.3 were introduced by James in 1972 [18], and they are very useful to characterize super-reflexive and non-super-reflexive spaces. The James Characterization of super-reflexive spaces [18] says that a Banach space is super-reflexive if and only if its closed unit ball does not have finite tree property. Next, we generalize the James Characterization of super-reflexive spaces to that of super-weakly compact convex subsets in terms of finite tree property(Theorem 1.1 or Theorem 3.6). For this purpose, we need the following lemma which was motivated by work of James [18].

Lemma 3.5 Suppose that $C \subset X$ is a nonempty convex set. If C has finite tree property, then there exist $\varepsilon > 0$ and an infinite $\varepsilon-$tree A in a Banach space E such that $A \overset{f.r}{\hookrightarrow} C$.

Proof We apply the finite tree property of C to construct a Banach space E which contains an infinite $\varepsilon-$tree for some $\varepsilon > 0$, and then show this tree is finitely representable in C.

Since C has finite tree property, there exists a subset G of the form

$$G = \Big\{x^{(n)}_{\varepsilon_1,\cdots,\varepsilon_k} : n \in \mathbb{N}, 1 \le k \le n, \varepsilon_i = 1,2 \ \text{ and } \ 1 \le i \le k\Big\}$$

in C such that for every $n \in \mathbb{N}$ and $1 \le k \le n-1$, for all $\varepsilon_i = 1,2$ and $i = 1,2,\cdots,k$

$$x^{(n)}_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_k} = \frac{1}{2}(x^{(n)}_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_k,1} + x^{(n)}_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_k,2}) \tag{3.3}$$

$$\Big\|x^{(n)}_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_k,1} - x^{(n)}_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_k,2}\Big\| \ge \varepsilon \tag{3.4}$$

Choose an infinite sequence of symbols

$$S \equiv \Big\{\xi_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n} : n \in \mathbb{N}, \varepsilon_i = 1,2 \text{ and } i = 1,2,\cdots,n\Big\}$$

and let $Y = \text{span} S$. Applying a diagonal argument to the number set

$$\Big\{\Big\| \sum_{\substack{\varepsilon_i = 1,2 \\ 1 \le i \le k \\ 1 \le k \le n}} \lambda_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_k} x^{(n)}_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_k}\Big\| : n \in \mathbb{N}\Big\}$$

we can obtain a subsequence $k_n \subset \mathbb{N}$ such that for every $r \in \mathbb{N}$ and every subset of 2^r rational numbers $\{\lambda_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_r} : \varepsilon_i = 1,2\}$, the following limit exists

$$\lim_{n\to\infty} \Big\| \sum_{\varepsilon_i=1,2;\ 1\le i\le r} \lambda_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_i} x^{(k_n)}_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_i}\Big\|$$

Set

$$A = \Big\{ \sum_{\varepsilon_i=1,2;\ 1\le i\le r} \lambda_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_i} \xi_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_i} :\ \forall r \in \mathbb{N}, \lambda_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_i} \in \mathbb{Q}\Big\}$$

and let

$$\Big| \sum_{\varepsilon_j=1,2;\ 1\le j\le r} \lambda_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_j} \xi_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_j}\Big| = \lim_{n\to\infty} \Big\| \sum_{\varepsilon_j=1,2;\ 1\le j\le r} \lambda_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_j} x^{(k_n)}_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_j}\Big\| \tag{3.5}$$

Then the equality above defines a function $|\cdot|$ on A. We extend $|\cdot|$ to Y , which is still denoted by $|\cdot|$. It is easy to see that $|\cdot|$ is a semi-norm on Y. Let $N = \{x \in X : |x| = 0\}$. Then the quotient space $F \equiv Y/N$ endowed with the quotient $\|\cdot\|_F$ introduced by $|\cdot|$ is a normed space. We denotes by E the completion of F, and by $\overline{\xi}$ the quotient vector of ξ, $i.e$, $\overline{\xi} = \xi + N$ for every ξ in Y. To show that

$$A \equiv \Big\{\overline{\xi}_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n} : n \in \mathbb{N}, \varepsilon_i = 1,2 \ \text{ and } \ i = 1,2,\cdots,n\Big\}$$

is an infinite $\varepsilon-$tree, it suffices to note the (3.3), (3.4) and (3.5), and which in turn imply for every $m \in \mathbb{N}, \varepsilon_i = 1,2$ and $i = 1,2,\cdots,m$

$$\Big\|\overline{\xi}_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_m} - \frac{1}{2}(\overline{\xi}_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_m,1} + \overline{\xi}_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_m,2})\Big\|_F$$

$$= \lim_{n\to\infty} \left\| x^{(k_n)}_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m} - \frac{1}{2}(x^{(k_n)}_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m,1} + x^{(k_n)}_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m,2}) \right\| \tag{3.6}$$
$$= 0$$

and

$$\left\| \overline{\xi}_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_m,1} - \overline{\xi}_{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_m,2} \right\|_F = \lim_{n\to\infty} \left\| x^{(k_n)}_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m,1} - x^{(k_n)}_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m,2} \right\| \geq \varepsilon \tag{3.7}$$

Clearly, the infinite tree A is finitely representable in C by observing the definition of the semi-norm $|\cdot|$ on Y □

Now, we restate and prove Theorem 1.1 as follows.

Theorem 3.6 A closed bounded convex subset C of a Banach space is super-weakly compact if and only if it does not possess finite tree property.

Proof Necessity. Suppose, to the contrary, that C has finite tree property. Then applying Lemma 3.5 to C, we can obtain an infinite (bounded) $\varepsilon-$tree A in a Banach space and which is finitely representable in C. It is clear that A is not weakly compact. This in turn contradicts the super-weak compactness of C.

Sufficiency. Suppose that C is not super-weakly compact. Then there exists a closed bounded non-weakly compact set D of a Banach space X such that it is finitely representable in C. Applying the James Characterization of non-weakly compact subsets to D (see, [15]), we know that there exist $\theta > 0$, a sequence $\{x_n\}$ in D and $\{x_n^*\}$ in S_{X^*} such that for all $m, n \in \mathbb{N}$

$$< x_m^*, x_n > = \begin{cases} \theta \ , & m \leq n; \\ 0 \ , & n < m. \end{cases}$$

By hypothesis, for every $n \in \mathbb{N}$, the $(2^n - 1)-$simplex $S_{2^n} \equiv co\{x_1, x_2, \cdots, x_{2^n}\}$ is finitely representable in C. Thus, there is a $(1 + \frac{1}{2})-$affine embedding T_n from S_{2^n} to C. Let $y_i^{(n)} = T_n(x_i),\ i = 1, 2, \cdots, 2^n$. It is easy to see that for $1 \leq k \leq 2^n - 1$

$$dist\Big(co\{y_1^{(n)}, y_2^{(n)}, \cdots, y_k^{(n)}\}, co\{y_{k+1}^{(n)}, y_{k+2}^{(n)}, \cdots, y_{2^n}^{(n)}\}\Big) \geq \frac{\theta}{2}$$

By the argument which is like one of that in [18], we see that C has finite tree property □

Corollary 3.7 A bounded closed convex subset C of a Banach space X is not super-weakly compact if and only if there exists $\theta > 0$ such that for every $n \in \mathbb{N}$ there is $\{x_i\}_{i=1}^n \subset C$ such that for all $1 \leq k \leq n - 1$

$$dist\Big(co\{x_1, x_2, \cdots, x_k\}, co\{x_{k+1}, x_{k+2}, \cdots, x_n\}\Big) > \theta.$$

Proof The proof is contained in the proof of the sufficiency of Theorem 3.6.

The following results are easy consequences of Corollary 3.7.

Corollary 3.8 Every compact set of a Banach spaces is super-weakly compact.

Corollary 3.9 A Banach space is super-reflexive if and only if its closed unit ball is super-weakly compact.

Corollary 3.10 Every bounded subset of super-reflexive Banach spaces is relatively super-weakly compact.

4 A Characterization of Super-Weakly Compact Convex Sets by $\varepsilon-$Uniformly Convex Functions

In this section, in terms of $\varepsilon-$uniformly convex functions we establish a new geometric characterization of super-weakly compact convex subsets, and which in turn implies that both $C\times D$ and $C-D$ are super-weakly compact whenever the two convex subsets C and D are super-weakly compact. The main idea of this section was motivated by Enflo [10] in showing that every super-reflexive space admits an equivalent uniformly convex norm.

Suppose that $C\subset X$ is a nonempty closed bounded convex set with $0\in C$, and that ρ is the Minkowski functional generated by C. Then ρ is an extended non-negative real-valued and positively homogenous convex function on X with its effective domain $\text{dom}\rho\equiv\{x\in X:\rho(x)<\infty\}=\cup_{\lambda>0}\lambda C$. Now, we give definition of a generalized $\varepsilon-$partition of $z\in\text{dom}\rho$ as follows.

Definition 4.1 With the set C and the function ρ as above, let $\varepsilon>0$ and $z\in\text{dom}\rho$ be given. A pair (x_1,x_2) of $(\text{dom}\rho)^2$ is said to be a $(1,\varepsilon,\rho)-$partition of z if it satisfies

$$\rho(x_1)=\rho(x_2),\|\frac{x_1}{\rho(x_1)}-\frac{x_2}{\rho(x_2)}\|\geq\varepsilon\ \text{ and }\ x_1+x_2=z$$

We denote by

$$\mathscr{P}_1(z,\varepsilon)=\{(x_1,x_2)\in(\text{dom}\rho)^2:(x_1,x_2)\text{ is a }(1,\varepsilon,\rho)-\text{partition of }z\}$$

We call (x_1,x_2,x_3,x_4) in $(\text{dom}\rho)^4$ a $(2,\varepsilon,\rho)-$partition of z if it satisfies that

$$\rho(x_1)=\rho(x_2),\ \rho(x_3)=\rho(x_4),\ \|\frac{x_1}{\rho(x_1)}-\frac{x_2}{\rho(x_2)}\|\geq\varepsilon,\ \|\frac{x_3}{\rho(x_3)}-\frac{x_4}{\rho(x_4)}\|\geq\varepsilon$$

and

$$(x_1+x_2,x_3+x_4)\in\mathscr{P}_1(z,\varepsilon)$$

We also denote by

$$\mathscr{P}_2(z,\varepsilon)=\{(x_1,x_2,x_3,x_4)\in(\text{dom}\rho)^4:(x_1,x_2,x_3,x_4)\text{ is a }(2,\varepsilon,\rho)-\text{partition of }z\}$$

Inductively, we say that $(x_1,x_2,\cdots,x_{2^n})\in(\text{dom}\rho)^{2^n}$ is an $(n,\varepsilon,\rho)-$partition of z, or equivalently, $(x_1,x_2,\cdots,x_{2^n})\in\mathscr{P}_n(z,\varepsilon)$, if it satisfies that

$$\rho(x_{2k-1})=\rho(x_{2k}),\|\frac{x_{2k-1}}{\rho(x_{2k-1})}-\frac{x_{2k}}{\rho(x_{2k})}\|\geq\varepsilon\ \text{ for }\ k=1,2,\cdots,2^{n-1}$$

and

$$(x_1+x_2,x_3+x_4,\cdots,x_{2^n-1}+x_{2^n})\in\mathscr{P}_{n-1}(z,\varepsilon)$$

For an $(n,\varepsilon,\rho)-$partition $(x_1,x_2,\cdots,x_{2^n})$ of z, and for every $1\leq k\leq n$, we call $\Big(\sum_{i=2^{n-k}(j-1)+1}^{2^{n-k}j}x_i\Big)_{j=1}^{2^k}$, a $k-$part of the $(n,\varepsilon,\rho)-$partition of z. Clearly, a $k-$part of an $(n,\varepsilon,\rho)-$partition of z is again a $(k,\varepsilon,\rho)-$partition of z.

Lemma 4.2 Suppose that C is a closed bounded convex set of X with $\dim(\text{span}C) \geq 3$. Then there exist $x_0 \in C$ and $\varepsilon > 0$ such that for every non-zero $z \in C_{x_0} \equiv \cup_{\lambda>0}\lambda(C - x_0)$ and every positive integer n, there exists an $(n, \varepsilon, \rho)-$partition of z in C_{x_0}, where ρ is the Minkowski functional generated by $C - x_0$. In particular, if C is symmetric, then we can put $x_0 = 0$.

Proof Since $\dim(\text{span}C) \geq 3$, there are four affinely independent vectors $\{x_i\}_{i=1}^4$ in C such that the $3-$simplex $S_3 \equiv co\{x_i\}_{i=1}^4$ has nonempty interior relative to the affine subspace $A \equiv \text{aff}\{x_i\}_{i=1}^4$ of 3 dimensions. Choose any x_0 in the relative interior of S_3. We know that there is $\delta > 0$ such that $B(x_0, \delta) \cap A \subset S_3$. Thus, the linear subspace $A_{x_0} \equiv A - x_0$ is contained in C_{x_0}.

Let ρ be the Minkowski functional generated by $C - x_0$, $\varepsilon = \frac{\delta}{3}$ and let $z \in C_{x_0} \setminus \{0\}$. Without loss of generality we assume that $\rho(z) = 1$. Now, for every $n \in \mathbb{N}$, we want to produce an $(n, \varepsilon, \rho)-$partition of z.

Let $L_z = \text{span}\{S_3 - x_0, z\}$ and $S_z = \cup_{\lambda>0}\lambda co\{S_3 - x_0, z\}$. Since ρ is continuous on S_z, the subdifferential mapping $\partial\rho : S_z : \ \to 2^{L_z^*}$ is nonempty-valued everywhere in S_z. Choose any $x_1^* \in \partial\rho(z)$. Then $< x_1^*, x > \leq \rho(x)$ for all $x \in S_z$ and $< x_1^*, z >= \rho(z)$. We extend x_1^* to the whole space X and which is still denoted by x_1^*. Then choose $x_2^* \in X^*$ with $\|x_2^*\| = 1$ such that $< x_2^*, z >= \|z\|$. Now, put

$$H_i = \{x \in X, < x_i^*, x >= 0\}, \quad i = 1, 2$$

Then the set $H_1 \cap H_2 \cap (S_3 - x_0)$ $(\subset C - x_0)$ contains two vectors $\pm u_1$ with $\|u_1\| \geq \delta$. We define two functions f and g: $[-1, 1] \to \mathbb{R}$ by

$$f(\alpha) = \rho(z + (\alpha z + u_1)), \quad g(\alpha) = \rho(z - (\alpha z + u_1))$$

Note

$$< x_1^*, z \pm u_1 >=< x_1^*, z >= \rho(z) = 1$$

We know

$$f(1) = \rho(2z + u_1) \geq < x_1^*, 2z + u_1 >= 2, \quad f(-1) = \rho(u_1) \leq 1$$

$$g(-1) = \rho(2z - u_1) \geq < x_1^*, 2z - u_1 >= 2, \quad g(1) = \rho(-u_1) \leq 1$$

Therefore, there exists $\alpha_1 \in (-1, 1)$ such that $f(\alpha_1) = g(\alpha_1)$, that is

$$\rho(z + (\alpha_1 z + u_1)) = \rho(z - (\alpha_1 z + u_1))$$

Set

$$z_1 = \frac{z + (\alpha_1 z + u_1)}{2}, \quad z_2 = \frac{z - (\alpha_1 z + u_1)}{2}$$

Then we obtain

$$z_1 + z_2 = z, \ \rho(z_1) = \rho(z_2) \leq \frac{3}{2}$$

and

$$\|\frac{z_1}{\rho(z_1)} - \frac{z_2}{\rho(z_2)}\| \geq \frac{2}{3}\|\alpha_1 z + u_1\| \geq \frac{2}{3}\max\{|\alpha_1|\|z\|, \|u_1\| - |\alpha_1|\|z\|\} \geq \frac{\delta}{3} = \varepsilon$$

that is, (z_1, z_2) is a $(1, \varepsilon, \rho)-$partition of z.

Next, let $y_i = \frac{z_i}{\rho(z_i)}, i = 1, 2$. We substitute y_i for z and repeat the procedure above to obtain

$$\alpha_{1,i} \in (-1, 1), \ \ \pm u_{1,i} \in C_{x_0} \ \text{ with } \ \|u_{1,i}\| \geq \delta$$

and

$$y_{i,1} = \frac{y_i + (\alpha_{1,i} y_i + u_{1,i})}{2}, y_{i,2} = \frac{y_i - (\alpha_{1,i} y_i + u_{1,i})}{2}$$

for $i = 1, 2$, such that

$$y_{i,1} + y_{i,2} = y_i, \ \rho(y_{i,1}) = \rho(y_{i,2}) \leq \frac{3}{2}$$

and such that

$$\left\|\frac{y_{i,1}}{\rho(y_{i,1})} - \frac{y_{i,2}}{\rho(y_{i,2})}\right\| \geq \varepsilon \ \text{ for } \ i = 1, 2$$

Let $z_{i,j} = \rho(z_i) y_{i,j}, \ i, j = 1, 2$. Thus for $i = 1, 2$

$$z_i = (y_{i,1} + y_{i,2})\rho(z_i) = z_{i,1} + z_{i,2}, \ \rho(z_{i,1}) = \rho(z_{i,2})$$

and

$$\left\|\frac{z_{i,1}}{\rho(z_{i,1})} - \frac{z_{i,2}}{\rho(z_{i,2})}\right\| = \left\|\frac{y_{i,1}}{\rho(y_{i,1})} - \frac{y_{i,2}}{\rho(y_{i,2})}\right\| \geq \varepsilon,$$

this says that $(z_{1,1}, z_{1,2}, z_{2,1}, z_{2,2})$ is a $(2, \varepsilon, \rho)-$partition of z.

Inductively, for every $n \in \mathbb{N}$, we can obtain an $(n, \varepsilon, \rho)-$partition $(z_{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n} : \varepsilon_i = 1, 2, i = 1, 2, \cdots, n)$ of z $\square$

Lemma 4.3 With the sets C, C_{x_0}, the $\varepsilon > 0$ and the Minkowski functional ρ as in Lemma 4.2, if, in addition, C is super-weakly compact, then there exist $0 < \delta < \varepsilon$ and $n \in \mathbb{N}$ such that for all $z \in C_{x_0}$ and for all $(n, \varepsilon, \rho)-$partition $(x_1, x_2, \cdots, x_{2^n})$ of z, the following inequality holds

$$\sum_{i=1}^{2^n} \rho(x_i) \geq (1 + \delta)\rho(z) \tag{4.1}$$

Proof Suppose, to the contrary, that for every $n \in \mathbb{N}$ and $\delta = 2^{-n}$, there exist $z_0 \in C_{x_0}$ and an $(n, \varepsilon, \rho)-$partition $(x_1, x_2, \cdots, x_{2^n})$ of z_0 such that

$$\sum_{i=1}^{2^n} \rho(x_i) < (1 + 2^{-n})\rho(z_0)$$

We assume again that $\rho(z_0) = 1$. Next, we want to produce an $(n, \varepsilon)-$tree in $2(C - x_0)$, and thus which contradicts the super-weak compactness of C.

Let

$$A = \{x_j^{(k)} = 2^k \sum_{i=2^{n-k}(j-1)+1}^{2^{n-k}j} x_i, \ k = 1, 2, \cdots, n \ \text{ and } \ j = 1, 2, \cdots, 2^k\}$$

We claim that

$$A \subset 2(C - x_0) \text{ and } A \text{ is an } (n, \varepsilon) - \text{tree.}$$

By the definition of $(n, \varepsilon, \rho)-$partition of z_0, we see that for all $1 \leq k \leq n$ and $1 \leq j \leq 2^{k-1}$

$$\rho(x_{2j-1}^{(k)}) = \rho(x_{2j}^{(k)}), \quad \sum_{j=1}^{2^k} x_j^{(k)} = 2^k z_0, \quad \| \frac{x_{2j-1}^{(k)}}{\rho(x_{2j-1}^{(k)})} - \frac{x_{2j}^{(k)}}{\rho(x_{2j}^{(k)})} \| \geq \varepsilon \tag{4.2}$$

and by the definition of the set A, for $1 \leq k \leq n-1$ and $1 \leq j \leq 2^{k-1}$

$$x_j^{(k)} = \frac{x_{2j-1}^{(k+1)} + x_{2j}^{(k+1)}}{2} \tag{4.3}$$

Let $k = 1$ in (4.2) and (4.3). We observe that

$$\rho(x_1^{(1)}) = \rho(x_2^{(1)}) \geq 1, \ x_1^{(1)} + x_2^{(1)} = 2z_0, \ x_1^{(1)} = \frac{x_1^{(2)} + x_2^{(2)}}{2}$$

and

$$\|x_1^{(1)} - x_2^{(1)}\| \geq \varepsilon$$

Let $k = 2$ in $(4.2), (4.3)$ and note the results above. We see that

$$\rho(x_1^{(2)}) = \rho(x_2^{(2)}) \geq 1, \ \rho(x_3^{(2)}) = \rho(x_4^{(2)}) \geq 1, \ x_j^{(2)} = \frac{x_{2j-1}^{(3)} + x_{2j}^{(3)}}{2}, \ j = 1, 2, 3, 4$$

and

$$\|x_1^{(2)} - x_2^{(2)}\| \geq \varepsilon, \quad \|x_3^{(2)} - x_4^{(2)}\| \geq \varepsilon$$

Inductively, for every $1 \leq k \leq n$, we have for $j = 1, 2, \cdots, 2^{k-1}$

$$\rho(x_{2j-1}^{(k)}) = \rho(x_{2j}^{(k)}) \geq 1, \quad \|x_{2j-1}^{(k)} - x_{2j}^{(k)}\| \geq \varepsilon$$

Thus, we have proven that A is an $(n, \varepsilon)-$tree in C_{x_0}. It remains to show that $A \subset 2(C - x_0)$.

Since $x_j^{(n)} = 2^n x_j$ and since $\rho(x_j^{(n)}) \geq 1$ for $j = 1, 2, \cdots, 2^n$, we have $\rho(x_j) \geq 2^{-n}$. Thus for each $1 \leq i \leq 2^n$,

$$\rho(x_i) + (2^n - 1)2^{-n} \leq \sum_{j=1}^{2^n} \rho(x_j) < 1 + 2^{-n}$$

This explains that $\rho(x_i) < 2^{-n+1}$, and therefore

$$\rho(x_i^{(n)}) = \rho(2^n x_i) < 2$$

i.e. $\{x_i^{(n)}\}_{i=1}^{2^n} \subset 2(C - x_0)$ and further $A \subset 2(C - x_0)$ $\square$

Lemma 4.4 Suppose that C is a nonempty super-weakly compact set. Then there exist $x_0 \in C$ and $\varepsilon_0 > 0$ satisfying that for every $0 < \varepsilon < \varepsilon_0$ there exist a function f on C_{x_0} and $0 < \gamma < \min\{\frac{1}{8}, \frac{\varepsilon}{1+\varepsilon}\}$ such that

i) $f(x) \geq 0, f(\alpha x) = \alpha f(x)$ for all $\alpha \geq 0$ and $f(x) = 0 \Leftrightarrow x = 0$;

ii) $(1 - \gamma)\rho(x) \leq f(x) \leq (1 - \frac{\gamma}{3})\rho(x)$, where ρ denotes the Minkowski functional generated by $C - x_0$;

iii) There exists $\delta > 0$ satisfying $\rho(x) = \rho(y) = 1$ and $\|x - y\| \geq \varepsilon$ imply $f(x+y) < f(x) + f(y) - \delta$;

iv) In particular, if C is symmetric, then we can put $x_0 = 0$ and therefore ρ is a lower semi-continuous norm on span$C$ and there exists a function f on span$C$ such that i),ii) and iii) hold.

Proof It is easy to observe that these assertions are true for $\dim(\text{span}C) < \infty$. Therefore, it suffices to consider the case for $\dim(\text{span}C) = \infty$.

By Lemma 4.2, there exist $x_0 \in C$ and $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$, every $z \in C_{x_0}$ and every $0 < \varepsilon \leq \varepsilon_0$, there is an $(n, \varepsilon, \rho)-$partition of z in C_{x_0}. Applying Lemma 4.3, we know that for every $0 < \varepsilon \leq \varepsilon_0$, there exist $0 < \gamma < \min\{\frac{1}{8}, \frac{\varepsilon}{1+\varepsilon}\}$ and $n \in \mathbb{N}$ such that for all $z \in C_{x_0}$ and for all $(n, \varepsilon, \rho)-$partition $(x_1, x_2, \cdots, x_{2^n})$ of z

$$\sum_{i=1}^{2^n} \rho(x_i) \geq (1 + \gamma)\rho(z) \tag{4.4}$$

Now, we fix such $z, \varepsilon, n, \gamma$ and let

$$f(z) = \inf\Big\{ \sum_{i=1}^{2^m} \frac{\rho(x_i)}{1 + \frac{\gamma}{2}(1 + \frac{1}{4} + \cdots + \frac{1}{4^m})} : 0 \leq m \leq n$$

$$(x_1, x_2, \cdots, x_{2^m}) \text{ is an } (m, \varepsilon, \rho) - \text{partition of } z\Big\} \tag{4.5}$$

Clearly, f satisfies i) and ii). It follows from (4.4) and (4.5) that we can assume $0 \leq m \leq n - 1$.

Suppose that $x, y \in C_{x_0}$ with $\rho(x) = \rho(y) = 1$ and with $\|x - y\| \geq \varepsilon$. Then (x, y) is a $(1, \varepsilon, \rho)-$partition of $x+y$. Let $0 < r < \frac{\gamma}{4^{2n}}$ and let $(u_1, \cdots, u_{2^k})$ be a $(k, \varepsilon, \rho)-$partition of x and $(v_1, v_2, \cdots, v_{2^l})$ be an $(l, \varepsilon, \rho)-$partition of y with $0 \leq k \leq l \leq n - 1$ such that

$$f(x) > \frac{\sum_{i=1}^{2^k} \rho(u_i)}{1 + \frac{\gamma}{2}(1 + \frac{1}{4} + \cdots + \frac{1}{4^k})} - r, \quad f(y) > \frac{\sum_{i=1}^{2^l} \rho(v_i)}{1 + \frac{\delta}{2}(1 + \frac{1}{4} + \cdots + \frac{1}{4^l})} - r \tag{4.6}$$

We denote by $(w_1, w_2, \cdots, w_{2^k})$ the $k-$part of the $(l, \varepsilon, \rho)-$partition of y. Then

$$f(y) > \frac{\sum_{i=1}^{2^l} \rho(v_i)}{1+\frac{\gamma}{2}(1+\frac{1}{4}+\cdots+\frac{1}{4^l})} - r \geq \frac{\sum_{i=1}^{2^k} \rho(w_i)}{1+\frac{\gamma}{2}(1+\frac{1}{4}+\cdots+\frac{1}{4^l})} - r$$

$$\geq \frac{\sum_{i=1}^{2^k} \rho(w_i)}{1+\frac{\delta}{2}(1+\frac{1}{4}+\cdots+\frac{1}{4^{k+1}}+\frac{1}{3\cdot 4^{k+1}})} - r \tag{4.7}$$

It follows that from (4.6), (4.7) and ii) which we have proven

$$1 = \rho(x) \leq \sum_{i=1}^{2^k} \rho(u_i) \leq 1+\gamma, \quad 1 \leq \sum_{i=1}^{2^k} \rho(w_i) \leq 1+\gamma \tag{4.8}$$

Note $(u_1, u_2, \cdots, u_{2^k}, w_1, w_2, \cdots, w_{2^k})$ is a $(k+1, \varepsilon, \rho)-$partition. We have

$$f(x+y) \leq \frac{\sum_{i=1}^{2^k} \rho(u_i) + \sum_{i=1}^{2^k} \rho(w_i)}{1+\frac{\gamma}{2}(1+\frac{1}{4}+\cdots+\frac{1}{4^{k+1}})} \tag{4.9}$$

It follows that from these inequalities above

$$f(x)+f(y)-f(x+y) \geq \sum_{i=1}^{2^k} \rho(u_i)\Big(\frac{1}{1+\frac{\gamma}{2}\sum_{i=0}^{k+1} 4^{-i}} - \frac{1}{1+\frac{\gamma}{2}\sum_{i=0}^{k+1} 4^{-i}}\Big)$$

$$-\sum_{i=1}^{2^k} \rho(w_i)\Big(\frac{1}{1+\frac{\gamma}{2}\sum_{i=0}^{k+1} 4^{-i}} - \frac{1}{1+\frac{\gamma}{2}\sum_{i=0}^{k+1} 4^{-i}+\frac{1}{3\cdot 4^{k+1}}}\Big) - 2r$$

This and $0 < r < \frac{\gamma}{4^{2n}}, 0 < \gamma < \frac{1}{8}$ imply

$$f(x)+f(y)-f(x+y) \geq \frac{\gamma}{2} \cdot \frac{1}{4^{k+3}} \geq \frac{\gamma}{2} \cdot \frac{1}{4^{n+2}}$$

Therefore, we finish the proof by letting $\delta = \frac{\gamma}{2} \cdot 4^{-(n+2)}$ □

Lemma 4.5 Suppose that C is a nonempty super-weakly compact convex set in X. Then there exist $\varepsilon_0 > 0$ and $x_0 \in C$ such that for every $0 < \varepsilon < \varepsilon_0$, there are $0 < \gamma < \min\{\frac{1}{8}, \frac{\varepsilon}{1+\varepsilon}\}$, $\delta > 0$ and an extended real-valued Minkowski functional p_ε with $\mathrm{dom} p_\varepsilon = C_{x_0}$ such that

i) $(1-\gamma)\rho(x) \leq p_\varepsilon(x) \leq (1-\frac{\gamma}{3})\rho(x)$, where ρ is the Minkowski functional generated by $C - x_0$;

ii) $\rho(x) = \rho(y) = 1$ with $\|x-y\| \geq \varepsilon$ imply $p_\varepsilon(x+y) \leq p_\varepsilon(x) + p_\varepsilon(y) - \varepsilon\delta$;

iii) For every $0 < \beta \leq 1$, $\rho(x_n) \to \beta, \rho(y_n) \to \beta$ and $p_\varepsilon(x_n)+p_\varepsilon(y_n)-p_\varepsilon(x_n+y_n) \to 0$ together imply $\overline{\lim}_{n\longrightarrow\infty} \|x_n - y_n\| < \beta\varepsilon$;

iv) In particular, if C is symmetric, then we can put $x_0 = 0$. Therefore ρ is a lower semi-continuous norm on $Y \equiv \mathrm{span} C$, and for every $0 < \varepsilon < \varepsilon_0$ there exists a norm p_ε on $\mathrm{span} C$ satisfying i), ii) and iii).

Proof Applying Lemma 4.4, we know that there exist $x_0 \in C$ and $\varepsilon_0 > 0$, such that for every $0 < 5\varepsilon_1 \equiv \varepsilon < \varepsilon_0$, there are a non-negative real-valued positively homogenous function f on C_{x_0} and $0 < \gamma < \min\{\frac{1}{8}, \frac{\varepsilon_1}{1+\varepsilon_1}\}$ satisfying

a) $(1-\gamma)\rho(x) \le f(x) \le (1-\frac{\gamma}{3})\rho(x)$ for all $x \in C_{x_0}$;

b) there exists $\delta_1 > 0$ such that $\rho(x) = \rho(y) = 1$ and $\|x-y\| \ge \varepsilon_1$ imply $f(x+y) < f(x) + f(y) - \delta_1$.

Without loss of generality, we can assume $C - x_0 \subset B_X$. Therefore $\|\cdot\| \le \rho$ on X. Now, let

$$p_\varepsilon(x) = \begin{cases} \inf\left\{ \sum\limits_{i=1}^{n} f(x_i) : \ n \in N, x_i \in C_{x_0} \text{ with } \sum\limits_{i=1}^{n} x_i = x \right\} &, \quad \text{if } x \in C_{x_0} \\ +\infty &, \quad \text{otherwise.} \end{cases}$$

Clearly, p_ε is an extended non-negative-valued positively homogenous subadditive function (hence, a Minkowski functional) on X with dom$p_\varepsilon = C_{x_0}$ and it satisfies i) of the lemma. Given $0 < a \le \min\{\gamma(\frac{1}{3}-\gamma), \frac{\delta_1\varepsilon_1}{4}\}$. Let $x = \sum\limits_{i=1}^{n} x_i,\ y = \sum\limits_{i=1}^{m} y_i$ with $x_i, y_i \in C_{x_0}$ such that

$$p_\varepsilon(x) > \sum_{i=1}^{n} f(x_i) - a, \quad p_\varepsilon(y) > \sum_{i=1}^{m} f(y_i) - a \tag{4.10}$$

We may assume without loss of generality that $m \le n$, and that $\rho(x_i) = \rho(y_i)$ for $i = 1, 2, \cdots, m$. Otherwise, say, $\rho(x_i) < \rho(y_i)$ for some $i \le m$, we set $s_i = \frac{\rho(x_i)}{\rho(y_i)}$. Then $\rho(x_i) = \rho(s_i y_i)$ and note that $f(y_i) = f(s_i y_i) + f((1-s_i)y_i)$. We substitute the two vectors $s_i y_i$ and $(1-s_i)y_i$ for y_i and renumber the new sequence $\{y_1, y_2, \cdots, y_{i-1}, s_i y_i, (1-s_i)y_i, y_{i+1}, \cdots, y_m\}$. Combining a) and (4.10), we can claim

$$1 \le \sum_{i=1}^{n} \rho(x_i) < 1 + \gamma, \quad 1 \le \sum_{j=1}^{m} \rho(y_j) < 1 + \gamma \tag{4.11}$$

Note $\rho(x_i) = \rho(y_i), i = 1, 2, \cdots, m$. Thus $\sum\limits_{i=m+1}^{n} \rho(x_i) < \gamma$, and which implies $\sum\limits_{i=m+1}^{n} \|x_i\| < \gamma$.

Set

$$J_1 = \Big\{ i : \ 1 \le i \le m, \|x_i - y_i\| \le \varepsilon_1 \rho(x_i) \Big\}, \quad J_2 = \Big\{ i : \ 1 \le i \le m, \|x_i - y_i\| > \varepsilon_1 \rho(x_i) \Big\}$$

Then

$$\begin{aligned} \|x-y\| &\le \sum_{i \in J_1} \|x_i - y_i\| + \sum_{i \in J_2} \|x_i - y_i\| + \sum_{i=m+1}^{n} \|x_i\| \\ &\le \varepsilon_1 \sum_{i \in J_1} \rho(x_i) + \sum_{i \in J_2} \|x_i - y_i\| + \gamma \\ &\le \varepsilon_1 (1+\gamma) + \sum_{i \in J_2} \|x_i - y_i\| + \gamma \end{aligned}$$

$$< 2\varepsilon_1 + \sum_{i \in J_2} \|x_i - y_i\|$$

Therefore, $\|x - y\| \geq \varepsilon$ implies $\sum\limits_{i \in J_2} \|x_i - y_i\| > 3\varepsilon_1$.

It follows from

$$x + y = \sum_{i \in J_2} (x_i + y_i) + \sum_{i \notin J_2} x_i + \sum_{i \notin J_2} y_i$$

that

$$p_\varepsilon(x + y) \leq \sum_{i \in J_2} f(x_i + y_i) + \sum_{i \notin J_2} f(x_i) + \sum_{i \notin J_2} f(y_i) \tag{4.12}$$

Inequalities (4.10) and (4.12) together imply

$$p_\varepsilon(x) + p_\varepsilon(y) - p_\varepsilon(x+y) \geq \sum_{i=1}^{n} f(x_i) - a + \sum_{i=1}^{m} f(y_i) - a - \sum_{i \in J_2} f(x_i + y_i) - \sum_{i \notin J_2} f(x_i) - \sum_{i \notin J_2} f(y_i)$$

$$= \sum_{i \in J_2} \Big(f(x_i) + f(y_i) - f(x_i + y_i) \Big) - 2a \tag{4.13}$$

Since $\rho(x_i) = \rho(y_i)$ for $i = 1, 2, \cdots, m$, we see that

$$\|\frac{x_i}{\rho(x_i)} - \frac{y_i}{\rho(y_i)}\| \geq \varepsilon_1 \quad \text{for all} \quad i \in J_2$$

This and b) imply

$$f(\frac{x_i}{\rho(x_i)} + \frac{y_i}{\rho(y_i)}) \leq f(\frac{x_i}{\rho(x_i)}) + f(\frac{y_i}{\rho(y_i)}) - \delta_1$$

Therefore

$$f(x_i) + f(y_i) - f(x_i + y_i) > \delta_1 \rho(x_i) \;\; \text{for all} \;\; i \in J_2 \tag{4.14}$$

This and (4.13) give

$$p_\varepsilon(x) + p_\varepsilon(y) - p_\varepsilon(x + y) > \delta_1 \sum_{i \in J_2} \rho(x_i) - 2a \tag{4.15}$$

Thus

$$3\varepsilon_1 \leq \sum_{i \in J_2} \|x_i - y_i\| \leq \sum_{i \in J_2} (\|x_i\| + \|y_i\|) \leq 2 \sum_{i \in J_2} \rho(x_i)$$

that is,

$$\sum_{i \in J_2} \rho(x_i) \geq \frac{3}{2}\varepsilon_1 \tag{4.16}$$

Finally, it follows from $0 < a < \frac{\delta_1 \ \varepsilon_1}{4}$, (4.15) and (4.16), that

$$p_\varepsilon(x) + p_\varepsilon(y) - p_\varepsilon(x + y) > \frac{3}{2}\varepsilon_1\delta - 2a > \delta_1\varepsilon_1 \equiv \delta\varepsilon$$

which explains that ii) holds. It remains to show iii).

For every $0 < \beta \leq 1$, let sequences x_n, y_n $(n = 1, 2, \cdots)$satisfy the condition of iii). Now, put $u_n = \frac{\beta}{\rho(x_n)} x_n$ and $v_n = \frac{\beta}{\rho(y_n)} y_n$. It is not difficult to show that

$$p_\varepsilon(u_n + v_n) - p_\varepsilon(x_n + y_n) \longrightarrow 0$$

Therefore

$$p_\varepsilon(u_n) + p_\varepsilon(v_n) - p_\varepsilon(u_n + v_n) \longrightarrow 0$$

Note $\rho(\frac{u_n}{\beta}) = \rho(\frac{v_n}{\beta}) = 1$. It follows from ii) that we have just proven,

$$\overline{\lim}_{n \longrightarrow \infty} \| \frac{u_n}{\beta} - \frac{v_n}{\beta} \| < \varepsilon$$

that is

$$\overline{\lim}_{n \longrightarrow \infty} \|x_n - y_n\| = \overline{\lim}_{n \longrightarrow \infty} \|u_n - v_n\| < \beta\varepsilon \ \square$$

Definition 4.6 Suppose f is a convex function on a nonempty convex set C of a Banach space X and $\varepsilon > 0$. Then

i) We say that f is $\varepsilon-$uniformly convex on C, if there exists $\delta > 0$ such that $x, y \in C$ with $\|x - y\| \geq \varepsilon$ implies $\frac{1}{2}[f(x) + f(y)] - f(\frac{x+y}{2}) > \delta$;

ii) f is said to be uniformly convex on C if f is $\varepsilon-$uniformly convex for all $\varepsilon > 0$.

Theorem 4.7 Suppose that C is a nonempty super-weakly compact convex set in a Banach space X. Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, there is a bounded $\varepsilon-$uniformly convex function f_ε on C.

Proof By Lemma 4.5, we can find $x_0 \in C$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, there exist $\delta > 0$ and a Minkowski functional p_ε with dom$p_\varepsilon = C_{x_0}$ which is bounded on $C - x_0$, satisfying

$$p_\varepsilon(x + y) \leq p_\varepsilon(x) + p_\varepsilon(y) - \varepsilon\delta$$

whenever $\rho(x) = \rho(y) = 1$ and $\|x - y\| \geq \frac{\varepsilon}{2}$, where ρ still denotes the Minkowski functional generated by $C - x_0$.

Let $f_\varepsilon(x) = p_\varepsilon^2(x) + \rho^2(x)$. Now we claim that $g_\varepsilon(x) \equiv f_\varepsilon(x - x_0)$ is $\varepsilon-$uniformly convex on C. Obviously, it suffices to show that the function f_ε has the property on $C - x_0$.

Suppose, to the contrary, that there are two sequences $\{x_n\}$ and $\{y_n\}$ in $C - x_0$ with $\|x_n - y_n\| \geq \varepsilon$, such that

$$\frac{1}{2}[f_\varepsilon(x_n) + f_\varepsilon(y_n)] - f_\varepsilon(\frac{x_n + y_n}{2}) \longrightarrow 0$$

which in turn implies that

$$p_\varepsilon(x_n) - p_\varepsilon(y_n) \longrightarrow 0, \quad \rho(x_n) - \rho(y_n) \longrightarrow 0$$

and

$$p_\varepsilon(\frac{x_n + y_n}{2}) - \frac{1}{2}[p_\varepsilon(x_n) + p_\varepsilon(y_n)] \longrightarrow 0$$

We can assume that

$$\rho(x_n) \longrightarrow r, \quad \rho(y_n) \longrightarrow r$$

for some $0 < r \le 1$. Let $x_n' = \frac{r}{\rho(x_n)} x_n, y_n' = \frac{r}{\rho(y_n)} y_n$ for all $n \in \mathbb{N}$. Then it is not difficult to show

$$p_\varepsilon(\frac{x_n' + y_n'}{2}) - p_\varepsilon(\frac{x_n + y_n}{2}) \longrightarrow 0$$

Therefore

$$p_\varepsilon(\frac{u_n + v_n}{2}) - \frac{1}{2}[p_\varepsilon(u_n) + p_\varepsilon(v_n)] \longrightarrow 0$$

where $u_n = \frac{1}{r} x_n', \ v_n = \frac{1}{r} y_n'$ satisfying $\rho(u_n) = \rho(v_n) = 1$. By hypothesis on p_ε, we obtain

$$\overline{\lim}_{n \longrightarrow \infty} \|u_n - v_n\| < \frac{\varepsilon}{2}$$

On the other hand,

$$\overline{\lim}_{n\to\infty} \|u_n - v_n\| = \overline{\lim}_{n\to\infty} \frac{1}{r}\|x_n' - y_n'\| = \overline{\lim}_{n\to\infty} \frac{1}{r}\|x_n - y_n\| \ge \frac{\varepsilon}{r} \ge \varepsilon$$

and this is a contradiction □

Theorem 4.8 Suppose that C is a nonempty bounded closed convex set of X. If there is $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exists a bounded $\varepsilon-$uniformly convex function f_ε on C, then C is super-weakly compact.

Proof Suppose that C is not super-weakly compact. Then by Corollary 3.7, there exists $\theta > 0$ such that for every $n \in \mathbb{N}$, there exist $x_i^{(n)} \in C$ for $i = 1, 2, \cdots, 2^n$ satisfying

$$\text{dist}\Big(co\{x_1^{(n)}, x_2^{(n)}, \cdots, x_k^{(n)}\}, co\{x_{k+1}^{(n)}, x_{k+2}^{(n)}, \cdots, x_{2^n}^{(n)}\}\Big) > \theta$$

for all $1 \le k < 2^n$. Clearly, $\dim(\text{span}C) = \infty$. Now, fix any $0 < \varepsilon < \min\{\theta, \varepsilon_0\}$, and let f_ε be a bounded $\varepsilon-$uniformly convex function which means that there exists $\delta > 0$ such that $x, y \in C$ with $\|x - y\| \ge \varepsilon$ implies

$$\frac{1}{2}[f_\varepsilon(x) + f_\varepsilon(y)] - f_\varepsilon(\frac{x+y}{2}) > \delta$$

Let $-\infty < \alpha = \inf_C f_\varepsilon$ and $\sup_C f_\varepsilon = \beta < \infty$. Finally let $n \in \mathbb{N}$ such that $\beta - n\delta < \alpha$. Since $\|x_i^{(n)} - x_j^{(n)}\| \ge \theta > \varepsilon$ for all $1 \le i \ne j \le 2^n$, we know that for all $1 \le i \ne j \le 2^n$

$$f_\varepsilon(\frac{x_i^{(n)} + x_j^{(n)}}{2}) < \frac{1}{2}(f_\varepsilon(x_i^{(n)}) + f_\varepsilon(x_j^{(n)})) - \delta$$

Note also that $\|\frac{x_1^{(n)}+x_2^{(n)}}{2} - \frac{x_3^{(n)}+x_4^{(n)}}{2}\| > \varepsilon$. We again see that

$$\begin{aligned} f_\varepsilon(\frac{x_1^{(n)} + x_2^{(n)} + x_3^{(n)} + x_4^{(n)}}{4}) &< f_\varepsilon(\frac{x_1^{(n)} + x_2^{(n)}}{2}) + f_\varepsilon(\frac{x_3^{(n)} + x_4^{(n)}}{2}) - \delta \\ &< \frac{1}{4}\Big(f_\varepsilon(x_1^{(n)}) + f_\varepsilon(x_2^{(n)}) + f_\varepsilon(x_3^{(n)}) + f_\varepsilon(x_4^{(n)})\Big) - 2\delta \end{aligned}$$

Inductively, we have

$$f_\varepsilon(\frac{\sum_{i=1}^{2^n} x_i^{(n)}}{2^n}) \le \frac{1}{2^n}\Big(\sum_{i=1}^{2^n} f(x_i^{(n)})\Big) - n\delta \le \beta - n\delta < \alpha = \inf_C f_\varepsilon$$

This is a contradiction □

The following result directly follows from Theorem 4.7 and 4.8.

Corollary 4.9 A nonempty closed bounded convex set C of X is super-weakly compact if and only if for every $\varepsilon > 0$, there exists a bounded $\varepsilon-$uniformly convex function f_ε on C.

The following property is a consequence of Corollary 3.7.

Proposition 4.10 Suppose that $C \subset X$ is a super-weakly compact convex set of a Banach space X and D is a nonempty set of a Banach space Y. Let $T : C \longrightarrow D$ be a uniformly continuous affine mapping. Then $TC \subset D$ is relatively super-weakly compact.

Proposition 4.11 Suppose that X, Y are two Banach spaces and that $A \subset X$, $B \subset Y$ are two super-weakly compact convex sets. Then $A \times B$ is also super-weakly compact in $X \times Y$.

Proof Without loss of generality we can assume that $X \times Y$ is equipped with the sup-norm, i.e. $\|(x, y)\| = \max\{\|x\|, \|y\|\}$ for all $(x, y) \in X \times Y$. Let f_ε be a bounded $\varepsilon-$uniformly convex function on A and g_ε a bounded $\varepsilon-$uniformly convex function on B. It is easy to see that $h_\varepsilon(x, y) = f_\varepsilon(x)+g_\varepsilon(y)$ is again a bounded $\varepsilon-$uniformly convex function on $A \times B$. Thus by Theorem 4.8, $A \times B$ is super-weakly compact □

Proposition 4.12 Suppose that $C, D \subset X$ are two super-weakly compact convex sets. Then $C - D$ is also super-weakly compact.

Proof By Proposition 4.10 and 4.11, it suffices to note $T : C \times D \longrightarrow X$, defined by $T(x, y) = x - y$, is a uniformly continuous affine mapping □

5 Renormings of Super-Weakly Compact Convex Sets

We have shown in Section 4 that a nonempty closed bounded convex set C of a Banach space X is super-weakly compact if and only if for every $\varepsilon > 0$ there exists a bounded $\varepsilon-$uniformly convex function on C. In this section, we will show that the super-weakly compact convex set C can always be renormed to have the geometric property that acts much like the closed unit ball of a uniformly convex Banach space. First, we need some more preparations.

For an extended real-valued Minkowski functional ρ on X, we denote by $B_\rho(r)$, the set $\{x \in X : \rho(x) \le r\}$.

Definition 5.1 Suppose that C is a nonempty convex set of a Banach space $(X, \|\cdot\|)$.

i) The norm $\|\cdot\|$ is said to be uniformly convex on C, if for every $r>0$ and every $x_0\in C$, it satisfies that

$$\|x_n+y_n\|\longrightarrow 2r \ \text{ implies } \ x_n-y_n\longrightarrow 0$$

whenever $\{x_n\},\{y_n\}$ are two sequences in $B_{\|\cdot\|}(r)\cap(C-x_0)$;

ii) The space X is called uniformly convex, if $\|\cdot\|$ is uniformly convex on X.

Lemma 5.2 Suppose that C is a nonempty symmetric closed convex set of a Banach space $(X,\|\cdot\|)$. If C is super-weakly compact, then there is $\varepsilon_0>0$ such that for every $0<\varepsilon<\varepsilon_0$, there exists a norm $|\cdot|$ on $Y\equiv\text{span}C$ such that

i) $(1-\varepsilon)\rho(x)\le|x|\le\rho(x)$;

ii) For every $r>0$, $|x_n|+|y_n|-|x_n+y_n|\to 0$ implies $\|x_n-y_n\|\to 0$, whenever $x_n,y_n\in B_\rho(r)\cap C$ with $\rho(x_n)\to r$ and $\rho(y_n)\to r$, where ρ denotes the norm(Minkowski functional) generated by C.

Proof Applying Lemma 4.5, there exists $\varepsilon_0>0$ such that for every $0<\varepsilon<\varepsilon_0$, there are $0<\gamma<\min\{\frac{1}{8},\frac{\varepsilon}{1+\varepsilon}\}$, $\delta>0$ and a norm p_ε on Y satisfying

a) $(1-\gamma)\rho(x)\le p_\varepsilon(x)\le(1-\frac{\gamma}{3})\rho(x)$;

b) for each $0<r\le 1$ and sequences $\{x_n\},\{y_n\}\subset B_\rho(r)$ with $\rho(x_n)\to r,\rho(y_n)\to r$, we have $\overline{\lim}_{n\to\infty}\|x_n-y_n\|<r\varepsilon$, whenever $p_\varepsilon(x_n+y_n)-p_\varepsilon(x_n)-p_\varepsilon(y_n)\to 0$.

Let $\varepsilon_n=\frac{\varepsilon}{2^n}$ and denote by the norms $p_n\equiv p_{\varepsilon_n}$ for $n=1,2,\cdots$. Finally, let

$$|x|=\sum_{n=1}^{\infty}2^{-n}p_n(x),\quad x\in Y \tag{5.1}$$

Clearly, $|\cdot|$ is a norm on Y satisfying $(1-\varepsilon)\rho(x)\le|x|\le\rho(x)$. We want to show that $|\cdot|$ has the desired property.

Suppose that $\{x_m\},\{y_m\}\subset B_\rho(r)$ with $\rho(x_m)\to r,\rho(y_m)\to r$ such that $|x_m|+|y_m|-|x_m+y_m|\to 0$. We claim that $\|x_m-y_m\|\to 0$. Without loss of generality we can assume that $r=1$. Suppose, to the contrary, that there exist $a>0$ and a subsequence of $\{x_m-y_m\}$ which is still denoted by $\{x_m-y_m\}$ such that for all sufficiently large $m\in\mathbb{N}$, $\|x_m-y_m\|>a$. Choose $j\in\mathbb{N}$ with $\varepsilon_j<a$. Note that

$$|x_m|+|y_m|-|x_m+y_m|\longrightarrow 0$$

implies that for all $i\in\mathbb{N}$

$$p_i(x_m)+p_i(y_m)-p_i(x_m+y_m)\longrightarrow 0$$

Thus, by hypothesis on p_j, we have

$$\overline{\lim}_{m\longrightarrow\infty}\|x_m-y_m\|<\varepsilon_j<a$$

This is a contradiction $\Box$

Lemma 5.3(Enflo, [10]) Suppose that X is a normed space and that $x, y \in X$ with $\|x\| = \|y\| = 1$. Let $f(\alpha) = \|\alpha x - y\|$. Then $f(\alpha) \geq \frac{1}{2} f(1)$ for all $\alpha \in \mathbb{R}$.

Lemma 5.4 Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a linear space E, and that $C \subset E$ is a convex set. Then $\|\cdot\|_2$ is uniformly convex on C if the following assumptions are satisfied

i) $\|x\|_2 \leq \|x\|_1 \leq 2\|x\|_2$ for all $x \in E$;

ii) for every $r > 0$ and $x_0 \in C$, $\|x_n\|_2 + \|y_n\|_2 - \|x_n + y_n\|_2 \to 0$ implies $\|x_n - y_n\|_1 \to 0$, whenever $x_n, y_n \in B_{\|\cdot\|_1}(r) \cap (C - x_0)$ with $\|x_n\|_1 \to r$ and $\|y_n\|_1 \to r$.

Proof For every $r > 0$ and every $x_0 \in C$, suppose that $x_n, y_n \in B_{\|\cdot\|_2}(r) \cap (C - x_0)$ such that $\|x_n + y_n\|_2 \to 2r$. We claim that $\|x_n - y_n\|_2 \to 0$. Indeed, let $\{\alpha_n\}, \{\beta_n\} \subset [\frac{1}{2}, 1]$ be two sequences with $\|\alpha_n x_n\|_1 = \|x_n\|_2$ and $\|\beta_n y_n\|_1 = \|y_n\|_2$. Then, it is easy to see that $\alpha_n x_n, \beta_n y_n \in C - x_0$ and

$$\|x_n + y_n\|_2 - (\|x_n\|_2 + \|y_n\|_2) \leq \|\alpha_n x_n + \beta_n y_n\|_2 - (\|\alpha_n x_n\|_2 + \|\beta_n y_n\|_2) \leq 0$$

Therefore, it follows from $\|x_n\|_2 + \|y_n\|_2 - \|x_n + y_n\|_2 \longrightarrow 0$ that

$$\|\alpha_n x_n\|_2 + \|\beta_n y_n\|_2 - \|\alpha_n x_n + \beta_n y_n\|_2 \longrightarrow 0$$

By the assumption of the lemma, we have

$$\|\alpha_n x_n - \beta_n y_n\|_1 \longrightarrow 0$$

Let $x_n' = \frac{1}{r} x_n, y_n' = \frac{1}{r} y_n$ and $\gamma_n = \frac{\beta_n}{\alpha_n}$. Due to Lemma 5.3 and i), we obtain

$$\|\alpha_n x_n - \beta_n y_n\|_1 \geq \|\alpha_n x_n - \beta_n y_n\|_2 \geq \frac{1}{2}\|x_n - \gamma_n y_n\|_2 \geq \frac{1}{4}\|x_n - y_n\|_2$$

Thus

$$\|x_n - y_n\|_2 \to 0 \ \Box$$

In the following, we show that the Grothendieck's lemma is still valid for super-weakly compact convex sets.

Lemma 5.5 A nonempty closed convex set C of a Banach space X is super-weakly compact if and only if for every $\varepsilon > 0$ there exists a super-weakly compact convex set C_ε in X such that $C \subset C_\varepsilon + \varepsilon B_X$.

Proof It suffices to show sufficiency. Suppose that C is not super-weakly compact. By Corollary 3.7, we can find $0 < \theta < 1$ such that for every $n \in \mathbb{N}$, there exist $x_i \in C$ for $i = 1, 2, \cdots, n$ such that for every $1 \leq k < n$,

$$\text{dist}\Big(co\{x_1, x_2, \cdots, x_k\}, co\{x_{k+1}, x_{k+2}, \cdots, x_n\}\Big) > \theta$$

Let $\varepsilon = \frac{\theta}{4}$. We have a super-weakly compact convex set C_ε such that $C \subset C_\varepsilon + \varepsilon B_X$. Let

$$x_i = y_i + z_i, \ i = 1, 2, \cdots, n$$

where $y_i \in C_\varepsilon$ and $\|z_i\| \leq \varepsilon$. One checks easily that for every $1 \leq k < n$,

$$\text{dist}\Big(co\{y_1, y_2, \cdots, y_k\}, co\{y_{k+1}, y_{k+2}, \cdots, y_n\}\Big) \geq \varepsilon$$

By Corollary 3.7 again, C_ε is not super-weakly compact which is a contradiction $\square$

It is well-known that the Davis-Figiel-Johnson-Pelzyński Lemma [5] "provides a way of making reflexive Banach spaces from weakly compact sets of arbitrary Banach spaces. This lemma has (at least) two virtues. A number of basic facts about Banach spaces are easy consequence of it and its proof is striking elementary" (see, [8] and therein). We first state the lemma as follows.

Lemma 5.6(Davis, Figiel, Johnson, Pelzyński) Suppose that $(X, \|\cdot\|)$ is a Banach space with closed unit ball B_X. Let W be a convex symmetric bounded set of X. For each positive integer n, let $U_n = 2^n W + 2^{-n} B_X$. Denote by $\|\cdot\|_n$ the Minkowski functional generated by U_n, i.e.

$$\|x\|_n = \inf\{\alpha > 0 : \ x \in \alpha U_n\}$$

For $x \in X$, let $|||x|||$ be given by $|||x||| = \Big(\sum_{n=1}^\infty \|x\|_n^2\Big)^{\frac{1}{2}}$ and let $Y = \{x \in X : \ |||x||| < \infty\}$. Denote by C the $|||\cdot|||-$closed unit ball of Y. Let $J : Y \to X$ be the natural inclusion. Then

i) $W \subset C$;

ii) $(Y, |||\cdot|||)$ is a Banach space and J is continuous;

ii) $J^{**} : Y^{**} \to X^{**}$ is one-to-one and $Y = (J^{**})^{-1}(X)$; and

iv) $(Y, |||\cdot|||)$ is reflexive if and only if W is relatively weakly compact in X.

We should mention here that there is still an important and elegant consequence quite hidden in the lemma that we have not discovered for over 30 years, though many useful and beautiful properties and applications of it have been founded (such as, the factorization of weakly compact operators and every weakly compact subsets can be weak-to-weak continuously embedded into a reflexive space, etc.). Now, we present the consequence in the following.

Lemma 5.7 With the Banach spaces X and Y, the subsets W and C, the norms $\|\cdot\|$, $|||\cdot|||$ and $\|\cdot\|_n$ $(n = 1, 2, \cdots)$ as in the previous lemma, then

i) the identity mapping $I : Y \to Y$, restricted to W, is uniformly $\|\cdot\|-$continuous;

ii) the separability of $(Y, |||\cdot|||)$ coincides with that of $(Y, \|\cdot\|)$;

iii) every relatively weakly compact set of X is (linearly) isomorphic to a weakly compact set of a reflexive space;

iv) if W is relatively super-weakly compact, then C is again super-weakly compact in X.

Proof i) For every positive integer m, let

$$P_m(x) = \Big(\sum_{n=1}^m \|x\|_n^2\Big)^{\frac{1}{2}} \quad \text{for} \quad x \in Y$$

Then P_m is uniformly $\|\cdot\|-$continuous, since every $\|\cdot\|_n$ is uniformly $\|\cdot\|-$continuous on X. Note for every positive integer n and $x \in W$, $\|x\|_n < 2^{-n}$. We know that P_m uniformly converges to $|||\cdot|||$ on W, and further which implies $|||\cdot|||$ is uniformly $\|\cdot\|-$continuous on W.

ii) It is trivial to see that $(Y, \|\cdot\|)$ is separable if $(Y, |||\cdot|||)$ is separable. Conversely, if $(Y, \|\cdot\|)$ is separable, then $(Y, \|\cdot\|_n)$ is again separable for every positive integer n by noting $\|\cdot\|$ and $\|\cdot\|_n$ are equivalent on Y. Therefore the direct sum $\sum_{n=1}^{\infty} \oplus(Y, \|\cdot\|_n)$ equipped with the norm $|||(x_n)||| = \left(\sum_{n=1}^{\infty} \|x_n\|_n^2\right)^{\frac{1}{2}}$, is a separable space. We complete the proof of ii) by observing that $(Y, |||\cdot|||)$ is isometric to a subspace of $\sum_{n=1}^{\infty} \oplus(Y, \|\cdot\|_n)$.

iii) This is just a direct consequence of i) we have just proved, since a subset K is relatively weakly compact if and only the closed convex hull of $\{K \cup -K\}$ is weakly compact.

iv) It suffices to note Lemma 5.4 and to note $C \subset 2^n W + 2^{-n} B_X$ for every positive integer n □

Theorem 5.8 Suppose that K is a nonempty super-weakly compact convex set of a Banach space $(X, \|\cdot\|)$. Then there exists a reflexive Banach space $(Y, |||\cdot|||)$ such that

i) $K - K \subset B_Y \subset X$;

ii) The topology of $|||\cdot|||$ is stronger than that of $\|\cdot\|$ on Y;

iii) $|||\cdot|||$ is uniformly $\|\cdot\|-$continuous and uniformly convex on $K - K$;

iv) $|||\cdot|||$ is uniformly convex with respect to $\|\cdot\|$ on Y, that is, for any two $|||\cdot|||-$bounded sequences $\{x_n\}$ and $\{y_n\}$ in Y,

$$2(|||x_n|||^2 + |||y_n|||^2) - |||x_n + y_n|||^2 \to 0 \text{ implies } \|x_n - y_n\| \to 0.$$

Proof Without loss of generality we assume that $0 \in K$. Let $W = K - K$. Then, by Proposition 4.12, W is again super-weakly compact. Applying the Davis-Figiel-Johnson-Pelzyński Lemma and Lemma 5.7 to produce a reflexive Banach space $(X_1, \|\cdot\|_1)$ such that the closed unit ball $B_{X_1}(\equiv W_1)$ of X_1 is also super-weakly compact in X and $W \subset W_1$. Lemma 5.2 explains that there is $\varepsilon_0 > 0$ such that for any fixed $0 < \varepsilon < \varepsilon_0$, there exists a norm $|||\cdot|||_1$ on X_1 satisfying

a) $(1-\varepsilon)\|x\|_1 \geq |||x|||_1 \leq \|x\|_1$ for every $x \in X_1$;

b) for every $r > 0$ and for any two sequences $\{x_n\}$, $\{y_n\}$ in rB_{X_1} with $\|x_n\|_1 \to r$ and $\|y_n\|_1 \to r$, $2(|||x_n|||_1^2 + |||y_n|||_1^2) - |||x_n + y_n|||_1^2 \to 0$ implies $\|x_n - y_n\| \to 0$.

Since $\|\cdot\|_1$ and $|||\cdot|||_1$ are equivalent on X_1 and since $\|\cdot\|_1$ is uniformly $\|\cdot\|-$continuous from Lemma 5.7, thus $|||\cdot|||_1$ is again uniformly $\|\cdot\|-$continuous on W. Now, for every $x_0 \in W$ and for any two sequences $\{x_n\}$, $\{y_n\}$ in $rB_{X_1} \cap \{C - x_0\}$ satisfying the assumptions of b), thus we have $\|x_n - y_n\| \to 0$, and which in turn implies $|||x_n - y_n|||_1 \to 0$. This together with a) and Lemma 5.4 give that $|||\cdot|||_1$ is uniformly convex on W.

Starting with the super-weakly compact set W_1 of X and repeating the construction above, we obtain again a reflexive space $(X_2, \|\cdot\|_2)$ with its closed unit ball $W_2 \supset W_1$ and an equivalent norm $|||\cdot|||_2$ of $\|\cdot\|_2$ on X_2 such that it is uniformly continuous and uniformly convex on W_1. Finally, let $|||\cdot||| = \sqrt{|||\cdot|||_1^2 + |||\cdot|||_2^2}$ on $Y \equiv X_1$. Then it is easy to check that $|||\cdot|||$ is an equivalent norm of $|||\cdot|||_1$ on Y such that it satisfies the desired properties $\square$

Corollary 5.9 Suppose C is a closed bounded convex set of a Banach space X. Then it is super-weakly compact if and only if there exists a uniformly continuous and uniformly convex function on C.

Proof Sufficiency is obvious by Theorem 4.8.

Necessity. We can assume that C is symmetric. Otherwise, we substitute $K \equiv \overline{co}\{C \cup -C\}$ for C. Since $co\{C \cup -C\}$ is the image of the super-weakly compact convex set $C \times (-C) \times [0,1]$ under the affine mapping $T: \ X^2 \to X$, defined by $T(x, y, \lambda) = \lambda x + (1-\lambda)y$, $co\{C \cup -C\}$ is relatively super-weakly compact. Now, by Theorem 5.8, there exists a norm $|\cdot|$ on span$K$ which is uniformly continuous and uniformly convex on K. We observe that $f = |\cdot|^2$ is a function with the desired properties $\square$

6 Final Remarks

Remark 6.1 Normal structure of super-weakly compact convex sets under Renormings

We begin with discussion of normal structure of super-weakly compact convex sets under renormings. After giving some extensions of Odell-Schlumprecht and Hájek-Johanis' renorming theorems, we finally point out some questions on this topic.

Suppose that C is a nonempty closed bounded convex set of a Banach space X. Let $d_C(x) = \sup_{y \in C} \|x - y\|$ for every $x \in C$.

Definition 6.1.1 With the set C and the function d_C as above, then

i) A point $x_0 \in C$ is said to be a diametral point if $d_C(x_0) = \text{diam}C$;

ii) We call C having normal structure if for every closed convex subset D of C containing at least two points has one non-diametral point of D.

Recall that a Banach space X is uniformly convex if and only if the convexity modulus δ_X of X is always proper positive-valued on $(0, \infty)$, where

$$\delta_X(\varepsilon) = \inf\left\{1 - \|\frac{x+y}{2}\|, \|x\|, \|y\| \leq 1, \|x - y\| \geq \varepsilon\right\}$$

For a closed convex set C of X, we define a function δ_C by

$$\delta_C(x_0, r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\|\frac{x+y}{2} - x_0\| : x, y \in C, \|x - x_0\| \leq r, \|y - x_0\| \leq r \text{ and } \|x - y\| \geq \varepsilon\right\}$$

for all $x_0 \in C, r > 0$ and $\varepsilon > 0$. Then it is clear that a norm $\|\cdot\|$ is uniformly convex on C if and only if δ_C is proper positive-valued on $C \times \mathbb{R}^{+2}$.

Definition 6.1.2 A convex set $C \subset X$ is said to be uniformly convexifiable if there exists a norm $|||\cdot|||$ on span$C$ such that it is uniformly continuous and uniformly convex on C.

Keeping this in mind, we immediately have the following consequence from Theorem 5.8.

Theorem 6.1.3 A closed bounded convex set of a Banach space is super-weakly compact if and only if it is uniformly convexifiable.

Theorem 6.1.4 Suppose that C is a super-weakly compact convex set of a Banach space $(X, \|\cdot\|)$. Then for every $\varepsilon > 0$, there exists a norm $|||\cdot|||$ on span$C$ such that

i) $\|x\| \leq |||x||| \leq \|x\| + \varepsilon$ for all $x \in C$;

ii) C has normal structure with respect to $|||\cdot|||$.

Proof By Theorem 5.8, there exists a norm $\|\cdot\|_1$ on span$C$ such that it is uniformly continuous and uniformly convex on C. For any fixed $\varepsilon > 0$, choose $\delta > 0$ such that $\sup_{x\in C} \|x\|_1 < \frac{\varepsilon}{\delta}$, and let $|||\cdot||| = \|\cdot\| + \delta\|\cdot\|_1$. Then $|||\cdot|||$ is also uniformly continuous and uniformly convex on C, and it satisfies i). It remains to show $(C, |||\cdot|||)$ has normal structure.

We can assume that C is symmetric. Now, note $C \pm C = 2C$. For every closed convex subset D of C with diam$D \equiv d > 0$, let $0 < \varepsilon < d$ and let $x, y \in D$ with $|||x - y||| \geq \varepsilon$. Then for every $u \in D$, we have $x - u, y - u \in 2C, |||x - u||| \leq d$ and $|||y - u||| \leq d$. Thus

$$1 - \frac{1}{d}|||\frac{(x-u)+(y-u)}{2}||| \geq \delta_{(D-u)}(0, d, \varepsilon) \equiv \alpha > 0$$

This implies

$$d_D(\frac{x+y}{2}) \equiv \sup_{u\in D} |||\frac{x+y}{2} - u||| \leq (1-\alpha)d < d$$

That is, $z = \frac{x+y}{2}$ is a non-diametral point of D □

Theorem 6.1.5 Suppose C is a super-weakly compact convex set of a Banach space $(X, \|\cdot\|)$. Then for every $\varepsilon > 0$ there exists a norm $|||\cdot|||$ on span$C$ such that

i) $|||\cdot|||$ is $\|\cdot\|-$uniformly continuous on C;

ii) $\|x\| \leq |||x||| \leq \|x\| + \varepsilon$ for all $x \in C$;

iii) every $|||\cdot|||-$non-expansive mapping $T: C \to C$ has a fixed point.

Proof We can still assume that C is symmetric. By Theorem 5.8, there exists a reflexive Banach space $(Y, |||\cdot|||)$ with $C \subset C - C \subset B_Y \subset X$ such that $|||\cdot|||$ is uniformly $\|\cdot\|-$continuous and uniformly convex on C. Obviously, we can claim that ii) holds. Theorem 6.1.4 tells us that $(C, |||\cdot|||)$ has normal structure. Thus, Kirk's theorem[22] guarantees that every $|||\cdot|||-$non-expansive mapping from C to C has a fixed point □

Remark 6.2 Extensions of recent renorming theorems for reflexive spaces

In 1998, E.Odell and T.Schlumprecht[24] gave an affirmative answer of the long-standing question "whether there exists a property of a geometric nature which is equivalent to reflexivity of Banach spaces" for separable spaces, that is the following theorem.

Theorem (Odell, Schlumprecht) A separable Banach space X is reflexive if and only if there exists an equivalent norm $|\cdot|$ on X such that for every sequence $\{x_n\} \subset X$,

$$\lim_m \lim_n |\frac{x_n + x_m}{2}| = \lim_n |x_n|$$

implies that $\{x_n\}$ is convergent in norm.

Recently, P.Hájek and M.Johanis, through introducing a new convexity property of Day's norm on $C_0(\kappa)$, showed the following renorming characterization of general reflexive spaces[13].

Theorem (Hájek, Johanis) A Banach space X is reflexive if and only if there exists an equivalent norm $|\cdot|$ on X such that for every sequence $\{x_n\} \subset X$,

$$\lim_m \lim_n |\frac{x_m + x_n}{2}| = \lim_n |x_n|$$

implies that $\{x_n\}$ weakly converges.

As an application of Lemma 5.7, we extend and localize both the Odell-Schlumprecht Theorem and Hájek-Johanis Theorem for reflexive spaces to those for weakly compact convex subsets. The following result is an extension version of the Odell-Schlumprecht renorming theorem.

Theorem 6.2.1 Suppose K is a nonempty closed bounded separable convex set of a Banach space $(X, \|\cdot\|)$. Let $W = K - K$ and $Y = \text{span}W$. Then

i) K is weakly compact if and only if there exists a norm $|\cdot|$ on Y (not necessarily equivalent to $\|\cdot\|$), which is $\|\cdot\|-$continuous on W, such that $|\cdot|$ has the asymptotic property on K, that is, every sequence $\{x_n\} \subset K$ with $\lim_m \lim_n |\frac{x_m+x_n}{2}| = \lim_n |x_n|$ is necessarily $\|\cdot\|-$convergent in K;

ii) In particular, if W has nonempty interior, then we can further claim that $|\cdot|$ is an equivalent norm of $\|\cdot\|$ with the asymptotic property on the whole space X.

Proof It suffices to show i). Sufficiency. Without loss of generality we assume $0 \in K$. Assume $|\cdot|$ is a norm on Y and which restricted to W is $\|\cdot\|-$continuous, such that $\lim_m \lim_n |\frac{x_m+x_n}{2}| = \lim_n |x_n|$ implies that $\{x_n\}$ is $\|\cdot\|-$convergent in K for every sequence $\{x_n\} \subset K$. By the James' Characterization for weakly compact subsets, we need only to show that every linear functional $x^* \in X^*$ attains its maximum on K.

Given $x^* \in X^*$, we can assume that $\sup_K x^* \equiv \sup\{< x^*, x >: x \in K\} > 0$ (Otherwise we have $\sup_K x^* = 0 =< x^*, 0 >$). Since W is also $\|\cdot\|-$bounded and since $|\cdot|$ is $\|\cdot\|-$continuous on W, there exists $r > 0$ such that $W \subset \{x \in Y : |x| \leq r\} \equiv B_{|\cdot|}(r)$.

Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$g(s) = \sup\{< x^*, x >: x \in K, |x| \le s\}$$

Clearly, g is a continuous and positively homogenous function on $\mathbb{R}^+$. Let $s_0 = \min\{s > 0 : g(s) = \sup\limits_K x^*\}$, and let $\{x_n\} \subset K \cap B_{|\cdot|}(s_0)$ such that $\lim\limits_n < x^*, x_n >= \sup\limits_K x^*$. By definition of s_0 we know $|x_n| \to s_0$, and yet,

$$\lim_m \lim_n |\frac{x_m + x_n}{2}| = \lim_n |x_n| = s_0$$

Therefore $\{x_n\}$ is $\|\cdot\|-$convergent in K, say, $\lim\limits_n x_n = x_0 \in K$. So we have $\sup\limits_K x^* =< x^*, x_0 >$.

Necessity. Suppose that K is a nonempty weakly compact separable convex set of $(X, \|\cdot\|)$. Then $\overline{Y}$ (the $\|\cdot\|-$closure of Y) is $\|\cdot\|-$separable. By Lemma 5.7, there exists a reflexive space $(Z, |||\cdot|||)$ with $|||x||| = \left(\sum_{n=1}^{\infty} \|x_n\|_n^2\right)^{\frac{1}{2}}$ and with $W \subset C \equiv \{x \in X : |||x||| \le 1\}$ satisfying $|||\cdot|||$ is $\|\cdot\|-$continuous on W. By noting the construction of $\|\cdot\|_n$, we assert that $Y \subset Z \subset \overline{Y}$. Separability of $(\overline{Y}, \|\cdot\|)$ and Lemma 5.7 together imply that $(Z, |||\cdot|||)$ is separable. Applying the Odell-Schlumprecht Theorem to the space $(Z, |||\cdot|||)$, there is a norm $|\cdot|$ which is equivalent to $|||\cdot|||$ on Z such that $\lim\limits_m \lim\limits_n |\frac{x_m+x_n}{2}| = \lim\limits_n |x_n|$ implies $\{x_n\}$ converges in the norm $|\cdot|$ whenever $\{x_n\}$ is a bounded sequence in Z. Clearly, the norm $|\cdot|$ is $\|\cdot\|-$continuous on W and it has the desired properties when it is restricted to K, since $|\cdot|$ stronger than $\|\cdot\|$ on Y $\square$

Analogously, we can prove the following extension version of the Hájek-Johanis Theorem.

Theorem 6.2.2 Suppose K is a nonempty closed bounded convex set of a Banach space $(X, \|\cdot\|)$. Let $W = K - K$ and $Y = \text{span}W$. Then

i) K is weakly compact if and only if there exists a norm $|\cdot|$ on Y (not necessarily equivalent to $\|\cdot\|$), which is $\|\cdot\|-$continuous on W, such that $|\cdot|$ has the weakly asymptotic property on K, that is, every sequence $\{x_n\} \subset K$ with $\lim\limits_m \lim\limits_n |\frac{x_m+x_n}{2}| = \lim\limits_n |x_n|$ is necessarily weakly convergent in K;

ii) In particular, if W has nonempty interior, then we can further claim that $|\cdot|$ is an equivalent norm of $\|\cdot\|$ with the weakly asymptotic property on the whole space X.

Remark 6.3 Cheng, Wu, Xue and Yao [3] showed that a Banach space is uniformly convexifiable if and only if it admits a continuous uniformly convex function on some nonempty open convex set of the space. Thus, Corollary 5.9 is a perfect generalization of this result.

Remark 6.4 From Theorem 5.8 we see that for every super-weakly compact convex set C of a Banach space $(X, \|\cdot\|)$, there exists a reflexive Banach space $(Y, |||\cdot|||)$ with $C - C \subset B_Y \subset X$ such that $|||\cdot|||$ is relatively uniformly convex with respect to $\|\cdot\|$ on Y. But we do not know that whether such space Y is super-reflexive. Therefore Problem 3 in the first section remains open.

References

[1] Y.Benyamini and J.Lindenstrauss, Geometric nonlinear functional analysis, Vol. 1, American Mathematical Society Colloquium Publications, 48. American Mathematical Society, Providence, RI, 2000

[2] F.E.Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. U.S.A. 54(1965), 1041–1044

[3] L. Cheng, C. Wu, X. Xue and X. Yao, Convex functions, subdifferentiability and renormings, Acta Math. Sinica (N.S.). 14(1998), no. 1, 47–56

[4] J.A.Clarkson, Uniformly convex spaces, Trans. Amer. Math. Soc. 40 (1936), 396–414

[5] W.J.Davis, T.Figiel, W.B.Johnson and A.Pelczynski, Factoring weakly compact operators, J.Func.Anal. 17(1974), 311-327

[6] R.Deville, G.Godefroy and V.Zizler, Smoothness and renormings in Banach spaces, Pitman Monographs and Surveys in Pure and Applied Mathematics, 64. Longman Sci & Tech. John Wiley & Sons, Inc, New York, 1993

[7] J.Diestel, Geometry of Banach spaces—selected topics, Lecture Notes in Mathematics, Vol. 485, Springer-Verlag, Berlin-New York, 1975

[8] J.Diestel and J.J.Uhl, Vector measures, Mathematical Surveys, No. 15, American Mathematical Society, Providence, R.I, 1977

[9] J.Diestel, Sequences and series in Banach spaces. Graduate Texts in Mathematics, 92, Springer-Verlag, New York, 1984

[10] P.Enflo, Banach spaces which can be given an equivalent uniformly convex norm, Israel J. Math. 13(1972),281-288

[11] K.Gobel and W.A.Krik, Topics in metric fixed point theory. Cambridge Studies in Advanced Mathematics, 28, Cambridge University Press, Cambridge, 1990

[12] G.Godefroy, Renormings of Banach spaces, Handbook of the geometry of Banach spaces, Vol. I, 781–835, North-Holland, Amsterdam, 2001

[13] P.Hájek and M.Johanis, Characterization of reflexivity by equivalent renorming, J. Funct. Anal. 211(2004), no. 1, 163–172

[14] V.I.Istratescu, Strict convexity and complex strict convexity, Theory and applications, Lecture Notes in Pure and Applied Mathematics, 89. Marcel Dekker, Inc, New York, 1984

[15] R.C.James, Weak compactness and reflexivity, Israel J. Math. 2(1964), 101-119

[16] R.C.James, Uniformly non square Banach spaces, Ann. Math.(2). 80(1964),542-550

[17] R.C.James, Super-reflexive Banach spaces, Canad. J. Math. 24(1972), 896–904

[18] R.C.James, Some self-dual properties of normed linear spaces, In proc. Sympos. Infinite Dimensional Topology, Ann. Math. Studies. 69(1972), 159-175

[19] R. C.James and J.J.Schäffer, Super-reflexivity and the girth of spheres, Israel J. Math. 11(1972), 398–404

[20] W.B.Johnson and J.Lindenstrauss, Handbook of the geometry of Banach spaces, Vol. I. Edited by North-Holland Publishing Co., Amsterdam, 2001

[21] G.Kasparov and G.Yu, The coarse Geometric Novikov conjecture and uniform convexity, Adv.Math., to appear

[22] W.A.Kirk, A fixed point theorem for mappings which do not increase distances, Amer.Math.Monthly. 72(1965), 1004-1006

[23] J.Lindenstrauss and L.Tzafriri, Classical Banach spaces II, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], 97, Springer-Verlag, Berlin-New York, 1979

[24] E.Odell and T.Schlumprecht, Asymptotic properties of Banach spaces under renormings, J. Amer. Math. Soc. 11(1998), no. 1, 175–188

[25] G.Piser, Martingales with values in uniformly convex spaces, Israel J.Math. 20(1975), 326-350