ON TRANSONIC SHOCKS IN A NOZZLE WITH VARIABLE END PRESSURES

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In the book [8], Courant and Friedrichs described the following transonic shock phenomena in a de Laval nozzle: Given the appropriately large receiver pressure p_r, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle a shock front intervenes and the gas is compressed and slowed down to subsonic speed. The position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes p_r. When the end pressure p_r varies and lies in an appropriate scope, in general, it is expected that a curved transonic shock is still formed in a nozzle. In this paper, we solve this problem for the two-dimensional steady Euler system with a variable exit pressure in a nozzle whose divergent part is an angular sector. Both existence and uniqueness are established.

Keywords: Steady Euler system, supersonic flow, subsonic flow, transonic shock, nozzle

Mathematical Subject Classification: 35L70, 35L65, 35L67, 76N15

§1. Introduction and main results

This paper concerns with the transonic shock problem in a nozzle when the given variable end pressure at the exit of the nozzle lies in an appropriate scope. In [22-25], the authors have studied the well-posedness or ill-posedness of a transonic shock for the supersonic flow through a general 2-D or 3-D slowly-varying nozzle with an appropriately large exit pressure. However, the end pressure or the position of the shock in [22-25] are either induced by the appropriate boundary conditions on the exit or determined by the ordinary differential equations which are resulted from the assumptions on symmetric properties of the supersonic incoming flow, the nozzle walls and the end pressure. In this paper, under the more natural and physical boundary condition (i.e. the variable exit pressure in a suitable scope), we will study the transonic shock problem when a supersonic flow goes through a 2-D curved nozzle with a straight diverging part. In particular, we will verify the following transonic shock phenomena for the steady Euler flow as illustrated in [8]: Given the appropriately large receiver pressure $p_e(x)$, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle a shock front intervenes

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX and the gas is compressed and slowed down to subsonic speed, moreover, the position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes $p_e(x)$.

To simplify the presentation, we only consider the isentropic gases. In fact, by a slight modification, our discussions are also available to the non-isentropic case. The steady isentropic Euler system in two dimensional space is

$$\begin{cases} \partial_1(\rho u_1) + \partial_2(\rho u_2) = 0, \\ \partial_1(P + \rho u_1^2) + \partial_2(\rho u_1 u_2) = 0, \\ \partial_1(\rho u_1 u_2) + \partial_2(P + \rho u_2^2) = 0, \end{cases} \tag{1.1}$$

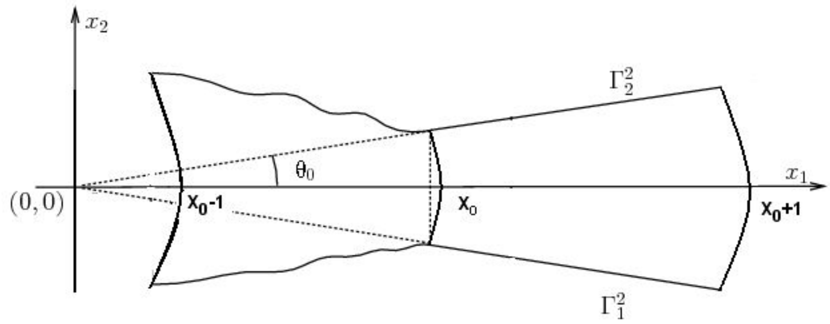
where $u = (u_1, u_2), \rho$ and P are the velocity, the density and the pressure respectively. Moreover, the pressure function $P = P(\rho)$ is smooth with $P'(\rho) > 0$ for $\rho > 0$, and $c(\rho) = \sqrt{P'(\rho)}$ being the sound speed.

For the ideal polytropic gas, the equation of state is given by

$$P = A\rho^\gamma,$$

here A and γ are positive constants, and $1 < \gamma < 3$ (especially $\gamma \approx 1.4$ with respect to the air).

Assume that the nozzle walls Γ_1 and Γ_2 are $C^{4,\alpha}-$regular for $X_0 - 1 \le r = \sqrt{x_1^2 + x_2^2} \le X_0 + 1$ ($X_0 > 0$ is a fixed constant and the constant $\alpha \in (0,1)$) and Γ_i consists of two curves Γ_i^1 and Γ_i^2 with Γ_1^1 and Γ_2^1 including the walls for the converging part of the nozzle, while Γ_1^2 and Γ_2^2 being straight line segments so that the divergent part of the nozzle is part of a symmetric angular sector. Assume that Γ_i^2 is represented by $x_2 = (-1)^i tg\theta_0 x_1$ with $x_1 > 0$ and $X_0 < r < X_0+1$, where $0 < \theta_0 < \frac{\pi}{2}$ is sufficiently small. Furthermore, it is assumed that the $C^{4,\alpha}-$smooth supersonic incoming flow $(\rho_0^-(x), u_{1,0}^-(x), u_{2,0}^-(x))$ is symmetric near $r = X_0$ so that $\rho_0^-(x) = \rho_0^-(r)$ and $u_{i,0}^-(x) = \dfrac{U_0^-(r)x_i}{r}(i = 1, 2)$ near $r = X_0$(this assumption can be easily realized by the hyperbolicity of the supersonic incoming flow and the symmetric property of the nozzle walls for $X_0 < r < X_0 + 1$, one can see [13]).



Suppose that the possible shock Σ and the flow field state behind Σ are denoted by $x_1 = \eta(x_2)$ and $(\rho^+(x), u_1^+(x), u_2^+(x))$ respectively. Then the Rankine-Hugoniot conditions on Σ become

$$\begin{cases} [\rho u_1] - \eta'(x_2)[\rho u_2] = 0, \\ [P + \rho u_1^2] - \eta'(x_2)[\rho u_1 u_2] = 0, \\ [\rho u_1 u_2] - \eta'(x_2)[P + \rho u_2^2] = 0. \end{cases} \tag{1.2}$$

In addition, $P^+(x)$ satisfies the physical entropy condition (see [8]):

$$P^+(x) > P^-(x) \qquad on \qquad x_1 = \eta(x_2). \tag{1.3}$$

On the exit of the nozzle, the end pressure is prescribed by

$$P^+(x) = P_e + \varepsilon P_0(\theta) \qquad on \qquad r = X_0 + 1, \tag{1.4}$$

here $\varepsilon > 0$ is sufficiently small, $\theta = arctan\frac{x_2}{x_1}$, $P_0(\theta) \in C^{3,\alpha}[-\theta_0, \theta_0]$ with

$$P_0'(\pm\theta_0) = P_0^{(3)}(\pm\theta_0) = 0, \qquad \|P_0(arctan\frac{x_2}{x_1})\|_{C^{3,\alpha}\{(x_1,x_2):\sqrt{x_1^2+x_2^2}=X_0+1,|x_2|\le x_1 tan\theta_0\}} \le C,$$

the constant P_e denotes the end pressure for which a symmetric shock lies at the position $r = r_0$ with $r_0 \in (X_0, X_0+1)$ and the supersonic incoming flow is given by $(\rho_0^-(r), U_0^-(r))$, for more details, one can see Proposition 2.1 in §2.

Since the flow is tangent to the nozzle walls $x_2 = (-1)^i tg\theta_0 x_1 (i = 1, 2)$, then

$$u_2^+ = (-1)^i tg\theta_0 u_1^+ \qquad on \qquad x_2 = (-1)^i tg\theta_0 x_1. \tag{1.5}$$

Finally, X_0 and θ_0 are assumed to be suitably large and small respectively so that

$$X_0\theta_0 = 1 \qquad and \qquad \frac{\eta_0}{2} < \theta_0 < \eta_0 \tag{1.6}$$

with $\eta_0 > 0$ being a suitably small constant.

It is noted here that the assumption (1.6) implies that the nozzle wall $\Gamma_i^2 : x_2 = (-1)^i tg\theta_0 x_1$ is close to the line segment $x_2 = (-1)^i$ for $X_0 \le r \le X_0 + 1$.

As will be shown in §2, under the above assumptions on the nozzle and the symmetric supersonic incoming flow near the throat of the nozzle, there exists a unique symmetric transonic shock solution for the given constant end pressure. Furthermore, the position of the shock location, $r = r_0$, depends monotonically on the given end pressure. This solution will be the background solution. Let $(P_0^+(r), U_0^+(r))$ be the subsonic part of the background solution for $r_0 < r < X_0 + 1$, which can be extended into the domain $\{r : X_0 \le r \le X_0 + 1\}$ and the extension will be denoted by $(\hat{P}_0^+(r), \hat{U}_0^+(r))$. For more details, one can see Proposition 2.1 and Remark 2.2 in §2.

The first main result in this paper is

Theorem 1.1. (Uniqueness)

Let the assumptions above hold and $M_0^-(X_0) \equiv \frac{U_0^-(X_0)}{c(\rho_0^-(X_0))} > \sqrt{\frac{2^{\gamma+1}-2}{\gamma}}$. *Then there exists a constant* $\varepsilon_0 = \frac{1}{X_0^3}$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$, *the problem (1.1)-(1.5) has no more than one solution* $(P^+(x), u_1^+(x), u_2^+(x); \eta(x_2))$ *with the following properties:*

(i). $\eta(x_2) \in C^{4,\alpha}[x_2^1, x_2^2]$, *and*

$$\|\eta(x_2) - \sqrt{r_0^2 - x_2^2}\|_{L^\infty[x_2^1,x_2^2]} \le C_0 X_0 \varepsilon, \qquad \|\big(\eta(x_2) - \sqrt{r_0^2 - x_2^2}\big)'\|_{C^{3,\alpha}[x_2^1,x_2^2]} \le C_0\varepsilon,$$

where $(x_1^i, x_2^i)(i = 1, 2)$ *stands for the intersection points of* $x_1 = \eta(x_2)$ *with* $x_2 = (-1)^i tg\theta_0 x_1$ *for* $i = 1, 2$, *and* C_0 *is a positive constant depending on* α *and the supersonic incoming flow.*

(ii). $(P^+(x),\ u_1^+(x), u_2^+(x)) \in C^{3,\alpha}(\bar{\Omega}_+)$, *and*

$$\|(P^+, u_1^+, u_2^+) - (\hat{P}_0^+, \hat{u}_{1,0}^+, \hat{u}_2^+)\|_{C^{3,\alpha}(\bar{\Omega}_+)} \le C_0\varepsilon,$$

where Ω_+ *is the subsonic region given by*

$$\Omega_+ = \{(x_1, x_2) : \eta(x_2) < x_1 < \sqrt{(X_0+1)^2 - x_2^2}, |x_2| < tg\theta_0 x_1\},$$

and $(\hat{P}_0^+, \hat{u}_{1,0}^+, \hat{u}_2^+) = (\hat{P}_0^+(r), \hat{U}_0^+(r)\frac{x}{r})$.

Remark 1.1. *Besides the uniqueness result described by Theorem 1.1, it will be also shown that the position of the shock depends on the given end pressure monotonically. This will be stated more precisely in Proposition 3.2 in §3. In addition, the order* $X_0\varepsilon$ *in the assumption of* $\|\eta(x_2) - \sqrt{r_0^2 - x_2^2}\|_{L^\infty[x_2^1,x_2^2]}$

comes essentially from the relation between the shock position and the end pressure (see Proposition 5.3 and Remark 5.2 in §5*). This implies that the shock position will move with the order $X_0 O(\varepsilon)$ when the end pressure changes in an order $O(\varepsilon)$ in (1.4).*

Remark 1.2. *Due to the corners in the subsonic region, the requirement of $C^{3,\alpha}$ regularity for the uniqueness in Theorem 1.1 seems stringent. However, based on such a uniqueness, we can obtain the existence of a transonic solution in the same regularity class, see Theorem 1.2 below and Appendix.*

Remark 1.3. *The previous uniqueness results in [22, 24-25] require that either the transonic shock curve goes through a fixed point or one solution has special symmetry.*

Remark 1.4. *The assumption that $r_0 \in (X_0, X_0+1)$ is made to ensure a positive distance between the shock position and the exit of the nozzle, this will be used in the analysis in* §4.

Based on Theorem 1.1, we can establish the following existence theorem.

Theorem 1.2. (Existence of a transonic shock solution)

For the nozzle and the supersonic incoming flow defined as above, the problem (1.1) with (1.2)-(1.5) has a unique transonic shock solution such that $(\rho^+, u_1^+, u_2^+; \eta(x_2))$ satisfies the estimates in Theorem 1.1.

Remark 1.5. *It should be noted that similar results as in Theorem 1.1 and Theorem 1.2 hold for the 2-D Euler flows with the pressure depending on both the density and the entropy. However, neither the uniqueness nor the existence as in Theorem 1.1 and Theorem 1.2 holds for irrotational flows, see [22, 24].*

Remark 1.6. *Theorem 1.1 and Theorem 1.2 can be extended into the 3-D nozzle case, which will be given in our forthcoming paper [15].*

Remark 1.7. *For nozzles with part of symmetric angular sector as the converging part, one can also establish a corresponding theory of uniqueness and existence of a transonic shock lying in the converging part of the nozzle as Theorem 1.1 and 1.2. However, it is shown in [25] that such a transonic shock is dynamically unstable.*

It is noted that there have been many works on the steady transonic problems (see [3-9], [12], [14], [18-28] and the references therein). In particular, in [25], it is shown that under the same assumptions as in this paper on the nozzle and the supersonic incoming flows, there exist two constant pressures P_1 and P_2 with $P_1 < P_2$, such that if the exit constant pressure $P_e \in (P_1, P_2)$, then the symmetric transonic shock exists uniquely in the diverging part of the nozzle, and the position and the strength of the shock is completely determined by P_e. More importantly, we establish the dynamically asymptotic stability of such a transonic shock for the unsteady Euler system in both 2-D and 3-D cases. Various related results, such as uniqueness with additional assumptions, non-existence, compatibility conditions, and many useful analytical tools on transonic shocks in slightly curved nozzles with given appropriate end pressure at the exit of the nozzle for either steady irrotational flows or steady Euler flows have been established in [22-25], see also [5-7, 26].

Next we would like to comment on the proofs of the main results in this paper. In should be noted that in almost all the previous works mentioned above except [15, 22], the uniqueness is obtained under the additional assumption that the shock curve goes through a fixed point in advance [7, 22-26]. This condition is crucial in the proofs. However, for non-flat nozzles, this additional condition may lead the transonic shock problem to be over-determined [22]. In the present paper, we have found a new way to determine the position of the transonic shock and remove the undesired assumption that the shock curve goes through a fixed point so that the transonic shock problem as described by Courant-Friedrichs is well-posed. Besides the analytical tools developed in [22-25], the new key ingredients in this paper are to establish the monotonic property of the pressure along the nozzle walls and to estimate the gradients of the solution instead of the solution itself so that one can avoid the difficulties induced by the unknown position of the shock, for more detailed explanations, see §5. It follows from this that the position of the shock can be uniquely determined in Theorem 1.1 when the end pressure is given, and the continuous dependence and monotonic property of the end pressure on the position of the shock are also derived. With these crucial results, we can complete the proofs on Theorem 1.1 and Theorem 1.2.

The rest of the paper is organized as follows. In §2, for reader's convenience, the description of the background solution is given although it has been done in [25]. In §3, we reformulate the 2-D problem (1.1) with the boundary conditions (1.2)-(1.5) so that one can obtain a weakly coupled second order elliptic equation for the density ρ with mixed boundary conditions, a 2×2 first order system for the angular velocity U_2 and an algebraic equation on (ρ, u_1, u_2) along a streamline. In §4, using the decomposition techniques in §3, we establish some a priori estimates on the derivatives of difference between two possible solutions. In §5, based on the estimates given in §4, and through looking for an ordinary differential inequality along a nozzle wall, we can derive that the end pressure on the wall is monotonic with respect to the position of the shock, thus the position of the shock can be uniquely determined by the end pressure and the proof of the uniqueness result in Theorem 1.1 is then completed. In §6, along the nozzle walls, by establishing the continuous dependence of the shock position on the end pressure, we can determine the position of the shock and complete the proof on Theorem 1.2. Finally, in Appendix, we will give a proof on the existence of a transonic shock $C^{3,\alpha}$ solution when the transonic shock is assumed to go through a fixed point and the end pressure is suitably adjusted by a constant.

In what follows, we will use the following convention:

$O(\varepsilon)$ means that there exists a generic constant $C_1 > 0$ independent of X_0 and ε such that $\|O(\varepsilon)\|_{C^{2,\alpha}} \leq C_1\varepsilon$.

$O(\frac{1}{X_0^m})(m > 0)$ means that there exists a generic constant $C_2 > 0$ independent of X_0 and ε such that $\|O(\frac{1}{X_0^m})\|_{C^{1,\alpha}} \leq \frac{C_2}{X_0^m}$.

§2. **The background solution.**

In this section, we will describe the transonic solution to the problem (1.1) with (1.2)-(1.5) when the end pressure is a suitable constant P_e under the assumptions on the nozzle walls and the supersonic incoming flow in §1. Such a solution is called the background solution and can be obtained by solving the related ordinary differential equations. In fact, the related analysis has been given in Section 147 of [8] and the details can be seen in [22, 25]. But for the reader's convenience and later computations in this paper, we still outline it here.

Proposition 2.1. (Existence of a transonic shock for the constant end pressure) *For the 2-D nozzle and the supersonic incoming flow given in §1, there exist two constant pressures P_1 and P_2 with $P_1 < P_2$, which are determined by the incoming flow and the nozzle, such that if the end pressure $P_e \in (P_1, P_2)$, then the system (1.1) has a symmetric transonic shock solution*

$$(P, u_1, u_2) = \begin{cases} (P_0^-(r), u_{1,0}^-(x), u_{2,0}^-(x)), & \textit{for } r < r_0, \\ (P_0^+(r), u_{1,0}^+(x), u_{2,0}^+(x)), & \textit{for } r > r_0, \end{cases}$$

here $u_{i,0}^{\pm}(x) = U_0^{\pm}(r)\frac{x_i}{r}(i = 1, 2)$, and $(P_0^{\pm}(r), U_0^{\pm}(r))$ are $C^{4,\alpha}-$ smooth. Moreover, the position $r = r_0$ with $X_0 < r_0 < X_0 + 1$ and the strength of the shock are determined by P_e.

Proof. Let $r = r_0$ be the location of the shock which to be found. It follows from (1.1) that the supersonic incoming flow $(\rho_0^-(r), U_0^-(r))(X_0 < r < r_0)$ and the subsonic flow $(\rho_0^+(r), U_0^+(r))(r_0 < r < X_0 + 1)$ satisfy respectively

$$\begin{cases} \frac{d}{dr}(r\rho_0^{\pm}U_0^{\pm}) = 0, \\ \frac{1}{2}(U_0^{\pm})^2 + h(\rho_0^{\pm}) = \frac{1}{2}(U_0^{\pm}(r_0))^2 + h(\rho_0^{\pm}(r_0)), \end{cases} \tag{2.1}$$

here $h(\rho_0^{\pm})$ is the enthalpy given by $h'(\rho_0^{\pm}) = \frac{c^2(\rho_0^{\pm})}{\rho_0^{\pm}}$.

The corresponding Rankine-Hugoniot conditions across the shock $r = r_0$ are

$$\begin{cases} [\rho_0 U_0] = 0, \\ [\rho_0 U_0^2 + P_0] = 0. \end{cases} \tag{2.2}$$

As in the [22, 25], the proof of the Proposition can be divided into four steps.

Step 1. For the supersonic state $(\rho_0^-(r_0), U_0^-(r_0))$, there exists a unique subsonic state $(\rho_0^+(r_0), U_0^+(r_0))$ satisfying (2.2).

The proof can be found in Section 147 of [8], so it is omitted here.

Step 2. For a given supersonic state $(\rho_0^-(X_0), U_0^-(X_0))$, (2.1) has a unique supersonic solution $(\rho_0^-(r), U_0^-(r))$ for $r \in [X_0, X_0+1]$.

In fact, it follows from (2.1) that

$$\begin{cases} f_1(\rho_0^-, U_0^-, r) \equiv r\rho_0^-(r)U_0^-(r) - C_0 = 0, \\ f_2(\rho_0^-, U_0^-, r) \equiv \frac{1}{2}(U_0^-(r))^2 + h(\rho_0^-(r)) - C_1^- = 0 \end{cases}$$

with $C_0 = X_0\rho_0^-(X_0)U_0^-(X_0)$ and $C_1^- = \frac{1}{2}(U_0^-(X_0))^2 + h(\rho_0^-(X_0))$.

Since

$$\begin{cases} \dfrac{d\rho_0^-}{dr} = -\dfrac{\rho_0^-(U_0^-)^2}{r((U_0^-)^2 - c^2(\rho_0^-))}, \\ \dfrac{dU_0^-}{dr} = \dfrac{U_0^-c^2(\rho_0^-)}{r((U_0^-)^2 - c^2(\rho_0^-))} \end{cases}$$

and

$$\frac{d((U_0^-)^2 - c^2(\rho_0^-))}{dr} = \frac{\big(2P'(\rho_0^-) + \rho_0^-P''(\rho_0^-)\big)(U_0^-)^2}{r((U_0^-)^2 - c^2(\rho_0^-))},$$

then one has

$$(U_0^-(r))^2 - c^2(\rho_0^-(r)) \geq \frac{1}{2}\Big((U_0^-(X_0))^2 - c^2(\rho_0^-(X_0))\Big) > 0 \qquad \text{for } X_0 \leq r \leq X_0+1. \tag{2.3}$$

In addition $\dfrac{\partial(f_1, f_2)}{\partial(\rho_0^-, U_0^-)} = r\big((U_0^-(r))^2 - c^2(\rho_0^-(r))\big)$ and $\dfrac{\partial(f_1, f_2)}{\partial(\rho_0^-, U_0^-)}\Big|_{\rho_0^-(X_0), U_0^-(X_0), X_0} > 0$. This, together with the implicit function theorem and (2.3), yields that (2.1) has a unique supersonic solution $(\rho_0^-(r), U_0^-(r))$ for $r \in [X_0, X_0+1]$.

Step 3. For a given subsonic state $(\rho_0^+(X_0), U_0^+(X_0))$, (2.1) has a unique subsonic solution $(\rho_0^+(r), U_0^+(r))$ for $r \in [X_0, X_0+1]$.

The proof is similar to that in Step 2, so is omitted.

Step 4. The shock position r_0 is a continuously decreasing function of P_e when the end pressure P_e lies in an appropriate scope.

In fact, it follows from (2.1) and (2.2) that for $r \in [X_0, X_0+1]$

$$\begin{cases} r\rho_0^\pm(r)U_0^\pm(r) \equiv C_0, \\ \frac{1}{2}(U_0^\pm(r))^2 + h(\rho_0^\pm(r))) \equiv C_1^\pm, \end{cases} \tag{2.4}$$

here $C_1^\pm$ are the Bernoulli's constants. Note that C_1^- and C_1^+ are different in general, moreover, C_1^+ depends on the end pressure $P_0^+(X_0+1) = P_e$.

Especially,

$$\begin{cases} X_0\rho_0^\pm(X_0)U_0^\pm(X_0) \equiv C_0, \\ \frac{1}{2}(U_0^\pm(X_0))^2 + h(\rho_0^\pm(X_0)) \equiv C_1^\pm. \end{cases} \tag{2.5}$$

Next we derive the dependence of r_0 on the end pressure $P_0^+(X_0+1) = P_e$.

It follows from the first equation in (2.4) and the second equation in (2.5) that

$$\begin{cases} \dfrac{d(\rho_0^\pm(r_0)U_0^\pm(r_0))}{d\rho_0^+(X_0)} = -\rho_0^\pm(r_0)U_0^\pm(r_0)\dfrac{dr_0}{r_0 d\rho_0^+(X_0)}, \\ U_0^+(r_0)\dfrac{dU_0^+(r_0)}{d\rho_0^+(X_0)} + \dfrac{c^2(\rho_0^+(r_0))}{\rho_0^+(r_0)}\dfrac{d\rho_0^+(r_0)}{d\rho_0^+(X_0)} = \dfrac{dC_1^+}{d\rho_0^+(X_0)}. \end{cases} \tag{2.6}$$

In addition, by $C_1^+=\frac{C_0^2}{2X_0^2(\rho_0^+(X_0+1))^2}+h(\rho_0^+(X_0+1))$, the second equation in (2.2) and (2.6), one has

$$[\rho_0U_0^2]\frac{dr_0}{r_0d\rho_0^+(X_0+1)}=\rho_0^+(r_0)\frac{dC_1^+}{d\rho_0^+(X_0+1)}=\frac{\rho_0^+(r_0)(c_0^2(X_0+1)-(U_0^+(X_0+1))^2)}{\rho_0^+(X_0+1)}. \tag{2.7}$$

Since $[\rho_0U_0^2]<0$ due to $[\rho_0U_0^2+P_0]=0$ and $[P_0]>0$, then (2.7) implies that r_0 is a continuous and strictly decreasing function of the end pressure $P_0^+(X_0+1)$.

Next, we complete the proof on Proposition 2.1.

For $r_0\in[X_0,X_0+1)$, it follows from Step 2 that there exists a unique supersonic flow in $[X_0,r_0]$. Moreover, due to Step 1 and Step 3, there exists a unique shock at r_0 and a unique subsonic flow in $[r_0,X_0+1]$. Thus one can define a function $F(r_0)=P_0^+(X_0+1)$ for $r_0\in(X_0,X_0+1)$. By Step 4, $F(r_0)$ is a strictly decreasing and continuous differentiable function on $P_0^+(X_0+1)$. When $r_0=X_0$ or $r_0=X_0+1$, one can obtain two different end pressures P_1 and P_2 with $P_1<P_2$. Therefore, by the monotonicity of $F(r_0)$, one can obtain a symmetric transonic shock for $P_0^+(X_0+1)\equiv P_e\in(P_1,P_2)$.

Remark 2.1. *By the assumption (1.6) and the equation (2.4), one can easily conclude that there exists a constant $C>0$ independent of X_0 such that for $X_0\le r\le X_0+1$*

$$\left|\frac{d^kU_0^+(r)}{dr^k}\right|+\left|\frac{d^kP_0^+(r)}{dr^k}\right|\le\frac{C}{X_0^k},\qquad k=1,2,3,4. \tag{2.8}$$

Remark 2.2. *It follows from Step 2 and Remark 2.1 that one can extend the subsonic flow $(\rho_0^+(r),U_0^+(r))$ into $(\hat\rho_0^+(r),\,\hat U_0^+(r))$ defined for $r\in(X_0,X_0+1)$ and satisfies (2.1) on (X_0,X_0+1). This extension will be often used later on.*

§3. The reformulation on problem (1.1) with (1.2)-(1.5)

In this section, the nonlinear problem (1.1) with (1.2)-(1.5) will be reformulated so that one can obtain a second order elliptic equation for $\rho^+(x)$ and two 2×2 first order systems for the radial speed U_1^+ and the angular speed U_2^+. To this end, as in [22, 25], we firstly derive the relations between (P^+,U_1^+) and U_2^+ on the shock Σ. It is more convenient to use the polar coordinates

$$\begin{cases} x_1=rcos\theta,\\ x_2=rsin\theta\end{cases} \tag{3.1}$$

and decompose (u_1^+,u_2^+) as

$$\begin{cases} u_1^+=U_1^+cos\theta-U_2^+sin\theta,\\ u_2^+=U_1^+sin\theta+U_2^+cos\theta.\end{cases} \tag{3.2}$$

Then, (1.1) and (1.2) become respectively

$$\begin{cases} \partial_r(\rho^+U_1^+)+\frac{1}{r}\partial_\theta(\rho^+U_2^+)+\frac{\rho^+U_1^+}{r}=0,\\ \partial_r(\rho^+(U_1^+)^2+P^+)+\frac{1}{r}\partial_\theta(\rho^+U_1^+U_2^+)+\frac{\rho^+((U_1^+)^2-(U_2^+)^2)}{r}=0,\\ \partial_r(\rho^+U_1^+U_2^+)+\frac{1}{r}\partial_\theta(P^++\rho^+(U_2^+)^2)+\frac{2}{r}\rho^+U_1^+U_2^+=0\end{cases} \tag{3.3}$$

and

$$\begin{cases} [\rho U_1]-\frac{\tilde r'(\theta)}{\tilde r(\theta)}[\rho U_2]=0,\\ [\rho U_1^2+P]-\frac{\tilde r'(\theta)}{\tilde r(\theta)}[\rho U_1U_2]=0,\\ [\rho U_1U_2]-\frac{\tilde r'(\theta)}{\tilde r(\theta)}[P+\rho U_2^2]=0,\end{cases} \tag{3.4}$$

where $r=\tilde{r}(\theta)$ stands for the shock Σ in the coordinate (r,θ).

In addition, for any C^1 solution, (3.3) is equivalent to

$$\begin{cases} \partial_r(\rho^+U_1^+)+\frac{1}{r}\partial_\theta(\rho^+U_2^+)+\frac{\rho^+U_1^+}{r}=0, \\ U_1^+\partial_rU_1^+ + \frac{U_2^+}{r}\partial_\theta U_1^+ + \frac{\partial_rP^+}{\rho^+}-\frac{(U_2^+)^2}{r}=0, \\ U_1^+\partial_rU_2^+ + \frac{U_2^+}{r}\partial_\theta U_2^+ + \frac{1}{r}\frac{\partial_\theta P^+}{\rho^+}+\frac{U_1^+U_2^+}{r}=0. \end{cases} \tag{3.5}$$

It follows from (3.4) that on $r=\tilde{r}(\theta)$

$$\begin{cases} (\rho^+-\hat{\rho}_0^+(r_0))\hat{U}_0^+(r_0)+\hat{\rho}_0^+(r_0)(U_1^+-\hat{U}_0^+(r_0))=g_1, \\ (P^+-\hat{P}_0^+(r_0))+(\rho^+-\hat{\rho}_0^+(r_0))(U_1^+)^2+2\hat{\rho}_0^+(r_0)\hat{U}_0^+(r_0)(U_1^+-\hat{U}_0^+(r_0))=g_2, \end{cases}$$

here

$$\begin{aligned} g_1&=\frac{(\rho^+)^2U_1^+(U_2^+)^2}{[P]+\rho^+(U_2^+)^2}-(\rho_0^-(r_0)U_0^-(r_0)-\rho_0^-U_0^-)-(\rho^+-\hat{\rho}_0^+(r_0))(U_1^+-\hat{U}_0^+(r_0)), \\ g_2&=\frac{(\rho^+U_1^+U_2^+)^2}{[P]+\rho^+(U_2^+)^2}-(P_0^-(r_0)-P^-)-(\rho_0^-(r_0)(U_0^-(r_0))^2-\rho_0^-(U_0^-)^2)-\hat{\rho}_0^+(r_0)(U_1^+-\hat{U}_0^+(r_0))^2. \end{aligned}$$

Thus, a direct computation yields that on $r=\tilde{r}(\theta)$

$$\begin{cases} U_1^+-\hat{U}_0^+(r_0)=\tilde{g}_1((U_2^+)^2,P_0^--P_0^-(r_0),U_0^--U_0^-(r_0)), \\ P^+-\hat{P}_0^+(r_0)=\tilde{g}_2((U_2^+)^2,P_0^--P_0^-(r_0),U_0^--U_0^-(r_0)) \end{cases} \tag{3.6}$$

with $\tilde{g}_i(0,0,0)=0$ for $i=1,2$, here the quantities on the right hand sides of the above will be suitably small.

For convenience, the following transformation

$$\begin{cases} y_1=r, \\ y_2=X_0\theta, \end{cases} \tag{3.7}$$

will be used to change the fixed walls into $y_2=\pm1$ respectively.

In addition, the superscripts "+" will be neglected for convenience in case no confusions.

Then, (3.5) and (3.4) can be rewritten respectively as

$$\begin{cases} \partial_{y_1}(\rho U_1)+\frac{X_0}{y_1}\partial_{y_2}(\rho U_2)+\frac{\rho U_1}{y_1}=0, \\ U_1\partial_{y_1}U_1+\frac{X_0U_2}{y_1}\partial_{y_2}U_1+\frac{\partial_{y_1}P}{\rho}-\frac{U_2^2}{y_1}=0, \\ U_1\partial_{y_1}U_2+\frac{X_0U_2}{y_1}\partial_{y_2}U_2+\frac{X_0}{y_1}\frac{\partial_{y_2}P}{\rho}+\frac{U_1U_2}{y_1}=0. \end{cases} \tag{3.8}$$

and

$$\begin{cases} [\rho U_1]-\frac{X_0\xi'(y_2)}{\xi(y_2)}[\rho U_2]=0, \\ [\rho U_1^2+P]-\frac{X_0\xi'(y_2)}{\xi(y_2)}[\rho U_1U_2]=0, \\ [\rho U_1U_2]-\frac{X_0\xi'(y_2)}{\xi(y_2)}[P+\rho U_2^2]=0, \end{cases} \tag{3.9}$$

where $\xi(y_2)=\tilde{r}(\frac{y_2}{X_0})$.

Note that the transformation between the coordinate systems (x_1, x_2) and (y_1, y_2) keeps the equivalence of $C^{4,\alpha}$ norm independent of X_0. So from now on, we will use the coordinate system (y_1, y_2) instead of (x_1, x_2).

Let $y_2 = y_2(y_1, \beta)$ be the characteristics starting from the point $(\xi(\beta), \beta)$ for the first order differential operator $U_1\partial_{y_1} + \frac{X_0 U_2}{y_1}\partial_{y_2}$. Then

$$\begin{cases} \dfrac{dy_2(y_1,\beta)}{dy_1} = \dfrac{X_0}{y_1}(\dfrac{U_2}{U_1})(y_1, y_2(y_1,\beta)), \\ y_2(\xi(\beta),\beta) = \beta, \qquad \beta \in [-1,1]. \end{cases} \tag{3.10}$$

It follows from the second and the third equations in (3.8) that the following Bernoulli's law holds

$$\left(\frac{1}{2}(U_1)^2 + \frac{1}{2}(U_2)^2 + h(\rho)\right)(y_1, y_2(y_1,\beta)) = G_0(\beta) \tag{3.11}$$

with

$$G_0(\beta) = \frac{1}{2}(U_1(\xi(\beta),\beta))^2 + \frac{1}{2}(U_2(\xi(\beta),\beta))^2 + h(\rho)(\xi(\beta),\beta).$$

Next, we derive the governing problems for U_1 and U_2.

It follows from (3.11) that

$$\begin{cases} U_1\partial_{y_2}U_1 + U_2\partial_{y_2}U_2 + \dfrac{\partial_{y_2}P}{\rho} = \frac{d}{d\beta}G_0(\beta)\partial_{y_2}\beta(y_1,y_2), & (3.12) \\ U_1\partial_{y_1}U_1 + U_2\partial_{y_1}U_2 + \frac{\partial_{y_1}P}{\rho} = -\dfrac{X_0}{y_1}\left(\dfrac{U_2}{U_1}\right)(y_1,y_2(y_1,\beta))\frac{d}{d\beta}G_0(\beta)\partial_{y_2}\beta(y_1,y_2), & (3.13) \end{cases}$$

here $\beta(y_1, y_2)$ denotes the inverse function of $y_2 = y_2(y_1, \beta)$.

Combining (3.12)-(3.13) with the first equation and the third equation in (3.8) yields

$$\begin{cases} \partial_{y_1}U_1 = h_1(P, U_1, U_2, \partial_{y_1}P, \partial_{y_2}P), \\ \partial_{y_2}U_1 = h_2(P, U_1, U_2, \partial_{y_1}P, \partial_{y_2}P) \end{cases} \tag{3.14}$$

and

$$\begin{cases} \partial_{y_1}U_2 = h_3(P, U_1, U_2, \partial_{y_1}P, \partial_{y_2}P), \\ \partial_{y_2}U_2 = h_4(P, U_1, U_2, \partial_{y_1}P, \partial_{y_2}P), \\ U_2(y_1, \pm 1) = 0, \end{cases} \tag{3.15}$$

here $h_i = \frac{\Delta_i}{\Delta_0}(1 \le i \le 4)$ with

$$
\begin{aligned}
\Delta_0 =&\frac{(U_1)^2+(U_2)^2}{y_1},\\
\Delta_1 =&\frac{U_2}{y_1}\bigg(\frac{X_0\partial_{y_2}P}{y_1\rho}+\frac{U_1U_2}{y_1}\bigg)-\frac{U_1}{y_1}\bigg(\frac{\partial_{y_1}P}{\rho}+\frac{X_0U_2}{y_1U_1}(\frac{d}{d\beta}G_0)(\beta(y_1,y_2))\partial_{y_2}\beta(y_1,y_2)\bigg)\\
&-\frac{(U_2)^2}{y_1}\bigg(\frac{U_1}{y_1}+\frac{1}{\rho c^2(\rho)}\big(U_1\partial_{y_1}P+\frac{X_0U_2}{y_1}\partial_{y_2}P\big)\bigg),\\
\Delta_2 =&\bigg(\frac{U_1}{y_1}+\frac{(U_2)^2}{y_1U_1}\bigg)\bigg((\frac{d}{d\beta}G_0)(\beta(y_1,y_2))\partial_{y_2}\beta(y_1,y_2)-\frac{\partial_{y_2}P}{\rho}\bigg)+\frac{U_2}{X_0}\bigg(\frac{U_1}{\rho c^2(\rho)}(U_1\partial_{y_1}P+\frac{X_0U_2}{y_1}\partial_{y_2}P)\\
&-\frac{X_0U_2}{y_1U_1}(\frac{d}{d\beta}G_0)(\beta(y_1,y)))\partial_{y_2}\beta(y_1,y_2)\bigg)+\frac{(U_2)^2}{X_0U_1}\bigg(\frac{X_0\partial_{y_2}P}{y_1\rho}+\frac{U_1U_2}{y_1}\bigg)+(\frac{(U_1)^2}{y_1}-\frac{\partial_{y_1}P}{\rho})\frac{U_2}{X_0},\\
\Delta_3 =&\frac{U_1U_2}{y_1\rho c^2(\rho)}(U_1\partial_{y_1}P+\frac{X_0U_2}{y_1}\partial_{y_2}P)-\frac{X_0U_1}{y_1^2\rho}\partial_{y_2}P-\frac{U_2}{y_1}\bigg(\frac{\partial_{y_1}P}{\rho}+\frac{X_0U_2}{y_1U_1}(\frac{d}{d\beta}G_0)(\beta(y_1,y_2))\partial_{y_2}\beta(y_1,y_2)\bigg),\\
\Delta_4 =&-\frac{(U_1)^2}{\rho c^2(\rho)}(\frac{U_1}{X_0}\partial_{y_1}P+\frac{U_2}{y_1}\partial_{y_2}P)-U_2\bigg(\frac{\partial_{y_2}P}{y_1\rho}+\frac{U_1U_2}{X_0y_1}\bigg)+\frac{U_2}{y_1}(\frac{d}{d\beta}G_0)(\beta(y_1,y_2))\partial_{y_2}\beta(y_1,y_2)\\
&-\bigg(\frac{(U_1)^2}{X_0y_1}-\frac{\partial_{y_1}P}{X_0\rho}\bigg)U_1.
\end{aligned}
$$

Next, we derive the governing problem for the density ρ.
It follows from $U_2=0$ on $y_2=\pm1$ and the third equation in (3.8) that

$$
\partial_{y_2}\rho=0 \qquad on \qquad y_2=\pm1. \tag{3.16}
$$

Furthermore, applying the first order operator $U_1\partial_{y_1}+\frac{X_0}{y_1}U_2\partial_{y_2}$ to the first equation in (3.8) and subsequently subtracting $\partial_{y_1}\big(\rho\times\{\text{the second equation in (3.8)}\}\big)$ and $\partial_{y_2}\big(\rho\times\{\text{the third equation in (3.8)}\}\big)$ yield the following boundary value problem for the density

$$
\begin{cases}
\partial_{y_1}\big((U_1^2-c^2(\rho))\partial_{y_1}\rho+\frac{X_0U_1U_2}{y_1}\partial_{y_2}\rho\big)+\frac{X_0}{y_1}\partial_{y_2}\big(U_1U_2\partial_{y_1}\rho+\frac{X_0}{y_1}((U_2)^2-(c(\rho))^2)\partial_{y_2}\rho\big)=\\
\qquad -\partial_{y_1}\big(\frac{X_0}{y_1}\rho U_1\partial_{y_2}U_2+\frac{\rho U_1^2}{y_1}-\frac{X_0\rho U_2}{y_1}\partial_{y_2}U_1+\frac{\rho U_2^2}{y_1}\big)-\frac{X_0}{y_1}\partial_{y_2}\big(\rho U_2\partial_{y_1}U_1-\rho U_1\partial_{y_1}U_2\big)\\
\qquad +(\partial_{y_1}U_1+\frac{X_0}{y_1}\partial_{y_2}U_2)(U_1\partial_{y_1}\rho+\frac{X_0}{y_1}U_2\partial_{y_2}\rho+\rho\partial_{y_1}U_1+\frac{X_0}{y_1}\rho\partial_{y_2}U_2+\frac{\rho U_1}{y_1}),\\
P(\rho)-\hat{P}_0^+(r_0)=\tilde{g}_2((U_2)^2,P^--P_0^-(r_0),U^--U_0^-(r_0)) \qquad on \qquad y_1=\xi(y_2),\\
\partial_{y_2}\rho=0 \qquad on \qquad y_2=\pm1,\\
P(\rho)=P_e+\varepsilon P_0(\frac{y_2}{X_0}) \qquad on \qquad y_1=X_0+1.
\end{cases} \tag{3.17}
$$

In addition, due to the third equation of (3.9), it holds that

$$
\xi'(y_2)=\frac{\xi(y_2)[\rho U_1U_2]}{X_0[\rho U_2^2+P]}. \tag{3.18}
$$

Therefore, in order to prove Theorem 1.1, it suffices to show

Theorem 3.1. *Let the assumptions of Theorem 1.1 hold. Then for $\varepsilon<\frac{1}{X_0^3}$, the free boundary value problem (3.14)-(3.15) and (3.17) with (3.9) has no more than one solution $(P(y),U_1(y),U_2(y);\xi(y_2))$ satisfying the following estimates with a uniform constant $C>0$ (depending on α and the supersonic incoming flow):*

(i). $\xi(y_2)\in C^{4,\alpha}[-1,1]$, and

$$
\|\xi(y_2)-r_0\|_{L^\infty[-1,1]}\le CX_0\sqrt{X_0}\varepsilon, \qquad \|\xi'(y_2)\|_{C^{3,\alpha}[-1,1]}\le C\varepsilon.
$$

(ii).

$$(P(y), U_1(y), U_2(y)) \in C^{3,\alpha}(\bar{\omega}_+)$$

satisfies

$$\|\partial_{y_1}^k\big((P, U_1)(y) - (\hat{P}_0^+, \hat{U}_0^+)(y_1)\big)\|_{C^\alpha(\bar{\omega}_+)} \le \frac{C}{X_0^2}, \quad k = 0, 1, 2, 3$$

and

$$\|\partial_{y_2}(P, U_1)\|_{C^{2,\alpha}(\bar{\omega}_+)} + \|U_2\|_{C^{3,\alpha}(\bar{\omega}_+)} \le C\varepsilon,$$

where $\omega_+ = \{(y_1, y_2) : \xi(y_2) < y_1 < X_0 + 1, -1 < y_2 < 1\}$.

Remark 3.1. *It is noted that (i) and (ii) in Theorem 3.1 are less restrictive than (i) and (ii) in Theorem 1.1.*

To show Theorem 3.1, as in [25], one may reduce the free boundary problem (3.14)-(3.15) and (3.17) with (3.9) into a fixed boundary value problem. Indeed, set

$$\begin{cases} z_1 = \dfrac{y_1 - \xi(y_2)}{X_0 + 1 - \xi(y_2)}, \\ z_2 = y_2. \end{cases} \tag{3.19}$$

Then the domain ω_+ becomes

$$E_+ = \{(z_1, z_2) : 0 < z_1 < 1, -1 < z_2 < 1\}. \tag{3.20}$$

For convenience, one sets

$$\begin{cases} D_0 \equiv \dfrac{1}{y_1} = \dfrac{1}{\xi(z_2) + z_1(X_0 + 1 - \xi(z_2))}, \\ D_1 \equiv \partial_{y_1} = \dfrac{1}{X_0 + 1 - \xi(z_2)}\partial_{z_1}, \\ D_2 \equiv \dfrac{X_0}{y_1}\partial_{y_2} = \dfrac{X_0(z_1 - 1)\xi'(z_2)}{(\xi(z_2) + z_1(X_0 + 1 - \xi(z_2)))(X_0 + 1 - \xi(z_2))}\partial_{z_1} + \dfrac{X_0}{\xi(z_2) + z_1(X_0 + 1 - \xi(z_2))}\partial_{z_2}. \end{cases} \tag{3.21}$$

Then, (3.9) and (3.14)-(3.15) can be rewritten respectively as

$$\begin{cases} [\rho U_1](\xi(z_2), z_2) = \dfrac{X_0\xi'(z_2)}{\xi(z_2)}[\rho U_2](\xi(z_2), z_2), \\ [\rho U_1^2 + P](\xi(z_2), z_2) = \dfrac{X_0\xi'(z_2)}{\xi(z_2)}[\rho U_1 U_2](\xi(z_2), z_2), \\ [\rho U_1 U_2](\xi(z_2), z_2) = \dfrac{X_0\xi'(z_2)}{\xi(z_2)}[P + \rho U_2^2](\xi(z_2), z_2), \end{cases} \tag{3.22}$$

$$\begin{cases} D_1 U_1 = \widetilde{h}_1(P, U_1, U_2, D_1 P, D_2 P), \\ D_2 U_1 = \widetilde{h}_2(P, U_1, U_2, D_1 P, D_2 P) \end{cases} \tag{3.23}$$

and

$$\begin{cases} D_1 U_2 = \widetilde{h}_3(P, U_1, U_2, D_1 P, D_2 P), \\ D_2 U_2 = \widetilde{h}_4(P, U_1, U_2, D_1 P, D_2 P), \\ U_2(z_1, \pm 1) = 0, \end{cases} \tag{3.24}$$

here $\widetilde{h}_i (i = 1, 2, 3, 4)$ has the same property as h_i.

In addition, (3.17) becomes

$$\begin{cases} D_1\big((U_1^2-c^2(\rho))D_1\rho+U_1U_2D_2\rho\big)+D_2\big(U_1U_2D_1\rho+(U_2^2-c^2(\rho))D_2\rho\big)= \\ \qquad -D_1(D_0\rho U_1^2+D_0\rho U_2^2)+D_0\rho U_1D_2U_2-D_1(\rho U_1)D_2U_2+D_2(\rho U_1)D_1U_2 \\ \qquad -D_0\rho U_2D_2U_1+D_1(\rho U_2)D_2U_1-D_2(\rho U_2)D_1U_1 \\ \qquad +(D_1U_1+D_2U_2)(U_1D_1\rho+U_2D_2\rho+\rho D_1U_1+\rho D_2U_2+D_0\rho U_1), \\ P(\rho)-\hat{P}_0^+(r_0)=\tilde{g}_2((U_2)^2,P^--P_0^-(r_0),U^--U_0^-(r_0)) \qquad on \qquad z_1=0, \\ \partial_{z_2}\rho=0 \qquad on \qquad z_2=\pm1, \\ P(\rho)=P_e+\varepsilon P_0(\frac{z_2}{X_0}) \qquad on \qquad z_1=1. \end{cases} \tag{3.25}$$

We conclude this section by noting that as a by-product of the analysis for Theorem 3.1 and Theorem 1.1, one can further obtain more estimates on the location of the shock and its monotonic dependence on the end pressure. Indeed, suppose that the problem (1.1) with (1.2)-(1.3) and (1.5) has two $C^{3,\alpha}$ solutions $(\rho,U_1,U_2;\xi_1)$ and $(q,V_1,V_2;\xi_2)$ which satisfy the exit pressure conditions $P_e+\varepsilon P_{01}(\theta)$ and $P_e+\varepsilon P_{02}(\theta)$ at $r=X_0+1$ respectively and admit the estimates in Theorem 1.1. In terms of the transformations (3.1), (3.7) and (3.19), the end conditions of $P(\rho)$ and $P(q)$ can be written as $P(\rho)=P_e+\varepsilon P_{01}(\frac{z_2}{X_0})$ and $P(q)=P_e+\varepsilon P_{02}(\frac{z_2}{X_0})$. Then we can arrive at

Proposition 3.2. *Under the assumptions of Theorem 1.1, if $(\rho,U_1,U_2;\xi_1)$ and $(q,V_1,V_2;\xi_2)$ are defined as above, then the following estimates hold*

$$\|\xi_1'(z_2)-\xi_2'(z_2)\|_{C^{2,\alpha}[-1,1]}\le C\big(\frac{1}{X_0^2}|\xi_1(1)-\xi_2(1)|+\varepsilon\|\frac{1}{X_0}(P_{01}'(\frac{z_2}{X_0})-P_{02}'(\frac{z_2}{X_0}))\|_{C^{1,\alpha}[-1,1]}\big) \tag{3.26}$$

and

$$\begin{aligned} &\|\rho-q\|_{C^{2,\alpha}(E_+)}+\|(U_1,U_2)-(V_1,V_2)\|_{C^{2,\alpha}(E_+)} \\ &\qquad \le C\big(\frac{1}{X_0}|\xi_1(1)-\xi_2(1)|+\varepsilon\|\frac{1}{X_0}(P_{01}'(\frac{z_2}{X_0})-P_{02}'(\frac{z_2}{X_0}))\|_{C^{1,\alpha}[-1,1]}\big), \end{aligned} \tag{3.27}$$

where E_+ is given in (3.20).

Furthermore, if $P_{01}(\frac{z_2}{X_0})=P_{02}(\frac{z_2}{X_0})+C_1$ with the constant $C_1>0$, then

$$\xi_1(z_2)<\xi_2(z_2). \tag{3.28}$$

§4. A priori estimates on the solutions of (3.22)-(3.25).

In this section, we establish some key a priori estimates on the gradients of the difference between two solutions $(\rho,U_1,U_2;\xi_1(z_2))$ and $(q,V_1,V_2;\xi_2(z_2))$ to the problem (3.22)-(3.25) with exit pressure conditions $P_e+\varepsilon P_{01}(\frac{z_2}{X_0})$ and $P_e+\varepsilon P_{02}(\frac{z_2}{X_0})$ respectively. These estimates will play crucial roles in proving Theorem 1.1 and Theorem 1.2.

For convenience, let $Q=P(q)$ denote the pressure for the density q, and (D_0,D_1,D_2) and $(\widetilde{D_0},\widetilde{D_1},\widetilde{D_2})$ will denote the expressions as in (3.21) corresponding to $(\rho,U_1,U_2;\xi_1(z_2))$ and $(q,V_1,V_2;\xi_2(z_2))$ respectively.

Set

$$\begin{cases} (W_i,W_3)(z)=(U_i,\rho)\big(\xi_1(z_2)+z_1(X_0+1-\xi_1(z_2)),z_2\big) \\ \qquad\qquad -(V_i,q)\big(\xi_2(z_2)+z_1(X_0+1-\xi_2(z_2)),z_2\big), \qquad i=1,2, \\ W_4=\xi_1(z_2)-\xi_2(z_2), \\ M_j=\partial_{z_1}W_j, \qquad j=1,2,3, \\ N_k=\partial_{z_2}W_k, \qquad k=1,2,3,4. \end{cases}$$

A key technical point here is that we will focus on the estimates on $M_j (j = 1, 2, 3)$ and $N_k (k = 1, 2, 3, 4)$ directly, which will be established in a series of Lemmas (Lemma 4.1-Lemma 4.5). In the following lemmas, we always assume that the assumptions in Theorem 3.1 hold.

Lemma 4.1.

$$\begin{cases} D_0 - \widetilde{D_0} = O(\frac{1}{X_0^2})W_4, \\ D_1 - \widetilde{D_1} = O(1)W_4\partial_{z_1}, \\ D_2 - \widetilde{D_2} = O(\varepsilon)W_4\partial_{z_1} + O(1)N_4\partial_{z_1} + O(\frac{1}{X_0})W_4\partial_{z_2}. \end{cases} \tag{4.1}$$

Proof. We only check the estimate on $D_1 - \widetilde{D_1}$ since the other terms can be treated similarly. Indeed,

$$\begin{aligned} D_1 - \widetilde{D_1} =& \frac{\xi_1(z_2) - \xi_2(z_2)}{(X_0 + 1 - \xi_1(z_2))(X_0 + 1 - \xi_2(z_2))}\partial_{z_1} \\ =& \frac{W_4}{(X_0 + 1 - \xi_1(z_2))(X_0 + 1 - \xi_2(z_2))}\partial_{z_1}, \end{aligned}$$

so the second inequality in (4.1) holds since $\|\frac{1}{(X_0 + 1 - \xi_1(z_2))(X_0 + 1 - \xi_2(z_2))}\|_{C^{1,\alpha}} \leq C$ and $\xi_i(z_2)$ is a small perturbation of r_0 due to the assumptions.

Lemma 4.2. (**Estimate of** N_4) *It holds that*

$$\|N_4\|_{C^{2,\alpha}} \leq C\varepsilon\big(\|(W_1, W_2, W_3)\|_{C^{1,\alpha}} + \|W_4\|_{L^\infty} + \|(N_1, \varepsilon^{-1}N_2, N_3)\|_{C^{1,\alpha}}\big). \tag{4.2}$$

Proof. It follows from (3.18) that

$$\begin{aligned} X_0\xi_1'(z_2)[\rho U_2^2 + P](\xi_1(z_2), z_2) = \xi_1(z_2)[\rho U_1U_2](\xi_1(z_2), z_2), \\ X_0\xi_2'(z_2)[qV_2^2 + Q](\xi_2(z_2), z_2) = \xi_2(z_2)[qV_1V_2](\xi_2(z_2), z_2). \end{aligned}$$

The difference of these two equations yields

$$\begin{cases} N_4'(z_2) = O(\varepsilon)\cdot(W_1, W_2, W_3, W_4) + O(\varepsilon)\cdot(N_1, \varepsilon^{-1}N_2, N_3, X_0^{-1}N_4), \\ N_4(\pm 1) = 0, \end{cases}$$

here and what follows, for notational convenience, $O(\varepsilon)\cdot A$ denotes the inner product of vectors A and $O(\varepsilon)$ whose component is of order ε. (4.2) then follows from this initial value problem for an ODE.

Lemma 4.3. (**Estimates of** M_i)

$$\|(M_1, M_3)\|_{C^{1,\alpha}} \leq \frac{C}{X_0}\|(W_1, \varepsilon X_0W_2, W_3, W_4)\|_{C^{1,\alpha}} + C\varepsilon\|(N_1, \varepsilon^{-1}N_2, N_3, N_4)\|_{C^{1,\alpha}}, \tag{4.3}$$

$$\|M_2\|_{C^{1,\alpha}} \leq C\varepsilon\big(\|(W_1, (\varepsilon X_0)^{-1}W_2, W_3, W_4)\|_{C^{1,\alpha}} + \|(\varepsilon N_1, N_2, \varepsilon^{-1}N_3, (\varepsilon X_0)^{-1}N_4)\|_{C^{1,\alpha}}\big). \tag{4.4}$$

Proof. Rewrite (3.8) as

$$\begin{cases} \widetilde{D_1}(\rho U_1) + \widetilde{D_2}(\rho U_2) + \widetilde{D_0}(\rho U_1) = O(\frac{1}{X_0})W_4 + O(\varepsilon)N_4 \\ \widetilde{D_1}P + \rho U_1\widetilde{D_1}U_1 + \rho U_2\widetilde{D_2}U_1 - \widetilde{D_0}(\rho U_2^2) = O(\frac{1}{X_0})W_4 + O(\frac{\varepsilon}{X_0})N_4 \\ \rho U_1\widetilde{D_1}U_2 + \rho U_2\widetilde{D_2}U_2 + \widetilde{D_2}P + \widetilde{D_0}(\rho U_1U_2) = O(\varepsilon)W_4 + O(\frac{1}{X_0})N_4. \end{cases} \tag{4.5}$$

Then tedious but elementary computations using the assumptions in Theorem 3.1 show that

$$\begin{cases} U_1\widetilde{D_1}W_3+\rho\widetilde{D_1}W_1=O(X_0^{-1})\cdot(W_1,\varepsilon X_0W_2,W_3,W_4)+O(\varepsilon)\cdot(M_2,\varepsilon M_3)+O(\varepsilon)\cdot(\varepsilon^{-1}N_2,N_3,N_4),\\ c^2(\rho)\widetilde{D_1}W_3+\rho U_1\widetilde{D_1}W_1=O(X_0^{-1})\cdot(W_1,\varepsilon X_0W_2,W_3,W_4)+O(\varepsilon^2)M_1+O(\varepsilon)N_1+O(\frac{\varepsilon}{X_0})N_4,\\ \rho U_1\widetilde{D_1}W_2=O(\varepsilon)\cdot(W_1,(\varepsilon X_0)^{-1}W_2,W_3,W_4)+O(\varepsilon)\cdot(\varepsilon M_2,M_3)+O(\varepsilon)N_2+O(1)N_3+O(\frac{1}{X_0})N_4, \end{cases} \tag{4.6}$$

With respect to the variables $(\widetilde{D_1}W_1,\widetilde{D_1}W_2,\widetilde{D_1}W_3)$, the determinant of the coefficient matrix in (4.6) is $\rho^2U_1(c^2(\rho)-U_1^2)>0$ for subsonic states, then a direct computation yields Lemma 4.3.

Lemma 4.4. (**Estimates of N_1 and N_2**) *For $i=1,2$,*

$$\|N_i\|_{C^{1,\alpha}}\le C\varepsilon\|(W_1,W_2,W_3,W_4)\|_{C^{1,\alpha}}+\frac{C}{X_0}\|N_4\|_{C^{2,\alpha}}+C\|N_3\|_{C^{1,\alpha}}. \tag{4.7}$$

Proof. First, we deal with the term $(\frac{d}{d\beta}G_0)(\beta(z_1,z_2))D_2\beta(z_1,z_2)$ contained in (3.23) and (3.24).

Let $z_2^1(s;z)$ and $z_2^2(s;z)$ be the characteristics of (3.10) going through (z_1,z_2) with $z_2^1(0;z)=\beta$ and $z_2^2(0;z)=\tilde\beta$ corresponding to (U_1,U_2) and (V_1,V_2) respectively, i.e.,

$$\begin{cases} \frac{dz_2^1(s;z)}{ds}=\frac{X_0(X_0+1-\xi_1(z_2^1))}{A_1}U_2\big(\xi_1(z_2^1)+s(X_0+1-\xi_1(z_2^1)),z_2^1\big),\\ z_2^1(z_1;z)=z_2,\qquad z_2^1(0;z)=\beta, \end{cases}$$

where

$$A_1=(\xi_1(z_2^1)+s(X_0+1-\xi_1(z_2^1)))U_1+U_2X_0(s-1)\xi_1'(z_2^1).$$

Similarly, one can define $z_2^2(s;z)$ corresponding to (V_1,V_2), ξ_2 and $\tilde\beta$.

Set $l(s;z)=z_2^1(s;z)-z_2^2(s;z)$. Then one has

$$\begin{cases} \frac{dl}{ds}=O(\varepsilon)l+O(\varepsilon)W_1(s,z_2^1)+O(1)W_2(s,z_2^1)+O(\varepsilon)W_4(z_2^1)+O(\varepsilon^2)N_4(z_2^1)\\ l(0;z)=\beta-\widetilde\beta,\qquad l(z_1;z)=0. \end{cases} \tag{4.8}$$

Since the solution has the $C^{3,\alpha}-$regularities, then all the coefficients in (4.8) are in $C^{2,\alpha}$, this will lead to the $C^{2,\alpha}$ estimate of $\beta-\tilde\beta$.

By (4.8), one has

$$\|\beta-\widetilde\beta\|_{L^\infty}\le C\|(\varepsilon W_1,W_2,\varepsilon W_4,\varepsilon^2N_4)\|_{L^\infty}$$

and

$$\begin{cases} \beta-\widetilde\beta=\int_0^{z_1}\big(O(\varepsilon)W_1(t,z_2^1)+O(1)W_2(t,z_2^1)+O(\varepsilon)W_4(z_2^1)+O(\varepsilon^2)N_4(z_2^1)+O(\varepsilon)l(t;z)\big)dt,\\ l(s;z)=\int_{z_1}^s\big(O(\varepsilon)W_1(t,z_2^1)+O(1)W_2(t,z_2^1)+O(\varepsilon)W_4(z_2^1)+O(\varepsilon^2)N_4(z_2^1)+O(\varepsilon)l(t;z)\big)dt, \end{cases} \tag{4.9}$$

and

$$\|\partial_{z_1}(\beta,\tilde\beta)\|_{C^{2,\alpha}}\le C\varepsilon,\qquad \|\partial_{z_2}(\beta,\tilde\beta)\|_{C^{2,\alpha}}\le C.$$

This yields

$$\|\beta-\widetilde\beta\|_{C^{2,\alpha}}\le C\|(\varepsilon W_1,W_2,\varepsilon W_4,\varepsilon^2N_4)\|_{C^{2,\alpha}}. \tag{4.10}$$

In addition, set

$$\begin{cases} B_1=(\frac{d}{d\beta}G_0)(\beta(z_1,z_2))D_2\beta(z_1,z_2),\\ B_2=(\frac{d}{d\tilde\beta}\widetilde{G_0})(\widetilde\beta(z_1,z_2))\widetilde{D_2}\widetilde\beta(z_1,z_2). \end{cases}$$

Then, direct computations using (3.11) and (3.21) show that

$$B_1 - B_2 = O(\frac{\varepsilon}{X_0})W_4 + O(\varepsilon^2)N_4 + O(\varepsilon^2)\partial_{z_1}(\beta - \widetilde{\beta}) + O(\varepsilon)\partial_{z_2}(\beta - \widetilde{\beta}) \\ + O(\varepsilon)\cdot(W_1, W_2, W_3)(0,\beta) + O(1)\cdot(N_1, \varepsilon N_2, N_3)(0,\beta) + O(\varepsilon)(\beta - \widetilde{\beta}).$$

Thus, by use of (4.10), the following estimate holds

$$\begin{aligned} &\|B_1 - B_2\|_{C^{1,\alpha}} \\ &\le C\varepsilon(\|(W_1, W_2, W_3, X_0^{-1}W_4, \varepsilon N_4)\|_{C^{1,\alpha}} + \|\beta - \widetilde{\beta}\|_{C^{2,\alpha}}) + C\|(N_1, \varepsilon N_2, N_3)(0, z_2)\|_{C^{1,\alpha}[-1,1]} \\ &\le C\varepsilon\|(W_1, W_2, W_3, X_0^{-1}W_4, \varepsilon N_4)\|_{C^{2,\alpha}} + C\|(N_1, \varepsilon N_2, N_3)(0, z_2)\|_{C^{1,\alpha}[-1,1]}. \end{aligned} \tag{4.11}$$

Note that the third equation in R-H conditions (3.9) implies that

$$\frac{X_0\xi_1''(z_2)}{\xi_1(z_2)} - \frac{X_0(\xi_1'(z_2))^2}{(\xi_1(z_2))^2} = \frac{1}{[\rho U_2^2 + P]}\Big(\partial_\tau(\rho U_1 U_2) - \frac{X_0\xi_1'(z_2)\partial_\tau[\rho U_2^2 + P]}{\xi_1(z_2)}\Big),$$

here ∂_τ denotes the tangent derivative of $z_1 = \xi_1(z_2)$ and similar expression holds for $\xi_2(z_2)$.

This, together with the first two equations in (3.9), yields for $i = 1, 3$,

$$N_i(0, z_2) = O(\varepsilon)\cdot(W_1, W_2, W_3, X_0{}^{-1}W_4) + O(\varepsilon)N_2 + O(\frac{1}{X_0})N_4. \tag{4.12}$$

It follows from (4.11), (4.12) and (4.3)-(4.5) that

$$\begin{aligned} &\|B_1 - B_2\|_{C^{1,\alpha}} \\ &\le C\varepsilon\|(W_1, W_2, W_3, X_0^{-1}W_4)\|_{C^{2,\alpha}} + \frac{C}{X_0}\|N_4\|_{C^{1,\alpha}} + C\varepsilon\|N_2\|_{C^{1,\alpha}} \\ &\le C\varepsilon\big(\|(W_1, W_2, W_3, X_0^{-1}W_4)\|_{C^{1,\alpha}} + \|(N_1, N_2, N_3)\|_{C^{1,\alpha}}\big) + \frac{C}{X_0}\|N_4\|_{C^{2,\alpha}}. \end{aligned} \tag{4.13}$$

By (3.23) and (3.24), one has

$$\begin{cases} \widetilde{D_1}W_1 = & O(\frac{1}{X_0})\cdot\big(W_1, \varepsilon X_0 W_2, W_3, W_4\big) + O(\varepsilon)\cdot\big(N_3, X_0^{-1}N_4, \varepsilon^{-1}M_3, B_1 - B_2\big), \\ \widetilde{D_2}W_1 = & O(\frac{1}{X_0})\cdot\big(\varepsilon W_1, W_2, \varepsilon X_0 W_3, W_4\big) + O(1)\cdot\big(N_3, X_0^{-1}N_4, \varepsilon M_3, B_1 - B_2\big) \end{cases} \tag{4.14}$$

and

$$\begin{cases} \widetilde{D_1}W_2 = & O(\varepsilon)\cdot\big(W_1, (\varepsilon X_0)^{-1}W_2, W_3, W_4\big) + O(1)\cdot\big(N_3, X_0^{-1}N_4, \varepsilon M_3, \varepsilon^2(B_1 - B_2)\big), \\ \widetilde{D_2}W_2 = & O(\frac{1}{X_0})\cdot\big(W_1, \varepsilon X_0 W_2, W_3, W_4\big) + O(\varepsilon)\cdot\big(N_3, N_4, \varepsilon^{-1}M_3, B_1 - B_2\big). \end{cases} \tag{4.15}$$

We can now estimate N_2.

Since $W_2(z_1, \pm 1) = 0$, then there exists $z_2 = z_2(z_1)$ such that $N_2(z_1, z_2(z_1)) = 0$ holds. So, (4.15) implies that

$$\begin{cases} \partial_{z_1}N_2 = O(\varepsilon)\cdot\big(W_1, W_2, W_3, W_4\big) + O(\varepsilon)\cdot\big(N_1, (\varepsilon X_0)^{-1}N_2, (\varepsilon X_0)^{-1}N_3, N_4, M_3\big) \\ \qquad\qquad + O(1)\cdot\big(\varepsilon\partial_{z_1}N_3, \partial_{z_2}N_3, X_0^{-1}\partial_{z_2}N_4\big) + O(\varepsilon^2)\cdot\big(B_1 - B_2, \partial_{z_2}(B_1 - B_2)\big), \\ \partial_{z_2}N_2 = O(\varepsilon)\cdot\big(W_1, W_2, W_3, W_4\big) + O(\varepsilon)M_3 + O(\frac{1}{X_0})\cdot\big(N_1, N_2, N_3, N_4\big) \\ \qquad\qquad + O(\varepsilon)\cdot\big(\varepsilon^{-1}\partial_{z_1}N_3, \partial_{z_2}N_3, \partial_{z_2}N_4, (B_1 - B_2), \partial_{z_2}(B_1 - B_2)\big), \\ N_2(z_1, z_2(z_1)) = 0. \end{cases}$$

It should be emphasized here that instead of estimating W_2 in (4.15) directly, one differentiates (4.15) with respect to z_2 and uses the structure of the background solution to derive the desired system for N_2 (with order $O(\varepsilon)$ coefficients for W_i $(i=1,2,3,4)$ and M_3). This will make it possible to obtain a control on $\|N_2\|_{C^{1,\alpha}}$ in terms of $\varepsilon\|(W_1,W_2,W_3,W_4)\|_{C^{1,\alpha}}$, $X_0^{-1}(\|N_1\|_{C^{1,\alpha}}+\|N_4\|_{C^{2,\alpha}})$, and $\|N_3\|_{C^{1,\alpha}}$. Indeed, it follows from this system for N_2, (4.3)-(4.4), (4.13), and a direct computation that

$$\|N_2\|_{C^{1,\alpha}} \le C\|\nabla_z N_2\|_{C^{\alpha}} \le C\varepsilon\|(W_1,W_2,W_3,W_4)\|_{C^{1,\alpha}} + \frac{C}{X_0}(\|N_1\|_{C^{1,\alpha}}+\|N_4\|_{C^{2,\alpha}}) + C\|N_3\|_{C^{1,\alpha}}. \tag{4.16}$$

Next, note that (4.14) shows that

$$\begin{cases} \partial_{z_1}N_1 = O(\varepsilon)\cdot\big(W_1,W_2,W_3,W_4\big) + O(\frac{1}{X_0})\cdot\big(\varepsilon X_0 M_3, N_1, \varepsilon X_0 N_2, N_3, N_4\big) \\ \qquad\qquad +O(1)\partial_{z_1}N_3 + O(\varepsilon)\big(\partial_{z_2}N_3, {X_0}^{-1}\partial_{z_2}N_4, B_1-B_2, \partial_{z_2}(B_1-B_2)\big), \\ \partial_{z_2}N_1 = O(\varepsilon)\cdot\big(W_1,W_2,W_3,W_4\big) + O(\frac{1}{X_0})\cdot\big(\varepsilon X_0 M_3, N_1, N_2, \varepsilon X_0 N_3, N_4\big) \\ \qquad\qquad +O(\varepsilon)\partial_{z_1}N_3 + O(1)\partial_{z_2}N_3 + O(\frac{1}{X_0})\partial_{z_2}N_4 + O(\varepsilon)(B_1-B_2) + O(1)\partial_{z_2}(B_1-B_2), \\ N_1(z_1,\pm1)=0, \end{cases}$$

where $N_1(0,\pm1)=0$ follows from the compatibility condition derived in [22].

Then one can estimate N_1 as above to obtain

$$\|N_1\|_{C^{1,\alpha}} \le C\varepsilon\|(W_1,W_2,W_3,W_4)\|_{C^{1,\alpha}} + \frac{C}{X_0}(\|N_2\|_{C^{1,\alpha}}+\|N_4\|_{C^{2,\alpha}}) + C\|N_3\|_{C^{1,\alpha}}.$$

Combining this with (4.16) shows Lemma 4.4.

Finally, we estimate N_3.

Lemma 4.5. (Estimate of N_3) *N_3 satisfies*

$$\begin{aligned} \|N_3\|_{C^{1,\alpha}} \le\; & C\varepsilon\|(W_1,W_2,W_3,W_4)\|_{C^{1,\alpha}} + \frac{C}{X_0}\big(\|(N_1,N_2)\|_{C^{1,\alpha}} + \|N_4\|_{C^{2,\alpha}}\big) \\ & + C\varepsilon\|\frac{1}{X_0}(P_{01}'(\frac{z_2}{X_0}) - P_{02}'(\frac{z_2}{X_0}))\|_{C^{1,\alpha}[-1,1]}. \end{aligned} \tag{4.17}$$

Proof. Due to (3.25), one has by a direct computation that

$$\begin{cases} \widetilde{D_1}\big((U_1^2-c^2(\rho)\widetilde{D_1}W_3 + U_1U_2\widetilde{D_2}W_3\big) + \widetilde{D_2}\big(U_1U_2\widetilde{D_1}W_3 + (U_2^2-c^2(\rho))\widetilde{D_2}W_3\big) \\ = O(\frac{1}{X_0^2})\cdot\big(W_1,\varepsilon X_0 W_2, W_3, W_4\big) + O(\frac{1}{X_0})\cdot\big(M_1,\varepsilon X_0 M_2, M_3\big) + O(\varepsilon)N_1 + O(\frac{1}{X_0})N_2 \\ \qquad +O(\varepsilon)N_3 + O(\frac{\varepsilon}{X_0})N_4 + O(\frac{1}{X_0})\partial_{z_2}N_4, \\ W_3(1,z_2) = P^{-1}(P_e+\varepsilon P_{01}(\frac{z_2}{X_0})) - P^{-1}(P_e+\varepsilon P_{02}(\frac{z_2}{X_0})), \\ N_3(z_1,\pm1)=0. \end{cases}$$

It follows from this and (4.12) that

$$\begin{cases} \widetilde{D_1}\big((U_1^2-c^2(\rho))\widetilde{D_1}N_3 + U_1U_2\widetilde{D_2}N_3\big) + \widetilde{D_2}\big(U_1U_2\widetilde{D_1}N_3 + (u_2^2-c^2(\rho))\widetilde{D_2}N_3\big) \\ = \partial_{z_1}\big(O(\varepsilon)\cdot(M_3,N_3)\big) + \partial_{z_2}\big(O(\varepsilon)\cdot(N_1,(\varepsilon X_0)^{-1}N_2,N_3,X_0^{-1}N_4) + O(\frac{1}{X_0})\partial_{z_2}N_4\big) \\ \qquad +O(\varepsilon)\cdot\big(W_1,W_2,W_3,W_4\big) + O(\varepsilon)\cdot\big(M_1,M_2,\varepsilon M_3\big) + O(X_0^{-2})\cdot\big(N_1,\varepsilon X_0N_2,N_3,N_4\big) \\ \qquad +O(\frac{1}{X_0})\partial_{z_1}N_1 + O(\varepsilon)\partial_{z_1}N_2 + O(\frac{1}{X_0})\partial_{z_1}N_3, \\ N_3(0,z_2) = O(\varepsilon)\cdot\big(W_1,W_2,W_3,X_0^{-1}W_4\big) + O(\varepsilon)N_2 + O(\frac{1}{X_0})N_4, \\ N_3(1,z_2) = O(\varepsilon)W_3(1,z_2) + O(\frac{\varepsilon}{X_0})(P_{01}'(\frac{z_2}{X_0}) - P_{02}'(\frac{z_2}{X_0})), \\ N_3(z_1,\pm1)=0. \end{cases} \tag{4.18}$$

As in [22], it can be verified that the compatible condition holds at the corner points $(0, \pm 1)$ and $(1, \pm 1)$. Furthermore, these compatible conditions guarantee the $C^{1,\alpha}$ regularity of solution.

So by the regularity estimates of second order elliptic equations with divergence forms in [1-2], one can arrive at from (4.18) and (4.3)-(4.4) that

$$\begin{aligned}\|N_3\|_{C^{1,\alpha}} \leq C\varepsilon\|(W_1, W_2, W_3, W_4)\|_{C^{1,\alpha}} + \frac{C}{X_0}\big(\|(N_1, N_2)\|_{C^{1,\alpha}} + \|N_4\|_{C^{2,\alpha}}\big)\\ + C\varepsilon\|\frac{1}{X_0}(P'_{01}(\frac{z_2}{X_0}) - P'_{02}(\frac{z_2}{X_0}))\|_{C^{1,\alpha}[-1,1]},\end{aligned}$$

which shows Lemma 4.5.

Finally, we point out that all the estimates above can be improved.

Remark 4.1. *Let $(\rho, U_1, U_2; \xi_1(z_2)) = (\hat{\rho}_0^+(r_0 + z_1(X_0 + 1 - r_0)), \hat{U}_0^+(r_0 + z_1(X_0 + 1 - r_0)), O; r_0)$ be the background solution and $(q, V_1, V_2; \xi_2(z_2))$ be any solution to the problem (3.22)-(3.25) as before. Then the corresponding estimates in Lemma 4.2 and Lemma 4.3–Lemma 4.5 can be improved $C^{3,\alpha}$ and $C^{2,\alpha}$ respectively. This fact is used to get the high regularity estimates in Theorem 1.2. Indeed, by Proposition 2.1, in this case, $(\rho, U_1, U_2; \xi_1(z_2))$ is $C^{4,\alpha}$-smooth. It follows that the coefficients of $l(s; z)$ in (4.8) are $C^{3,\alpha}$-smooth. Hence, one may obtain the desired higher regularity estimates just following the proofs of Lemma 4.2–Lemma 4.5.*

Based on Lemma 4.1–Lemma 4.5, the uniqueness result of Theorem 3.1 (and thus Theorem 1.1) will be proved in next section.

§5. **Proof of the Theorem 1.1.**

Due to the equivalence between Theorem 1.1 and Theorem 3.1, it suffices to prove Theorem 3.1 only.

In §4, we have established a priori estimates for the gradients of the solution instead of the solution itself. If trying to derive a priori estimates on the solution itself, one then can obtain from (4.6) and (4.15) that

$$\|M_3\|_{C^{1,\alpha}} \leq C_1\|N_2\|_{C^{1,\alpha}} + \text{some positive terms with small coefficients}$$

and

$$\|N_2\|_{C^{1,\alpha}} \leq C_2\|M_3\|_{C^{1,\alpha}} + \text{some positive terms with small coefficients},$$

with C_1 and C_2 being some order one positive constants. Thus, it seems difficult to get any useful information on M_3 and N_2. To overcome this difficulty, we derive the gradient estimates on the solution instead of the solution itself. Furthermore, we also estimate N_3 instead of M_3 from the corresponding second order elliptic equation (4.18) to avoid the difficulties caused by the constant P_e in the variable end pressure. Combining these estimates with properties of the background solution, we can derive the monotonic and continuous dependence between the shock position and the exit pressure along the nozzle wall, which will be crucial in proving Theorem 1.1 and Theorem 1.2.

Assume that there exist two solutions $(\rho, U_1, U_2; \xi_1)$ and $(q, V_1, V_2; \xi_2)$ to the problem (3.22)-(3.25). First, we intend to show $\xi_1(1) = \xi_2(1)$ holds by contradiction.

Otherwise, without loss of generality, one may assume

$$\xi_1(1) < \xi_2(1). \tag{5.1}$$

Under this assumption, it will be shown that the corresponding end pressures are different, which is contradictory with (1.4). Indeed, we have first

Lemma 5.1. *Under the assumption (5.1) and $M_0^-(X_0) \equiv \frac{U_0^-(X_0)}{c(\rho_0^-(X_0))} > \sqrt{\frac{2^{\gamma+1} - 2}{\gamma}}$, it holds that*

$$\rho(\xi_1(1), 1) > q(\xi_2(1), 1). \tag{5.2}$$

Proof. Note that the background supersonic solution $(\rho_0^-(y_1), U_0^-(y_1))$ satisfies

$$\begin{cases} \frac{d\rho_0^-}{dy_1} &= \frac{\rho_0^-(M_0^-)^2}{y_1(1-(M_0^-)^2)}, \\ \frac{dU_0^-}{dy_1} &= -\frac{U_0^-}{y_1(1-(M_0^-)^2)}, \\ \frac{dM_0^-}{dy_1} &= -\frac{M_0^-(1+\frac{\gamma-1}{2}(M_0^-)^2)}{y_1(1-(M_0^-)^2)}, \end{cases} \tag{5.3}$$

here $M_0^-(y_1) = \frac{U_0^-(y_1)}{c(\rho_0^-(y_1))}$ denotes the Mach number of the supersonic incoming flow.

This yields that for large X_0 and $y \in (X_0-1, X_0+1)$,

$$(\rho_0^-, U_0^-, M_0^-)(y_1) = (\rho_0^-, U_0^-, M_0^-)(X_0) + O(\frac{1}{X_0}).$$

In addition,

$$\begin{cases} \frac{d(\rho_0^-(U_0^-)^2 + P_0^-)}{dy_1} = -\frac{\rho_0^-(U_0^-)^2}{y_1} < 0, \\ \frac{d(\rho_0^- U_0^-)}{dy_1} = -\frac{\rho_0^- U_0^-}{y_1} < 0. \end{cases} \tag{5.4}$$

Next, we analyze the relation between the density $\rho(y_1, 1)$ and the shock position $(y_1, 1)$. Since $U_2 = 0$ for $y_2 = \pm 1$, then the Rankine-Hugoniot conditions (3.9) imply that

$$[\rho U_1](y_1, 1) = 0, \qquad [\rho U_1^2 + P](y_1, 1) = 0.$$

This yields for the polytropic gas,

$$A(\rho(y_1,1))^{\gamma+1} - B(y_1)\rho(y_1,1) + C(y_1) = 0, \tag{5.5}$$

with $B(y_1) = P_0^-(y_1) + \rho_0^-(y_1)(U_0^-(y_1))^2$ and $C(y_1) = (\rho_0^-(y_1)U_0^-(y_1))^2$.

It follows from (5.5) that

$$\frac{d\rho(y_1,1)}{dy_1} = \frac{\rho(y_1,1)B'(y_1) - C'(y_1)}{\rho(y_1,1)(c^2(\rho(y_1,1)) - U_1^2)}.$$

In addition, (5.4) shows that

$$\rho(y_1,1)B'(y_1) - C'(y_1) = \frac{\rho_0^-(y_1)(U_0^-(y_1))^2}{y_1}(2\rho_0^-(y_1) - \rho(y_1,1)).$$

Next, elementary calculations show that $2\rho_0^-(y_1) < \rho(y_1,1)$ for $y_1 \in (X_0, X_0+1)$. Indeed, set

$$f(x) = Ax^{\gamma+1} - B(y_1)x + C(y_1), \tag{5.6}$$

then by the expressions of $B(y_1)$ and $C(y_1)$, it holds that

$$f(\rho_0^-(y_1)) = 0, \qquad f'(\rho_0^-(y_1)) < 0, \qquad f''(x) > 0 \text{ for } x > 0, \qquad f(+\infty) = +\infty,$$

so there exists a unique point $\rho(y_1,1) > \rho_0^-(y_1)$ such that

$$f(\rho(y_1,1)) = 0,$$

namely, (5.5) holds.

On the other hand, noting that $M_0^-(y_1) > M_0^-(X_0) > 1$ due to (5.3), one has for large X_0

$$f(2\rho_0^-(y_1)) = \rho_0^-(y_1)P_0^-(y_1)(\frac{2^{\gamma+1}-2}{\gamma} - (M_0^-(y_1))^2) < 0,$$

where one has used the assumption $M_0^-(X_0) > \sqrt{\frac{2^{\gamma+1}-2}{\gamma}}$.

Thus, one concludes that

$$\rho(y_1, 1) > 2\rho_0^-(y_1).$$

This implies that $\frac{d\rho(y_1,1)}{dy_1} < 0$, and consequently

$$\rho(\xi_1(1), 1) > q(\xi_2(1), 1).$$

Remark 5.1. *It follows from the assumption (5.1) and Lemma 5.1 that* $W_3(0,1) > 0$ *holds in* §4. *This property will play an important role in proving Theorem 3.1.*

Next, we establish some estimates which will be used to derive the monotonic property of the shock position on the end pressure.

Lemma 5.2. *For* $\varepsilon_0 \le \frac{1}{X_0^3}$ *in Proposition 3.2, the following estimates hold*

$$\begin{cases} \|(W_1, W_3, X_0^{-1}W_4, M_1, M_3)\|_{C^{1,\alpha}} \le C|W_3(0,1)| + C\varepsilon\|\frac{1}{X_0}\big(P'_{01}(\frac{z_2}{X_0}) - P'_{02}(\frac{z_2}{X_0})\big)\|_{C^{1,\alpha}[-1,1]}, \\ \|(W_2, M_2)\|_{C^{1,\alpha}} \le \frac{C}{X_0}|W_3(0,1)| + C\varepsilon\|\frac{1}{X_0}\big(P'_{01}(\frac{z_2}{X_0}) - P'_{02}(\frac{z_2}{X_0})\big)\|_{C^{1,\alpha}[-1,1]}, \\ \sum_{i=1}^{4}\|N_i\|_{C^{1,\alpha}} \le \frac{C}{X_0}|W_3(0,1)| + C\varepsilon\|\frac{1}{X_0}\big(P'_{01}(\frac{z_2}{X_0}) - P'_{02}(\frac{z_2}{X_0})\big)\|_{C^{1,\alpha}[-1,1]}. \end{cases} \tag{5.8}$$

Proof. As in the derivation of (5.5), one may obtain that

$$A(\rho(\xi_1(1),1))^{\gamma+1} - B(\xi_1(1))\rho(\xi_1(1),1) + C(\xi_1(1)) = 0. \tag{5.9}$$

Same expression holds for (q, V_1, ξ_2).

Then,

$$\begin{aligned} &A(\rho(\xi_1(1),1))^{\gamma+1} - \rho(\xi_1(1),1)A(q(\xi_2(1),1))^{\gamma} - q(\xi_2(1),1)V_1^2(\xi_2(1),1)(\rho(\xi_1(1),1) - q(\xi_2(1),1)) \\ &= \rho(\xi_1(1),1)(B(\xi_1(1)) - B(\xi_2(1))) - (C(\xi_1(1)) - C(\xi_2(1))). \end{aligned}$$

This, together with (5.4) and the definitions of $B(y_1)$ and $C(y_1)$, yields

$$a_0W_3(0,1) = \frac{\rho_0^-(\tilde{\xi})(U_0^-)^2(\tilde{\xi})}{\tilde{\xi}}(2\rho_0^-(\tilde{\xi}) - \rho(\xi_1(1),1))W_4(1), \tag{5.10}$$

for some $\tilde{\xi}$ between $\xi_1(1)$ and $\xi_2(1)$, and

$$\begin{aligned} a_0 &= \frac{\rho(\xi_1(1),1)(P(\rho(\xi_1(1),1)) - P(q(\xi_2(1),1)))}{\rho(\xi_1(1),1) - q(\xi_2(1),1)} - q(\xi_2(1),1)V_1^2(\xi_2(1),1) \\ &\ge \frac{q(\xi_2(1),1)(P(\rho(\xi_1(1),1)) - P(q(\xi_2(1),1)))}{\rho(\xi_1(1),1) - q(\xi_2(1),1)} - q(\xi_2(1),1)V_1^2(\xi_2(1),1) \\ &> 0. \end{aligned}$$

Then under the assumptions of lemma 5.1, it follows from (5.10) and the R-H conditions for (ρ, U_1) and (q, V_1) respectively that

$$|W_1(0,1)| \leq C|W_3(0,1)|, \qquad |W_4(1)| \leq CX_0|W_3(0,1)|, \qquad |W_3(0,1)| \leq \frac{C}{X_0}|W_4(1)|. \tag{5.11}$$

Since

$$\begin{aligned}
\|W_1\|_{C^{1,\alpha}} \leq& |W_1(0,1)| + \|M_1\|_{C^{1,\alpha}} + \|N_1\|_{C^{1,\alpha}}\\
\|W_2\|_{C^{1,\alpha}} \leq& \|M_2\|_{C^{1,\alpha}} + \|N_2\|_{C^{1,\alpha}},\\
\|W_3\|_{C^{1,\alpha}} \leq& |W_3(0,1)| + \|M_3\|_{C^{1,\alpha}} + \|N_3\|_{C^{1,\alpha}},\\
\|W_4\|_{C^{1,\alpha}} \leq& |W_4(1)| + \|N_4\|_{C^{1,\alpha}},
\end{aligned}$$

then one has by (5.11) that

$$\begin{cases}
\|W_1\|_{C^{1,\alpha}} \leq & C|W_3(0,1)| + \|M_1\|_{C^{1,\alpha}} + \|N_1\|_{C^{1,\alpha}},\\
\|W_2\|_{C^{1,\alpha}} \leq & \|M_2\|_{C^{1,\alpha}} + \|N_2\|_{C^{1,\alpha}},\\
\|W_3\|_{C^{1,\alpha}} \leq & |W_3(0,1)| + \|M_3\|_{C^{1,\alpha}} + \|N_3\|_{C^{1,\alpha}},\\
\|W_4\|_{C^{1,\alpha}} \leq & CX_0|W_3(0,1)| + \|N_4\|_{C^{1,\alpha}}.
\end{cases} \tag{5.12}$$

On the other hand, it follows from lemma 4.2-lemma 4.5 that

$$\begin{aligned}
\|(N_1,N_2,N_3)\|_{C^{1,\alpha}} + \|N_4\|_{C^{2,\alpha}} \leq& C\varepsilon(\|(W_1,W_2,W_3,W_4)\|_{C^{1,\alpha}} + \|(M_1,M_2,M_3)\|_{C^{1,\alpha}})\\
&+ \frac{C\varepsilon}{X_0}\|P'_{01}(\frac{z_2}{X_0}) - P'_{02}(\frac{z_2}{X_0})\|_{C^{1,\alpha}[-1,1]},\\
\|(M_1,M_3)\|_{C^{1,\alpha}} \leq& \frac{C}{X_0}\|(W_1,\varepsilon X_0 W_2, W_3, W_4)\|_{C^{1,\alpha}} + C\varepsilon\|M_2\|_{C^{1,\alpha}}\\
&+ \frac{C}{X_0}\big(\|(N_1,\varepsilon X_0 N_2, N_3)\|_{C^{1,\alpha}} + \|N_4\|_{C^{2,\alpha}}\big),\\
\|M_2\|_{C^{1,\alpha}} \leq& C\varepsilon\|(W_1,W_3,W_4)\|_{C^{1,\alpha}} + \frac{C}{X_0}\|W_2\|_{C^{1,\alpha}}\\
&+ C\varepsilon(\|(M_1,M_3,N_1,N_2)\|_{C^{1,\alpha}} + \|N_4\|_{C^{2,\alpha}}) + \frac{C}{X_0}\|N_2\|_{C^{1,\alpha}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\|(N_1,N_2,N_3)\|_{C^{1,\alpha}} + \|N_4\|_{C^{2,\alpha}} \leq C\varepsilon\|(W_1,W_2,W_3,W_4)\|_{C^{1,\alpha}} + C\varepsilon\|\frac{1}{X_0}\big(P'_{01}(\frac{z_2}{X_0}) - P'_{02}(\frac{z_2}{X_0})\big)\|_{C^{1,\alpha}[-1,1]},\\
&\|(M_1,M_3)\|_{C^{1,\alpha}} \leq \frac{C}{X_0}\|(W_1,W_3,W_4)\|_{C^{1,\alpha}} + C\varepsilon\|W_2\|_{C^{1,\alpha}} + C\varepsilon\|\frac{1}{X_0}\big(P'_{01}(\frac{z_2}{X_0}) - P'_{02}(\frac{z_2}{X_0})\big)\|_{C^{1,\alpha}[-1,1]},\\
&\|M_2\|_{C^{1,\alpha}} \leq C\varepsilon\|(W_1,W_3,W_4)\|_{C^{1,\alpha}} + \frac{C}{X_0}\|W_2\|_{C^{1,\alpha}} + C\varepsilon\|\frac{1}{X_0}\big(P'_{01}(\frac{z_2}{X_0}) - P'_{02}(\frac{z_2}{X_0})\big)\|_{C^{1,\alpha}[-1,1]}.
\end{aligned}$$

Consequently, combining this with (5.12) yields (5.8), which completes the proof of Lemma 5.2.
Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1.

Under the assumption of $P_{01}(\frac{z_2}{X_0}) = P_{02}(\frac{z_2}{X_0})$, it suffices to prove

$$W_1 = W_2 = W_3 = W_4 = 0.$$

It can be verified directly that (4.6) may be rewritten as

$$\begin{cases} U_1\widetilde{D_1}W_3+\rho\widetilde{D_1}W_1=a_1W_4+O(\frac{1}{X_0})\cdot(W_1,\varepsilon X_0W_2,W_3)+O(\varepsilon)\cdot(M_2,\varepsilon M_3)\\ \qquad\qquad +O(1)N_2+O(\varepsilon)\cdot(N_3,N_4),\\ c^2(\rho)\widetilde{D_1}W_3+\rho U_1\widetilde{D_1}W_1=a_2W_4+O(\frac{1}{X_0})\cdot(W_1,\varepsilon X_0W_2,W_3)+O(\varepsilon^2)M_1\\ \qquad\qquad +O(\varepsilon)N_1+O(\frac{\varepsilon}{X_0})N_4,\end{cases} \tag{5.13}$$

where

$$\begin{aligned} a_1=&-\frac{1}{(X_0+1-\xi_1(z_2))(X_0+1-\xi_2(z_2))}\partial_{z_1}(\rho U_1)\\ &+\frac{(1-z_1)\rho U_1^2}{(\xi_1(z_2)+z_1(X_0+1-\xi_1(z_2)))(\xi_2(z_2)+z_1(X_0+1-\xi_2(z_2)))}+O(\frac{\varepsilon}{X_0}),\\ a_2=&-\frac{1}{(X_0+1-\xi_1(z_2))(X_0+1-\xi_2(z_2))}(c^2(\rho)\partial_{z_1}\rho+\rho U_1\partial_{z_1}U_1)+O(\frac{\varepsilon}{X_0}).\end{aligned}$$

Then, it follows from (3.21) and (5.13) that for every $z_2\in[-1,1]$,

$$\begin{aligned}\partial_{z_1}W_3=&a(z)W_4+O(\frac{1}{X_0})\cdot(W_1,\varepsilon X_0W_2,W_3)+O(\varepsilon)\cdot(\varepsilon M_1,M_2,\varepsilon M_3)+O(1)N_2\\ &+O(\varepsilon)\cdot(N_1,N_3,N_4),\end{aligned} \tag{5.14}$$

where

$$\begin{aligned} a(z)=&\frac{(X_0+1-\xi_2(1))}{c^2(\rho)-U_1^2}(a_2-U_1a_1)\\ =&-\frac{\partial_{z_1}\rho}{X_0+1-\xi_1(z_2)}\\ &-\frac{(X_0+1-\xi_2(1))(1-z_1)\rho U_1^3}{(c^2(\rho)-U_1^2)(\xi_1(z_2)+z_1(X_0+1-\xi_1(z_2)))(\xi_2(z_2)+z_1(X_0+1-\xi_2(z_2)))}+O(\frac{\varepsilon}{X_0}).\end{aligned}$$

Under the assumptions of Theorem 3.1, we have

$$\begin{aligned}&\partial_{z_1}\rho>0,\qquad \partial_{z_1}\rho=O(\frac{1}{X_0}),\qquad U_1>0,\qquad U_1=O(1),\\ &c^2(\rho)-U_1^2>0,\qquad c^2(\rho)-U_1^2=O(1).\end{aligned}$$

Hence, $a(z)$ is a negative function in the subsonic domain. Then it follows from Remark 5.1, Lemma 5.2 and (5.14) that for every $z_2\in[-1,1]$

$$\partial_{z_1}W_3=a(z)W_4(z_2)+b(z)W_3(0,1) \tag{5.15}$$

with $\|b(z)\|_{L^\infty}\le O(\frac{1}{X_0})$.

In addition, $W_4(1)<0$ due to assumption (5.1). This means that the term $a(z)W_4(1)$ is always non-negative. Therefore, along the line $z_2=1$, (5.15) yields

$$\begin{cases}\partial_{z_1}W_3\ge b(z_1,1)W_3(0,1),\\ W_3(0,1)>0.\end{cases}$$

Thus, for suitably large X_0

$$W_3(z_1,1)>C_1W_3(0,1)>0$$

for some constant $C_1 > 0$. However, this contradicts to $W_3(1,1) = 0$ due to the end pressure condition (1.4).

This implies that the assumption (5.1) is not right. Thus, $\xi_1(1) = \xi_2(1)$ holds, namely, $W_4(0,1) = 0$.

As a consequence of this and (5.11), $W_3(0,1) = 0$. This, together with Lemma 5.2, yields

$$W_1 = W_2 = W_3 = W_4 \equiv 0,$$

which completes the proof of Theorem 3.1.

Proof of Proposition 3.2

(3.26) and (3.27) in Proposition 3.2 follow immediately from Lemma 5.2. So it suffices to show (3.28)-(3.29).

By $P_{01}(\frac{z_2}{X_0}) = P_{02}(\frac{z_2}{X_0}) + C_1$, we get $\|P'_{01}(\frac{z_2}{X_0}) - P'_{02}(\frac{z_2}{X_0})\|_{C^{1,\alpha}[-1,1]} = 0$ in (5.8). Thus, it follows from the third inequality in (5.8) and (5.11) that there exits a generic constant $C > 1$ such that

$$\frac{1}{C}|W_3(0,1)| \le |W_3(0,z_2)| \le C|W_3(0,1)|.$$

First, we claim that $\xi_1(1) < \xi_2(1)$ holds. Otherwise, it follows from the proof of Lemma 5.1 that $W_4(1) \ge 0$ and $W_3(0,1) \le 0$. This, together with (5.15), shows that

$$\begin{cases} \partial_{z_1} W_3 \le b(z_1,1)W_3(0,1) \qquad on \qquad z_2 = 1, \\ W_3(0,1) \le 0 \end{cases} \tag{5.16}$$

with $\|b(z_1,1)\|_{L^\infty} = O(\frac{1}{X_0})$.

Hence $W_3(1,1) \le 0$ for suitably large X_0, which contradicts to $P_{01}(\frac{z_2}{X_0}) > P_{02}(\frac{z_2}{X_0})$. Similarly, one can obtain $\xi_1(-1) < \xi_2(-1)$.

Next, we show that $\xi_1(z_2) < \xi_2(z_2)$ for $z_2 \in [-1,1]$. Note that $W_4(z_2) = W_4(1) + N_4(\tilde{z}_2)(z_2 - 1)$ for some $\tilde{z}_2 \in [z_2,1]$. By (5.8) and (5.11), $\|N_4\|_{L^\infty} \le CX_0^{-1}|W_3(0,1)| \le CX_0^{-2}|W_4(1)|$. Hence $W_4(z_2) < 0$ holds for all $z_2 \in [-1,1]$ for suitably large X_0 since $W_4(1) < 0$. So we complete the proof of Proposition 3.2.

In the end of this section, based on Lemma 5.1 and Remark 2.1-Remark 2.2 in §2, one can estimate the differences of two shock positions and the related subsonic flows in the domain $\{(r,\theta) : X_0 \le r \le X_0 + 1, -\theta_0 \le \theta \le \theta_0\}$ corresponding to two different background transonic shock solutions.

Proposition 5.3. *Let $(\hat{\rho}^+_{0,i}, \hat{U}^+_{0,i})(r)$, $i = 1,2$, $r \in [X_0, X_0+1]$ be two subsonic flows with corresponding shock location $r_{0,i}$ and constant exit pressures $P_{i,e}$ $(i = 1,2)$ respectively as described in Remark 2.2. Then for large X_0, it holds that*

$$\begin{cases} \|(\hat{P}^+_{0,2}(r), \hat{U}^+_{0,2}(r)) - (\hat{P}^+_{0,1}(r), \hat{U}^+_{0,1}(r))\|_{C^{4,\alpha}[X_0,X_0+1]} \le C|P_{2,e} - P_{1,e}|, \\ |r_{0,2} - r_{0,1}| \le CX_0|P_{2,e} - P_{1,e}|. \end{cases} \tag{5.17}$$

Remark 5.2. *It follows from Proposition 5.3 that if the difference of two end pressures is of order $O(\varepsilon)$, then the differences of related shock positions and extended subsonic flows will be of order $X_0O(\varepsilon)$ and $O(\varepsilon)$ respectively. In addition, it also implies that the assumptions in Theorem 1.1 are plausible although the actual shock position and further the related background transonic flow are not known in advance for such an end pressure condition $P_e + O(\varepsilon)$.*

Proof. Without loss of generality, we assume that $X_0 < r_{0,2} < r_{0,1} < X_0 + 1$. Then

$$\frac{X_0 - r_{0,1}}{X_0 + 1 - r_{0,1}} < \frac{X_0 - r_{0,2}}{X_0 + 1 - r_{0,2}} < 0.$$

Denoted by L the interval $[\frac{X_0 - r_{0,2}}{X_0 + 1 - r_{0,2}}, 1]$. As in §4, we set

$$\begin{cases} W_1(z_1) = \hat{U}^+_{0,2}(r_{0,2} + z_1(X_0+1-r_{0,2})) - \hat{U}^+_{0,1}(r_{,1} + z_1(X_0+1-r_{0,1})), \qquad z_1 \in L, \\ W_3(z_1) = \hat{\rho}^+_{0,2}(r_{0,2} + z_1(X_0+1-r_{0,2})) - \hat{\rho}^+_{0,1}(r_{0,1} + z_1(X_0+1-r_{0,1})), \qquad z_1 \in L, \\ W_4 = r_{0,2} - r_{0,1}. \end{cases}$$

Since $(\hat{\rho}^+_{0,i}(r), \hat{U}^+_{0,i}(r))(i=1,2)$ satisfy

$$\begin{cases} \dfrac{d\hat{\rho}^+_{0,i}(r)}{dr} = -\dfrac{\hat{\rho}^+_{0,i}(r)(\hat{U}^+_{0,i}(r))^2}{r((\hat{U}^+_{0,i}(r))^2 - c^2(\hat{\rho}^+_{0,i}(r)))}, & r \in [X_0, X_0+1] \\ \dfrac{d\hat{U}^+_{0,i}(r)}{dr} = \dfrac{\hat{U}^+_{0,i}(r)c^2(\hat{\rho}^+_{0,i}(r))}{r((\hat{U}^+_{0,i}(r))^2 - c^2(\hat{\rho}^+_{0,i}(r)))}, & r \in [X_0, X_0+1], \end{cases}$$

then it follows from Remark 2.2 and a direct computation that

$$\begin{cases} \dfrac{dW_1}{dz_1} = O(X_0^{-1}) \cdot (W_1, W_3) + b_1(z_1)W_4, & z_1 \in L, \\ \dfrac{dW_3}{dz_1} = O(X_0^{-1}) \cdot (W_1, W_3) + b_3(z_1)W_4, & z_1 \in L, \end{cases} \tag{5.18}$$

where

$$\begin{cases} b_1(z_1) = -\dfrac{X_0+1}{(r_{0,2} + z_1(X_0+1-r_{0,2}))(r_{0,1} + z_1(X_0+1-r_{0,1}))} \dfrac{\hat{U}^+_{0,1}c^2(\hat{\rho}^+_{0,1})}{(\hat{U}^+_{0,1})^2 - c^2(\hat{\rho}^+_{0,1})} \\ b_3(z_1) = \dfrac{X_0+1}{(r_{0,2} + z_1(X_0+1-r_{0,2}))(r_{0,1} + z_1(X_0+1-r_{0,1}))} \dfrac{\hat{\rho}^+_{0,1}(\hat{U}^+_{0,1})^2}{(\hat{U}^+_{0,1})^2 - c^2(\hat{\rho}^+_{0,1})}. \end{cases}$$

Obviously, $b_3(z_1) < 0$ and $b_i(z_1) = O(X_0^{-1})(i=1,3)$ for large X_0.
As a consequence of $W_4 < 0$, Lemma 5.1 and (5.11), one has

$$W_3(0) > 0, \qquad |W_1(0)| \le C|W_3(0)|, \qquad |W_4| \le CX_0|W_3(0)|, \qquad |W_3(0)| \le \frac{C}{X_0}|W_4|. \tag{5.19}$$

Thus, it follows from (5.18)-(5.19) and the positivity of $b_3(z_1)W_4$ that

$$\begin{cases} W_3(z_1) \ge C(W_3(0) - X_0^{-1}\|W_1\|_{L^\infty[0,1]}), & z_1 \in [0,1], \\ \|W_1\|_{L^\infty[0,1]} \le C(X_0^{-1}\|W_3\|_{L^\infty[0,1]} + W_3(0)), \\ \|W_3\|_{L^\infty[0,1]} \le C(X_0^{-1}\|W_1\|_{L^\infty[0,1]} + W_3(0)). \end{cases}$$

This yields that

$$\begin{cases} W_3(z_1) \ge C(W_3(0) - X_0^{-2}\|W_3\|_{L^\infty}), & z_1 \in [0,1], \\ \|W_3\|_{L^\infty[0,1]} \le CW_3(0). \end{cases}$$

Therefore, $W_3(z_1) \ge CW_3(0)$ for $z_1 \ge 0$ and further $W_3(0) \le CW_3(1)$ hold true.
On the other hand, by (5.18), one has

$$\begin{cases} \|W_1\|_{C^{3,\alpha}(L)} \le C(|W_1(0)| + X_0^{-1}|W_4| + X_0^{-1}\|W_3\|_{C^{2,\alpha}(L)}), \\ \|W_3\|_{C^{3,\alpha}(L)} \le C(|W_3(0)| + X_0^{-1}|W_4| + X_0^{-1}\|W_1\|_{C^{2,\alpha}(L)}). \end{cases} \tag{5.20}$$

Combining (5.20) with (5.19) and $0 < W_3(0) \le CW_3(1)$ yields

$$\|(W_1, W_3)\|_{C^{3,\alpha}(L)} \le C|W_3(1)|. \tag{5.21}$$

This, together with $|W_4| \le CX_0|W_3(0)|$ in (5.19), shows

$$|W_4| \le CX_0|W_3(1)|,$$

which is the second inequality in (5.17).

Next, we prove the first estimate in (5.17).

In fact, Remark 2.1 in §2 implies that

$$\|((\hat{P}^+_{0,1})', (\hat{U}^+_{0,1})')\|_{C^{3,\alpha}[X_0,X_0+1]} \leq CX_0^{-1}.$$

Thus,

$$\begin{aligned}
&\|\hat{P}^+_{0,2}(r) - \hat{P}^+_{0,1}(r)\|_{C^{3,\alpha}[X_0,X_0+1]}\\
\leq&\|\hat{P}^+_{0,2}\big(r_{0,2} + \frac{r-r_{0,2}}{X_0+1-r_{0,2}}(X_0+1-r_{0,2})\big) - \hat{P}^+_{0,1}\big(r_{0,1} + \frac{r-r_{0,2}}{X_0+1-r_{0,2}}(X_0+1-r_{0,1})\big)\|_{C^{3,\alpha}[X_0,X_0+1]}\\
&+\|\hat{P}^+_{0,1}(r_{0,1} + \frac{r-r_{0,2}}{X_0+1-r_{0,2}}(X_0+1-r_{0,1})) - \hat{P}^+_{0,1}(r_{0,1} + \frac{r-r_{0,1}}{X_0+1-r_{0,1}}(X_0+1-r_{0,1}))\|_{C^{3,\alpha}[X_0,X_0+1]}\\
\leq&C(\|W_3\|_{C^{3,\alpha}(L)} + \|(\hat{P}^+_{0,1})'\|_{C^{3,\alpha}[X_0,X_0+1]}|W_4|)\\
\leq&C|W_3(1)|.
\end{aligned}\tag{5.22}$$

Analogously, $\hat{U}^+_{0,2} - \hat{U}^+_{0,1}$ can be estimated. Thus the proof of Proposition 5.3 is completed.

§6. **Proof of Theorem 1.2**

In this section, based on Theorem 1.1 and the related estimates given in §4-§5, we will show the existence result in Theorem 1.2. First, note that if the transonic shock is required to go through some fixed point on the wall, then as in [26], one can prove that the problem (1.1) with (1.2)-(1.3) and (1.5) has a unique transonic shock solution when the end pressure $P_e + \varepsilon P_0(\theta)$ in (1.4) is adjusted by an appropriate constant. It follows from this that if one can show that there exists a point at the wall such that the shock goes through this point and the corresponding adjustment constant on the end pressure is zero, then Theorem 1.2 will be proved.

Next we state an existence result when the shock is assumed to go through a fixed point on the wall, whose proof will be given in Appendix.

Consider the 2-D nozzle and the supersonic incoming flow as given in §1. Let $(x_1^0, x_1^0 tan\theta_0)$ be a given point on the wall of the nozzle with $r_0 = x_1^0\sqrt{1+tan^2\theta_0} \in (X_0, X_0+1)$. Denote by $P_e \in (P_1, P_2)$ the constant exit pressure when the shock position is given by $r = r_0$ with P_1 and P_2 being given in Proposition 2.1 of §2. Then one has

Theorem 6.1. *Under the assumptions as in Theorem 1.1, there exists a constant C_0 such that the transonic shock problem (1.1) with (1.2), (1.3) and (1.5) has a solution with the following properties*

$$(\rho^+, u_1^+, u_2^+; \eta(x_2)) \in C^{3,\alpha}, \tag{6.1}$$

$$\eta(x_1^0 tan\theta_0) = x_1^0, \tag{6.2}$$

$$P^+ = P_e + \varepsilon P_0(\theta) + C_0 \qquad on \qquad r = X_0+1. \tag{6.3}$$

Moreover, the solution satisfies the analogous estimates in Theorem 1.1. In particular, $|\eta(x_2) - r_0| \leq C\varepsilon$ holds true.

In terms of the coordinates (y_1, y_2) in (3.7), Theorem 6.1 can be restated equivalently as follows

Theorem 6.1′. *Under the assumptions in Theorem 6.1, there exists an appropriate constant C_0 such that the free boundary value problem (3.14)-(3.15), (3.9), and (3.14) has a $C^{3,\alpha}$ solution $(\rho, U_1, U_2; \xi)$ satisfying*

$$\xi(1) = r_0, \tag{6.4}$$

$$P(\rho) = P_e + \varepsilon P_0(\frac{y_2}{X_0}) + C_0 \qquad on \qquad y_1 = X_0+1. \tag{6.5}$$

Moreover, the solution admits the same estimates as in Theorem 3.1. In particular, $|\eta(y_2) - r_0| \le C\varepsilon$.

It follows from Proposition 3.2 that the adjustment constant C_0 in Theorem 6.1 (Theorem 6.1′) depends continuously on the position where the shock intersects with the wall of the nozzle. More precisely, one has

Lemma 6.1. (Continuity and uniqueness)

(i) Assume that two variable exit pressures $\tilde{P}_1$ *and* $\tilde{P}_2$ *have the form (1.4) and satisfy* $\tilde{P}_1 = \tilde{P}_2 + C_0$ *with a constant* C_0*. Let* $(\rho, U_1, U_2; \xi_1)$ *and* $(q, V_1, V_2; \xi_2)$ *be solutions to the free boundary value problems (3.14)-(3.15), (3.9) and (3.14) corresponding to the exit pressure* $\tilde{P}_1$ *and* $\tilde{P}_2$ *respectively and satisfy the corresponding estimates in Theorem 3.1. Then*

$$|C_0| \le \frac{C}{X_0}|\xi_1(1) - \xi_2(1)|. \tag{6.6}$$

with a uniform constant C*.*

(ii). If the transonic shock goes through a fixed point on the wall of the nozzle, then the corresponding exit pressure is uniquely determined. Namely, if there exist two constants C_1 *and* C_2 *such that the end pressures of two solutions are* $\tilde{P}_1 = P_e + \varepsilon P_0(\theta) + C_1$ *and* $\tilde{P}_2 = P_e + \varepsilon P_0(\theta) + C_2$*, then* $C_1 \equiv C_2$*.*

Remark 6.1. *By Theorem 3.1, the solutions corresponding to the variable exit pressure* $\tilde{P}_1$ *and* $\tilde{P}_2$ *are unique respectively.*

Based on Lemma 6.1, we now prove Theorem 1.2.

Proof of Theorem 1.2.

Denote by $\bar{P}_1 = P_e - \sqrt{X_0}\varepsilon$ and $\bar{P}_2 = P_e + \sqrt{X_0}\varepsilon$ the exit pressures of the symmetric transonic shock solutions with corresponding shocks at $y_1 = r_1$ and $y_1 = r_2$ respectively. Then it follows from Remark 5.1 and the uniqueness result in Theorem 1.1 that $r_1 > r_2$.

For each fixed point $(y_1*, 1)$ with $y_1* \in [r_2, r_1]$, it follows from Theorem 6.1′ that there exists a constant C_0 such that problem (1.1) with (1.2)-(1.3), (1.5) and the end pressure $P = P_e + \varepsilon P_0(\theta) + C_0$ has a unique solution $(\rho, U_1, U_2; \xi(y_2))$ which satisfies $\xi(1) = y_1*$ and the estimates in Theorem 3.1.

If $y_1* = r_2$, then this corresponding adjustment constant, C_0, must be positive. Indeed, if not, then $C_0 \le 0$. Applying the estimate (3.27) to $(\rho, U_1, U_2; \xi(y_2))$ and the background solution $(q, V_1, V_2; r_2)$ which corresponds to the constant end pressure $\bar{P}_2$, and noting that $W_4(1) = 0$, one has

$$\|W_3\|_{L^\infty} \le C\varepsilon \|\frac{1}{X_0}P_0'(\frac{z_2}{X_0})\|_{C^{1,\alpha}} < C\varepsilon. \tag{6.7}$$

On the other hand, $W_3(1,1) = \bar{P}_2 - (P_e + \varepsilon P_0(\frac{1}{X_0})) \ge C\sqrt{X_0}\varepsilon > C\varepsilon$, which contradicts to (6.7) for large X_0. Hence, $C_0 > 0$. Similarly, for $y_1* = r_2$, the corresponding adjustment constant, C_0, must be negative. It follows from Theorem 6.1 and Lemma 6.1 that C_0 is a Lipschitz continuous function of y_1*, i.e. $C_0 = C_0(y_1*)$. We have shown that $C_0(r_2) > 0$ and $C_0(r_1) < 0$, thus there exists a $y_1^0 \in (r_2, r_1)$ such that $C_0(y_1^0) = 0$. Consequently, it follows from Theorem 6.1 that the problem (1.1)-(1.5) has a transonic shock solution $(P(y), U_1(y), U_2(y); \xi(y))$ and the transonic shock passes through $(y_1^0, 1)$. By Theorem 1.1 such a solution is unique. Thus Theorem 1.2 is proved.

Appendix

In this section, we will focus on the proof on Theorem 6.1′. In [26], for almost parallel nozzle walls and a special exit pressure boundary condition, when the shock is required to go through a fixed point, it is proved that the problem (1.1) with the related boundary conditions has a solution in some weighted Hölder space if the exit pressure is adjusted by an appropriate constant. It should be noted that the exit boundary in [26] is straight, this makes it possible to straighten out both the solid walls and the exit of the nozzle simultaneously by a Langrange transformation. This ingredient plays an important role in the proof of the main result in [26]. However, in our case, the exit of the nozzle is curved, so it is related to the solution itself under a Langrange transformation. Thus, in order to overcome this difficulty and also obtain higher regularities (than weighted Hölder regularity) of the solution, we will use a different method. In particular the reformation of the system (1.1) in §3 will be used.

Before starting to prove Theorem 6.1′, we now state a regularity result for the Laplacian equation with mixed boundary conditions satisfying suitable compatibility conditions at the corners.

Lemma A.1. *Let*

$$\begin{cases} \Delta u(x_1,x_2)=f(x_1,x_2) \quad in \quad \Omega=(-1,1)\times(-1,1), \\ u(x_1,x_2)=g(x_1,x_2) \quad on \quad x_2=\pm 1, \\ \partial_{x_1}u(x_1,x_2)=0 \quad on \quad x_1=\pm 1, \end{cases} \tag{A.1}$$

where $f\in C^{2,\alpha}(\bar\Omega), g\in C^{4,\alpha}(\bar\Omega)$ *and* $\partial_{x_1}g(\pm1,\pm1)=\partial^3_{x_1}g(\pm1,\pm1)=0, \partial_{x_1}f(\pm1,x_2)=0$, *then the equation (A.1) has a unique solution* $u(x_1,x_2)\in C^{4,\alpha}(\bar\Omega)$.

Proof. First it is noted that the Dirichlet boundary is not empty, so it follows from [16] that (A.1) has a unique solution

$$u\in C^{4,\alpha}(\Omega)\cap C^{4,\alpha}(\{\pm1\}\times(-1,1))\cap C^{4,\alpha}((-1,1)\times\{\pm1\})\cap C^0(\bar\Omega).$$

To obtain the higher regularities of the solution at the corners, one can use the standard reflection method such as in [1-2] or [6]. Without loss of generality, we deal only with the corner $(-1,-1)$ as an example since the treatments on other corners are the same. Set

$$U(x_1,x_2)=\begin{cases} u(x_1,x_2), & -1\le x_1<1, \\ u(-2-x_1,x_2), & -3<x_1\le -1, \end{cases}$$

$$F(x_1,x_2)=\begin{cases} f(x_1,x_2), & -1\le x_1<1, \\ f(-2-x_1,x_2), & -3<x_1\le -1, \end{cases}$$

$$G(x_1,x_2)=\begin{cases} g(x_1,x_2), & -1\le x_1<1, \\ g(-2-x_1,x_2), & -3<x_1\le -1. \end{cases}$$

Then it follows from the compatibility conditions of f and g that $F(x)\in C^{2,\alpha}, G(x)\in C^{4,\alpha}$ and $U(x),F(x),G(x)$ satisfy

$$\begin{cases} \Delta U(x_1,x_2)=F(x_1,x_2) \quad in \quad \Omega=(-3,1)\times(-1,1), \\ U(x_1,x_2)=G(x_1,x_2) \quad on \quad x_2=\pm1. \end{cases} \tag{A.2}$$

So it follows from the local regularity estimates in [11], that $U(x)\in C^{4,\alpha}$ in a small neighborhood of the point $(-1,-1)$. Hence, $u(x)\in C^{4,\alpha}(\bar\Omega)$ admits the following estimate

$$\|u\|_{C^{4,\alpha}(\bar\Omega)}\le C(\|g\|_{C^{4,\alpha}(\bar\Omega)}+\|f\|_{C^{2,\alpha}(\bar\Omega)}).$$

Thus, Lemma A.1. is proved.

Lemma A.2. *If the system (3.8), with (3.9) and (1.4)-(1.5), has a solution* $(\rho(y),U_1(y),U_2(y),\xi(y_2))$ *with* $(\rho(y),U_1(y),U_2(y))\in C^{3,\alpha}$ *and* $\xi(y_2)\in C^{4,\alpha}$, *then the following compatible conditions at the corners hold*

$$\begin{cases} \partial_{y_2}\rho(y_1,\pm1)=0, \partial^3_{y_2}\rho(y_1,\pm1)=0, \\ \partial_{y_2}U_1(y_1,\pm1)=0, \\ U_2(y_1,\pm1)=0, \partial^2_{y_2}U_2(y_1,\pm1)=0, \\ \xi'(\pm1)=0, \xi^{(3)}(\pm1)=0. \end{cases} \tag{A.3}$$

Proof. It follows from boundary condition (1.5), the jumping condition and (3.8) that

$$U_2(y_1,\pm 1)=0, \partial_{y_2}\rho(y_1,\pm 1)=0, \xi'(\pm 1)=0.$$

Applying $\xi'(y_2)\partial_{y_1}+\partial_{y_2}$ to the first and the second equations in (3.9) yields

$$\partial_{y_2}U_1(\xi(\pm 1),\pm 1)=0, \partial_{y_2}\rho(\xi(\pm 1),\pm 1)=0.$$

Thus, in terms of the second equation in (3.8), $\partial_{y_2}U_1(y_1,\pm 1)$ satisfies

$$\begin{cases} U_1\partial_{y_1}(\partial_{y_2}U_1)+(\partial_{y_1}U_1+\frac{X_0}{y_1}\partial_{y_2}U_2)\partial_{y_2}U_1=0 \qquad on \qquad y_2=\pm 1, \\ \partial_{y_2}U_1(\xi(\pm 1),\pm 1)=0, \end{cases}$$

which implies $\partial_{y_2}U_1(y_1,\pm 1)=0$.

In addition, differentiating the first equation of (3.8) with respect to y_2, one can get

$$\partial^2_{y_2}U_2(y_1,\pm 1)=0.$$

And taking $\xi'(y_2)\partial_{y_1}+\partial_{y_2}$ on the third equation of (3.9) twice yields

$$\partial^3_{y_2}\xi(\pm 1)=0.$$

The other equalities can be obtained similarly, and then the proof of Lemma A.2 is completed.

Next, we reformulate the problem in Theorem 6.1′ for easy presentation.

Let $(\rho,U_1,U_2;\xi)$ be a solution to (3.8)-(3.9) such that $\xi(1)=r_0$. In terms of the transformation (3.19), the domain

$$\omega_+=\{(y_1,y_2):\xi(y_2)<y_1<X_0+1,-1<y_2<1\}$$

is transformed into

$$E_+=\{(z_1,z_2):0<z_1<1,-1<z_2<1\}. \tag{A.4}$$

Set $w=\frac{U_2}{U_1}$. Then it follows from a direct computation that the system (3.8) with (1.4), (3.6), (3.16) and (3.18) is equivalent to the following problem

$$\begin{cases} \xi'(z_2)=\dfrac{\xi(z_2)(\rho U_1U_2)(\xi(z_2),z_2)}{X_0((\rho U_2^2)(\xi(z_2),z_2)+P(\xi(z_2),z_2)-P_0^-(\xi(z_2)))}, \\ \xi(1)=r_0 \end{cases} \tag{A.5}$$

and

$$\begin{cases} \partial_1 w+\lambda_1\partial_2\rho=F_1(\rho,U_1,U_2,w;\xi), \\ \partial_2 w-\lambda_2\partial_1\rho=F_2(\rho,U_1,U_2,w;\xi), \\ P(\rho)=\hat{P}_0^+(r_0)+\tilde{g_2}((U_2)^2,P^--P_0^-(r_0),U^--U_0^-(r_0)) \qquad on \qquad z_1=0, \\ P(\rho)=P_e+\varepsilon P_0(\frac{z_2}{X_0}) \qquad on \qquad z_1=1, \\ w=0 \qquad on \qquad z_2=\pm 1, \end{cases} \tag{A.6}$$

where

$$\lambda_1=\frac{X_0}{r_0}\frac{(X_0+1-r_0)}{\hat{\rho}^+(r_0)}\frac{c^2(\hat{\rho}^+(r_0))}{(\hat{U}_0^+(r_0))^2}, \qquad \lambda_2=\frac{r_0}{X_0}\frac{1}{\hat{\rho}^+(r_0)(X_0+1-r_0)}\Big(\frac{c^2(\hat{\rho}^+(r_0))}{(\hat{U}_1^+(r_0))^2}-1\Big), \tag{A.7}$$

and

$$
\begin{aligned}
F_1(\rho,U_1,U_2,w;\xi)=&\frac{w}{\rho}\partial_{z_1}\rho-\frac{X_0(1-z_1)\xi'(z_2)}{\rho(\xi(z_2)+z_1(X_0+1-\xi(z_2)))}(\frac{c^2(\rho)}{U_1^2}-w^2)\partial_{z_1}\rho\\
&-\Big(\frac{X_0(X_0+1-\xi(z_2))}{\rho(\xi(z_2)+z_1(X_0+1-\xi(z_2)))}(\frac{c^2(\rho)}{U_1^2}-w^2)-\lambda_1\Big)\partial_{z_2}\rho, \qquad (A.8)\\
F_2(\rho,U_1,U_2,w;\xi)=&\Big(\frac{(\xi(z_2)+z_1(X_0+1-\xi(z_2))}{X_0(X_0+1-\xi(z_2))\rho}(\frac{c^2(\rho)}{U_1^2}-1)-\lambda_2\Big)\partial_{z_1}\rho\\
&-\frac{w}{\rho}\Big(\frac{(z_1-1)\xi'(z_2)}{X_0+1-\xi(z_2)}\partial_{z_1}+\partial_{z_2}\Big)\rho-\frac{(z_1-1)\xi'(z_2)}{X_0+1-\xi(z_2)}\partial_{z_1}w-\frac{1+w^2}{X_0}. \qquad (A.9)
\end{aligned}
$$

After determination of w and ρ, U_1 can be obtained through the Bernoulli's law

$$
\left\{
\begin{aligned}
&\{\big(U_1+\frac{X_0(z_1-1)\xi'(z_2)U_2}{\xi(z_2)+z_1(X_0+1-\xi(z_2))}\big)\partial_{z_1}+\frac{X_0(X_0+1-\xi(z_2))U_2}{\xi(z_2)+z_1(X_0+1-\xi(z_2))}\partial_{z_2}\}(\tfrac{1}{2}U_1^2(1+w^2)+h(\rho))=0,\\
&(\tfrac{1}{2}U_1^2(1+w^2)+h(\rho))(\xi(z_2),z_2)=\big(\tfrac{1}{2}(\hat{U}_0^++\widetilde{g}_1)^2+\tfrac{1}{2}w^2(\hat{U}_0^++\widetilde{g}_1)^2\\
&\qquad\qquad\qquad +h(\hat{\rho}_0^+(r_0)+\widetilde{g}_2)(\xi(z_2),z_2),
\end{aligned}
\right. \qquad (A.10)
$$

here $\tilde{g}_i(i=1,2)$ in (A.6) and (A.10) have the analogous expressions as in (3.6).

With a slight abuse of notations, we still set

$$
\left\{
\begin{aligned}
&\rho(z)=\rho(\xi(z_2)+z_1(X_0+1-\xi(z_2)),z_2),\\
&U_i(z)=U_i(\xi(z_2)+z_1(X_0+1-\xi(z_2)),z_2),\quad i=1,2.
\end{aligned}
\right.
$$

Now we begin to prove Theorem 6.1′. This will be achieved by using of the contractible mapping theorem. To this end, we introduce the iteration spaces as

$$
S_\sigma=\{\xi(z_2)\in C^{4,\alpha}[-1,1]:\|\xi-r_0\|_{C^{4,\alpha}[-1,1]}\le\sigma,\xi(1)=r_0,\xi'(\pm1)=\xi^{(3)}(\pm1)=0\} \qquad (A.11)
$$

and

$$
\begin{aligned}
\Xi_\delta=&\{(\rho(z),U_1(z),U_2(z)):\|(\rho_1U_1)-(\hat{\rho}_0^+,\hat{U}_0^+)(r_0+z_1(X_0+1-r_0))\|_{C^{3,\alpha}(E_+)}+\|U_2\|_{C^{3,\alpha}(E_+)}\le\delta,\\
&\partial_{z_2}U_1(z_1,\pm1)=0,U_2(z_1,\pm1)=\partial_{z_2}^2U_2(z_1,\pm1)=0,\\
&\partial_{z_2}\rho(z_1,\pm1)=\partial_{z_2}^3\rho(z_1,\pm1)=0\}, \qquad (A.12)
\end{aligned}
$$

where $\sigma>0$ and $\delta>0$ will be determined later on.

The proof of Theorem 6.1′ will be divided into five steps.

Step 1. Approximate Shock.

For every $(q,V_1,V_2)\in\Xi_\delta$, the approximate shock is defined as follows

$$
\left\{
\begin{aligned}
&\xi'(z_2)=\frac{\xi(z_2)}{X_0}\frac{(qV_1V_2)(0,z_2)}{P(q(0,z_2))-P_0^-(\xi(z_2))+(qV_2^2)(0,z_2)},\\
&\xi(1)=r_0.
\end{aligned}
\right. \qquad (A.13)
$$

Obviously, (A.13) has a unique solution $\xi=\xi(z_2)\in C^{4,\alpha}([-1,1])$, moreover, one has

$$
\xi'(\pm1)=0,\qquad \xi^{(3)}(\pm1)=0, \qquad (A.14)
$$

and

$$
\|\xi(z_2)-r_0\|_{C^{k,\alpha}[-1,1]}\le C\|V_2\|_{C^{k-1,\alpha}(E_+)},\qquad k=2,3,4. \qquad (A.15)
$$

If $\delta > 0$ is chosen such that

$$C\delta \leq \sigma, \tag{A.16}$$

then (A.15) yields

$$\|\xi(z_2) - r_0\|_{C^{4,\alpha}[-1,1]} \leq \sigma. \tag{A.17}$$

namely, $\xi(z_2) \in S_\sigma$.

Step 2. Approximate ρ and w.

In this Step, we will look for the solution $(\rho(z), w(z))$ to the following problem

$$\begin{cases} \partial_1 w + \lambda_1 \partial_2 \rho = F_1(q, V_1, V_2, \frac{V_2}{V_1}; \xi), \\ \partial_2 w - \lambda_2 \partial_1 \rho = F_2(q, V_1, V_2, \frac{V_2}{V_1}; \xi), \\ \rho = \hat{\rho}_0^+(r_0) + \tilde{g}_2(V_2^2(0, z_2), P_0^-(\xi(z_2)) - P_0^-(r_0), U_0^-(\xi(z_2)) - U_0^-(r_0)) \quad on \quad z_1 = 0, \\ P(\rho) = P_e + \varepsilon P_0(\frac{z_2}{X_0}) + C_0 \quad on \quad z_1 = 1, \\ w = 0 \quad on \quad z_2 = \pm 1, \end{cases} \tag{A.18}$$

where F_1 and F_2 are defined by (A.8) and (A.9), and C_0 is a constant to be adjusted so that (A.18) has a solution.

Let (ρ_1, w_1) and (ρ_2, w_2) solve

$$\begin{cases} \partial_1 w_1 + \lambda_1 \partial_2 \rho_1 = F_1(q, V_1, V_2, \frac{V_2}{V_1}; \xi), \\ \partial_2 w_1 - \lambda_2 \partial_1 \rho_1 = 0, \\ \rho_1 = 0 \quad on \quad z_1 = 0, \\ \rho_1 = 0 \quad on \quad z_1 = 1, \\ w_1 = 0 \quad on \quad z_2 = -1, \\ w_1 = 0 \quad on \quad z_2 = 1 \end{cases} \tag{A.19}$$

and

$$\begin{cases} \partial_1 w_2 + \lambda_1 \partial_2 \rho_2 = 0, \\ \partial_2 w_2 - \lambda_2 \partial_1 \rho_2 = F_2(q, V_1, V_2, \frac{V_2}{V_1}; \xi), \\ \rho_2 = \hat{\rho}_0^+(r_0) + \tilde{g}_2(V_2^2(0, z_2), P_0^-(\xi(z_2)) - P_0^-(r_0), U_0^-(\xi(z_2)) - U_0^-(r_0)) \quad on \quad z_1 = 0, \\ P(\rho_2) = P_e + \varepsilon P_0(\frac{z_2}{X_0}) + C_0 \quad on \quad z_1 = 1, \\ w_2 = 0 \quad on \quad z_2 = -1, \\ w_2 = 0 \quad on \quad z_2 = 1 \end{cases} \tag{A.20}$$

respectively. Set $w = w_1 + w_2$ and $\rho = \rho_1 + \rho_2$. Then (ρ, w) solves (A.18).

To solve the elliptic systems (A.19) and (A.20), one may introduce potential functions $\phi_1(z)$ and $\phi_2(z)$ as follows

$$\partial_1 \phi_1 = w_1, \quad \partial_2 \phi_1 = \lambda_2 \rho_1, \quad \phi_1(0, 0) = 0 \tag{A.21}$$

and

$$\partial_1 \phi_2 = -\lambda_1 \rho_2, \quad \partial_2 \phi_2 = w_2, \quad \phi_2(0, 0) = 0. \tag{A.22}$$

Then (A.19) becomes

$$\begin{cases} \partial_1^2 \phi_1 + \frac{\lambda_1}{\lambda_2} \partial_2^2 \phi_1 = F_1(q, V_1, V_2, \frac{V_2}{V_1}; \xi) \quad in \quad E_+, \\ \phi_1 = 0 \quad on \quad \partial E_+ \end{cases} \tag{A.23}$$

while (A.20) is changed into

$$\begin{cases} \frac{\lambda_2}{\lambda_1}\partial_1^2\phi_2 + \partial_2^2\phi_2 = F_2(q, V_1, V_2, \frac{V_2}{V_1}; \xi) \qquad in \qquad E_+, \\ \partial_1\phi_2 = -\lambda_1(\hat{\rho}_0^+(r_0) + \tilde{g}_2(V_2^2(0, z_2), P_0^-(\xi(z_2)) - P_0^-(r_0), U_0^-(\xi(z_2)) - U_0^-(r_0))) \qquad on \qquad z_1 = 0, \\ \partial_1\phi_2 = -\lambda_1 P^{-1}(P_e + \varepsilon P_0(\frac{z_2}{X_0}) + C_0) \qquad on \qquad z_1 = 1, \\ \partial_2\phi_2 = 0 \qquad on \qquad z_2 = -1, \\ \partial_2\phi_2 = 0 \qquad on \qquad z_2 = 1, \\ \phi_2(0,0) = 0. \end{cases} \tag{A.24}$$

First, due to (A.8), one can check that $F_1(q, V_1, V_2, \frac{V_2}{V_1}; \xi) \in C^{2,\alpha}(E_+)$ and $F_1(z_1, \pm 1) = \partial_2^2 F_1(z_1, \pm 1) = 0$, so the compatible conditions for (A.23) are satisfied. Then, similar to the proof of Lemma A.1, (A.23) has a unique solution $\phi_1(z) \in C^{4,\alpha}(E_+)$ and admits the following estimate

$$\begin{aligned} \|w_1\|_{C^{3,\alpha}(E_+)} + \|\rho_1\|_{C^{3,\alpha}(E_+)} \leq \|\phi_1\|_{C^{4,\alpha}(E_+)} &\leq C\|F_1(q, V_1, V_2, \frac{V_2}{V_1}; \xi)\|_{C^{2,\alpha}(E_+)} \\ &\leq O(\frac{1}{X_0} + \delta + \sigma)\|V_2\|_{C^{3,\alpha}(E_+)} + O(\delta)\|q - \hat{\rho}_0^+(r_0 + z_1(X_0 + 1 - r_0))\|_{C^{3,\alpha}(E_+)} \\ &\quad + O(\delta + \sigma)\|V_1 - \hat{U}_0^+(r_0 + z_1(X_0 + 1 - r_0))\|_{C^{3,\alpha}(E_+)} + O(\delta)\|\xi - r_0\|_{C^{4,\alpha}[-1,1]} \\ &\leq O(\frac{1}{X_0} + \delta + \sigma)\delta. \end{aligned} \tag{A.25}$$

Furthermore, the following compatible conditions hold

$$\partial_2^2\phi_1(z_1, \pm 1) = \partial_1\partial_2^2\phi_1(z_1, \pm 1) = \partial_2^4\phi_1(z_1, \pm 1) = 0. \tag{A.26}$$

Next we solve the problem (A.24).

It follows from (A.9) and $(q, V_1, V_2) \in \Xi_\sigma$ that

$$F_2(q, V_1, V_2, \frac{V_2}{V_1}; \xi) \in C^{2,\alpha}(E_+), \qquad \partial_1\phi_2(0, z_2) \in C^{3,\alpha}[-1,1], \qquad \partial_1\phi_2(1, z_2) \in C^{3,\alpha}[-1,1]$$

and

$$\partial_2 F_2(q, V_1, V_2, \frac{V_2}{V_1}; \xi)(z_1, \pm 1) = 0, \quad \partial_2^k(\partial_1\phi_2)(0, \pm 1)) = \partial_2^k(\partial_1\phi_2)(1, \pm 1)) = 0, \quad k = 1, 3.$$

In addition, it can be verified directly that the background solution $\big(\hat{\rho}_0^+(r_0 + z_1(X_0 + 1 - r_0)), \hat{U}_0^+(r_0 + z_1(X_0 + 1 - r_0)), 0; r_0\big)$ satisfies

$$\begin{cases} \frac{\lambda_2}{\lambda_1}\partial_1^2\hat{\phi}_2 + \partial_2^2\hat{\phi}_2 = F_2(\hat{\rho}_0^+(r_0 + z_1(X_0 + 1 - r_0)), \hat{U}_0^+(r_0 + z_1(X_0 + 1 - r_0)), 0, 0, r_0) \qquad in \qquad E_+, \\ \partial_1\hat{\phi}_2 = -\lambda_1\hat{\rho}_0^+(r_0) \qquad on \qquad z_1 = 0, \\ \partial_1\hat{\phi}_2 = -\lambda_1 P^{-1}(P_e) \qquad on \qquad z_1 = 1, \\ \partial_2\hat{\phi}_2 = 0 \qquad on \qquad z_2 = -1, \\ \partial_2\hat{\phi}_2 = 0 \qquad on \qquad z_2 = 1, \\ \hat{\phi}_2(0,0) = 0, \end{cases} \tag{A.27}$$

where $\partial_1\hat{\phi}_2 = -\lambda_1\hat{\rho}_0^+(r_0 + z_1(X_0 + 1 - r_0))$, $\partial_2\hat{\phi}_2 = 0$.

So as in [11], the solvability condition for (A.24) is

$$\int\int_{E_+}\Big(F_2(q,V_1,V_2,\frac{V_2}{V_1};\xi)-F_2\big(\hat{\rho}_0^+(r_0+z_1(X_0+1-r_0)),\hat{U}_0^+(r_0+z_1(X_0+1-r_0)),0,0,r_0\big)\Big)dz$$
$$=\lambda_2\int_{-1}^{1}\Big(\tilde{g_2}\big(V_2(0,z_2),P_0^-(\xi(z_2))-P_0^-(r_0),U_0^-(\xi(z_2))-U_0^-(r_0)\big)+P^{-1}(P_e)$$
$$-P^{-1}\big(P_e+\varepsilon P_0(\frac{z_2}{X_0})+C_0)\big)dz_2. \tag{A.28}$$

It is easy to check that (A.28) has a unique solution C_0. Hence, (A.24) has a unique solution $\phi_2\in C^{4,\alpha}(E_+)$ with the following estimate

$$\|w_2\|_{C^{3,\alpha}(E_+)}+\|\rho_2-\hat{\rho}_0^+(r_0+z_1(X_0+1-r_0))\|_{C^{3,\alpha}(E_+)}+|C_0|$$
$$\le C\|\phi_2-\hat{\phi}_2\|_{C^{4,\alpha}(E_+)}$$
$$\le C(\|F_2(q,V_1,V_2,\frac{V_2}{V_1};\xi)-F_2(\hat{\rho}_0^+(r_0+z_1(X_0+1-r_0)),\hat{U}_0^+(r_0+z_1(X_0+1-r_0)),0,0,r_0)\|_{C^{2,\alpha}(E_+)}$$
$$+\|\tilde{g_2}(V_2^2(0,z_2),P_0^-(\xi(z_2))-P_0^-(r_0),U_0^-(\xi(z_2))-U_0^-(r_0))\|_{C^{3,\alpha}(E_+)}+\varepsilon\|P_0(\frac{z_2}{X_0})\|_{C^{3,\alpha}[-1,1]})$$
$$\le O(\frac{1}{X_0}+\delta)\|\xi-r_0\|_{C^{4,\alpha}[-1,1]}+O(\delta)\|V_2\|_{C^{3,\alpha}(E_+)}+O(\frac{1}{X_0}+\delta)\|V_1-\hat{U}_0^+(r_0+z_1(X_0+1-r_0))\|_{C^{3,\alpha}(E_+)}$$
$$+O(\frac{1}{X_0}+\delta+\sigma)\|q-\hat{\rho}_0^+(r_0+z_1(X_0+1-r_0))\|_{C^{3,\alpha}(E_+)}+C\varepsilon\|P_0(\frac{z_2}{X_0})\|_{C^{3,\alpha}(E_+)}$$
$$\le O(\frac{1}{X_0}+\delta+\sigma)(\sigma+\delta)+C\varepsilon. \tag{A.29}$$

Meanwhile, it follows from (A.20)-(A.24) that the following compatible conditions hold

$$\partial_2\phi_2(z_1,\pm1)=\partial_2^3\phi_2(z_1,\pm1)=\partial_1\partial_2\phi_2(z_1,\pm1)=\partial_1\partial_2^3\phi_2(z_1,\pm1)=0. \tag{A.30}$$

Due to (A.21), (A.22), (A.25)-(A.26) and (A.29)-(A.30), it holds that

$$\|\rho-\hat{\rho}_0^+(r_0+z_1(X_0+1-r_0))\|_{C^{3,\alpha}(E_+)}+\|w\|_{C^{3,\alpha}(E_+)}+|C_0|\le O(\frac{1}{X_0}+\delta+\sigma)\delta+O(\frac{1}{X_0}+\delta)\sigma+C\varepsilon \tag{A.31}$$

and

$$\partial_2\rho(z_1,\pm1)=\partial_2^3\rho(z_1,\pm1)=0,\qquad w(z_1,\pm1)=\partial_2^2w(z_1,\pm1)=0. \tag{A.32}$$

Step 3. Approximate U_1.
By (A.10), U_1 is obtained by solving

$$\begin{cases}\{\big(V_1+\frac{X_0(z_1-1)\xi'(z_2)V_2}{\xi(z_2)+z_1(X_0+1-\xi(z_2))}\big)\partial_{z_1}+\frac{X_0(X_0+1-\xi(z_2))V_2}{\xi(z_2)+z_1(X_0+1-\xi(z_2))}\partial_{z_2}\}\big(\frac12U_1^2(1+w^2)+h(\rho)\big)=0,\\ \big(\frac12U_1^2(1+w^2)+h(\rho)\big)(0,z_2)=\\ \qquad\frac12\big(1+w^2(0,z_2)\big)\Big(\hat{U}_0^+(r_0)+\tilde{g_1}\big(V_2^2(0,z_2),P_0^-(\xi(z_2))-P_0^-(r_0),U_0^-(\xi(z_2))-U_0^-(r_0)\big)\Big)^2\\ \qquad+h\big(\hat{\rho}_0^+(r_0)+\tilde{g_2}(V_2^2(0,z_2),P_0^-(\xi(z_2))-P_0^-(r_0),U_0^-(\xi(z_2))-U_0^-(r_0)\big).\end{cases} \tag{A.33}$$

It follows from the characteristics method and the analysis in §4 that (A.33) has a unique solution $U_1 = U_1(z) \in C^{3,\alpha}(E_+)$ such that

$$
\begin{aligned}
&\|U_1 - \hat{U}_0^+(r_0 + z_1(X_0 + 1 - r_0))\|_{C^{3,\alpha}(E_+)} \\
&\leq O(1)(\|\rho - \hat{\rho}_0^+(r_0 + z_1(X_0 + 1 - r_0))\|_{C^{3,\alpha}(E_+)} + O(\frac{1}{X_0} + \delta)\|\xi - r_0\|_{C^{4,\alpha}[-1,1]} + O(\delta + \sigma)\|w\|_{C^{3,\alpha}(E_+)} \\
&\leq O(\frac{1}{X_0} + \delta + \sigma)\delta + O(\frac{1}{X_0} + \delta)\sigma + C\varepsilon. \qquad (A.34)
\end{aligned}
$$

Due to (A.32) and the definition of Ξ_δ, one can check from (A.33) by following the proof of Lemma A.2 that

$$\partial_2 U_1(z_1, \pm 1) = 0. \qquad (A.35)$$

Step 4. A mapping on Ξ_δ.

Note that the coefficients of ε in (A.31) and (A.34) depend only on the background solution and then are uniformly bounded. Hence, one can select proper constants $\sigma = O(1)\varepsilon > 0$ and $\delta = O(1)\varepsilon > 0$ such that the solution $(\rho, U_1, U_2; \xi)$ obtained in Step 1-Step 3 satisfies

$$\|\xi - r_0\|_{C^{4,\alpha}[-1,1]} \leq \sigma.$$

and

$$\|(\rho_1 U_1) - (\hat{\rho}_0^+, \hat{U}_0^+)(r_0 + z_1(X_0 + 1 - r_0))\|_{C^{3,\alpha}(E_+)} + \|U_2\|_{C^{3,\alpha}(E_+)} \leq \delta.$$

This, together with (A.14), (A.32) and (A.35), shows that

$$\xi \in S_\sigma, \qquad (\rho(z_1, z_2), U_1(z_1, z_2), U_2(z_1, z_2)) \in \Xi_\delta.$$

Therefore, for each $(q, V_1, V_2) \in \Xi_\delta$, by use of Step 1-Step 3, we can define a mapping T from Ξ_δ into itself by

$$T(q, V_1, V_2) = (\rho, U_1, U_2). \qquad (A.36)$$

In order to prove Theorem 6.1$'$, it suffices to show that the mapping T is contractible in $C^{2,\alpha}(E_+)$.

Step 5. Contractible estimate on the mapping T.

For any given $(\tilde{\rho}, \tilde{U}_1, \tilde{U}_2)$ and $(\tilde{q}, \tilde{V}_1, \tilde{V}_2)$ in Ξ_δ, set

$$T(\tilde{\rho}, \tilde{U}_1, \tilde{U}_2) = (\rho, U_1, U_2), \qquad T(\tilde{q}, \tilde{V}_1, \tilde{V}_2) = (q, V_1, V_2).$$

The corresponding approximate shocks $\xi_1(z_2)$ and $\xi_2(z_2)$ can be obtained from (A.12).

As in §4, define $W_i (i = 1, 2, 3), M_j (j = 1, 2, 3), N_k (k = 1, 2, 3)$ corresponding to (ρ, U_1, U_2) and (q, V_1, V_2), and define $\tilde{W}_i (i = 1, 2, 3, 4), \tilde{M}_j (j = 1, 2, 3), \tilde{N}_k (k = 1, 2, 3, 4)$ in terms of $(\tilde{\rho}, \tilde{U}_1, \tilde{U}_2; \xi_1)$ and $(\tilde{q}, \tilde{V}_1, \tilde{V}_2; \xi_2)$. In addition, set $W_5 = \frac{U_2}{U_1} - \frac{V_2}{V_1}$ and $\tilde{W}_5 = \frac{\tilde{U}_2}{\tilde{U}_1} - \frac{\tilde{V}_1}{\tilde{V}_2}$.

We first establish some estimates on T.

By (A.13), one has

$$
\begin{cases}
\tilde{N}_4(z_2) = O(\delta)\tilde{W}_1 + O(1)\tilde{W}_2 + O(\delta)\tilde{W}_3 + O(\frac{\delta}{X_0})\tilde{W}_4 \qquad in \qquad (-1, 1), \\
\tilde{W}_4(1) = 0.
\end{cases} \qquad (A.37)
$$

This implies that

$$\|\tilde{W}_4\|_{C^{3,\alpha}[-1,1]} \leq C\delta\|\tilde{W}_1\|_{C^{2,\alpha}(\bar{E}_+)} + C\|\tilde{W}_2\|_{C^{2,\alpha}(\bar{E}_+)} + C\delta\|\tilde{W}_3\|_{C^{2,\alpha}(\bar{E}_+)}. \qquad (A.38)$$

By (A.18) and (A.28), one has

$$
\begin{cases}
\partial_1 W_5+\lambda_1\partial_2 W_3=O(\frac{\sigma}{X_0})\tilde{W}_1+O(\frac{\varepsilon+\sigma}{X_0}+\delta^2)\tilde{W}_3+O(\delta^2)\tilde{W}_4+O(\frac{1}{X_0})\tilde{W}_5\\
\qquad\qquad\qquad +O(\delta)\tilde{M}_3+O(\delta)\tilde{N}_3+O(\frac{1}{X_0})\tilde{N}_4,\\
\partial_2 W_5-\lambda_2\partial_1 W_3=O(\frac{1}{X_0^2}+\frac{1}{X_0})\tilde{W}_1+O(\frac{1}{X_0}+\delta^2)\tilde{W}_3+O(\frac{\delta}{X_0}+\frac{\sigma\delta}{X_0}+\sigma\delta)\tilde{W}_4+O(\frac{\sigma}{X_0}+\delta)\tilde{W}_5\\
\qquad\qquad\qquad +O(\delta)\tilde{M}_3+O(\delta)\tilde{N}_3+O(\sigma)\partial_{z_1}\tilde{W}_5+O(\frac{\delta}{X_0})\tilde{N}_4,\\
W_3=O(\sigma)\tilde{W}_1+O(\sigma)\tilde{W}_2+O(\delta\sigma)\tilde{W}_3+O(\frac{1}{X_0}+\sigma\delta)\tilde{W}_4+O(\delta)\tilde{N}_4 \qquad on \qquad z_1=0,\\
W_3=O(\sigma)\tilde{W}_1+O(\sigma)\tilde{W}_2+O(\delta\sigma)\tilde{W}_3+O(\frac{1}{X_0}+\sigma\delta)\tilde{W}_4+O(\delta)\tilde{N}_4 \qquad on \qquad z_1=1,\\
W_5=0 \qquad on \qquad z_2=-1,\\
W_5=0 \qquad on \qquad z_2=1.
\end{cases}
\tag{A.39}
$$

So following the arguments in Step 2, one can arrive at

$$
\begin{aligned}
&\|W_3\|_{C^{2,\alpha}(\bar{E}_+)}+\|W_5\|_{C^{2,\alpha}(\bar{E}_+)}\le C(\frac{1}{X_0}+\sigma)\|\tilde{W}_1\|_{C^{2,\alpha}(\bar{E}_+)}+C\sigma\|\tilde{W}_2\|_{C^{2,\alpha}(\bar{E}_+)}\\
&+C(\frac{1}{X_0}+\delta)\|\tilde{W}_3\|_{C^{2,\alpha}(\bar{E}_+)}+C(\frac{1}{X_0}+\delta)\|\tilde{W}_4\|_{C^{2,\alpha}[-1,1]}+C(\frac{1}{X_0}+\sigma+\delta)\|\tilde{W}_5\|_{C^{2,\alpha}(\bar{E}_+)}.
\end{aligned}
$$

It follows from (A.38) and the expression of W_5 that

$$
\begin{aligned}
\|W_3\|_{C^{2,\alpha}(\bar{E}_+)}+\|W_5\|_{C^{2,\alpha}(\bar{E}_+)}\le\ & C(\frac{1}{X_0}+\sigma)\|\tilde{W}_1\|_{C^{2,\alpha}(\bar{E}_+)}+C(\frac{1}{X_0}+\sigma+\delta)\|\tilde{W}_2\|_{C^{2,\alpha}(\bar{E}_+)}\\
&+C(\frac{1}{X_0}+\delta)\|\tilde{W}_3\|_{C^{2,\alpha}(\bar{E}_+)}.
\end{aligned}
\tag{A.40}
$$

In addition, due to (A.33), one can calculate to obtain

$$
W_1=O(\delta)W_2+O(1)W_3+O(\sigma\delta)\tilde{W}_1+O(\sigma)\tilde{W}_2+O(\sigma\delta)\tilde{W}_3+O(\frac{1}{X_0}+\sigma\delta)\tilde{W}_4+O(\delta)\tilde{N}_4+O(\delta)(\beta-\tilde{\beta}),
\tag{A.41}
$$

where β and $\tilde{\beta}$ stand for the starting points from the transonic shock of two characteristics respectively, whose definitions are given in (4.8).

As in Lemma 4.4, one can obtain the following estimate

$$
\|\beta-\tilde{\beta}\|_{C^{2,\alpha}(\bar{E}_+)}\le C(\delta\|\tilde{W}_1\|_{C^{2,\alpha}(\bar{E}_+)}+\|\tilde{W}_2\|_{C^{2,\alpha}(\bar{E}_+)}+\delta\|\tilde{W}_4\|_{C^{2,\alpha}[-1,1]}).
\tag{A.42}
$$

Then, it follows from (A.41) and (A.42) that

$$
\begin{aligned}
\|W_1\|_{C^{2,\alpha}(\bar{E}_+)}\le\ & C\delta\|W_2\|_{C^{2,\alpha}(\bar{E}_+)}+C\|W_3\|_{C^{2,\alpha}(\bar{E}_+)}+C(\sigma\delta+\delta^2)\|\tilde{W}_1\|_{C^{2,\alpha}(\bar{E}_+)}+C(\sigma+\delta)\|\tilde{W}_2\|_{C^{2,\alpha}(\bar{E}_+)}\\
&+C\sigma\delta\|\tilde{W}_3\|_{C^{2,\alpha}(\bar{E}_+)}+C(\frac{1}{X_0}+\delta)\|\tilde{W}_4\|_{C^{3,\alpha}[-1,1]}.
\end{aligned}
$$

This, together with (A.38) and (A.40), yields

$$
\|W_1\|_{C^{2,\alpha}(\bar{E}_+)}\le C(\frac{1}{X_0}+\sigma)\|(\tilde{W}_1,\tilde{W}_3)\|_{C^{2,\alpha}(\bar{E}_+)}+C(\frac{1}{X_0}+\sigma+\delta)\|\tilde{W}_2\|_{C^{2,\alpha}(\bar{E}_+)}.
\tag{A.43}
$$

Combining (A.38) with (A.43) yields

$$\|(W_1, W_2, W_3)\|_{C^{2,\alpha}(\bar{\Omega})} \leq C(\frac{1}{X_0} + \sigma)\|(\tilde{W}_1, \tilde{W}_3)\|_{C^{2,\alpha}(\bar{\Omega})} + C(\frac{1}{X_0} + \sigma + \delta)\|\tilde{W}_2\|_{C^{2,\alpha}(\bar{\Omega})} + \frac{C}{X_0}\|\tilde{W}_4\|_{C^{2,\alpha}[-1,1]}.$$

Thus, combining this with the estimate (A.38), we arrive at

$$\|(W_1, W_2, W_3)\|_{C^{2,\alpha}(\bar{\Omega})} \leq C(\frac{1}{X_0} + \sigma)\|(\tilde{W}_1, \tilde{W}_3)\|_{C^{2,\alpha}(\bar{\Omega})} + C(\frac{1}{X_0} + \sigma + \delta)\|\tilde{W}_2\|_{C^{2,\alpha}(\bar{\Omega})}. \tag{A.44}$$

This shows that the mapping T is contractible in $C^{2,\alpha}(E_+)$ for suitably small δ, σ and X_0^{-1}. In fact, as stated in Step 4, we can choose $\sigma = O(1)\varepsilon > 0$ and $\delta = O(1)\varepsilon > 0$.

Therefore, the system (A.5), (A.6) and (A.10) has a unique solution

$$(\rho(z), U_1(z), U_2(z); \xi(z_2))$$

when the exit pressure condition in (A.6) is adjusted by a unique constant C_0 (determined by the integral equality (A.28)). Since the coordinate transformation (A.4) is reversible and keeps the equivalence of $C^{4,\alpha}$ norms between the two coordinates (z_1, z_2) and (y_1, y_2) for $r_0 \in (X_0, X_0 + 1)$ and suitably small $\sigma = O(\varepsilon)$, then we finish the proof of Theorem 6.1′ and Theorem 6.1.

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