THE TRANSONIC SHOCK IN A NOZZLE FOR 2-D AND 3-D COMPLETE EULER SYSTEMS

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Abstract

In this paper, we study a transonic shock problem for the Euler flows through a class of 2-D or 3-D nozzles. The nozzle is assumed to be symmetric in the diverging (or converging) part. If the supersonic incoming flow is symmetric near the divergent (or convergent) part of the nozzle, then, as indicated in [11], there exist two constant pressures P_1 and P_2 with $P_1 < P_2$ such that for given constant exit pressure $P_e \in (P_1, P_2)$, a symmetric transonic shock exists uniquely in the nozzle, and the position and the strength of the shock is completely determined by P_e. Moreover, it is shown in this paper that such a transonic shock solution is unique under the restriction that the shock goes through the fixed point at the wall in the Multi-dimensional setting. Furthermore, we establish the global existence, stability and the long time asymptotic behavior of a unsteady symmetric transonic shock under the exit pressure P_e when the initial unsteady shock lies in the symmetric diverging part of the 2-D or 3-D nozzle. On the other hand, it is shown that a unsteady symmetric transonic shock is structurally unstable in a global-in-time sense if it lies in the symmetric converging part of the nozzle.

Keywords: Steady Euler equation, unsteady Euler equation, supersonic flow, subsonic flow, transonic shock, nozzle, Cauchy-Riemann equation

Mathematical Subject Classification: 35L70, 35L65, 35L67, 76N15

§1. Introduction and the main results

This is a continuation of our studies on the transonic shock problem in a nozzle [28-30]. In [29-30], under the assumptions that the flow is steady, isentropic and irrotational, we use the potential equation to study the well-posedness or ill-posedness of a transonic shock to the steady flow through a general 2-D or 3-D slowly variable nozzle with a large exit pressure induced by the appropriate boundary condition on the exit. In [28], the ill-posedness results in [29-30] was extended to the 2-D complete Euler flow case when the nozzle is arbitrary but slightly curved. However, for a suitably curved 2-D nozzle with symmetric supersonic incoming flows, as indicated in Section 147 of [11], it is shown in Theorem 5.2 of [28] that there exist two constant pressures P_1 and P_2 with $P_1 < P_2$ which depend only on the incoming flow and the shape of the nozzle, such that if the exit pressure $P_e \in (P_1, P_2)$, then for the 2-D complete steady Euler system, a unique symmetric transonic

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shock exists in the diverging (or converging) part of the nozzle. In this paper, we first study the well-posedness of the steady transonic shock problem when a steady symmetric supersonic incoming flow goes through a slightly curved 2-D or 3-D nozzle whose diverging (or converging) part is symmetric with an appropriately given constant exit pressure at the exit of the nozzle. Although the existence and uniqueness in the symmetric class can be established quite easily by theory for ordinary differential equations, the uniqueness of such a symmetric transonic shock in the multi-dimensional setting requires more delicate analysis. Next, we focus on the unsteady transonic shock problem. More precisely, for symmetric unsteady supersonic incoming flows through a symmetric De Laval nozzle with an appropriate constant pressure at the exit of the nozzle, we will establish the global existence, stability and the long time asymptotic behavior of a unsteady symmetric transonic shock in a nozzle when the initial shock lies in the diverging part. On the other hand, it is shown that a unsteady symmetric transonic shock is structurally unstable in a global-in-time sense if it lies in the converging part as observed in physical experiments and numerical computations.

The m-dimensional complete compressible Euler system can be written as

$$\begin{cases} \partial_t \rho + div(\rho u) = 0, \\ \partial_t(\rho u) + div(\rho u \otimes u) + \nabla P = 0, \\ \partial_t \left(\rho\left(e + \dfrac{|u|^2}{2}\right)\right) + div\left(\left(\rho\left(e + \dfrac{|u|^2}{2}\right) + P\right) u\right) = 0, \end{cases} \tag{1.1}$$

where $u = (u_1, \cdots, u_m)$ is the velocity, ρ, P, e and S represent the density, the pressure, the internal energy and the specific entropy respectively. Moreover, the equations of states, $P = P(\rho, S)$ and $e = e(\rho, S)$, are assumed to be smooth such that $\partial_\rho P(\rho, S) > 0$ and $\partial_S e(\rho, S) > 0$ for $\rho > 0$. For convenience, we sometimes write the equations of states as $\rho = \rho(P, S)$ and $e = e(P, S)$. In the case of the ideal polytropic gases, the equations of states read as

$$P = A\rho^\gamma e^{\frac{S}{c_v}}, \qquad and \qquad e = \frac{P}{(\gamma - 1)\rho},$$

here A, c_v and γ are positive constants, and $1 < \gamma < 2$. The sound speed c is given by $c^2 = \partial_\rho P(\rho, S)$.

In the case of steady flows, the system (1.1) is reduced to

$$\begin{cases} div(\rho u) = 0, \\ div(\rho u \otimes u) + \nabla P = 0, \\ div\left(\left(\rho\left(e + \dfrac{|u|^2}{2}\right) + P\right) u\right) = 0. \end{cases} \tag{1.2}$$

We now describe the classes of nozzles and supersonic incoming flows we are going to study. Let X_0 be any fixed positive constant. First, for 2-D case, it is assumed that the walls of the nozzle are given by two curves Γ_1 and Γ_2, which are C^4-regular for $r \in [X_0, X_0 + 1]$ with $r = |x| \equiv \sqrt{x_1^2 + x_2^2}$. Furthermore, we assume that Γ_i can be decomposed into two curves Π_i^1 and Π_i^2 such that Π_1^1 and Π_2^1 include the converging part of the nozzle while Π_1^2 and Π_2^2 form a two-dimensional angular section with its vertex at the origin $(0, 0)$, more precisely, Π_i^2 is given by

$$x_2 = (-1)^i x_1\, tg\alpha_0 \qquad for \qquad r \in \left(X_0 + \frac{1}{4}, X_0\right), \alpha_0 \in \left(0, \frac{\pi}{2}\right),$$

so that Π_1^2 and Π_2^2 form a portion of the diverging part of the nozzle, see Figure 1.

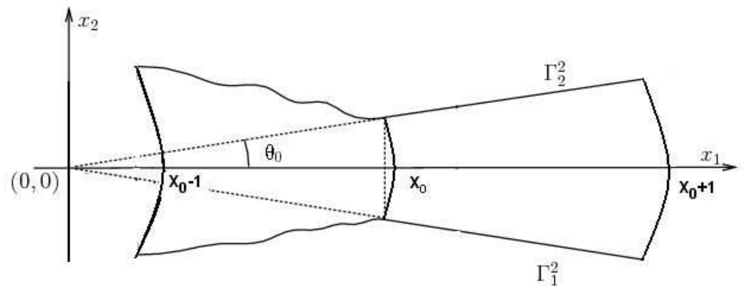

Figure 1

Similarly, for the 3-Dimensional case, the wall of the nozzle, Γ, is assumed to be C^4-regular for $r \in [X_0, X_0+1]\,(r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2})$, such that Γ is a disjoint union of Π_1 and Π_2 with Π_2 being part of a circular cone surface given by

$$x_2^2 + x_3^2 = x_1^2\, tg^2\alpha_0 \qquad for \qquad x_1 > 0, \quad r \in \left[X_0 + \frac{1}{4}, X_0 + 1\right],$$

where α_0 is a positive constant, $\alpha_0 \in \left(0, \frac{\Pi}{2}\right)$. See Figure 2.

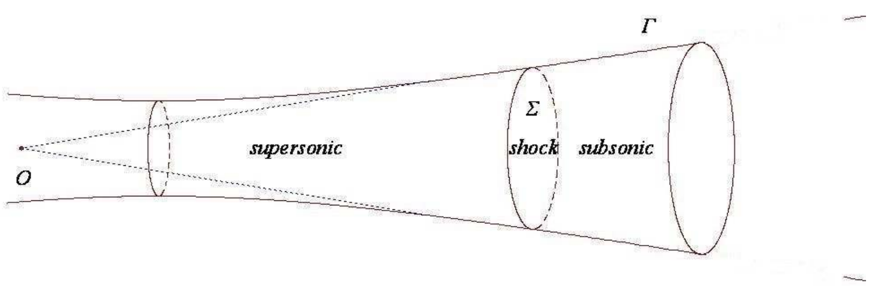

Figure 2

We will assume that the steady supersonic incoming flow, $(\rho_0^-, u_0^-, S_0^-)(x)$, is C^3-smooth and symmetric near $r = X_0 + \frac{1}{2}$, i.e.,

$$\rho_0^-(x) = \rho_0^-(r), \quad u_0^-(x) = U_0^-(r)\frac{x}{r}, \quad S_0^- = constant \quad near \quad r = X_0 + \frac{1}{2}.$$

We now focus on the well-posedness of steady transonic shock solution. First, by an analyzing some systems of ordinary differential equations as in Section 147 of [11], one can obtain the following existence and uniqueness of symmetric transonic shock solutions:

Theorem 1.1. (Existence) *Let m-dimensional nozzle and the steady supersonic incoming flow be given as above. Then there exist two constant pressures P_1 and P_2 with $P_1 < P_2$, which are determined by the coming flow and the nozzle, such that if the end pressure $P_e \in (P_1, P_2)$, then the system (1.1) has a unique symmetric transonic shock solution*

$$(P, u, S) = \begin{cases} (P_0^-(r), u_0^-(x), S_0^-), & \text{for } r < r_0, \\ (P_0^+(r), u_0^+(x), S_0^+), & \text{for } r > r_0, \end{cases}$$

here $u_0^+(x) = U_0^+(r)\frac{x}{r}$, S_0^+ is a constant, and $(P_0^+(r), U_0^+(r))$ is C^3-smooth. Moreover, the position $r = r_0$ with $r_0 \in (X_0 + \frac{1}{2}, X_0 + 1)$ and the strength of the shock are uniquely determined by P_e.

Remark 1.1. *Although the proof of Theorem 1.1 can be carried out as sketched for the 2-D case in [28], yet for completeness, we still give a detailed proof, which yields more useful estimates of the solutions that will be used in the later analysis.*

Next, we turn to the uniqueness of the symmetric transonic shock solution constructed in Theorem 1.1 in a large class which is not necessarily symmetric. Assume that the shock Σ is given by $x_1 = \xi(x')$ with $x' = (x_2, \cdots, x_m)$, and the flow behind the shock is denoted by $(\rho^+, u^+, S^+)(x)$. The Rankine-Hugoniot conditions on Σ read:

$$\begin{cases} [(1, -\nabla_{x'}\xi(x')) \cdot \rho u] = 0, \\ [((1, -\nabla_{x'}\xi(x')) \cdot \rho u)u] + (1, -\nabla_{x'}\xi(x'))^t\,[P] = 0, \\ [(1, -\nabla_{x'}\xi(x')) \cdot \left(\rho\left(e + \frac{1}{2}|u|^2\right) + P\right) u] = 0, \end{cases} \tag{1.3}$$

where $\rho = \rho(P, S)$. Then entropy condition requires (see [11])

$$P^+(x) > P^-(x) \qquad on \qquad \Sigma. \tag{1.4}$$

At the exit of the nozzle, one poses the following end pressure condition

$$P^+(x) = P_e \qquad for \qquad |x| = r = X_0 + 1, \tag{1.5}$$

here the constant pressure P_e is given as in Theorem 1.1. A natural boundary condition on the wall of the nozzle, Γ, is the no-flow condition, which reads as

$$u^+ \cdot \left(tg\,\alpha_0, -\frac{x'}{|x'|}\right) = 0 \qquad on \quad \Pi_2, \tag{1.6}$$

for 3-D, and

$$u_2^+ = f_i'(x_1)u_1^+ \qquad on \qquad \Pi_2^i, \tag{1.7}$$

where $x_2 = f_i(x_1) \equiv (-1)^i x_1\, tg\,\alpha_0$ for 2-D. Let Ω_+ be the subsonic region, i.e.,

$$\Omega_+ = \{x : \ \xi(x') < x_1 < \sqrt{(X_0+1)^2 - |x'|^2}, \ |x'|^2 < x_1^2\, tg^2\,\alpha_0\},$$

D is the projection of Σ onto x'-plane, $L = \bar{\Sigma} \cap \Gamma$, and $x^0 \in \Pi_2$ be a fixed point. Finally, we assume that X_0 is suitably large and α_0 is sufficiently small so that

$$\left(X_0 + \frac{1}{2}\right) tg\,\alpha_0 = 1, \qquad \frac{\eta}{2} < \alpha_0 < \eta_0 \tag{1.8}$$

hold, where η_0 is a small constant. We note that the condition (1.8) implies that Π_2 is close to the cylinder $|x'| = 1$ for $r \in [X_0 + \frac{1}{4}, X_0 + 1]$.

We now can state our main uniqueness theorem.

Theorem 1.2. (Uniqueness)

Let the assumptions in Theorem 1.1 and (1.8) hold. Then the steady transonic shock problem, (1.2)-(1.7), has no more than one pair of solution $(P^+(x), u^+(x), S^+(x); \xi(x'))$ with the following properties:

(i) There exists positive constants $\delta_0 \in (0,1)$ and ε such that $\xi \in C^{3,\delta_0}(\bar{D})$, $x^0 \in \bar{\Pi}_2 \cap \bar{\Sigma}$, (i.e., $x_1^0 = \xi(x')$), and

$$\left\| \xi(x') - \sqrt{r_0^2 - |x'|^2} \right\|_{C^{3,\delta_0}(\bar{D})} \leq \varepsilon, \tag{1.9}$$

where r_0 is given as in Theorem 1.1, and ε depends only on η_0 and the incoming supersonic flow.

(ii) $(P^+, u^+, S^+)(x) \in C^{2,\delta_0}(\bar{\Omega}_+) \cap C^3(\Omega_+)$ such that

$$\|(P^+(x) - \hat{P}_0^+(r), u^+(x) - \hat{u}_0^+(x), S^+(x) - S_0^+)\|_{C^{2,\delta_0}(\bar{\Omega}_+)} \leq \varepsilon, \tag{1.10}$$

here $\hat{u}_0^+(x) = \hat{U}_0^+(r)\frac{x}{r}$, and $(\hat{P}_0^+(r), \hat{U}_0^+(r))$ stands for the extension $(P_0^+(r), U_0^+(r))$ in Ω_+.

Remark 1.2. *For 3-D, the uniqueness holds with less regularity. Indeed, it suffices to assume that $\xi \in C^{2,\delta_0}(\bar{D})$ such that*

$$\left\| \xi(x') - \sqrt{r_0^2 - |x'|^2} \right\|_{C^{2,\delta_0}(\bar{D})} \leq \varepsilon \tag{1.9'}$$

and $(P^+, S^+)(x) \in C^{2,\delta_0}(\bar{\Omega}_+) \cap C^3(\Omega_+)$, $u^+(x) \in C^{1,\delta_0}(\bar{\Omega}_+) \cap C^2(\Omega_+)$ such that

$$||(P^+(x) - \hat{P}_0^+(r), S^+(x) - S_0^+)||_{C^{2,\delta_0}(\bar{\Omega}_+)} + ||u^+(x) - \hat{u}_0^+(x)||_{C^{1,\delta_0}(\bar{\Omega}_+)} \leq \varepsilon. \tag{1.10'}$$

This is explained in more details in Appendix B.

Remark 1.3. *It can be shown that the compatibility of the boundary conditions (1.3) and (1.6)-(1.7) holds at the corner $L = \bar{\Sigma} \cap \Pi_2$ (see Lemma 4.2 in §4 and Lemma 6.1 in §6). Thus, the assumptions on the regularities of the solution $(P^+(x), u^+(x), S^+(x); \xi(x'))$ in Theorem 1.2 are plausible. This follows from Remark 1.1 in [28] (or one can see [2-3], [19-20], and the references therein) for 2-D. In the 3-D case, an explanation is given in the Appendix B. It is interesting that such a compatibility condition is satisfied naturally for any $C^1(\bar{\Omega}_+)$-regular solution in contrast to the general unsteady shocks [23-24].*

Remark 1.4. *It can be verified (see §2) that $(P_0^+(r), U_0^+(r))$ in Theorem 1.1 can be the domain $\{x : X_0 + \frac{1}{4} \leq r \leq X_0 + 1, |x'| \leq x_1 \, tg\, \alpha_0\}$, so that $(\hat{P}_0^+(r), \hat{U}_0^+(r))$ in Theorem 1.2 is well-defined.*

Remark 1.5. *Consider a general 2-D nozzle. Let the diverging part of the walls of nozzle, given by $\Gamma_i : x_2 = f_i(x_1)$, $i = 1, 2$, be curved slightly and intersect the shock surface Σ at the point $x^i = (x_1^i, x_2^i)$. Then a necessary condition for the existence of a weak transonic shock solution $(P^+(x), u^+(x), S^+(x); \xi(x_2)) \in C^1(\bar{\Omega}_+)$ is that $f_i''(x_1^i) = 0$. This implies that, in general, one cannot expect the existence of a $C^1(\bar{\Omega}_+)-$regular transonic shock solution in the diverging part of a 2-D De Laval nozzle. The proof of this fact is given in Appendix A.*

Remark 1.6. *It follows from the proof of Theorem 1.2 that one can actually obtain a more general uniqueness result even if the supersonic coming flow is not symmetric and the nozzle walls are general but slightly curved for $r_0 + \delta < r < X_0 + 1$ with a fixed constant $\delta > 0$.*

Remark 1.7. *In order to illustrate the validity of C^{1,δ_0} regularity of the solution to (1.2), we require that the function $G(M_0^-) \neq 0$, where $G(M_0^-) = \frac{(2-\gamma)(M_0^-)^2}{\mu^2(M_0^-)} + \frac{2-\gamma}{2}\big(\mu^2(M_0^-) - 1\big) + \frac{3\mu(M_0^-) - 1}{\mu(M_0^-) - 1}$ with $M_0^- = \frac{U_0^-(r_0)}{c(\rho^-(r_0), S_0^-)}$ and $\mu(M_0^-) = \frac{U_0^+(r_0)}{U_0^-(r_0)}$, see Lemma 6.1 of §6 for more details. It follows from this and Appendix B that the assumptions on the regularities of solution $(P^+(x), u_1^+(x), u_2^+(x), u_3^+(x), S^+(x); \xi(x_2, x_3))$ in Theorem 1.2 are plausible.*

Remark 1.8. *For the unsteady multidimensional compressible Euler systems, A.Majda in [23-24] has shown the existence and stability of a multidimensional shock under the appropriate compatibility conditions on the discontinuous initial data along the initial shock curve. But for the steady transonic multidimensional Euler system (1.2), the compatibility condition will be satisfied naturally for any $C^1(\bar{\Omega}_+)-$regular solution (see Lemma 6.1 and Remark 6.2). This is an interesting fact.*

Analogously, we can study the well-posedness of steady transonic shock in a nozzle with a symmetric converging part. Indeed, consider a m-D ($m = 2, 3$) nozzle whose wall contains a straight section given by

$$|x'| = |x_1| tg\,\alpha_0, \quad x_1 < 0, \quad \alpha_0 \in \left(0, \frac{\pi}{2}\right), \quad for \quad X_0 \le r \equiv |x| < X_0 + 1.$$

In addition, we assume that the supersonic incoming flow is C^3-smooth, isentropic, and symmetric near $r = X_0 + \frac{3}{4}$, which can represented as $(P_0^-(x), U_0^-(x)) = (P_0^-(r), -U_0^-(r)\frac{x}{r})$ near $r = X_0 + \frac{3}{4}$. Then as a counter part of Theorem 1.1-1.2, we can show

Theorem 1.3. *Let the nozzle and the supersonic incoming flow be as described above. Assume further that the flow is isentropic. Then for suitably large $X_0 > 0$, there exist two constant pressures P_1 and P_2 with $P_1 < P_2$, such that for $P_e \in (p_1, p_2)$, the steady transonic shock problem (1.2)-(1.8) has a unique solution given by*

$$(P, u)(x) = \begin{cases} (P_0^-(r), u_0^-(x)), & r > r_0, \\ (P_0^+(r), u_0^+(x)), & r < r_0, \end{cases}$$

where $U_0^+(x) = -U_0^+(r)\frac{x}{r}$, $r_0 \in (X_0, X_0 + \frac{3}{4})$ is uniquely determined by P_e, and $(P_0^+(r), U_0^+(r))$ is C^3-smooth.

Remark 1.9. *In Theorem 1.3, the uniqueness is in the class which can be described analogously as in Theorem 1.2.*

Remark 1.10. *The proof of Theorem 1.3 is similar to that of Theorem 1.1-1.2, for completeness, we give the sketch in Appendix C.*

Next, we turn to the problem of dynamical stability of a steady symmetric transonic shock, constructed in Theorem 1.1 and Theorem 1.4, under small generic unsteady symmetric perturbations for simplicity in presentation, we will only study the isentropic flows. We start with transonic shocks in a symmetric expanding nozzle. Thus suppose that the initial flow is a small perturbation of the steady symmetric transonic shock solution, $(\rho_0^\pm(r), U_0^\pm(r))$ for $r \in [X_0 + \frac{1}{4}, X_0 + 1]$, given in Theorem 1.1, i.e.

$$\rho^\pm(0, r) = \rho_0^\pm(r) + \varepsilon\rho_1^\pm(r), \qquad U^\pm(0, r) = U_0^\pm(r) + \varepsilon U_1^\pm(r), \quad r \in [X_0 + \frac{1}{4}, X_0 + 1], \tag{1.11}$$

where $(\rho_0^\pm(r), U_0^\pm(r))$ is defined in Theorem 1.1 with $\rho_0^\pm(r) = (\frac{P_0^\pm(r)}{A})^{\frac{1}{\gamma}}$, and $(\rho_1^-(r), U_1^-(r)) \in C_0^2(X_0 + \frac{1}{4}, r_0)$ and $(\rho_1^\pm(r), U_1^\pm(r)) \in C_0^2(r_0, X_0 + 1)$, and $\varepsilon > 0$ is a suitably small constant. We will impose the following unsteady boundary condition at the entry and the exit of the nozzle:

$$(\rho^-, U^-)\left(t, r = X_0 + \frac{1}{4}\right) = (\rho_0^-, U_0^-)\left(X_0 + \frac{1}{4}\right) + \varepsilon(\rho_2^-, U_2^-)(t), \tag{1.12}$$

and

$$\rho^+(t, r = X_0 + 1) = \rho_e + \varepsilon\rho_2^+(t), \tag{1.13}$$

here $(\rho_2^-(t), U_2^-(t); \rho_2^+(t)) \in C_0^2(0, +\infty)$ and $\rho_e = (\frac{P_e}{A})^{\frac{1}{\gamma}}$ with P_e as given in Theorem 1.1.

Let the unsteady shock front Σ be denoted by $r = r(t)$ and the flow field before and behind the shock be given by $(\rho^-, U^-)(t, r)$ and $(\rho^+, U^+)(t, r)$ respectively. It then follows from (1.1) for isentropic flows for $r \gtrless r(t)$,

$$\begin{cases} \partial_t\rho^\pm + \partial_r(\rho^\pm U^\pm) + \frac{m-1}{r}\rho^\pm U^\pm = 0, \\ \partial_t(\rho^\pm U^\pm) + \partial_r(\rho^\pm (U^\pm)^2 + P^\pm) + \frac{m-1}{r}\rho^\pm (U^\pm)^2 = 0. \end{cases} \tag{1.14}$$

On the shock front Σ, the Rankine-Hugoniot conditions become

$$\begin{cases} [\rho] r'(t) - [\rho U] = 0, \\ [\rho U] r'(t) - [\rho U^2 + P] = 0. \end{cases} \tag{1.15}$$

In addition, $(\rho^\pm, U^\pm)(t,r)$ should satisfy Lax's entropy condition ([27]):

$$\lambda_1(\rho^+, U^+)(t, r(t)+0) < r'(t) < \lambda_1(\rho^-, U^-)(t, r(t)-0), \qquad r' < \lambda_2(\rho^+, U^+)(t, r(t)+0) \tag{1.16}$$

with $\lambda_1(\rho, U) = U - c(\rho)$ and $\lambda_2(\rho, U) = U + c(\rho)$.

Then we have the following nonlinear stability result for a transonic shock in an expending nozzle:

Theorem 1.4. (Global Existence and Dynamical Stability) *Consider the problem of unsteady transonic shocks in an expanding symmetric nozzle as described above. Assume that $X_0 > 0$ is suitably large and the steady transonic shock $(\rho_0^\pm, U_0^\pm)(r)$, $r \in [X_0 + \frac{1}{4}, X_0 + 1]$ is weak in the sense that $0 < \min(U_0^\pm(r_0) - c(\rho_0^\pm(r_0))) \le \max(U^\pm(r_0) - c(\rho_0^\pm(r_0))) < \delta_0$ for a suitably small positive constant δ_0. Then there exists a positive constant ε_0 such that for $\varepsilon \le \varepsilon_0$, the initial-boundary value problem (1.11)-(1.16) has a unique global $(\rho^\pm, U^\pm; r(t))$ with the property that $r(t) \in C^2[0, +\infty)$, and $(\rho^\pm, U^\pm)(t,x)$ is C^2-smooth for $r \gtrless r(t)$. Furthermore, the location $r = r(t)$ of the shock front and the flow field after the shock, $(\rho^+, U^+)(t,r)$ tend to $r = r_0$ and $(\hat\rho_0^+, \hat U_0^+)(r)$ respectively with a rate of decay as $(1+t)^{-2}$, here $(\hat\rho_0^+, \hat U_0^+)(r)$ denotes the extension of $(\rho_0^+, U_0^+)(r)$ for $r \in [X_0 + \frac{1}{4}, X_0 + 1]$.*

Remark 1.11. *Theorem 1.4 shows that a weak steady symmetric transonic shock in an expanding symmetric nozzle globally (in time) nonlinear stable for generic unsteady symmetric perturbations with prescribed pressure condition at the exit of the nozzle. Furthermore, it is remarkable that the solution is globally (in time) piecewise smooth and there are no other discontinuities in the solution besides the main perturbed transonic shock, which are in sharp contrast to the theory if Cauchy problems in [21-22, 1].*

Remark 1.12. *The boundary condition (1.12) guarantees the global existence of a shock. Otherwise, other singularities may form (see [16], [33] and so on).*

Remark 1.13. *Since the isentropic compressible Euler systems (1.14) are used to describe the transonic flow, then it is plausible to require that the shock is weak in the sense that although $U_0^-(r_0) > c(\rho_0^-(r_0))$ and $U_0^+(r_0) < c(\rho_0^+(r_0)))$, $U_0^-(r_0) - c(\rho_0^-(r_0))$ and $c(\rho_0^+(r_0)) - U_0^+(r_0)$ are suitably small.*

Remark 1.14. *The rate of decay to the steady transonic shock stated in Theorem 1.4 is not optimal. In fact, it follows from the proof of Theorem 1.4 that for any positive m, there exists a positive constant ε_0 depending only on m such that if $\varepsilon < \varepsilon_0$, then the solutions, $(\rho^+(t,r), U^+(t,r); r(t))$ in Theorem 1.4, tends to $(\hat\rho_0^+(r), \hat U_0^+(r); r_0)$ as t approaches to infinity with a rate of order $(1+t)^{-m}$.*

Finally, we study the instability of a m-D steady symmetric transonic shock in a symmetric converging nozzle as given in Theorem 1.3 under generic unsteady small perturbations. For convenience, in this part of the presentation, we will use the variable $\tilde r = -r$ instead of Ω, and denote the states before and behind the shock by $(\tilde\rho^-, \tilde U^-)(t, \tilde r)$ and $(\tilde\rho^+, \tilde U^+)(t, \tilde r)$ respectively.

As in Theorem 1.4, the initial data is assumed to be a small perturbation of the steady symmetric transonic flow $(\rho_0^\pm(\tilde r), U_0^\pm(\tilde r))$ for $\tilde r \in [-X_0 - \frac{3}{4}, -X_0]$, i.e.,

$$(\tilde\rho^\pm, \tilde U^\pm)(0, \tilde r) = (\rho_0^\pm, U_0^\pm)(\tilde r) + \varepsilon(\tilde\rho_1^\pm, \tilde U_1^\pm)(\tilde r), \quad \tilde r \in \left[-X_0 - \frac{3}{4}, -X_0\right], \tag{1.17}$$

where ε is small positive constant, $(\rho_0^\pm, U_0^\pm)(\tilde r)$ is given in Theorem 1.3 with $\rho_0^\pm(\tilde r) = (\frac{P_0^\pm(\tilde r)}{A})^{\frac{1}{\gamma}}$, and $(\tilde\rho_1^-, \tilde U_1^-) \in C_0^2(-X_0 - \frac{3}{4}, -r_0)$ and $(\tilde\rho_1^+, \tilde U_1^+) \in C_0^2(-r_0, -X_0)$.

In addition, the boundary conditions at the entrance and the exit of the nozzle are imposed as:

$$(\tilde\rho^-, \tilde U^-)\left(t, -X_0 - \frac{1}{4}\right) = (\rho_0^-, U_0^-)\left(-X_0 - \frac{3}{4}\right) + \varepsilon(\tilde\rho_2, \tilde U_2)(t) \tag{1.18}$$

and

$$\tilde\rho^+(t, -X_0) = \rho_e + \varepsilon\tilde\rho_2^+(t) \tag{1.19}$$

here $(\tilde\rho_2^-, \tilde U_2^-; \tilde\rho_2^+) \in C_0^2(0, \infty)$ and $\rho_e = (\frac{P_e}{A})^{\frac{1}{\gamma}}$ with P_e given in Theorem 1.3. Denote by $\tilde r = \tilde r(t)$ the unsteady shock front $\tilde\Sigma$. Then it follows from (1.1) that

$$\begin{cases} \partial_t\tilde\rho^\pm + \partial_{\tilde r}(\tilde\rho^\pm\tilde U^\pm) + \dfrac{m-1}{\tilde r}\tilde\rho^\pm\tilde U^\pm = 0, & \tilde r \gtrless \tilde r(t), \\ \partial_t(\tilde\rho^\pm\tilde U^\pm) + \partial_{\tilde r}(\tilde\rho^\pm(\tilde U^\pm)^2 + \tilde P^\pm) + \dfrac{m-1}{\tilde r}\tilde\rho^\pm(\tilde U^\pm)^2 = 0, & r \gtrless \tilde r(t). \end{cases} \tag{1.20}$$

Across the shock front $\tilde{\Sigma}$, the Rankine-Hugoniot conditions are

$$\begin{cases} [\tilde{\rho}]\tilde{r}'(t) - [\tilde{\rho}\tilde{U}] = 0, \\ [\tilde{\rho}\tilde{U}]\tilde{r}'(t) - [\tilde{\rho}\tilde{U}^2 + \tilde{P}] = 0, \end{cases} \tag{1.21}$$

and the Lax's geometrical entropy conditions become

$$\lambda_1(\tilde{\rho}^+, \tilde{U}^+)(t, \tilde{r}(t) + 0) < \tilde{r}'(t) < \lambda_1(\tilde{\rho}^-, \tilde{U}^-)(t, \tilde{r}(t) - 0), \qquad \tilde{r}'(t) < \lambda_2(\tilde{\rho}^+, \tilde{U}^+)(t, \tilde{r}(t) + 0), \tag{1.22}$$

with $\lambda_1(\tilde{\rho}, \tilde{U}) = \tilde{U} - c(\tilde{\rho})$ and $\lambda_2(\tilde{\rho}, \tilde{U}) = \tilde{U} + c(\tilde{\rho})$.

Then we have the following instability result:

Theorem 1.5. (Dynamical Instability) *Let $(\rho_0^\pm, U_0^\pm)(\tilde{r})$ denote a m-D symmetric steady transonic shock solution in a symmetric converging nozzle as described in Theorem 1.3. Assume that $X_0 > 0$ is sufficiently large and the strength of the transonic shock is suitably weak. Then there exist appropriately choosen perturbations $\varepsilon(\tilde{\rho}_1^\pm, \tilde{U}_1^\pm)(\tilde{r})$ and $\varepsilon(\tilde{\rho}_2^-(t), \tilde{U}_2^-(t); \tilde{\rho}_2^+(t))$ of the initial-boundary value such that the solution to the problem (1.17)-(1.22) is asymptotically unstable in the sense that there is no uniform constant $C_0 > 0$ independent of ε such that*

$$\|(\tilde{\rho}^+, \tilde{U}^+)(t, \cdot) - (\hat{\rho}_0^+, \hat{U}^+)(\cdot)\|_{C[\tilde{r}(t), -X_0]} + |\tilde{r}(t) + r_0| + |\tilde{r}'(t)| \le C_0\varepsilon \qquad \textit{for all} \quad t \ge 0. \tag{1.23}$$

It should be noted that there have been many studies on m-Dimensional steady transonic shock waves (see [5-8], [11-12], [17], [25-26], [28-31], [35], and the references therein). In particular, for a flat nozzle of the form $(-N_1, N_2) \times (0, b; 0, b)$ in 3-D, the existence and uniqueness of a transonic shock for the steady compressible Euler are established under the assumptions that the shock front goes through a fixed point and the pressure condition is given with freedom one. However, as conjectured by Courant-Friedrich's in [11], such transonic shock phenomena occur in a class of physically interesting nozzles, such as the De Laval nozzle whose wall cannot be flat, and physically relevant condition at the exit of a nozzle should be a given suitably large pressure. Furthermore, it is of great important to study the effects of geometry of the nozzle and boundary condition, in particular, how to determine the shape and location of the transonic shock front [11]. In [29-30], for 2-D and 3-D steady potential equation, we have established the uniqueness of the transonic shock wave pattern as conjectured by Courant-Friedrich's for general slightly curved finite nozzles with arbitrarily given large pressure at the exit of the nozzle, proved the existence of transonic shock wave solutions in such a nozzle for a class pressures induced by appropriate boundary conditions at the exit of the nozzle, and more surprisingly, the problem is ill-posed in general by showing no such piecewise smooth transonic shock wave pattern for a class of nozzles, which include both De Laval type nozzles and the flat nozzles, for arbitrarily given large pressure at the exit. The ill-posedness results for the potential in [29-30] were extended to the transonic shock problem for the full steady compressible Euler system (1.2) for flat nozzles or slightly curved nozzles with given pressure at the exit in [28]. In this paper, Theorem 1.1, Theorem 1.2 and Theorem 1.3 yield the existence and uniqueness of a steady transonic shock wave pattern for a special class of m-D nozzle with appriately pressure given at the exit of the nozzle.

The studies on the unsteady transonic shocks began with the works of Liu ([21-22]), where he studied the dynamical stability of transonic shock in a duct by Glimm's method for a quasi-one dimensional model of the form:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = -\dfrac{a'(x)}{a(x)}\rho u, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P) = -\dfrac{a'(x)}{a(x)}\rho u^2, \\ \partial_t(\rho E) + \partial_x(\rho E u + P u) = -\dfrac{a'(x)}{a(x)}(\rho E u + P u), \end{cases} \tag{1.24}$$

where $E = e + \frac{|u|^2}{2}$ is the total energy and $a(x)$ is the cross section of the duct. It is shown in [21-22] that flows along the expanding part of the nozzle are asymptotically stable, while flows with standing shock waves in a contracting duct are dynamically unstable by studying weak solutions to Cauchy problems for (1.24) based on Glimm's random choice method. For some recent generalization of the results in [21-22], see [1]. However, these results are different from the results in Theorem 1.4-1.5 in this paper due to the boundary conditions and the structures of the solutions.

We now comment on the proofs of the main results. First, we note that the steady compressible Euler system (1.2) is hyperbolic-elliptic in the subsonic region, it is challenging to investigate even the fixed boundary value problem for such systems. Thus, to prove the m-D uniqueness of a free-boundary value problem, our main strategy is to decompose the m-D full system (1.2) into a second order elliptic equation on the pressure P^+ with some mixed boundary conditions and $m+1$ first order equations on u^+ and S^+ by using the Bernoulli's law. Based on this decomposition and the Rankine-Hugoniot relations, we are able to use the theory for second order elliptic equations and the characteristic method to estimate $(P^+(x)-\hat{P}_0^+(r), u_1^+(x)-\hat{u}_{1,0}^+(x), S^+(x)-S_0^+)$ in the subsonic region $\bar{\Omega}_+$ in terms of $(u_2^+-\hat{u}_{2,0}^+, \cdots, u_m^+-\hat{u}_{m,0}^+)(x)$, which can be estimated by its values on the shock surface Σ and the system (1.2) using the method of characteristics. It should be noted that on the shock surface Σ, $(u_2^+-\hat{u}_{2,0}^+, \cdots, u_m^+-\hat{u}_{m,0}^+)(x)$ is governed by the Cauchy-Riemann system with a natural boundary condition on L, the intersection of Σ with the wall of the nozzle, so its estimate on Σ can be obtained without much difficulties. These will lead to the proof of Theorem 1.2. Next, we turn to the study on unsteady transonic shocks. In the case of the divergent duct, the keys to the asymptotic stability of the symmetric steady transonic shock are some global (in time) uniform decay estimates for $(\rho^+(r,t)-\hat{\rho}_0^+(r), U^+(r,t)-\hat{U}_0^+(r), r(t)-r_0)$ and its derivatives which can be established by making use of the properties of the background solution $(P_0^\pm(r), U_0^\pm(r))$ given in Theorem 1.1. The strategy is similar to that in [18, 32-33]. While for converging nozzle, one of the crucial elements of the analysis for the dynamical instability of the transonic shock is that we can derive an ordinary differential equation on the shock position $\tilde{r}(t)+r_0$ to show that $\tilde{r}(t)$ increases rapidly in time, as motivated by the work in [21], which can yield the unstable phenomena.

The rest of the paper is organized as follows. In §2, we prove Theorem 1.1 and study some useful properties of the steady symmetric shock solutions. In §3, we reformulate the 2-D problem (1.2) with the boundary conditions (1.3)-(1.7) by some useful decomposition of the 4×4 two dimensional full Euler system. In §4, we establish some a priori estimates on the difference $(P^+(x)-\hat{P}_0^+(r), U_1^+(x)-\hat{U}_0^+(r), U_2^+(x), S^+(x)-S_0^+; \xi(x_2)-\sqrt{r_0^2-x_2^2})$ based on the decompositions in §3, which yields the proof of Theorem 1.2 for 2-D. The reformulation of the 3-D problem and the decomposition of the 5×5 full Euler system are given in §5. In §6, using the decompositions in §5, we derive some a priori estimates on $(P^+(x)-\hat{P}_0^+(r), u_1^+(x)-\hat{u}_1^+(x), u_2^+(x)-\hat{u}_2^+(x), u_3^+(x)-\hat{u}_3^+(x), S^+(x)-S_0^+; \xi(x_2,x_3)-\sqrt{r_0^2-x_2^2-x_3^2})$, which yields the proof of Theorem 1.3 for 3-D as in §4. In §7, we give a reformulation on the problem (1.14) with the boundary conditions (1.11)-(1.13) and (1.15)-(1.16). Subsequently, we complete the proof on Theorem 1.4 in §8. Finally, we prove Theorem 1.5 in §9. In Appendix A, the stated fact in Remark 1.4 will be shown. In Appendix B, we will give a detailed explanation on the regularity assumption of solution $(P^+(x), u_1^+(x), u_2^+(x), u_3^+(x), S^+(x); \xi(x_2,x_3))$ in Theorem 1.2. In Appendix C, we will give a proof on Theorem 1.3

In what follows, we will use the following convention:

$O(Y)$ means that there exists a generic constant C such that $|O(Y)|\le CY$, here C is independent of ε and η_0.

§2. The existence of steady symmetric transonic shock solution

In this section, we will sketch the proof of the existence of a steady symmetric transonic solution in Theorem 1.1, and list some important properties of such solutions which will be used later. Details of the analysis can be found in [11, 20, 25, 34, 28].

The proof of Theorem 1.1. Since we are looking for piecewise smooth solutions of (1.2) separated by a transonic shock. We may assume the entropy are piecewise constant S_0^- and S_0^+ before and after the shock. Due to the symmetric properties of the incoming flow and the nozzle, we can look for symmetric solutions of the form $(\rho, u, S)(x) = (\rho_0^\pm(r), U_0^\pm(r)\frac{x}{r}, S_0^\pm)$ for $r \gtrless r_0$. Then the full steady Euler system is reduced to

$$\begin{cases} \frac{d}{dr}(r^{m-1}\rho_0^\pm U_0^\pm) = 0, \\ \frac{d}{dr}\left(\frac{1}{2}(U_0^\pm)^2 + h(\rho_0^\pm, S_0^\pm)\right) = 0, \end{cases} \tag{2.1}$$

where $h(\rho, S)$ is the enthalpy such that $\partial_\rho h(\rho, S) = \dfrac{c^2(\rho, S)}{\rho}$ and $c^2(\rho, S) = \partial_\rho P(\rho, S)$.

Let the location of the shock be given by $r = r_0$ with $r_0 \in [X_0+\frac{1}{2}, X_0+1]$. The Rankine-Hugoniot conditions at $r = r_0$ are

$$\begin{cases} [\rho_0 U_0] = 0, \\ [\rho_0 U_0^2 + P_0] = 0, \\ [\rho_0(\frac{1}{2}U_0^2 + e_0)U_0 + P_0 U_0] = 0. \end{cases} \tag{2.2}$$

Now we divide the proof of Theorem 1.1. II into four steps.

Step 1. For the given supersonic state $(\rho_0^-(r_0), U_0^-(r_0), S_0^-)$, then it follows from (2.2) that there exists a unique subsonic state $(\rho_0^+(r_0), U_0^+(r_0), S_0^+)$ such that (2.2) holds.

This is given in [11, 27] so is omitted here.

Step 2. (2.1) has a unique supersonic solution $(\rho_0^-(r), U_0^-(r), S_0^-)$ for $r \in [X_0 + \frac{1}{4}, X_0 + 1]$.

In fact, due to the radial symmetries of both the nozzle for $r \in [X_0 + \frac{1}{4}, X_0 + 1]$ and $(\rho_0^-, U_0^-, S_0^-)(x)$ at $r = X_0+\frac{1}{2}$, the unique smooth solution to (2.1) should be radial symmetric and satisfies the following relations:

$$\begin{cases} f_1(\rho_0^-, U_0^-, r) \equiv r^{m-1}\rho_0^-(r)U_0^-(r) - C_0 = 0, \\ f_2(\rho_0^-, U_0^-, r) \equiv \dfrac{1}{2}(U_0^-(r))^2 + h(\rho_0^-(r), S_0^-) - C_1 = 0 \end{cases}$$

with $C_0 = (X_0 + \frac{1}{2})^{m-1}\rho_0^-(X_0 + \frac{1}{2})U_0^-(X_0 + \frac{1}{2})$ and $C_1 = \frac{1}{2}(U_0^-(X_0 + \frac{1}{2}))^2 + h(\rho_0^-(X_0 + \frac{1}{2}), S_0^-)$.

Since

$$\begin{cases} \dfrac{dU_0^-}{dr} = \dfrac{(m-1)C_0 c^2(\rho_0^-, S_0^-)}{r^m \rho_0^-((U_0^-)^2 - c^2(\rho_0^-, S_0^-))}, \\ \dfrac{d((U_0^-)^2 - c^2(\rho_0^-, S_0^-))}{dr} = \dfrac{(m-1)\big(2\partial_\rho P(\rho_0^-, S_0^-) + \rho_0^-\partial_\rho^2 P(\rho_0^-, S_0^-)\big)U_0^-}{r^3((U_0^-)^2 - c^2(\rho_0^-, S_0^-))}, \end{cases}$$

then one has

$$(U_0^-(r))^2 - c^2(\rho_0^-(r), S_0^-) \geq (U_0^-(X_0 + \frac{1}{2}))^2 - c^2(\rho_0^-(X_0 + \frac{1}{2}), S_0^-) > 0 \qquad \text{for } r \geq X_0 + \frac{1}{2}. \tag{2.3}$$

This implies that one the interval of existence of $(\rho_0^-, U_0^-, S_0^-)(r)$, $U_0^-(r)$ and $(U_0^-(r))^2 - c^2(\rho_0^-(r), S_0^-)$ are increasing in r, which, in return, implies that $\frac{dU_0^-}{dr}$ is bounded a priorily. This, together (2.3), yields that (2.1) has a unique supersonic solution $(\rho_0^-(r), U_0^-(r), S_0^-)$ for $r \in [X_0 + \frac{1}{2}, X_0 + 1]$.

Step 3. (2.1) has a unique subsonic solution $(\rho_0^+(r), U_0^+(r), S_0^+)$ for $r \in [r_0 - \delta_0, X_0 + 1]$, here $\delta_0 > 0$ is a fixed and small constant. If the assumption (1.8) holds, then the subsonic solution $(\rho_0^+(r), U_0^+(r), S_0^+)$ of (2.1) exists uniquely for $r \in [X_0 + \frac{1}{4}, X_0 + 1]$.

This can be proved as in Step 2.

Step 4. The end pressure $P_e = P_0^+(X_0 + 1)$ is a decreasing function of the shock position $r = r_0$ for $r_0 \in [X_0 + \frac{1}{2}, X_0 + 1]$.

Indeed, for $r_0 \in [X_0 + \frac{1}{2}, X_0 + 1]$, let $(\rho_0^+(r), U_0^+(r), S_0^+(r)) = (\rho_0^+(r), U_0^+(r), S_0^+(r_0))$ for $r \in [r_0, X_0 + 1]$ be the unique subsonic solution given in Step 3. It follows from (2.1) and (2.2) that

$$\begin{cases} r^{m-1}\rho_0^+(r)U_0^+(r) \equiv C_0, \\ \dfrac{1}{2}(U_0^+(r))^2 + h(\rho_0^+(r), S_0^+(r)) \equiv C_1 \end{cases} \tag{2.4}$$

for $r \in [r_0, X_0+1]$, with $h(\rho,S) = e(\rho,S) + \frac{P(\rho,S)}{\rho}$, and C_0 and C_1 are positive constants determined by the incoming supersonic flow. In particular, the end pressure $P_e \equiv P_0^+(X_0+1)$ is the unique solution of

$$F(P_e, S_0^+(r_0)) \equiv \frac{C_0^2}{2(X_0+1)^{2m-2}(\rho_0^+(P_e,S_0^+(r_0)))^2} + h(\rho_0^+(P_e,S_0^+(r_0)), S_0^+(r_0)) - C_1 = 0 \tag{2.5}$$

Note that

$$\frac{\partial F}{\partial P} = \frac{1}{\rho_0^+(X_0+1)}\left(1 - \frac{(U_0^+(X_0+1))^2}{C^2(X_0+1)}\right) > 0, \tag{2.6}$$

and

$$\frac{\partial F}{\partial S} = \left(\frac{(U_0^+(X_0+1))^2}{\rho_0^+(X_0+1)} + \frac{\gamma-1}{\gamma}\frac{P_e}{\rho_0^+(X_0+1)}\right)\left(-\frac{\partial\rho}{\partial S}(X_0+1)\right) > 0. \tag{2.7}$$

Hence

$$\frac{dP_e}{dr_0} = -\left(\frac{\partial F}{\partial S}\right)^{-1}\left(\frac{\partial F}{\partial P}\right)\frac{dS_0^+(r_0)}{dr_0} < 0, \tag{2.8}$$

provided that

$$\frac{dS_0^+(r_0)}{dr_0} > 0. \tag{2.9}$$

One need to verify (2.9). Since (2.4) holds at $r = r_0$ and C_1 and C_0 are independent of r_0, one can get from direct computations that

$$\begin{cases} \dfrac{d}{dr_0}(\rho_0^\pm(r_0)U_0^\pm(r_0)) = -\dfrac{m-1}{r_0}\rho_0^\pm(r_0)U_0^\pm(r_0), \\ \rho_0^+(r_0)U_0^+(r_0)\dfrac{d}{dr_0}U_0^+(r_0) = -\rho_0^+(r_0)T_0^+(r_0)\dfrac{dS_0^\pm(r_0)}{dr_0} - \dfrac{dP_0^+(r_0)}{dr_0}, \end{cases} \tag{2.10}$$

with $T > 0$ being the absolute temperature. Thus,

$$-\rho_0^+(r_0)T_0^+(r_0)\frac{dS_0^+(r_0)}{dr_0} = \left[\rho_0U_0\frac{dU}{dr_0}\right](r_0) + \left[\frac{dP_0}{dr_0}\right](r_0) \tag{2.11}$$

Since $\frac{dS_0^-(r_0)}{dr_0} = 0$. On the other hand, it follows from (2.2) and (2.10) that

$$\frac{m-1}{r_0}[\rho_0U_0^2] = \left[\rho_0U_0\frac{dU_0}{dr_0}\right](r_0) + \left[\frac{dP_0}{dr_0}(r_0)\right] \tag{2.12}$$

Hence, one obtains from (2.11)-(2.12) that

$$\frac{dS_0^+(r_0)}{dr_0} = -\frac{m-1}{r_0}\frac{1}{\rho_0^+(r_0)T_0^+(r_0)}[\rho_0U_0^2](r_0) > 0 \tag{2.13}$$

here one has used the entropy condition $[P_0](r_0) > 0$ and $[\rho_0U_0^2 + P_0](r_0) = 0$. Thus, we have shown that the end pressure P_e is a strictly increasing function of the shock position $r = r_0$.

We can now complete the proof of Theorem 1.1.

For $r_0 \in [X_0+\frac{1}{2}, X_0+1]$, by Step 2, there exists a unique supersonic flow in $[X_0+\frac{1}{2}, r_0]$. Moreover, it follows from Step 1 and Step 3 that there exist a unique shock at r_0 and a unique subsonic flow in $[r_0, X_0+1]$. Thus the function $F(r_0) = P_0^+(X_0+1)$ is well-defined for $r_0 \in [X_0+\frac{1}{2}, X_0+1]$. By Step 4, $F(r_0)$ is a strictly decreasing and continuous function on $P_0^+(X_0+1)$. When $r_0 = X_0+\frac{1}{2}$ or $r_0 = X_0+1$, one can obtain two different end pressures P_2 and P_1 with $P_1 < P_2$. Therefore, by the monotonicity of $F(r_0)$, one can obtain a unique symmetric transonic shock for $P_e \equiv P_0^+(X_0+1) \in (P_1, P_2)$. Hence, Theorem 1.1 is proved.

Remark 2.1. *By the assumption (1.7) and the proof of Theorem 1.1, it can be checked easily that there exists a constant $\delta(\eta_0) > 0$ with $\delta(\eta_0) \to 0$ as $\eta_0 \to 0$ such that for $r_0 \le r \le X_0 + 1$*

$$\left|\frac{d^k U_0^+(r)}{dr^k}\right| + \left|\frac{d^k P_0^+(r)}{dr^k}\right| \le \delta(\eta_0), \qquad k = 1, 2, 3.$$

Remark 2.2. *It follows from the derivation in Step 2 that one can get an extension $(\hat\rho_0^+(r), \hat U_0^+(r))$ of $(\rho_0^+(r), U_0^+(r))$ for $r \in (X_0 + \frac{1}{2}, X_0 + 1)$.*

§3. **The reformulation of the 2-D problem**

To prove Theorem 1.2 in the 2-D case as in [28] and [35], we reformulate the nonlinear problem (1.2)-(1.7) so that one can obtain a second order elliptic equation on P^+ and a 2×2 system on the angular velocity U_2^+.

First, due to the Bernoulli's law, for any C^1 solution, the system (1.2) in Ω_+ is equivalent to

$$\begin{cases} \partial_1(\rho^+ u_1^+) + \partial_2(\rho^+ u_2^+) = 0, \\ (u_1^+\partial_1 + u_2^+\partial_2)\Big(\dfrac{1}{2}|u^+|^2 + h(\rho^+, S^+)\Big) = 0, \\ u_1^+\partial_1 u_2^+ + u_2^+\partial_2 u_2^+ + \dfrac{\partial_2 P^+}{\rho^+} = 0, \\ u_1^+\partial_1 S^+ + u_2^+\partial_2 S^+ = 0. \end{cases} \tag{3.1}$$

Next we derive a second order equation on the pressure P^+ from (3.1).

By the state equation of gas dynamics, we can assume $\rho = \rho(P, S)$ and $e = e(P, S)$.

For simplicity, set $D = u_1^+\partial_1 + u_2^+\partial_2$. Then it follows from the first equation in (3.1) that

$$D^2\rho^+ + \rho^+ D(\partial_1 u_1^+ + \partial_2 u_2^+) - \frac{(D\rho^+)^2}{\rho^+} = 0.$$

Since

$$D\partial_i u_i^+ = \partial_i D u_i^+ - (\partial_i u_i^+)^2 - \partial_1 u_2^+ \partial_2 u_1^+, \qquad i = 1, 2,$$

then combining these with (1.2) yields

$$D^2\rho^+ - \rho^+\Big(\partial_1(\frac{\partial_1 P^+}{\rho^+}) + \partial_2(\frac{\partial_2 P^+}{\rho^+})\Big) - \frac{2(D\rho^+)^2}{\rho^+} - \frac{2}{u_1^+}(\partial_2 u_2^+ \partial_1 P^+ - \partial_2 u_1^+ \partial_2 P^+) = 0. \tag{3.2}$$

Additionally, in terms of $DS^+ = 0$ in (3.1), one can derive that

$$D\rho^+(P^+, S^+) = \partial_P \rho^+ DP^+ \qquad \textit{and} \qquad D^2\rho^+ = \partial_P^2 \rho^+ (DP^+)^2 + \partial_P \rho^+ D^2 P^+.$$

Thus (3.2) becomes

$$\begin{aligned} &\partial_P\rho^+ D^2 P^+ - \rho^+\Big(\partial_1(\frac{\partial_1 P^+}{\rho^+}) + \partial_2(\frac{\partial_2 P^+}{\rho^+})\Big) + \Big(\partial_P^2\rho^+ - \frac{2(\partial_P\rho^+)^2}{\rho^+}\Big)(DP^+)^2 \\ &- \frac{2}{u_1^+}(\partial_2 u_2^+ \partial_1 P^+ - \partial_2 u_1^+ \partial_2 P^+) = 0. \end{aligned} \tag{3.3}$$

Furthermore, (3.3) can be rewritten as

$$\partial_1\Big(\big(\frac{(u_1^+)^2}{c^2(\rho^+,S^+)}-1\big)\partial_1P^+ + \frac{u_1^+u_2^+}{c^2(\rho^+,S^+)}\partial_2P^+\Big) + \partial_2\Big(\frac{u_1^+u_2^+}{c^2(\rho^+,S^+)}\partial_1P^+ + \big(\frac{(u_2^+)^2}{c^2(\rho^+,S^+)}-1\big)\partial_2P^+\Big)$$
$$-\Big(\partial_1\big(\frac{u_1^+}{c^2(\rho^+,S^+)}\big)+\partial_2\big(\frac{u_2^+}{c^2(\rho^+,S^+)}\big)\Big)DP^+ + \frac{\partial_1\rho^+}{\rho^+}\partial_1P^+ + \frac{\partial_2\rho^+}{\rho^+}\partial_2P^+$$
$$+\Big(\partial_P^2\rho^+ - \frac{2(\partial_P\rho^+)^2}{\rho^+}\Big)(DP^+)^2 - \frac{2}{u_1^+}(\partial_2u_2^+\partial_1P^+ - \partial_2u_1^+\partial_2P^+) = 0. \tag{3.4}$$

Next, we derive a Dirichlet boundary condition for P^+ on the shock Σ.
It follows from (1.3) that

$$\begin{cases} \xi'(x_2) = \dfrac{[\rho u_1u_2]}{[P+\rho u_2^2]}, \\ \xi(x_2^1) = x_1^0. \end{cases} \tag{3.5}$$

Substituting (3.5) into (1.3) yields on Σ

$$\begin{cases} G_1(P^+,u_1^+,u_2^+,S^+) \equiv [\rho u_1u_2][\rho u_2] - [\rho u_1][P+\rho u_2^2] = 0, \\ G_2(P^+,u_1^+,u_2^+,S^+) \equiv [\rho u_1u_2]^2 - [P+\rho u_1^2][P+\rho u_2^2] = 0, \\ G_3(P^+,u_1^+,u_2^+,S^+) \equiv \Big[\rho u_1\Big(\dfrac12|u|^2+h(\rho,S)\Big)\Big][P+\rho u_2^2] \\ \qquad\qquad - \Big[\rho u_2\Big(\dfrac12|u|^2+h(\rho,S)\Big)\Big][\rho u_1u_2] = 0. \end{cases} \tag{3.6}$$

To derive the relations between (P^+,S^+) and (u_1^+,u_2^+) on Σ, we use the polar coordinates

$$\begin{cases} x_1 = r\cos\theta, \\ x_2 = r\sin\theta \end{cases} \tag{3.7}$$

and the decomposition

$$\begin{cases} u_1 = U_1\cos\theta - U_2\sin\theta, \\ u_2 = U_1\sin\theta + U_2\cos\theta. \end{cases} \tag{3.8}$$

Then, (1.2) takes the form

$$\begin{cases} \partial_r(\rho U_1) + \partial_\theta\left(\dfrac{\rho U_2}{r}\right) + \dfrac{\rho U_1}{r} = 0, \\ \partial_r(\rho U_1^2+P) + \dfrac1r\partial_\theta(\rho U_1U_2) + \dfrac{\rho(U_1^2-U_2^2)}{r} = 0, \\ \partial_r(\rho U_1U_2) + \dfrac1r\partial_\theta(P+\rho U_2^2) + \dfrac2r\rho U_1U_2 = 0, \\ \partial_r\left(\rho U_1\left(\dfrac12|u|^2+h(\rho,S)\right)\right) + \dfrac{\partial_\theta}{r}\left(\rho U_2\left(\dfrac12|u|^2+h(\rho,S)\right)\right) + \dfrac{U_1}{r}\left(\rho\left(\dfrac12|u|^2+h(\rho,S)\right)\right) = 0. \end{cases} \tag{3.9}$$

In addition, for any C^1 solution, (3.9) is equivalent to

$$\begin{cases} \partial_r(\rho U_1) + \partial_\theta\left(\frac{\rho U_2}{r}\right) + \frac{\rho U_1}{r} = 0, \\ U_1\partial_rU_1 + \frac{U_2}{r}\partial_\theta U_1 + \frac{\partial_rP}{\rho} - \frac{U_2^2}{r} = 0, \\ U_1\partial_rU_2 + \frac{U_2}{r}\partial_\theta U_2 + \frac1r\frac{\partial_\theta P}{\rho} + \frac{U_1U_2}{r} = 0, \\ U_1\partial_rS + \frac{U_2}{r}\partial_\theta S = 0. \end{cases} \tag{3.10}$$

Denote the shock Σ by $r=\tilde{r}(\theta)$ in the polar coordinates. Then, the R-H conditions become

$$\begin{cases} [\rho U_1] - \dfrac{\tilde{r}'(\theta)}{\tilde{r}(\theta)}[\rho U_2] = 0, \\ [\rho U_1^2 + P] - \dfrac{\tilde{r}'(\theta)}{\tilde{r}(\theta)}[\rho U_1 U_2] = 0, \\ [\rho U_1 U_2] - \dfrac{\tilde{r}'(\theta)}{\tilde{r}(\theta)}[P + \rho U_2^2] = 0, \\ \left[\rho U_1\left(\dfrac{1}{2}|u|^2 + h(\rho,S)\right)\right] - \dfrac{\tilde{r}'(\theta)}{\tilde{r}(\theta)}\left[\rho U_2\left(\dfrac{1}{2}|u|^2 + h(\rho,S)\right)\right] = 0. \end{cases} \tag{3.11}$$

Thus (3.6) is reduced to

$$\begin{cases} \tilde{G}_1(P^+,U_1^+,U_2^+,S^+) \equiv [\rho U_1U_2][\rho U_2] - [\rho U_1][P+\rho U_2^2] = 0, \\ \tilde{G}_2(P^+,U_1^+,U_2^+,S^+) \equiv [\rho U_1U_2]^2 - [P+\rho U_1^2][P+\rho U_2^2] = 0, \\ \tilde{G}_3(P^+,U_1^+,U_2^+,S^+) \equiv \left[\rho U_1\left(\dfrac{1}{2}|u|^2 + h(\rho,S)\right)\right][P+\rho U_2^2] \\ \qquad\qquad - \left[\rho U_2\left(\dfrac{1}{2}|u|^2 + h(\rho,S)\right)\right][\rho U_1 U_2] = 0. \end{cases} \tag{3.12}$$

Due to the radial symmetry of the data and the nozzle, the incoming supersonic flow must be symmetric and $(P^-,U_1^-,U_2^-,S^-) \equiv (P_0^-,U_0^-,0,S_0^-)$. Then it follows from (2.2) and (3.12) and a direct computation that on $r=\tilde{r}(\theta)$,

$$\begin{cases} \rho_0^+(r_0)(U_1^+ - U_0^+(r_0)) + \partial_P\rho_0^+(r_0)U_0^+(r_0)(P^+ - P_0^+(r_0)) + \partial_S\rho_0^+(r_0)U_0^+(r_0)(S^+ - S_0^+) = g_1, \\ 2\rho_0^+(r_0)U_0^+(r_0)(U_1^+ - U_0^+(r_0)) + \Big(1 + \partial_P\rho_0^+(r_0)(U_0^+(r_0))^2\Big)(P^+ - P_0^+(r_0)) \\ \qquad + \partial_S\rho_0^+(r_0)(U_0^+(r_0))^2(S^+ - S_0^+) = g_2, \\ \Big(\rho_0^+(r_0)e_0^+(r_0) + \dfrac{3}{2}\rho_0^+(r_0)(U_0^+(r_0))^2 + P_0^+(r_0)\Big)(U_1^+ - U_0^+(r_0)) \\ \qquad + \Big(\dfrac{1}{2}\partial_P\rho_0^+(r_0)(U_0^+(r_0))^2 + 1 + \partial_P(\rho_0^+e_0^+)(r_0)\Big)U_0^+(r_0)(P^+ - P_0^+(r_0)) \\ \qquad + \Big(\partial_S(\rho_0^+e_0^+)(r_0) + \dfrac{1}{2}\partial_S\rho_0^+(r_0)(U_0^+(r_0))^2\Big)U_0^+(r_0)(S^+ - S_0^+) = g_3, \end{cases} \tag{3.13}$$

where
$g_i = g_i\big((U_2^+)^2, (U_1^+ - U_0^+(r_0))^2, (P^+ - P_0^+(r_0))^2, (P^+ - P_0^+(r_0))(S^+ - S_0^+), (S^+ - S_0^+)^2, (U_1^+ - U_0^+(r_0))(P^+ - P_0^+(r_0)), (U_1^+ - U_0^+(r_0))(S^+ - S_0^+), P_0^- - P_0^-(r_0), U_0^- - U_0^-(r_0)\big)$ $(i=1,2,3)$ is smooth on its arguments and $g_i(0,0,0,0,0,0,0,0,0)=0$.

Furthermore, it can be verified that the determinant Δ of coefficient matrix in (3.13) satisfies $\Delta \neq 0$. Indeed, for the polytropic gas, one has by a direct computation that

$$\begin{aligned} \Delta &= \det\begin{pmatrix} \rho_0^+ & \partial_P\rho_0^+U_0^+ & \partial_S\rho_0^+U_0^+ \\ \rho_0^+U_0^+ & 1 & 0 \\ \rho_0^+e_0^+ + \rho_0^+(U_0^+)^2 + P_0^+ & (\partial_P(\rho_0^+e_0^+)+1)U_0^+ & \partial_S(\rho_0^+e_0^+)U_0^+ \end{pmatrix}(r_0) \\ &= \partial_S\rho_0^+(r_0)U_0^+(r_0)\det\begin{pmatrix} 0 & 1 \\ \rho_0^+e_0^+ + P_0^+ - \rho_0^+(U_0^+)^2\partial_P(\rho_0^+e_0^+) & (\partial_P(\rho_0^+e_0^+)+1)U_0^+ \end{pmatrix}(r_0) \\ &= -\Big(\partial_S\rho_0^+U_0^+(\rho_0^+e_0^+ + P_0^+)(1 - \frac{(U_0^+)^2}{c^2(\rho_0^+)})\Big)(r_0) > 0. \quad \Big(\text{by use of } e = \frac{P}{(\gamma-1)\rho},\ \partial_\rho e = \frac{P}{\rho^2} \text{ and } \partial_S\rho < 0\Big) \end{aligned}$$

Thus, on Σ, it follows from the implicit function theorem that

$$\begin{cases} U_1^+ - U_0^+(r_0) = \tilde{g}_1((U_2^+)^2, P_0^- - P_0^-(r_0), U_0^- - U_0^-(r_0)), \\ P^+ - P_0^+(r_0) = \tilde{g}_2((U_2^+)^2, P_0^- - P_0^-(r_0), U_0^- - U_0^-(r_0)), \\ S^+ - S_0^+ = \tilde{g}_3((U_2^+)^2, P_0^- - P_0^-(r_0), U_0^- - U_0^-(r_0)). \end{cases} \tag{3.14}$$

An important property of $\tilde{g}_i$ is

$$\tilde{g}_i = O((U_2^+)^2) + O(P_0^- - P_0^-(r_0)) + O(U_0^- - U_0^-(r_0)).$$

Roughly speaking, this implies, on the shock, the influence of U_2^+ on U_1^+, P^+ and S^+ can be almost "neglected".

Next, we derive the boundary conditions of P^+ on the fixed boundaries $\Gamma_i : \theta = (-1)^i\theta_0$.

In fact, in terms of the polar coordinates, the boundary condition (1.7) is equivalent to

$$U_2^+ = 0 \qquad on \qquad \theta = \pm\theta_0. \tag{3.15}$$

Thus the third equation in (3.10) implies that

$$\partial_n P^+ \equiv \partial_\theta P^+ = 0 \qquad on \qquad \theta = \pm\theta_0, \tag{3.16}$$

here ∂_n represents the derivative along the outer normal direction of the nozzle wall.

Consequently, P^+ in Ω_+ can be determined by the following boundary value problem

$$\begin{cases} \partial_1\Big(\big(\frac{(u_1^+)^2}{c^2(\rho^+,S^+)} - 1\big)\partial_1 P^+ + \frac{u_1^+u_2^+}{c^2(\rho^+,S^+)}\partial_2 P^+\Big) + \partial_2\Big(\frac{u_1^+u_2^+}{c^2(\rho^+,S^+)}\partial_1 P^+ + \big(\frac{(u_2^+)^2}{c^2(\rho^+,S^+)} - 1\big)\partial_2 P^+\Big) \\ \quad -\Big(\partial_1\big(\frac{u_1^+}{c^2(\rho^+,S^+)}\big) + \partial_2\big(\frac{u_2^+}{c^2(\rho^+,S^+)}\big)\Big)DP^+ + \frac{\partial_1\rho^+}{\rho^+}\partial_1 P^+ + \frac{\partial_2\rho^+}{\rho^+}\partial_2 P^+ \\ \quad +\Big(\partial_P^2\rho^+ - \frac{2(\partial_P\rho^+)^2}{\rho^+}\Big)(DP^+)^2 - \frac{2}{u_1^+}(\partial_2 u_2^+\partial_1 P^+ - \partial_2 u_1^+\partial_2 P^+) = 0, \\ P^+ - P_0^+(r_0) = \tilde{g}_2((U_2^+)^2, P_0^- - P_0^-(r_0), U_0^- - U_0^-(r_0)) \qquad \text{on} \qquad r = \tilde{r}(\theta), \\ \partial_n P^+ = 0 \qquad \text{on} \qquad \theta = \pm\theta_0, \\ P^+ = P_e \qquad \text{on} \qquad r = X_0 + 1. \end{cases} \tag{3.17}$$

Next, we derive an algebraic relation for P^+, U_1^+, U_2^+ and S^+ so that we can determine U_1^+ in terms of P^+, U_2^+ and S^+.

It follows from the second equation in (3.1) and the boundary conditions (1.7) and (3.14) that

$$\begin{cases} \Big(U_1^+\partial_r + \frac{U_2^+}{r}\partial_\theta\Big)\Big(\frac{1}{2}|U^+|^2 + h(\rho^+,S^+)\Big) = 0, \\ U_1^+ = U_0^+(r_0) + \tilde{g}_1((U_2^+)^2, P_0^- - P_0^-(r_0), U_0^- - U_0^-(r_0)) \qquad \text{on} \qquad r = \tilde{r}(\theta), \\ P^+ = P_0^+(r_0) + \tilde{g}_2((U_2^+)^2, P_0^- - P_0^-(r_0), U_0^- - U_0^-(r_0)) \qquad \text{on} \qquad r = \tilde{r}(\theta), \\ S^+ = S_0^+ + \tilde{g}_3((U_2^+)^2, P_0^- - P_0^-(r_0), U_0^- - U_0^-(r_0)), \\ U_2^+ = 0 \qquad on \qquad \theta = \pm\theta_0. \end{cases} \tag{3.18}$$

Let $\theta = \theta(r,\beta)$ be the characteristics starting from the point $(\tilde{r}(\beta),\beta)$ for the first order differential operator $U_1^+\partial_r + \frac{U_2^+}{r}\partial_\theta$, that is, $\theta(r,\beta)$ satisfies

$$\begin{cases} \dfrac{d\theta(r,\beta)}{dr} = \frac{1}{r}\big(\dfrac{U_2^+}{U_1^+}\big)(r,\theta(r,\beta)), \\ \theta(\tilde r(\beta),\beta) = \beta, \qquad \beta\in[-\theta_0,\theta_0]. \end{cases} \tag{3.19}$$

Integrating the first order equation in (3.18) along $\theta=\theta(r,\beta)$ and noting that $\theta=\pm\theta_0$ is the characteristics of $U_1^+\partial_r + \frac{U_2^+}{r}\partial_\theta$ starting from the point $(\tilde r(\pm\theta_0),\pm\theta_0)$, then we have in Ω_+

$$\left(\frac{1}{2}|U^+|^2 + h(\rho^+,S^+)\right)(r,\theta(r,\beta)) = G_0(\tilde r(\beta),\beta,U_2^+(\tilde r(\beta),\beta)) \tag{3.20}$$

with

$$\begin{aligned} G_0(\tilde r(\beta),\beta,U_2^+(\tilde r(\beta),\beta)) &= e\big(P_0^+(r_0)+\tilde g_2, S_0^+ + \tilde g_3\big)(\tilde r(\beta),\beta) + \frac{1}{2}\big(U_0^+(r_0)+\tilde g_1\big)^2(\tilde r(\beta),\beta) \\ &\quad + \frac{1}{2}(U_2^+)^2(\tilde r(\beta),\beta)) + \left(\frac{P_0^+(r_0)+\tilde g_2}{\rho\big(P_0^+(r_0)+\tilde g_2, S_0^+ + \tilde g_3\big)}\right)(\tilde r(\beta),\beta). \end{aligned}$$

Here it is noted that $U_2^+(\tilde r(\beta),\beta)$ has not been determined yet.

Finally, we determine U_2^+.

It follows from (3.19) that

$$\begin{cases} \dfrac{d}{dr}\big(\dfrac{\partial\theta}{\partial\beta}\big) = \frac{1}{r}\partial_\theta\big(\dfrac{U_2^+}{U_1^+}\big)(r,\theta(r,\beta))\dfrac{\partial\theta}{\partial\beta}, \\ \dfrac{\partial\theta}{\partial\beta}(\tilde r(\beta),\beta) = 1 - \dfrac{\tilde r'(\beta)}{\tilde r(\beta)}\big(\dfrac{U_2^+}{U_1^+}\big)(\tilde r(\beta),\beta), \qquad \beta\in[-\theta_0,\theta_0]. \end{cases} \tag{3.21}$$

By (3.20),

$$\left(\partial_\theta\left(\frac{1}{2}|U^+|^2 + h(\rho^+,S^+)\right)\right)(r,\theta(r,\beta))\frac{\partial\theta}{\partial\beta} = \frac{d}{d\beta}G_0(\tilde r(\beta),\beta,U_2^+(\tilde r(\beta),\beta)) \tag{3.22}$$

and

$$\begin{aligned} &\left(\partial_r\left(\frac{1}{2}|U^+|^2 + h(\rho^+,S^+)\right)\right)(r,\theta(r,\beta)) \\ =& -\left(\left(\partial_\theta\left(\frac{1}{2}|U^+|^2 + h(\rho^+,S^+)\right)\right)\partial_\theta P^+ + \partial_S\big(e+\frac{P}{\rho}\big)(P^+,S^+)\partial_\theta S^+\right)(r,\theta(r,\beta))\frac{\partial\theta}{\partial r} \\ =& -\frac{1}{r}\left(\frac{U_2^+}{U_1^+}\right)(r,\theta(r,\beta))\partial_\beta G_0(\tilde r(\beta),\beta,U_2^+(\tilde r(\beta),\beta))\partial_\theta\beta(r,\theta), \end{aligned} \tag{3.23}$$

here $\beta(r,\theta)$ represents the inverse function of $\theta=\theta(r,\beta)$.

In addition, the first equation and the third equation in (3.10) can be rewritten as

$$\begin{cases} \partial_r U_1^+ + \frac{1}{r}\partial_\theta U_2^+ = -\dfrac{U_1^+}{r} - \dfrac{1}{\rho^+}\big(U_1^+\partial_r\rho^+ + \dfrac{U_2^+}{r}\partial_\theta\rho^+\big), \\ U_1^+\partial_r U_2^+ + \dfrac{U_2^+}{r}\partial_\theta U_2^+ = -\frac{1}{r}\dfrac{\partial_\theta P^+}{\rho^+} - \dfrac{U_1^+U_2^+}{r}. \end{cases} \tag{3.24}$$

Combining (3.23) with (3.24) gives

$$\begin{cases} \partial_r U_2^+ = h_1(P^+,U_1^+,U_2^+,S^+,\partial_r P^+,\partial_\theta P^+,\partial_r S^+,\partial_\theta S^+), \\ \partial_\theta U_2^+ = h_2(P^+,U_1^+,U_2^+,S^+,\partial_r P^+,\partial_\theta P^+,\partial_r S^+,\partial_\theta S^+), \\ U_2^+(r_0,-\theta_0) = 0, \end{cases} \tag{3.25}$$

here $h_1 = \frac{\Delta_1}{\Delta_0}, h_2 = \frac{\Delta_2}{\Delta_0}$ with

$$
\begin{aligned}
&\Delta_0 = r^{-1}|U^+|^2, \\
&\Delta_1 = -\frac{U_1^+}{r^2\rho^+}\partial_\theta P^+ + \frac{U_1^+U_2^+}{r\rho^+}(U_1^+\partial_r\rho^+ + \frac{U_2^+}{r}\partial_\theta\rho^+) - \frac{U_2^+}{r}\Big(\partial_P\big(e+\frac{P}{\rho}\big)(P^+,S^+)\partial_r P^+ \\
&\quad + \partial_S\big(e+\frac{P}{\rho}\big)(P^+,S^+)\partial_r S^+ + \frac{U_2^+}{rU_1^+}(\frac{d}{d\beta}G_0)(\beta(r,\theta))\partial_\theta\beta(r,\theta)\Big), \\
&\Delta_2 = -\frac{(U_1^+)^3}{r} - \frac{(U_1^+)^2}{\rho^+}(U_1^+\partial_r\rho^+ + \frac{U_2^+}{r}\partial_\theta\rho^+) - U_2^+(\frac{1}{r}\frac{\partial_\theta P^+}{\rho^+} + \frac{U_1^+U_2^+}{r}) \\
&\quad + U_1^+\Big(\partial_P\big(e+\frac{P}{\rho}\big)(P^+,S^+)\partial_r P^+ + \partial_S\big(e+\frac{P}{\rho}\big)(P^+,S^+)\partial_r S^+ + \frac{U_2^+}{rU_1^+}(\frac{d}{d\beta}G_0)(\beta(r,\theta))\partial_\theta\beta(r,\theta)\Big)
\end{aligned}
$$

Additionally, it follows from (3.10) and (3.14) that S^+ satisfies the following equation

$$
\begin{cases}
(U_1^+\partial_r + \frac{U_2^+}{r}\partial_\theta)S^+ = 0, \\
S^+(\tilde r(\beta),\beta) = S_0^+(r_0) + \tilde g_3((U_2^+)^2, P_0^- - P_0^-(r_0), U_0^- - U_0^-(r_0))(\tilde r(\beta),\beta).
\end{cases}
\tag{3.26}
$$

Furthermore, (3.5) can be rewritten as

$$
\begin{cases}
\tilde r'(\theta) = \dfrac{\tilde r(\theta)[\rho U_1 U_2]}{[P+\rho U_2^2]}, \\
\tilde r(-\theta_0) = r_0.
\end{cases}
\tag{3.27}
$$

In order to show Theorem 1.2, we need only to treat the uniqueness problem (3.17)-(3.20) and (3.25)-(3.27). This will be done in next section.

§4. **The Uniqueness in 2-D**

We now prove the uniqueness of solutions stated in Theorem 1.2 for 2-D. It will be more convenient to change the domain Ω_+ with a free boundary Σ into a fixed domain $Q_+ = \{y : X_0 < y_1 < X_0+1, -\theta_0 < y_2 < \theta_0\}$. To this end, set

$$
\begin{cases}
y_1 = X_0 + \dfrac{r - \tilde r(\theta)}{X_0 + 1 - \tilde r(\theta)}, \\
y_2 = \theta.
\end{cases}
\tag{4.1}
$$

For simplicity, in Q_+, we still write (P^+, U_1^+, U_2^+, S^+) as the state of fluid behind the shock in the new coordinates $y = (y_1, y_2)$.

Noting that

$$
\partial_r = \frac{1}{X_0 + 1 - \tilde r(y_2)}\partial_{y_1}, \qquad \partial_\theta = \frac{(X_0+1-y_1)\tilde r'(y_2)}{\tilde r(y_2) - (X_0+1)}\partial_{y_1} + \partial_{y_2}.
$$

Then the equation (3.17) can be changed as follows

$$\begin{cases} \tilde{D}_1\Big((\frac{(u_1^+)^2}{c^2(\rho^+,S^+)}-1)\tilde{D}_1P^+ + \frac{u_1^+u_2^+}{c^2(\rho^+,S^+)}\tilde{D}_2P^+\Big) + \tilde{D}_2\Big(\frac{u_1^+u_2^+}{c^2(\rho^+,S^+)}\tilde{D}_1P^+ + (\frac{(u_2^+)^2}{c^2(\rho^+,S^+)}-1)\tilde{D}_2P^+\Big) \\ \quad -\Big(\tilde{D}_1(\frac{u_1^+}{c^2(\rho^+,S^+)}) + \tilde{D}_2(\frac{u_2^+}{c^2(\rho^+,S^+)})\Big)\tilde{D}P^+ + \frac{\tilde{D}_1\rho^+}{\rho^+}\tilde{D}_1P^+ + \frac{\tilde{D}_2\rho^+}{\rho^+}\tilde{D}_2P^+ \\ \quad +\Big(\partial_P^2\rho^+ - \frac{2(\partial_P\rho^+)^2}{\rho^+}\Big)(\tilde{D}P^+)^2 - \frac{2}{u_1^+}(\tilde{D}_2u_2^+\tilde{D}_1P^+ - \tilde{D}_2u_1^+\tilde{D}_2P^+) = 0, \\ P^+ - P_0^+(r_0) = \tilde{g}_2((U_2^+)^2, P_0^- - P_0^-(r_0), U_0^- - U_0^-(r_0)) \qquad \text{on} \qquad y_1 = X_0, \\ \partial_\theta P^+ = 0 \qquad\qquad \text{on} \qquad y_2 = \pm\theta_0, \\ P^+ = P_e \qquad\qquad \text{on} \qquad y_1 = X_0+1. \end{cases} \tag{4.2}$$

with

$$\begin{aligned} & r(y) = \tilde{r}(y_2) + (X_0+1-\tilde{r}(y_2))(y_1-X_0), \\ & u_1^+ = U_1^+\cos y_2 - U_2^+\sin y_2, \qquad u_2^+ = U_1^+\sin y_2 + U_2^+\cos y_2, \\ & \tilde{D}_1 = \left(\frac{\cos y_2}{X_0+1-\tilde{r}(y_2)} + \frac{(X_0+1-y_1)\tilde{r}'(y_2)\sin y_2}{r(y)(X_0+1-\tilde{r}(y_2))}\right)\partial_{y_1} - \frac{\sin y_2}{r(y)}\partial_{y_2}, \\ & \tilde{D}_2 = \left(\frac{\sin y_2}{X_0+1-\tilde{r}(y_2)} - \frac{(X_0+1-y_1)\tilde{r}'(y_2)\cos y_2}{r(y)(X_0+1-\tilde{r}(y_2))}\right)\partial_{y_1} + \frac{\cos y_2}{r(y)}\partial_{y_2}, \\ & \tilde{D} = \left(\frac{U_1^+}{X_0+1-\tilde{r}(y_2)} - \frac{(X_0+1-y_1)\tilde{r}'(y_2)U_2^+}{r(y)(X_0+1-\tilde{r}(y_2))}\right)\partial_{y_1} + \frac{U_2^+}{r(y)}\partial_{y_2}. \end{aligned}$$

Additionally, it follows from the equation (3.18) that

$$\tilde{D}\left(\frac{1}{2}|U^+|^2 + h(\rho^+,S^+)\right) = 0. \tag{4.3}$$

The characteristics $y_2 = y_2(y_1,\beta)$ of $\tilde{D}$ starting from the point (X_0,β) of (4.3) is given by

$$\begin{cases} \frac{dy_2}{dy_1} = \frac{(X_0+1-\tilde{r}(y_2))U_2^+}{r(y)U_1^+ - (X_0+1-y_1)\tilde{r}'(y_2)U_2^+}, \\ y_2(X_0,\beta) = \beta. \end{cases} \tag{4.4}$$

Thus it follows from (4.3) that

$$\left(\frac{1}{2}|U^+|^2 + h(\rho^+,S^+)\right)(y_1, y_2(y_1,\beta)) = G_0(\tilde{r}(\beta),\beta,U_2^+(X_0,\beta)). \tag{4.5}$$

As in the derivation of (3.24), one can obtain from (4.3), (4.4) and (3.24) that

$$\begin{cases} \partial_{y_i}U_2^+ = \tilde{H}_i(P^+,U_1^+,U_2^+,S^+,\partial_{y_1}P^+,\partial_{y_2}P^+,\partial_{y_1}S^+,\partial_{y_2}S^+), \quad i=1,2, \\ U_2^+(X_0,-\theta_0) = 0, \end{cases} \tag{4.6}$$

here $\tilde{H}_i = \dfrac{det(\tilde{A}_i)}{det(\tilde{A}_0)}$ for $i = 1, 2$, the 4×4 matrix $\tilde{A}_0 = (l_1, l_2, l_3, l_4)$ is defined as

$$
\begin{aligned}
l_1 &= \left(0, U_2^+, \frac{(X_0+1-y_1)\tilde{r}'(y_2)}{r(y)(\tilde{r}(y_2)-(X_0+1))}, \frac{U_1^+}{X_0+1-\tilde{r}(y_2)} + \frac{(X_0+1-y_1)\tilde{r}'(y_2)U_2^+}{r(y)(\tilde{r}(y_2)-(X_0+1))}\right)^T, \\
l_2 &= \left(U_2^+, 0, \frac{1}{r(y)}, \frac{U_2^+}{r(y)}\right)^T, \\
l_3 &= \left(0, U_1^+, \frac{1}{X_0+1-\tilde{r}(y_2)}, 0\right)^T, \\
l_4 &= \left(U_1^+, 0, 0, 0\right)^T
\end{aligned}
$$

and $\tilde{A}_i (i = 1, 2)$ denotes the 4×4 matrix which is obtained from $\tilde{A}_0$ by replacing the $i-$column in $\tilde{A}_0$ with the vector $\tilde{l} = (\tilde{l}_{01}, \tilde{l}_{02}, \tilde{l}_{03}, \tilde{l}_{04})^T$ defined as

$$
\begin{aligned}
\tilde{l}_{01} &= \frac{d}{d\beta} G_0(\tilde{r}(\beta), \beta, U_2^+(X_0,\beta)\partial_{y_2}\beta(y) - \partial_{y_2} h(\rho^+, S^+), \\
\tilde{l}_{02} &= -\frac{d}{d\beta} G_0(\tilde{r}(\beta), \beta, U_2^+(X_0,\beta)\partial_{y_2}\beta(y)\frac{dy_2(y_1,\beta)}{dy_1} - \partial_{y_1} h(\rho^+, S^+), \\
\tilde{l}_{03} &= -\frac{U_1^+}{r(y)} - \frac{1}{\rho^+}\left(U_1^+\partial_r\rho^+ + \frac{U_2^+}{r(y)}\partial_\theta\rho^+\right), \\
\tilde{l}_{04} &= -\frac{1}{r(y)}\frac{\partial_\theta P^+}{\rho^+} - \frac{U_1^+U_2^+}{r(y)},
\end{aligned}
$$

where $\beta = \beta(y)$ is an inverse function of $y_2 = y_2(y_1, \beta)$.

In addition, S^+ solves the following problem

$$
\begin{cases}
\tilde{D}S^+ = 0, \\
S^+(X_0, y_2) = S_0^+ + \tilde{g}_3((U_2^+)^2, P_0^- - P_0^-(r_0), U_0^- - U_0^-(r_0))(X_0, y_2).
\end{cases}
\tag{4.7}
$$

Finally, (3.27) can be rewritten as

$$
\begin{cases}
\tilde{r}'(y_2) = \dfrac{\tilde{r}(y_2)[\rho U_1 U_2]}{[P + \rho U_2^2]}, \\
\tilde{r}(-\theta_0) = r_0.
\end{cases}
\tag{4.8}
$$

To validate the regularity assumption in Theorem 1.2, we now give two lemmas to ensure the compatibility relations of any $C^1(\bar{\Omega}_+)$ solution at the corned points formed by the shock curve and the nozzle walls.

Lemma 4.1. (**Orthogonality**) Under the assumptions in Theorem 1.2, we have

$$\tilde{r}'(\pm\theta_0) = 0.$$

Namely, the shock curve is perpendicular to the walls of the nozzle.

Proof. This fact follows from the third equation in (3.11) and the boundary condition (3.15) directly since the jump of the pressure is non-zero.

Lemma 4.2. (**Compatibility**) If the assumptions in Theorem 1.2 hold, then

$$\partial_\theta P^+(x^i) = 0, \qquad i = 1, 2.$$

In particular, the first order compatibility condition of the problem (4.2) at the point $x^i = (x_1^i, x_2^i)$ is satisfied with $x^1 = (r_0\cos\theta_0, -r_0\sin\theta_0)$ and $x^2 = (\tilde r(\theta_0)\cos\theta_0, \tilde r(\theta_0)\sin\theta_0)$.

Proof. By Lemma 4.1, $U_2^\pm(r, \pm\theta_0) = 0$ and (3.11), one has

$$\begin{cases} [\rho U_1](\tilde r(\pm\theta_0), \pm\theta_0) = 0, \\ [\rho U_1^2 + P](\tilde r(\pm\theta_0), \pm\theta_0) = 0, \\ [(\rho e(\rho, S) + \frac{1}{2}\rho U_1^2 + P(\rho, S))U_1](\tilde r(\pm\theta_0), \pm\theta_0) = 0 \end{cases} \tag{4.9}$$

and

$$\begin{cases} \partial_\theta[\rho U_1](\tilde r(\pm\theta_0), \pm\theta_0) = 0, \\ \partial_\theta[P(\rho, S) + \rho U_1^2](\tilde r(\pm\theta_0), \pm\theta_0) = 0, \\ \partial_\theta[(\rho e(\rho, S) + \frac{1}{2}\rho U_1^2 + P(\rho, S))U_1](\tilde r(\pm\theta_0), \pm\theta_0) = 0. \end{cases}$$

This implies at the points $(\tilde r(\pm\theta_0), \pm\theta_0)$ that

$$\begin{cases} \rho^+\partial_\theta U_1^+ + U_1^+\partial_\theta\rho^+ = 0, \\ 2\rho^+U_1^+\partial_\theta U_1^+ + (c^2(\rho^+, S^+) + (U_1^+)^2)\partial_\theta\rho^+ + \partial_S P(\rho^+, S^+)\partial_\theta S^+ = 0, \\ \Big(\rho^+ e(\rho^+, S^+) + \frac{3}{2}\rho^+(U_1^+)^2 + P(\rho^+, S^+)\Big)\partial_\theta U_1^+ + U_1^+\Big(e(\rho^+, S^+) + \rho^+\partial_\rho e(\rho^+, S^+) + \frac{1}{2}(U_1^+)^2 \\ \qquad +\partial_\rho P(\rho^+, S^+)\Big)\partial_\theta\rho^+ + U_1^+\Big(\rho^+\partial_S e(\rho^+, S^+) + \partial_S P(\rho^+, S^+)\Big)\partial_\theta S^+ = 0. \end{cases} \tag{4.10}$$

For the polytropic gas, the determinant Δ of coefficients in (4.10) satisfies $\Delta = (\rho^+)^2U_1^+\partial_S e^+(c^2(\rho^+) - (U_1^+)^2) \neq 0$. Thus,

$$\partial_\theta\rho^+(\tilde r(\pm\theta_0), \pm\theta_0) = \partial_\theta U_1^+(\tilde r(\pm\theta_0), \pm\theta_0) = \partial_\theta S^+(\tilde r(\pm\theta_0), \pm\theta_0) = 0.$$

Consequently, $\partial_\theta P^+(\tilde r(\pm\theta_0), \pm\theta_0) = 0$ and the compatibility condition holds.

Now we are ready to prove Theorem 1.2 in the 2-D case.

Suppose that the problem (4.2)-(4.4) and (4.6)-(4.8) has another solution $(P^+, U_1^+, U_2^+, S^+; \tilde r(y_2))$ with the corresponding regularities in Theorem 1.2.

Set

$$\begin{aligned} &W_1(y) = P^+(y) - \hat P_0^+\big(r_0 + (X_0 + 1 - r_0)(y_1 - X_0)\big), W_2(y) = U_1^+(y) - \hat U_0^+\big(r_0 + (X_0 + 1 - r_0)(y_1 - X_0)\big), \\ &W_3(y) = U_2^+(y), W_4(y) = S^+(y) - S_0^+, \qquad \Xi(y_2) = \tilde r(y_2) - r_0. \end{aligned}$$

By (4.8), Lemma 4.2, the Remark 2.1 in §2 and the assumptions in Theorem 1.2, one obtains after a careful computation that

$$\begin{cases} \Xi'(y_2) = a_0(y_2)\Xi(y_2) + \sum_{i=1}^4 a_i(y_2)W_i(\tilde r(y_2), y_2) \\ \Xi(-\theta_0) = 0, \end{cases} \tag{4.12}$$

with $a_0(y_2) \in C^{1,\delta_0}[-\theta_0, \theta_0], a_i(y_2) \in C^{2,\delta_0}[-\theta_0, \theta_0](1 \le i \le 4)$ satisfying

$$\|a_0\|_{C^{1,\delta_0}} + \|a_1\|_{C^{2,\delta_0}} + \|a_3\|_{C^{2,\delta_0}} + \|a_4\|_{C^{2,\delta_0}} \le C(\varepsilon + \delta(\eta_0)), \qquad \|a_2\|_{C^{2,\delta_0}} \le C.$$

It follows from the Granwall's inequality, Lemma 4.2 and (3.14) that

$$|\Xi(y_2)| \le C(\varepsilon + \delta(\eta_0))(\|W_1\|_{L^\infty(Q_+)} + \|W_3\|_{L^\infty(Q_+)} + \|W_4\|_{L^\infty(Q_+)}) + C\|W_2\|_{L^\infty(Q_+)}. \tag{4.13}$$

Thus, (4.12)-(4.13) implies that

$$\|\Xi(y_2)\|_{C^1[-\theta_0,\theta_0]}\leq C(\varepsilon+\delta(\eta_0))(\|W_1\|_{L^\infty(Q_+)}+\|W_3\|_{L^\infty(Q_+)}+\|W_4\|_{L^\infty(Q_+)})+C\|W_2\|_{L^\infty(Q_+)}$$

and

$$\|\Xi(y_2)\|_{C^{2,\delta_0}[-\theta_0,\theta_0]}\leq C(\varepsilon+\delta(\eta_0))(\|W_1\|_{C^{1,\delta_0}(Q_+)}+\|W_3\|_{C^{1,\delta_0}(Q_+)}+\|W_4\|_{C^{1,\delta_0}(Q_+)})+C\|W_2\|_{C^{1,\delta_0}(Q_+)}, \quad (4.14)$$

here $\delta(\eta_0)>0$ is a generic constant with $\delta(\eta_0)\to 0$ as $\eta_0\to 0$.

Based on (4.14) and the assumptions in Theorem 1.2, one can estimate W_1 by (4.2).

Indeed, (4.2) implies that

$$\begin{cases} \tilde{D}_1\Big(\big(\frac{(u_1^+)^2}{c^2(\rho^+,S^+)}-1\big)\tilde{D}_1W_1+\frac{u_1^+u_2^+}{c^2(\rho^+,S^+)}\tilde{D}_2W_1\Big)+\tilde{D}_2\Big(\frac{u_1^+u_2^+}{c^2(\rho^+,S^+)}\tilde{D}_1W_1+\big(\frac{(u_2^+)^2}{c^2(\rho^+,S^+)}-1\big)\tilde{D}_2W_1\Big)\\ \qquad =F(\tilde{r}(y_2),\tilde{r}'(y_2),\tilde{r}''(y_2),P^+,\nabla P^+,U_1^+,\nabla U_1^+,U_2^+,\nabla U_2^+,S^+,\nabla S^+),\\ W_1=\tilde{g}_2((U_2^+)^2,P_0^- - P_0^-(r_0),U_0^- - U_0^-(r_0)) \qquad \text{on} \qquad y_1=X_0,\\ \partial_\theta W_1=0 \qquad \text{on} \qquad y_2=\pm\theta_0,\\ W_1=0 \qquad \text{on} \qquad y_1=X_0+1, \end{cases}$$

here

$$\begin{aligned} F=&\sum_{k=1,2;j=1,2,3,4}\partial_{y_k}(b_{0j}^k(y)W_j)+\sum_{k=1,2}\partial_{y_k}(b_{05}^k(y)\Xi'(y_2))+\sum_{k=1,2;j=1,2,3,4}b_{1j}^k(y)\partial_{y_k}W_j+\sum_{j=1}^4 b_{2j}(y)W_j\\ &+b_{31}(y)\Xi(y_2)+b_{32}(y)\Xi'(y_2) \end{aligned}$$

with $b_{ij}^l(y),b_{ij}\in C^{1,\delta_0}(\bar{Q}_+)$ and $\|b_{ij}^l(y)\|_{C^{1,\delta_0}(\bar{Q}_+)}+\|b_{ij}(y)\|_{C^{1,\delta_0}(\bar{Q}_+)}\leq C(\varepsilon+\delta(\eta_0))$.

Due to Lemma 4.2, it follows from the known regularity estimates on second order elliptic equations of divergence form with corned boundaries and mixed boundary conditions (see [2-3], [19-20] and so on) that

$$\begin{aligned} \|W_1\|_{C^{1,\delta_0}}\leq& C\Big(\|\tilde{g}_2\|_{C^{1,\delta_0}}+\sum_{k=1,2;j=1,2,3,4}\|b_{0j}^kW_j\|_{C^{\delta_0}}+\sum_{k=1,2;j=1,2,3,4}\|b_{1j}^k\partial_{y_k}W_j\|_{C^{\delta_0}}\\ &+\sum_{k=1,2}\|b_{05}^k(y)\Xi'(y_2)\|_{C^{\delta_0}}+\sum_{1\leq j\leq 4}\|b_{2j}W_j\|_{C^{\delta_0}}+\|b_{31}(y)\Xi(y_2)+b_{32}(y)\Xi'(y_2)\|_{C^{\delta_0}}\Big)\\ \leq& C(\varepsilon+\delta(\eta_0))(\|W_1\|_{C^{1,\delta_0}}+\|W_2\|_{C^{1,\delta_0}}+\|W_3\|_{C^{1,\delta_0}}+\|W_4\|_{C^{1,\delta_0}}+\|\Xi(y_2)\|_{C^{1,\delta_0}}). \end{aligned} \quad (4.15)$$

Substituting (4.14) into (4.15) yields

$$\|W_1\|_{C^{1,\delta_0}}\leq C(\varepsilon+\delta(\eta_0))(\|W_2\|_{C^{1,\delta_0}}+\|W_3\|_{C^{1,\delta_0}}+\|W_4\|_{C^{1,\delta_0}}). \quad (4.16)$$

Next, we estimate W_3.

By (4.4), we obtain

$$\|y_2(y_1,\beta)-\beta)\|_{C^{1,\delta_0}[X_0,X_0+1;-\theta_0,\theta_0]}\leq C(\sum_{i=1}^4\|W_i\|_{C^{1,\delta_0}}+\|\Xi(y_2))\|_{C^{1,\delta_0}})\leq C\sum_{i=1}^4\|W_i\|_{C^{1,\delta_0}}. \quad (4.17)$$

It follows from (4.6) that W_3 satisfies

$$\begin{cases} \partial_{y_i}W_3=\bar{H}_i(y), \quad i=1,2,\\ W_3(0,0)=0. \end{cases} \quad (4.18)$$

here $\bar{H}_i(y)$ has such a form

$$\bar{H}_i(y) = d_1^i(y)\partial_{y_1}W_1 + d_2^i(y)\partial_{y_2}W_1 + d_3^i(y)\partial_{y_1}W_4 + d_4^i(y)\partial_{y_2}W_4 + \sum_{k=5}^{8} d_k^i(y)W_{k-4} + d_9^i(y)(y_2(y_1,\beta)-\beta)$$
$$+ d_{10}^i(y)\Xi(y_2) + d_{11}^i(y)\Xi'(y_2) + d_{12}^i(y)\partial_\beta(y_2(y_1,\beta)-\beta) + d_{13}^i(y)\partial_2 U_2^+(X_0,\beta),$$

with $\beta = \beta(y)$ being the inverse function of $y_2 = y_2(y_1,\beta)$, $d_k^i(y) \in C^{1,\delta_0}$ for $1 \le k \le 13$ and

$$\sum_{k=5}^{13} \|d_k^i\|_{C^{1,\delta_0}} \le C(\varepsilon + \delta(\eta_0)).$$

Thus, combining the equation (4.18) with the estimate (4.17) yields

$$\|W_3\|_{C^{1,\delta_0}} \le C(\|\bar{H}_1\|_{C^{\delta_0}} + \|\bar{H}_2\|_{C^{\delta_0}}) \le C(\|W_1\|_{C^{1,\delta_0}} + \|W_2\|_{C^{1,\delta_0}} + \|W_4\|_{C^{1,\delta_0}})$$
$$+ C(\varepsilon + \delta(\eta_0))\|W_3\|_{C^{1,\delta_0}}.$$

For sufficiently small ε and η_0, one has

$$\|W_3\|_{C^{1,\delta_0}} \le C(\|W_1\|_{C^{1,\delta_0}} + \|W_2\|_{C^{1,\delta_0}} + \|W_4\|_{C^{1,\delta_0}}). \tag{4.19}$$

Next, we derive the estimate on W_2.
By (4.5) and the estimates above, we obtain

$$\|W_2\|_{C^{1,\delta_0}} \le C(\|W_1\|_{C^{1,\delta_0}} + \|W_4\|_{C^{1,\delta_0}}) + C\varepsilon\big(\|W_3\|_{C^{1,\delta_0}} + \|y_2(y_1,\beta)-\beta\|_{C^{1,\delta_0}}\big)$$
$$\le C(\varepsilon + \delta(\eta_0))\big(\|W_2\|_{C^{1,\delta_0}} + \|W_3\|_{C^{1,\delta_0}}\big) + C\|W_4\|_{C^{1,\delta_0}}. \tag{4.20}$$

Finally, it follows from the equation (4.7) that

$$\|W_4\|_{C^{1,\delta_0}} \le C(\varepsilon + \delta(\eta_0))(\|W_3\|_{C^{1,\delta_0}} + \|y_2(y_1,\beta)-\beta\|_{C^{1,\delta_0}}) \le C(\varepsilon + \delta(\eta_0))\sum_{k=1}^{4}\|W_k\|_{C^{1,\delta_0}}. \tag{4.21}$$

Combining (4.16) and (4.19)-(4.21) yields

$$\sum_{k=1}^{4}\|W_k\|_{C^{1,\delta_0}} \le C(\varepsilon + \delta(\eta_0))\sum_{k=1}^{4}\|W_k\|_{C^{1,\delta_0}}.$$

Thus, for small ε and η_0 we arrive at

$$W_1 = W_2 = W_3 = W_4 = 0.$$

It follows from (4.13) that

$$\Xi(y_2) = 0.$$

Therefore, we can obtain $P^+(y) = \hat{P}_0^+\big(r_0 + (X_0+1-r_0)(y_1 - X_0)\big), U_1^+(y) = \hat{U}_0^+\big(r_0 + (X_0+1-r_0)(y_1 - X_0)\big), U_2^+(y) = 0, S^+(y) = S_0^+$ and $\tilde{r}(y_2) = r_0$ immediately. This leads to the proof on Theorem 1.2 in 2-D case.

§5. The reformulation of 3-D Problem

As for the 2-dimension problem in §3, we will use the Bernoulli's law to reformulate the nonlinear problem (1.2) with the boundary conditions (1.3)-(1.7) as a second order elliptic equation on P^+ and four first order equations for $u^+ = (u_1^+, u_2^+, u_3^+)$ and S^+.

First, for any C^1-solution to (1.2) in Ω_+, it holds that

$$\begin{cases} divu^+ + \dfrac{D\rho^+}{\rho^+} = 0, \\ Du^+ + \dfrac{\nabla P^+}{\rho^+} = 0, \\ D\left(\dfrac{1}{2}|u^+|^2 + h(\rho^+, S^+)\right) = 0, \end{cases} \tag{5.1}$$

here $D = u_1^+\partial_1 + u_2^+\partial_2 + u_3^+\partial_3$, and $\rho^+ = \rho(P^+, S^+)$.

Without loss of generality, we consider only the polytropic gases. Then the last equation in (5.1) is equivalent to

$$u^+ \cdot Du^+ + \frac{\gamma}{\gamma - 1}\left(\frac{DP^+}{\rho^+} - \frac{P^+}{(\rho^+)^2}D\rho^+\right) = 0. \tag{5.2}$$

Combining (5.2) with the second, third and fourth equations in (5.1) yields

$$DP^+ = \frac{\gamma P^+}{\rho^+}D\rho^+. \tag{5.3}$$

By (5.2), the first equation in (5.1) can be rewritten as

$$divu^+ \frac{DP^+}{\gamma P^+} = 0. \tag{5.4}$$

Thus it follows from (5.4) and (5.1) that

$$\nabla \cdot (\frac{\nabla P^+}{\rho^+}) - D(\frac{DP^+}{\gamma P^+}) + \sum_{i,j=1}^{3} \partial_i u_j^+ \partial_j u_i^+ = 0. \tag{5.5}$$

It is easy to verify that the equation (5.5) on P^+ is a second order elliptic equation for the subsonic flow. Note that the third term in (5.5) is of the order $O(|\nabla u^+|^2)$, which can be almost "neglected".

Next we derive a Dirichlet boundary condition for P^+ on the shock Σ as in §3.

In fact, it follows from the third and fourth equations in (1.8) that

$$\begin{cases} \partial_i \xi(x_2, x_3) = \frac{\Delta_{i-1}}{\Delta_0}, \quad i = 2, 3, \\ \xi(x_2^0, x_3^0) = x_1^0 \end{cases} \tag{5.6}$$

with

$$\begin{aligned} \Delta_1 &= [\rho u_1 u_2][P + \rho u_3^2] - [\rho u_1 u_3][\rho u_2 u_3], \\ \Delta_2 &= [\rho u_1 u_3][P + \rho u_2^2] - [\rho u_1 u_2][\rho u_2 u_3], \\ \Delta_0 &= [P + \rho u_2^2][P + \rho u_3^2] - [\rho u_2 u_3]^2, \end{aligned}$$

here $x^0 = (x_1^0, x_2^0, x_3^0) \in \Gamma_2$ is defined in Theorem 1.3.

Substituting (5.6) into the other equations in (1.8) yields on Σ

$$\begin{cases} G_1(P^+,u^+,S^+) & \equiv [\rho u_1]\Delta_0-[\rho u_2]\Delta_1-[\rho u_3]\Delta_2=0,\\ G_2(P^+,u^+,S^+) & \equiv [P+\rho u_1^2]\Delta_0-[\rho u_1u_2]\Delta_1-[\rho u_1u_3]\Delta_2=0,\\ G_3(P^+,u^+,S^+) & \equiv [\rho u(\frac{1}{2}|u|^2+h(\rho,S))\cdot(\Delta_0,\Delta_1,\Delta_2)]=0.\end{cases} \tag{5.7}$$

As in §3, it follows from a direct computation and the implicit function theorem that on Σ

$$\begin{cases} u_1^+-u_{1,0}^+(x)=\tilde g_1(u_2^+-u_{2,0}^+,u_3^+-u_{3,0}^+,P_0^--P_0^-(r_0),u_0^--u_0^-(r_0)),\\ P^+-P_0^+(r_0)=\tilde g_2(u_2^+-u_{2,0}^+,u_3^+-u_{3,0}^+,P_0^--P_0^-(r_0),u_0^--u_0^-(r_0)),\\ S^+-S_0^+=\tilde g_3(u_2^+-u_{2,0}^+,u_3^+-u_{3,0}^+,P_0^--P_0^-(r_0),u_0^--u_0^-(r_0)),\end{cases} \tag{5.8}$$

here $u_{i,0}^+=U_0^+(r_0)\frac{x_i}{r_0}(i=1,2,3)$ and $\tilde g_j(0,0,0,0)=0$. Thus, by the assumption (1.8) and the Remark 2.1, we can conclude that $\tilde g_i$ satisfies

$$\tilde g_i=(O(\varepsilon)+C(\eta_0))\left(O(u_2^+-\hat u_{2,0}^+)+O(u_3^+-\hat u_{3,0}^+)+O(\xi(x_2,x_3)-\sqrt{r_0^2-x_2^2-x_3^2})\right),$$

here the generic constant $C(\eta_0)\to 0$ as $\eta_0\to 0$. This fact also illustrates that on the shock, the influence of $u_2^+-\hat u_{2,0}^+$ and $u_3^+-\hat u_{3,0}^+$ on $u_1^+-\hat u_{1,0}^+$, $P^+-\hat P_0^+$ and $S^+-S_0^+$ can be almost "neglected".

Next, we derive the boundary condition of P^+ on the cone surface $\Gamma_2: x_2^2+x_3^2=x_1^2tg^2\alpha_0$.

To this end, it is convenient to use the standard spherical coordinates (r,θ,α), and the corresponding velocity decomposition

$$\begin{cases} U_1^+=\cos\alpha u_1^++\sin\alpha\cos\theta u_2^++\sin\alpha\sin\theta u_3^+,\\ U_2^+=-\sin\theta u_2^++\cos\theta u_3^+,\\ U_3^+=\sin\alpha u_1^+-\cos\alpha\cos\theta u_2^+-\cos\alpha\sin\theta u_3^+,\end{cases}$$

with $0\le\theta<2\pi$ and $0\le\alpha\le\alpha_0$.

Then the system (5.1) becomes

$$\begin{cases} \partial_r(\rho^+U_1^+)+\frac{1}{r\sin\alpha}\partial_\theta(\rho^+U_2^+)-\frac{1}{r}\partial_\alpha(\rho^+U_3^+)+\frac{2\rho^+U_1^+}{r}-\frac{\rho^+U_3^+}{r}ctg\alpha=0,\\ \partial_r(\rho^+(U_1^+)^2+P^+)+\frac{1}{r\sin\alpha}\partial_\theta(\rho^+U_1^+U_2^+)-\frac{1}{r}\partial_\alpha(\rho^+U_1^+U_3^+)+\frac{2\rho^+(U_1^+)^2}{r}-\frac{\rho^+((U_2^+)^2+(U_3^+)^2)}{r}\\ \qquad -\frac{\rho^+U_1^+U_3^+}{r}ctg\alpha=0,\\ \partial_r(\rho^+U_1^+U_2^+)+\frac{1}{r\sin\alpha}\partial_\theta(\rho^+(U_2^+)^2+P^+)-\frac{1}{r}\partial_\alpha(\rho^+U_2^+U_3^+)+\frac{3\rho^+U_1^+U_2^+}{r}-\frac{2\rho^+U_2^+U_3^+}{r}ctg\alpha=0,\\ \partial_r(\rho^+U_1^+U_3^+)+\frac{1}{r\sin\alpha}\partial_\theta(\rho^+U_2^+U_3^+)-\frac{1}{r}\partial_\alpha(\rho^+(U_3^+)^2+P^+)+\frac{3\rho^+U_1^+U_3^+}{r}+\frac{\rho^+((U_2^+)^2-(U_3^+)^2)}{r}ctg\alpha=0,\\ \partial_r\left(\left(\frac{1}{2}|U^+|^2+h(\rho^+,S^+)\right)\rho^+U_1^+\right)+\frac{1}{r\sin\alpha}\partial_\theta\left(\left(\frac{1}{2}|U^+|^2+h(\rho^+,S^+)\right)\rho^+U_2^+\right)\\ \qquad -\frac{1}{r}\partial_\alpha\left(\left(\frac{1}{2}|U^+|^2+h(\rho^+,S^+)\right)\rho^+U_3^+\right)+\frac{2}{r}\rho^+U_1^+\left(\frac{1}{2}|U^+|^2+h(\rho^+,S^+)\right)\\ \qquad -\frac{1}{r}ctg\alpha\rho^+U_3^+\left(\frac{1}{2}|U^+|^2+h(\rho^+,S^+)\right)=0.\end{cases} \tag{5.9}$$

Correspondingly, Γ_2 becomes $\alpha=\alpha_0$ and the boundary condition (1.6) reduce to

$$U_3^+=0 \qquad on \qquad \alpha=\alpha_0. \tag{5.10}$$

Thus, it follows from the fourth equation in (5.9) that

$$\partial_n P^+ \equiv \partial_\alpha P^+ = \rho^+ (U_2^+)^2 ctg\alpha_0 \qquad on \qquad \Gamma_2, \tag{5.11}$$

here n represents the outer normal of the surface Γ_2.

It follows from the analysis above that P^+ should solve the following problem

$$\begin{cases} \nabla\cdot(\frac{\nabla P^+}{\rho^+}) - D(\frac{DP^+}{\gamma P^+}) + \sum_{i,j=1}^{3} \partial_i u_j^+ \partial_j u_i^+ = 0, \\ P^+ - P_0^+(r_0) = \tilde{g}_2(u_2^+ - u_{2,0}^+, u_3^+ - u_{3,0}^+, P_0^- - P_0^-(r_0), u_0^- - u_0^-(r_0)) \qquad \text{on} \qquad x_1 = \xi(x_2, x_3), \\ \partial_n P^+ = \rho^+ (U_2^+)^2 ctg\alpha_0 \qquad on \qquad \Gamma_2, \\ P^+ = \tilde{P}_e \qquad \text{on} \qquad r = X_0 + 1. \end{cases} \tag{5.12}$$

In addition, by (5.1), (1.6), and (5.8), we arrive at the following first order equations on u_1^+ and S^+

$$\begin{cases} Du_1^+ + \frac{\partial_1 P^+}{\rho^+} = 0, \\ u_1^+ - u_{1,0}^+(x) = \tilde{g}_1(u_2^+ - u_{2,0}^+, u_3^+ - u_{3,0}^+, P_0^- - P_0^-(r_0), u_0^- - u_0^-(r_0)) \qquad \text{on} \qquad x_1 = \xi(x_2, x_3), \\ u_1^+ x_1 tg^2\alpha_0 - u_2^+ x_2 - u_3^+ x_3 = 0 \qquad on \qquad \sqrt{x_2^2 + x_3^2} = x_1 tg\alpha_0 \end{cases} \tag{5.13}$$

and

$$\begin{cases} DS^+ = 0, \\ S^+ - S_0^+ = \tilde{g}_3(u_2^+ - u_{2,0}^+, u_3^+ - u_{3,0}^+, P_0^- - P_0^-(r_0), u_0^- - u_0^-(r_0)) \qquad \text{on} \qquad x_1 = \xi(x_2, x_3), \\ u_1^+ x_1 tg^2\alpha_0 - u_2^+ x_2 - u_3^+ x_3 = 0 \qquad on \qquad \sqrt{x_2^2 + x_3^2} = x_1 tg\alpha_0. \end{cases} \tag{5.14}$$

It remains to determine $u_2^+ - u_{2,0}^+$ and $u_3^+ - u_{3,0}^+$. Once the values of u_2^+ and u_3^+ on the shock are known, then we can solve the problems (5.13) and (5.14) by the characteristics method to estimate $u_1^+ - u_{1,0}^+$ and $S^+ - S_0^+$. Furthermore, by the third and fourth equation in (5.1), one can estimate $u_2^+ - u_{2,0}^+$ and $u_3^+ - u_{3,0}^+$ in Ω_+ as well.

We now derive a system on u_2^+ and u_3^+ on the shock.

By (5.2)-(5.4), one has

$$\partial_2 u_2^+ + \partial_3 u_3^+ = -\frac{DP^+}{\gamma P^+} + \frac{1}{(u_1^+)^2}\Big(\frac{DP^+}{\rho^+} + u_2^+ Du_2^+ + u_3^+ Du_3^+ + u_1^+ u_2^+ \partial_2 u_1^+ + u_1^+ u_3^+ \partial_3 u_1^+\Big). \tag{5.15}$$

In addition, it follows from (5.6) that $\partial_3\Big((\frac{\Delta_1}{\Delta_0})(\xi(x_2,x_3),x_2,x_3)\Big) = \partial_2\Big((\frac{\Delta_2}{\Delta_0})(\xi(x_2,x_3),x_2,x_3)\Big)$.

This implies

$$(\partial_3\Delta_1 - \partial_2\Delta_2)(\xi(x_2,x_3),x_2,x_3) = \Big(\frac{\Delta_1}{\Delta_0}\partial_3\Delta_0 - \frac{\Delta_2}{\Delta_0}\partial_2\Delta_0 + \partial_1(\frac{\Delta_2}{\Delta_0})\Delta_1 - \partial_1(\frac{\Delta_1}{\Delta_0})\Delta_2\Big)(\xi(x_2,x_3),x_2,x_3).$$

This, together with a direct computation making use of (5.1), yields

$$\partial_3 u_2^+ - \partial_2 u_3^+ = F(u_2^+\nabla u^+, u_3^+\nabla u^+, \nabla P^+, \nabla S^+, \nabla P_0^-, \nabla u_0^-), \tag{5.16}$$

here $F(0,0,0,0,0,0) = 0$.

By the boundary conditions (1.6) and (5.8), we have that on the intersection line $l = \{x_1 = \xi(x_2, x_3)\} \cap \Gamma_2$

$$\frac{x_2}{\sqrt{x_2^2+x_3^2}}(u_2^+ - u_{2,0}^+) + \frac{x_3}{\sqrt{x_2^2+x_3^2}}(u_3^+ - u_{3,0}^+) = \tilde{g}_0(u_2^+ - u_{2,0}^+, u_3^+ - u_{3,0}^+, P_0^- - P_0^-(r_0), u_0^- - u_0^-(r_0)),$$

here the function $\tilde{g}_0$ has the same property as in (5.8).

Thus, on $x_1 = \xi(x_2, x_3)$, we have

$$\begin{cases} \partial_2 u_2^+ + \partial_3 u_3^+ = -\frac{D^+P^+}{\gamma P^+} + \frac{1}{(u_1^+)^2}\left(\frac{DP^+}{\rho^+} + u_2^+ Du_2^+ + u_3^+ Du_3^+ - u_1^+ u_2^+ \partial_2 u_1^+ - u_1^+ u_3^+ \partial_3 u_1^+\right), \\ \partial_3 u_2^+ - \partial_2 u_3^+ = F(u_2^+ \nabla u^+, u_3^+ \nabla u^+, \nabla P^+, \nabla S^+, \nabla P_0^-, \nabla u_0^-), \\ \frac{x_2}{\sqrt{x_2^2+x_3^2}}(u_2^+ - u_{2,0}^+) + \frac{x_3}{\sqrt{x_2^2+x_3^2}}(u_3^+ - u_{3,0}^+) = \tilde{g}_0\big(u_2^+ - u_{2,0}^+, u_3^+ - u_{3,0}^+, P_0^- - P_0^-(r_0), \\ \qquad\qquad u_0^- - u_0^-(r_0)\big) \qquad \text{on} \qquad l, \end{cases} \tag{5.17}$$

here it should be noted that the position of the intersection l can be exactly estimated in terms of the $C^1(\bar{\Omega}_+)$ regularity of (P^+, u^+, S^+) and the compatibility condition (see Lemma 6.1 for details).

By (5.17), we can obtain some useful estimates on $u_2^+ - u_{2,0}^+$ and $u_3^+ - u_{3,0}^+$ on the shock (see Lemma 6.2). Then it follows from the third and fourth equation in (5.1) that u_2^+ and u_3^+ can be determined by the following problems respectively,

$$\begin{cases} Du_i^+ + \frac{\partial_i P^+}{\rho^+} = 0, \\ u_i^+ = u_i^+(\xi(x_2, x_3), x_2, x_3) \qquad \textit{on} \qquad x_1 = \xi(x_2, x_3), \\ x_1 tg^2\alpha_0 u_1^+ - x_2 u_2^+ - x_3 u_3^+ = 0 \qquad \textit{on} \qquad \Gamma_2, \end{cases} \tag{5.18}$$

here $i = 2, 3$.

Therefore, in order to prove Theorem 1.3, one needs only to study the uniqueness problem of solutions to the equations (5.6), (5.12)-(5.14) and (5.17)-(5.19). This will be done in §6.

Remark 5.1. *By the references [4] and so on, if the Cauchy-Riemann equation (5.17) has a C^2 solution, then the solution is unique. Namely, the boundary condition in (5.17) is enough to give a priori estimate on (u_2^+, u_3^+).*

Remark 5.2. *We can obtain a pressure boundary condition on the general curved nozzle wall Γ for the system (1.2).*

Indeed, let U be any C^1-smooth solution to (1.2).

If Γ is represented by $\alpha = f(r, \theta)$ with $f(r, \theta) \in C^2$, then the boundary condition (1.6) can be written as

$$U_1^+ \partial_r f + U_2^+ \frac{\partial_\theta f}{r \sin\alpha} + \frac{U_3^+}{r} = 0 \qquad \textit{on} \qquad \Gamma. \tag{5.19}$$

Then (5.19) implies that

$$U_1^+ \partial_r U_1^+ \partial_r f + U_1^+ \partial_r U_2^+ \frac{\partial_\theta f}{r \sin\alpha} + U_1^+ \frac{\partial_r U_3^+}{r} = h_0(\partial_\alpha U^+, \nabla f, \nabla^2 f) \tag{5.20}$$

with $h_0(\partial_\alpha U^+, \nabla f, \nabla^2 f) = -U_1^+\big(\partial_\alpha U_3^+ \frac{\partial_r f}{r} - \frac{U_3^+}{r^2} + \partial_\alpha U_1^+ (\partial_r f)^2 + U_1^+ \partial_r^2 f + \partial_\alpha U_2^+ \frac{\partial_r f \partial_\theta f}{r \sin\alpha} + U_2^+ \partial_r(\frac{\partial_\theta f}{r \sin\alpha})\big)$.

It follows from the equations for the momentum in (5.19) and (5.20) that

$$\partial_r f \partial_r P^+ + \frac{\partial_\theta f}{(r \sin\alpha)^2} \partial_\theta P^+ - \frac{1}{r^2} \partial_\alpha P^+ = H_0(\rho^+, U_1^+, U_2^+, U_3^+, \nabla_{\theta,\alpha} U^+, \nabla f, \nabla^2 f) \qquad \textit{on} \qquad \Gamma. \tag{5.21}$$

Moreover, for the small curved nozzle wall Γ (i.e. $|\nabla^\beta_{r,\theta}(f-\alpha_0)|$ is small for $0 \le |\beta| < 2$, here $\alpha_0 > 0$ is a small constant), (5.21) is a strictly oblique derivative boundary condition on P^+. Thus we can extend Theorem 1.2 to more general curved nozzles.

§6. **The Uniqueness in 3-D**

We now prove Theorem 1.2 for 3-D case. As in §4, we transform the domain Ω_+ with a free boundary Σ into a fixed domain $Q_+ = \{y : X_0 < y_1 < X_0 + 1, y_2^2 + y_3^2 < 1\}$ by the following transformation

$$\begin{cases} y_1 = X_0 + \dfrac{x_1 - \xi(x_2, x_3)}{\sqrt{(X_0+1)^2 - x_2^2 - x_3^2} - \xi(x_2, x_3)}, \\ y_i = \dfrac{x_i}{x_1 tg\alpha_0}, \qquad i = 2, 3. \end{cases} \tag{6.1}$$

For simplicity in presentation, in Q_+, we still denote by $(P^+, u_1^+, u_2^+, u_3^+, S^+)$ and $\zeta(y)$ the state of fluid behind the shock and the shock surface equation $\xi(x_2(y), x_3(y))$ in the new coordinates $y = (y_1, y_2, y_3)$ respectively.

With the notation $\tilde{\partial}_i \equiv \partial_{x_i} = \sum_{j=1}^{3} \partial_{x_i} y_j \partial_{y_j} (i = 1, 2, 3)$, the equation (5.12) can be rewritten as follows

$$\begin{cases} \sum_{i=1}^{3} \tilde{\partial}_i (\dfrac{\tilde{\partial}_i P^+}{\rho^+}) - \tilde{D}(\dfrac{\tilde{D} P^+}{\gamma P^+}) + \sum_{i,j=1}^{3} \tilde{\partial}_i u_j^+ \tilde{\partial}_j u_i^+ = 0, \\ P^+ - P_0^+(r_0) = \tilde{g}_2(u_2^+ - u_{2,0}^+, u_3^+ - u_{3,0}^+, P_0^- - P_0^-(r_0), u_0^- - u_0^-(r_0)) \qquad \text{on} \qquad y_1 = X_0, \\ \partial_{\tilde{n}} P^+ = \rho^+ (U_2^+)^2 ctg\alpha_0 \qquad \textit{on} \qquad \sqrt{y_2^2 + y_3^2} = y_1, \\ P^+ = \tilde{P}_e \qquad \text{on} \qquad y_1 = X_0 + 1. \end{cases} \tag{6.2}$$

with $\tilde{D} = u_1^+ \tilde{\partial}_1 + u_2^+ \tilde{\partial}_2 + u_3^+ \tilde{\partial}_3$ and $\partial_{\tilde{n}} = tg\alpha_0 \tilde{\partial}_1 - y_2 \tilde{\partial}_2 - y_3 \tilde{\partial}_3$.

Additionally, (5.6) becomes

$$\begin{cases} \nabla_y \zeta(y) = \nabla_y x_2 \frac{\Delta_1}{\Delta_0} + \nabla_y x_3 \frac{\Delta_2}{\Delta_0}, \\ \zeta(y^0) = x_1^0, \end{cases} \tag{6.3}$$

here

$$\begin{aligned} & y^0 = (X_0, \frac{x_2^0}{x_1^0 tg\alpha_0}, \frac{x_3^0}{x_1^0 tg\alpha_0}), \\ & x_i(y) = y_i x_1(y) tg\alpha_0, \qquad \partial_{y_1} x_i(y) = y_i tg\alpha_0 \partial_{y_1} x_1(y), \qquad i = 2, 3, \\ & x_1(y) = \frac{A_0(y) + \sqrt{A_0^2(y) + B_0(y)((y_1 - X_0)^2 (X_0+1)^2 - A_0^2(y))}}{B_0(y)}, \\ & A_0(y) = (X_0 + 1 - y_1)\zeta(y), \qquad B_0(y) = 1 + (y_1 - X_0)^2 (y_2^2 + y_3^2) tg^2\alpha_0, \\ & \partial_{y_1} x_1(y) = \frac{x_1(y) A(y) B^2(y) \Delta_0}{(x_1(y) A(y) B(y) + (x_1(y) - \zeta(y))(x_2^2(y) + x_3^2(y))\Delta_0 + A(y)(x_1(y) - A(y))(x_2(y)\Delta_1 + x_3(y)\Delta_2)}, \\ & A(y) = \sqrt{(X_0+1)^2 - x_2^2(y) - x_3^2(y)}, \qquad B(y) = A(y) - \zeta(y) \end{aligned}$$

with analogous expressions for $\partial_{y_i} x_j(y) (i, j = 2, 3)$. Roughly speaking, $\partial_{y_i} x_j(y) = \delta_{ij} + O(\eta_0) + O(\varepsilon)(i, j = 1, 2, 3)$ holds.

Correspondingly, (5.13)-(5.14) can be rewritten as follows

$$\begin{cases} \tilde{D}^+u_1^+ + \frac{\tilde{\partial}_1 P^+}{\rho^+} = 0, \\ u_1^+ - u_{1,0}^+(x(y)) = \tilde{g}_1(u_2^+ - u_{2,0}^+, u_3^+ - u_{3,0}^+, P_0^- - P_0^-(r_0), u_0^- - u_0^-(r_0)) \quad \text{on} \quad y_1 = X_0, \\ tg\alpha_0 u_1^+ - y_2 u_2^+ - y_3 u_3^+ = 0 \quad on \quad \sqrt{y_2^2 + y_3^2} = y_1 \end{cases} \tag{6.4}$$

and

$$\begin{cases} \tilde{D}^+S^+ = 0, \\ S^+ - S_0^+ = \tilde{g}_3(u_2^+ - u_{2,0}^+, u_3^+ - u_{3,0}^+, P_0^- - P_0^-(r_0), u_0^- - u_0^-(r_0)) \quad \text{on} \quad y_1 = X_0, \\ tg\alpha_0 u_1^+ - y_2 u_2^+ - y_3 u_3^+ = 0 \quad on \quad \sqrt{y_2^2 + y_3^2} = y_1. \end{cases} \tag{6.5}$$

Define $\tilde{\zeta}(y_2, y_3) = \zeta(X_0, y_2, y_3)$ and $\tilde{u}_i^+(y_2, y_3) = u_i^+(\tilde{\zeta}(y_2, y_3), x_2(X_0, y_2, y_3), x_3(X_0, y_2, y_3))$ for $i = 2, 3$. Then a direct computation yields

$$\partial_{y_i}\tilde{u}_j^+(y_2, y_3) = \partial_1 u_j^+ \partial_{y_i}\tilde{\zeta} + \partial_2 u_j^+ \partial_{y_i} x_2(X_0, y_2, y_3) + \partial_3 u_j^+ \partial_{y_i} x_3(X_0, y_2, y_3), \qquad i, j = 2, 3.$$

Thus, the system (5.17) becomes

$$\begin{cases} \partial_{y_2}\tilde{u}_2^+ + \partial_{y_3}\tilde{u}_3^+ = f_1(y_2, y_3), \\ \partial_{y_3}\tilde{u}_2^+ - \partial_{y_2}\tilde{u}_3^+ = f_2(y_2, y_3), \\ y_2(\tilde{u}_2^+ - \hat{u}_{2,0}^+(x(y))) + y_3(\tilde{u}_3^+ - \hat{u}_{3,0}^+(x(y))) = \tilde{g}_0(u_2^+ - u_{2,0}^+, u_3^+ - u_{3,0}^+, P_0^- - P_0^-(r_0), \\ \qquad u_0^- - u_0^-(r_0)) \quad \text{on} \quad \tilde{l}. \end{cases} \tag{6.6}$$

here $\tilde{l} = \{y_1 = X_0\} \cap \{y_2^2 + y_3^2 = 1\}$ and

$$\begin{aligned} f_1(y_2, y_3) = \Big(& \partial_{y_3}x_3(\partial_1 u_2^+ \partial_{y_2}\zeta + \partial_3 u_2^+ \partial_{y_2}x_3) + \partial_{y_2}x_2(\partial_1 u_3^+ \partial_{y_3}\zeta + \partial_2 u_3^+ \partial_{y_3}x_2) - \partial_{y_2}x_2\partial_{y_3}x_3\big(\frac{\tilde{D}^+P^+}{\gamma P^+} \\ & - \frac{1}{(u_1^+)^2}\big(\frac{\tilde{D}P^+}{\rho^+} + u_2^+\tilde{D}u_2^+ + u_3^+\tilde{D}u_3^+ - u_1^+u_2^+\tilde{\partial}_2 u_1^+ - u_1^+u_3^+\tilde{\partial}_3 u_1^+\big)\big) + (1 - \partial_{y_3}x_3)\partial_{y_2}u_2^+ \\ & + (1 - \partial_{y_2}x_2)\partial_{y_3}u_3^+\Big)(\tilde{\zeta}(y_2, y_3), x_2(X_0, y_2, y_3), x_3(X_0, y_2, y_3)), \\ f_2(y_2, y_3) = \Big(& \partial_{y_2}x_2(\partial_1 u_2^+ \partial_{y_3}\zeta + \partial_2 u_2^+ \partial_{y_3}x_2) - \partial_{y_3}x_3(\partial_1 u_3^+ \partial_{y_2}\zeta + \partial_2 u_3^+ \partial_{y_2}x_2) \\ & - \partial_{y_2}x_2\partial_{y_3}x_3 F(u_2^+\tilde{\nabla}u^+, u_3^+\tilde{\nabla}u^+, \tilde{\nabla}P^+, \tilde{\nabla}S^+, \tilde{\nabla}P^-, \tilde{\nabla}u_0^-) + (1 - \partial_{y_2}x_2)\partial_{y_3}\tilde{u}_2^+ \\ & - (1 - \partial_{y_3}x_3)\partial_{y_3}\tilde{u}_3^+\Big)(\tilde{\zeta}(y_2, y_3), x_2(X_0, y_2, y_3), x_3(X_0, y_2, y_3)). \end{aligned}$$

We notice that $f_i(y_2, y_3)(i = 1, 2)$ is of the "quadratic" error (i.e. $|f_i| + |\nabla_{y_2,y_3}f_i| \le C(\varepsilon + \delta(\eta_0))\sum_{i=1}^{3}(|\nabla u_i^+| + |\nabla^2 u_i^+|) + C(|\nabla P^+| + |\nabla^2 P^+|) + C(\varepsilon + \delta(\eta_0))(|\nabla S^+| + |\nabla^2 S^+|) + C(|\nabla P^-| + |\nabla S^-| + |\nabla^2 P^-| + |\nabla^2 S^-|)$). More precisely, it follows from the second, the third, the fourth equations in (5.1) and the first equality in (5.8) that $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$ can be expressed the functions of $P^+, u_1^+, u_2^+, u_3^+, S^+, \nabla_{y_2,y_3}\tilde{u}_2^+, \nabla_{y_2,y_3}\tilde{u}_3^+, \tilde{\nabla}P^+, \tilde{\nabla}S^+$, $\tilde{\zeta}(y_2, y_3)$ and $\nabla_{y_2,y_3}\tilde{\zeta}(y_2, y_3)$ with $|f_i| + |\nabla_{y_2,y_3}f_i| \le C(\varepsilon + \delta(\eta_0))\Big(\sum_{i=2}^{3}(|\nabla_{y_2,y_3}u_i^+| + |\nabla_{y_2,y_3}^2 u_i^+|) + |S^+| + |\nabla_{y_2,y_3}S^+| + |\nabla_{y_2,y_3}^2 S^+| + |\tilde{\zeta}| + |\nabla_{y_2,y_3}\tilde{\zeta}| + |\nabla_{y_2,y_3}^2\tilde{\zeta}|\Big) + C(|\nabla P^+| + |\nabla^2 P^+|)$.

Similarly, (5.18) can be written as

$$
\begin{cases}
\tilde{D}u_i^+ + \dfrac{\tilde{\partial}_i P^+}{\rho^+} = 0, \\
u_i^+ = \tilde{u}_i^+(y_2, y_3) \qquad \text{on} \qquad y_1 = X_0, \\
tg\alpha_0 u_1^+ - y_2 u_2^+ - y_3 u_3^+ = 0 \qquad \text{on} \qquad \sqrt{y_2^2 + y_3^2} = y_1
\end{cases}
\tag{6.7}
$$

with $i = 2, 3$.

Next we show that the compatibility relation of the solution to (1.2) on the intersection curve l holds true.

For this end, we define the function $G(M_0^-) = \dfrac{(2-\gamma)(M_0^-)^2}{\mu^2(M_0^-)} + \dfrac{2-\gamma}{2}\big((\mu^2(M_0^-) - 1\big) + \dfrac{3\mu(M_0^-) - 1}{\mu(M_0^-) - 1}$, where $M_0^- = \dfrac{U_0^-(r_0)}{c(\rho^-(r_0), S_0^-)}$ stands for the Mach number of the supersonic incoming flow, further $\mu(M_0^-) = \dfrac{U_0^+(r_0)}{U_0^-(r_0)}$ can be determined by M_0^- (see (6.16) below).

Lemma 6.1. *If the solution $(P^+, u_1^+, u_2^+, u_3^+, S^+)$ of (1.2) is of $C^1(\bar{\Omega}_+)-$regular, and $G(M_0^-) \neq 0$, then we have on the intersection curve $l = \{x_1 = \xi(x_2, x_3)\} \cap \Gamma_2$*

$$U_2^+ = 0.$$

Moreover, if the equation of the shock Σ is given by $r = \tilde{r}(\theta, \alpha)$ in the spherical coordinates, then

$$\tilde{r}(\theta, \alpha_0) \equiv r_0.$$

Remark 6.1. *If $1 < \gamma \leq 2$ and the transonic shock is weak, then it follows from the proof of Lemma 6.1 that $G(M_0^-) \neq 0$ always holds.*

Proof. First we show that the shock surface is perpendicular to the fixed boundary, i.e., $\partial_\alpha \tilde{r}(\theta, \alpha_0) = 0$. Indeed, in the spherical coordinate, the R-H conditions (1.3) become

$$
\begin{cases}
[\rho U_1] - \frac{1}{\tilde{r}\sin\alpha}[\rho U_2]\partial_\theta \tilde{r} + \frac{1}{\tilde{r}}[\rho U_3]\partial_\alpha \tilde{r} = 0, \\
[\rho U_1^2 + P] - \frac{1}{\tilde{r}\sin\alpha}[\rho U_1 U_2]\partial_\theta \tilde{r} + \frac{1}{\tilde{r}}[\rho U_1 U_3]\partial_\alpha \tilde{r} = 0, \\
[\rho U_1 U_2] - \frac{1}{\tilde{r}\sin\alpha}[\rho U_2^2 + P]\partial_\theta \tilde{r} + \frac{1}{\tilde{r}}[\rho U_2 U_3]\partial_\alpha \tilde{r} = 0, \\
[\rho U_1 U_3] - \frac{1}{\tilde{r}\sin\alpha}[\rho U_2 U_3]\partial_\theta \tilde{r} + \frac{1}{\tilde{r}}[\rho U_3^2 + P]\partial_\alpha \tilde{r} = 0, \\
[(\rho e + \frac{1}{2}\rho|U|^2 + P)U_1] - \frac{1}{\tilde{r}\sin\alpha}[(\rho e + \frac{1}{2}\rho|U|^2 + P)U_2]\partial_\theta \tilde{r} + \frac{1}{\tilde{r}}[(\rho e + \frac{1}{2}\rho|U|^2 + P)U_3]\partial_\alpha \tilde{r} = 0.
\end{cases}
\tag{6.8}
$$

Since the fixed boundary condition (1.6) implies

$$U_3^+ = 0 \qquad \text{on} \qquad \alpha = \alpha_0,$$

then the fourth equation in (5.9) and $[P] \neq 0$ yield

$$\partial_\alpha \tilde{r}(\theta, \alpha_0) = 0. \tag{6.9}$$

Obviously, (6.9) shows that the shock surface is perpendicular to the fixed boundary Γ_2.

Next, we show $U_2^+ = 0$ on l.

By (6.8), (6.9), $U_2^- \equiv 0$ for $r \in [X_0 + \frac{1}{4}, X_0 + 1]$ and $U_3^\pm = 0$ on Γ_2, we can arrive at on the intersection curve l

$$
\begin{cases}
[\rho U_1][\rho U_2^2 + P] - [\rho U_1 U_2][\rho U_2] = 0, \\
[\rho U_1^2 + P][\rho U_2^2 + P] - [\rho U_1 U_2]^2 = 0, \\
[(\rho e + \frac{1}{2}\rho|U|^2 + P)U_1][\rho U_2^2 + P] - [(\rho e + \frac{1}{2}\rho|U|^2 + P)U_2][\rho U_1 U_2] = 0
\end{cases}
$$

and

$$
\begin{cases}
\partial_\alpha(\rho^+U_1^+) + \Big(ctg\alpha_0\rho^+U_2^+ - \partial_\alpha(\rho^+U_2^+)\Big)\dfrac{\rho^+U_1^+U_2^+}{\rho^+(U_2^+)^2+[P]} = 0,\\
\partial_\alpha(\rho^+(U_1^+)^2+P^+) + \Big(ctg\alpha_0\rho^+U_1^+U_2^+ - \partial_\alpha(\rho^+U_1^+U_2^+)\Big)\dfrac{\rho^+U_1^+U_2^+}{\rho^+(U_2^+)^2+[P]} = 0,\\
\partial_\alpha(\rho^+U_1^+U_2^+) + \Big(ctg\alpha_0(\rho^+(U_2^+)^2+[P]) - \partial_\alpha(\rho^+(U_2^+)^2+P^+)\Big)\dfrac{\rho^+U_1^+U_2^+}{\rho^+(U_2^+)^2+[P]} = 0,\\
\partial_\alpha\big((\rho^+e^+ + \frac{1}{2}\rho^+|U^+|^2+P^+)U_1^+\big) + \Big(ctg\alpha_0(\rho^+e^+ + \frac{1}{2}\rho^+|U^+|^2+P^+)U_2^+\\
\qquad -\partial_\alpha\big((\rho^+e^+ + \frac{1}{2}\rho^+|U^+|^2+P^+)U_2^+\big)\Big)\dfrac{\rho^+U_1^+U_2^+}{\rho^+(U_2^+)^2+[P]} = 0.
\end{cases}
\tag{6.10}
$$

In order to guarantee $P^+ \in C^1(\bar{\Omega}_+)$, due to (5.11), one must require that

$$
\partial_\alpha P^+ = \rho^+(U_2^+)^2 ctg\alpha_0 \qquad on \qquad l. \tag{6.11}
$$

Substituting (6.11) into (6.10) yields

$$
\begin{cases}
U_1^+(1+O((U_2^+)^2))\partial_\alpha\rho^+ + \rho^+\partial_\alpha U_1^+ = -\frac{2(\rho^+)^2U_1^+(U_2^+)^2}{[P]}ctg\alpha_0 + O((U_2^+)^3),\\
(U_1^+)^2(1+O((U_2^+)^2))\partial_\alpha\rho^+ + 2\rho^+U_1^+(1+O((U_2^+)^2))\partial_\alpha U_1^+ = -\rho^+(U_2^+)^2\big(1+\frac{2\rho^+(U_1^+)^2}{[P]}\big)ctg\alpha_0\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +O((U_2^+)^3),\\
\partial_\alpha U_2^+ = -U_2^+ctg\alpha_0 + O((U_2^+)^2),\\
-\frac{\gamma P^+}{(\gamma-1)(\rho^+)^2}\partial_\alpha\rho^+ + U_1^+\partial_\alpha U_1^+ = -\frac{(U_2^+)^2}{\gamma-1}ctg\alpha_0 + O((U_2^+)^3).
\end{cases}
\tag{6.12}
$$

For convenience, we use the notations that equations $\rho=\rho(P,S), e=e(P,S)$ and $c^2(P,S)=\frac{\gamma P}{\rho(P,S)}$. Then it follows from (6.12) that

$$
\begin{cases}
U_1^+(1+O((U_2^+)^2))\partial_S\rho^+\partial_\alpha S^+ + \rho^+\partial_\alpha U_1^+ + \rho^+(U_2^+)^2ctg\alpha_0\big(\frac{2\rho^+U_1^+}{[P]} + \frac{U_1^+}{c^2(P^+,S^+)}\big) = O((U_2^+)^3),\\
(U_1^+)^2\partial_S\rho^+(1+O((U_2^+)^2))\partial_\alpha S^+ + 2\rho^+U_1^+(1+O((U_2^+)^2))\partial_\alpha U_1^+ + \rho^+(U_2^+)^2ctg\alpha_0\Big(1+\frac{2\rho^+(U_1^+)^2}{[P]}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\frac{(U_1^+)^2}{c^2(P^+,S^+)}\Big) = O((U_2^+)^3),\\
-\frac{c^2(P^+,S^+)}{(\gamma-1)\rho^+}\partial_S\rho^+\partial_\alpha S^+ + U_1^+\partial_\alpha U_1^+ = O((U_2^+)^3).
\end{cases}
\tag{6.13}
$$

Now we claim that $U_2^+ = 0$ on l. Indeed, regard (6.13) as a system for $(\partial_\alpha S^+, \partial_\alpha U_1^+, (U_2^+)^2)$, suffices to verify that the determinant $\Delta \neq 0$ with

$$
\Delta = \det\begin{pmatrix}
U_1^+\partial_S\rho^+ & \rho^+ & \rho^+\big(\frac{2\rho^+U_1^+}{[P]} + \frac{U_1^+}{c^2(P^+,S^+)}\big)\\
(U_1^+)^2\partial_S\rho^+ & 2\rho^+U_1^+ & \rho^+\Big(1+\frac{2\rho^+(U_1^+)^2}{[P]} + \frac{(U_1^+)^2}{c^2(P^+,S^+)}\Big)\\
-\frac{c^2(P^+,S^+)}{(\gamma-1)\rho^+}\partial_S\rho^+ & U_1^+ & 0
\end{pmatrix}
$$

A direct computation yields

$$
\Delta = \frac{\rho^+\partial_S\rho^+c^2(\rho^+,S^+)}{\gamma-1}\Big((2-\gamma)(M^+)^2 + \frac{(3U_0^+-U_0^-)(r_0)}{(U_0^+-U_0^-)(r_0)} + O(\varepsilon)\Big),
$$

here the Mach number $M_0^+ = \dfrac{U_0^+(r_0)}{c(\rho^+(r_0), S^+)}$.

Due to the Rankine-Hugoniot conditions (2.2) and the equation of state, one has

$$\begin{cases} \dfrac{\rho_0^+(r_0)}{\rho_0^-(r_0)} = \dfrac{U^-(r_0)}{U^+(r_0)}, \\ [P_0](r_0) = (\rho_0^- U_0^-)(r_0)(U_0^- - U_0^+)(r_0), \\ \dfrac{\gamma}{\gamma-1}\dfrac{[P_0](r_0)}{\rho_0^+(r_0)} - \dfrac{\gamma}{\gamma-1}\dfrac{P_0^-(r_0)[\rho_0](r_0)}{(\rho_0^+\rho_0^-)(r_0)} + \frac{1}{2}(U_0^+(r_0))^2 - \frac{1}{2}(U_0^-(r_0))^2 = 0. \end{cases}$$

It follows from this that

$$\begin{cases} \mu^2(M_0^-) + \dfrac{2}{\gamma+1}(\gamma + \dfrac{1}{(M_0^-)^2})\mu(M_0^-) + \dfrac{2(\gamma-1)}{\gamma+1}\left(1 + \dfrac{1}{(\gamma-1)(M_0^-)^2}\right) = 0, \\ (M_0^+)^2 = \dfrac{(M_0^-)^2}{\mu^2(M_0^-)} + \frac{1}{2}\big((\mu^2(M_0^-) - 1\big), \end{cases} \tag{6.14}$$

here $\mu(M_0^-) = \dfrac{U_0^+(r_0)}{U_0^-(r_0)}$.

Therefore,

$$\Delta = \frac{\rho^+\partial_S\rho^+ c^2(\rho^+, S^+)}{\gamma-1}G(M_0^-) + O(\varepsilon) \neq 0,$$

and then $U_2^+ \equiv 0$ holds true on l.

Furthermore, it follows from the last equation in (6.8) and $[P] \neq 0$ that

$$\partial_\theta \tilde{r}(\theta, \alpha_0) = 0,$$

which implies $\tilde{r}(\theta, \alpha_0) \equiv r_0$ in terms of $x_1^0 = \xi(x_2^0, x_3^0)$.

Remark 6.2. *It follows from the proof of Lemma 6.1 that the shock surface $\Sigma : x_1 = \xi(x_2, x_3)$ is perpendicular to the cone surface. More generally, for an arbitrary but slight curved nozzle wall Γ: $\alpha = f(r, \theta)$(as in Remark 5.2) and the solution $(U_1^+, U_2^+, U_3^+, P^+, S^+; \xi(x_2, x_3)) \in C^1(\bar{\Omega}_+)$, the shock surface Σ is still perpendicular to Γ.*

Indeed, as in (5.20), on the fixed boundary $\alpha = f(r, \theta)$, we have

$$U_1^+\partial_r f + U_2^+\frac{\partial_\theta f}{r\sin\alpha} + \frac{U_3^+}{r} = 0.$$

Thus it follows from the second, the third and the fourth equations in (6.8) that on $\tilde{l} = \Gamma \cap \Sigma$

$$\begin{cases} k_0[\rho U_1^2] - k_1[\rho U_1 U_2] + [P] = 0, \\ k_0[\rho U_1 U_2] - k_1[\rho U_2^2] - \frac{\partial_\theta \tilde{r}}{\tilde{r}\sin\alpha}[P] = 0, \\ -\tilde{r}\partial_r f k_0[\rho U_1^2] + (-\frac{\partial_\theta f}{\sin\alpha}k_0 + \tilde{r}\partial_r f k_1)[\rho U_1 U_2] + \frac{\partial_\theta f}{\sin\alpha}k_1[\rho U_2^2] + \frac{\partial_\alpha \tilde{r}}{\tilde{r}}[P] = 0 \end{cases} \tag{6.15}$$

with $k_0 = 1 - \partial_r f \partial_\alpha \tilde{r}$ and $k_1 = \frac{\partial_\theta \tilde{r}}{\tilde{r}\sin\alpha} + \frac{\partial_\theta f \partial_\alpha \tilde{r}}{\tilde{r}\sin\alpha}$.

By the first and the second equations in (6.15), we can obtain on $\tilde{l}$

$$\begin{cases} [\rho U_1 U_2] = \dfrac{k_1}{k_0}[\rho U_2^2] + \dfrac{\partial_\theta \tilde{r}}{k_0 \tilde{r}\sin\alpha}[P], \\ [\rho U_1^2] = (\dfrac{k_1}{k_0})^2[\rho U_2^2] + \dfrac{k_1\partial_\theta \tilde{r}}{k_0^2 \tilde{r}\sin\alpha}[P] - \dfrac{1}{k_0}[P]. \end{cases}$$

Substituting the expressions above into the third equation in (6.15) yields

$$(\tilde{r}\partial_r f - \frac{\partial_\theta f \partial_\theta \tilde{r}}{\tilde{r}\sin^2\alpha} + \frac{\partial_\alpha \tilde{r}}{\tilde{r}})[P] = 0.$$

Since $[P] \neq 0$, then

$$\tilde{r}\partial_r f - \frac{\partial_\theta f \partial_\theta \tilde{r}}{\tilde{r}\sin^2\alpha} + \frac{\partial_\alpha \tilde{r}}{\tilde{r}} = 0 \qquad on \qquad \tilde{l}.$$

In addition, a direct computation derives

$$\begin{aligned}&(\partial_1(\alpha - f(r,\theta)), \partial_2(\alpha - f(r,\theta)), \partial_3(\alpha - f(r,\theta))) \cdot (\partial_1(r - \tilde{r}(\theta,\alpha)), \partial_2(r - \tilde{r}(\theta,\alpha)), \partial_3(r - \tilde{r}(\theta,\alpha)))\\ &= -\partial_r f + \frac{\partial_\theta f \partial_\theta \tilde{r}}{\tilde{r}^2\sin^2\alpha} - \frac{\partial_\alpha \tilde{r}}{\tilde{r}^2}.\end{aligned}$$

Therefore, the shock surface Σ is perpendicular to the nozzle wall Γ.

Remark 6.3. *To guarantee the solution $(U_1^+, U_2^+, U_3^+, P^+, S^+; \xi(x_2, x_3)) \in C^1(\bar{\Omega}_+)$ in Remark 6.2, as in Remark 1.4, one should give some restrictions on the nozzle wall Γ.*

Next, we study the boundary value problem (6.6). To this end, we need a Lemma.

Lemma 6.2. *Let $B(0,1)$ be the disk centered at the origin $O = (0,0)$ with the radius 1. If $w_1(x), w_2(x) \in C^{2,\delta}(\bar{B})$ satisfy*

$$\begin{cases} \partial_1 w_1 + \partial_2 w_2 = f_1(x), \\ \partial_2 w_1 - \partial_1 w_2 = f_2(x), \\ x_1 w_1 + x_2 w_2 = g(x) \qquad\qquad on \qquad \partial B, \end{cases} \tag{6.16}$$

here $x = (x_1, x_2)$, $f_1(x), f_2(x) \in C^{1,\delta}(\bar{B})$, $g(x) \in C^{2,\delta}(\bar{B})$, $0 < \delta < 1$, then it holds that

$$\|w_1\|_{C^{2,\delta}(\bar{B})} + \|w_2\|_{C^{2,\delta}(\bar{B})} \leq C(\|f_1\|_{C^{1,\delta}(\bar{B})} + \|f_2\|_{C^{1,\delta}(\bar{B})} + \|g\|_{C^{2,\delta}(\bar{B})}). \tag{6.17}$$

Proof. Set

$$\begin{cases} \Delta\varphi_i = (-1)^{i+1} f_i(x), \\ \varphi_i = 0 \qquad on \qquad \partial B, \qquad\qquad i = 1, 2. \end{cases}$$

Then the following estimate holds

$$\|\varphi_1\|_{C^{3,\delta}(\bar{B})} + \|\varphi_2\|_{C^{3,\delta}(\bar{B})} \leq C(\|f_1\|_{C^{1,\delta}(\bar{B})} + \|f_2\|_{C^{1,\delta}(\bar{B})}) \tag{6.18}$$

Decompose w_1 and w_2 as

$$w_1 = \tilde{w}_1 + \partial_1\varphi_1 - \partial_2\varphi_2, \qquad w_2 = \tilde{w}_2 + \partial_2\varphi_1 + \partial_1\varphi_2.$$

Then

$$\begin{cases} \partial_1 \tilde{w}_1 + \partial_2 \tilde{w}_2 = 0, \\ \partial_2 \tilde{w}_1 - \partial_1 \tilde{w}_2 = 0, \\ x_1 \tilde{w}_1 + x_2 \tilde{w}_2 = \tilde{g}(x) \qquad\qquad on \qquad \partial B, \end{cases} \tag{6.19}$$

with $\tilde{g}(x) = g(x) - x_1(\partial_1\varphi_1 - \partial_2\varphi_2) - x_2(\partial_2\varphi_1 + \partial_1\varphi_2)$.

Define

$$W_1(x) = x_1\tilde{w}_1 + x_2\tilde{w}_2, \qquad W_2(x) = x_2\tilde{w}_1 - x_1\tilde{w}_2.$$

Then

$$\begin{cases} \partial_1 W_1 - \partial_2 W_2 = 0, \\ \partial_2 W_1 + \partial_1 W_2 = 0, \\ W_1 = \tilde{g}(x) \qquad\qquad on \qquad \partial B, \\ W_2(0,0) = 0. \end{cases}$$

Thus the standard estimates for Cauchy-Riemann equation yield

$$\|W_1\|_{C^{2,\delta}(\bar{B})} + \|W_2\|_{C^{2,\delta}(\bar{B})} \le C\|\tilde{g}\|_{C^{2,\delta}(\bar{B})},$$

which implies

$$\|\tilde{w}_1\|_{L^\infty(B)} + \|\tilde{w}_2\|_{L^\infty(B)} \le C\|\tilde{g}\|_{C^{2,\delta}(\bar{B})}. \tag{6.20}$$

By (6.19), we can assume $\tilde{w}_1 = \partial_1\varphi$ and $\tilde{w}_2 = \partial_2\varphi$ with $\varphi(0,0) = 0$ such that

$$\begin{cases} \Delta\varphi = 0, \\ \partial_r\varphi = \tilde{g} \qquad on \qquad \partial B, \\ \varphi(0,0) = 0. \end{cases}$$

As a consequence of the elliptic estimate and (6.20), one has

$$\|\varphi\|_{C^{3,\delta}(\bar{B})} \le C(\|\varphi\|_{L^\infty(B)} + \|\tilde{g}\|_{C^{2,\delta}(\bar{B})}) \le C(\|\bar{w}_1\|_{L^\infty(B)} + \|\bar{w}_2\|_{L^\infty(\bar{B})} + \|\tilde{g}\|_{C^{2,\delta}(\bar{B})}) \le C\|\tilde{g}\|_{C^{2,\delta}(\bar{B})}. \tag{6.21}$$

We are now ready to prove Theorem 1.2 in the 3-D case by modifying the ideas in §4.

Suppose that the problem (6.2)-(6.7) has the solution $(P^+, u_1^+, u_2^+, u_3^+, S^+; \zeta(y))$ with the corresponding regularities in Theorem 1.2.

Set

$$W_1(y) = P^+(y) - \hat{P}_0^+(\bar{r}(y)), \qquad W_i(y) = u_i^+(y) - \hat{u}_{i,0}^+(\bar{r}(y)), i = 2, 3, 4, \qquad W_5 = S^+(y) - S_0^+,$$
$$\Xi(y) = \zeta(y) - \sqrt{r_0^2 - (\bar{x}_2(y))^2 - (\bar{x}_2(y))^2}$$

with $\bar{r}(y) = \sqrt{\sum_{i=1}^{3}(\bar{x}_i(y))^2}$ and $\bar{x}(y) = (\bar{x}_1(y), \bar{x}_2(y), \bar{x}_3(y))$ given by the following transformation

$$\begin{cases} y_1 = X_0 + \dfrac{x_1 - \sqrt{r_0^2 - x_2^2 - x_3^2}}{\sqrt{(X_0+1)^2 - x_2^2 - x_3^2} - \sqrt{r_0^2 - x_2^2 - x_3^2}}, \\ y_i = \dfrac{x_i}{x_1 tg\alpha_0}, \qquad i = 2, 3. \end{cases}$$

As in §4, making use of (6.3), Lemma 6.1, Remark 2.1 and the assumptions in Theorem 1.2, we can obtain

$$\|\Xi(y)\|_{C^{2,\delta_0}} \le C(\varepsilon + \delta(\eta_0))\|(W_1, W_2, W_5)\|_{C^{1,\delta_0}(Q_+)} + C\|(W_3, W_4)\|_{C^{1,\delta_0}(Q_+)}, \tag{6.22}$$

here $\delta(\eta_0) > 0$ is a generic constant and $\delta(\eta_0) \to 0$ as $\eta_0 \to 0$.

Similarly, it follows from (6.2) , Lemma 6.1 and (6.22) that

$$\begin{aligned} \|W_1\|_{C^{2,\delta_0}} &\le C(\varepsilon + \delta(\eta_0))\Big(\sum_{i=1}^{5}\|W_i\|_{C^{1,\delta_0}} + \|\Xi(y)\|_{C^{2,\delta_0}} + \|(W_3, W_4)(X_0, \cdot, \cdot)\|_{C^{2,\delta_0}(\bar{B})}\Big) \\ &\le C(\varepsilon + \delta(\eta_0))\Big(\sum_{i=1}^{5}\|W_i\|_{C^{1,\delta_0}} + \|(W_3, W_4)(X_0, \cdot, \cdot)\|_{C^{2,\delta_0}(\bar{B})}\Big). \end{aligned} \tag{6.23}$$

Next, W_2 and W_5 can be estimated by the characteristics method and the equations (6.4) and (6.5) as

$$\|W_2\|_{C^{1,\delta_0}} + \|W_5\|_{C^{2,\delta_0}} \le C\|W_1\|_{C^{2,\delta_0}} + C(\varepsilon + \delta(\eta_0))\Big(\|(W_2, W_3)\|_{C^{1,\delta_0}} + \|(W_3, W_4)(X_0, \cdot, \cdot)\|_{C^{2,\delta_0}(\bar{B})}\Big). \tag{6.24}$$

In addition, by Lemma 6.2 and (6.6)-(6.7), one has

$$\|(W_3,W_4)(X_0,\cdot,\cdot)\|_{C^{2,\delta_0}(\bar{B})}\le C(\varepsilon+\delta(\eta_0))\sum_{i=2}^{4}\|W_i\|_{C^{1,\delta_0}}+C\|(W_1,W_2)\|_{C^{2,\delta_0}},\tag{6.25}$$

and

$$\|(W_3,W_4)\|_{C^{1,\delta_0}}\le\|(W_3,W_4)(X_0,\cdot,\cdot)\|_{C^{1,\delta_0}(\bar{B})}+C(\varepsilon+\delta(\eta_0))\Big(\sum_{i=2}^{4}\|W_i\|_{C^{1,\delta_0}}+\|W_5\|_{C^{2,\delta_0}}\Big)+C\|W_1\|_{C^{2,\delta_0}}.\tag{6.26}$$

It follows from (6.22)-(6.26) that

$$\|(W_1,W_5)\|_{C^{2,\delta_0}}+\sum_{k=2}^{4}\|W_k\|_{C^{1,\delta_0}}\le C(\varepsilon+\delta(\eta_0))\left(\|(W_1,W_5)\|_{C^{2,\delta_0}}+\sum_{k=2}^{4}\|W_k\|_{C^{1,\delta_0}}\right).$$

Thus, for small ε and η_0, we arrive at

$$W_1=W_2=W_3=W_4=W_5=0.$$

This and (6.22) show

$$\Xi(y)=0.$$

Therefore, $P^+(y)=\hat{P}_0^+(\bar{r}(y)), u_1^+(y)=\hat{u}_{i,0}^+(\bar{r}(y))(i=1,2,3), S^+(y)=S_0^+$ and $\zeta(y)=r_0$. This completes the proof of Theorem 1.2 for 3-D case.

§7. The reformulation of the dynamical problem

In this section, we start to reformulate the dynamical problem (1.14) with (1.11)-(1.13) and (1.15)-(1.16). Since the system $(1.14)_-$ is hyperbolic with respect to $r-$direction and $t-$direction, then it follows from the finite propagation property that $(1.14)_-$ has a global C^2 solution $(\rho^-(t,r),U^-(t,r))$ in the domain $\Omega_-=\{(t,r):0\le t<\infty, X_0+\frac{1}{4}<r<r(t)\}$, especially $(\rho^-,U^-)\equiv(\hat{\rho}_0^-(r),\hat{U}_0^-(r))$ for $t\ge t_0$ ($t_0>0$ is some fixed constant) and $|\nabla_{t,r}^k(\rho^--\rho_0^-(r))|+|\nabla_{t,r}^k(U^--U_0^-(r))|\le C\varepsilon$ for $k=0,1,2$, here $(\hat{\rho}_0^-(r),\hat{U}_0^-(r))$ represents the extension of $(\rho_0^-(r),U_0^-(r)))$ in $[X_0+\frac{1}{4},X_0+1]$.

The system $(1.14)_+$ has two eigenvalues $\lambda_1(\rho^+,U^+)=U^+-c(\rho^+)$ and $\lambda_2(\rho^+,U^+)=U^++c(\rho^+)$. The corresponding Riemann invariants are $w_1=U^+-F(\rho^+)$ and $w_2=U^++F(\rho^+)$ with $F'(\rho)=\frac{c(\rho)}{\rho}$. In this case, it follows from $(1.14)_+$, (1.11) and (1.13) that

$$\begin{cases}\partial_t w_i+\lambda_i(w)\partial_r w_i=(-1)^{i+1}\dfrac{(w_1+w_2)c(w)}{2r}, & \\ w_i(0,r)=w_{i,0}^+(r)+w_{i,0}(\varepsilon,r), & i=1,2,\\ \rho^+(w)=\rho_e & on\qquad r=X_0+1\end{cases}\tag{7.1}$$

with $w_{i,0}^+(r)=U_0^+(r)+(-1)^iF(\rho_0^+(r))$, $w_{i,0}(\varepsilon,r)=\varepsilon\{U_1^+(r)+(-1)^i\big(\int_0^1F'(\rho_0^+(r)+\varepsilon(1-\theta)\rho_1^+(r))d\theta\big)\rho_1^+(r)\}$, $i=1,2$, $c(w)=c(\rho^+(w))$, $\rho^+(w)=F^{-1}(\dfrac{w_2-w_1}{2})$, and F^{-1} represents the inverse function of $F(\rho^+)=\dfrac{w_2-w_1}{2}$.

On the shock $r=r(t)$, by use of (1.15), one has

$$\begin{cases}r'(t)=\dfrac{[\rho U]}{[\rho]}=\dfrac{\big(\rho^+(w)(w_1+w_2)-2\rho^-U^-\big)(t,r(t))}{2(\rho^+(w)-\rho^-)(t,r(t))},\\ r(0)=r_0\end{cases}\tag{7.2}$$

and

$$G(w) = [\rho U]^2 - [\rho][\rho U^2 + P] = 0. \tag{7.3}$$

To simplify (7.3), we will use the following fact

$$(\hat{\rho}_0^- \hat{U}_0^-)(r(t)) = (\hat{\rho}_0^+ \hat{U}_0^+)(r(t)). \tag{7.4}$$

Indeed, (1.14) implies that

$$r\hat{\rho}_0^\pm \hat{U}_0^\pm = r_0 \rho_0^\pm(r_0) U_0^\pm(r_0).$$

Thus it follows from the Rankine-Hugoniot condition $\rho_0^-(r_0)U_0^-(r_0) = \rho_0^+(r_0)U_0^+(r_0)$ that (7.4) holds. Next we analyze the boundary condition (7.3) for $t \geq t_0$.
(7.3) can be rewritten as

$$[\rho U^2 + P] = \frac{[\rho U]^2}{[\rho]}. \tag{7.5}$$

Note that

$$\begin{aligned}[\rho U^2 + P] = &\big(\rho^+(U^+)^2 + P^+\big)(t, r(t)) - \big(\hat{\rho}_0^+(\hat{U}_0^+)^2 + \hat{P}_0^+\big)(r(t)) \\ &+ \big(\hat{\rho}_0^+(\hat{U}_0^+)^2 + \hat{P}_0^+\big)(r(t)) - \big(\hat{\rho}_0^-(\hat{U}_0^-)^2 + \hat{P}_0^-\big)(r(t))\end{aligned}$$

and

$$\begin{aligned}&\big(\hat{\rho}_0^+(\hat{U}_0^+)^2 + \hat{P}_0^+\big)(r(t)) - \big(\hat{\rho}_0^-(\hat{U}_0^-)^2 + \hat{P}_0^-\big)(r(t)) = \big(\hat{\rho}_0^+(\hat{U}_0^+)^2 + \hat{P}_0^+\big)(r(t)) - \big(\hat{\rho}_0^+(\hat{U}_0^+)^2 + \hat{P}_0^+\big)(r_0) \\ &\qquad\qquad - \big(\hat{\rho}_0^-(\hat{U}_0^-)^2 + \hat{P}_0^-\big)(r(t)) + \big(\hat{\rho}_0^-(\hat{U}_0^-)^2 + \hat{P}_0^-\big)(r_0) \\ &= \Big(\int_0^1 \{\partial_r(\hat{\rho}_0^+(\hat{U}_0^+)^2 + \hat{P}_0^+)(\theta r_0 + (1-\theta)r(t)) - \partial_r(\hat{\rho}_0^-(\hat{U}_0^-)^2 + \hat{P}_0^-)(\theta r_0 + (1-\theta)r(t))\} d\theta\Big)(r(t) - r_0) \\ &= \Big(\int_0^1 \Big(\frac{\hat{\rho}_0^- \hat{U}_0^-(\hat{U}_0^- - \hat{U}_0^+)}{r}\Big)(\theta r_0 + (1-\theta)r(t)) d\theta\Big)(r(t) - r_0) \\ &= B_0(r(t) - r_0) + B_1(t)(r(t) - r_0)^2 \qquad \text{(by the second equations in (1.14))}\end{aligned}$$

with

$$\begin{aligned}B_0 &= \frac{(\rho_0^- U_0^-(U_0^- - U_0^+))(r_0)}{r_0} > 0, \\ B_1(t) &= \int_0^1 (1-\theta) d\theta \int_0^1 \partial_r\Big(\frac{\hat{\rho}_0^- \hat{U}_0^-(\hat{U}_0^- - \hat{U}_0^+)}{r}\Big)\big((1 - \theta_1 + \theta_1\theta)r_0 + \theta_1(1-\theta)r(t)\big) d\theta_1.\end{aligned}$$

It follows from (7.4), (7.5) and Taylor's formula that on $r = r(t)$

$$\begin{aligned}\rho^+ - \hat{\rho}_0^+ = &-\frac{2\hat{\rho}_0^+ \hat{U}_0^+}{c^2(\hat{\rho}_0^+) + (\hat{U}_0^+)^2}(U^+ - \hat{U}_0^+) - \frac{B_0}{c^2(\hat{\rho}_0^+) + (\hat{U}_0^+)^2}(r(t) - r_0) \\ &+ f((\rho^+ - \hat{\rho}_0^+)^2, (U^+ - \hat{U}_0^+)^2, (\rho^+ - \hat{\rho}_0^+)(U^+ - \hat{U}_0^+), (r(t) - r_0)^2),\end{aligned} \tag{7.6}$$

here $f(0,0,0,0) = 0$ and $f \in C^2$ on its arguments.
Hence, due to (7.6), it holds that on $r = r(t)$

$$w_2 - \hat{w}_{2,0}^+ = A_0(w_1 - \hat{w}_{1,0}^+) - \tilde{B}_0(r(t) - r_0) + f_1((w_1 - \hat{w}_{1,0}^+)^2, (w_2 - \hat{w}_{2,0}^+)^2, (w_1 - \hat{w}_{1,0}^+)(w_2 - \hat{w}_{2,0}^+), (r(t) - r_0)^2), \tag{7.7}$$

here $f_1(0,0,0,0) = 0$, $f_1 \in C^2$ on its arguments, $\hat{w}_{i,0}^+(r) = \hat{U}_0^+(r) + (-1)^i F(\hat{\rho}_0^+(r))$, $i = 1, 2$,

$A_0 = \left(\frac{U_0^+(r_0) - c(\rho_0^+(r_0))}{U_0^+(r_0) + c(\rho_0^+(r_0))}\right)^2$ and $\tilde{B}_0 = \frac{2c(\rho_0^+(r_0))B_0}{\rho_0^+(r_0)(c(\rho_0^+(r_0)) + U_0^+(r_0))^2}$.

Obviously,

$$0 < A_0 < 1. \tag{7.8}$$

Thanks to (7.4) and (7.6), (7.2) can be rewritten as

$$r'(t) = A_1(U^+ - \hat{U}_0^+) - \bar{B}_0(r(t) - r_0) + f_2\Big((w_1 - \hat{w}_{1,0}^+)^2, (w_2 - \hat{w}_{2,0}^+)^2, (w_1 - \hat{w}_{1,0}^+)(w_2 - \hat{w}_{2,0}^+),$$
$$(w_1 - \hat{w}_{1,0}^+)(r(t) - r_0), (w_2 - \hat{w}_{2,0}^+)(r(t) - r_0), (r(t) - r_0)^2\Big) \tag{7.9}$$

with $A_1 = \frac{\rho_0^+(r_0)(c^2(\rho_0^+(r_0)) - (U_0^+(r_0))^2)}{[\rho_0](c^2(\rho_0^+(r_0)) + (U_0^+(r_0))^2)}$, $\bar{B}_0 = \frac{B_0 U_0^+(r_0)}{[\rho_0](c^2(\rho_0^+(r_0)) + (U_0^+(r_0))^2)} > 0$ and $f_2(0,0,0,0,0,0) = 0$.

Here we emphasize that $\bar{B}_0 > 0$ will play a crucial role to derive the decay estimate on the solution $(\rho^+ - \hat{\rho}_0^+, U^+ - \hat{U}_0^+; r(t) - r_0)$ in §8.

In addition, on the boundary $r = X_0 + 1$,

$$w_1 - w_{1,0}^+ = w_2 - w_{2,0}^+ + f_3(w_2 - w_{2,0}^+) + g_0(t) \tag{7.10}$$

with $f_3(0) = f_3'(0) = 0$ and $g_0(t) \in C_0^2(0, \infty)$.

Hence in order to prove Theorem 1.4, by the local existence of solution in [18] (since (7.8) implies that the boundary conditions (7.7) and (7.10) are dissipative), we need only to solve the problem (7.1) with (7.7), (7.9), (7.10) and with the initial data $(w_i(t,r) - \hat{w}_{i,0}^+(r))|_{t=t_0} (i = 1, 2)$ and $(r(t) - r_0)|_{t=t_0}$ in the domain $\{(t,r) : t \geq t_0, r(t) \leq r \leq X_0 + 1\}$. Here the initial can be regarded as suitable small in the sense that

$$\sum_{|\alpha| \leq 1} \sup_{r(t_0) \leq r \leq X_0+1} |\nabla_{t,r}^\alpha (w_i(t_0, r) - \hat{w}_{i,0}^+(r))| \leq C\varepsilon, \qquad |r(t_0) - r_0| \leq C\varepsilon, \qquad |r'(t_0)| + |r''(t_0)| \leq C\varepsilon, \tag{7.11}$$

which can be derived from the results on the local existence and stability in [18].

§8. **Global Dynamical Stability**

To prove Theorem 1.4 for $m = 2$, we will give a uniform estimate on w and its derivatives.

Lemma 8.1. *Set $D_T = \{t_0 \leq t \leq T, r(t) \leq r \leq X_0 + 1\}$ for any large $T > t_0$. If $w \in C^2(D_T)$ satisfies (7.1),(7.7)- (7.11), then there exist two positive constants C_0 and $\tilde{C}_0$ independent of ε and T, such that $|w_i - \hat{w}_{i,0}^+(r)| + |\nabla_{t,r}(w_i - \hat{w}_{i,0}^+(r))| \leq \frac{C_0\varepsilon}{(1+t)^2}$ in D_T for $|\alpha| \leq 1$, $i = 1, 2$, and $|\partial_t^j(r(t) - r_0)| \leq \frac{\tilde{C}_0\varepsilon}{(1+t)^2}$ in $[t_0, T]$ for $0 \leq j \leq 2$.*

Proof. We shall use the reflected characteristics method together with (7.9) to obtain the needed estimates (the reflected characteristics method has been used in Lemma 2.1 of Chapter 5 of [18]). Because the background solution $(\rho_0^+(r), U_0^+(r))$ is not a constant state, a more delicate treatment than that in [18] and [32] is needed here. In addition, by the local existence result of the solution in [18] and the continuity induction, in order to prove Lemma 8.1, it suffices to show that

For some positive constants $C_0, \tilde{C}_0, C_1, C_2$ and C_3, if $|w_i - \hat{w}_{i,0}^+(r)| \leq \frac{C_0\varepsilon}{(1+t)^2}$, $|\partial_t(w_i - \hat{w}_{i,0}^+(r))| \leq \frac{C_1\varepsilon}{(1+t)^2}$ and $|\partial_r(w_i - \hat{w}_{i,0}^+(r))| \leq \frac{C_2\varepsilon}{(1+t)^2}$ in D_T, $i = 1, 2$; $|r(t) - r_0|, |r'(t)| \leq \frac{\tilde{C}_0\varepsilon}{(1+t)^2}$ and $|r''(t)| \leq \frac{C_3\varepsilon}{(1+t)^2}$ in $[t_0, T]$, then there exist positive constants $C_0', \tilde{C}_0', C_1', C_2'$ and C_3' ($C_i' < C_i$ and $\tilde{C}_0' < \tilde{C}_0$) such that $|w_i - \hat{w}_{i,0}^+(r)| \leq \frac{C_0'\varepsilon}{(1+t)^2}$, $|\partial_t(w_i - \hat{w}_{i,0}^+(r))| \leq \frac{C_1'\varepsilon}{(1+t)^2}$ and $|\partial_r(w_i - \hat{w}_{i,0}^+(r))| \leq \frac{C_2'\varepsilon}{(1+t)^2}$ in D_T, $i = 1, 2$; $|r(t) - r_0|, |r'(t)| \leq \frac{\tilde{C}_0'\varepsilon}{(1+t)^2}$ and $|r''(t)| \leq \frac{C_3'\varepsilon}{(1+t)^2}$ in $[0, T]$. (8.1)

Below we denote by C various strictly positive constants independent of ε, $T, C_0, \tilde{C}_0$ and X_0. For $(t,r) \in D_T, t<T$, let $\gamma_j^-(s,t,r)$ $(j=1,2)$ be the backward j-th characteristic curve through (t,r), i.e.,

$$\begin{cases} \dfrac{d\gamma_j^-(s,t,r)}{ds} = \lambda_j(w(s,\gamma_j^-(s,t,r))), \qquad s \le t, \\ \gamma_j^-(s,t,r)|_{s=t} = r. \end{cases} \tag{8.2}$$

By the assumptions in (8.1) and the entropy condition (1.16), one has

$$|\frac{d\gamma_j^-(s,t,r)}{ds} - \lambda_j(\hat{w}_0^+(\gamma_j^-(s,t,r)))| \le \frac{CC_0\varepsilon}{(1+s)^2} \qquad in \qquad D_T. \tag{8.3}$$

If $\{(s,\gamma_2^-(s,t,r))\} \cap \{(s,r) : r = r(s)\} = (\Gamma_2(t,r), \xi_2(t,r))$, $\{(s,\gamma_1^-(s,t,r))\} \cap \{(s,r) : r = X_0+1\} = (\Gamma_1(t,r), \xi_1(t,r))$, then it follows from (7.1) and Remark 2.1 that

$$|w_i(t,r) - \hat{w}_{i,0}^+(r)| \le |(w_i - \hat{w}_{i,0}^+)(\Gamma_i(t,r), \xi_i(t,r))| + C\delta(X_0) \int_{\Gamma_i(t,r)}^t \sum_{i=1}^2 |w_i(s,\gamma_i^-(s,t,r)) - \hat{w}_{i,0}^+(\gamma_i^-(s,t,r))| ds. \tag{8.4}$$

If $\{(s,\gamma_1^-(s,\Gamma_2(t,r),\xi_2(t,r))\} \cap \{(s,r) : r = X_0+1\} = (\pi_1(t,r), \eta_1(t,r))$ and $\{(s,\gamma_2^-(s,\Gamma_1(t,r),\ \xi_1(t,r))\} \cap \{(s,r) : r = r(s)\} = (\pi_2(t,r), \eta_2(t,r))$, then by the characteristics method (7.7), and (7.10), we get for small $\varepsilon > 0$

$$\begin{aligned} |w_1(t,r) - \hat{w}_{1,0}^+(r)| &\le |(w_2 - \hat{w}_{2,0}^+)(\Gamma_1(t,r), \xi_1(t,r))| + |f_3(w_2 - \hat{w}_{2,0}^+)(\Gamma_1(t,r), \xi_1(t,r)))| \\ &\quad + C\delta(X_0) \int_{\Gamma_1(t,r)}^t \sum_{i=1}^2 |w_i(s,\gamma_1^-(s,t,r)) - \hat{w}_{i,0}^+(\gamma_1^-(s,t,r))| ds \\ &\le (1+C\varepsilon)\Big(C\delta(X_0) \sum_{i=1}^2 \int_{\pi_2(t,r)}^{\Gamma_1(t,r)} |w_i(s,\gamma_2^-(s,\Gamma_1(t,r),\xi_1(t,r))) - \hat{w}_{i,0}^+(\gamma_2^-(s,\Gamma_1(t,r),\xi_1(t,r)))| ds \\ &\quad + |(w_2 - \hat{w}_{2,0}^+)(\pi_2(t,r), \eta_2(t,r))| \Big) + C\delta(X_0) \int_{\Gamma_1(t,r)}^t \sum_{i=1}^2 |w_i(s,\gamma_1^-(s,t,r)) - \hat{w}_{i,0}^+(\gamma_1^-(s,t,r))| ds \\ &\le (1+C\varepsilon)|A_0(w_1 - \hat{w}_{1,0}^+)(\pi_2(t,r), \eta_2(t,r))| + (1+C\varepsilon)\tilde{B}_0|r(\pi_2(t,r)) - r_0| + \frac{C\delta(X_0)C_0\varepsilon}{(1+t)^2} \\ &\le (1+C\varepsilon)\tilde{B}_0|r(\pi_2(t,r)) - r_0| + \frac{(A_0 + C\delta(X_0) + CC_0\varepsilon)C_0\varepsilon}{(1+t)^2}, \end{aligned} \tag{8.5}$$

here we have used the following relations (for large t)

$$1 - \frac{C(1+\delta(X_0)+\varepsilon)}{1+t} \le \frac{\Gamma_i(t,r)}{t} \le 1 + \frac{C(1+\delta(X_0)+\varepsilon)}{1+t} \tag{8.6}$$

and

$$1 - \frac{C(1+\delta(X_0)+\varepsilon)}{1+t} \le \frac{\pi_i(t,r)}{t} \le 1 + \frac{C(1+\delta(X_0)+\varepsilon)}{1+t}. \tag{8.7}$$

Similarly, one can obtain

$$\begin{aligned} |w_2(t,r) - \hat{w}_{2,0}^+(r)| &\le A_0|(w_1 - \hat{w}_{1,0}^+)(\Gamma_2(t,r), \xi_2(t,r))| + \frac{C(\delta(X_0) + (C_0+\tilde{C}_0)\varepsilon)(C_0+\tilde{C}_0)\varepsilon}{(1+t)^2} \\ &\quad + \tilde{B}_0|r(\Gamma_2(t,r)) - r_0| \\ &\le \tilde{B}_0|r(\pi_1(t,r)) - r_0| + \left(A_0 + \frac{C(C_0+\tilde{C}_0)(\delta(X_0)+(C_0+\tilde{C}_0)\varepsilon)}{C_0}\right)\frac{C_0\varepsilon}{(1+t)^2}. \end{aligned} \tag{8.8}$$

If $\{(s,\gamma_1^-(s,t,r))\}\cap\{(s,r): r=X_0+1\}=\emptyset$, or $\{(s,\gamma_2^-(s,t,r))\}\cap\{(s,r): r=r(s)\}=\emptyset$, or $\{(s,\gamma_1^-(s,\Gamma_2(t,r),\xi_2(t,r)))\}\cap\{(s,r): r=X_0+1\}=\emptyset$, or $\{(s,\gamma_2^-(s,\Gamma_1(t,r),\xi_1(t,r)))\}\cap\{(s,r): r=r(s)\}=\emptyset$, then by (8.3), (1.19) and the initial data (7.8) we can conclude

$$t\le C \qquad \textit{and} \qquad |w_i(t,r)-\hat{w}_{i,0}^+(r)|\le C\varepsilon. \tag{8.9}$$

In addition, it follows from (7.9) and (8.1) that

$$|(e^{\bar{B}_0 t}(r(t)-r_0))'|\le A_1(C_0+C(C_0+\tilde{C}_0)^2\varepsilon)\varepsilon\frac{e^{\bar{B}_0 t}}{(1+t)^2}.$$

Thus, for large t and small ε, one has

$$\begin{aligned}|r(t)-r_0|&\le C\varepsilon e^{-\bar{B}_0 t}+A_1(C_0+C(C_0+\tilde{C}_0)^2\varepsilon)\varepsilon\int_{t_0}^t\frac{e^{-\bar{B}_0(t-s)}}{(1+s)^2}ds\\&\le C\varepsilon e^{-\bar{B}_0 t}+A_1(C_0+C(C_0+\tilde{C}_0)^2\varepsilon)\varepsilon\int_0^t\frac{e^{-\bar{B}_0\tau}}{(1+t-\tau)^2}d\tau\\&\le C\varepsilon e^{-\bar{B}_0 t}+A_1(C_0+C(C_0+\tilde{C}_0)^2\varepsilon)\varepsilon\Big(\int_0^{\eta t}\frac{e^{-\bar{B}_0\tau}}{(1+t-\tau)^2}d\tau+\int_{\eta t}^t\frac{e^{-\bar{B}_0\tau}}{(1+t-\tau)^2}d\tau\Big)\\&\le C\varepsilon\big(e^{-\bar{B}_0 t}+e^{-\bar{B}_0\eta t}\big)+\frac{A_1(C_0+C(C_0+\tilde{C}_0)^2\varepsilon)\varepsilon}{\bar{B}_0(1+(1-\eta)t)^2}\\&\le\frac{C\varepsilon}{(1+t)^3}+\frac{A_1(C_0+C(C_0+\tilde{C}_0)^2\varepsilon)\varepsilon}{\bar{B}_0(1+(1-\eta)t)^2},\end{aligned}\tag{8.10}$$

here $\eta\in(0,1)$ is a suitably small constant.

Collecting (8.5)-(8.7) and (8.10)

$$|w_1(t,r)-\hat{w}_{1,0}^+(r)|\le(1+C\varepsilon)\tilde{B}_0\Big(\frac{C\varepsilon}{(1+t)^3}+\frac{A_1(C_0+C(C_0+\tilde{C}_0)^2\varepsilon)\varepsilon}{\bar{B}_0(1+(1-\eta)t)^2}\Big)+\frac{(A_0+C\delta(X_0)+CC_0\varepsilon)C_0\varepsilon}{(1+t)^2}. \tag{8.11}$$

In order to show (8.1), for large t, X_0 and small ε,η, we require

$$\frac{\tilde{B}_0A_1}{\bar{B}_0}+A_0<1. \tag{8.12}$$

In fact, (8.12) holds if and only if

$$c^2(\rho_0^+(r_0))<3(U_0^+(r_0))^2. \tag{8.13}$$

For the weak transonic shock (namely, $U_0^-(r_0)\sim c(\rho_0^-(r_0))$ and $U_0^+(r_0)\sim c(\rho_0^+(r_0))$ although $U_0^-(r_0)>c(\rho_0^-(r_0))$ and $U_0^+(r_0)<c(\rho_0^+(r_0))$), then (8.13) obviously holds (in fact, we only need $U_0^+(r_0)>\frac{\sqrt{3}}{3}c(\rho_0^+(r_0))$).

In this case, it follows from (8.11) that for large t, X_0 and small ε,

$$|w_i(t,r)-\hat{w}_{i,0}^+(r)|<\frac{C_0\varepsilon}{(1+t)^2}, \qquad i=1,2. \tag{8.14}$$

Thus, (8.14), together with (7.9) and (8.10), implies that there exists a constant $\tilde{C}_0>0$ independent of ε and T such that

$$|r(t)-r_0|<\frac{\tilde{C}_0\varepsilon}{(1+t)^2}, \qquad |r'(t)|<\frac{\tilde{C}_0\varepsilon}{(1+t)^2}. \tag{8.15}$$

Next we estimate $|\nabla_{t,r}(w_i(t,r)-\hat{w}^+_{i,0}(r))|$.

Set $\bar{w}_i=\partial_t(w_i-\hat{w}^+_{i,0}), i=1,2$. Then (7.1) yields

$$\partial_t\bar{w}_i+\lambda_i(w)\partial_r\bar{w}_i=g_i,\qquad i=1,2, \tag{8.16}$$

where

$$g_i=(-1)^i\frac{\partial_t(\lambda_i(w)-\lambda_i(\hat{w}^+_0))}{\lambda_i(w)}\left(\frac{(w_1+w_2)c(w)}{2r}-\frac{(\hat{w}^+_{1,0}+\hat{w}^+_{2,0})c(\hat{w}^+_0)}{2r}(-1)^i\left((\lambda_i(w)-\lambda_i(\hat{w}^+_0))\partial_r\hat{w}^+_{i,0}+\bar{w}_i\right)\right)$$
$$-\partial_t\left((\lambda_i(w)-\lambda_i(\hat{w}^+_0))\partial_r\hat{w}^+_{i,0}\right)+(-1)^{i+1}\partial_t\left(\frac{(w_1+w_2)c(w)}{2r}-\frac{(\hat{w}^+_{1,0}+\hat{w}^+_{2,0})c(\hat{w}^+_0)}{2r}\right),\qquad i=1,2.$$

It follows from (7.10) that

$$\bar{w}_1=\bar{w}_2+f_3'(w_2-\hat{w}^+_{2,0})\bar{w}_2+g_1(t) \tag{8.17}$$

with $g_1(t)\in C^2_0(0,\infty)$.

To get the boundary condition of $\bar{w}$ on $r=r(t)$, one should notice that the vector field $V=\partial_t+r'(t)\partial_r$ tangent to $r=r(t)$ can be expressed as follows

$$V=\frac{1}{\lambda_i(w)}\{(\lambda_i(w)-r'(t))\partial_t+r'(t)(\partial_t+\lambda_i(w)\partial_r)\}.$$

So on the shock $r=r(t)$, due to (7.1), (7.7) and the assumptions in (8.1), it holds that

$$\bar{w}_2=\frac{A_0\lambda_2(\hat{w}^+_0)(\lambda_1(\hat{w}^+_0)-r'(t))}{\lambda_1(\hat{w}^+_0)(\lambda_2(\hat{w}^+_0)-r'(t))}\bar{w}_1-\tilde{B}_0r'(t)+\bar{f}_1(\bar{w}_1,r'(t))\qquad on\qquad r=r(t) \tag{8.18}$$

with $\bar{f}_1(0,0)=0$ and $|\bar{f}_1(z_1,z_2)|\le\frac{\bar{C}\varepsilon}{(1+t)^2}|z_1|+\bar{C}(\delta(X_0)+\frac{\varepsilon}{(1+t)^2})|z_2|$, here and below the generic constant $\bar{C}$ may depend on C_0 and $\tilde{C}_0$ but is independent of ε, T and X_0.

By the assumptions in (8.1),

$$|g_i(t,r)|\le\frac{C_1\varepsilon}{(1+t)^2}\left(\frac{\bar{C}\varepsilon}{(1+t)^2}+\bar{C}\delta(X_0)\right). \tag{8.19}$$

Using the notations above, if $\gamma_1^-(s,t,r)$ and $\gamma_2^-(s,t,r)$ both intersect with fixed boundary and shock front, then by the characteristics method, (8.19), the boundary conditions (8.17), (8.18) and (8.15), as in (8.9)-(8.13), one can arrive at

$$\begin{aligned}|\bar{w}_1(t,r)|&\le|\big((1+f_3'(w_2-\hat{w}^+_{2,0}))\bar{w}_2\big)(\Gamma_1(t,r),\xi_1(t,r))|+\bar{C}(\delta(X_0)+\varepsilon)\int_{\Gamma_1(t,r)}^t\frac{C_1\varepsilon}{(1+s)^2}ds\\&\le(1+\bar{C}\varepsilon)|A_0\bar{w}_1(\pi_2(t,r),\eta_2(t,r))|+(1+\bar{C}\varepsilon)\tilde{B}_0|r'(\pi_2(t,r))|+\bar{C}(\delta(X_0)+\varepsilon)\frac{C_1\varepsilon}{(1+t)^2}+\frac{\bar{C}(\varepsilon+\delta(X_0))\varepsilon}{(1+t)^2}\\&\le\big(\frac{\tilde{B}_0A_1}{\bar{B}_0}+A_0+\bar{C}\delta(X_0)+\bar{C}\varepsilon\big)\frac{C_1\varepsilon}{(1+t)^2}+\frac{\bar{C}(\delta(X_0)+\varepsilon)\varepsilon}{(1+t)^2}.\end{aligned}$$

Similarly,

$$|\bar{w}_2(t,r)|\le\big(\frac{\tilde{B}_0A_1}{\bar{B}_0}+A_0+\bar{C}\delta(X_0)+\bar{C}\varepsilon\big)\frac{C_1\varepsilon}{(1+t)^2}+\frac{\bar{C}(\delta(X_0)+\varepsilon)\varepsilon}{(1+t)^2}.$$

Thanks to (8.12), (8.1) holds for $|\partial_t(w_i(t,r)-\hat{w}^+_{i,0}(r))|$ $(i=1,2)$.

Since for $i = 1, 2$,

$$\partial_r(w_i - \hat{w}_{i,0}^+) = \frac{(-1)^i}{\lambda_i(w)} \left(\frac{(w_1 + w_2)c(w)}{2r} - \frac{(\hat{w}_{1,0}^+ + \hat{w}_{2,0}^+)c(\hat{w}_0^+)}{2r} + (-1)^i \left((\lambda_i(w) - \lambda_i(\hat{w}_0^+))\partial_r \hat{w}_{i,0}^+ + \bar{w}_i \right) \right),$$

then a direct computation yields

$$|\partial_r(w_1 - \hat{w}_{1,0}^+)| + |\partial_r(w_2 - \hat{w}_{2,0}^+)| \le \frac{C_2'\varepsilon}{(1+t)^2},$$

here $C_2' > 0$ can be determined by $C_0, \tilde{C}_0$ and C_1.

Thus for the appropriately chosen large constant $C_2 > C_2'$ we can obtain

$$|\nabla_{t,r}(w_1 - \hat{w}_{1,0}^+)| + |\nabla_{t,r}(w_2 - \hat{w}_{2,0}^+)| < \frac{C_2\varepsilon}{(1+t)^2}.$$

Since $|r''(t)| \le C \sum_{i=1}^{2} (|\nabla_{t,r}(w_i - w_{i,0}^+)| + |(w_i - w_{i,0}^+)|) + C\delta(X_0)(|r'(t)| + |r(t) - r_0|)$ on $r = r(t)$, then by the estimates on $w_i - w_{i,0}^+$, $\nabla_{t,r}(w_i - w_{i,0}^+)$, $|r(t) - r_0|$ and $|r'(t)|$, it is easy to conclude

$$|r''(t)| \le \frac{C_3\varepsilon}{(1+t)^2},$$

here $C_3 > 0$ depends on $C_0, \tilde{C}_0$, C_1 and C_2.

Therefore (8.1) and Lemma 8.1 are proved.

Proof of Theorem 1.4. Since the local well-posedness of the solution is achieved in [18], while for any given t, the solution of (7.1) with the initial data (7.11) given on $t = t_0$ and the boundary conditions (7.7), (7.9) and (7.10) in $[t_0, t_0 + \frac{C}{\varepsilon}]$ can be obtained by use of the characteristics method. Therefore, by Lemma 8.1, we can get a smaller initial data for $w - w_0^+$ and $r(t) - r_0$ on $t = \frac{C}{\varepsilon}$, then the solution can be extended continuously to the whole domain. Thus, Theorem 1.3 is proved in the case $m = 2$. The case $m = 3$ can be treated similarly.

§9. **Dynamical Instability**

We now treat the instability of a transonic shock in a converging nozzle. To simplify the notations, we will neglect the notation "$\sim$" on $(\tilde{\rho}^\pm, \tilde{U}^\pm)$ in (1.17)-(1.22). By the hyperbolicity with respect to $\tilde{r}-$direction and $t-$direction, $(1.20)_-$ has a global C^2 solution $(\rho^-(t, \tilde{r}), U^-(t, \tilde{r}))$ in the domain $\tilde{\Omega}_- = \{(t, \tilde{r}) : 0 \le t < \infty, -X_0 - \frac{3}{4} < \tilde{r} < \tilde{r}(t)\}$, especially $(\rho^-, U^-) \equiv (\hat{\rho}_0^-(-\tilde{r}), \hat{U}_0^-(-\tilde{r}))$ for $t \ge t_0$ ($t_0 > 0$ is some fixed constant) and $|\nabla_{t,\tilde{r}}^k(\rho^- - \hat{\rho}_0^-(-\tilde{r}))| + |\nabla_{t,\tilde{r}}^k(U^- - \hat{U}_0^-(-\tilde{r}))| \le C\varepsilon$ for $k = 0, 1, 2$, here $(\hat{\rho}_0^-(-\tilde{r}), \hat{U}_0^-(-\tilde{r}))$ represents the extension of $(\rho_0^-(r), U_0^-(r)))$ in $[X_0, X_0 + 1]$. As in §7, we can reformulate the problem $(1.20)_+$ with (1.17), (1.19) and (1.21)-(1.22) as follows:

$$\begin{cases} \partial_t w_i + \lambda_i(w)\partial_{\tilde{r}} w_i = (-1)^{i+1} \dfrac{(w_1 + w_2)c(w)}{2\tilde{r}}, \\ w_i(0, \tilde{r}) = w_{i,0}^+(\tilde{r}) + w_{i,0}(\varepsilon, \tilde{r}), \qquad i = 1, 2, \\ \rho^+(w) = \rho_e + \tilde{\rho}_2^+(t) \qquad on \qquad \tilde{r} = -X_0 \end{cases} \tag{9.1}$$

with $w_{i,0}(0, \tilde{r}) = 0$ $(i = 1, 2)$.

On the shock $\tilde{r} = \tilde{r}(t)$, by (1.21), one has

$$\begin{cases} \tilde{r}'(t) = \dfrac{[\rho U]}{[\rho]}, \\ \tilde{r}(0) = -r_0 \end{cases} \tag{9.2}$$

and

$$G(w) = [\rho U]^2 - [\rho][\rho U^2 + P] = 0. \tag{9.3}$$

Since

$$\begin{aligned}[][\rho U^2 + P] = &\big(\rho^+(U^+)^2 + P^+\big)(t, \tilde{r}(t)) - \big(\hat{\rho}_0^+(\hat{U}_0^+)^2 + \hat{P}_0^+\big)(-\tilde{r}(t)) \\ &+ \big(\hat{\rho}_0^+(\hat{U}_0^+)^2 + \hat{P}_0^+\big)(-\tilde{r}(t)) - \big(\hat{\rho}_0^-(\hat{U}_0^-)^2 + \hat{P}_0^-\big)(-\tilde{r}(t))\end{aligned}$$

and

$$\begin{aligned}&\big(\hat{\rho}_0^+(\hat{U}_0^+)^2 + \hat{P}_0^+\big)(-\tilde{r}(t)) - \big(\hat{\rho}_0^-(\hat{U}_0^-)^2 + \hat{P}_0^-\big)(-\tilde{r}(t)) = \big(\hat{\rho}_0^+(\hat{U}_0^+)^2 + \hat{P}_0^+\big)(-\tilde{r}(t)) - \big(\hat{\rho}_0^+(\hat{U}_0^+)^2 + \hat{P}_0^+\big)(r_0) \\ &\qquad - \big(\hat{\rho}_0^-(\hat{U}_0^-)^2 + \hat{P}_0^-\big)(-\tilde{r}(t)) + \big(\hat{\rho}_0^-(\hat{U}_0^-)^2 + \hat{P}_0^-\big)(r_0) \\ &= -B_0(\tilde{r}(t) + r_0) + B_1(t)(\tilde{r}(t) + r_0)^2,\end{aligned}$$

with the constant B_0 and the function $B_1(t)$ being defined in §7. Then as in §7, on $\tilde{r} = \tilde{r}(t)$ and for $t \geq t_0$, the boundary condition (9.3) can be reduced to

$$\begin{aligned}\rho^+ - \hat{\rho}_0^+ = &-\frac{2\hat{\rho}_0^+\hat{U}_0^+}{c^2(\hat{\rho}_0^+) + (\hat{U}_0^+)^2}(U^+ - \hat{U}_0^+) + \frac{B_0}{c^2(\hat{\rho}_0^+) + (\hat{U}_0^+)^2}(\tilde{r}(t) + r_0) \\ &+ \tilde{f}((\rho^+ - \hat{\rho}_0^+)^2, (U^+ - \hat{U}_0^+)^2, (\rho^+ - \hat{\rho}_0^+)(U^+ - \hat{U}_0^+), (\tilde{r}(t) + r_0)^2),\end{aligned} \tag{9.4}$$

here $\tilde{f}(0,0,0,0) = 0$ and $\tilde{f} \in C^2$ on its arguments.

Analogously, on $\tilde{r} = \tilde{r}(t)$,

$$w_2 - \hat{w}_{2,0}^+ = A_0(w_1 - \hat{w}_{1,0}^+) + \tilde{B}_0(\tilde{r}(t) + r_0) + \tilde{f}_1((w_1 - \hat{w}_{1,0}^+)^2, (w_2 - \hat{w}_{2,0}^+)^2, (w_1 - \hat{w}_{1,0}^+)(w_2 - \hat{w}_{2,0}^+), (\tilde{r}(t) + r_0)^2), \tag{9.5}$$

and

$$\begin{cases} \tilde{r}'(t) = A_1(U^+ - \hat{U}_0^+) + \bar{B}_0(\tilde{r}(t) + r_0) + \tilde{f}_2\Big((w_1 - \hat{w}_{1,0}^+)^2, (w_2 - \hat{w}_{2,0}^+)^2, (w_1 - \hat{w}_{1,0}^+)(w_2 - \hat{w}_{2,0}^+), \\ \qquad (w_1 - \hat{w}_{1,0}^+)(\tilde{r}(t) + r_0), (w_2 - \hat{w}_{2,0}^+)(\tilde{r}(t) + r_0), (\tilde{r}(t) + r_0)^2\Big), \\ (\tilde{r}(t) + r_0)|_{t=0} = 0, \end{cases} \tag{9.6}$$

here $\tilde{f}_1(0,0,0,0) = 0$, $\tilde{f}_2(0,0,0,0,0,0) = 0$, $\tilde{f}_1, \tilde{f}_2 \in C^2$ on its arguments, and $A_0, \tilde{B}_0, A_1$ and $\bar{B}_0$ are given in §7.

Here it should be emphasized that the equation (9.6) on $\tilde{r}(t)$ and the equation (7.9) on $r(t)$ are very different because the coefficient of $\tilde{r}(t) + r_0$ in (9.6) is positive meanwhile it is negative in (7.9). This difference yields that (7.9) has a global decay solution but the solution of (9.6) blows up in general.

In addition, on the boundary $\tilde{r} = -X_0$,

$$w_1 - w_{1,0}^+ = (w_2 - w_{2,0}^+) + \tilde{f}_3(w_2 - w_{2,0}^+) + \tilde{g}_0(t) \tag{9.7}$$

with $\tilde{f}_3(0) = \tilde{f}_3'(0) = 0$ and $\tilde{g}_0(t) \in C_0^2(0, \infty)$.

Now we prove Theorem 1.5 by contradiction. To this end, we assume that for large $X_0 > 0$ and some weak background transonic shock, there exists a small constant $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ and any smooth perturbations $(\tilde{\rho}_1^{\pm}(\tilde{r}), \tilde{U}_1^{\pm}(\tilde{r});\ \tilde{\rho}_2^{-}(t), \tilde{U}_2^{-}(t); \tilde{\rho}_2^{+}(t))$ with supports in some fixed interval and satisfying $\sum_{k=0}^{2}\Big(|\frac{d^k}{d\tilde{r}^k}\tilde{\rho}_1^{\pm}(\tilde{r})| + |\frac{d^k}{d\tilde{r}^k}\tilde{U}_1^{\pm}(\tilde{r})| + |\frac{d^k}{dt^k}\tilde{\rho}_2^{-}(t)| + |\frac{d^k}{dt^k}\tilde{U}_2^{-}(t)| + |\frac{d^k}{dt^k}\tilde{\rho}_2^{+}(t)|\Big) \le 1$, then there exists a uniform constant $C_0 > 0$ such that

$$\sum_{k=0}^{2}\Big(|\nabla_{t,\tilde{r}}^k(\rho^{\pm} - \hat{\rho}_0^{\pm})| + |\nabla_{t,\tilde{r}}^k(U^{\pm} - \hat{U}_0^{\pm})| + |\frac{d^k}{dt^k}(\tilde{r}(t) + r_0)| \le C_0\varepsilon \qquad \text{holds for all } t \ge 0. \tag{9.8}$$

For simplicity, we choose $(\tilde{\rho}_1^{+}(\tilde{r}), \tilde{U}_1^{+}(\tilde{r}), \tilde{\rho}_2^{+}(t)) \equiv 0$. In addition, one can choose the perturbed initial-boundary values $(\tilde{\rho}_1^{-}(\tilde{r}), \tilde{U}_1^{-}(\tilde{r}),\ \tilde{\rho}_2^{-}(t))$ of the supersonic coming flow such that $|\tilde{r}(\bar{t}_0) + r_0| \ge \bar{C}_0\varepsilon$ for some fixed positive constants $\bar{C}_0$ and $\bar{t}_0$.

In a similar way as in §8, there exists a fixed $T_0 > 0$ (independent of ε) such that for $t \ge T_0$

$$|w_i(t,r) - \hat{w}_{i,0}^{+}(r)| \le (1 + C\varepsilon)\tilde{B}_0 C_0\varepsilon + (A_0 + C\delta(X_0) + CC_0\varepsilon)C_0\varepsilon, \qquad i = 1, 2, \tag{9.9}$$

with a constant $C > 0$ independent of ε and X_0.

If $t \le T_0$, one has

$$|w_i(t,r) - \hat{w}_{i,0}^{+}(r)| \le C(C_0^2\varepsilon^2 + \delta(X_0)|\tilde{r}(t) + r_0|) \le C(C_0\varepsilon_0 + \delta(X_0))C_0\varepsilon. \tag{9.10}$$

It follows from (9.6), (9.9) and (9.10) that

$$\begin{cases} \frac{d}{dt}((\tilde{r}(t) + r_0)^2) \ge 2\bar{B}_0(\tilde{r}(t) + r_0)^2 - 2A_1(\tilde{B}_0 + A_0 + C\delta(X_0) + C\varepsilon)C_0^2\varepsilon^2, \\ (\tilde{r}(t) + r_0)^2|_{t=\bar{t}_0} \ge \bar{C}_0^2\varepsilon^2. \end{cases} \tag{9.11}$$

Hence,

$$(\tilde{r}(t) + r_0)^2 \ge e^{2\bar{B}_0(t-\bar{t}_0)}B\varepsilon^2 \tag{9.12}$$

with $B = \bar{C}_0^2 - \dfrac{A_1(\tilde{B}_0 + A_0 + C\delta(X_0) + C\varepsilon)C_0^2}{\bar{B}_0}$.

Next we claim that the constant $\frac{A_1(\tilde{B}_0 + A_0)}{\bar{B}_0}$ can be very small for the weak background transonic shock such that $B > \frac{\bar{C}_0^2}{2}$ holds.

Indeed, a direct computation yields

$$\frac{A_1(\tilde{B}_0 + A_0)}{\bar{B}_0} = \frac{2c(\rho_0^{+}(r_0))(c(\rho_0^{+}(r_0)) - U_0^{+}(r_0))}{(c(\rho_0^{+}(r_0)) + U_0^{+}(r_0))U_0^{+}(r_0)} + \frac{r_0(c(\rho_0^{+}(r_0)) - U_0^{+}(r_0))^3}{(U_0^{+}(r_0))^2(c(\rho_0^{+}(r_0)) + U_0^{+}(r_0))(U_0^{-} - U_0^{+})(r_0)}.$$

Denote by $\sigma = (U_0^{-} - U_0^{+})(r_0) > 0$. Then σ is small when the transonic shock is weak. Due to the Rankine-Hugoniot condition on $r = r_0$, one has

$$P(\rho_0^{+}(r_0)) = P(\rho_0^{-}(r_0)) + (\rho_0^{-}U_0^{-})(r_0)\sigma.$$

This gives

$$c(\rho_0^{+}(r_0)) - c(\rho_0^{-}(r_0)) = \frac{(\rho_0^{-}U_0^{-})(r_0)\int_0^1 c'(\theta\rho_0^{-}(r_0) + (1-\theta)\rho_0^{+}(r_0))d\theta}{\int_0^1 c^2(\theta\rho_0^{-}(r_0) + (1-\theta)\rho_0^{+}(r_0))d\theta}\sigma$$

and

$$\frac{c(\rho_0^+(r_0)) - U_0^+(r_0)}{(U_0^- - U_0^+)(r_0)} = \frac{c(\rho_0^-(r_0)) - U_0^-(r_0)}{\sigma} + 1 + \frac{c(\rho_0^+(r_0)) - c(\rho_0^-(r_0))}{\sigma}$$
$$= \frac{c(\rho_0^-(r_0)) - U_0^-(r_0)}{\sigma} + 1 + \frac{(\rho_0^- U_0^-)(r_0)\int_0^1 c'(\theta\rho_0^-(r_0) + (1-\theta)\rho_0^+(r_0))d\theta}{\int_0^1 c^2(\theta\rho_0^-(r_0) + (1-\theta)\rho_0^+(r_0))d\theta}. \tag{9.13}$$

Next we treat the term $\frac{c(\rho_0^-(r_0)) - U_0^-(r_0)}{\sigma}$ in (9.13).

Set $U_0^-(r_0) = c(\rho_0^-(r_0)) + \mu$ and $F(\mu,\sigma) \equiv P(\rho_0^+(r_0)) + \rho_0^+(r_0)(U_0^+(r_0))^2 - P(\rho_0^-(r_0)) - \rho_0^-(r_0)(U_0^-(r_0))^2 = P(\frac{\rho_0^-(r_0)(c(\rho_0^-(r_0)) + \mu)}{c(\rho_0^-(r_0)) + \mu - \sigma}) - P(\rho_0^-(r_0)) - \rho_0^-(r_0)\big(c(\rho_0^-(r_0)) + \mu\big)\sigma$, here $\mu > 0$ will be estimated. A direct computation yields

$$\begin{aligned} &\partial_\mu F(\mu,\sigma) < 0, \\ &F(0,\sigma) = \Big(\rho_0^-(r_0) + \frac{P''(\rho_0^-(r_0))}{2}\big(\frac{\rho_0^-(r_0)}{c(\rho_0^-(r_0))}\big)^2\Big)\sigma^2 + O(\sigma^3) > 0, \\ &F(\sqrt{\sigma},\sigma) = -2\rho_0^-(r_0)\sigma^{\frac{3}{2}} + O(\sigma^2) < 0. \end{aligned}$$

Therefore there exists a unique $\mu \in (0,\sqrt{\sigma})$ such that $F(\mu,\sigma) = 0$. Consequently, $\frac{(c(\rho_0^+(r_0)) - U_0^+(r_0))^2}{(U_0^- - U_0^+)(r_0)}$ is bounded and $\frac{A_1(\tilde{B}_0 + A_0)}{\tilde{B}_0} > 0$ is small enough if the transonic shock is sufficiently weak. This implies that $B > \frac{\bar{C}_0^2}{2}$ holds. Then it follows from (9.12) that $\lim_{t\to\infty}(\tilde{r}(t) + r_0)^2 = \infty$. Obviously, this is contradictory with (9.8), so we complete the proof of Theorem 1.7.

Appendix A.

In this appendix, the two nozzle walls Γ_1 and Γ_2 are assumed to be small perturbations of two straight line segments $x_2 = -1$ and $x_2 = 1$ with $-1 \le x_1 \le 1$. More precisely, the equations of Γ_1 and Γ_2 are given by

$$x_2 = f_1(x_1) \qquad \text{and} \qquad x_2 = f_2(x_1) \tag{A.1}$$

respectively with

$$|\frac{d^k}{dx_1^k}(f_1(x_1) + 1)| \le \varepsilon \qquad \text{and} \qquad |\frac{d^k}{dx_1^k}(f_2(x_1) - 1)| \le \varepsilon \qquad for \qquad -1 \le x_1 \le 1, k \le 4, \qquad k \in \mathbb{N}\cup\{0\}, \tag{A.2}$$

here $\varepsilon > 0$ suitably small.

Suppose that the supersonic coming flow $(\rho^-(x), u_1^-(x), u_2^-(x), S^-(x))$ in the nozzle satisfies

$$\left\{ \begin{array}{lll} (\rho^-(x), u_1^-(x), u_2^-(x)) \in C^2(\Omega), & \partial_2 u_1^-(x) \equiv \partial_1 u_2^-(x), & S^-(x) \equiv S_0, \\ |\nabla_x^\alpha(\rho^-(x) - \rho_0)| + |\nabla_x^\alpha(u_1^-(x) - q_0)| + |\nabla_x^\alpha u_2^-(x)| \le C\varepsilon, & |\alpha| \le 2, & \end{array}\right. \tag{A.3}$$

here $\Omega = \{(x_1,x_2) : -1 < x_1 < 1, f_1(x_1) < x_2 < f_2(x_1)\}$ and $q_0 > c(\rho_0, S_0)$. Namely, the assumption (A.3) implies that the incoming flow is close to the uniform supersonic flow $(\rho_0, q_0, 0, S_0)$.

Across the shock $\Sigma : x_1 = \xi(x_2)$, the flow field is denoted by $(P^+(x), u_1^+(x), u_2^+(x), S^+(x))$. Then we have the following proposition which yields Remark 1.5.

Proposition. *Under the assumptions (A.1)-(A.3), for small $\varepsilon > 0$, if the weak transonic shock solution $(P^+(x), u_1^+(x), u_2^+(x), S^+(x); \xi(x_2))$ has the following regularities and estimates* *(i). $\xi(x_2) \in C^2[x_2^1, x_2^2]$, here (x_1^i, x_2^i) with $x_2^i = f_i(x_1^i)(i = 1, 2)$ stands for the intersection point of $x_1 = \xi(x_2)$ with $x_2 = f_i(x_1)$. Moreover*

$$\|\xi(x_2)\|_{C^2[x_2^1, x_2^2]} \leq C\varepsilon.$$

(ii). Denote by $\Omega_+ = \{(x_1, x_2) : \xi(x_2) < x_1 < 1, f_1(x_1) < x_2 < f_2(x_1)\}$, then $(P^+(x), u_1^+(x), u_2^+(x), S^+(x)) \in C^1(\bar{\Omega}_+)$ satisfies

$$\|P^+(x) - P_0^+\|_{C^1(\bar{\Omega}_+)} + \|u_1^+(x) - q_0^+\|_{C^1(\bar{\Omega}_+)} + \|u_2^+(x)\|_{C^1(\bar{\Omega}_+)} + \|S^+(x) - S_0^+\|_{C^1(\bar{\Omega}_+)} \leq C\varepsilon,$$

here the constants $(P_0^+, q_0^+, 0, S_0^+)$ are determined by the following relations

$$\begin{cases} \rho_0 q_0 = \rho(P_0^+, S_0^+) q_0^+, \qquad \rho_0 q_0^2 + P_0 = \rho(P_0^+, S_0^+)(q_0^+)^2 + P_0^+, \\ (\rho_0 e_0 + \frac{1}{2}\rho_0 q_0^2 + P_0) q_0 = (\rho(P_0^+, S_0^+) e(P_0^+, S_0^+) + \frac{1}{2}\rho(P_0^+, S_0^+)(q_0^+)^2 + P_0^+) q_0^+; \\ P_0 < P_0^+ \quad \text{and} \quad q_0^+ < c(P_0^+, S_0^+). \end{cases} \tag{A.4}$$

Then $f_i''(x_1^i) = 0$ holds.

Remark A.1. *The transonic shock is assumed to be weak in the sense that although $q_0 > c(P_0, S_0)$ and $q_0^+ < c(P_0^+, S_0^+)$), $q_0 - c(P_0, S_0)$ and $c(P_0^+, S_0^+) - q_0^+$ are small.*

Remark A.2. *All the assumptions in Proposition can be realized in some cases, one can see [28] for more details.*

Proof. First we show that the shock curve Σ is perpendicular to the fixed boundaries Γ_1 and Γ_2, namely, $\xi'(x_2^i) = -f_i'(x_1^i)(i = 1, 2)$ holds.

It follows from (1.7) and (1.3) that

$$[\rho(P, S)u_1](x_1^i, x_2^i)(1 - \xi'(x_2^i) f_i'(x_1^i)) = 0.$$

Thus by the "smallness" assumption in Proposition we have

$$[\rho(P, S)u_1](x_1^i, x_2^i) = 0. \tag{A.5}$$

(A.5) together with the second equation in (1.3), yields

$$[P](x_1^i, x_2^i) = -(\rho(P^+, S^+) u_1^+ [u_1])(x_1^i, x_2^i)(1 - \xi'(x_2^i) f_i'(x_1^i)). \tag{A.6}$$

Additionally, (1.3) yields

$$\xi'(x_2^i)[P](x_1^i, x_2^i) = f_i'(x_1^i)(\rho(P^+, S^+) u_1^+ [u_1])(x_1^i, x_2^i)(1 - \xi'(x_2^i) f_i'(x_1^i)). \tag{A.7}$$

Noting that $[P](x_1^i, x_2^i) \neq 0$ and $[u_1](x_1^i, x_2^i) \neq 0$, then combining (A.6) with (A.7) yields

$$\xi'(x_2^i) = -f_i'(x_1^i). \tag{A.8}$$

Next, we derive $f_i''(x_1^i) = 0$.

By (A.5), (A.8) and the third equation in (1.3), one has

$$[\frac{1}{2}|u|^2 + h(\rho, S)](x_1^i, x_2^i)(\rho(P^+, S^+) u_1^+)(x_1^i, x_2^i)(1 + (f_i'(x_1^i))^2) = 0.$$

This implies

$$[\frac{1}{2}|u|^2 + h(\rho, S)](x_1^i, x_2^i) = 0. \tag{A.9}$$

Taking $\partial_\tau = \xi'(x_2)\partial_1 + \partial_2$ on two sides of the equations (1.3), and noting that (A.5), (A.8) and (A.9), then at the points (x_1^i, x_2^i) we have

$$\left\{\begin{array}{l}\partial_\tau[\rho(P,S)u_1](x_1^i,x_2^i) + f_i'(x_1^i)\partial_\tau[\rho(P,S)u_2](x_1^i,x_2^i) = 0,\\ (f_i'(x_1^i))^2\partial_\tau[P+\rho(P,S)u_2^2](x_1^i,x_2^i) + 2f_i'(x_1^i)\partial_\tau[\rho(P,S)u_1u_2](x_1^i,x_2^i)\\ \qquad\qquad +\partial_\tau[P+\rho(P,S)u_1)^2](x_1^i,x_2^i) = 0,\\ \partial_\tau[\big(\rho(P,S)e(P,S)+\frac{1}{2}\rho(P,S)|u|^2+P\big)u_1](x_1^i,x_2^i)\\ \qquad +f_i'(x_1^i)\partial_\tau[\big(\rho(P,S)e(P,S)+\frac{1}{2}\rho(P,S)|u|^2+P\big)u_2](x_1^i,x_2^i) = 0.\end{array}\right. \tag{A.10}$$

Thus it follows from a direct computation that at the point (x_1^i, x_2^i)

$$\left\{\begin{array}{l}\partial_\tau u_1^+ + f_i'(x_1^i)\partial_\tau u_2^+ = \dfrac{1}{\rho(P^+,S^+)}\Big\{\partial_\tau(\rho^-u_1^-) + f_i'(x_1^i)\partial_\tau(\rho^-u_2^-) - (u_1^+ + f_i'(x_1^i)u_2^+)\partial_\tau\rho(P^+,S^+)\Big\},\\ \partial_\tau u_1^+ + f_i'(x_1^i)\partial_\tau u_2^+ = \dfrac{1}{2\rho(P^+,S^+)(u_1^+ + f_i'(x_1^i)u_2^+)}\Big\{(f_i'(x_1^i))^2\partial_\tau(P^- + \rho^-(u_2^-)^2) + 2f_i'(x_1^i)\partial_\tau(\rho^-u_1^-u_2^-)\\ \qquad +\partial_\tau(P^- + \rho^-(u_1^-)^2) - \Big(\big(1+(f_i'(x_1^i))^2\big)\partial_\tau P^+ + \big((f_i'(x_1^i))^2(u_2^+)^2 + 2f_i'(x_1^i)u_1^+u_2^+ + (u_1^+)^2\big)\partial_\tau\rho^+\Big\},\\ \partial_\tau u_1^+ + f_i'(x_1^i)\partial_\tau u_2^+ = \dfrac{1}{u_1^+}\Big\{\partial_\tau(e^- + \frac{1}{2}|u^-|^2 + \frac{P^-}{\rho^-}) - \partial_\tau\big(e(P^+,S^+) + \dfrac{P^+}{\rho(P^+,S^+)}\big)\Big\}.\end{array}\right. \tag{A.11}$$

Since

$$u_2^-(x_1^i,x_2^i) = f_i'(x_1^i)u_1^-(x_1^i,x_2^i)), \qquad u_1^-\partial_\tau u_1^- + u_2^-\partial_\tau u_2^- + \frac{c^2(P^-,S_0)}{\rho^-}\partial_\tau\rho^- \equiv 0,$$
$$\partial_\tau(e^- + \frac{1}{2}|u^-|^2 + \frac{P^-}{\rho^-}) \equiv 0,$$

then

$$(\partial_\tau(\rho^-u_1^-) + f_i'(x_1^i)\partial_\tau(\rho^-u_2^-))(x_1^i,x_2^i) = \Big((1+(f_i'(x_1^i))^2)u_1^-(x_1^i,x_2^i) - (\frac{c^2(P^-,S_0)}{u_1^-})(x_1^i,x_2^i)\Big)\partial_\tau\rho^-(x_1^i,x_2^i)$$

and

$$\begin{aligned}&\Big((f_i'(x_1^i))^2\partial_\tau(P^- + \rho^-(u_2^-)^2) + 2f_i'(x_1^i)\partial_\tau(\rho^-u_1^-u_2^-) + \partial_\tau(P^- + \rho^-(u_1^-)^2)\Big)(x_1^i,x_2^i)\\ &= (1+(f_i'(x_1^i))^2)\Big((u_1^-)^2 + (u_2^-)^2 - c^2(P^-,S_0)\Big)(x_1^i,x_2^i)\partial_\tau\rho^-(x_1^i,x_2^i).\end{aligned}$$

Substituting the above computations and the equation of state for the polytropic gas into (A.11) yields at the point (x_1^i, x_2^i)

$$\left\{\begin{array}{l}\partial_\tau u_1^+ + f_i'(x_1^i)\partial_\tau u_2^+ = \dfrac{1}{\rho(P^+,S^+)}\Big\{\big((1+(f_i'(x_1^i))^2)u_1^- - \frac{c^2(P^-,S_0)}{u_1^-}\big)\partial_\tau\rho^- - (u_1^+ + f_i'(x_1^i)u_2^+)\big(\frac{\partial_\tau P^+}{c^2(P^+,S^+)}\\ \qquad\qquad\qquad +\partial_S\rho(P^+,S^+)\partial_\tau S^+\big)\Big\},\\ \partial_\tau u_1^+ + f_i'(x_1^i)\partial_\tau u_2^+ = \dfrac{1}{2\rho(P^+,S^+)(u_1^+ + f_i'(x_1^i)u_2^+)}\Big\{(1+(f_i'(x_1^i))^2)\big((u_1^-)^2 + (u_2^-)^2 - c^2(P^-,S_0)\big)\partial_\tau\rho^-\\ \qquad -(1+(f_i'(x_1^i))^2)\big(1 - \dfrac{(1+(f_i'(x_1^i))^2)(u_1^+)^2}{c^2(P^+,S^+)}\big)\partial_\tau P^+ - (u_1^+ + f_i'(x_1^i)u_2^+)^2\partial_S\rho(P^+,S^+)\partial_\tau S^+\Big\},\\ \partial_\tau u_1^+ + f_i'(x_1^i)\partial_\tau u_2^+ = -\dfrac{1}{\rho(P^+,S^+)u_1^+}\Big(\partial_\tau P^+ - \dfrac{c^2(P^+,S^+)}{\gamma-1}\partial_S\rho(P^+,S^+)\partial_\tau S^+\Big).\end{array}\right. \tag{A.12}$$

Furthermore, (A.12) can be simplified at the point (x_1^i, x_2^i) as follows

$$\begin{cases} \big((1+(f_i'(x_1^i))^2)(u_1^-)^2 - c^2(P^-,S_0)\big)\frac{\partial_\tau\rho^-}{u_1^-} - \big((1+(f_i'(x_1^i))^2)\frac{u_1^+}{c^2(P^+,S^+)} - \frac{1}{u_1^+}\big)\partial_\tau P^+ \\ \qquad -\bigg((1+(f_i'(x_1^i))^2)u_1^+ + \frac{c^2(P^+,S^+)}{(\gamma-1)u_1^+}\bigg)\partial_S\rho(P^+,S^+)\partial_\tau S^+ = 0, \\ (1+(f_i'(x_1^i))^2)\big((1+(f_i'(x_1^i))^2)(u_1^-)^2 - c^2(P^-,S_0)\big)\frac{\partial_\tau\rho^-}{u_1^+} + (1+(f_i'(x_1^i))^2)(\frac{(1+(f_i'(x_1^i))^2)u_1^+}{c^2(P^+,S^+)} + \frac{1}{u_1^+})\partial_\tau P^+ \\ \qquad -(1+(f_i'(x_1^i))^2)\bigg((1+(f_i'(x_1^i))^2)u_1^+ + \frac{2c^2(P^+,S^+)}{(\gamma-1)u_1^+}\bigg)\partial_S\rho(P^+,S^+)\partial_\tau S^+ = 0. \end{cases} \tag{A.13}$$

Thus it follows from (A.13) and the assumptions in Proposition that at the point (x_1^i, x_2^i)

$$\begin{aligned} &(q_0^2 - c^2(P_0,S_0))\bigg(1 - \frac{q_0^+}{q_0} + \frac{c^2(P_0^+,S_0^+)(q_0-2q_0^+)}{(\gamma-1)q_0(q_0^+)^2} + O(\varepsilon)\bigg)\partial_\tau\rho^- \\ &+\bigg(\frac{2(q_0^+)^2}{c^2(P_0^+,S_0^+)} + \frac{3}{\gamma-1} - \frac{c^2(P_0^+,S_0^+)}{(\gamma-1)(q_0^+)^2} + O(\varepsilon)\bigg)\partial_\tau P^+ = 0. \end{aligned} \tag{A.14}$$

In addition, $\partial_\tau\rho^-(x_1^i,x_2^i) = -\bigg(\dfrac{\rho^-(u_1^-)^2}{c^2(P^-,S_0)}\bigg)(x_1^i,x_2^i)f_i''(x_1^i)$ and $\partial_\tau P^+(x_1^i,x_2^i) = -\big(\rho^+(u_1^+)^2\big)(x_1^i,x_2^i)f_i''(x_1^i)$ hold, thus by use of (A.14) we obtain

$$(A_0 + O(\varepsilon))f_i''(x_1^i) = 0, \tag{A.15}$$

here $A_0 = \rho_0^+(q_0^+)^2\bigg(\frac{2(q_0^+)^2}{c^2(P_0^+,S_0^+)} + \frac{3}{\gamma-1} - \frac{c^2(P_0^+,S_0^+)}{(\gamma-1)(q_0^+)^2}\bigg) + (q_0^2 - c^2(P_0,S_0))\bigg(1 - \frac{q_0^+}{q_0} + \frac{c^2(P_0^+,S_0^+)(q_0-2q_0^+)}{(\gamma-1)q_0(q_0^+)^2}\bigg)\frac{\rho_0 q_0^2}{c^2(P_0,S_0)}$.

For the weak transonic shock solution, one can easily derive that $A_0 > 0$ holds. Then we have from (A.15) that

$$f_i''(x_1^i) = 0.$$

Hence, the Proposition is proved.

Remark A.3. *It follows from the proof above that the weak transonic shock assumption in Proposition can be removed as long as $A_0 \neq 0$ holds.*

Appendix B.

Now we give some explanations on the regularity assumption of the solution $(P^+(x), u_1^+(x), u_2^+(x), u_3^+(x), S^+(x); \xi(x_2,x_3))$ in Theorem 1.2, see Remark 1.2.

For the C^1 solution, the energy equation in (5.9) can be rewritten as

$$U_1^+\partial_r S^+ + \frac{1}{r\sin\alpha}S^+\partial_\theta U_3^+ - \frac{1}{r}U_3^+\partial_\alpha S^+ = 0. \tag{B.1}$$

Set $D = U_1^+\partial_r + \frac{1}{r\sin\alpha}U_2^+\partial_\theta - \frac{1}{r}U_3^+\partial_\alpha$. Then for any $C^2(\Omega^+)$ solution, one can derive from (B.1) and (5.9) that

$$\begin{aligned} &D^2\rho^+ + \rho^+\bigg(\partial_r DU_1^+ + \frac{1}{r\sin\alpha}\partial_\theta DU_2^+ - \frac{1}{r}\partial_\alpha DU_3^+\bigg) + \rho^+\bigg([D,\partial_r]U_1^+ + [D,\frac{1}{r\sin\alpha}\partial_\theta]U_2^+ - [D,\frac{\partial_\alpha}{r}]U_3^+\bigg) \\ &+ D\rho^+\bigg(\partial_r U_1^+ + \frac{1}{r\sin\alpha}\partial_\theta U_2^+ - \frac{1}{r}\partial_\alpha U_3^+\bigg) + D\bigg(\frac{2\rho^+U_1^+}{r} - \frac{\rho^+U_3^+}{r}ctg\alpha\bigg) = 0 \end{aligned} \tag{B.2}$$

with the standard notation of commutator $[A,B] = AB - BA$.

It follows from (B.1) and the equation of state that

$$D\rho^+ = \frac{DP^+}{c^2(P^+,S^+)}, \qquad D^2\rho^+ = \frac{D^2P^+}{c^2(P^+,S^+)} + DP^+D\Big(\frac{1}{c^2(P^+,S^+)}\Big). \tag{B.3}$$

Substituting the momentum equations in (5.9) and (B.3) into (B.2) yields

$$\begin{aligned}
&\Big(\frac{(U_1^+)^2}{c^2(P^+,S^+)} - 1\Big)\partial_r^2 P^+ + \frac{1}{r^2\sin^2\alpha}\Big(\frac{(U_2^+)^2}{c^2(P^+,S^+)} - 1\Big)\partial_\theta^2 P^+ + \frac{1}{r^2}\Big(\frac{(U_3^+)^2}{c^2(P^+,S^+)} - 1\Big)\partial_\alpha^2 P^+ +\\
&+ \frac{2U_1^+U_2^+}{r\sin\alpha c^2(P^+,S^+)}\partial_{r\theta}^2 P^+ - \frac{2U_2^+U_3^+}{r^2\sin^2\alpha c^2(P^+,S^+)}\partial_{\theta\alpha}^2 P^+ - \frac{2U_1^+U_3^+}{rc^2(P^+,S^+)}\partial_{r\alpha}^2 P^+\\
&+ F(r,\theta,\alpha,U^+,\nabla U^+,P^+,\nabla P^+,S^+,\nabla S^+) = 0
\end{aligned} \tag{B.4}$$

with

$$\begin{aligned}
&F(r,\theta,\alpha,U^+,\nabla U^+,P^+,\nabla P^+,S^+,\nabla S^+) = \rho^+\partial_r\Big(\frac{(U_2^+)^2 + (U_3^+)^2}{r}\Big) - \frac{\rho^+}{r\sin\alpha}\partial_\theta\Big(\frac{U_1^+U_2^+}{r} - \frac{U_2^+U_3^+}{r}ctg\alpha\Big)\\
&+ \frac{\rho^+}{r}\partial_\alpha\Big(\frac{U_1^+U_3^+}{r} + \frac{(U_2^+)^2}{r}ctg\alpha\Big) + \rho^+\Big(\partial_r DU_1^+ + \frac{1}{r\sin\alpha}\partial_\theta DU_2^+ - \frac{1}{r}\partial_\alpha DU_3^+\Big)\\
&+ \rho^+\Big([D,\partial_r]U_1^+ + [D,\frac{1}{r\sin\alpha}\partial_\theta]U_2^+ - [D,\frac{\partial_\alpha}{r}]U_3^+\Big) + D\rho^+\Big(\partial_r U_1^+ + \frac{1}{r\sin\alpha}\partial_\theta U_2^+ - \frac{1}{r}\partial_\alpha U_3^+\Big)\\
&+ D\Big(\frac{2\rho^+U_1^+}{r} - \frac{\rho^+U_3^+}{r}ctg\alpha\Big) + DP^+D(\frac{1}{c^2(P^+,S^+)}) - \rho(P^+,S^+)\Big(\partial_r P^+\partial_r(\frac{1}{\rho(P^+,S^+)})\\
&+ \frac{1}{r^2\sin^2\alpha}\partial_\theta P^+\partial_\theta(\frac{1}{\rho(P^+,S^+)}) + \frac{1}{r^2}\partial_\alpha P^+\partial_\alpha(\frac{1}{\rho(P^+,S^+)})\Big) + \frac{1}{c^2(P^+,S^+)}\Big(DU_1^+\partial_r P^+ + D(\frac{U_2^+}{r\sin\alpha})\partial_\theta P^+\\
&- D(\frac{U_3^+}{r})\partial_\alpha P^+\Big).
\end{aligned} \tag{B.5}$$

It follows from the boundary condition (5.10) and Lemma 6.1 that $U_2^+ = U_3^+ = 0$ on the intersection curve l, the shock surface $r = \tilde{r}(\theta,\alpha)$ is perpendicular to the fixed boundaries $\alpha = \alpha_0$, and the compatibility condition holds on l. In this case, the principal part of second order elliptic equation (B.4) on l is

$$-\Big(1 - \frac{(U_1^+)^2}{c^2(P^+,S^+)}\Big)\partial_r^2 P^+ - \frac{1}{r^2\sin^2\alpha}\partial_\theta^2 P^+ - \frac{1}{r^2}\partial_\alpha^2 P^+,$$

which can be transformed into the Laplacian $-\partial_{\bar{r}}^2 - \partial_{\bar{\theta}}^2 - \partial_{\bar{\alpha}}^2$ on l by a dilation as follows

$$\begin{cases}
\bar{r} = \dfrac{c(P^+(\tilde{r}_0),S^+(\tilde{r}_0))}{\sqrt{c^2(P^+(\tilde{r}_0),S^+(\tilde{r}_0)) - (U_1^+)^2(P^+(\tilde{r}_0),S^+(\tilde{r}_0))}}r,\\
\bar{\theta} = \tilde{r}_0\sin\alpha_0\theta,\\
\bar{\alpha} = \tilde{r}_0\alpha,
\end{cases}$$

here we have used the facts that the intersection curve l is represented by $r = \tilde{r}_0$ and $(P^+(x), U_1^+(x), U_2^+(x), U_3^+(x), S^+(x))$ depends only on $\tilde{r}_0$ on l in Lemma 6.1.

Thus by the compatibility condition on l in Lemma 6.1 and the results in [3], we can assert the validity of the assumption of $P^+(x) \in C^{2,\delta_0}(\bar{\Omega})$ in Remark 1.2. The $C^{1,\delta_0}(\bar{\Omega})-$regularity of $(u_1^+(x), u_2^+(x), u_3^+(x))$ and $C^{2,\delta_0}(\bar{\Omega})-$regularity of $S^+(x)$ near the intersection curve l can be obtained from (5.13), (5.18), (5.19) and (5.14).

Appendix C.

In this appendix, we sketch the proof of the existence in Theorem 1.3. Although the existence proof is very similar to that on Theorem 1.1, we still give a detailed proof for the reader's convenience. Without loss of generality, we consider only the 2-D isentropic flow in Theorem 1.3.

Proof of Theorem 1.3

On two hand sides of the shock $r = r_0$ $(X_0 \le r_0 \le X_0 + \frac{3}{4})$, the supersonic incoming flow $(\rho_0^-(r),\, U_0^-(r))$ and the subsonic flow $(\rho_0^+(r), U_0^+(r))$ satisfy respectively

$$\begin{cases} \frac{d}{dr}(r\rho_0^{\pm}U_0^{\pm}) = 0, \\ \frac{1}{2}(U_0^{\pm})^2 + h(\rho_0^{\pm}) = \frac{1}{2}(U_0^{\pm}(r_0))^2 + h(\rho_0^{\pm}(r_0)), \end{cases} \tag{C.1}$$

here $h(\rho_0^{\pm})$ is the enthalpy with $h'(\rho_0^{\pm}) = \dfrac{c^2(\rho_0^{\pm})}{\rho_0^{\pm}}$.

The corresponding Rankine-Hugoniot conditions across the shock $r = r_0$ are

$$\begin{cases} [\rho_0 U_0] = 0, \\ [\rho_0 U_0^2 + P_0] = 0. \end{cases} \tag{C.2}$$

As in the proof of Theorem 1.1, the proof can be divided into four steps.

Step 1. For the supersonic incoming flow $(\rho_0^-(r_0), U_0^-(r_0))$, it follows from (C.2) that there exists a unique subsonic flow $(\rho_0^+(r_0), U_0^+(r_0))$, see [11, 27].

Step 2. For any given supersonic state $(\rho_0^-(X_0 + \frac{3}{4}), U_0^-(X_0 + \frac{3}{4}))$, (C.1) has a unique supersonic solution $(\rho_0^-(r), U_0^-(r))$ for $r \in [X_0, X_0 + \frac{3}{4}]$ for large X_0.

In fact, it follows from (C.1) that

$$\begin{cases} f_1(\rho_0^-, U_0^-, r) \equiv r\rho_0^-(r)U_0^-(r) - C_0 = 0, \\ f_2(\rho_0^-, U_0^-, r) \equiv \frac{1}{2}(U_0^-(r))^2 + h(\rho_0^-(r)) - C_1^- = 0 \end{cases}$$

with $C_0 = (X_0 + \frac{3}{4})\rho_0^-(X_0 + \frac{3}{4})U_0^-(X_0 + \frac{3}{4})$ and $C_1^- = \frac{1}{2}(U_0^-(X_0 + \frac{3}{4}))^2 + h(\rho_0^-(X_0 + \frac{3}{4}))$.

Since

$$\begin{cases} \dfrac{d\rho_0^-}{dr} = -\dfrac{\rho_0^-(U_0^-)^2}{r^2((U_0^-)^2 - c^2(\rho_0^-))}, \\ \dfrac{dU_0^-}{dr} = \dfrac{U_0^- c^2(\rho_0^-)}{r^2((U_0^-)^2 - c^2(\rho_0^-))} \end{cases}$$

and

$$\frac{d((U_0^-)^2 - c^2(\rho_0^-))}{dr} = \frac{\big(2P'(\rho_0^-) + \rho_0^- P''(\rho_0^-)\big)(U_0^-)^2}{r^2((U_0^-)^2 - c^2(\rho_0^-))}$$

then for large X_0, one has

$$(U_0^-(r))^2 - c^2(\rho_0^-(r)) \ge \frac{1}{2}\Big((U_0^-(X_0 + \frac{3}{4}))^2 - c^2(\rho_0^-(X_0 + \frac{3}{4}))\Big) > 0 \qquad \text{for } X_0 \le r \le X_0 + \frac{3}{4}. \tag{C.3}$$

In addition $\dfrac{\partial(f_1, f_2)}{\partial(\rho_0^-, U_0^-)} = r\big((U_0^-(r))^2 - c^2(\rho_0^-(r))\big)$ and $\dfrac{\partial(f_1, f_2)}{\partial(\rho_0^-, U_0^-)}\Big|_{\rho_0^-(X_0+\frac{3}{4}), U_0^-(X_0+\frac{3}{4}), X_0+\frac{3}{4}} > 0$. This, together with the implicit function theorem and (C.3), yields that (C.1) has a unique supersonic solution $(\rho_0^-(r), U_0^-(r))$ for $r \in [X_0, X_0 + \frac{3}{4}]$.

Step 3. (C.1) has a unique subsonic solution $(\rho_0^+(r), U_0^+(r))$ for $r \in [X_0, X_0 + \frac{3}{4}]$ and large X_0.

Since the proof is very similar to that in Step 2, we omit it.

Step 4. The shock position r_0 is a continuously decreasing function of P_e when the end pressure P_e lies in an appropriate scope.

In fact, from (C.1) and (C.2) we arrive at for $r \in [X_0, X_0 + \frac{3}{4}]$

$$\begin{cases} r\rho_0^{\pm}(r)U_0^{\pm}(r) \equiv C_0, \\ \frac{1}{2}(U_0^{\pm}(r))^2 + h(\rho_0^{\pm}(r))) \equiv C_1^{\pm}, \end{cases} \tag{C.4}$$

here $C_1^{\pm}$ are the Bernoulli's constants. We note that C_1^- and C_1^+ may be different in general, moreover, C_1^+ depends on the end pressure $P_e = P_0^+(X_0)$.

Especially,

$$\begin{cases} X_0\rho_0^{\pm}(X_0)U_0^{\pm}(X_0) \equiv C_0, \\ \frac{1}{2}(U_0^{\pm}(X_0))^2 + h(\rho_0^{\pm}(X_0)) \equiv C_1^{\pm}. \end{cases} \tag{C.5}$$

Next we derive the dependence of r_0 on the end pressure $P_e = P_0^+(X_0)$.

It follows from the first equation in (C.4) and the second equation in (C.5) that

$$\begin{cases} \dfrac{d(\rho_0^{\pm}(r_0)U_0^{\pm}(r_0))}{d\rho_0^+(X_0)} = -\rho_0^{\pm}(r_0)U_0^{\pm}(r_0)\dfrac{dr_0}{r_0 d\rho_0^+(X_0)}, \\ U_0^+(r_0)\dfrac{dU_0^+(r_0)}{d\rho_0^+(X_0)} + \dfrac{c^2(\rho_0^+(r_0))}{d\rho_0^+(r_0)}\dfrac{d\rho_0^+(r_0)}{\rho_0^+(X_0)} = \dfrac{dC_1^+}{d\rho_0^+(X_0)}. \end{cases} \tag{C.6}$$

In addition, due to $C_1^+ = \frac{C_0^2}{2X_0^2(\rho_0^+(X_0))^2} + h(\rho_0^+(X_0))$, the second equation in (C.2) and (C.6), one gets

$$[\rho_0 U_0^2]\frac{dr_0}{r_0 d\rho_0^+(X_0)} = \rho_0^+(r_0)\frac{dC_1^+}{d\rho_0^+(X_0)} = \frac{\rho_0^+(r_0)(c_0^2(X_0) - (U_0^+(X_0))^2)}{\rho_0^+(X_0)}. \tag{C.7}$$

Since $[\rho_0 U_0^2] < 0$ holds by use of $[\rho_0 U_0^2 + P_0] = 0$ and $[P_0] > 0$, we conclude that r_0 is a continuous and strictly decreasing function of the end pressure $P_0^+(X_0)$.

Now, the existence result in Theorem 1.3 can be proved in the same as that for Theorem 1.1. Furthermore, the uniqueness in Theorem 1.3 can be shown in a similar way as in §3 and §4 (even much simpler). Thus, the proof of Theorem 1.3 is considered complete.

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