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THE CAUCHY PROBLEM FOR 1D COMPRESSIBLE FLOWS WITH DENSITY-DEPENDENT VISCOSITY COEFFICIENTS

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Abstract. This paper concerns with Cauchy problems for the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity coefficients. Two cases are considered here, first, the initial density is assumed to be integrable on the whole real line. Second, the deviation of the initial density from a positive constant density is integrable on the whole real line. It is proved that for both cases, weak solutions for the Cauchy problem exist globally in time and the large time asymptotic behavior of such weak solutions are studied. In particular, for the second case, the phenomena of vanishing of vacuum and blow-up of the solutions are presented, and it is also shown that after the vanishing of vacuum states, the globally weak solution becomes a unique strong one. The initial vacuum is permitted and the results apply to the one-dimensional Saint-Venant model for shallow water.

1. **Introduction.**

Consider the one-dimensional (1D) compressible Navier-Stokes equations with density-dependent viscosity coefficients

$$\rho_t + (\rho u)_x = 0, \tag{1.1}$$

$$(\rho u)_t + (\rho u^2 + P(\rho))_x = (\mu(\rho)u_x)_x. \tag{1.2}$$

Here $\rho(x,t), u(x,t)$ and $P(\rho) = \rho^\gamma (\gamma \geq 1)$ stand for the fluid density, velocity and pressure respectively. For simplicity, the viscosity coefficient $\mu(\rho)$ is assumed to be $\mu(\rho) = \rho^\alpha$ with $\alpha > \frac{1}{2}$. The initial data is imposed as

$$(\rho, \rho u)|_{t=0} = (\rho_0, m_0). \tag{1.3}$$

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When the viscosity $\mu(\rho)$ is a positive constant, there has been a lot of investigations on the compressible Navier-Stokes equations, for smooth initial data or discontinuous initial data, one-dimensional or multidimensional problems (see [15, 12, 10, 11, 19, 17, 7] and the references therein). However, the studies in [12, 25, 18] show that the compressible Navier-Stokes equations with constant viscosity coefficients behave singularly in the presence of vacuum. By some physical considerations, Liu, Xin and Yang in [18] introduced the modified compressible Navier-Stokes equations with density-dependent viscosity coefficients for isentropic fluids. As presented in [18], in deriving the compressible Navier-Stokes equations from the Boltzmann equations by the Chapman-Enskog expansions, the viscosity depends on the temperature, and correspondingly depends on the density for isentropic cases. Meanwhile, an one-dimensional viscous Saint-Venant system for shallow water, which is derived rigorously by Gerbeau-Perthame recently (see [8]), is expressed exactly as (1.1)-(1.2) with $\mu(\rho) = \rho$ and $\gamma = 2$.

There are many literatures on mathematical studies on (1.1)-(1.2). If the initial density is assumed to be connected to vacuum with discontinuities, Liu, Xin and Yang first obtained in [18] the local existence of weak solutions. The global existence of weak solutions was obtained later by [13], [14], [22], [26] respectively. If the initial density connects to vacuum continuously, new difficulty is encountered since no positive lower bound for the density is available. This case is studied by [6],[24],[26] and [27] respectively. Most of these results concern with free boundary problems. Recently, initial-boundary-value problems for one-dimensional equations (1.1)-(1.2) with $\mu(\rho) = \rho^\alpha(\alpha > 1/2)$ was studied by Li, Li and Xin recently in [16] and the phenomena of vacuum vanishing and blow-up of solutions were found there. The global existence of weak solutions for the initial-boundary-value problems for spherically symmetric compressible Navier-Stokes equations with density-dependent viscosity was proved by Guo, Jiu and Xin in [9]. It should be noted that the results in [16] and [9] are valid for the viscous Saint-Venant system for shallow water.

A new entropy estimate was found in [1]-[3], when the authors studied the L^1 stability of weak solutions for the Korteweg's system (with the Korteweg stress tensor $k\rho\nabla\triangle\rho$ on the right hand side of the momentum equations) and the viscous shallow water equations (with an quadratic friction term $r\rho|\mathbf{U}|\mathbf{U}$ on the left hand side of the momentum equations) respectively. Their results were later improved by Mellet and Vasseur [20] to the case of more general compressible Navier-Stokes equations. The new entropy estimate provides some high regularity for the density but needs some more restrictions on the viscosity (see [20] for more details). Meanwhile, although L^1 stability guarantee the compactness arguments on the approximate solutions and is considered as one of the main steps to prove existence of weak solutions, the global existence of weak solutions of the compressible Navier-Stokes equations with density-dependent viscosity is still open in the multi-dimensional cases. The key issue now is how to construct approximate solutions satisfying the a priori estimates required in the L^1 stability analysis. It seems highly non-trivial to do so due to the degeneracy of the viscosities near vacuum and the additional entropy inequality to be hold in the construction of approximate solutions. Further and recent studies in this direction are referred to [4], [5].

This paper is concerned with the global existence, asymptotic behavior, the vanishing of the vacuum and the blow-up phenomena of the weak solutions to the Cauchy problem of (1.1)-(1.2). Two cases will be considered here. The first case is that the initial density ρ_0 belongs to $L^1(R)$; the other is that there exists a positive constant $\bar\rho$ such that $\rho_0-\bar\rho\in L^1(R)$. We will construct a class of approximate solutions satisfying the required estimates in the L^1 stability of weak solutions in [20] and furthermore prove the global existence of weak solutions for Cauchy problem of (1.1)-(1.2). This is motivated by the approach of [14], in which one-dimensional free boundary problem is considered, [9] and [16] in which the initial-boundary-value problem is considered. Moreover, the asymptotic behaviors of the weak solutions are investigated in this paper. More precisely, if the initial density $\rho_0\in L^1(R)$, then we will prove that the density tends to 0 as $t\to\infty$. If there exists a positive constant $\bar\rho$ such that $\rho_0-\bar\rho\in L^1(R)$, then we will prove that the density tends to $\bar\rho$ as $t\to\infty$. As a consequence, there exists a time $T_0>0$ such that when $t>T_0$, the vacuum states vanish and any global weak solution become a unique strong one. These will generalize the corresponding results for initial-boundary value problem in [16]. It should be noted that the initial vacuum is permitted in our results. Very recently, if the initial density is bounded away from zero (no vacuum), Mellet and Vasseur proved the existence and uniqueness of global strong solution to (1.1)-(1.2) in [21] for $0<\alpha<1/2$.

The subsequent contents of the paper are organized as follows. In Section 2 we will present the main results of this paper. In Section 3 we will show various a priori estimates of the solutions. Based on these and using similar approaches in [9] and [16], we obtain the global existence of weak solutions. In Section 4, the asymptotic behaviors of weak solutions will be discussed. In Section 5, we will focus on the case $\bar\rho>0$ and present the results on the vanishing of vacuum states and blow-up phenomena of the solutions. It will be shown that after vanishing the vacuum states, the global weak solution becomes a unique strong one.

2. **Main results.** We start with the assumptions on the initial data. The initial data is assumed to satisfy

$$\begin{cases} \rho_0\ge 0; \quad m_0=0 \ \ a.e.\text{on } \{x\in R|\rho_0(x)=0\};\\ (\rho_0^{\alpha-\frac12})_x\in L^2(R)\cap L^1(R), \frac{|m_0|^2}{\rho_0}\in L^1(R), \frac{|m_0|^{2+\delta}}{\rho_0^{1+\delta}}\in L^1(R), \end{cases} \tag{2.1}$$

where $\alpha>\frac12$ and$0<\delta<1$ is permitted to be small. Moreover, we assume that there exists a constant $\bar\rho\ge 0$ such that

$$\rho_0-\bar\rho\in L^1(R)\cap L^\infty(R). \tag{2.2}$$

Before stating the main results, we give the definition of weak solutions to (1.1)-(1.2).

Definition 2.1. A pair (ρ,u) is said to be a weak solution to (1.1)-(1.2) provided that

(1) $\rho\ge 0$ a.e., and

$$\rho-\bar\rho\in L^\infty(0,T;L^1(R)\cap L^\gamma(R))\cap C([0,\infty);W^{1,\infty}(R)^*),$$
$$(\rho^{\alpha-\frac12})_x\in L^\infty(0,T;L^2(R)), \sqrt{\rho}u\in L^\infty(0,T;L^2(R)),$$

where $W^{1,\infty}(R)^*$ is the dual space of $W^{1,\infty}(R)$;

(2) For any $t_2\ge t_1\ge 0$ and any $\zeta\in C^1(R\times[t_1,t_2])$, the mass equation (1.1) holds in the following sense:

$$\int_R \rho\zeta dx|_{t_1}^{t_2}=\int_{t_1}^{t_2}\int_R(\rho\zeta_t+\rho u\cdot\zeta_x)dxdt; \tag{2.3}$$

(3) For any $\psi \in C_0^\infty(R\times[0,T))$, it holds that

$$\int_R m_0\psi(0,\cdot)dx + \int_0^T\int_R [\sqrt{\rho}(\sqrt{\rho}u)\psi_t + ((\sqrt{\rho}u)^2+\rho^\gamma)\psi_x]dxdt$$
$$+ < \rho^\alpha u_x, \psi_x >= 0, \tag{2.4}$$

where the diffusion term makes sense when written as

$$< \rho^\alpha u_x, \psi >= -\int_0^T\int_R \rho^{\alpha-\frac12}\sqrt{\rho}u\psi_x dxdt$$
$$-\frac{2\alpha}{2\alpha-1}\int_0^T\int_R (\rho^{\alpha-\frac12})_x\sqrt{\rho}u\psi dxdt. \tag{2.5}$$

We remark that in the definition of the weak solution, (2.5) implies $\rho^\alpha u_x \in L^2(0,T;W^{-1,1}(R))$, which follows from the fact that $(\rho^{\alpha-\frac12})_x \in L^\infty(0,T;L^2(R), \sqrt{\rho}u \in L^\infty(0,T;L^2(R))$.

Our main results read as

Theorem 2.1. *Let $\gamma > 1$. Suppose that (2.1) and (2.2) hold. If $\bar\rho = 0$, then the Cauchy problem (1.1)-(1.3) admits a global weak solution $(\rho(x,t), u(x,t))$ satisfying*

$$\rho \in C(R\times(0,T)), \tag{2.6}$$

$$\sup_{t\in[0,T]}\int_R \rho dx + \max_{(x,t)\in R\times[0,T]}\rho \le C, \tag{2.7}$$

$$\sup_{t\in[0,T]}\int_R (|\sqrt{\rho}u|^2 + (\rho^{\alpha-\frac12})_x^2 + \frac{1}{\gamma-1}\rho^\gamma)dx$$
$$+\int_0^T\int_R ([(\rho^{\frac{\gamma+\alpha-1}{2}})_x]^2 + \Lambda(x,t)^2)dxdt \le C, \tag{2.8}$$

where C is an absolute constant depending on the initial data and $\Lambda(x,t) \in L^2(R\times(0,T))$ is a function satisfying

$$\int_0^T\int_R \Lambda\varphi dxdxt = -\int_0^T\int_R \rho^{\alpha-\frac12}\sqrt{\rho}u\varphi_x dxdt$$
$$-\frac{2\alpha}{2\alpha-1}\int_0^T\int_R (\rho^{\alpha-\frac12})_x\sqrt{\rho}u\varphi dxdt. \tag{2.9}$$

Theorem 2.2. *Let $\gamma > 1$ and $\gamma \ge \alpha - \frac12$. Suppose that (2.2) and (2.1) hold. If $\bar\rho > 0$, then the Cauchy problem (1.1)-(1.3) admits a global weak solution $(\rho(x,t), u(x,t))$ satisfying*

$$\rho \in C(R\times(0,T)).,\tag{2.10}$$

$$\sup_{t\in[0,T]}\int_R |\rho-\bar\rho|dx + \max_{(x,t)\in R\times[0,T]}\rho \le C, \tag{2.11}$$

$$\sup_{t\in[0,T]}\int_R (|\sqrt{\rho}u|^2 + (\rho^{\alpha-\frac12})_x^2 + \frac{1}{\gamma-1}(\rho^\gamma - (\bar\rho)^\gamma - \gamma(\bar\rho)^{\gamma-1}(\rho-\bar\rho))dx$$
$$+\int_0^T\int_R ([(\rho^{\frac{\gamma+\alpha-1}{2}})_x]^2 + \Lambda(x,t)^2)dxdt \le C, \tag{2.12}$$

where C is an absolute constant depending on the initial data and $\Lambda(x,t) \in L^2(R\times(0,T))$ is same as in (2.9).

Theorem 2.3. *Suppose that $(\rho(x,t),u(x,t))$ is a weak solution of the Cauchy problem (1.1)-(1.3) satisfying (2.6),(2.7) and (2.8). Then we have*

$$\lim_{t\to\infty}\sup_{x\in R}\rho=0. \tag{2.13}$$

Theorem 2.4. *Suppose that $(\rho(x,t),u(x,t))$ is a weak solution of the Cauchy problem (1.1)-(1.3) satisfying (2.10),(2.11) and (2.12). Then we have*

$$\lim_{t\to\infty}\sup_{x\in R}|\rho-\bar\rho|=0. \tag{2.14}$$

Based on Theorem 2.4, it is easy to deduce that under assumption of $\bar\rho>0$, there exists a time $T_0>0$ after which the density has a positive lower bound and the vacuum states vanish. Moreover, it will be shown that after the time $t=T_0$, the weak solution becomes a unique strong one. Precisely, we have

Theorem 2.5. *Suppose that the assumptions of Theorem 2.2 hold. Let $(\rho(x,t),u(x,t))$ be a weak solution of the Cauchy problem (1.1)-(1.3) satisfying (2.10),(2.11) and (2.12). Then for any $0<\rho_1<\bar\rho$, there exists a time T_0 such that*

$$0<\rho_1\le\rho(x,t)\le C,\quad (x,t)\in R\times[T_0,\infty), \tag{2.15}$$

where C is a constant same as in (2.11). Moreover, for $t\ge T_0$, the weak solution becomes a unique strong solution to (1.1)-(1.3), satisfying

$$\begin{aligned}&\rho-\bar\rho\in L^\infty(T_0,t;H^1(R)),\quad \rho_t\in L^\infty(T_0,t;L^2(R)),\\ &u\in L^2(T_0,t;H^2(R)),\quad u_t\in L^2(T_0,t;L^2(R))\end{aligned}$$

and

$$\sup_{x\in R}|\rho-\bar\rho|+\|\rho-\bar\rho\|_{L^p(R)}+\|u\|_{L^2(R)}\to 0 \tag{2.16}$$

as $t\to\infty$, where $1<p\le\infty$.

In addition, similar to [16], we can obtain some result on the blow-up phenomena of the solutions when the vacuum states vanish, which will be presented in Section 5.

3. **Existence of weak solutions.** The key to the proof of Theorem 2.1-2.2 is to construct smooth approximate solutions satisfying the a priori estimates required in the L^1 stability analysis in [20]. The crucial issue is to obtain lower and upper bounds of the density, as mentioned in the introduction. To this end, we study the following system as an approximate system of (1.1)-(1.2).

$$\rho_t+(\rho u)_x=0, \tag{3.1}$$

$$(\rho u)_t+(\rho u^2+P(\rho))_x=(\mu_\varepsilon(\rho)u_x)_x \tag{3.2}$$

where $\mu_\varepsilon(\rho)=\mu(\rho)+\varepsilon\rho^\theta,\varepsilon>0,\theta\in(0,1/2)$.

For any fixed $M>0$, we will construct the smooth solution of (3.1)-(3.2) in the truncated region $\Omega^M=\{x\in R|-M<x<M\}$ with the following initial condition

$$(\rho,\rho u)(x,0)=(\rho_{0\varepsilon},m_{0\varepsilon}), \tag{3.3}$$

and boundary conditions

$$u(x,t)|_{x=\pm M}=0, \tag{3.4}$$

where the initial data $\rho_{0\varepsilon}, m_{0\varepsilon} \in C^\infty(\Omega^M)$ satisfy

$$\begin{cases} \rho_{0\varepsilon} \to \rho_0 \text{ in } L^1(\Omega^M), (\rho_{0\varepsilon}^{\alpha-1/2})_x \to (\rho_0^{\alpha-1/2})_x \text{ in } L^2(\Omega^M); \\ (m_{0\varepsilon})^2(\rho_{0\varepsilon})^{-1} \to m_0^2\rho_0^{-1}, (m_{0\varepsilon})^{2+\delta}(\rho_{0\varepsilon})^{-1-\delta} \\ \to (m_0)^{2+\delta}(\rho_0)^{-1-\delta} \text{ in } L^1(\Omega^M) \end{cases} \tag{3.5}$$

as $\varepsilon \to 0$, where $\delta > 0$ is same as in (2.1), and

$$\rho_{0\varepsilon} \geq C_0\varepsilon^{1/2\alpha-2\theta} \tag{3.6}$$

for some constant C_0 independent of ε. The initial data can be regularized in various ways (for more details, please see [9] and [16]).

To make a priori estimates on the solutions of the approximate system (3.1)-(3.2), we transform (3.1)-(3.2) into Lagrangian. Set

$$\xi = \int_{-M}^x \rho(t,y)dy, \quad \tau = t,$$

where $x \in [-M, M], t > 0$. Then (3.1)-(3.2) become

$$\rho_\tau + \rho^2 u_\xi = 0, \tag{3.7}$$

$$u_\tau + (P(\rho))_\xi = (\rho\mu_\varepsilon(\rho)u_\xi)_\xi, \tag{3.8}$$

where $\xi \in \Omega_L = (0, \int_{-M}^M \rho(t,y)dy)$.

Two cases will be considered respectively in the following.

3.1. Case I: $\bar\rho = 0, \rho_0 \in L^1(R)$.

Since $\int_{-M}^M \rho(t,y)dy$ is invariant along with time t, in this case the volume $|\Omega_L|$ is uniformly bounded. We denote $L = \int_{-M}^M \rho(t,y)dy$. The system (3.7)-(3.8) is implemented with the following initial condition

$$(\rho, \rho u)(\xi, 0) = (\rho_{0\varepsilon}, m_{0\varepsilon}), \tag{3.9}$$

and boundary condition

$$u|_{\xi=0} = 0, \quad u|_{\xi=L} = 0. \tag{3.10}$$

Then, we have

Lemma 3.1. *(Energy Estimates) Suppose that $(\rho_\varepsilon, u_\varepsilon)$ are smooth solutions of (3.7)-(3.10). Then*

$$\frac{1}{2}\frac{d}{d\tau}\int_{\Omega_L} u_\varepsilon^2 d\xi + \frac{1}{\gamma-1}\frac{d}{d\tau}\int_{\Omega_L}\rho_\varepsilon^{\gamma-1}d\xi + \int_{\Omega_L}(\rho\mu_\varepsilon(\rho))u_{\varepsilon\xi}^2 d\xi = 0. \tag{3.11}$$

Transform back into the Eulerian coordinates, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^M}\rho u_\varepsilon^2 dx + \frac{1}{\gamma-1}\frac{d}{dt}\int_{\Omega^M}\rho_\varepsilon^\gamma dx + \int_{\Omega^M}\mu_\varepsilon(\rho)u_{\varepsilon x}^2 dx = 0. \tag{3.12}$$

Lemma 3.2. *(Entropy Estimates) Suppose that $(\rho_\varepsilon, u_\varepsilon)$ are smooth solutions of (3.7)-(3.10) with $\rho_\varepsilon > 0$. Then*

$$\begin{aligned} &\int_{\Omega_L}[u_\varepsilon^2 + (\rho_\varepsilon^\alpha)_\xi^2 + \varepsilon^2(\rho_\varepsilon^\theta)_\xi^2 + \frac{1}{\gamma-1}\rho_\varepsilon^{\gamma-1}]d\xi + \int_0^t\int_{\Omega_L}\rho_\varepsilon\mu_\varepsilon(\rho_\varepsilon)(u_{\varepsilon\xi})^2 d\xi dt \\ &+ \int_0^t\int_{\Omega_L}(\rho_\varepsilon^{\gamma+\alpha-2} + \varepsilon\rho_\varepsilon^{\gamma+\theta-2})(\rho_{\varepsilon\xi})^2 d\xi dt \\ &\leq \int_{\Omega_L}[u_{0\varepsilon}^2 + (\rho_{0\varepsilon}^\alpha)_\xi^2 + \varepsilon^2(\rho_{0\varepsilon}^\theta)_\xi^2 + \frac{1}{\gamma-1}\rho_{0\varepsilon}^{\gamma-1}]d\xi. \end{aligned} \tag{3.13}$$

Transforming into Eulerian coordinates, we have

$$\int_{\Omega^M}(|\sqrt{\rho_\varepsilon}u_\varepsilon|^2+[\frac{\alpha}{2\alpha-1}(\rho_\varepsilon^{\alpha-\frac{1}{2}})_x]^2+\varepsilon^2[\frac{\theta}{2\theta-1}(\rho_\varepsilon^{\theta-\frac{1}{2}})_x]^2+\frac{1}{\gamma-1}\rho_\varepsilon^\gamma)dx$$
$$+\int_0^t\int_{\Omega^M}(\mu_\varepsilon(\rho_\varepsilon)|u_{\varepsilon x}|^2+(\rho_\varepsilon^{\gamma+\alpha-3}+\varepsilon\rho_\varepsilon^{\gamma+\theta-3})(\rho_{\varepsilon x})^2)dxdt$$
$$\le C\int_{\Omega^M}(|\sqrt{\rho_{0\varepsilon}}u_{0\varepsilon}|^2+|\frac{\mu(\rho_{0\varepsilon})_x}{\sqrt{\rho_{0\varepsilon}}}|^2+\frac{1}{\gamma-1}\rho_{0\varepsilon}^\gamma)dx. \tag{3.14}$$

Proof. It follows from (3.7) that

$$\frac{1}{\alpha}(\rho_\varepsilon^\alpha)_\tau+\rho_\varepsilon^{1+\alpha}u_{\varepsilon\xi}=0.$$

That is

$$\rho_\varepsilon^{1+\alpha}u_{\varepsilon\xi}=-\frac{1}{\alpha}(\rho_\varepsilon^\alpha)_\tau. \tag{3.15}$$

Similarly, one has

$$\rho_\varepsilon^{1+\theta}u_{\varepsilon\xi}=-\frac{1}{\theta}(\rho_\varepsilon^\theta)_\tau. \tag{3.16}$$

Denote $\gamma_\varepsilon(\rho_\varepsilon)=\frac{1}{\alpha}\rho_\varepsilon^\alpha+\frac{\varepsilon}{\theta}\rho_\varepsilon^\theta$. It follows from (3.15), (3.16) and (3.8) that

$$u_{\varepsilon\tau}+(P(\rho_\varepsilon))_\xi+(\gamma_\varepsilon(\rho_\varepsilon))_{\tau\xi}=0. \tag{3.17}$$

Multiplying (3.17) by $(\gamma_\varepsilon(\rho_\varepsilon))_\xi$ and integrating the resulting equations over Ω_L, we get

$$\int_{\Omega_L}u_{\varepsilon\tau}(\gamma_\varepsilon(\rho_\varepsilon))_\xi d\xi+\int_{\Omega_L}(P(\rho_\varepsilon))_\xi(\gamma_\varepsilon(\rho_\varepsilon))_\xi d\xi$$
$$+\int_{\Omega_L}(\gamma_\varepsilon(\rho_\varepsilon))_{\tau\xi}(\gamma_\varepsilon(\rho_\varepsilon))_\xi d\xi=0. \tag{3.18}$$

Noticing that $\rho_\varepsilon\gamma_\varepsilon'(\rho_\varepsilon)=\mu_\varepsilon(\rho_\varepsilon)$, one has

$$\int_{\Omega_L}u_{\varepsilon\tau}(\gamma_\varepsilon(\rho_\varepsilon))_\xi d\xi=(\int_{\Omega_L}u_\varepsilon(\gamma_\varepsilon(\rho_\varepsilon))_\xi d\xi)_\tau-\int_{\Omega_L}u_\varepsilon(\gamma_\varepsilon(\rho_\varepsilon))_{\tau\xi}d\xi$$
$$=(\int_{\Omega_L}u_\varepsilon(\gamma_\varepsilon(\rho_\varepsilon))_\xi d\xi)_\tau+\int_{\Omega_L}u_{\varepsilon\xi}(\gamma_\varepsilon(\rho_\varepsilon))_\tau d\xi$$
$$=(\int_{\Omega_L}u_\varepsilon(\gamma_\varepsilon(\rho_\varepsilon))_\xi d\xi)_\tau-\int_{\Omega_L}\gamma_\varepsilon'(\rho_\varepsilon)\rho_\varepsilon^2(u_{\varepsilon\xi})^2d\xi$$
$$=(\int_{\Omega_L}u_\varepsilon(\gamma_\varepsilon(\rho_\varepsilon))_\xi d\xi)_\tau-\int_{\Omega_L}\rho_\varepsilon\mu_\varepsilon(\rho_\varepsilon)(u_{\varepsilon\xi})^2d\xi. \tag{3.19}$$

Substituting (3.18) into (3.19) yields

$$\frac{d}{d\tau}\int_{\Omega_L}(\gamma_\varepsilon(\rho_\varepsilon))_\xi^2d\xi+2\frac{d}{d\tau}\int_{\Omega_L}u_\varepsilon(\gamma_\varepsilon(\rho_\varepsilon))_\xi d\xi$$
$$+2\int_{\Omega_L}P'(\rho_\varepsilon)\gamma_\varepsilon'(\rho_\varepsilon)(\rho_{\varepsilon\xi})^2d\xi=2\int_{\Omega_L}\rho_\varepsilon\mu_\varepsilon(\rho_\varepsilon)(u_{\varepsilon\xi})^2d\xi. \tag{3.20}$$

It follows from (3.11) and (3.20) that (multiplying 10 on both sides of (3.11) and multiplying 2 on both sides of (3.20) and then summing over the resulting equations)

$$\frac{d}{d\tau}[\int_{\Omega_L}(2u_\varepsilon+\gamma_\varepsilon(\rho_\varepsilon)_\xi)^2d\xi+\int_{\Omega_L}(u_\varepsilon^2+(\gamma_\varepsilon(\rho_\varepsilon)_\xi)^2)d\xi+\frac{10}{\gamma-1}\int_{\Omega_L}\rho_\varepsilon^{\gamma-1}d\xi]$$
$$+4\int_{\Omega_L}P'(\rho_\varepsilon)\gamma_\varepsilon'(\rho_\varepsilon)(\rho_{\varepsilon\xi})^2d\xi+6\int_{\Omega_L}\rho_\varepsilon\mu_\varepsilon(\rho_\varepsilon)(u_{\varepsilon\xi})^2d\xi=0. \tag{3.21}$$

Noting that

$$(\rho_\varepsilon^{2\alpha-2}+\varepsilon^2\rho_\varepsilon^{2\theta-2})(\rho_{\varepsilon\xi})^2\le(\gamma_\varepsilon(\rho_\varepsilon)_\xi)^2\le 2(\rho_\varepsilon^{2\alpha-2}+\varepsilon^2\rho_\varepsilon^{2\theta-2})(\rho_{\varepsilon\xi})^2, \tag{3.22}$$

thanks to (3.21), we have

$$\int_{\Omega_L}[u_\varepsilon^2+(\rho_\varepsilon^\alpha)_\xi^2+\varepsilon^2(\rho_\varepsilon^\theta)_\xi^2+\frac{1}{\gamma-1}\rho_\varepsilon^{\gamma-1}]d\xi+\int_0^t\int_{\Omega_L}\rho_\varepsilon\mu_\varepsilon(\rho_\varepsilon)(u_{\varepsilon\xi})^2d\xi dt$$
$$+\int_0^t\int_{\Omega_L}(\rho_\varepsilon^{\gamma+\alpha-2}+\varepsilon\rho_\varepsilon^{\gamma+\theta-2})(\rho_{\varepsilon\xi})^2d\xi dt$$
$$\le C\int_{\Omega_L}[u_{0\varepsilon}^2+(\rho_{0\varepsilon}^\alpha)_\xi^2+\varepsilon^2(\rho_{0\varepsilon}^\theta)_\xi^2+\frac{1}{\gamma-1}\rho_{0\varepsilon}^{\gamma-1}]d\xi.$$

(3.13) is proved. Transforming into Eulerian, we get (3.14). The proof of the lemma is finished. □

The following lemma is about the upper and lower bound of the density.

Lemma 3.3. *There exist an absolutely constant C and a positive constant $C(\varepsilon)$ depending on ε such that*

$$0<C(\varepsilon)\le\rho_\varepsilon\le C. \tag{3.23}$$

Proof. Note that

$$\int_{\Omega_L}\rho_\varepsilon^{-1}d\xi=|\Omega^M|=2M.$$

By continuity of ρ_ε, there exists a $\xi_0(t)\in\Omega_L$ such that

$$\rho_\varepsilon(\xi_0(t),t)=\frac{|\Omega_L|}{|\Omega^M|}=\frac{|\Omega_L|}{2M}.$$

Applying (3.13), one has

$$\rho_\varepsilon^\alpha(\xi,t)=\rho_\varepsilon^\alpha(\xi_0(t),t)+\int_{\xi_0(t)}^{\xi}(\rho_\varepsilon^\alpha)_\xi d\xi$$
$$\le(\frac{|\Omega_L|}{2M})^\alpha+CE_\varepsilon(0)+|\Omega_L|, \tag{3.24}$$

where $E_\varepsilon(0)$ is defined by $E_\varepsilon(0)=\int_{\Omega_L}[u_{0\varepsilon}^2+(\rho_{0\varepsilon}^\alpha)_\xi^2+\varepsilon^2(\rho_{0\varepsilon}^\theta)_\xi^2+\frac{1}{\gamma-1}\rho_{0\varepsilon}^{\gamma-1}]d\xi$. Therefore, the density is bounded by an absolute constant. To prove the lower bound of the density, we let $v_\varepsilon = \rho_\varepsilon^{-1}$. Then we have

$$\begin{aligned}
v_\varepsilon(\xi,t) &\le \int_{\Omega_L} v_\varepsilon d\xi + \int_{\Omega_L} (v_\varepsilon)^2 |\rho_{\varepsilon\xi}| d\xi \\
&\le |\Omega^M| + C\int_{\Omega_L} (v_\varepsilon)^{1+\theta} |(\rho_\varepsilon^\theta)_\xi| d\xi \\
&\le 2M + C\max_{\xi\in\Omega_L} v_\varepsilon^{\theta+\frac12} \|(\rho_\varepsilon^\theta)_\xi\|_{L^2(\Omega_L)} (\int_{\Omega_L} v_\varepsilon(\xi,t)d\xi)^{\frac12} \\
&\le 2M + \frac12 \max_{\xi\in\Omega_L} v_\varepsilon + C\varepsilon^{-\frac{1}{1-2\theta}} \|(\rho_\varepsilon^\theta)_\xi\|_{L^2(\Omega_L)}^{\frac{2}{1-2\theta}},
\end{aligned}$$

which implies

$$\max_{\xi\in\Omega_L} v_\varepsilon(\xi,t) \le \tilde{C}(\varepsilon), \tag{3.25}$$

where $\tilde{C}(\varepsilon)$ is some constant depending on ε. Hence there exists a constant $C(\varepsilon) > 0$ such that $\rho(\varepsilon) \ge C(\varepsilon)$. The proof of the lemma is finished. □

3.2. **Case II:** $\bar\rho > 0, \rho_0 - \bar\rho \in L^1(R)$**.** Multiplying $\frac{\gamma}{\gamma-1}(\rho^{\gamma-1} - \bar\rho^{\gamma-1})$ on both sides of (3.1), noting that

$$\frac{\gamma}{\gamma-1} u\rho((\rho^{\gamma-1} - \bar\rho^{\gamma-1}))_x = u(\rho^\gamma)_x, \tag{3.26}$$

we can rewrite (3.1) as

$$\begin{aligned}
&\frac{1}{\gamma-1}\frac{\partial}{\partial t}(\rho^\gamma - \bar\rho^\gamma - \gamma\bar\rho^{\gamma-1}(\rho-\bar\rho)) + \frac{\gamma}{\gamma-1}(u\rho(\rho^{\gamma-1} - \bar\rho^{\gamma-1}))_x \\
&= u(\rho^\gamma)_x.
\end{aligned} \tag{3.27}$$

Let

$$j_\gamma(f) = f^\gamma - \bar\rho^\gamma - \gamma\bar\rho^{\gamma-1}(f-\bar\rho). \tag{3.28}$$

The following lemma will be needed later, of which proof is referred to [17] and we omit it here.

Lemma 3.4. *Let $f = f(x), x \in R$ be a measurable function. Then $j_\gamma(f) \in L^1(R)$ if and only if $(f-\bar\rho)\mathbf{1}_{\{|f-\bar\rho|\le\delta\}} \in L^2(R)$ and $(f-\bar\rho)\mathbf{1}_{\{|f-\bar\rho|\ge\delta\}} \in L^\gamma(R)$ for any $\delta \in (0,\bar\rho)$, where $\mathbf{1}_{\{|f-\bar\rho|\le\delta\}}$ and $\mathbf{1}_{\{|f-\bar\rho|\ge\delta\}}$ are the characteristic functions of $\{|f-\bar\rho| \le \delta\}$ and $\{|f-\bar\rho| \ge \delta\}$ respectively.*

Applying Lemma 3.4 and the assumption (2.2), we obtain that $j_\gamma(\rho_0) \in L^1(R)$ and hence $j_\gamma(\rho_{0\varepsilon}) \in L^1(R)$.

The following energy estimate is usual.

Lemma 3.5. *(Energy Estimates) Suppose that $(\rho_\varepsilon, u_\varepsilon)$ are smooth solutions of (3.1)-(3.4). Then*

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_{\Omega^M} \rho u^2 dx + \frac{1}{\gamma-1}\frac{d}{dt}\int_{\Omega^M} (\rho_\varepsilon^\gamma - \bar\rho^\gamma - \gamma\bar\rho^{\gamma-1}(\rho_\varepsilon - \bar\rho))dx \\
&+ \int_{\Omega^M} \mu_\varepsilon(\rho_\varepsilon) u_{\varepsilon x}^2 dx = 0.
\end{aligned} \tag{3.29}$$

Proof. By (3.1), (3.2), it is easy to get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^M}\rho u^2 dx+\int_{\Omega^M}(\rho^\gamma)_x u dx=-\int_{\Omega^M}\mu_\varepsilon(\rho)u_x^2 dx. \tag{3.30}$$

Using (3.27), we get (3.29) and the proof of the lemma is finished. □

Based on Lemma 3.5, similar to Lemma 3.2, we have

Lemma 3.6. *(Entropy Estimates) Suppose that $(\rho_\varepsilon, u_\varepsilon)$ are smooth solutions of (3.1)-(3.4) with $\rho_\varepsilon > 0$. Then*

$$\begin{aligned}
&\int_{\Omega^M}(|\sqrt{\rho_\varepsilon}u_\varepsilon|^2+[\frac{\alpha}{2\alpha-1}(\rho_\varepsilon^{\alpha-\frac{1}{2}})_x]^2+\varepsilon^2[\frac{\theta}{2\theta-1}(\rho_\varepsilon^{\theta-\frac{1}{2}})_x]^2)dx\\
&+\int_{\Omega^M}\frac{1}{\gamma-1}(\rho_\varepsilon^\gamma-\bar\rho_\varepsilon^\gamma-\gamma\bar\rho^{\gamma-1}(\rho_\varepsilon-\bar\rho))dx\\
&+\int_0^t\int_{\Omega^M}(\mu_\varepsilon(\rho_\varepsilon)|u_{\varepsilon x}|^2+(\rho_\varepsilon^{\gamma+\alpha-3}+\varepsilon\rho_\varepsilon^{\gamma+\theta-3})(\rho_{\varepsilon x})^2)dxdt\\
&\le C\int_{\Omega^M}(|\sqrt{\rho_{0\varepsilon}}u_{0\varepsilon}|^2+|\frac{\mu_\varepsilon(\rho_{0\varepsilon})_x}{\sqrt{\rho_{0\varepsilon}}}|^2+\frac{1}{\gamma-1}(\rho_{0\varepsilon}^\gamma-\bar\rho^\gamma-\gamma\bar\rho^{\gamma-1}(\rho_{0\varepsilon}-\bar\rho))dx,
\end{aligned} \tag{3.31}$$

Proof. Similar to the proof of (3.2), we get (3.20). Transforming (3.20) into Eulerian, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega^M}\rho_\varepsilon^{-1}(\gamma_\varepsilon(\rho_\varepsilon))_x^2dx+2\frac{d}{dt}\int_{\Omega^M}u_\varepsilon(\gamma_\varepsilon(\rho_\varepsilon))_x dx\\
&+2\int_{\Omega^M}\rho_\varepsilon^{-1}P'(\rho_\varepsilon)\gamma_\varepsilon'(\rho_\varepsilon)(\rho_{\varepsilon x})^2dx=2\int_{\Omega^M}\mu(\rho_\varepsilon)(u_{\varepsilon x})^2 d\xi.
\end{aligned} \tag{3.32}$$

Multiplying 10 on both sides of (3.29) and multiplying 2 on both sides of (3.32) and then summing over the resulting equations, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega^M}(\rho_\varepsilon u_\varepsilon^2+\rho_\varepsilon^{-1}(\gamma_\varepsilon(\rho_\varepsilon)_x)^2)dx+\frac{10}{\gamma-1}\frac{dt}{dt}\int_{\Omega^M}(\rho_\varepsilon^\gamma-\bar\rho^\gamma-\gamma\bar\rho^{\gamma-1}(\rho_\varepsilon-\bar\rho))dx\\
&+\frac{d}{dt}\int_{\Omega^M}\rho_\varepsilon(2u_\varepsilon+\rho_\varepsilon^{-1}(\gamma_\varepsilon(\rho_\varepsilon))_x)^2dx\\
&+4\int_{\Omega^M}\rho_\varepsilon^{-1}P'(\rho_\varepsilon)\gamma_\varepsilon'(\rho_\varepsilon)(\rho_{\varepsilon x})^2dx+6\int_{\Omega^M}\mu_\varepsilon(\rho_\varepsilon)(u_{\varepsilon x})^2dx=0,
\end{aligned} \tag{3.33}$$

which implies (3.32). The proof of the lemma is finished. □

Now we can prove the upper and lower bound of the approximate solutions as follows.

Lemma 3.7. *Let $\gamma > 1$ and $\gamma \ge \alpha - \frac{1}{2}$. There exist an absolutely constant C and a positive constant $C(\varepsilon)$ depending on ε such that*

$$0 < C(\varepsilon) \le \rho_\varepsilon \le C. \tag{3.34}$$

Proof. The approximate solutions satisfy

$$(\rho_\varepsilon)_t+(\rho_\varepsilon u_\varepsilon)_x=0,\quad x\in[-M,M], \tag{3.35}$$

which can be rewritten as

$$(\rho_\varepsilon-\bar\rho)_t+((\rho_\varepsilon-\bar\rho)u_\varepsilon)_x+(\bar\rho u_\varepsilon)_x=0,\quad x\in[-M,M]. \tag{3.36}$$

The boundary conditions are imposed as

$$u_\varepsilon|_{x=\pm M} = 0.$$

Thus we have

$$\int_{\Omega^M} |\rho_\varepsilon - \bar\rho| dx \le \int_R |\rho_0 - \bar\rho| dx \le C. \tag{3.37}$$

Now we prove the upper bound of the density. By continuity of ρ_ε, there exist $x_1, \bar x_1 \in \Omega^M$ such that

$$\begin{aligned}\rho_\varepsilon(x_1,t) &= \frac{1}{|\Omega^M|}\int_{\Omega^M} \rho_\varepsilon dx \\ &= \frac{1}{|\Omega^M|}\int_{\Omega^M} \rho_{0\varepsilon}(x)dx = \rho_{0\varepsilon}(\bar x_1). \end{aligned}\tag{3.38}$$

For $\delta \in (0,\bar\rho)$, if $|\rho_\varepsilon - \bar\rho| \le \delta$, then $|\rho_\varepsilon| \le \bar\rho + \delta$. If $|\rho_\varepsilon - \bar\rho| \ge \delta$, then there exists a constant $C = C(\delta)$ such that

$$|\rho_\varepsilon^s - \bar\rho^s| \le C|\rho_\varepsilon - \bar\rho|^s, \quad s>0. \tag{3.39}$$

In fact, since

$$\frac{|\rho^s - \bar\rho^s|}{|\rho-\bar\rho|^s} \to 1, \quad \rho \to \infty,$$

$$\frac{|\rho^s - \bar\rho^s|}{|\rho-\bar\rho|^s} \to 1, \quad \rho \to 0,$$

we obtain that there exist $\bar\rho_1, \bar\rho_2$ satisfying $0 < \bar\rho_1 < \bar\rho_2 < \infty$ such that

$$|\rho_\varepsilon^s - \bar\rho^s| \le 2|\rho_\varepsilon - \bar\rho|^s, \quad \rho_\varepsilon \in [0,\bar\rho_1] \cup [\bar\rho_2, \infty). \tag{3.40}$$

When $\rho_\varepsilon \in [\bar\rho_1, \bar\rho_2]$, we have

$$|\rho_\varepsilon^s - \bar\rho^s| \le C|\rho_\varepsilon - \bar\rho|^s \tag{3.41}$$

for some constant C since $|\rho_\varepsilon - \bar\rho| \ge \delta$. (3.40) and (3.41) show that (3.39) holds true.

For $\beta > 0$ which is to be determined later, applying Hölder inequality and the entropy estimate (3.33), we have the following estimates:

$$\begin{aligned}&(\rho_\varepsilon^{\alpha-\frac12} - \bar\rho^{\alpha-\frac12})^{2\beta} = (\rho_\varepsilon^{\alpha-\frac12}(x_1,t) - \bar\rho^{\alpha-\frac12})^{2\beta} + \int_{x_1}^x ((\rho_\varepsilon^{\alpha-\frac12} - \bar\rho^{\alpha-\frac12})^{2\beta})_x dx \\ &= (\rho_\varepsilon^{\alpha-\frac12}(x_1,t) - \bar\rho^{\alpha-\frac12})^{2\beta} + 2\beta\int_{x_1}^x (\rho_\varepsilon^{\alpha-\frac12} - \bar\rho^{\alpha-\frac12})^{2\beta-1}(\rho_\varepsilon^{\alpha-\frac12})_x dx \\ &\le C + 2\beta\Big(\int_{x_1}^x |(\rho_\varepsilon^{\alpha-\frac12} - \bar\rho^{\alpha-\frac12})|^{2(2\beta-1)}dx\Big)^{\frac12}\Big(\int_{x_1}^x [(\rho_\varepsilon^{\alpha-\frac12})_x]^2 dx\Big)^{\frac12} \\ &\le (2\beta+1)C + \int_{x_1}^x |(\rho_\varepsilon^{\alpha-\frac12} - \bar\rho^{\alpha-\frac12})|^{2(2\beta-1)}[\mathbf{1}|_{\{|\rho_\varepsilon-\bar\rho|<\frac{\bar\rho}{2}\}} + \mathbf{1}|_{\{|\rho_\varepsilon-\bar\rho|\ge\frac{\bar\rho}{2}\}}]dx \\ &\equiv (2\beta+1)C + I_1(t) + I_2(t). \end{aligned}\tag{3.42}$$

Here $t \in [0,T]$ is any fixed time. Noting that when $|\rho_\varepsilon - \bar\rho| < \frac{\bar\rho}{2}$, that is, $\frac{\bar\rho}{2} < |\rho_\varepsilon| < \frac32\bar\rho$, one has

$$|\rho_\varepsilon^{\alpha-\frac12} - \bar\rho^{\alpha-\frac12}| \le C|\rho_\varepsilon - \bar\rho|,$$

where $C = C(\bar\rho)$. Hence,

$$I_1(t) \le C\int_{x_1}^{x} |\rho_\varepsilon - \bar\rho|^{2(2\beta-1)}\mathbf{1}|_{\{|\rho_\varepsilon-\bar\rho|<\frac{\bar\rho}{2}\}}dx. \tag{3.43}$$

When $|\rho_\varepsilon - \bar\rho| \ge \frac{\bar\rho}{2}$, due to (3.39), one has

$$|\rho_\varepsilon^{\alpha-\frac12} - \bar\rho^{\alpha-\frac12}| \le C|\rho_\varepsilon-\bar\rho|^{\alpha-\frac12},$$

where $C = C(\bar\rho)$. Hence,

$$I_2(t) \le C\int_{x_1}^{x} |\rho_\varepsilon - \bar\rho|^{2(2\beta-1)(\alpha-\frac12)}\mathbf{1}|_{\{|\rho_\varepsilon-\bar\rho|\ge\frac{\bar\rho}{2}\}}dx. \tag{3.44}$$

Choose β such that $2(2\beta-1)=1$, i.e. $\beta=\frac34$. Then

$$I_1(t) \le C\int_{\Omega^M} |\rho_\varepsilon-\bar\rho|dx \le C, \tag{3.45}$$

and

$$I_2(t) \le C\int_{x_1}^{x} |\rho_\varepsilon - \bar\rho|^{(\alpha-\frac12)}\mathbf{1}|_{\{|\rho_\varepsilon-\bar\rho|\ge\frac{\bar\rho}{2}\}}dx. \tag{3.46}$$

Now we estimate I_2. Two cases are considered respectively in the following.

$I.0 < \alpha - \frac12 \le 1.$

Noting that $y^{\alpha-\frac12} \le Cy$ if $y \ge \frac{\bar\rho}{2}$. Thus

$$I_2(t) \le C\int_{\Omega^M} |\rho_\varepsilon-\bar\rho|dx \le C. \tag{3.47}$$

$II.1 < \alpha - \frac12 \le \gamma$

In this case, using the interpolation inequality, one has

$$I_2(t) \le C\Big(\int_{\Omega^M} |\rho_\varepsilon-\bar\rho|dx\Big)^{(\alpha-\frac12)\theta}\Big(\int_{\Omega^M} |\rho_\varepsilon-\bar\rho|^{\gamma}\mathbf{1}|_{\{|\rho_\varepsilon-\bar\rho|\ge\frac{\bar\rho}{2}\}}dx\Big)^{(\alpha-\frac12)(1-\theta)/\gamma} \le C, \tag{3.48}$$

where

$$\frac{1}{\alpha-\frac12} = \theta + \frac{1-\theta}{\gamma}.$$

In the last inequality of (3.48), Lemma 3.4 has been used. It should be noted that in this case we need that $\gamma \ge \alpha - \frac12$.

It follows from (3.45), (3.47), (3.48) that

$$(\rho_\varepsilon^{\alpha-\frac12} - \bar\rho^{\alpha-\frac12})^{2\beta} \le C, \tag{3.49}$$

which implies that

$$|\rho_\varepsilon| \le C. \tag{3.50}$$

The upper bound of the approximate solutions is proved. The proof of lower bound estimates is similar to Lemma 3.3 and we omit it here. □

Based on a priori estimates of Lemma 3.1-Lemma 3.7, applying similar approaches in [9], [16],[20] and the references therein, we can obtain that for any $T>0$ there exists a unique global smooth solutions of (3.1)-(3.4) satisfying

$$\rho_\varepsilon, \rho_{\varepsilon x}, \rho_{\varepsilon t}, u_\varepsilon, u_{\varepsilon x}, u_{\varepsilon t}, u_{\varepsilon xx} \in C^{\beta,\frac{\beta}{2}}([-M,M]\times[0,T])$$

for some $0<\beta<1$, and $\rho_\varepsilon \ge C(\varepsilon) > 0$ on $[-M,M]\times[0,T]$. Moreover, the estimates of Lemma 3.1-Lemma 3.3 and Lemma 3.5-Lemma 3.7 hold true for $\{\rho_\varepsilon, u_\varepsilon\}$.

We are ready to give sketch of proof of Theorem 2.1 and Theorem 2.2.

Sketch of proof of Theorem 2.1. For any fixed $M > 0$, completely similar to [9], [16], we can obtain that (up to a subsequence)

$$\rho_\varepsilon \to \rho \quad \text{in} \ \ C([0,T]\times[-M,M]), \tag{3.51}$$

$$(\rho_\varepsilon^{\alpha-\frac{1}{2}})_x \rightharpoonup (\rho^{\alpha-\frac{1}{2}})_x \ \ \text{weakly in} \ \ L^2((0,T)\times[-M,M]), \tag{3.52}$$

$$\rho_\varepsilon^\alpha u_{\varepsilon x} \rightharpoonup \Lambda \ \ \text{weakly in} \ \ L^2((0,T)\times[-M,M]), \tag{3.53}$$

as $\varepsilon \to 0$, for some functions $\rho \in C([0,T]\times[-M,M])$ and $\Lambda \in L^2((0,T)\times[-M,M])$ which satisfies

$$\int_0^T\int_{-M}^M \Lambda\varphi dxdxt = -\int_0^T\int_{-M}^M \rho^{\alpha-\frac{1}{2}}\sqrt{\rho}u\varphi_x dxdt$$
$$-\frac{2\alpha}{2\alpha-1}\int_0^T\int_{-M}^M (\rho^{\alpha-\frac{1}{2}})_x\sqrt{\rho}u\varphi dxdt. \tag{3.54}$$

To get the convergence of the term $\sqrt{\rho_\varepsilon}u_\varepsilon$, we apply similar approaches in [9], [16], [20]. More precisely, we have $\rho_\varepsilon u_\varepsilon$ converges strongly in $L^1((0,T)\times[-M,M])$ and $L^2(0,T;L^{1+\zeta}(-M,M))$ and almost everywhere to some function $m(x,t)$, where $\zeta > 0$ is some small positive number. Also, we can prove that $\sqrt{\rho_\varepsilon}u_\varepsilon$ converges strongly in $L^2((0,T)\times[-M,M])$ to $\frac{m}{\sqrt{\rho}}$ which is defined to be zero when $m = 0$ and there exists a function $u(x,t)$ such that $m(x,t) = \rho(x,t)u(x,t)$. Moreover, we have

$$\rho \in C([-M,M]\times(0,T)), \tag{3.55}$$

$$\sup_{t\in[0,T]}\int_{-M}^M \rho dx + \max_{(x,t)\in[-M,M]\times[0,T]} \rho \le C, \tag{3.56}$$

$$\sup_{t\in[0,T]}\int_{-M}^M (|\sqrt{\rho}u|^2 + (\rho^{\alpha-\frac{1}{2}})_x^2 + \frac{1}{\gamma-1}\rho^\gamma)dx$$
$$+\int_0^T\int_{-M}^M ([(\rho^{\frac{\gamma+\alpha-1}{2}})_x]^2 + \Lambda(x,t)^2)dxdt \le C, \tag{3.57}$$

where C is an absolute constant depending on the initial data.

Using a diagonal procedure, we obtain that the above converges (up to a subsequence) remain true for any $M > 0$ and the existence of weak solutions of (1.1)-(1.3) can be directly proved. Moreover, (2.6)-(2.9) hold true due to (3.55)-(3.57). The proof of Theorem 2.1 is finished. □

The proof of Theorem 2.2 is completely similar and we omit it here.

4. **Asymptotic behavior of weak solutions.** In this section, we will study the asymptotic behavior of the weak solutions. We assume that the solutions are smooth enough. The rigorous proof can be obtained by using the usual regularization procedure. It suffices to prove Theorem 2.4, since Theorem 2.3 can be proved in a completely similar way if we set $\bar\rho = 0$ in the following proof.

Proof of Theorem 2.4. For any $s \ge 1$, since $|\rho| \le C$, we have

$$|\rho^s - \bar\rho^s| \le C|\rho - \bar\rho|.$$

Hence

$$\int_R |\rho^s - \bar\rho^s|dx \le C\int_R |\rho-\bar\rho|dx \le C. \tag{4.1}$$

Similarly, we have

$$\int_R |\rho^s - \bar{\rho}^s|^\lambda dx \le C \int_R |\rho - \bar{\rho}|^\lambda dx \le C \tag{4.2}$$

for any $\lambda \ge 1$. Moreover, one has

$$\begin{aligned}&\int_R |[(\rho^s - \bar{\rho}^s)^\lambda]_x| dx = \lambda s \int_R |(\rho^s - \bar{\rho}^s)^{\lambda-1} \rho^{s-1} \rho_x| dx \\ &\le \frac{\lambda s(2\alpha - 1)}{2} (\int_R (\rho^s - \bar{\rho}^s)^{2(\lambda-1)} \rho^{2s+1-2\alpha} dx)^{\frac{1}{2}} (\int_R [(\rho^{\alpha-\frac{1}{2}})_x]^2 dx)^{\frac{1}{2}} \le C.\end{aligned}$$

Combining the fact that $(\rho^s - \bar{\rho}^s)^\lambda \in L^1(R)$ due to (4.2), we have, for any fixed t, that

$$\rho^s - \bar{\rho}^s \to 0 \tag{4.3}$$

as $|x| \to \infty$. By (2.7) (see also (3.14)), it holds that

$$\int_0^t \int_R [(\rho^{\frac{\gamma+\alpha-1}{2}})_x]^2 dx dt \le C, \tag{4.4}$$

where C is an absolute constant depending only on the initial data. Denote $b = \frac{\gamma+\alpha-1}{2}$. Then

$$\int_0^t \int_R [(\rho^b)_x]^2 dx dt \le C. \tag{4.5}$$

Choosing $s > b + 1$, one has

$$\begin{aligned}(\rho^s - \bar{\rho}^s)^2 &= \int_{-\infty}^x [(\rho^s - \bar{\rho}^s)^2]_x dx = 2 \int_{-\infty}^x (\rho^s - \bar{\rho}^s)(\rho^s)_x dx \\ &= 2s \int_{-\infty}^x (\rho^s - \bar{\rho}^s) \rho^{s-1} \rho_x dx = \frac{2s}{b} \int_{-\infty}^x (\rho^s - \bar{\rho}^s)(\rho^b)_x \rho^{s-b} dx \\ &\le C \|\rho^s - \bar{\rho}^s\|_{L^2(R)} \|(\rho^b)_x\|_{L^2(R)}.\end{aligned}$$

Consequently,

$$\int_0^t \sup_{x \in R} (\rho^s - \bar{\rho}^s)^4 dt \le C \sup_t \|\rho^s - \bar{\rho}^s\|_{L^2(R)}^2 \int_0^t \|(\rho^b)_x\|_{L^2(R)}^2 dt \le C. \tag{4.6}$$

Moreover, applying (4.2), one has

$$\begin{aligned}&\int_0^t \int_R (\rho^s - \bar{\rho}^s)^4 (\rho^s - \bar{\rho}^s)^{2l} dx dt \\ &\le \int_0^t [\sup_{x \in R} (\rho^s - \bar{\rho}^s)^4 \int_R (\rho^s - \bar{\rho}^s)^{2l} dx] dt \\ &\le \sup_t \int_R (\rho^s - \bar{\rho}^s)^{2l} dx \int_0^t \sup_{x \in R} (\rho^s - \bar{\rho}^s)^4 dt \le C,\end{aligned} \tag{4.7}$$

where $l \ge 1$ is any real number. Hence

$$\int_0^t \int_R (\rho^s - \bar{\rho}^s)^{4+2l} dx dt \le C. \tag{4.8}$$

Denote $f(t) = \int_R (\rho^s - \bar{\rho}^s)^{4+2l} dx$. Then $f \in L^1(0,\infty) \cap L^\infty(0,\infty)$. Furthermore, direct calculations show that

$$\begin{aligned}
&\frac{d}{dt} f(t) = (4+2l) \int_R (\rho^s - \bar{\rho}^s)^{3+2l} s\rho^{s-1} \rho_t dx \\
&= -(4+2l)s \int_R (\rho^s - \bar{\rho}^s)^{3+2l} \rho^{s-1} (\rho u)_x dx \\
&= (4+2l)(3+2l)s \int_R (\rho^s - \bar{\rho}^s)^{2+2l} (\rho^s)_x \rho^{s-1} \rho u dx \\
&+(4+2l)s \int_R (\rho^s - \bar{\rho}^s)^{3+2l} (s-1)\rho^{s-2} \rho_x \rho u dx \\
&= \frac{(4+2l)s^2(3+2l)}{b} \int_R (\rho^s - \bar{\rho}^s)^{2+2l} (\rho^b)_x \rho^{2s-b-1} u dx \\
&+\frac{(4+2l)s(s-1)}{b} \int_R (\rho^s - \bar{\rho}^s)^{3+2l} \rho^{s-b} (\rho^b)_x u dx \\
&\le C\|\sqrt{\rho} u\|_{L^2(R)} \|(\rho^b)_x\|_{L^2(R)}.
\end{aligned} \tag{4.9}$$

Hence

$$\frac{d}{dt} f \in L^2(0,\infty). \tag{4.10}$$

Combining the obtained fact that $f \in L^1(0,\infty) \cap L^\infty(0,\infty)$, one has

$$f(t) \to 0, \quad t \to \infty. \tag{4.11}$$

Letting $m \ge 1$ be any real number to be determined later, we have

$$\begin{aligned}
&|(\rho^s - \bar{\rho}^s)^m| = |\int_{-\infty}^x [(\rho^s - \bar{\rho}^s)^m]_x dx| = |m \int_{-\infty}^x (\rho^s - \bar{\rho}^s)^{m-1} (\rho^s)_x dx| \\
&= \frac{2ms}{2\alpha - 1} |\int_{-\infty}^x (\rho^s - \bar{\rho}^s)^{m-1} (\rho^{\alpha - \frac{1}{2}})_x \rho^{s-\alpha-\frac{1}{2}} dx| \\
&\le C\|(\rho^{\alpha-\frac{1}{2}})_x\|_{L^2(R)} |\int_{-\infty}^x (\rho^s - \bar{\rho}^s)^{2(m-1)} dx|^{\frac{1}{2}}.
\end{aligned} \tag{4.12}$$

Choose $2(m-1) = 4+2l$ to get

$$\sup_{x \in R} |(\rho^s - \bar{\rho}^s)^m| \le C f^{\frac{1}{2}}(t) \to 0 \tag{4.13}$$

as $t \to \infty$. Therefore, $\lim_{t\to\infty} \sup_x |\rho^s - \bar{\rho}^s| = 0$. Now we prove that $\lim_{t\to\infty} \sup_x |\rho - \bar{\rho}| = 0$. Using the fact that

$$\frac{|\rho^s - \bar{\rho}^s|}{|\rho - \bar{\rho}|^s} \to 1, \quad \rho \to 0,$$

we have that there exists a $\delta > 0$ such that

$$|\rho - \bar{\rho}|^s \mathbf{1}_{\{|\rho| \le \delta\}} \le 2|\rho^s - \bar{\rho}^s| \mathbf{1}_{\{|\rho| \le \delta\}}. \tag{4.14}$$

Moreover, when $|\rho| \ge \delta$, we have

$$|\rho - \bar{\rho}| \mathbf{1}_{\{|\rho| \ge \delta\}} \le sC(\delta, \bar{\rho}) |\rho^s - \bar{\rho}^s| \mathbf{1}_{\{|\rho| \ge \delta\}}. \tag{4.15}$$

It follows from (4.14) and(4.15) that

$$\begin{aligned}
&|\rho - \bar{\rho}|^s = |\rho - \bar{\rho}|^s \mathbf{1}_{\{|\rho| \le \delta\}} + |\rho - \bar{\rho}|^s \mathbf{1}_{\{|\rho| \ge \delta\}} \\
&\le 2|\rho^s - \bar{\rho}^s| \mathbf{1}_{\{|\rho| \le \delta\}} + s^s [C(\delta, \bar{\rho})]^s |\rho^s - \bar{\rho}^s|^s \mathbf{1}_{\{|\rho| \ge \delta\}}.
\end{aligned}$$

Therefore, we have

$$\sup_{x\in R}|\rho-\bar\rho|^s \le 2\sup_{x\in R}|\rho^s-\bar\rho^s|\mathbf{1}_{\{|\rho|\le\delta\}} + C(s,\delta,\bar\rho)\sup_{x\in R}|\rho^s-\bar\rho^s|^s\mathbf{1}_{\{|\rho|\ge\delta\}} \to 0,$$

as $t\to\infty$, which implies that $\lim_{t\to\infty}\sup_x|\rho-\bar\rho| = 0$. The proof of the lemma is finished. ☐

5. **Vanishing of vacuum states and blow-up phenomena.** In this subsection, we focus on the case $\bar\rho > 0$. We first give a sketch of proof of Theorem 2.5 and then give some remarks on the blow-up phenomena of the solutions when the vacuum states vanish. These results are similar to those in [16] in which the initial-boundary value problem and periodic problem are studied.

Sketch of proof of Theorem 2.5. From Theorem 2.2, it is easy to deduce that for any $0<\rho_1<\bar\rho$, there exists a time $T_0>0$ such that

$$0<\rho_1\le\rho(x,t)\le C, \quad (x,t)\in R\times[T_0,\infty). \tag{5.1}$$

Therefore, for $t\ge T_0$, the density is bounded away from the zero and the vacuum states vanish. Using estimates of (2.12) and standard linear parabolic theory, we can obtain that for $t\ge T_0$, the weak solution becomes a unique strong solution to (1.1)-(1.3), satisfying

$$\begin{cases} \rho-\bar\rho\in L^\infty(T_0,t;H^1(R)), \quad \rho_t\in L^\infty(T_0,t;L^2(R)),\\ u\in L^2(T_0,t;H^2(R)), \quad u_t\in L^2(T_0,t;L^2(R)). \end{cases} \tag{5.2}$$

The detail of the proof is referred to [16] and we omit it here. Furthermore, the asymptotic behaviors $\lim_{t\to\infty}\sup_{x\in R}|\rho-\bar\rho| = 0$ and $\lim_{t\to\infty}\|\rho-\bar\rho\|_{L^p} = 0$ for $1<p\le\infty$ follow directly from (2.14) and the estimate $\|\rho-\bar\rho\|_{L^1}\le C$. The asymptotic behavior on the velocity $\lim_{t\to\infty}\|u\|_{L^2}=0$ follows from the standard arguments, see [23] for instance. ☐

It should be remarked that we also have finite blow-up phenomena for the weak solutions of the Cauchy problem (1.1)-(1.3) at the time when the vacuum states vanish if the density contains vacuum states at least at one point. These are similar as in [16] in which the 1D initial-boundary value problem and periodic problem are investigated. To be more precise, we note that, if the density contains vacuum states at least at one point, due to the facts that $\rho\in C(R\times[0,T])$ for any $T>0$ and $\lim_{t\to\infty}\sup_{x\in R}|\rho-\bar\rho| = 0$, there exists some critical time $T_1\in[0,T_0)$ with $T_0>0$ given by (5.1) and a nonempty subset $\Omega^0\subset R$ such that

$$\begin{cases} \rho(x,T_1)=0, \ \forall x\in\Omega^0,\\ \rho(x,T_1)>0, \ \forall x\in R\backslash\Omega^0,\\ \rho(x,t)>0, \ \forall (x,t)\in R\times(T_1,T_0]. \end{cases} \tag{5.3}$$

From (5.2), it is easy to know that for any $\delta>0$,

$$\int_{T_1+\delta}^{T_0}\|u_x\|_{L^\infty}ds<\infty.$$

However, one has the following blow-up result of the solution.

Theorem 5.1. *Let (ρ,u) be any global weak solution, which contains vacuum states at least at one point for some time, to the Cauchy problem (1.1)-(1.2) satisfying (2.11)-(2.12) with $\bar\rho>0$. Let $T_0>0$ and $T_1\in[0,T_0)$ be the time such that (5.1) and (5.3) holds respectively.*

Then, the solution (ρ, u) blows up as vacuum states vanish in the following sense: for any $\eta > 0$, it holds

$$\lim_{t\to T_1^+}\int_t^{T_1+\eta}\|u_x\|_{L^\infty}ds=\infty. \tag{5.4}$$

On the other hand, if there exists some $T_2 \in (0, T_0)$ such that the weak solution (ρ, u) satisfies

$$\|u\|_{L^1(0,T_2;W^{1,\infty}(R))}<\infty,$$

then, there is a time $T_3 \in [T_2, T_0)$ such that

$$\lim_{t\to T_3^-}\int_0^{t}\|u_x\|_{L^\infty}ds=\infty. \tag{5.5}$$

The proof of Theorem 5.1 is completely similar to that in [16]. For completeness, we just give a sketch of proof here.

Proof. It suffices to prove (5.4) since the proof of (5.5) is similar. If (5.4) is not true, then there exists a fixed constant $\eta > 0$, such that

$$\int_{T_1}^{T_1+\eta}\|u_x\|_{L^\infty}ds<\infty. \tag{5.6}$$

Thanks to (5.2) and (5.6), the particle path $x(s) = X(s; t, x)$ through $(x, t) \in R \times (T_1, T_1 + \eta]$ can be well defined by solving

$$\begin{cases} \frac{\partial}{\partial s}X(s;t,x)=u(X(s;t,x),s), & T_1\le s<T_1+\eta,\\ X(t;t,x)=x, & T_1\le t<T_1+\eta, x\in R. \end{cases} \tag{5.7}$$

Then by the continuity equation (1.1), one has

$$\rho(x,t)=\rho(X(T_1;t,x),T_1)\exp\{-\int_{T_1}^{t}u_y(y,s)|_{y=X(s;t,x)}ds\} \tag{5.8}$$

for any $(x, t) \in R \times (T_1, T_1 + \eta]$. It follows from (5.6) and (5.7) that for $x_1 \in \Omega_0$ defined by (5.3), which satisfies $\rho(x_1, T_1) = 0$, there exists a trajectory $x = x_1(t) \in R$ for $t \in [T_1, T_1 + \eta]$ so that $X(T_1; t, x_1(t)) = x_1$. Thus, due to (5.8) and (5.6), one has that $\rho(x_1(t), t) = 0$ for all $t \in (T_1, T_1 + \eta]$, which is a contradiction to (5.3). (5.4) is then proved and the proof of the theorem is finished. □

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