Title: Global Existence and Asymptotic Behavior for the Compressible Navier-Stokes Equations with a Non-autonomous External Force and a Heat Source

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Global existence and asymptotic behavior for the compressible Navier-Stokes equations with a non-autonomous external force and a heat source[1]

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Abstract

In this paper, we prove the global existence and asymptotic behavior, as time tends to infinity, of solutions in H^i $(i = 1, 2)$ to the initial boundary value problem of the compressible Navier-Stokes equations of one-dimensional motion of a viscous heat conducting gas in a bounded region with a non-autonomous external force and a heat source. Some new ideas and more delicate estimates are used to prove these results.

Key words: global existence, asymptotic behavior, a uniform priori estimates

1 Introduction

This paper is concerned with global existence and asymptotic behavior to the equations of one-dimensional motion of a viscous heat conducting gas in a bounded region with a non-autonomous external force and a heat source. The Lagrangian form of the conservation laws of mass, momentum, and energy for a one-dimensional gas is (see, e.g., [1, 29, 30])

$$u_t - v_x = 0, \tag{1.1}$$

$$v_t = \sigma_x + f(\int_0^x udy, t), \tag{1.2}$$

$$e_t = -Q_x + \sigma v_x + g(\int_0^x udy, t). \tag{1.3}$$

Here subscripts indicate partial differentiations, u, v, σ, e, Q and θ denote specific volume, velocity, stress, internal energy, heat flux and absolute temperature, respectively. f, g denote the non-autonomous external force and the heat source, respectively.

For simplicity, we only consider the polytropic viscous ideal gas, i.e.,

$$e = c_v\theta, \quad \sigma = -p + \mu\frac{v_x}{u}, \quad Q = -k\frac{\theta_x}{u}, \quad p = R\frac{\theta}{u} \tag{1.4}$$

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with suitable positive constants c_v, R, μ and k.

We consider problem (1.1)–(1.3) in the region $\{0 \le x \le 1, t \ge 0\}$ under initial conditions

$$u(x,0) = u_0(x), v(x,0) = v_0(x), \theta(x,0) = \theta_0(x) \;\; on \; [0,1] \tag{1.5}$$

and the boundary conditions of the form

$$v(0,t) = v(1,t) = 0, Q(0,t) = Q(1,t) = 0, \;\; \forall t \ge 0. \tag{1.6}$$

Throughout the paper we suppose:

$$\int_0^1 u_0(x)dx \equiv \overline{u_0}, \tag{1.7}$$

$$0 < C_0^{-1} \le u_0(x) \le C_0, \;\; \forall \;\; 0 \le x \le 1 \tag{1.8}$$

where C_0 is a positive constant. Furthermore, we assume that for any $u(x,\cdot) \in L^\infty(R^+, L^1[0,1])$ with $\xi(x,t) \equiv \int_0^x u(y,t)dy$ and $\hat{f}(x,t) \equiv \int_0^t f(\int_0^x u(y,s)dy, s)ds) =$ $\int_0^t f(\xi(x,s),s)ds$, the non-autonomous external force $f = f(\xi(x,t),t)$ and heat source $g = g(\xi(x,t),t)$ satisfy the following conditions

$$\begin{aligned}
&f(\xi(x,\cdot),\cdot) \in L^\infty(R^+, L^2[0,1]) \cap L^2(R^+, L^\infty[0,1]) \cap L^1(R^+, L^1[0,1]),\\
&\hat{f}(x,\cdot) \in L^1(R^+, L^2[0,1]), f_\xi(\xi(x,\cdot),\cdot) \in L^2(R^+, L^2[0,1]),\\
&f_t(\xi(x,\cdot),\cdot) \in L^2(R^+, L^2[0,1]), \qquad (1.9)\\
&g(\xi,t) > 0, g(\xi(x,\cdot),\cdot) \in L^\infty(R^+, L^2[0,1]) \cap L^2(R^+, L^2[0,1]) \cap L^1(R^+, L^\infty[0,1]),\\
&g_\xi(\xi(x,\cdot),\cdot) \in L^2(R^+, L^2[0,1]). \qquad (1.10)
\end{aligned}$$

Before stating and proving our results, let us first recall the related results in the literature.

For the case of ideal gas (1.4) or real gas with $f \equiv 0, g \equiv 0$, the global existence and asymptotic of smooth (generalized) solutions to the system have been investigated by many authors, e.g., see [2-7, 9-15, 24] on the initial boundary value problems and the Cauchy problem. Moreover, Zheng and Qin [28] obtained the existence of maximal attractor for the problem (1.1)-(1.3) and (1.5),(1.6) for $f \equiv 0, g \equiv 0$. Qin et al [24] established the global existence and large-time behavior of solutions in $H^i (i = 2,4)$ for the Cauchy problem. Qin [19] established the existence and exponential stability of a C_0-semigroup in the subspace of $H^i \times H^i \times H^i (i = 1,2)$ for a viscous ideal gas in a bounded domain in R and in a bounded annular domain $G_n = \{x \in R^n | 0 < a < |x| < b\} (n = 2,3)$ in R^n for a viscous spherically symmetric ideal gas. This result improved those in [13] for an ideal gas and in [3] for the viscous spherically symmetric ideal gas in G_n. As it is known, the constitutive equations of a real gas are well approximated within moderate ranges of u and θ by the model of an ideal gas. However, under very high temperatures, it becomes inadequate.

For $f \ne 0, g \equiv 0$, when the temperature θ is a constant, Qin and Zhao [23] have proved the global existence and asymptotic behavior of solutions in H^2, Piotr Boguslaw Mucha [8] obtained the exponential stability under various boundary conditions. Shigemori Yanagi [26] have proved the existence of classical solutions. Zhang and Fang [27] also obtained the global existence, asymptotic behavior, exponential stability with the free boundary condition for the isentropic compressible Navier-Stokes equations. Moreover, we would like to refer the works in [16-30] for the related models.

The methods used in this paper come from those in Qin [16-24], in which the global existence, asymptotic behavior, exponential stability and the existence of universal attractor of solutions to equations for a nonlinear one-dimensional viscous heat-conducting real gas were proved.

The notation in this paper will be as follows:
$L^{\bar{p}}, 1 \leq \bar{p} \leq +\infty, W^{m,\bar{p}}, m \in N, H^1 = W^{1,2}, H_0^1 = W_0^{1,2}$ denote the usual (Sobolev) spaces on $(0,1)$. In addition, $\|\cdot\|_B$ denotes the norm in the space B; we also put $\|\cdot\| = \|\cdot\|_{L^2}$. We denote by $C^k(I,B), k \in N_0$, the space of k-times continuously differentiable functions from $I \subseteq R$ into a Banach space B, and likewise by $L^{\bar{p}}(I,B), 1 \leq \bar{p} \leq +\infty$ the corresponding Lebesgue spaces. Subscripts t and x denote the (partial) derivatives with respect to t and x, respectively. We use $C_i (i = 1,2)$ to denote the generic positive constant depending only on the H^i-norm $(i = 1,2)$ of the initial data and $\min_{x\in[0,1]} \theta_0(x), \min_{x\in[0,1]} u_0(x)$.

We now in a position to state our main theorems.

Theorem 1.1 *Assume conditions (1.5)-(1.10) hold. Then for any* $(u_0, v_0, \theta_0) \in H^1[0,1] \times H_0^1[0,1] \times H^1[0,1]$ *with* $\theta_0(x) > 0$ *for any* $x \in [0,1]$, *and that the compatibility conditions hold. Then problem (1.1)-(1.4) admits a unique, uniformly bounded, global-in-time solution* $(u, v, \theta) \in H^1[0,1] \times H_0^1[0,1] \times H^1[0,1]$ *such that*

$$0 < C_1^{-1} \leq u(x,t) \leq C_1, \forall (x,t) \in [0,1] \times [0,+\infty), \tag{1.11}$$

$$\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 + \int_0^t (\|u_x\|^2 + \|v_x\|^2 + \|v_t\|^2 + \|v_{xx}\|^2 + \|\theta_x\|^2 + \|\theta_t\|^2 + \|\theta_{xx}\|^2)(s)ds \leq C_1, \quad \forall t > 0. \tag{1.12}$$

Moreover, as $t \to +\infty$, *we have*

$$\begin{aligned} &\|u(t) - \bar{u}\|_{H^1} \longrightarrow 0, \|v(t)\|_{H^1} \longrightarrow 0, \\ &\|\theta(t) - \bar{\theta}\|_{H^1} \longrightarrow 0, \|\theta(t) - \bar{\theta}\|_{L^\infty} \longrightarrow 0 \end{aligned} \tag{1.13}$$

where $\bar{u} = \int_0^1 u(x,t)dx = \int_0^1 u_0(x)dx, \bar{\theta} = \int_0^1 \theta(x,t)dx$.

Theorem 1.2 *Assume conditions (1.5)-(1.10) hold. Then for any* $(u_0, v_0, \theta_0) \in H^2[0,1] \times H_0^2[0,1] \times H^2[0,1]$, *problem (1.1)-(1.4) admits a unique, uniformly bounded, global-in-time solution* $(u, v, \theta) \in H^2[0,1] \times H_0^2[0,1] \times H^2[0,1]$ *such that*

$$\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + \int_0^t (\|u_x\|_{H^1}^2 + \|v_x\|_{H^2}^2 + \|v_t\|_{H^1}^2 + \|\theta_x\|_{H^2}^2 + \|\theta_t\|_{H^1}^2)(s)ds \leq C_1, \quad \forall t > 0. \tag{1.14}$$

Moreover, as $t \to +\infty$, *we have*

$$\|u(t) - \bar{u}\|_{H^2} \longrightarrow 0, \|v(t)\|_{H^2} \longrightarrow 0, \|\theta(t) - \bar{\theta}\|_{H^2} \longrightarrow 0 \tag{1.15}$$

where $\bar{u} = \int_0^1 u(x,t)dx = \int_0^1 u_0(x)dx, \bar{\theta} = \int_0^1 \theta(x,t)dx$.

We organize our present paper as follows.We will complete the proofs of Theorem 1.1 in Section 2 and Section 3. In Section 4 and Section 5 we will complete the proof of Theorem 1.2.

2 Global Existence in H^1

The proof of global existence of a solution in Theorem 1.1 is based on a priori estimates that can be used to continue a local solution globally in time. The existence and uniqueness of local solutions (with positive u and θ) can be obtained by the linearization of the problem (1.1)-(1.6), and by use of the Banach contraction mapping theorem (cf. [24]). In this section, we assume that assumptions in Theorem 1.1 hold.

We begin with the following lemma.

Lemma 2.1 *The following estimates hold,*

$$\theta(x,t) > 0, \ \ \forall (x,t) \in [0,1] \times [0,\infty), \tag{2.1}$$

$$\int_0^1 u(x,t)dx = \int_0^1 u_0(x)dx \equiv \overline{u_0}, \ \ \forall t > 0, \tag{2.2}$$

$$\int_0^1 \Big(v^2 + c_v(\theta - \log\theta - 1) + R(u - \log u - 1)\Big)dx + \int_0^t\int_0^1 (\frac{v_x^2}{u\theta} + \frac{\theta_x^2}{u\theta^2} + \frac{g}{\theta})dxds \le e_0, \ \ \forall t > 0. \tag{2.3}$$

Proof Inequality (2.1) is a consequence of the generalized maximum principle applied to the following equation, which is equivalent to (1.3),

$$c_v\theta_t - k[\frac{\theta_x}{u}]_x + \frac{R\theta v_x}{u} = \frac{\mu v_x^2}{u} + g(\int_0^x udy, t) \tag{2.4}$$

by considering the positivity of θ_0 and g in (1.10).

Integrating (1.1) with respect to x and t, by conditions (1.6)-(1.7), we can easily get (2.2).

Multiplying (1.2) by v, combining with (1.3) and integrating the resultant over $[0,1]$, we can get

$$\int_0^1 (vv_t + c_v\theta_t)dx = \int_0^1 (\sigma_x v + \sigma v_x + k(\frac{\theta_x}{u})_x + fv + g)dx. \tag{2.5}$$

That is,

$$\frac{d}{dt}\int_0^1 (c_v\theta + \frac{v^2}{2})dx = \int_0^1 (k(\frac{\theta_x}{u})_x + (\sigma v)_x + fv + g)dx = \int_0^1 (fv + g)dx. \tag{2.6}$$

Multiplying (1.3) by θ^{-1} and integrating the resultant over $[0,1]$, we can get

$$\int_0^1 c_v\frac{\theta_t}{\theta}dx = \int_0^1 [k(\frac{\theta_x}{u})_x\frac{1}{\theta} + \frac{\sigma v_x}{\theta} + \frac{g}{\theta}]dx. \tag{2.7}$$

That is,

$$\frac{d}{dt}\int_0^1 (c_v\log\theta + R\log u)dx = \int_0^1 (k\frac{\theta_x^2}{u\theta^2} + \mu\frac{v_x^2}{u\theta} + \frac{g}{\theta})dx. \tag{2.8}$$

Adding (2.6) and (2.8) up, we get

$$\frac{d}{dt}\int_0^1 (c_v\theta + \frac{v^2}{2} - c_v \log\theta - R\log u)dx + \int_0^1 (k\frac{\theta_x^2}{u\theta^2} + \mu\frac{v_x^2}{u\theta} + \frac{g}{\theta})dx$$
$$= \int_0^1 (fv + g)dx. \tag{2.9}$$

Integrating (2.9) over $[0,t]\times[0,1]$, and using (2.2) and estimates

$$\theta - 1 - \log\theta \geq 0, \quad for \ \theta > 0, \tag{2.10}$$
$$u - 1 - \log u \geq 0, \quad for \ u > 0, \tag{2.11}$$

we have

$$\begin{aligned}
&\int_0^1 [\frac{1}{2}v^2 + c_v(\theta - \log\theta - 1) + R(u - \log u - 1)]dx \\
&+ \int_0^t \int_0^1 [\frac{k\theta_x^2}{u\theta^2} + \frac{\mu v_x^2}{u\theta} + \frac{g}{\theta}]dxds \\
&\leq C_1 + \int_0^t \int_0^1 (fv+g)dxds \\
&\leq C_1 + \frac{1}{2}\int_0^t \int_0^1 \|f\|_{L^2[0,1]}(\|v\|^2 + 1)ds + \int_0^t \|g\|_{L^1[0,1]}ds \\
&\leq C_1 + \frac{1}{2}\int_0^t \|f\|_{L^2[0,1]}\|v\|^2 ds.
\end{aligned} \tag{2.12}$$

Applying the Gronwall inequality to (2.12) and using (1.9) yields the desired estimate (2.3). □

The next lemma was given when $f = g \equiv 0$ in [2, 14-17] for a viscous heat-conducting real gas and in [6] for a viscous polytropic ideal gas.

Lemma 2.2 *For any $t \geq 0$, there exists one point $x_1 = x_1(t) \in [0,1]$ such that the solution $u(x,t)$ to problem (1.1)-(1.6) possesses the following expression:*

$$u(x,t) = D(x,t)Z(t)\left[1 + \frac{R}{\mu}\int_0^t \frac{\theta(x,s)}{D(x,s)Z(s)}ds\right] \tag{2.13}$$

where

$$\begin{aligned}
D(x,t) &= u_0(x)\exp\Big\{\frac{1}{\mu}\left[\int_{x_1(t)}^x vdy - \int_0^x v_0 dy + \frac{1}{\overline{u_0}}\int_0^1 (u_0\int_0^x v_0 dy)dx\right] \\
&\quad + \frac{1}{\overline{u_0}}\int_0^t\int_0^1 v(x,s)\int_0^s f(\int_0^x u(z,\tau)dz,\tau)d\tau dxds \\
&\quad - \int_{x_1(t)}^x \int_0^t f(\int_0^y u(z,s)dz,s)dsdy\Big\},
\end{aligned} \tag{2.14}$$

$$Z(t) = \exp\left[-\frac{1}{\mu\overline{u_0}}\int_0^t\int_0^1 (v^2 + R\theta)dxds\right]. \tag{2.15}$$

Proof Let

$$h(x,t)=\int_0^x v_0(y)dy+\int_0^t \sigma(x,s)ds. \tag{2.16}$$

Then we infer from (1.1)-(1.2)

$$\begin{aligned} h_x &= v_0(x)+\int_0^t \sigma_x(x,s)ds = v(x,t)-\int_0^t f(\int_0^x u(y,\tau)dy,\tau)ds, && (2.17)\\ h_t &= \sigma(x,t)=-p+\mu\frac{v_x}{u}. && (2.18)\end{aligned}$$

By (1.1) and (2.18), we get

$$(uh)_t = u_t h + u h_t = v_x h - up + \mu v_x. \tag{2.19}$$

Integrating (2.19) over $[0,1]\times[0,t]$, by (1.6), we can conclude

$$\begin{aligned}\int_0^1 uhdx &= \int_0^1 u_0h_0dx+\int_0^t\int_0^1 (v_xh-up+\mu v_x)dxds\\ &= \int_0^1 u_0h_0dx-\int_0^t\int_0^1 [v^2+up-v\int_0^s fd\tau](x,s)dxds. && (2.20)\end{aligned}$$

Applying the mean value theorem to (2.20) and using (2.2), we derive that there exists one point $x_1=x_1(t)\in[0,1]$ such that

$$\int_0^1 uhdx = h(x_1(t),t)\int_0^1 u(x,t)dx=\overline{u_0}h(x_1(t),t) \tag{2.21}$$

with $\overline{u_0}=\int_0^1 u_0(x)dx$. Thus it follows from (2.16) and (2.21) that

$$\begin{aligned}&\int_0^t \sigma(x_1(t),s)ds = h(x_1(t),t)-\int_0^{x_1(t)} v_0(y)dy\\ &=\frac{1}{\overline{u_0}}\int_0^1 h(x,t)u(x,t)dx-\int_0^{x_1(t)} v_0(y)dy\\ &=\frac{1}{\overline{u_0}}[\int_0^1 u_0(x)h_0(x)dx-\int_0^t\int_0^1 (v^2+up)(x,s)dxds\\ &+\int_0^t\int_0^1 v(x,s)\int_0^s f(\int_0^x u(z,\tau)dz,\tau)d\tau dxds]-\int_0^{x_1(t)} v_0(y)dy. && (2.22)\end{aligned}$$

By (1.1) and (1.4), we can rewrite (1.2) as

$$v_t-\mu[\log u]_{xt}=-(\frac{R\theta}{u})_x+f. \tag{2.23}$$

Integrating (2.23) over $[x_1(t),x]\times[0,t]$, we can infer

$$\mu\log u - R\int_0^t \frac{\theta}{u}ds=\mu\log u_0+\int_0^t \sigma(x_1(t),s)ds+\int_{x_1(t)}^x (v-v_0)dy-\int_{x_1(t)}^x\int_0^t fdsdy. \tag{2.24}$$

Inserting (2.22) in (2.24), we obtain

$$\mu\log u - R\int_0^t \frac{\theta}{u}ds = \mu\log u_0 - \frac{1}{\overline{u_0}}\int_0^t\int_0^1 (v^2+up)dxds + \frac{1}{\overline{u_0}}\int_0^1 u_0h_0dx$$
$$+\frac{1}{\overline{u_0}}\int_0^t\int_0^1 v\int_0^s f(\int_0^x u(z,\tau)dz,\tau)d\tau dsdx + \int_{x_1(t)}^x v(y,t)dy$$
$$-\int_0^x v_0(y)dy - \int_{x_1(t)}^x\int_0^t fdsdy. \tag{2.25}$$

With the definitions of $D(x,t)$ and $Z(t)$, we infer from (2.25) that

$$\frac{Z(t)}{B(x,t)} = \frac{1}{u(x,t)}\exp[\frac{R}{\mu}\int_0^t \frac{\theta(x,s)}{u(x,s)}ds]. \tag{2.26}$$

Multiplying (2.26) by $R\theta/u$, and integrating the result with respect to t, we have

$$\exp[\frac{R}{\mu}\int_0^t \frac{\theta(x,s)}{u(x,s)}ds] = 1 + \frac{R}{\mu}\int_0^t \frac{\theta(x,s)Z(s)}{D(x,s)}ds$$

which along with (2.26) gives (2.13). □

Lemma 2.3

$$0 < C_1^{-1} \le u(x,t) \le C_1, \ \ \forall (x,t)\in[0,1]\times[0,+\infty). \tag{2.27}$$

Proof Let

$$M_u(t) = \max_{x\in[0,1]} u(x,t), \ \ m_u(t) = \min_{x\in[0,1]} u(x,t),$$
$$M_\theta(t) = \max_{x\in[0,1]} \theta(x,t), \ \ m_\theta(t) = \min_{x\in[0,1]} \theta(x,t).$$

It follows from (2.3) and the convexity of the function $-\log y$ that

$$\int_0^1 \theta dx - \log\int_0^1 \theta dx - 1 \le \int_0^1 (\theta - \log\theta - 1)dx \le C_1$$

which implies that there exist a point $b(t)\in[0,1]$ and two positive constants r_1, r_2 such that

$$0 < r_1 \le \int_0^1 \theta(x,t)dx = \theta(b(t),t) \le r_2 \tag{2.28}$$

with r_1, r_2 being two positive roots of the equation $y - \log y - 1 = C_1$.
By Young's inequality, Hölder's inequality, (1.7), (1.9) and Lemma 2.1, we have

$$\Big|\int_{x_1}^x vdy - \int_0^x v_0dy + \frac{1}{\overline{u_0}}\int_0^1 (u_0\int_0^x v_0dy)dx\Big|$$
$$\le \|v\|_{L^1[0,1]} + 2\|v_0\|_{L^1[0,1]} \le C_1, \tag{2.29}$$
$$\Big|\int_{x_1(t)}^x\int_0^t f(\int_0^y u(z,s)dz,s)dsdy\Big| \le \|f\|_{L^1(R^+,L^1[0,1])} \le C_1, \tag{2.30}$$
$$\Big|\frac{1}{\overline{u_0}}\int_0^t\int_0^1 v(x,s)\int_0^s f(\int_0^x u(z,\tau)dz,\tau)d\tau dxds\Big| \le C_1\int_0^t \|v(x,s)\|\|\hat{f}(x,s)\|ds$$
$$\le C_1\int_0^t \|\hat{f}(x,s)\|ds \le C_1. \tag{2.31}$$

Thus we easily deduce from Lemmas 2.1-2.2, (2.28)-(2.30) that

$$0 < C_1^{-1} \le D(x,t) \le C_1, \ \ \forall (x,t) \in [0,1] \times [0,+\infty). \tag{2.32}$$

On the other hand, we readily infer from (2.3), (2.28) that for $0 \le s \le t$,

$$\begin{aligned}
&Rr_1(t-s) \le R\int_s^t \int_0^1 \theta dx d\tau \le \int_s^t \int_0^1 (v^2 + R\theta)(x,\tau) dx d\tau \\
&\le \int_s^t [\|v\|^2 + \int_0^1 R\theta dx](\tau) d\tau \\
&\le (e_0 + Rr_2)(t-s)
\end{aligned}$$

which, combined with (2.15), implies that for any $0 \le s \le t$,

$$\begin{aligned}
C_1^{-1} e^{-a_2(t-s)} \le Z(t) Z^{-1}(s) &= \exp\left[-\frac{1}{\mu \overline{u_0}} \int_s^t \int_0^1 (v^2 + R\theta) dx ds \right] \\
&\le C_1 e^{-a_1(t-s)}
\end{aligned} \tag{2.33}$$

with $a_1 = \frac{r_1 R}{\mu \overline{u_0}}$ and $a_2 = \frac{r_2 R + e_0}{\mu \overline{u_0}}$.
We have from (2.28)

$$\begin{aligned}
|\theta^{1/2}(x,t) - \theta^{1/2}(b(t),t)| &\le C\Big| \int_{b(t)}^x \theta^{-1/2} \theta_x dx \Big| \le C\Big(\int_0^1 \frac{\theta_x^2}{u\theta} dx \Big)^{\frac{1}{2}} \Big(\int_0^1 u\theta^2 dx \Big)^{\frac{1}{2}} \\
&\le C V^{\frac{1}{2}}(t) M_u^{\frac{1}{2}}(t)
\end{aligned}$$

where $V(t) = \int_0^1 \frac{\theta_x^2}{u\theta^2} dx$ satisfies $\int_0^\infty V(s) ds < +\infty$, by (2.3) in Lemma 2.1, which gives

$$C_1^{-1} - C_1 V(t) M_u(t) \le \theta(x,t) \le C_1 + CV(t) M_u(t). \tag{2.34}$$

Thus we conclude from Lemma 2.2 and (2.28)-(2.34)

$$\begin{aligned}
u(x,t) &= D(x,t)\Big[Z(t) + \frac{R}{\mu} \int_0^t (\theta(x,s) D^{-1}(x,s) Z(t) Z^{-1}(s) ds \Big] \\
&\le C_1 \Big[e^{-a_1 t} + \int_0^t (1 + V(s) M_u(s)) e^{-a_1(t-s)} ds \Big] \\
&\le C_1 + C_1 \int_0^t M_u(s) V(s) ds
\end{aligned}$$

whence

$$M_u(t) \le C_1 + C_1 \int_0^t M_u(s) V(s) ds$$

which, by using Gronwall's inequality and (2.3), implies

$$M_u(t) \le C_1. \tag{2.35}$$

So, from (2.34)-(2.35),

$$C_1^{-1} - C_1 V(t) \le \theta(x,t) \le C_1 + C_1 V_1(t). \tag{2.36}$$

Similarly, noting that as $t \to +\infty$,

$$\int_0^t V(s)e^{-a_2(t-s)}ds \le e^{-a_2 t/2}\int_0^{t/2} V(s)ds + \int_{t/2}^t V(s)ds \to 0,$$

we infer from (2.31)-(2.36) and Lemma 2.2 that there exists a large time $t_0 > 0$ such that as $t \ge t_0$, $\forall x \in [0,1]$,

$$\begin{aligned} u(x,t) &= D(x,t)\Big[Z(t) + \frac{R}{\mu}\int_0^t \theta D^{-1}(x,s)Z(t)Z^{-1}(s)ds\Big] \\ &\ge C_1^{-1}\Big[e^{-a_2 t} + \int_0^t (C_1^{-1} - C_1 V(s))e^{-a_2(t-s)}ds\Big] \\ &\ge C_1^{-1} - C_1\int_0^t V(s)e^{-a_2(t-s)}ds \ge (2C_1)^{-1}. \end{aligned} \tag{2.37}$$

Noting that $D(x,t) \ge 1, Z(t) \ge 1$ and $\theta(x,t) > 0$, we infer from Lemma 2.2 that

$$u(x,t) \ge D(x,t)Z(t) \ge C_1^{-1}e^{-a_2 t} \ge C_1^{-1}e^{-a_2 t_0},\ \ \forall (x,t) \in [0,1]\times[0,t_0]$$

which with (2.37) and (2.35) gives (2.27). □

Corollary 2.1 *There holds that for any $t > 0$,*

$$\int_0^t \|v(s)\|^2_{L^\infty[0,1]} \le C_1. \tag{2.38}$$

Proof By Lemmas 2.1-2.4, we can derive

$$\begin{aligned} \int_0^t \|v(s)\|^2_{L^\infty[0,1]}ds &\le \int_0^t (\int_0^1 |v_x|dx)^2 ds \\ &\le \int_0^t \int_0^1 \frac{v_x^2}{\theta}dx \int_0^1 \theta dx ds \le C_1. \end{aligned}$$

□

Lemma 2.4 *The following estimate holds*

$$\|\theta(t)\|^2 + \|v(t)\|^4_{L^4[0,1]} + \int_0^t (\|\theta_x\|^2 + \|vv_x\|^2)(s)ds \le C_1,\ \ \forall t > 0. \tag{2.39}$$

Proof Multiplying (2.5) by $c_v\theta + \frac{v^2}{2}$, and integrating the resulting equation by parts, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|c_v\theta + \frac{v^2}{2}\|^2 + \int_0^1 (kc_v\frac{\theta_x^2}{u} + \mu\frac{v^2v_x^2}{u})dx \\ &= \int_0^1 \Big[c_v p v\theta_x + pv^2 v_x - \mu\frac{c_v v v_x\theta_x}{u} - k\frac{\theta_x v v_x}{u} + (vf+g)(c_v\theta + \frac{v^2}{2})\Big]dx. \end{aligned} \tag{2.40}$$

Integrating (2.40) over [0,t], by Lemma 2.3 and Young's inequality, we can obtain

$$\begin{aligned}
&\|c_v\theta + \frac{v^2}{2}\|^2 + \int_0^t (\|vv_x\|^2 + \|\theta_x\|^2)(s)ds \\
\le\ & C_1\Big[\int_0^t\int_0^1 (|pv(\theta_x| + |vv_x|) + |vv_x\theta_x| + |v\theta f| + |v^3 f| + |g\theta| + |v^2 g|)dxds\Big] \\
\le\ & \varepsilon\int_0^t \|\theta_x(s)\|^2 ds + C_1\int_0^t [\|vv_x\|^2 + \|\theta v\|^2 + \|f\|^2 + \|g\|_{L^\infty[0,1]}(\|\theta\|_{L^1[0,1]}\|v\|^2)](s)ds \\
&+ \int_0^t (\|f\|^2_{L^\infty[0,1]}\|v\|^4_{L^4[0,1]} + \|v\|^2)(s)ds. \qquad (2.41)
\end{aligned}$$

Thus, by (1.7)-(1.9) and Corollary 2.1, for $\varepsilon > 0$ small enough,

$$\begin{aligned}
&\|\theta(t)\|^2 + \|v(t)\|^4_{L^4[0,1]} + \int_0^t \|\theta_x(s)\|^2 ds \\
&\le C_1 + C_1\int_0^t [\|vv_x\|^2 + \|\theta\|^2\|v\|^2_{L^\infty[0,1]} + \|f\|^2_{L^\infty[0,1]}\|v\|^4_{L^4[0,1]}](s)ds. \qquad (2.42)
\end{aligned}$$

Multiplying (1.2) by v^3, and integrating it over $[0,1]$, we can conclude

$$\frac{d}{dt}\int_0^1 \frac{v^4}{4}dx = \int_0^1 (3v^2 v_x(R\frac{\theta}{u} - \mu\frac{v_x}{u}) + fv^3)dx. \qquad (2.43)$$

Integrating (2.43) over $[0,t]$ and by Young's inequality, we can get

$$\begin{aligned}
\|v(t)\|^4_{L^4[0,1]} + \int_0^t \|vv_x\|^2(s)ds \ &\le\ C_1 + C_1\int_0^t\int_0^1 (|\theta v^2 v_x| + |fv^3|)dxds \\
&\le\ C_1 + \frac{1}{2}\int_0^t \|vv_x\|^2(s)ds + C_1\int_0^t [\|v\|^2_{L^\infty[0,1]}\|\theta\|^2 \\
&\quad + \|f\|^2_{L^\infty[0,1]}\|v\|^4_{L^4[0,1]} + \|v\|^2](s)ds
\end{aligned}$$

whence

$$\|v(t)\|^4_{L^4[0,1]} + \int_0^t \|vv_x\|^2(s)ds \le C_1 + C_1\int_0^t [\|v\|^2_{L^\infty[0,1]}\|\theta\|^2 + \|f\|^2_{L^\infty[0,1]}\|v\|^4_{L^4[0,1]}](s)ds. \qquad (2.44)$$

Multiplying (2.44) by a large constant and adding it to (2.42) yields

$$\begin{aligned}
&\|\theta(t)\|^2 + \|v(t)\|^4_{L^4[0,1]} + \int_0^t (\|\theta_x\|^2 + \|vv_x\|^2)(s)ds \\
&\le C_1 + C_1\int_0^t [\|v\|^2_{L^\infty[0,1]} + \|f\|^2_{L^\infty[0,1]}](\|\theta\|^2 + \|v\|^4_{L^4[0,1]})ds. \qquad (2.45)
\end{aligned}$$

Applying the Gronwall inequality to (2.45), and using Corollary 2.1, (1.7) and the following estimates,

$$\begin{aligned}
\int_0^t \|f\|^2_{L^\infty[0,1]}ds \ &\le\ C\int_0^t (\|f\|\|f_x\| + \|f\|^2)ds \le C\int_0^t (\|f\|^2 + \|f_x\|^2) \\
&\le\ C\int_0^t (\|f\|^2 + \|f_\xi\|^2) \le C_1, \\
\int_0^t \|g\|_{L^\infty[0,1]}ds \ &\le\ C\int_0^t (\|g\|^{1/2}\|g_x\|^{1/2} + \|g\|)ds \le C\int_0^t (\|g\| + \|g_x\|) \\
&\le\ C\int_0^t (\|g\| + \|g_\xi\|) \le C_1,
\end{aligned}$$

we can conclude (2.39). □

Lemma 2.5 *The following estimates holds*

$$\|u_x(t)\|^2 + \int_0^t \int_0^1 \theta u_x^2 dxds \le C_1, \quad \forall t > 0. \tag{2.46}$$

Proof By (1.2), we get

$$\Big(v - \mu\frac{u_x}{u}\Big)_t - R\frac{\theta u_x}{u^2} = -R\frac{\theta_x}{u} + f. \tag{2.47}$$

Multiplying (2.47) by $v - \mu\frac{u_x}{u}$ and integrating over $[0,1]\times[0,t]$, by Lemmas 2.1-2.4 and Corollary 2.1, we obtain

$$\begin{aligned}
& \frac{1}{2}\|v - \mu\frac{u_x}{u}\|^2 + \int_0^t\int_0^1 \mu R\frac{u_x^2\theta}{u^3}dxds \\
= \;& \frac{1}{2}\|v_0 - \mu\frac{u_{0x}}{u}\|^2 + \int_0^t\int_0^1 \Big(R\frac{u_x\theta v}{u^2} + (f - R\frac{\theta_x}{u})(v - \mu\frac{u_x}{u})\Big)dxds \\
\le \;& C_1 + C_1\int_0^t\int_0^1 \theta(\varepsilon u_x^2 + C(\varepsilon)v^2)dxds + C_1\int_0^t(\|\theta_x\|^2 + \|v\|^2)dxds \\
& +\varepsilon\int_0^t\int_0^1 \theta u_x^2 dxds + C(\varepsilon)\int_0^t\int_0^1 \frac{\theta_x^2}{\theta}dxds + \int_0^t\int_0^1 f(v - \mu\frac{u_x}{u})dxds \\
\le \;& C_1 + C_1\varepsilon\int_0^t\int_0^1 \theta u_x^2 dxds + C_1\int_0^t \|v\|^2_{L^\infty[0,1]}\int_0^1 \theta dxds \\
& +C_1\int_0^t\int_0^1 (\frac{\theta_x^2}{\theta^2} + \theta_x^2)dxds + C_1\int_0^t(\|f\|^2 + \|v\|^2)ds \\
& +\varepsilon\sup_{0\le s\le t}\|u_x(s)\|\int_0^t \|f\|_{L^\infty[0,1]}ds \\
\le \;& C_1 + C_1\varepsilon\int_0^t\int_0^1 \theta u_x^2 dxds + \varepsilon\sup_{0\le s\le t}\|u_x(s)\|^2
\end{aligned}$$

which with (2.2) yields

$$\sup_{0\le s\le t}\|u_x(s)\|^2 \le C_1 + C_1\varepsilon\int_0^t\int_0^1 \theta u_x^2 dxds + \varepsilon\sup_{0\le s\le t}\|u_x(s)\|^2. \tag{2.48}$$

Thus taking $\varepsilon > 0$ sufficiently small in (2.48) gives (2.46). □

Lemma 2.6 *The following estimates hold,*

$$\|v(t)\|^2 + \int_0^t \|v_x(s)\|^2 ds \le C_1, \quad \forall t > 0, \tag{2.49}$$

$$\|v_x(t)\|^2 + \int_0^t (\|v_{xx}\|^2 + \|v_t\|^2)(s)ds \le C_1\Big(1 + \sup_{0\le s\le t}\|\theta(s)\|_{L^\infty}\Big), \quad \forall t > 0. \tag{2.50}$$

Proof Multiplying (1.2) by v, v_{xx} and v_t, respectively, and then integrating the resultants over $[0,1]\times[0,t]$, using Lemmas 2.1-2.5 and Corollary 2.1, we deduce

$$\begin{aligned}
&\|v(t)\|^2 + 2\mu\int_0^t\int_0^1 \frac{v_x^2}{u}dxds\\
&\le C_1 + C_1\int_0^t\int_0^1 (-R\frac{\theta_x v}{u} + R\frac{\theta u_x v}{u^2} + fv)dxds\\
&\le C_1\int_0^t (\|\theta_x\|^2 + \|v\|^2 + \int_0^1 \theta u_x^2 dx + \|v\|^2_{L^\infty[0,1]}\int_0^1 \theta dx + \|f\|^2)ds\\
&\le C_1,
\end{aligned} \tag{2.51}$$

$$\begin{aligned}
&\frac{1}{2}\|v_x(t)\|^2 + \mu\int_0^t\int_0^1 \frac{v_{xx}^2}{u}dxds\\
= \ &\frac{1}{2}\|v_{0x}\|^2 + \int_0^t\int_0^1 \Big(R(\frac{\theta}{u})_x v_{xx} + \mu\frac{v_x u_x v_{xx}}{u^2} - fv_{xx}\Big)dxds\\
\le \ &C_1 + C_1\int_0^t\int_0^1 \Big(|v_x u_x v_{xx}| + |R(\frac{\theta}{u})_x v_{xx}| + |fv_{xx}|\Big)dxds\\
\le \ &C_1 + \varepsilon\int_0^t \|v_{xx}\|^2 ds + C_1\int_0^t \Big(\|v_x\|^2_{L^\infty[0,1]}\|u_x\|^2 + \|\theta_x\|^2\\
&+\|\theta\|_{L^\infty[0,1]}\|\theta^{1/2}u_x\|^2 + \|f\|^2\Big)ds\\
\le \ &\varepsilon\int_0^t \|v_{xx}\|^2 ds + C_1\int_0^t \Big(\|v_x\|\|v_{xx}\|\|u_x\|^2\Big)ds + C_1(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty})\\
\le \ &2\varepsilon\int_0^t \|v_{xx}(s)\|^2 ds + C_1\int_0^t \|v_x\|^2 ds + C_1(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty[0,1]})
\end{aligned}$$

i.e., for sufficiently small $\varepsilon > 0$,

$$\|v_x(t)\|^2 + \int_0^t \|v_{xx}(s)\|^2 ds \le C_1(1+\sup_{0\le s\le t}\|\theta(s)\|_{L^\infty[0,1]}) \tag{2.52}$$

and

$$\begin{aligned}
&\|v_x(t)\|^2 + \int_0^t \|v_t(s)\|^2 ds\\
&\le C_1 + C_1\int_0^t (\|p_x\|^2 + \|f\|^2 + \|v_x\|^3_{L^3})(s)ds\\
&\le C_1 + C_1\int_0^t (\|\theta u_x\|^2 + \|\theta_x\|^2 + \|v_x\|^{5/2}\|v_{xx}\|^{1/2} + \|f\|^2)ds\\
&\le C_1 + C_1\int_0^t [\sup_{0\le s\le t}\|\theta\|_{L^\infty}\int_0^1 \theta u_x^2 dx + \|\theta_x\|^2 + \|f\|^2]ds\\
&\quad +C_1\Big(\int_0^t \|v_x\|^{10/3}ds\Big)^{3/4}\Big(\int_0^t \|v_{xx}\|^2 ds\Big)^{1/4}\\
&\le C_1\Big(1+\sup_{0\le s\le t}\|\theta\|_{L^\infty}\Big) + \varepsilon\sup_{0\le s\le t}\|v_x\|^2\Big(\int_0^t \|v_x\|^2 ds\Big)^{3/2} + C_1(\varepsilon)\Big(\int_0^t \|v_{xx}\|^2 ds\Big)^{1/2}\\
&\le C_1\Big(1+\sup_{0\le s\le t}\|\theta\|_{L^\infty}\Big) + \varepsilon\sup_{0\le s\le t}\|v_x\|^2
\end{aligned}$$

which along with (2.52) yields estimate (2.49). Here we have used the following interpolation inequalities,

$$
\begin{aligned}
&\|v_x(t)\|_{L^\infty[0,1]} \leq C\|v_x(t)\|^{1/2}\|v_{xx}(t)\|^{1/2},\\
&\|v_x(t)\|_{L^3}^3 \leq C\|v_x(t)\|^{5/2}\|v_{xx}(t)\|^{1/2} + C\|v_x(t)\|^3 \leq C\|v_x(t)\|^{5/2}\|v_{xx}(t)\|^{1/2},\\
&\bar{v_x} = \int_0^1 v_x dx = 0.
\end{aligned}
$$

□

Lemma 2.7 *The following estimates hold*

$$
\|\theta_x(t)\|^2 + \int_0^t \|\theta_{xx}\|^2 ds \leq C_1\Big(1 + \sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}\Big)^2, \quad \forall t > 0. \tag{2.53}
$$

Proof Multiplying (1.3) by θ, integrating the resulting equation over $[0,1]\times[0,t]$, we get

$$
\begin{aligned}
&\frac{1}{2}c_v\|\theta_x\|^2 + \int_0^t\int_0^1 k\frac{\theta_{xx}^2}{u}dxds\\
= \;& \frac{1}{2}c_v\|\theta_{0x}\|^2 + \int_0^t\int_0^1 k\frac{\theta_x u_x\theta_{xx}}{u^2}dxds - \int_0^t\int_0^1(\sigma v_x + g)\theta_{xx})dxds\\
\leq \;& C_1 + \varepsilon\int_0^t \|\theta_{xx}\|^2 ds + C_1\int_0^t\int_0^1(\theta_x^2u_x^2 + \theta^2v_x^2 + v_x^4 + g^2)dxds.
\end{aligned}
$$

That is, for small $\varepsilon > 0$,

$$
\begin{aligned}
&\|\theta_x(t)\|^2 + \int_0^t \|\theta_{xx}(s)\|^2 ds\\
\leq \;& C_1 + C_1\int_0^t\Big((\|\theta_x\|\|\theta_{xx}\| + \|\theta_x\|^2)\|u_x\|^2 + \|v_x\|^3\|v_{xx}\|\Big)(s)ds\\
&+C_1\sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty}^2\int_0^t\|v_x(s)\|^2ds + \int_0^t\|g\|^2ds\\
\leq \;& C_1(1 + \sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty})^2 + \frac{1}{2}\int_0^t\|\theta_{xx}(s)\|^2ds + C_1\int_0^t\|\theta_x(s)\|^2(\|u_x\|^4 + \|u_x\|^2)(s)ds\\
&+C_1\int_0^t\|v_{xx}(s)\|^2ds + C_1\int_0^t\|v_x(s)\|^6ds\\
\leq \;& C_1(1 + \sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty})^2 + C_1\int_0^t\|v_{xx}(s)\|^2ds + \frac{1}{2}\int_0^t\|\theta_{xx}(s)\|^2ds\\
&+C_1\sup_{0\leq s\leq t}(\|u_x\|^4 + \|u_x\|^2)\int_0^t\|\theta_x(s)\|^2ds + C_1\sup_{0\leq s\leq t}\|v_x(s)\|^4\int_0^t\|v_x(s)\|^2ds\\
\leq \;& C_1(1 + \sup_{0\leq s\leq t}\|\theta(s)\|_{L^\infty[0,1]})^2.
\end{aligned}
$$

□

Lemma 2.8 *The following estimates hold*

$$\|\theta(t)\|_{L^\infty} \le C_1, \ \ \forall t > 0, \tag{2.54}$$

$$\begin{aligned} &\|\theta_x(t)\|^2 + \|v_x(t)\|^2 + \|u_x(t)\|^2 + \int_0^t (\|\theta_x\|^2 + \|vv_x\|^2 + \|u_x\|^2 \\ &+\|v_x\|^2 + \|v_t\|^2 + \|v_{xx}\|^2 + \|\theta_t\|^2 + \|\theta_{xx}\|^2)(s)ds \le C_1, \ \ \forall t > 0. \end{aligned} \tag{2.55}$$

Proof With the help of the Gagliardo-Nirenberg interpolation inequality, Young's inequality and Lemmas 2.1-2.7, we derive

$$\begin{aligned} \|\theta(t)\|_{L^\infty} &\le C\|\theta_x(t)\|^{2/3}\|\theta(t)\|^{1/3}_{L^1[0,1]} + C\|\theta(t)\|_{L^1[0,1]} \le C(\|\theta_x\|^{2/3} + 1) \\ &\le C(1 + \sup_{0\le s\le t} \|\theta(s)\|^{2/3}_{L^\infty[0,1]}) \\ &\le \varepsilon \sup_{0\le s\le t} \|\theta(s)\|_{L^\infty[0,1]} + C_1. \end{aligned} \tag{2.56}$$

Taking the supremum on the right-hand side of (2.56), and picking ε sufficiently small, and we can get the inequality (2.54).

Multiplying (1.3) by θ_t, and then integrating the resultants over $[0,1] \times [0,t]$, using Lemmas 2.1-2.7, we infer

$$\begin{aligned} &\int_0^1 k\frac{\theta_x^2}{u}dx + \int_0^t\int_0^1 c_v\theta_t^2 dxds \\ = &\int_0^1 k\frac{\theta_{0x}^2}{u_0}dx + \int_0^t\int_0^1 \Big(- k\frac{\theta_x^2 v_x}{u^2} + (-R\frac{\theta}{u} + \mu\frac{v_x}{u})v_x\theta_t + g\theta_t\Big)dxds \\ \le\ & C_1 + C_1\int_0^t\int_0^1 (\theta_x^2|v_x| + \theta|v_x\theta_t| + v_x^2|\theta_t| + g|\theta_t|)dxds \\ \le\ & C_1 + \varepsilon\int_0^t \|\theta_t(s)\|^2 ds + C_1\int_0^t (\|v_x\|^2 + \|\theta_x\|^4_{L^4[0,1]} \\ &+\|\theta\|^2_{L^\infty[0,1]}\|v_x\|^2 + \|v_x\|^4_{L_4[0,1]} + \|g\|^2)(s)ds \\ \le\ & C_1 + \varepsilon\int_0^t \|\theta_t(s)\|^2 ds + C_1\int_0^t (\|\theta_x\|^3\|\theta_{xx}\| + \|\theta_x\|^4 \\ &+\|\theta\|^2_{L^\infty[0,1]}\|v_x\|^2 + \|v_x\|^3\|v_{xx}\| + \|g\|^2)(s)ds \\ \le\ & C_1 + \varepsilon\int_0^t \|\theta_t\|^2 ds + C_1\sup_{0\le s\le t}(\|\theta_x\|^2 + \|\theta_x\|^4)\int_0^t \|\theta_x(s)\|^2 ds \\ &+C_1\sup_{0\le s\le t}\|v_x(s)\|^4\int_0^t \|v_x(s)\|^2 ds + C_1\int_0^t (\|\theta_{xx}\|^2 + \|v_x\|^2 + \|v_{xx}\|^2 + \|g\|^2)(s)ds \\ \le\ & C_1 + \varepsilon\int_0^t \|\theta_t(s)\|^2 ds \end{aligned}$$

whence, for small $\varepsilon > 0$,

$$\|\theta_x(t)\|^2 + \int_0^t \|\theta_t(s)\|^2 ds \le C_1$$

which, combined with Lemmas 2.1-2.7 and (2.54), gives the estimate (2.56). □

3 Asymptotic Behavior in H^1

In this section we will complete the proof of (1.13) and assume that the assumptions in Theorem 1.1 are valid. We begin with the following lemma.

Lemma 3.1 *Suppose that y and h are nonnegative functions on $[0, +\infty)$, y' is locally integrable, and y and h satisfy*

$$\forall t \geq 0, \ \ y'(t) \leq A_1 y^2(t) + A_2 + h(t),$$
$$\forall T > 0, \ \int_0^T y(s)ds \leq A_3, \ \ \int_0^T h(s)ds \leq A_4$$

where A_1, A_2, A_3, A_4 are positive constants independent of t and T. Then for any $r > 0, \ t \geq 0$,

$$y(t + r) \leq (\frac{A_3}{r} + A_2 r + A_4) \cdot e^{A_1 A_2}.$$

Moreover,

$$\lim_{t \to +\infty} y(t) = 0.$$

Proof. See, e.g., [23]. □

Lemma 3.2 *There holds that*

$$\lim_{t \to +\infty} \|u(t) - \bar{u}\|_{H^1} = 0. \tag{3.1}$$

Proof Differentiating (1.1) with respect to x, multiplying the result by u_x, then integrating it over $[0, 1]$, by Young's inequality, we can deduce

$$\frac{d}{dt}\|u_x(t)\|^2 \leq \|v_{xx}(t)\|^2 + \|u_x(t)\|^2 \leq \frac{1}{2} + \frac{1}{2}\|u_x(t)\|^4 + \|v_{xx}(t)\|^2$$

which along with (1.12) and Lemma 3.1 gives

$$\lim_{t \to +\infty} \|u_x(t)\|^2 = 0. \tag{3.2}$$

On the other hand, by (2.2) and the imbedding theorem, we can deduce

$$\|u(t) - \bar{u}\| \leq C_1 \|u_x(t)\|$$

which with (3.2) gives (3.1). □

Lemma 3.3 *There holds that*
$$\lim_{t\to+\infty}\|v(t)\|_{H^1}=0. \tag{3.3}$$

Proof Equation (1.2) can be rewritten as

$$v_t=\Big(-R\frac{\theta}{u}+\int_0^x f dy\Big)_x+\mu\Big(\frac{v_x}{u}\Big)_x. \tag{3.4}$$

Denote

$$\hat p=\hat p(x,t)=R\frac{\theta}{u}-\int_0^x f dy,$$
$$\hat\sigma=\hat\sigma(x,t)=-R\frac{\theta}{u}+\int_0^x f dy+\mu\frac{v_x}{u}.$$

Then
$$v_t=\Big(-\hat p+\mu\frac{v_x}{u}\Big)_x=\hat\sigma_x. \tag{3.5}$$

Put

$$\hat p^*=\hat p^*(x,t)=\hat p(x,t)-\int_0^1\hat p(x,t)dx,$$
$$\hat\sigma^*=\hat\sigma^*(x,t)=\hat\sigma(x,t)-\int_0^1\hat\sigma(x,t)dx.$$

Then

$$\int_0^1\hat p^*(x,t)dx=0,\quad \int_0^1\hat\sigma^*(x,t)dx=0, \tag{3.6}$$
$$(\hat p^*)_x=\hat p_x,\quad (\hat\sigma^*)_x=\hat\sigma_x. \tag{3.7}$$

Noting (3.5) and integrating by parts, we see that

$$\begin{aligned}\|\hat p^*\|^2 &= (\hat p^*,\hat p^*)=\Big(-\hat p^*_x,\int_0^x\hat p^* dy\Big)\\ &= \Big(-\hat p_x,\int_0^x\hat p^* dy\Big)=\Big(v_t-\mu(\frac{v_x}{u})_x,\int_0^x\hat p^* dy\Big)\\ &= \Big(v_t,\int_0^x\hat p^* dy\Big)-\Big(\mu(\frac{v_x}{u})_x,\int_0^x\hat p^* dy\Big)\\ &\le (\int_0^1 v_t^2dx)^{1/2}\Big(\int_0^1(\int_0^x\hat p^* dy)dx\Big)^{1/2}+\mu\int_0^1\frac{v_x}{u}\hat p^* dx\\ &\le \varepsilon\|\hat p^*\|^2+C(\varepsilon)\|v_t\|^2+C_1(\varepsilon\|\hat p^*\|^2+C(\varepsilon)\|v_x\|^2)\\ &\le (C_1+1)\varepsilon\|\hat p^*\|^2+C_1(\|v_t\|^2+\|v_x\|^2)\end{aligned} \tag{3.8}$$

where $(\cdot,\cdot)$ denotes the inner product on $L^2[0,1]$.

Taking $\varepsilon>0$ small enough in (3.8) and integrating it over $[0,t]$, combining the result with (1.12), we can deduce
$$\int_0^t\|\hat p^*\|^2(s)ds\le C_1. \tag{3.9}$$

On the other hand, noting that $\frac{df}{dt} = f_\xi v + f_t$, we arrive at

$$\begin{aligned}\frac{d}{dt}\|\hat{p}^*\|^2 &= 2(\hat{p}^*, \hat{p}_t^*) = 2\Big(\hat{p}_x^*, -\int_0^x \hat{p}_t^* dy\Big) \\ &= 2\Big(v_t, \int_0^x \hat{p}_t^* dy\Big) - 2\Big(\mu(\frac{v_x}{u})_x, \int_0^x \hat{p}_t^* dy\Big) \\ &= 2\Big(v_t, \int_0^x \hat{p}_t^* dy\Big) + 2\mu\Big(\frac{v_x}{u}, \hat{p}_t^*\Big) \\ &\le C_1(\|v_t\|^2 + \|v_x\|^2 + \|\hat{p}_t^*\|^2) \\ &\le C_1(\|v_t\|^2 + \|v_x\|^2 + \|\theta_t\|^2 + \|\frac{df}{dt}\|^2) \\ &\le C_1(\|v_t\|^2\|v_x\|^2 + \|\theta_t\|^2 + \|f_\xi\|^2 + \|f_t\|^2)\end{aligned}$$

which combined with (3.9), Lemma 3.1 and (1.12), we can get

$$\lim_{t\to+\infty} \|\hat{p}^*(t)\|^2 = 0. \tag{3.10}$$

By (3.9) and (1.12), we have

$$\int_0^t \|\hat{\sigma}^*\|^2(s)ds \le C_1 \int_0^t (\|\hat{p}^*\|^2 + \|u_x\|^2)(s)ds \le C_1. \tag{3.11}$$

Noting (3.5) and integrating by parts, we see that

$$\begin{aligned}\frac{d}{dt}\|\hat{\sigma}^*\|^2 &= 2(\hat{\sigma}^*, \hat{\sigma}_t^*) = 2\Big(\hat{\sigma}_x^*, -\int_0^x \hat{\sigma}_t^* dy\Big) \\ &= 2\Big(v_t, -\int_0^x \hat{\sigma}_t^* dy\Big) \\ &\le C_1\Big(\|v_t\|^2 + \Big\|\int_0^x \hat{\sigma}_t^* dy\Big\|^2\Big)\end{aligned} \tag{3.12}$$

where

$$\int_0^x \hat{\sigma}_t^* dy = \int_0^x \Big(\hat{\sigma}_t - \int_0^1 \hat{\sigma}_t(\xi, t)d\xi\Big)dy \tag{3.13}$$

and

$$\begin{aligned}\hat{\sigma}_t(x,t) &= \Big(-R\frac{\theta}{u} + \int_0^x f dy + \mu\frac{v_x}{u}\Big)_t \\ &= -R\frac{\theta_t}{u} + R\frac{\theta u_t}{u^2} + \int_0^x f_t dy + \mu\frac{v_{xt}}{u} - \mu\frac{v_x^2}{u^2} \\ &= -R\frac{\theta_t}{u} + R\frac{\theta u_t}{u^2} + \int_0^x f_t dy + \mu\Big(\frac{v_t}{u}\Big)_x + \mu\frac{v_t u_x - v_x^2}{u^2}.\end{aligned}$$

Noting

$$v_t(0,t) = v_t(1,t) = 0, \quad \forall t > 0,$$

we deduce that for any $x \in [0,1]$,

$$\Big|\int_0^x \hat{\sigma}_t dy\Big| \leq C_1(\|\theta_t\| + \|v_x\| + \|f_t\|_{L^1[0,1]} + |v_t| + \|v_t\|), \tag{3.14}$$

$$\Big|\int_x^1 \hat{\sigma}_t dy\Big| \leq C_1(\|\theta_t\| + \|v_x\| + \|f_t\|_{L^1[0,1]} + |v_t| + \|v_t\|). \tag{3.15}$$

Thus it follows from Fubini's theorem and (3.13)-(3.15) that

$$\begin{aligned}
\int_0^x \hat{\sigma}_t^* dy &= \int_0^x \Big(\hat{\sigma}_t - \int_0^1 \hat{\sigma}_t(\xi,t)d\xi\Big)dy \\
&= \int_0^x \hat{\sigma}_t dy - \int_0^1 \int_x^1 \hat{\sigma}_t dy d\xi \\
&\leq C_1(\|\theta_t\| + \|v_x\| + \|f_t\|_{L^1[0,1]} + |v_t| + \|v_t\|)
\end{aligned}$$

which implies

$$\Big\|\int_0^x \hat{\sigma}_t^* dy\Big\|^2 \leq C_1(\|\theta_t\|^2 + \|v_x\|^2 + \|f_t\|^2_{L^1[0,1]} + \|v_t\|^2). \tag{3.16}$$

By (3.12) and (3.16), we deduce

$$\frac{d}{dt}\|\hat{\sigma}^*\|^2 \leq C_1(\|\theta_t\|^2 + \|v_x\|^2 + \|f_t\|^2 + \|v_t\|^2)$$

which with (3.11), Lemma 3.1 and (1.12) yields

$$\lim_{t\to+\infty} \|\hat{\sigma}^*(t)\|^2 = 0. \tag{3.17}$$

Noting that

$$\int_0^1 \Big(\frac{v}{u}\Big)_x dx = 0,$$

we have

$$\frac{v_x}{u} = \Big(\frac{v_x}{u}\Big)^* + \int_0^1 \frac{v_x}{u} dx = \frac{1}{\mu}(\hat{\sigma}^* + \hat{p}^*) + \int_0^1 \frac{vu_x}{u^2} dx$$

which with Lemma 2.4 gives

$$\|v_x\| \leq C_1\|\frac{v_x}{u}\| \leq C_1(\|\hat{\sigma}^*\| + \|\hat{p}^*\| + \|v\|\|u_x\|). \tag{3.18}$$

By (3.18), (3.17), (3.10) and Lemma 3.2, we derive

$$\lim_{t\to+\infty} \|v_x(t)\| = 0. \tag{3.19}$$

By (1.6) and the Poincaré inequality, we deduce

$$\|v(t)\|_{H^1} \leq C_1\|v_x(t)\|$$

which, together with (3.19), gives (3.3). □

Lemma 3.4 *The following estimates hold,*

$$\lim_{t\to+\infty}\|\theta(t)-\bar\theta\|_{H^1}=0,\quad \lim_{t\to+\infty}\|\theta(t)-\bar\theta\|_{L^\infty}=0. \tag{3.20}$$

Proof Multiplying (1.3) by θ_t, and then integrating the resultant over $[0,1]$, we get

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 k\frac{\theta_x^2}{u}dx+\int_0^1 c_v\theta_t^2dx\\
=&\int_0^1\Big(-k\frac{\theta_x^2v_x}{u^2}+(-R\frac{\theta}{u}+\mu\frac{v_x}{u})v_x\theta_t+g\theta_t\Big)dx\\
\le&\int_0^1(\theta_x^2|v_x|+\theta|v_x\theta_t|+v_x^2|\theta_t|+g|\theta_t|)dx\\
\le&\ \varepsilon\|\theta_t\|^2+C_1(\|v_x\|^2+\|\theta_x\|_{L^4}^4+\|\theta\|^2_{L^\infty[0,1]}\|v_x\|^2+\|v_x\|_{L_4}^4+\|g\|^2)\\
\le&\ \varepsilon\|\theta_t\|^2+C_1(\|\theta_x\|^3\|\theta_{xx}\|+\|\theta_x\|^4+\|\theta\|^2_{L^\infty[0,1]}\|v_x\|^2+\|v_x\|^3\|v_{xx}\|+\|g\|^2)\\
\le&\ \varepsilon\|\theta_t\|^2+C_1(\|\theta_x\|^2+\|\theta_{xx}\|^2+\|v_x\|^2+\|v_{xx}\|^2+\|g\|^2)
\end{aligned}$$

which, by taking $\varepsilon>0$ small enough, along with (1.12) and Lemma 3.1, we derive

$$\lim_{t\to+\infty}\|\theta_x(t)\|=0. \tag{3.21}$$

By the Poincaré inequality, we deduce

$$\|\theta(t)-\bar\theta\|_{H^1}\le C_1\|\theta_x(t)\|,\|\theta(t)-\bar\theta\|_{L^\infty[0,1]}\le C_1\|\theta_x(t)\|$$

which, together with (3.21), gives (3.20). □

4 Global Existence in H^2

In this section we will complete the proof of (1.14) and assume that the assumptions in Theorem 1.2 are valid. We begin with the following lemma. The first lemma concerns the uniform global (in time) positive lower bound (independent of t) of the absolute temperature θ.

Lemma 4.1 *If $(u_0,v_0,\theta_0)\in H^1$, then the generalized global solution $(u(t),v(t),\theta(t))$ to the problem (1.1)-(1.6) satisfies*

$$0<C_1^{-1}\le\theta(x,t),\quad \forall(x,t)\in[0,1]\times[0,+\infty). \tag{4.1}$$

Proof We use the idea in [20] to prove (4.1). If (4.1) is not true, that is, $\inf_{(x,t)\in[0,1]\times[0,+\infty)}$ $\theta(x,t)=0$, then there exists a sequence $(x_n,t_n)\in[0,1]\times[0,+\infty)$ such that

$$\theta(x_n,t_n)\to0,\quad as\ n\to+\infty. \tag{4.2}$$

If the sequence t_n has a subsequence, denoted also by t_n, converging to $+\infty$, then by the asymptotic behavior results in Lemma 3.4, we know that

$$\theta(x_n,t_n)\to\bar\theta>0,\quad as\ n\to+\infty$$

which contradicts with (4.2).

If the sequence t_n is bounded, i.e., there exists a constant $M > 0$, independent of n, such that for any $n = 1, 2, 3, \cdots, 0 < t_n \le M$, then there exists a point $(x^*, t^*) \in [0,1] \times [0, M]$ such that $(x_n, t_n) \to (x^*, t^*)$ as $n \to +\infty$. On the other hand, by (4.2) and the continuity of solutions, we conclude that $\theta(x_n, t_n) \to \theta(x^*, t^*) = 0$ as $n \to +\infty$, which contradicts with (2.1). Thus the proof is complete. □

Lemma 4.2 *There holds that*

$$\|v_t(t)\|^2 + \int_0^t \|v_{tx}(s)\|^2 ds \le C_2, \ \ \forall t > 0, \tag{4.3}$$

$$\|v_{xx}(t)\| \le C_2, \ \ \forall t > 0. \tag{4.4}$$

Proof Differentiating (1.2) with respect to t, multiplying the result by v_t, then integrating it over $[0, 1]$ and integrating by part, by (1.1), (1.7) and Young's inequality, we can derive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_t(t)\|^2 + \int_0^1 \mu \frac{v_{tx}^2}{u} dx \\ = \ & \int_0^1 v_{tx}(R\frac{\theta_t}{u} - R\frac{\theta u_t}{u^2} + \mu\frac{v_x^2}{u^2} - \int_0^x \frac{df}{dt} dy)dx \\ \le \ & \varepsilon\|v_{tx}\|^2 + C_2(\|\theta_t\|^2 + \|v_x\|^2 + \|v_x\|_{L^\infty}^2\|v_x\|^2 + \|\frac{df}{dt}\|^2) \\ \le \ & \varepsilon\|v_{tx}\|^2 + C_2(\|\theta_t\|^2 + \|v_x\|^2 + \|v_{xx}\|^2 + \|f_t\|^2 + \|f_\xi\|^2). \end{aligned} \tag{4.5}$$

Integrating (4.5) over $[0, t]$, by (1.12) and (1.9), we can derive (4.3).

Equation (1.2) can be written as

$$v_{xx} = \frac{u}{\mu}(v_t + R\frac{\theta_x}{u} - R\frac{\theta u_x}{u^2} + \mu\frac{v_x u_x}{u^2} - f). \tag{4.6}$$

By (4.6), (4.3) (1.12) and (1.9), we can get

$$\begin{aligned} \|v_{xx}(t)\| \ = \ & \frac{1}{\mu}\Big(\int_0^1 u^2(v_t + R\frac{\theta_x}{u} - R\frac{\theta u_x}{u^2} + \mu\frac{v_x u_x}{u^2} - f)^2 dx\Big)^{1/2} \\ \le \ & C_2(\|v_t\| + \|\theta_x\| + \|\theta u_x\| + \|v_x\|_{L^\infty}\|u_x\| + \|f\|) \\ \le \ & \varepsilon\|v_{xx}(t)\| + C_2(\|v_t(t)\| + \|v_x(t)\| + \|\theta_x(t)\| + \|u_x(t)\| + \|f(t)\|) \\ \le \ & \varepsilon\|v_{xx}(t)\| + C_2 \end{aligned} \tag{4.7}$$

which gives (4.4). □

Lemma 4.3 *The following estimates hold*

$$\|u_{xx}(t)\|^2 + \int_0^t \|u_{xx}(s)\|^2 ds \le C_2, \ \ \forall t > 0, \tag{4.8}$$

$$\int_0^t \|v_{xxx}(s)\|^2 ds \le C_2, \ \ \forall t > 0. \tag{4.9}$$

Proof Differentiating (1.2) with respect to x, using (1.1) ($u_{txx} = v_{xxx}$), we can deduce

$$\begin{aligned} v_{tx} &= -p_{xx} + \mu\Big(\frac{v_x}{u}\Big)_{xx} + f_x \\ &= -(p_u u_x + p_\theta \theta_x)_x + \mu\Big(\frac{u_{xx}}{u}\Big)_t - 2\mu\frac{v_{xx}u_x}{u^2} + 2\mu\frac{v_x u_x^2}{u^3} + f_x \\ &= -p_{uu}u_x^2 - p_u u_{xx} - 2p_{u\theta}\theta_x u_x - p_{\theta\theta}\theta_x^2 - p_\theta\theta_{xx} \\ &\quad +\mu\Big(\frac{u_{xx}}{u}\Big)_t - 2\mu\frac{v_{xx}u_x}{u^2} + 2\mu\frac{v_x u_x^2}{u^3} + f_x, \end{aligned}$$

i.e.

$$\begin{aligned} \mu\Big(\frac{u_{xx}}{u}\Big)_t - p_u u_{xx} &= v_{tx} + p_{uu}u_x^2 + 2p_{u\theta}\theta_x u_x + p_{\theta\theta}\theta_x^2 + p_\theta\theta_{xx} \\ &\quad +2\mu\frac{v_{xx}u_x}{u^2} - 2\mu\frac{v_x u_x^2}{u^3} - f_\xi u. \end{aligned} \tag{4.10}$$

Multiplying (4.10) by u_{xx}/u, and by Young's inequality Lemma 4.1, we can deduce that

$$\begin{aligned} &\frac{d}{dt}\Big\|\frac{u_{xx}}{u}\Big\|^2 + C_1\int_0^1 \frac{u_{xx}^2}{u^2}dx \le \frac{d}{dt}\Big\|\frac{u_{xx}}{u}\Big\|^2 + \int_0^1 R\frac{\theta}{u}\frac{u_{xx}^2}{u^2}dx \\ \le\ & \varepsilon\Big\|\frac{u_{xx}}{u}\Big\|^2 + C_1(\|v_{tx}\|^2 + \int_0^1(\theta^2 u_x^4 + u_x^2\theta_x^2 + \theta_{xx}^2 \\ & v_{xx}^2 u_x^2 + v_x^2 u_x^4 + f_\xi^2)dx) \\ \le\ & \varepsilon\Big\|\frac{u_{xx}}{u}\Big\|^2 + C_1(\|v_{tx}\|^2 + \|u_x\|^2_{L^\infty[0,1]}(\|u_x\|^2 + \|\theta_x\|^2) \\ & +\|\theta_{xx}\|^2 + \|u_x\|^2_{L^\infty}\|v_{xx}\|^2 + \|v_x\|^2_{L^\infty[0,1]}\|u_x\|^2_{L^\infty[0,1]}\|u_x\|^2 + \|f_\xi\|^2) \\ \le\ & \varepsilon\Big\|\frac{u_{xx}}{u}\Big\|^2 + C_2(\|v_{tx}\|^2 + \|u_x\|^2 + \|u_x\|\|u_{xx}\| + \|\theta_{xx}\|^2 + \|f_\xi\|^2) \\ \le\ & 2\varepsilon\Big\|\frac{u_{xx}}{u}\Big\|^2 + C_2(\|v_{tx}\|^2 + \|u_x\|^2 + \|\theta_{xx}\|^2 + \|f_\xi\|^2) \end{aligned} \tag{4.11}$$

which, by taking $\varepsilon > 0$ small enough and using Lemmas 2.1-2.8, Lemmas 4.1-4.2, gives (4.8).

Differentiating (1.2) with respect to x, by (1.1), we can derive

$$\begin{aligned} v_{xxx} &= \frac{u}{\mu}\Big(v_{tx} + R\frac{\theta_{xx}}{u} - 2R\frac{\theta_x u_x}{u^2} - R\frac{\theta u_{xx}}{u^2} + 2R\frac{\theta u_x^2}{u^3} \\ &\quad +\mu\frac{u_{xx}u_t}{u^2} + 2\mu\frac{v_{xx}u_x}{u^2} - 2\mu\frac{v_x u_x^2}{u^3} + f_x\Big). \end{aligned}$$

Thus

$$\begin{aligned} \int_0^t \|v_{xxx}(s)\|^2 ds &\le C_2\int_0^t\int_0^1 (v_{tx}^2 + \theta_{xx}^2 + \theta_x^2 u_x^2 + \theta^2 u_{xx}^2 + \theta^2 u_x^4 \\ &\quad +u_{xx}^2 v_x^2 + v_{xx}^2 u_x^2 + v_x^2 u_x^4 + f_x^2)dxds \end{aligned}$$

$$
\begin{aligned}
&\le C_2\int_0^t(\|v_{tx}\|^2+\|u_{xx}\|^2+\|u_x\|^2+\|f_\xi\|^2)ds\\
&\quad +C_2\int_0^t(\|u_x\|^2_{L^\infty}(\|\theta_x\|^2\|v_{xx}\|^2)\\
&\quad +\|v_x\|^2_{L^\infty}\|u_{xx}\|^2+\|u_x\|^2_{L^\infty}\|v_x\|^2_{L^\infty}\|u_x\|^2)ds\\
&\le C_2\int_0^t(\|v_{tx}\|^2+\|u_{xx}\|^2+\|u_x\|^2+\|f_\xi\|^2)ds\\
&\quad +C_2\int_0^t(\|v_x\|^2_{L^\infty}+\|u_x\|^2_{L^\infty})ds\\
&\le C_2
\end{aligned}
$$

which gives (4.9). □

Lemma 4.4 *The following estimates hold,*

$$\|\theta_t(t)\|^2+\int_0^t\|\theta_{xt}\|^2ds\le C_2,\quad \forall t>0, \tag{4.12}$$

$$\|\theta_{xx}(t)\|\le C_2,\quad \forall t>0, \tag{4.13}$$

$$\int_0^t\|\theta_{xxx}(s)\|^2\le C_2,\quad \forall t>0. \tag{4.14}$$

Proof Differentiating (1.3) with respect to t and multiplying the resultant by θ_t, we can derive

$$
\begin{aligned}
&\frac{c_v}{2}\frac{d}{dt}\|\theta_t(t)\|^2+\int_0^1 k\frac{\theta_{xt}^2}{u}dx\\
&=\int_0^1\Big(k\frac{\theta_x\theta_{xt}v_x}{u^2}-R\frac{\theta_t^2v_x+\theta\theta_tv_{xt}}{u^2}+R\frac{\theta v_x^2\theta_t}{u^2}\\
&+\mu\frac{2v_xv_{xt}\theta_t}{u}-\mu\frac{v_x^3\theta_t}{u^2}+\frac{dg}{dt}\theta_t\Big)dx\\
&\le \varepsilon\|\theta_{xt}\|^2+C_2\int_0^1(\theta_x^2v_x^2+|\theta_t^2v_x|+|\theta\theta_t(v_{xt}+v_x^2)|+|v_xv_{xt}\theta_t|+|v_x^3\theta_t|+|\frac{dg}{dt}\theta_t|)dx\\
&\le \varepsilon\|\theta_{xt}\|^2+C_2\Big(\|v_x\|^2_{L^\infty}\|\theta_x\|^2+\|\theta_t\|^4_{L^4}+\|v_x\|^2+\|\theta\|_{L^\infty}(\|v_{tx}\|^2+\|v_x\|^4_{L^4}+\|\theta_t\|^2)\\
&+\|v_x\|^2_{L^\infty}(\|v_{xt}\|^2+\|\theta_t\|^2)+\|v_x\|_{L^\infty}(\|v_x\|^4_{L^4}+\|\theta_t\|^2)+\|g_t\|^2+\|g_\xi\|^2+\|\theta_t\|^2\Big)\\
&\le \varepsilon\|\theta_{xt}\|^2+C_2(\|\theta_x\|^2+\|v_x\|^2+\|v_{tx}\|^2+\|\theta_t\|^2)+C_2(\|v_x\|^4_{L^4}+\|\theta_t\|^4_{L^4})\\
&+C_2(\|g_t\|^2+\|g_\xi\|^2)\\
&\le \varepsilon\|\theta_{xt}\|^2+C_2(\|\theta_x\|^2+\|v_x\|^2+\|v_{tx}\|^2+\|\theta_t\|^2)\\
&+C_2(\|v_x\|^3\|v_{xx}\|+\|v_x\|^4+\|\theta_t\|^3\|\theta_{tx}\|+\|\theta_t\|^4)+C_2(\|g_t\|^2+\|g_\xi\|^2)\\
&\le \varepsilon\|\theta_{xt}(t)\|^2+C_2(\|\theta_x\|^2+\|v_x\|^2+\|v_{tx}\|^2+\|\theta_t\|^2)+C_2(\|g_t\|^2+\|g_\xi\|^2).
\end{aligned} \tag{4.15}
$$

Taking $\varepsilon>0$ small enough, integrating (4.15) over $[0,t]$, and using Lemmas 2.1-2.8, Lemmas 4.1-4.3, we can get (4.12).
(1.3) can be rewritten as

$$\theta_{xx}=\frac{u}{k}\Big(c_v\theta_t+k\frac{\theta_xu_x}{u^2}+R\frac{\theta v_x}{u}-\mu\frac{v_x^2}{u}-g\Big).$$

Thus

$$\begin{aligned}
\|\theta_{xx}(t)\|^2 &\le C_2(\|\theta_t\|^2+\|\theta_x u_x\|^2+\|\theta v_x\|^2+\|v_x\|^4_{L^4[0,1]}+\|g\|^2)\\
&\le C_2(\|\theta_t\|^2+\|\theta_x\|^2_{L^\infty[0,1]}\|u_x\|^2+\|\theta\|^2_{L^\infty[0,1]}\|v_x\|^2+\|v_x\|^2+\|v_x\|^3\|v_{xx}\|+\|g\|^2)\\
&\le \varepsilon\|\theta_{xx}\|^2+C_2(\|\theta_t\|^2+\|\theta_x\|^2+\|v_x\|^2+\|g\|^2) \qquad (4.16)
\end{aligned}$$

which, combine with (4.12), implies

$$\|\theta_{xx}(t)\| \le C_2(\|\theta_t(t)\|+1)\le C_2.$$

Differentiating (1.3) with respect to x, we can derive

$$\begin{aligned}
\theta_{xxx} &= \frac{u}{k}\Big(c_v\theta_{tx}+k\frac{\theta_{xx}u_x}{u^2}+(k\frac{\theta_x u_x}{u^2}+R\frac{\theta v_x}{u}-\mu\frac{v_x^2}{u}-g)_x\Big)\\
&= \frac{u}{k}\Big(c_v\theta_{tx}+k\frac{\theta_{xx}u_x}{u^2}+k\frac{\theta_{xx}u_x+\theta_x u_{xx}}{u^2}\\
&\quad -2k\frac{\theta_x u_x^2}{u^3}-R\frac{\theta_x v_x+\theta v_{xx}}{u}+R\frac{\theta v_x u_x}{u^2}+2\mu\frac{v_x v_{xx}}{u}-\mu\frac{v_x^2 u_x}{u^2}+g_x\Big). \qquad (4.17)
\end{aligned}$$

Integrating (4.17) over $[0,1]\times[0,t]$, we get

$$\begin{aligned}
\int_0^t\|\theta_{xxx}(s)\|^2ds &\le C_2\int_0^t(\|\theta_{tx}\|^2+\|u_x\|^2_{L^\infty}\|\theta_{xx}\|^2+\|\theta_x\|^2_{L^\infty}\|u_{xx}\|^2\\
&\quad +\|\theta_x\|^2_{L^\infty}\|u_x\|^2_{L^\infty}\|u_x\|+\|v_x\|^2_{L^\infty}\|\theta_x\|^2+\|\theta\|^2_{L^\infty}\|v_{xx}\|^2\\
&\quad +\|\theta\|^2_{L^\infty}\|v_x\|^2_{L^\infty}\|u_x\|^2+\|v_x\|^2_{L^\infty}\|v_{xx}\|^2\\
&\quad +\|v_x\|^2_{L^\infty}\|u_x\|^2_{L^\infty}\|v_x\|^2+\|g_\xi\|^2)ds\\
&\le \int_0^t(\|\theta_{tx}\|^2+\|\theta_{xx}\|^2+\|u_{xx}\|^2\\
&\quad +\|u_x\|^2+\|\theta_x\|^2+\|v_{xx}\|^2+\|v_x\|^2+\|g_\xi\|^2)ds
\end{aligned}$$

which, along with (4.8), (4.12), Lemmas 2.1-2.8, Lemmas 4.1-4.3, gives estimate (4.14).□

5 Asymptotic Behavior in H^2

In this section we will complete the proof of (1.15) and assume that the assumptions in Theorem 1.2 are valid.

Lemma 5.1 *The following estimate holds*

$$\lim_{t\to+\infty}\|u(t)-\bar{u}\|_{H^2}=0 \qquad (5.1)$$

where $\bar{u}=\int_0^1 u(x,t)dx=\int_0^1 u_0(x)ds$.

Proof Differentiating (1.1) with respect to twice x, multiplying the result by u_{xx}, then integrating it over $[0,1]$, by Young's inequality, we can deduce

$$\frac{d}{dt}\|u_{xx}\|^2 \leq \|v_{xxx}\|^2 + \|u_{xx}\|^2 \leq \frac{1}{2} + \frac{1}{2}\|u_{xx}\|^4 + \|v_{xxx}\|^2$$

which, along with Lemmas 3.1 and 4.3, gives

$$\lim_{t\to+\infty} \|u_{xx}(t)\|^2 = 0. \tag{5.2}$$

□

Lemma 5.2 *The following estimate holds*

$$\lim_{t\to+\infty} \|v(t)\|_{H^2} = 0. \tag{5.3}$$

Proof Noting

$$\frac{d}{dt}\|f\|^2 = \frac{d}{dt}\int_0^1 f^2 dx = 2\int_0^1 f \cdot \frac{df}{dt} dx \leq C_2(\|f\|^2 + \|f_t\|^2 + \|f_\xi\|^2). \tag{5.4}$$

By (5.4), (1.9) and Lemma 3.1, we can conclude

$$\lim_{t\to+\infty} \|f(t)\|^2 = 0. \tag{5.5}$$

Similarly, we can get

$$\lim_{t\to+\infty} \|g(t)\|^2 = 0. \tag{5.6}$$

By (3.4), (3.5), we have

$$\begin{aligned} \hat{p}_x &= \Big(R\frac{\theta}{u}\Big)_x - f = R\frac{\theta_x}{u} - R\frac{\theta u_x}{u^2} - f \\ &= -v_t + \mu\Big(\frac{v_x}{u}\Big)_x. \end{aligned} \tag{5.7}$$

Hence

$$\|\hat{p}_x(t)\| \leq C_2(\|\theta_x\| + \|u_x\| + \|f\|)$$

which implies

$$\lim_{t\to+\infty} \|\hat{p}_x(t)\|^2 = 0. \tag{5.8}$$

Differentiating (5.7) with respect to t, multiplying the result by v_t, then integrating it over $[0,1]$, and integrating by parts, using(1.1),(1.7)-(1.10), we can get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v_t(t)\|^2 + \int_0^1 \mu\frac{v_{xt}^2}{u}dx &= \int_0^1 \hat{p}_t v_{tx} dx + \int_0^1 \mu\frac{v_x^2 v_{tx}}{u^2}dx \\ &\leq \varepsilon\|v_{tx}\|^2 + \|\hat{p}_t\|^2 + \|v_x\|_{L^4}^4 \\ &\leq \varepsilon\|v_{tx}\|^2 + \|\theta_t\|^2 + \|\frac{df}{dt}\|^2 + \|\theta\|_{L^\infty}^2\|v_x\|^2 + \|v_x\|^3\|v_{xx}\| + \|v_x\|^4. \end{aligned}$$

Thus for small $\varepsilon > 0$,

$$\frac{d}{dt}\|v_t(t)\|^2 + \|v_{tx}(t)\|^2 \le C_2(\|\theta_t(t)\|^2 + \|v_x(t)\|^2 + \|f_t\|^2 + \|f_\xi\|^2). \tag{5.9}$$

Hence we infer from (5.9), (1.14), (1.9) and Lemma 3.1,

$$\lim_{t\to+\infty} \|v_t(t)\|^2 = 0. \tag{5.10}$$

By (5.7) and (1.14), we can conclude

$$\|v_{xx}(t)\| \le C_2(\|v_t(t)\| + \|\hat{p}_x(t)\| + \|v_x(t)\|)$$

which with (5.10), (5.8), (1.13) gives

$$\lim_{t\to+\infty} \|v_{xx}(t)\|^2 = 0. \tag{5.11}$$

Thus (5.3) follows from (5.11) and (1.13). □

Lemma 5.3 *There holds that*

$$\lim_{t\to+\infty} \|\theta(t) - \bar{\theta}\|_{H^2} = 0. \tag{5.12}$$

Proof By (4.15), we can get

$$\begin{aligned} & \frac{d}{dt}\|\theta_t(t)\|^2 + \|\theta_{xt}(t)\|^2 \\ \le\ & C_2(\|\theta_x\|^2 + \|v_x\|^2 + \|v_{tx}\|^2 + \|\theta_t\|^2) + C_2\|\frac{dg}{dt}\|^2 \\ \le\ & C_2(\|\theta_x(t)\|^2 + \|v_x(t)\|^2 + \|v_{tx}(t)\|^2 + \|\theta_t(t)\|^2) + C_2(\|g_t\|^2 + \|g_\xi\|^2) \end{aligned} \tag{5.13}$$

which combines with (1.14) and Lemma 3.1, we can conclude

$$\lim_{t\to+\infty} \|\theta_t(t)\| = 0. \tag{5.14}$$

By (4.16), we get

$$\|\theta_{xx}(t)\|^2 \le C_2(\|\theta_t(t)\|^2 + \|\theta_x(t)\|^2 + \|v_x(t)\|^2 + \|g\|^2) \tag{5.15}$$

which with (5.6), (5.13), (1.13) gives

$$\lim_{t\to+\infty} \|\theta_{xx}(t)\| = 0. \tag{5.16}$$

Thus (5.12) follows from (5.16) and (1.13). □

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