A BLOW-UP CRITERION FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we obtain a blow up criterion for strong solutions to the 3-D compressible Naveri-Stokes equations just in terms of the gradient of the velocity, similar to the Beal-Kato-Majda criterion for the ideal incompressible flow. The key ingredients in our analysis are the a priori super-norm estimate of the momentum by a Moser-iteration and an estimate of the space-time square mean of the gradient of the density. In addition, initial vacuum is allowed in our case.

1. introduction

We shall study the following isentropic compressible $Navier-Stokes$ equations in $3-\mathcal{D}$ case:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathrm{u}) = 0, \\ \dfrac{(\partial \rho u)}{\partial t} + \mathrm{div}(\rho \mathrm{u} \otimes \mathrm{u}) - \mu \triangle \mathrm{u} - (\mu+\lambda)\nabla(\mathrm{div} \mathrm{u}) + \nabla \mathrm{P}(\rho) = 0 \end{cases} \tag{1.1}$$

Where ρ, u, P denotes the density, velocity and pressure respectively. The pressure-density state equation is given by

$$P(\rho) = a\rho^{\gamma} \quad (a > 0, \gamma > 1) \tag{1.2}$$

μ and λ are shear viscosity and bulk viscosity respectively satisfying the physical condition:

$$\mu > 0, \lambda + \frac{2}{N}\mu \geq 0 \tag{1.3}$$

Lions [1] [2], Feireisl [3][11] et. established the global existence of weak solutions to the problem $(1.1)-(1.3)$, where vacuum is allowed initially. The global existence to the compressible Navier-Stokes equations is obtained by Matsumura[18]and Nishida under the condition that the initial data is a small perturbation of a non-vacuum constant. It is also shown by Xin[21] that there is no global in time regular solutions in R^3 to the compressible Naiver-Stokes equations provided the initial density is compactly supported.

There are many results concerning the existence of strong solutions to the Navier-Stokes equations, only local existence results have been established, see [15], [16],[17],[20]. V.A.Solonnikov proved in [19] that for C^2 pressure laws and initial data satisfies for some $q > N$,

$$0 < m \leq \rho_0(x) \leq M < \infty, \quad and \quad \rho_0 \in W^{1,q}(T^N) \tag{1.4}$$

$$u_0 \in W^{2-\frac{2}{q},q}(T^N)^N \tag{1.5}$$

there exists a local unique strong solution (ρ, u) to $(1.5)-(1.6)$ for periodic data, such that

$$\begin{aligned} &\rho \in L^\infty(0,T;W^{1,q}(T^N)), \quad \rho_t \in L^q((0,T)\times T^N) \\ &u \in L^q(0,T;W^{2,q}(T^N)), \quad u_t \in L^q((0,T)\times T^N)^N \end{aligned} \tag{1.6}$$

Later, it was shown in [15] that if Ω is either a bounded domain or the whole space, the initial data ρ_0 and u_0 satisfy

$$0 \leq \rho_0 \in W^{1,\tilde{q}}(\Omega), \quad u_0 \in H_0^1(\Omega)\cap H^2(\Omega) \tag{1.7}$$

for some $\tilde{q} \in (3,\infty)$ and the compatibility condition:

$$-\mu\triangle u_0 - (\lambda+\mu)\nabla \mathrm{div} \mathrm{u}_0 + \nabla \mathrm{P}(\rho_0) = \rho_0^{1/2}\mathrm{g} \quad \text{for} \quad \text{some} \quad \mathrm{g} \in \mathrm{L}^2(\Omega) \tag{1.8}$$

then there exists a positive time $T_1 \in (0,\infty)$ and a unique strong solution (ρ, u) to the isentropic problem, such that

$$\begin{aligned} &\rho \in C([0,T_1];W^{1,q_0}(\Omega)), \quad \rho_t \in C([0,T_1];L^{q_0}(\Omega)) \\ &u \in C([0,T_1];D_0^1\cap D^2(\Omega))\cap L^2(0,T_1;D^{2,q_0}(\Omega)) \\ &u_t \in L^2(0,T_1;D_0^1(\Omega)) \quad \sqrt{\rho}u_t \in L^\infty(0,T_1;L^2(\Omega)) \end{aligned} \tag{1.9}$$

Furthermore, one has the following blow-up criterion: if T^* is the maximal time of existence of the strong solution (ρ, u) and $T^* < \infty$, then

$$\sup_{t\to T^*}(\|\rho\|_{W^{1,q_0}} + \|u\|_{D_0^1}) = \infty \tag{1.10}$$

where $q_0 = \min(6,\tilde{q})$.

Throughout this paper, we use the following notations for the standard homogeneous and inhomogeneous Sobolev spaces.

$$D^{k,r}(\Omega) = \{u \in L^1_{loc}(\Omega) : \|\nabla^k u\|_{L^r} < \infty\},$$

$$W^{k,r} = L^r \cap D^{k,r}, H^k = W^{k,2}, \quad D^k = D^{k,2}$$

$$D_0^1 = \{u \in L^6(\Omega) : \|\nabla u\|_{L^2} < \infty \quad \textit{and} \quad u = 0 \quad \textit{on} \quad \partial\Omega\},$$

$$H_0^1 = L^2 \cap D_0^1, \quad \|u\|_{D^{k,r}} = \|\nabla^k u\|_{L^r}$$

Very recently, Fan and Jiang [14] proved a blow-up criterion for such strong solutions. i.e, when $7\mu > 9\lambda$,

$$\lim_{T\to T^*}\{\sup_{0\le T<T^*}\|\rho\|_{L^\infty} + \int_0^T (\|\rho\|_{W^{1,q_0}} + \|\nabla\rho\|_{L^2}^4)dt\} = \infty \tag{1.11}$$

Here they only require a sufficient regularity of density ρ to admit the global existence of strong solutions, as (1.11) revealed.

In this paper, we assume that

$$\begin{aligned} &\mu+\lambda \ge 0, \quad N = 2, \quad \Omega = T^2 \\ &\mu+\lambda = 0, \quad N = 3, \quad \Omega \subset R^3 \end{aligned} \tag{1.12}$$

Here and thereafter C always denotes a generic constant depending only on Ω, T and initial data.

For the initial boundary value problem, we have the following result:

Theorem 1.1. *Let Ω be either a three dimensional bounded domain or two dimensional torus. $Q_T = (0,T)\times\Omega$. Assume that the initial data satisfy $(1.7)-(1.8)$ and (1.12) holds. Let (ρ,u) be a strong solution of the problem $(1.1)-(1.3)$ satisfying the regularity (1.9). If $T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T\to T^*}\int_0^T \|\nabla u\|_{L^\infty(\Omega)}dt = \infty \tag{1.13}$$

In the case of initial value problem, it holds that

Theorem 1.2. *Let $\Omega = R^3$. Assume that the initial data satisfy*

$$\rho_0 \in H^1(R^3)\cap W^{1,\tilde q}(R^3), \quad u_0 \in D_0^1(R^3)\cap D^2(R^3) \tag{1.14}$$

for some $\tilde q$ with $3 < \tilde q < \infty$ and the compatibility condition (1.8). Let (ρ,u) be a strong solution to the problem $(1.1)-(1.3)$, and satisfy

$$\begin{aligned} &\rho \in C([0,T_1], H^1(R^3)\cap W^{1,q_0}(R^3)), \quad \rho_t \in C([0,T_1], L^2(R^3)\cap L^{q_0}(R^3)) \\ &u \in C([0,T_1], D_0^1(R^3)\cap D^2(R^3))\cap L^2(0,T_1; D^{2,q_0}(R^3)) \\ &u_t \in L^2(0,T_1; D_0^1), \quad \sqrt{\rho}u_t \in L^\infty(0,T_1; L^2(R^3)) \end{aligned} \tag{1.15}$$

where $q_0 = \min(6, \tilde{q})$. *If* $T^* < \infty$ *is the maximal time of existence, then*

$$\lim_{T \to T^*} \int_0^T \|\nabla u\|_{L^\infty(\Omega)} dt = \infty \tag{1.16}$$

Remark 1.1 The blow up criterion (1.10) involves both the density and velocity. It may be natural to expect the higher regularity of velocity if density is regular enough. (1.11) shows that sufficient regularity of the gradient of density indeed guarantees the global existence of strong solutions. The main difficulty in our case is to control the gradient of density, which is not a priori known and coupled with the second derivative of velocity.

We develop some new estimates under the condition (2.1). In fact, the key estimates in our analysis are both L^∞ bound of ρu and $L^\infty(0, T; L^2(\Omega))$ norm of $\nabla \rho$. The Super-norm estimate for the momentum is obtained by a Moser-iteration based on the a priori energy bounds motivated by an analysis in [12]. To control the $L^\infty(0, T; L^2(\Omega))$ norm of $\nabla \rho$, our key observation is that the space-time square mean of the convection term $F = \rho u_t + \rho u \cdot \nabla u$ is controlled by that of $\nabla \rho$ (see Lemma 2.3). This, in turn, gives the desired $L^\infty(0, T; L^2(\Omega))$ estimate on $\nabla \rho$, and thus the $L^2(0, T; H^2(\Omega))$ of u. Then the higher order regularity can be obtained by using the equations and the compatibility condition (1.8).

Remark 1.2 There are many results concerning blow-up criteria of the incompressible flow. In their well-known paper [4], Beal-Kato-Majda established a blow-up criterion for the incompressible Euler equations. One can get global smooth solution if $\int_0^T \|\omega\|_{L^\infty} dt$ is bounded. It's worth noting that only the vorticity ω plays an important role in the global existence of smooth solutions. Moreover, as pointed out by Constantin[9], the solution is smooth if and only if $\int_0^T \|((\nabla u)\xi) \cdot \xi\|_{L^\infty}$ is bounded, where ξ is the unit vector in the direction of ω. It turns out that the solution becomes smooth either the asymmetric or symmetric part of ∇u is controlled. Later, Constantin[7], Fefferman and Majda showed a sufficient geometric condition to control the breakdown of smooth solutions of incompressible Euler involving the Lipschitz regularity of the direction of the vorticity. It is also shown by Constantin[8] and Fefferman that the solution of incompressible Navier-Stokes equations is smooth if the direction of vorticity is well behaved.

Recently, in[5], assuming that the added stress tensor is given in a proper form, and using an idea of J.-Y. Chemin and N. Masmoudi [6], Constantin, P. and Fefferman, C., Titi, E. S. and Zarnescu, A obtain a logarithmic bound for $\int_0^T \|\nabla u\|_{L^\infty} dt$. to conclude that the solution to Navier-Stokes-Fokker-Planck system exists for all time and is smooth.

In our paper, we establish a similar criterion to Beal-Kato-Majda. Our blow up criteria involve both the symmetric and asymmetric part of ∇u, as compressibility and vorticity are two key issues in the formation of singularity of compressible Navier-Stokes.

Remark 1.3 The condition (1.12) is assumed to obtain $L^\infty(0,T;L^\infty(\Omega))$ norm of the momentum ρu by a Moser-Iteration. However, in $2-D$ periodic case, using an estimate motivated by Desjardin[10], we can show (1.13) holds for natural physical constraint: $\mu+\lambda\geq 0$.

Remark 1.4 We will study the blow up criteria for smooth solution of compressible Navier-Stokes in another paper[13].

2. Proof of Theorem 1.1

Let (ρ, u) be a strong solution to the problem $(1.1)-(1.3)$. We assume that the opposite holds, i.e

$$\lim_{T\to T^*}\int_0^T \|\nabla u\|_{L^\infty(\Omega)} dt \leq C < \infty \tag{2.1}$$

First, the standard energy estimate yields

$$\sup_{0\leq t\leq T} \|\rho^{1/2}u(t)\|_{L^2} + \int_0^T \|u\|_{H^1}^2 dt \leq C, \quad 0\leq T<T^* \tag{2.2}$$

By assumption (2.1) and the conservation of mass, the L^∞ bounds of density follows immediately,

Lemma 2.1. *Assume that*

$$\int_0^T \|divu\|_{L^\infty} dt \leq C, \quad 0\leq T<T^* \tag{2.3}$$

then

$$\|\rho\|_{L^\infty(Q_T)} \leq C, \quad 0\leq T<T^* \tag{2.4}$$

Proof. It follows from the conservation of mass that for $\forall q>1$,

$$\partial_t(\rho^q) + \mathrm{div}(\rho^{\mathrm{q}}\mathrm{u}) + (\mathrm{q}-1)\rho^{\mathrm{q}}\mathrm{divu} = 0 \tag{2.5}$$

Integrate (2.5) over Ω to obtain,

$$\partial_t \int_\Omega \rho^q dx \leq (q-1)\|\nabla u\|_{L^\infty(\Omega)} \int_\Omega \rho^q dx \tag{2.6}$$

i.e

$$\partial_t \|\rho\|_{L^q} \leq \frac{q-1}{q} \|\nabla u\|_{L^\infty(\Omega)} \|\rho\|_{L^q} \tag{2.7}$$

Which implies immediately

$$\|\rho\|_{L^q}(t) \leq C \tag{2.8}$$

with C independent of q, so our lemma follows.

□

In the next proposition, we derive bound on L^∞ norm of momentum ρu.

Proposition 2.2. *Under condition* (2.3)*, it holds that*

$$\|\rho u\|_{L^\infty(Q_T)} \leq C(\|\rho_0\|_{L^\infty}, \|u_0\|_{L^\infty}, \|\nabla u\|_{L^1 L^\infty}, T), \quad 0 \leq T < T^* \tag{2.9}$$

Proof. Let p be a fixed positive large number. Obviously,

$$\int_\Omega \rho_0 |u_0|^{p+2} dx \leq c_0^{p+2} \tag{2.10}$$

Without losing of generality, we assume $\mu = 1$.

Multiplying $|u|^p u$ on both sides of the momentum equations in (1.1) yields

$$\begin{aligned} &\frac{1}{p+2}\frac{d}{dt}\int_\Omega \rho |u|^{p+2} dx + \int_\Omega |\nabla u|^2 |u|^p dx + p\int_\Omega |u|^p (\nabla |u|)^2 dx \\ &= \int_\Omega a\rho^\gamma |u|^p divu dx + p\int_\Omega a\rho^\gamma |u|^{p-1} u \cdot \nabla |u| dx \\ &= I + II \end{aligned} \tag{2.11}$$

First, it follows from (2.4) and Hölder inequality that

$$\begin{aligned} |I| &\leq C(\int_\Omega |u|^p |\nabla u|^2 dx)^{1/2} (\int_\Omega \rho |u|^{p+2} dx)^{\frac{p}{2(p+2)}} \\ &\leq \frac{1}{2}\int_\Omega |u|^p |\nabla u|^2 dx + C(\int_\Omega \rho |u|^{p+2} dx)^{\frac{p}{p+2}} \end{aligned} \tag{2.12}$$

Similarly,

$$\begin{aligned} |II| &\leq Cp(\int_\Omega |u|^p |\nabla |u||^2 dx)^{1/2} (\int_\Omega \rho |u|^{p+2} dx)^{\frac{p}{2(p+2)}} \\ &\leq \frac{p}{2}\int_\Omega |u|^p (\nabla |u|)^2 + Cp(\int_\Omega \rho |u|^{p+2} dx)^{\frac{p}{p+2}} \end{aligned} \tag{2.13}$$

Therefore,

$$\begin{aligned}&\frac{1}{p+2}\partial_t\int_\Omega \rho|u|^{p+2}dx+\frac{1}{2}\int_\Omega |u|^p|\nabla u|^2dx+\frac{p}{2}\int_\Omega |u|^p(\nabla|u|)^2dx\\&\leq C(p+1)(\int_\Omega \rho|u|^{p+2}dx)^{\frac{p}{p+2}}\end{aligned}\tag{2.14}$$

Integrating (2.14) over $(0,T)$ yields

$$\begin{aligned}&\frac{1}{p+2}\int_\Omega \rho|u|^{p+2}(T)dx+\frac{1}{2}\int_{Q_T} |u|^p|\nabla u|^2dx+\frac{p}{2}\int_{Q_T} |u|^p(\nabla|u|)^2dx\\&\leq C(p+1)\int_0^T(\int_\Omega \rho|u|^{p+2}dx)^{\frac{p}{p+2}}dt+\frac{1}{p+2}\int_\Omega\rho_0|u_0|^{p+2}dx\\&\leq C(p+1)(\int_{Q_T}\rho|u|^{p+2}dxdt)^{\frac{p}{p+2}}T^{\frac{2}{p+2}}+\frac{1}{p+2}\int_\Omega\rho_0|u_0|^{p+2}dx\\&\leq C(p+1)(\int_{Q_T}\rho|u|^{p+2}dxdt)^{\frac{p}{p+2}}+\frac{1}{p+2}c_0^{p+2}\end{aligned}\tag{2.15}$$

It follows that

$$\int_\Omega \rho|u|^{p+2}(T)dx\leq C(p,c_0,T)$$

Moreover, it follows from Hölder inequality and (2.15) that

$$\begin{aligned}\int_{Q_T}\rho|u|^{\frac{5}{3}(p+2)}dxdt&=\int_{Q_T}(\rho^{\frac{3}{2}}|u|^{p+2})^{\frac{2}{3}}|u|^{p+2}dxdt\\&\leq\int_0^T(\int_\Omega\rho^{\frac{3}{2}}|u|^{p+2}dx)^{\frac{2}{3}}(\int_\Omega|u|^{3(p+2)}dx)^{\frac{1}{3}}dt\\&\leq C\int_0^T(\int_\Omega\rho|u|^{p+2}dx)^{\frac{2}{3}}(\int_\Omega|u|^{3(p+2)})^{\frac{1}{3}}dt\\&\leq C(\sup_{0\leq t\leq T}\int_\Omega\rho|u|^{p+2}dx)^{\frac{2}{3}}\int_0^T(\int_\Omega(|u|^{\frac{p+2}{2}})^6dx)^{\frac{1}{3}}dt\\&\leq C(\sup_{0\leq t\leq T}\int_\Omega\rho|u|^{p+2}dx)^{\frac{2}{3}}\int_0^T\int_\Omega(\nabla|u|^{\frac{p+2}{2}})^2dxdt\\&\leq (C(p+2)^2(\int_{Q_T}\rho|u|^{p+2})^{\frac{p}{p+2}}+c_0^{p+2})^{\frac{5}{3}}\\&\leq C(p+2)^{\frac{10}{3}}(\int_{Q_T}\rho|u|^{p+2}dxdt)^{\frac{5}{3}}+C(p+2)^{\frac{10}{3}}+C_0^{\frac{5}{3}(p+2)}\end{aligned}\tag{2.16}$$

Here $C_0=2c_0$. Set

$$r=\frac{5}{3},p+2=r^k,c_2=\frac{10}{3},c_1=C(|\rho_0|_{L^\infty},|u_0|_{L^\infty},\|\nabla u\|_{L^1L^\infty},T)\tag{2.17}$$

We conclude from above the following reverse Holder inequality.

$$\int_{Q_T} \rho|u|^{r^{k+1}} dxdt \le c_1 r^{c_2 k} (\int_{Q_T} \rho|u|^{r^k} dxdt)^r + c_1 r^{c_2 k} + C_0^{r^{k+1}} \tag{2.18}$$

Define

$$A(k) = \int_{Q_T} \rho|u|^{r^k} dxdt, \quad f(k) = c_1 r^{c_2 k}, \quad g(k) = C_0^{r^k} \tag{2.19}$$

Then (2.18) could be written as

$$A(k+1) \le f(k)A(k)^r + f(k) + g(k)^r \tag{2.20}$$

Write

$$B(k) = max(A(k), g(k), 1), \quad F(k) = 3f(k) \tag{2.21}$$

Without lose of generality, we assume that $f(k) \ge 1$. Hence

$$\begin{aligned} B(k+1) &= max(A(k+1), g(k+1), 1) \\ &\le max(f(k)A(k)^r + f(k) + g(k)^r, g(k)^r, 1) \\ &\le max(f(k)B(k)^r + f(k)B(k)^r + f(k)B(k)^r, g(k)^r, 1) \\ &\le 3f(k)B(k)^r = F(k)B(k)^r \end{aligned} \tag{2.22}$$

By induction, we obtain

$$\begin{aligned} B(k+1) &\le F(k)B(k)^r \\ &\le F(k)(F(k-1)B(k-1)^r)^r = F(k)F(k-1)^r B(k-1)^{r^2} \\ &\le \dots\dots \\ &\le F(k)F(k-1)^r F(k-2)^{r^2} \dots F(2)^{r^{k-2}} B(2)^{r^k} \end{aligned} \tag{2.23}$$

Consequently,

$$\begin{aligned} B(k+1)^{\frac{1}{r^{k+1}}} &\le F(k)^{\frac{1}{r^{k+1}}} F(k-1)^{\frac{1}{r^k}} \dots F(2)^{\frac{1}{r^3}} B(2)^{\frac{1}{r^2}} \\ &= (3c_1)^{\sum_{i=3}^{k+1} r^{-i}} r^{c_2 \sum_{i=0}^{k-2} (k-i) r^{i-k-1}} B(2)^{\frac{1}{r^2}} \end{aligned} \tag{2.24}$$

And

$$\sum_{i=3}^{k+1} r^{-i} + \sum_{i=0}^{k-2} (k-i) r^{i-k-1} < \infty \tag{2.25}$$

$$B(2) = max(A(2), g(2), 1) = max((\int_{Q_T} \rho|u|^{r^2} dxdt)^{\frac{1}{r^2}}, C_0^{r^2}, 1) < \infty \tag{2.26}$$

By definition

$$(\int_{Q_T} \rho|u|^{r^k} dxdt)^{\frac{1}{r^k}} \le B(k)^{\frac{1}{r^k}} \le C \tag{2.27}$$

Thus we can conclude , $\forall q>1$, there exists a k,α with $0<\alpha\leq 1$, $r^{k-1}\leq q<r^k$

$$\|\rho u\|_{L^q(Q_T)}\leq \|\rho u\|^{\alpha}_{L^{r^{k-1}}(Q_T)}\|\rho u\|^{1-\alpha}_{L^{r^k}(Q_T)}$$

Note that

$$(\int_{Q_T}\rho^{r^k}|u|^{r^k}dxdt)^{\frac{1}{r^k}}\leq \|\rho\|^{\frac{r^k-1}{r^k}}_{L^\infty(Q_T)}(\int_{Q_T}\rho|u|^{r^k}dxdt)^{\frac{1}{r^k}}\leq C$$

Consequently,

$$\|\rho u\|_{L^\infty(Q_T)}\leq C(|\rho_0|_{L^\infty},|u_0|_{L^\infty},\|\nabla u\|_{L^1L^\infty},T) \tag{2.28}$$

Moreover, for $\alpha,\beta>0$, $p\geq 1$

$$\|\rho^\alpha|u|^\beta\|_{L^\infty(Q_T)}\leq C(\alpha,\beta),\quad \|\rho^\alpha|u|^\beta\|_{L^p(Q_T)}\leq C(\alpha,\beta,p) \tag{2.29}$$

□

The next lemma shows a connection between a convection term and the gradient of the density, which will play an important role in deriving the desired bounds on $\nabla\rho$.

Lemma 2.3. *Let $F=\rho u_t+\rho u\cdot\nabla u$. Then it holds that*

$$\int_{Q_T}F^2dxdt\leq C\int_{Q_T}|\nabla\rho|^2dxdt+C,\quad 0\leq T<T^*$$

Proof. Note that

$$\int_{Q_T}F^2dxdt\leq C^*(\|\rho\|_{L^\infty(Q_T)})\int_{Q_T}\rho u_t^2dxdt+2\int_{Q_T}|\rho u\cdot\nabla u|^2dxdt \tag{2.30}$$

It follows from proposition 2.1 that

$$\int_{Q_T}F^2dxdt\leq C^*(\|\rho\|_{L^\infty(Q_T)})\int_{Q_T}\rho u_t^2dxdt+C(\|\rho u\|^2_{L^\infty(Q_T)})\int_{Q_T}|\nabla u|^2dxdt \tag{2.31}$$

Multiplying the momentum equation by u_t and integrating show that

$$\int_\Omega\rho u_t^2dx+\int_\Omega\rho u\cdot\nabla u\cdot u_tdx+\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla u|^2dx=\int_\Omega P\mathrm{divu_tdx} \tag{2.32}$$

Note that

$$\int_\Omega P\mathrm{divu_tdx}=\partial_\mathrm{t}\int_\Omega\mathrm{Pdivudx}-\int_\Omega\mathrm{P_tdivudx}, \tag{2.33}$$

and

$$P_t+\mathrm{div(Pu)}+(\gamma-1)\mathrm{Pdivu}=0$$

One gets

$$\begin{aligned}\int_\Omega P\mathrm{divu_t dx} &= \partial_t \int_\Omega P\mathrm{divudx} + \int_\Omega \mathrm{div(Pu)divudx} + (\gamma-1)\int_\Omega \mathrm{P(divu)^2dx} \\ &= \partial_t \int_\Omega P\mathrm{divudx} - \int_\Omega \mathrm{(Pu)}\cdot\nabla\mathrm{divudx} + (\gamma-1)\int_\Omega \mathrm{P(divu)^2dx}\end{aligned} \tag{2.34}$$

Combing (2.32) and (2.34) yields

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla u|^2 dx(T) + \int_{Q_T} \rho u_t^2 dxdt + \int_{Q_T} \rho u\cdot\nabla u\cdot u_t dxdt \\ &= \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx + \int_\Omega P\mathrm{divudx(T)} - \int_\Omega \mathrm{P_0 divu_0 dx} \\ &- \int_{Q_T} Pu\cdot\nabla\mathrm{divudxdt} + (\gamma-1)\int_{\mathrm{Q_T}} \mathrm{P(divu)^2dxdt}\end{aligned} \tag{2.35}$$

Direct estimates show that

$$\int_\Omega P\mathrm{divudx(T)} \le \frac{1}{4}\int_\Omega \mathrm{|\nabla u|^2dx(T) + C} \tag{2.36}$$

$$\begin{aligned}\int_{Q_T} \rho u\cdot\nabla u\cdot u_t dxdt &\le \frac{1}{2}\int_{Q_T} \rho u_t^2 + \int_{Q_T} \rho|u\cdot\nabla u|^2 dxdt \\ &\le \frac{1}{2}\int_{Q_T} \rho u_t^2 + C(\|\rho|u|^2\|_{L^\infty(Q_T)})\int_{Q_T} |\nabla u|^2 dxdt \\ &= \frac{1}{2}\int_{Q_T} \rho u_t^2 + C\end{aligned} \tag{2.37}$$

On the other hand, it follows from $F = \triangle u - \nabla P$ and (2.2) that

$$\begin{aligned}\int_{Q_T} Pu\cdot\nabla\mathrm{divudxdt} &= \int_{Q_T} Pu\cdot\nabla\mathrm{div}\triangle^{-1}\nabla\mathrm{Pdxdt} + \int_{\mathrm{Q_T}} \mathrm{Pu}\cdot\nabla\mathrm{div}\triangle^{-1}\mathrm{Fdxdt} \\ &\le C\int_{Q_T} |\nabla\rho|^2 dxdt + \epsilon\int_{Q_T} F^2 dxdt + C\end{aligned} \tag{2.38}$$

Consequently,

$$\int_{Q_T} \rho u_t^2 dxdt + \frac{1}{2}\int_\Omega |\nabla u|^2 dx(T) \le C\int_{Q_T} |\nabla\rho|^2 dxdt + 2\epsilon\int_{Q_T} F^2 dxdt + C \tag{2.39}$$

Choosing ϵ as $2C^*\epsilon < 1$, we may conclude

$$\int_{Q_T} F^2 dxdt \le C\int_{Q_T} |\nabla\rho|^2 dxdt + C$$

Which completes the proof of Lemma 2.3.

$\square$

We are now ready to show the desired $L^\infty(0,T;L^2(\Omega))$ estimate of $\nabla\rho$.

Proposition 2.4. *Under the assumption* (2.1), *it holds that*

$$\sup_{0\le t\le T}\int_\Omega |\nabla\rho|^2dx \le C \quad ,0\le T<T^* \tag{2.40}$$

$$\int_{Q_T}\rho u_t^2dxdt+\sup_{0\le t\le T}\int_\Omega |\nabla u|^2dx \le C \quad ,0\le T<T^* \tag{2.41}$$

$$\int_0^T \|u\|^2_{H^2(\Omega)}dt\le C, \quad 0\le T<T^* \tag{2.42}$$

Proof. Differentiating the mass equation in (1.1) with respect to x_i,

$$\partial_t(\partial_i\rho)+\mathrm{div}(\partial_\mathrm{i}\rho \mathrm{u})+\mathrm{div}(\rho\partial_\mathrm{i}\mathrm{u})=0$$

Which can be multiplied by $2\partial_i\rho$ to obtain

$$\partial_t|\partial_i\rho|^2+\mathrm{div}(|\partial_\mathrm{i}\rho|^2\mathrm{u})+|\partial_\mathrm{i}\rho|^2\mathrm{divu}+2\partial_\mathrm{i}\rho\rho\partial_\mathrm{i}\mathrm{divu}+2\partial_\mathrm{i}\rho\partial_\mathrm{i}\mathrm{u}\cdot\nabla\rho=0 \tag{2.43}$$

Integrating over Ω and using $\mathrm{divu}=\mathrm{div}\triangle^{-1}\nabla\mathrm{P}+\mathrm{div}\triangle^{-1}\mathrm{F}$ show that

$$\begin{aligned}\partial_t\int_\Omega|\partial_i\rho|^2dx&=-\int_\Omega|\partial_i\rho|^2\mathrm{divudx}-2\int_\Omega\rho\partial_\mathrm{i}\rho\partial_\mathrm{i}\mathrm{div}\triangle^{-1}\nabla\mathrm{Pdx}\\&-\int_\Omega 2\rho\partial_i\rho\partial_i\mathrm{div}\triangle^{-1}\mathrm{Fdx}-\int_\Omega 2\partial_\mathrm{i}\rho\partial_\mathrm{i}\mathrm{u}\cdot\nabla\rho\mathrm{dx}\\&=-(A_1+A_2+A_3+A_4)\end{aligned} \tag{2.44}$$

Each term on the right hand side of (2.44) can be estimated as follows:

$$|A_1(t)|\le\|divu\|_{L^\infty}(t)\int_\Omega|\partial_i\rho|^2dx\le\|divu\|_{L^\infty}(t)\int_\Omega|\nabla\rho|^2dx \tag{2.45}$$

$$|A_2(t)|\le C\|\nabla\rho\|_{L^2}\|\nabla P\|_{L^2}\le C\int_\Omega|\nabla\rho|^2dx \tag{2.46}$$

$$|A_3(t)|\le C\|\nabla\rho\|_{L^2}\|F\|_{L^2} \tag{2.47}$$

$$|A_4(t)|\le C\|\nabla u\|_{L^\infty}(t)\int_\Omega|\nabla\rho|^2dx \tag{2.48}$$

Consequently,

$$\partial_t\int_\Omega|\nabla\rho|^2dx\le C(\|\nabla u\|_{L^\infty}(t)+1)\int_\Omega|\nabla\rho|^2dx+C\int_\Omega F^2dx \tag{2.49}$$

This, together with Gronwall's inequality yields

$$\begin{aligned}\int_\Omega |\nabla\rho|^2 dx(t) &\le Ce^{C\int_0^t(\|\nabla u\|_{L^\infty}(s)+1)ds}\Big(\int_\Omega |\nabla\rho_0|^2dx+\int_0^t\Big(\int_\Omega F^2(s)dx\Big)e^{-C\int_0^s(\|\nabla u\|_{L^\infty}(\tau)+1)d\tau}ds\Big)\\ &\le C\int_0^t\int_\Omega F^2dxds+C\\ &\le C\int_0^t\int_\Omega |\nabla\rho|^2dxds+C\end{aligned}\tag{2.50}$$

Hence

$$\sup_{0\le t\le T}\int_\Omega |\nabla\rho|^2dx\le C\tag{2.51}$$

Next, it follows from (2.31), (2.39) and (2.51) that

$$\int_{Q_T}\rho u_t^2dxdt+\sup_{0\le t\le T}\int_\Omega|\nabla u|^2dx\le C\tag{2.52}$$

This, together with $\triangle u=\rho u_t+\rho u\cdot\nabla u+\nabla P$, shows that

$$\begin{aligned}\|u\|_{L^2(0,T;H^2(\Omega))}&\le \|\rho u_t\|_{L^2(Q_T)}+\|\rho u\cdot\nabla u\|_{L^2(Q_T)}+\|\nabla P\|_{L^2(Q_T)}\\ &\le C+C\|\nabla u\|_{L^2(Q_T)}+C\|\nabla\rho\|_{L^2(Q_T)}\le C\end{aligned}\tag{2.53}$$

□

Next, we proceed to improve the regularity class of ρ and u. To this end, we first derive some bounds on derivatives of u based on above estimates.

Proposition 2.5. *Under the condition* (2.1), *it holds that*

$$\sup_{0\le t\le T}\|\rho^{1/2}u_t(t)\|_{L^2}^2+\int_{Q_T}|\nabla u_t|^2dxdt\le C,\quad 0\le T<T^*\tag{2.54}$$

$$\sup_{0\le t\le T}\|u\|_{H^2}\le C,\quad 0\le T<T^*\tag{2.55}$$

Proof. Differentiating the momentum equations in (1.1) with respect to time t yields

$$\rho u_{tt}+\rho u\cdot\nabla u_t-\triangle u_t+\nabla p_t=-\rho_t(u_t+u\cdot\nabla u)-\rho u_t\cdot\nabla u\tag{2.56}$$

Taking the inner product of the above equation with u_t in $L^2(\Omega)$ and integrating by parts, one gets

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\frac12\rho u_t^2dx+\int_\Omega|\nabla u_t|^2dx-\int_\Omega P_t\mathrm{divu_t dx}\\ &=-\int_\Omega(\rho u\cdot\nabla[(u_t+u\cdot\nabla u)u_t]+\rho(u_t\cdot\nabla u)\cdot u_t)dx\end{aligned}\tag{2.57}$$

The last term on the left-hand side of (2.57) can be rewritten as (using (2.33)):

$$
\begin{aligned}
-\int_\Omega P_t \mathrm{divu_t dx} = {} & \frac{d}{dt}\int_\Omega \frac{\gamma}{2}P(\mathrm{divu})^2\mathrm{dx} + \int_\Omega \nabla\mathrm{P}\cdot(\mathrm{udivu_t})\mathrm{dx} \\
& + \frac{\gamma}{2}\int_\Omega (-Pu\cdot\nabla(\mathrm{divu})^2 + (\gamma-1)\mathrm{P}(\mathrm{divu})^3)\mathrm{dx}
\end{aligned}
\tag{2.58}
$$

It follows from (2.57) and (2.58) that

$$
\begin{aligned}
& \frac{d}{dt}\int_\Omega (\frac{1}{2}\rho u_t^2 + \frac{\gamma}{2}P(\mathrm{divu})^2)\mathrm{dx} + \int_\Omega |\nabla \mathrm{u_t}|^2\mathrm{dx} \\
& \le \int_\Omega (2\rho|u||u_t||\nabla u_t| + \rho|u||u_t||\nabla u|^2 + \rho|u|^2|u_t||\nabla^2 u| + \rho|u|^2||\nabla u||\nabla u_t| \\
& + \rho|u_t|^2|\nabla u| + |\nabla P||u||\nabla u_t| + \gamma P|u||\nabla u||\nabla^2 u| + \gamma^2 P|\nabla u|^3)dx \\
& \equiv \sum_{i=0}^{8} F_i
\end{aligned}
\tag{2.59}
$$

Now, we estimate each F_i separately, where the Sobolev inequality and Hölder inequality will be frequently used.

$$
\begin{aligned}
|F_1| &= \int_\Omega 2\rho|u||u_t||\nabla u_t|dx \\
&\le C\|\rho^{1/2}u\|_{L^\infty(Q_T)}\|\rho^{1/2}u_t\|_{L^2}\|\nabla u_t\|_{L^2} \\
&\le \epsilon\|\nabla u_t\|_{L^2}^2 + C\|\rho^{1/2}u_t\|_{L^2}^2
\end{aligned}
\tag{2.60}
$$

Due to (2.40) and (2.41), one has

$$
\sup_{0\le t\le T}\int_\Omega |\nabla u|^2 dx \le C
$$

Thus, it follows from Hölder inequality, Sobolev imbedding and interpolation inequality that

$$
\begin{aligned}
|F_2| &= \int_\Omega \rho|u||u_t||\nabla u|^2 dx \\
&\le C\|\rho^{1/2}u\|_{L^\infty(Q_T)}\int_\Omega \rho^{1/2}|u_t||\nabla u|^2 dx \\
&\le C\|\rho^{1/2}u_t\|_{L^3}\|\nabla u\|_{L^3}^2 \\
&\le C\|\rho^{1/2}u_t\|_{L^2}^{1/2}\|\rho^{1/2}u_t\|_{L^6}^{1/2}\|\nabla u\|_{L^2}\|\nabla u\|_{L^6} \\
&\le \epsilon\|\rho^{1/2}u_t\|_{L^2}\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^6}^2 \\
&\le \epsilon\|\nabla u_t\|_{L^2}^2 + C\|\rho^{1/2}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 \\
&\le \epsilon\|\nabla u_t\|_{L^2}^2 + C\|\rho^{1/2}u_t\|_{L^2}^2 + C\|u\|_{H^2}^2,
\end{aligned}
\tag{2.61}
$$

$$\begin{aligned}|F_3| &= \int_\Omega \rho|u|^2|u_t||\nabla^2 u|dx \\ &\le C\|\rho^{1/2}u_t\|_{L^2}\|\nabla^2 u\|_{L^2} \\ &\le C\|\rho^{1/2}u_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2,\end{aligned} \tag{2.62}$$

$$\begin{aligned}|F_4| &= \int_\Omega \rho|u|^2|\nabla u||\nabla u_t|dx \\ &\le \epsilon\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2,\end{aligned} \tag{2.63}$$

$$\begin{aligned}|F_5| &= \int_\Omega \rho|u_t|^2|\nabla u|dx \\ &\le C\|\rho u_t^2\|_{L^2}\|\nabla u\|_{L^2} \\ &\le C\|\rho^{1/2}u_t\|_{L^4}^2 \\ &\le \epsilon\|\rho^{1/2}u_t\|_{L^6}^2 + C\|\rho^{1/2}u_t\|_{L^2}^2 \\ &\le \epsilon\|u_t\|_{L^6}^2 + C\|\rho^{1/2}u_t\|_{L^2}^2 \\ &\le \epsilon\|\nabla u_t\|_{L^2}^2 + C\|\rho^{1/2}u_t\|_{L^2}^2,\end{aligned} \tag{2.64}$$

$$\begin{aligned}|F_6| &= \int_\Omega |\nabla P||u||\nabla u_t|dx \\ &\le C\int_\Omega |\nabla\rho||\nabla u_t|dx \\ &\le \epsilon\|\nabla u_t\|_{L^2}^2 + C\|\nabla\rho\|_{L^2}^2,\end{aligned} \tag{2.65}$$

$$\begin{aligned}|F_7| &= \int_\Omega \gamma P|u||\nabla u||\nabla^2 u|dx \\ &\le C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2},\end{aligned} \tag{2.66}$$

Finally, noting that $\nabla u \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;L^\infty(\Omega))$,

$$L^3(Q_T) \subset L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;L^\infty(\Omega))$$

Hence

$$\begin{aligned}|F_8| &= \int_\Omega \gamma^2 P|\nabla u|^3 dx \\ &\le C\int_\Omega |\nabla u|^3 dx \\ &\le C\|\nabla u\|_{L^\infty(\Omega)}\int_\Omega |\nabla u|^2 dx \\ &\le C\|\nabla u\|_{L^\infty(\Omega)},\end{aligned} \tag{2.67}$$

Collecting all the estimates for F_i, we conclude

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega}(\frac{1}{2}\rho u_t^2+\frac{\gamma}{2}p(\mathrm{divu})^2)\mathrm{dx}+\int_{\Omega}|\nabla \mathrm{u_t}|^2\mathrm{dx}\\&\leq 5\epsilon\int_{\Omega}|\nabla u_t|^2dx+C(\|\rho^{1/2}u_t\|_{L^2}^2+\|u\|_{H^2}^2+\|\nabla\rho\|_{L^2}^2+\|\nabla u\|_{L^\infty})\end{aligned}\tag{2.68}$$

Thanks to the compatibility condition:

$$\rho_0(x)^{\frac{1}{2}}(\rho_0(x)^{\frac{1}{2}}u_t(t=0,x)+\rho_0^{\frac{1}{2}}u_0\cdot\nabla u_0(x)-\rho_0^{\frac{1}{2}}g)=0\tag{2.69}$$

it holds that

$$\rho_0(x)^{\frac{1}{2}}u_t(t=0,x)=\rho_0^{\frac{1}{2}}u_0\cdot\nabla u_0(x)-\rho_0^{\frac{1}{2}}g\in L^2(\Omega)\tag{2.70}$$

Therefore, for arbitrary small ϵ, (2.68) yields

$$\sup_{0\leq t\leq T}\|\rho^{1/2}u_t(t)\|_{L^2}^2+\int_{Q_T}|\nabla u_t|^2dxdt\leq C,\quad 0\leq T<T^*\tag{2.71}$$

Moreover,

$$\triangle u=\rho u_t+\rho u\cdot\nabla u+\nabla P\in L^\infty L^2$$

Hence,

$$\sup_{0\leq T<T^*}\|u\|_{H^2}^2\leq C\tag{2.72}$$

Our lemma follows immediately.

□

Finally, the following lemma gives bounds of derivatives of the density and the second derivatives of the velocity.

Lemma 2.6. *Under the condition* (2.1)*, it holds that*

$$\sup_{0\leq t\leq T}(\|\rho_t(t)\|_{L^{q_0}}+\|\rho\|_{W^{1,q_0}})\leq C,\quad 0\leq T<T^*$$

$$\int_0^T\|u(t)\|_{W^{2,q_0}}^2dt\leq C,\quad 0\leq T<T^*,q_0=min(6,\tilde{q})$$

Proof. It follows from (2.71) and (2.72) that

$$u_t\in L^2(0,T;L^6(\Omega)),\nabla u\in L^6(Q_T)$$

$$F\in L^2(0,T;L^6(\Omega))$$

Differentiating the mass equation in (1.1) with respect to x_i, one gets

$$\partial_t(\partial_i\rho)+\mathrm{div}(\partial_\mathrm{i}\rho\mathrm{u})+\mathrm{div}(\rho\partial_\mathrm{i}\mathrm{u})=0$$

Multiplying above identity by $q_0|\partial_i\rho|^{q_0-2}\partial_i\rho$, one gets that for some $q_0 = min(6, \tilde{q})$,

$$\begin{aligned}&\partial_t|\partial_i\rho|^{q_0} + \mathrm{div}(|\partial_i\rho|^{q_0}u) + (q_0-1)|\partial_i\rho|^{q_0}\mathrm{div}u\\&+ q_0|\partial_i\rho|^{q_0-2}\partial_i\rho\rho\partial_i\mathrm{div}u + q_0|\partial_i\rho|^{q_0-2}\partial_i\rho\partial_i u\cdot\nabla\rho = 0\end{aligned} \tag{2.73}$$

This, together with $\mathrm{div}u = \mathrm{div}\triangle^{-1}\nabla P + \mathrm{div}\triangle^{-1}F$ shows that

$$\begin{aligned}\partial_t\int_\Omega|\partial_i\rho|^{q_0}dx &= -(q_0-1)\int_\Omega|\partial_i\rho|^{q_0}\mathrm{div}u\,dx - q_0\int_\Omega\rho|\partial_i\rho|^{q_0-2}\partial_i\rho\partial_i\mathrm{div}\triangle^{-1}\nabla P dx\\&-\int_\Omega q_0\rho|\partial_i\rho|^{q_0-2}\partial_i\rho\partial_i\mathrm{div}\triangle^{-1}F dx - q_0\int_\Omega|\partial_i\rho|^{q_0-2}\partial_i\rho\partial_i u\cdot\nabla\rho dx\\&= -(B_1+B_2+B_3+B_4)\end{aligned} \tag{2.74}$$

Which can be estimated as:

$$|B_1(t)| \le (q_0-1)\|\nabla u\|_{L^\infty}(t)\int_\Omega|\partial_i\rho|^{q_0}dx \le C\|\nabla u\|_{L^\infty}(t)\int_\Omega|\nabla\rho|^{q_0}dx \tag{2.75}$$

$$|B_2(t)| \le C\||\nabla\rho|^{q_0-1}\|_{L^{\frac{q_0}{q_0-1}}}\|\nabla P\|_{L^{q_0}} \le C\int_\Omega|\nabla\rho|^{q_0}dx \tag{2.76}$$

$$\begin{aligned}|B_3(t)| &\le C\||\nabla\rho|^{q_0-1}\|_{L^{\frac{q_0}{q_0-1}}}\|F\|_{L^{q_0}}\\&\le C\|\nabla\rho\|_{L^{q_0}}^{q_0-1}\|F\|_{L^{q_0}}\end{aligned} \tag{2.77}$$

$$|B_4(t)| \le C\|\nabla u\|_{L^\infty}(t)\int_\Omega|\nabla\rho|^{q_0}dx \tag{2.78}$$

It follows from (2.74) – (2.78) that

$$\partial_t\|\nabla\rho\|_{L^{q_0}} \le C(\|\nabla u\|_{L^\infty}(t)+1)\|\nabla\rho\|_{L^{q_0}} + C\|F\|_{L^{q_0}} \tag{2.79}$$

Hence,

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^{q_0}} \le C$$

Therefore, due to this, (2.71) and interpolation inequality, one has

$$\rho_t = -(u\cdot\nabla\rho + \rho\mathrm{div}u) \in L^\infty L^{q_0} \tag{2.80}$$

Recall that

$$\triangle u = F + \nabla P \in L^2L^{q_0}$$

gives

$$\int_0^T \|u\|^2_{W^{2,q_0}(\Omega)} dt \leq C \tag{2.81}$$

This will close the estimates and guarantee to have an extension of the strong solution.

In fact, in view of Proposition 2.4 – 2.5 and lemma 2.6, the functions $(\rho, u)|_{t=T^*} = \lim_{t\to T^*}(\rho, u)$ satisfy the conditions imposed on the initial data $(1.7)-(1.8)$ at the time $t = T^*$ Furthermore,

$$\rho^{\frac{1}{2}} u_t + \rho^{\frac{1}{2}} u \cdot \nabla u \in L^\infty L^2$$

$$-\triangle u + \nabla P|_{t=T^*} = \lim_{t\to T^*} \rho^{\frac{1}{2}}(\rho^{\frac{1}{2}} u_t + \rho^{\frac{1}{2}} u \cdot \nabla u) \triangleq \rho^{\frac{1}{2}} g|_{t=T^*} \tag{2.82}$$

Where $g|_{t=T^*} \in L^2(\Omega)$. Therefore, we can take $(\rho, u)|_{t=T^*}$ as the initial data and apply the local existence theorem [15] to extend our local strong solution beyond T^*. This contradicts the assumption on T^*.

□

Note that after some minor modifications, the above ideas also works in both periodic case and $\Omega = R^3$, so theorem 1.2 holds.

When $\Omega = T^2$, using an estimate from Desjardin[10],

$$\begin{aligned}
&\int_0^T (\|\rho^{\frac{1}{2}} u_t\|^2_{L^2(T^2)} + \|\rho^{\frac{1}{2}} u \cdot \nabla u\|^2_{L^2(T^2)}) ds + \sup_{0\leq t\leq T} \|\nabla u(t,)\|^2_{L^2(T^2)} \\
&\leq \exp(C \exp(\int_0^T \|\rho\|_{L^\infty(T^2)} \delta(s) ds))
\end{aligned} \tag{2.83}$$

Where

$$\delta(s) = 1 + \|\nabla u\|_{L^2(T^2)} \in L^1(0, T) \tag{2.84}$$

it follows from $(2.83) - (2.84)$ that

$$\int_{Q_T} F^2 dx dt + \sup_{0\leq t\leq T} \|\nabla u\|^2_{L^2(T^2)} \leq C \tag{2.85}$$

Then we can do a similar estimate step by step as three dimensional case to obtain a higher regularity of (ρ, u). We omit the detail for simplicity.

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