On the Dynamics of Navier-Stokes Equations for a Shallow Water Model

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Abstract. In this paper, we study a free boundary problem of one-dimensional compressible Navier-stokes equations with a density-dependent viscosity, which include, in particular, a shallow water model. Under some suitable assumptions on the initial data, we obtain the global existence, uniqueness and the large time behavior of weak solutions. In particular, it is shown that a stationary wave pattern connecting a gas to the vacuum continuously is asymptotically stable for small initial general perturbations.

1 Introduction

We study a free boundary problem of one-dimensional compressible Navier-stokes equations with a density-dependent viscosity, which can be written in Eulerian coordinates as

$$\begin{cases} \rho_\tau + (\rho u)_\xi = 0, \\ (\rho u)_\tau + (\rho u^2 + P(\rho))_\xi = (\mu(\rho) u_\xi)_\xi - \rho g, \end{cases} \tag{1}$$

in a domain $O := \{(\xi,\tau) : 0 < \xi < l(\tau), \tau > 0\}$ with the boundary function l satisfying

$$l'(\tau) = u|_{\xi=l(\tau)}, \qquad for\ \ \tau > 0, \tag{2}$$

where ρ, u, $P(\rho) = A\rho^2$ and g are the density, velocity, pressure and gravitational constant respectively, and the viscosity $\mu(\rho)$ is assumed to be $\mu(\rho) = C\rho$, A and $C > 0$ are constants.

In this paper, the initial conditions are

$$(\rho, u)|_{\tau=0} = (\rho_0, u_0)(\xi), \qquad for\ \ \xi \in (0, l_0),\ \ l|_{\tau=0} = l_0, \tag{3}$$

and the boundary conditions are

$$u|_{\xi=0} = 0, \rho|_{\xi=l(\tau)} = 0, \qquad for\ \ \tau > 0. \tag{4}$$

These systems include, in particular, a simple model of the one-dimensional shallow water system describeing vertically averaged flows in three–dimensional shallow domains in term of the mean velocity u and the variation of the depth ρ due to the free surface, which is widely used in geophysical flows [8]. Our main concern here is the global existence, uniqueness and the large time behavior of weak solutions to the above initial boundary value problem.

Let $(\rho_\infty, 0)$ be a stationary solution to the equation (1) with the boundary conditions (4). Then

$$(P(\rho_\infty))_\xi = -\rho_\infty g, \tag{5}$$

in an interval $\xi \in (0, l_\infty)$ with the end l_∞ satisfying

$$\rho_\infty(l_\infty) = 0, \tag{6}$$

$$\int_0^{l_\infty} \rho_\infty d\xi = m := \int_0^{l_0} \rho_0 d\xi. \tag{7}$$

It follows from (5) that there exists a unique solution (ρ_∞, l_∞) to the stationary system $(5) - (7)$ satisfying $(\rho_\infty)_\xi < 0$ and $l_\infty < +\infty$.

When the viscosity $\mu(\rho)$ is a positive constant, there have been many investigations on the compressible Navier-Stokes equations for sufficiently smooth data or discontinuous initial data, one-dimensional or multi-dimensional problems. For instance, the one-dimensional problem was addressed by Kazhikhov in [15] for the sufficiently smooth data, and by Serre in [27] and Hoff [11] for discontinuous initial data where the data were uniformly away from the vacuum. In [21], Matsumura and Nishida showed the global existence and the large-time behavior of solutions for sufficiently small data in multi-dimensional case if the data were small perturbation of an uniform non-vacuum state. However, for large data, many important problems, for example, the existence of global solution in the case of heat-conducting gases and the uniqueness of weak solutions are still open. The first general result was obtained by Lions in [17], in which he used the method of weak convergence to obtain global weak solutions provided the specific heat ratio γ is appropriately large, for example, $\gamma \geq \dfrac{3N}{N+2}$, $N = 2, 3$. There have been many generalizations of this results, see [11, 17, 31] and references therein. The free boundary problem for the one-dimensional compressible Navier-Stokes equations were investigated in [1, 2, 24], where the global existence of weak solutions was proved. In particular, in [19], Luo, Xin and Yang studied the regularity and the behavior of solutions near the interfaces between the gas and vacuum, and give a quite precise description on growth rate of the free boundary.

However, the studies in [12, 18, 30] showed that the compressible Navier-Stokes equations with constant viscosity coefficients behave singularity in the presence of vacuum. As pointed out in [18], one of the main reasons for this came from the independence of the kinematic viscosity coefficient on the density. In fact, if one derives the compressible Navier-Stokes equations from the Boltzmann equation by exploiting Chapman-Enskog expansion up to the second order, as in [9], one can find the viscosity is not constant but a function of the temperature. For isentropic flows, this dependence is translated to the dependence on the density by the law of Boyle and Gay-Lussac for ideal gas as discussed in Liu [18].

In recent years, there have been many studies for the compressible Navier-Stokes equations with the density-dependent viscosity in both one-dimensional and higher dimensional setting. Bresch, Desjardins, and Lin [3] showed the L^1 stability of weak solutions for the Korteweg system with the Korteweg stress tensor $\kappa\rho\nabla\triangle\rho$, and their result was later improved in [4] to include the case of vanishing capillarity ($\kappa = 0$) but with an additional quadratic friction term $r\rho|u|u$. In their papers, a new entropy estimate is established which provided some high order regularity for the density [3, 4]. Recently, Mellet and Vasseuer [22] proved the L^1 stability results of [3, 4] for the case $r = \kappa = 0$. Nevertheless, the global existence of weak solutions of the compressible Navier-Stokes equations with density-dependent viscosity is still open in the multi-dimensional cases except for spherical symmetric case, see [10]. The key issue now is how to construct approximate solutions satisfying the a priori estimates required in the L^1 stability analysis. It seems highly nontrivial to do so due to the degeneracy of the viscosities near vacuum and the additional entropy inequality to be held in constructions of approximate solutions.

In contrast to the higher dimensional case, there are fruitful studies in the one-dimensional setting. Suppose that $\mu = c\rho^\theta$ with c and θ being positive constants. When the initial density connects to vacuum with discontinuities, Liu, Xin and Yang obtained the local existence of weak solutions to Navier-Stokes equations with vacuum [20]. The global existence and uniqueness of weak solutions when $0 < \theta < 1/3$ was obtained by Okada in [25]. Later, it was generalized to the cases for $0 < \theta < 1/2$ and $0 < \theta < 1$ in [32, 13] respectively. Very recently, if the initial density is bounded away from zero (no vacuum), Mellet and Vasseur proved the existence and uniqueness of global strong solutions in [23] for $0 < \alpha < 1/2$. One of the key estimates for all these results is the uniform positive lower bound for densities in constructions of approximate solutions.

If the density function connects to vacuum continuously, there is no positive lower bound for the density function and the viscosity coefficient vanishes at vacuum. This degeneracy in the viscosity coefficient gives rise to new difficulties in the analysis. A local existence result was obtained in [33] with $\theta > 1/2$, global existence results were studied by [6, 26, 28, 29, 34]. In [28], Zhang and Fang obtained the global existence and uniqueness of weak solution when the initial data is a small perturbation to the stationary solution as long as $\theta \in (0, \gamma - 1) \bigcap (0, \gamma/2]$, where $\gamma > 1$ is the adiabatic constant of polytropic gas and also proved the weak solution tended to the stationary one. For $\mu(\rho) = \rho^\theta$ $(\theta > 1/2)$, Li, Li and Xin in [16] studied this case for both bounded spatial domains or periodic domains and showed that for any global entropy weak solution, any (possibly existing) vacuum state must vanish within finite time, and furthermore, after the vanishing of vacuum states, the global entropy weak solution becomes a strong solution. These results were generalized in [14] to the Cauchy problem for one-dimensional compressible flows, where two cases were considered: the initial density assumed to be integrable on the whole real line and the deviation of the initial density from a positive constant density being integrable on the whole real line.

In this paper, we study the free boundary value problem $(1) - (4)$ and obtain the upper and lower bounds of the density function uniformly in time. Also we show that the upper bound of the velocity function is finite, and obtained one of the important features of this problem, that is, the interface separating the gas and vacuum propagates with finite speed. For the large time behavior, we can show that the solution tends to the stationary one as time tends to infinity. One of the key new ingredient is a new global space-time square estimate.

The rest of this paper is organized as follows. In section 2, we convert the free boundary problem to a fixed boundary problem by using Lagrangian transformation, then give the definition of weak solutions and state the main result in this paper. In section 3, we give a series of a prior estimates which will be used to obtain the global existence of weak solutions. In section 4, we will prove the uniqueness of the weak solution. Large time behavior of solutions is studied in section 5, where we show that the solution to the free boundary problem tends to a stationary one as time goes to infinity.

2 Formulation and main result

Since the free boundary $l(\tau)$ is a particle path, it can be converted to a fixed boundary in the lagrangian coordinates.

Introduce the following coordinates transformation

$$x = \int_0^{\xi} \rho(y,\tau)dy, \ \ t = \tau.$$

Set

$$X(\tau) = x(l(\tau),\tau) = \int_0^{l(\tau)} \rho(y,\tau)d\tau.$$

It follows from $(1)_1$, (3) and (4) that

$$\frac{dX(\tau)}{d\tau} = 0, \tag{8}$$

i.e., $X(\tau)$ is independent of τ.

Set

$$X = \int_0^{l_0} \rho(y,0)dy, \tag{9}$$

where $\int_0^{l_0} \rho(y,0)dy$ is the total mass.

Rescaling if necessary, the problem(1), (3) and (4) can be transformed to the following fixed boundary problem:

$$\begin{cases} \rho_t + \rho^2 u_x = 0, & t > 0, \\ u_t + (\rho^2)_x = (\rho^2 u_x)_x - g, & 0 < x < 1, \end{cases} \tag{10}$$

with the boundary conditions

$$u(0,t) = \rho(1,t) = 0, \ \ t \geq 0, \tag{11}$$

and the initial data

$$(\rho, u)(x,0) = (\rho_0(x), u_0(x)), \ \ x \in [0,1]. \tag{12}$$

Let $\rho_\infty(x)$ be the solution of the following stationary problem,

$$\begin{cases} (\rho_\infty^2)_x = -g, \\ \rho_\infty(1) = 0 \end{cases} \tag{13}$$

Then

$$\rho_\infty(x) = [g(1-x)]^{1/2}. \tag{14}$$

Throughout this paper, the initial data will be assumed to satisfy:

(A_1): $[N_1(1-x)]^{1/2} \leq \rho_0 \leq [N_2(1-x)]^{1/2}$, with some positive constant $0 < N_1 \leq N_2$ and $(1-x)^{1/2}(\rho_0)_x^2 \in L^1([0,1])$;

(A_2): $u_0 \in H^1([0,1]), u_0(0) = 0$;

(A_3): $(\rho_0^2(x) u_{0x})_x \in L^2([0,1])$.

Under assumptions $(A_1) - (A_3)$, we will prove the existence of a global weak solution to the initial boundary value problem (10)-(12). The weak solution is defined bellow:

Definition 2.1 A pair $(\rho(x,t),\ u(x,t))$ is called a global weak solution to the initial boundary value problem $(10) - (12)$ if for any large $T > 0$,

$$\rho, u \in L^\infty([0,1] \times [0,T]) \bigcap C^1([0,T]; L^2(0,1)), \tag{15}$$

$$\rho^2 u_x \in L^\infty([0,1]\times[0,T])\bigcap C^{1/2}([0,T];L^2(0,1)), \tag{16}$$

furthermore, the following equalities hold:

$$\rho_t + \rho^2 u_x = 0, \ \ a.e. \tag{17}$$

and

$$\int_0^\infty \int_0^1 (u\phi_t + (P(\rho) - \mu(\rho)\rho u_x)\phi_x - g\phi)dxdt + \int_0^1 u_0(x)\phi(x,0)dx = 0, \tag{18}$$

for and test function $\phi(x,t) \in C_0^\infty(\Omega)$ with $\Omega = \{(x,t) : 0 < x \le 1, t > 0\}$.

In what follows, $C(C(T))$ will be used to denote a generic positive constant depending only on initial data (or the given time T).

The main result in this paper can be stated as follows:

Theorem 2.1 *Let $(A_1)-(A_3)$ hold. There is a constant $\epsilon_0 > 0$, such that if*

$$\|u_0\|_{L^2} \le \epsilon, \int_0^1 (1-x)^{-1/2}(\rho_0 - \rho_\infty)^2 dx \le \epsilon, \tag{19}$$

for all $\epsilon \in [0, \epsilon_0]$, the free boundary problem $(10)-(12)$ has a unique weak solution $(\rho(x,t), u(x,t))$ satisfying

$$C_1[(1-x)]^{1/2} \le \rho(x,t) \le C_2[(1-x)]^{1/2}, (x,t) \in Q, \tag{20}$$

$$\sup_{t\ge 0}(\|u\|_{L^2}) \le C, \tag{21}$$

$$\sup_{t\ge 0} \|(1-x)^{1/2}(\rho_x)^2\|_{L^1} \le C(T), \tag{22}$$

$$\|u\|_{L^2(Q)} + \|\rho u_x\|_{L^2(Q)} \le C, \tag{23}$$

$$\|u\|_{L^\infty(Q)} \le C(T), \tag{24}$$

where $Q = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$, C_1 and C_2 are independent of $t > 0$. Furthermore, it holds that

$$\lim_{t\to\infty} \int_0^1 [\frac{1}{2}u^2(x,t) + (\rho^2(x,t) - \rho_\infty^2(x))^2]dx = 0, \tag{25}$$

$$\lim_{t\to\infty} \|\rho - \rho_\infty(\cdot,t)\|_{L^q} = 0, \forall q \in [1,\infty). \tag{26}$$

3 Some a priori estimates

In this section, we will use a standard finite difference approximation to obtain the existence of the weak solution. For this purpose, we first derive some a priori estimates to obtain the desired estimates on the solution. The key point is to obtain the uniform lower bound of the density function $\rho(x,t)$.

First, we derive some elementary equalities which follow from the equation directly. These equalities will be used frequently later.

Lemma 3.1 *Under the condition of Theorem 2.1, it holds that for* $0 < x < 1$, $t > 0$,

$$\rho_t = -\rho^2 u_x(x,t), \tag{27}$$

$$(\rho^2 u_x)(x,t) = \rho^2(x,t) - g(1-x) - \int_x^1 u_t(y,t)dy, \tag{28}$$

and

$$\rho(x,t) = \rho_0(x) + g(1-x)t + \int_0^t \int_x^1 u_t(y,s)dyds - \int_0^t \rho^2(x,s)ds. \tag{29}$$

Proof: (27) and (28) follow from (10) and (11) directly, where (29) is obtained by integrating (27) and then using (28).

Lemma 3.2 *(Uniform Energy estimate) Under the condition of Theorem 2.1, one has*

$$\int_0^1 (\frac{1}{2}u^2 + \int_{\rho_\infty}^{\rho} \frac{h^2 - \rho_\infty^2}{h^2} dh)dx + \int_0^t \int_0^1 \rho^2 {u_x}^2 dxds \le C\epsilon, \tag{30}$$

where C is independent of $t \ge 0$.

Proof: Multiplying $(10)_2$ by u, integrating the resulting equation over $[0,1]\times[0,t]$, and using integration by parts, one can obtain

$$\frac{d}{dt}\int_0^1 (\frac{1}{2}u^2 + \rho)dx + \int_0^1 \rho^2 {u_x}^2 dx + \int_0^1 gudx = 0, \tag{31}$$

where the boundary condition (11) is used.

Set $r(x,t) = \int_0^x \rho^{-1}(y,t)dy$. Then $r_t = u$. Hence (31) becomes

$$\frac{d}{dt}\int_0^1 (\frac{1}{2}u^2 + \rho + gr)dx + \int_0^1 \rho^2 {u_x}^2 dx = 0, \tag{32}$$

and

$$\frac{d}{dt}\int_0^1 (\frac{1}{2}u^2 + \rho - 2\rho_\infty + gr)dx + \int_0^1 \rho^2 {u_x}^2 dx = 0. \tag{33}$$

Due to (14), one has

$$\begin{aligned}
\int_0^1 gr dx &= \int_0^1 g \int_0^x \rho^{-1}(y,t)dydx \\
&= g\int_0^1 \int_y^1 \rho^{-1}(y,t)dxdy \\
&= \int_0^1 \frac{\rho_\infty^2}{\rho}dx,
\end{aligned}$$

and so

$$\begin{aligned}
\int_0^1 (\rho - 2\rho_\infty + gr)dx &= \int_0^1 (\rho - 2\rho_\infty + \frac{\rho_\infty^2}{\rho})dx \\
&= \int_0^1 \int_{\rho_\infty}^{\rho} \frac{h^2 - \rho_\infty^2}{h^2} dh dx \geq 0,
\end{aligned}$$

this yields

$$\begin{aligned}
&\int_0^1 (\frac{1}{2}u^2 + \int_{\rho_\infty}^{\rho} \frac{h^2 - \rho_\infty^2}{h^2}dh)dx + \int_0^t \int_0^1 \rho^2 {u_x}^2 dxds \\
&= \int_0^1 (\frac{1}{2}{u_0}^2 + \rho_0 - 2\rho_\infty + gr_0)dx.
\end{aligned}$$

It follows from (A_1) and (14) that

$$\begin{aligned}
&\int_{\rho_\infty}^{\rho_0} \frac{h^2 - \rho_\infty^2}{h^2}dh \\
&\leq C(1-x)^{-1}(\rho_0 - \rho_\infty)(\rho_0^2 - \rho_\infty^2) \\
&\leq C(1-x)^{-\frac{1}{2}}(\rho_0 - \rho_\infty)^2.
\end{aligned}$$

Taking into account (19), we have

$$
\begin{aligned}
&|\int_0^1 (\rho_0 - 2\rho_\infty + gr_0)dx| \\
&= |\int_0^1 \int_{\rho_\infty}^{\rho_0} \frac{h^2 - \rho_\infty^2}{h^2} dh dx| \\
&\le C\int_0^1 (1-x)^{-\frac{1}{2}}(\rho_0 - \rho_\infty)^2 \le C\epsilon.
\end{aligned}
$$

Hence, we obtain

$$
\int_0^1 (\frac{1}{2}u^2 + \int_{\rho_\infty}^{\rho} \frac{h^2 - \rho_\infty^2}{h^2} dh)dx + \int_0^t \int_0^1 \rho^2 {u_x}^2 dxds \le C\epsilon.
$$

This concludes the proof. $\square$

The uniform energy estimate yields also the smallness of $\sup\limits_{t\ge 0} \|u\|_{L^2(0,1)}$. Based on this result, we will get the uniform lower bound for density function $\rho(x,t)$.

To this end, one needs the following lemma:

Lemma 3.3 *Let $f \in C(\mathbb{R})$ and $y, b \in W^{1,1}(0,T)$. Let y satisfy the following equation*

$$
D_t y = f(y) + D_t b, \quad on \ \mathbb{R}^+ \tag{34}
$$

and moreover, $|b(t_2) - b(t_1)| \le N_0$ for any $0 \le t_1 < t_2$. Then
(1) if $f(z) \ge 0$, for $z \le \underline{z}$,

$$
\min\{y(0), \underline{z}\} - N_0 \le y(t) \ \ on \ \mathbb{R}^+. \tag{35}
$$

(2) if $f(z) \le 0$, for $z \ge \bar{z}$, then

$$
y(t) \le \max\{y(0), \bar{z}\} + N_0 \ \ on \ \mathbb{R}^+. \tag{36}
$$

Now we can derive some uniform lower and upper bounds for $\rho(x,t)$.

Lemma 3.4 *Under the conditions in Theorem 2.1, we have*

$$
C_1[(1-x)]^{1/2} \le \rho(x,t) \le C_2[(1-x)]^{1/2}, \tag{37}
$$

where $(x,t) \in Q := \{(x,t) : 0 \le x \le 1, t \ge 0\}$*,* C_1 *and* C_2 *are independent of* $t \ge 0$.

Proof: Due to (27) and (28),

$$\rho_t = g(1-x) - \rho^2(x,t) + \int_x^1 u_t(y,t)dy.$$

Let $Y(x,t) = \rho(x,t)(1-x)^{-1/2}$. Then

$$Y_t(x,t) = g(1-x)^{1/2} - Y^2(1-x)^{1/2} + (1-x)^{-1/2}\int_x^1 u_t dy. \tag{38}$$

Let $b(x,t) = (1-x)^{-1/2}\int_x^1 udy$. Then for any $0 < t_1 < t_2$,

$$\begin{aligned}|b(x,t_2) - b(x,t_1)| &= |(1-x)^{-1/2}\int_x^1 u(y,t_2) - u(y,t_1)dy| \\ &\le 2(1-x)^{-1/2}(1-x)^{1/2}\sup_{t\ge 0}(\int_0^1 u^2 dy)^{1/2} \\ &\le 2(C\epsilon)^{\frac{1}{2}},\end{aligned}$$

Let $f(Y) = g(1-x)^{1/2} - Y^2(1-x)^{1/2}$. Then $f(Y) \ge 0$ if $Y(x,t) \le g^{1/2}$. Then (35) in Lemma 3.3 yields

$$Y(x,t) \ge \min\{Y_0, g^{1/2}\} - 2(C\epsilon)^{1/2},$$

with $Y_0 = \rho_0(1-x)^{-1/2} \ge N_1$. Hence there exists ϵ_0 such that for $\epsilon \in (0,\epsilon_0)$, $Y(x,t) \ge C_1 > 0$.

On the other hand , if $Y(x,t) \ge g^{1/2}$, then $f(Y) \le 0$, so by using (36) in Lemma 3.3, we have

$$Y(x,t) \le \max\{Y_0, g^{1/2}\} + 2(C\epsilon)^{1/2},$$

here $Y_0 = \rho_0(1-x)^{-1/2} \le N_2$. Hence $Y(x,t) \le C_2$. $\square$

Based on this, the desired global space-time square estimate can be obtained.

Corollary 3.5 *Under the conditions of Theorem 2.1, it holds that*

$$\int_0^t \int_0^1 u^2(x,t)dxdt \le C, \tag{39}$$

where C is independent of $t \geq 0$.

Proof: $\forall x \in (1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}})$, since $u(0,t)=0$, we have

$$|u(x,t)| \leq \int_0^x |u_y(y,t)|dy.$$

Using Hölder inequality, (37) and $0 \leq x \leq 1$, we obtain

$$\begin{aligned} |u(x,t)|^2 &\leq \int_0^x |u_y(y,t)|^2 dy \cdot x \\ &\leq \sum_{i=1}^{n+1} \int_{1-\frac{1}{2^{i-1}}}^{1-\frac{1}{2^i}} |u_y(y,t)|^2 \rho^2(y,t) dy \cdot 2^i. \end{aligned} \tag{40}$$

Integrating (40) over $[0,1]$ yields

$$\begin{aligned} \int_0^1 |u(x,t)|^2 dx &= \sum_{n=0}^{\infty} \int_{1-\frac{1}{2^n}}^{1-\frac{1}{2^{n+1}}} |u(x,t)|^2 dx \\ &\leq \sum_{n=0}^{\infty} \sum_{i=1}^{n+1} \int_{1-\frac{1}{2^n}}^{1-\frac{1}{2^{n+1}}} \int_{1-\frac{1}{2^{i-1}}}^{1-\frac{1}{2^i}} |u_y(y,t)|^2 \rho^2(y) dy \cdot 2^i dx \\ &\leq \sum_{n=0}^{\infty} \sum_{i=1}^{n+1} 2^{i-(n+1)} \int_{1-\frac{1}{2^{i-1}}}^{1-\frac{1}{2^i}} |u_y(y,t)|^2 \rho^2(y) dy \\ &= \sum_{i=1}^{\infty} \sum_{n=i-1}^{\infty} 2^{i-(n+1)} \int_{1-\frac{1}{2^{i-1}}}^{1-\frac{1}{2^i}} |u_y(y,t)|^2 \rho^2(y) dy \\ &\leq C \sum_{i=1}^{\infty} \int_{1-\frac{1}{2^{i-1}}}^{1-\frac{1}{2^i}} |u_y(y,t)|^2 \rho^2(y) dy \\ &= C \int_0^1 |u_y(y,t)|^2 \rho^2(y) dy. \end{aligned}$$

This together with (30) shows

$$\int_0^\infty \int_0^1 |u(x,t)|^2 dx dt \leq C \int_0^\infty \int_0^1 \rho^2 {u_x}^2 dx dt \leq C\epsilon, \tag{41}$$

where C is independent of $t \geq 0$. $\square$

Lemma 3.6 *Under the same condition as in Theorem 2.1, it holds that*

$$\int_0^1 (1-x)^{\frac{1}{2}} (\rho_x)^2 dx \leq C(T), \tag{42}$$

$$\int_0^t\int_0^1(\rho_x)^2dxds\le C(T). \tag{43}$$

Proof: it follows from $(10)_1$ that

$$\rho_{xt}=-(\rho^2u_x)_x=-g-(\rho^2)_x-u_t. \tag{44}$$

Multiplying (44) by $\rho_x(1-x)^{\frac{1}{2}}$ and then integrating the resulting equality with respect to x over $[0,1]$, one gets

$$\begin{aligned}&\frac{d}{dt}\int_0^1\frac{1}{2}(1-x)^{\frac{1}{2}}(\rho_x)^2dx\\&=-\int_0^1 g(1-x)^{\frac{1}{2}}\rho_xdx-2\int_0^1(\rho_x)^2(1-x)^{\frac{1}{2}}\rho dx\\&-\int_0^1 u_t\rho_x(1-x)^{\frac{1}{2}}dx.\end{aligned} \tag{45}$$

Integrating (45) with respect to t over $[0,t]$, and integrating by parts, one obtains by using $(10)_1$ that

$$\begin{aligned}&\int_0^1\frac{1}{2}(1-x)^{\frac{1}{2}}(\rho_x)^2dx=\int_0^1\frac{1}{2}(1-x)^{\frac{1}{2}}(\rho_{0x})^2dx-\int_0^t\int_0^1 g(1-x)^{\frac{1}{2}}\rho_xdxds\\&-2\int_0^t\int_0^1(\rho_x)^2(1-x)^{\frac{1}{2}}\rho dxds-\int_0^t\int_0^1 u_t\rho_x(1-x)^{\frac{1}{2}}dxds\\&=\int_0^1\frac{1}{2}(1-x)^{\frac{1}{2}}(\rho_{0x})^2dx-\int_0^t\int_0^1 g(1-x)^{\frac{1}{2}}\rho_xdxds\\&-2\int_0^t\int_0^1(\rho_x)^2(1-x)^{\frac{1}{2}}\rho dxds+\int_0^t\int_0^1(1-x)^{\frac{1}{2}}\rho^2{u_x}^2dxds\\&-\frac{1}{2}\int_0^t\int_0^1(1-x)^{-\frac{1}{2}}\rho^2u_xudx+\int_0^1 u_0\rho_{0x}(1-x)^{\frac{1}{2}}dx-\int_0^1 u\rho_x(1-x)^{\frac{1}{2}}dx.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_0^1\frac{1}{2}(1-x)^{\frac{1}{2}}(\rho_x)^2dx+2\int_0^t\int_0^1(\rho_x)^2(1-x)^{\frac{1}{2}}\rho dxds\\&=\int_0^1\frac{1}{2}(1-x)^{\frac{1}{2}}(\rho_{0x})^2dx-\int_0^t\int_0^1 g(1-x)^{\frac{1}{2}}\rho_xdxds\\&+\int_0^t\int_0^1(1-x)^{\frac{1}{2}}\rho^2{u_x}^2dxds-\frac{1}{2}\int_0^t\int_0^1(1-x)^{-\frac{1}{2}}\rho^2u_xudxds\\&-\int_0^1(1-x)^{\frac{1}{2}}u\rho_xdx+\int_0^1 u_0\rho_{0x}(1-x)^{\frac{1}{2}}dx.\end{aligned} \tag{46}$$

Using Young's inequality, we have

$$
\begin{aligned}
&\int_0^1 \frac{1}{2}(1-x)^{\frac{1}{2}}(\rho_x)^2dx + 2\int_0^t\int_0^1 (\rho_x)^2(1-x)^{\frac{1}{2}}\rho dxds\\
&\leq \int_0^1 \frac{1}{2}(1-x)^{\frac{1}{2}}(\rho_{0x})^2dx + \frac{1}{2}\int_0^t\int_0^1 g^2dxds\\
&+\frac{1}{2}\int_0^t\int_0^1(1-x)^{\frac{1}{2}}{\rho_x}^2dxds + \int_0^t\int_0^1(1-x)^{\frac{1}{2}}\rho^2{u_x}^2dxds\\
&+\frac{1}{4}\int_0^t\int_0^1(1-x)^{-1}\rho^2u^2dxds + \frac{1}{4}\int_0^t\int_0^1 \rho^2{u_x}^2dxds\\
&+\int_0^1 u^2dx + \frac{1}{4}\int_0^1(1-x)^{\frac{1}{2}}(\rho_x)^2dx + \frac{1}{2}\int_0^1 {u_0}^2dx + \frac{1}{2}\int_0^1 \rho_{0x}^2(1-x)^{\frac{1}{2}}dx,
\end{aligned}
$$

which together with (30), (37) and (39) shows

$$
\begin{aligned}
&\frac{1}{4}\int_0^1(1-x)^{\frac{1}{2}}(\rho_x)^2dx + 2\int_0^t\int_0^1 (\rho_x)^2(1-x)^{\frac{1}{2}}\rho dxds\\
&\leq C(T) + \frac{1}{2}\int_0^t\int_0^1(1-x)^{\frac{1}{2}}(\rho_x)^2dxds.
\end{aligned} \tag{47}
$$

By using Gronwall's inequality, we obtain

$$\int_0^1(1-x)^{\frac{1}{2}}(\rho_x)^2dx \leq C(T), \tag{48}$$

$$\int_0^t\int_0^1(\rho_x)^2dxds \leq C(T). \quad \Box \tag{49}$$

Lemma 3.7 *Under the same assumptions as in Theorem 2.1, we have*

$$\int_0^1 {u_t}^2dx + \int_0^t\int_0^1 \rho^2u_{xt}^2dxds \leq C(T). \tag{50}$$

Proof: It follows from $(10)_2$ that

$$u_{tt} + (\rho^2)_{xt} = (\rho^2u_x)_{xt}. \tag{51}$$

Multiply (51) by $2u_t$ and integrate over $[0,1]\times[0,t]$ to get

$$
\begin{aligned}
&\int_0^1(u_t)^2dx + \int_0^t\int_0^1 2(\rho^2)_{xt}u_tdxds\\
&= \int_0^t\int_0^1 2(\rho^2u_x)_{xt}u_tdxds + \int_0^1(u_{0t})^2dx.
\end{aligned} \tag{52}
$$

Due to $u_{0t}=(\rho_0^2 u_{0x})_x+g-(\rho_0^2)_x$, and from the initial assumptions (A_2) and (A_3),

$$\int_0^1 (u_{0t})^2 dx \le C, \tag{53}$$

then we get

$$\int_0^1 (u_t)^2 dx + \int_0^t\int_0^1 2(\rho^2)_{xt} u_t dxds \le \int_0^t\int_0^1 2(\rho^2 u_x)_{xt} u_t dxds + C. \tag{54}$$

Integrating by parts, and using $(10)_1$ and (11), one obtains

$$\begin{aligned}
&\int_0^t\int_0^1 (\rho^2 u_x)_{xt} u_t dxds = -\int_0^t\int_0^1 (\rho^2 u_x)_t u_{xt} dxds + \int_0^t (\rho^2 u_x)_t u_t|_0^1 ds \\
&= -\int_0^t\int_0^1 \rho^2 u_{xt}^2 dxds - 2\int_0^t\int_0^1 \rho_t \rho u_x u_{xt} dxds \\
&= -\int_0^t\int_0^1 \rho^2 u_{xt}^2 dxds + 2\int_0^t\int_0^1 \rho^3 u_x^2 u_{xt} dxds.
\end{aligned} \tag{55}$$

Similarly,

$$\begin{aligned}
\int_0^t\int_0^1 2(\rho^2)_{xt} u_t dxds &= -2\int_0^t\int_0^1 (\rho^2)_t u_{xt} dxds + 2\int_0^t (\rho^2)_t u_t|_0^1 ds \\
&= \int_0^t\int 4\rho^3 u_x u_{xt} dxds.
\end{aligned} \tag{56}$$

It follows from (54), (55), (56), and the Cauchy-Schwartz inequality that

$$\begin{aligned}
&\int_0^1 (u_t)^2 dx + 2\int_0^t\int_0^1 \rho^2 u_{xt}^2 dxds \\
&\le C + 4\int_0^t\int_0^1 \rho^3 u_x^2 u_{xt} dxds - 4\int_0^t\int_0^1 \rho^3 u_x u_{xt} dxds \\
&\le C + \frac{1}{2}\int_0^t\int_0^1 \rho^2 u_{xt}^2 dxds + 8\int_0^t\int_0^1 \rho^4 u_x^4 dxds \\
&+ \frac{1}{2}\int_0^t\int_0^1 \rho^2 u_{xt}^2 dxds + 8\int_0^t\int_0^1 \rho^4 u_x^2 dxds,
\end{aligned} \tag{57}$$

which, together (30) and (37), yields

$$\int_0^1 (u_t)^2 dx + \int_0^t\int_0^1 \rho^2 u_{xt}^2 dxds \le C + 8\int_0^t\int_0^1 \rho^4 u_x^4 dxds. \tag{58}$$

Since

$$\int_0^t\int_0^1 \rho^4 u_x^4 dxds \le \int_0^t \max_{[0,1]}(\rho^2 u_x^2(\cdot,s))V(s)ds, \tag{59}$$

here

$$V(s)=\int_0^1 \rho^2 u_x^2(x,s)dx,$$

and

$$\begin{aligned}\rho^2u_x^2 &= (\rho^2 u_x)^2\rho^{-2}\\ &= \rho^{-2}[\int_x^1(-g-(\rho^2)_x-u_t(y,t)dy)]^2\\ &\le C\rho^{-2}[g^2(1-x)^2+\rho^2+(1-x)\int_0^1 u_t^2dx].\end{aligned} \tag{60}$$

Thus,

$$\max_{[0,1]}(\rho^2u_x^2(\cdot,s)) \le C + C\int_0^1 u_t^2dx, \tag{61}$$

due to (37). Hence,

$$\begin{aligned}&\int_0^1 u_t^2dx + \int_0^t\int_0^1 \rho^2 u_{xt}^2dxds \le C + 8\int_0^t (C + C\int_0^1 u_t^2 dx)V(s)ds\\ &\le C + C\int_0^t V(s)ds + C\int_0^t V(s)\int_0^1 u_t^2dxds.\end{aligned} \tag{62}$$

The uniform energy estimate (30) implies

$$\int_0^t\int_0^1 \rho^2u_x^2dxds \le C,$$

and so

$$\int_0^1 u_t^2dx + \int_0^t\int_0^1 \rho^2 u_{xt}^2dxds \le C(1+\int_0^t V(s)(\int_0^1 u_t^2dx)ds). \tag{63}$$

Then using Gronwall's inequality yields that

$$\int_0^1 u_t^2dx \le C(T)\exp(C(T)\int_0^t V(s)ds) \le C(T). \quad \Box \tag{64}$$

Remark 3.8 *It follows from the proof of Lemma 3.7 that*

$$\|\rho^2u_x^2\|_{L^\infty((0,1)\times[0,T])} \le C(T). \tag{65}$$

Lemma 3.9 *Under the conditions of Theorem 2.1, the following estimates hold true*

$$\int_0^1 |\rho_x(x,t)|dx \le C(T), \tag{66}$$

$$\|\rho^2 u_x\|_{L^\infty[0,1]\times[0,T]} \le C(T), \tag{67}$$

and

$$\int_0^1 |(\rho^2 u_x)_x|dx \le C(T). \tag{68}$$

Proof: Since

$$\int_0^1 (1-x)^{1/2}|\rho_x|^2 dx \le C(T), \tag{69}$$

due to (42), one can get that

$$\begin{aligned} \int_0^1 |\rho_x|dx &\le (\int_0^1 (1-x)^{\frac{1}{2}}|\rho_x|^2 dx)^{\frac{1}{2}}(\int_0^1 (1-x)^{-\frac{1}{2}}dx)^{\frac{1}{2}} \\ &\le C(\int_0^1 (1-x)^{\frac{1}{2}}|\rho_x|^2 dx)^{\frac{1}{2}} \le C(T). \end{aligned} \tag{70}$$

Note that

$$(\rho^2 u_x(x,t)) = \int_x^1 [-g - (\rho^2)_x - u_t(y,t)]dy. \tag{71}$$

It follows from (37) and (50) that

$$|\rho^2 u_x(x,t)| \le g + C + (\int_0^1 u_t^2 dx)^{\frac{1}{2}}(1-x)^{\frac{1}{2}} \le C(T). \tag{72}$$

Integrating

$$(\rho^2 u_x)_x = g + (\rho^2)_x + u_t, \tag{73}$$

with respect to x over $[0,1]$, using (37), (50) and (66), one can obtain

$$\begin{aligned} \int_0^1 |(\rho^2 u_x)_x|dx &\le \int_0^1 |g + (\rho^2)_x + u_t|dx \\ &\le g + (\int_0^1 |u_t|^2 dx)^{\frac{1}{2}} + 2\int_0^1 \rho|\rho_x|dx \\ &\le C(T). \quad \Box \end{aligned} \tag{74}$$

Lemma 3.10 *Under the conditions of Theorem 2.1, it holds that*

$$\int_0^1 |u_x(x,t)|dx \le C(T), \tag{75}$$

$$\|u(x,t)\|_{L^\infty([0,1]\times[0,T])} \le C(T). \tag{76}$$

Proof: (28) yields

$$u_x(x,t) = -g(1-x)\rho^{-2} + 1 - \rho^{-2}\int_x^1 u_t(y,t)dy. \tag{77}$$

It follows from this, (37), (50) and Hölder inequality that

$$\begin{aligned}\int_0^1 |u_x(x,t)|dx &\le C + \int_0^1 g(1-x)\rho^{-2}dx + \int_0^1 \rho^{-2}\int_x^1 |u_t(y,t)|dydx \\ &\le C + \int_0^1 \rho^{-2}(\int_x^1 |u_t^2|dx)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}dx \\ &\le C + C(T)\int_0^1 (1-x)^{-\frac{1}{2}}dx \\ &\le C(T).\end{aligned} \tag{78}$$

On the other hand, by using Sobolev's embedding theorem $W^{1,1}[0,1] \hookrightarrow L^\infty[0,1]$ and Cauchy-Schwartz inequality, one has

$$\|u(x,t)\|_{L^\infty([0,1]\times[0,T])} \le \int_0^1 |u(x,t)|dx + \int_0^1 |u_x(x,t)|dx \le C(T),$$

where (30) and (78) have been used. $\square$

Lemma 3.11 *Under the conditions of Theorem 2.1, we have for* $0 < s < t \le T$,

$$\int_0^1 |\rho(x,t) - \rho(x,s)|^2 dx \le C|t-s|, \tag{79}$$

$$\int_0^1 |u(x,t) - u(x,s)|^2 dx \le C(T)|t-s|, \tag{80}$$

$$\int_0^1 |(\rho^2 u_x)(x,t) - (\rho^2 u_x)(x,s)|^2 dx \le C(T)|t-s|. \tag{81}$$

Proof: First, it follows from equation $(10)_1$ and Hölder inequality that

$$\begin{aligned}\int_0^1 |\rho(x,t)-\rho(x,s)|^2dx &= \int_0^1 |\int_s^t \rho_t(x,\eta)d\eta|^2dx\\ &= \int_0^1 |\int_s^t (\rho^2u_x)(x,\eta)d\eta|^2dx\\ &\le |t-s|\int_s^t\int_0^1 (\rho^4u_x^2)(x,\eta)dxd\eta\\ &\le |t-s|\int_0^t \max_{[0,1]}\rho^2\int_0^1 \rho^2u_x^2dxd\eta \le C|t-s|,\end{aligned}$$

which implies (79).

Second, Hölder inequality and (50) give

$$\begin{aligned}\int_0^1 |u(x,t)-u(x,s)|^2dx &= \int_0^1 |\int_s^t u_t(x,\eta)d\eta|^2dx\\ &\le |t-s|\int_s^t\int_0^1 u_t^2(x,\eta)dxd\eta \le C(T)|t-s|.\end{aligned}$$

Finally, since

$$\begin{aligned}\int_0^1 |(\rho^2u_x)(x,t)-(\rho^2u_x)(x,s)|^2dx &= \int_0^1 |\int_s^t (\rho^2u_x)_t(x,\eta)d\eta|^2dx\\ &\le |t-s|\int_s^t\int_0^1 ((\rho^2u_x)_t(x,\eta))^2dxd\eta,\end{aligned}$$

and

$$(\rho^2u_x)_t(x,t)) = \rho^2u_{xt}+2\rho\rho_tu_x(x,t) = \rho^2u_{xt}-2\rho^3u_x^2(x,t)$$

due to $(10)_1$, thus it follows from (50), (65) and Cauchy-Schwartz inequality that

$$\begin{aligned}&\int_s^t[\int_0^1(\rho^2u_x)_t]^2dxd\eta\\ &\le C\int_0^t\int_0^1\rho^4u_{xt}^2dxd\eta + C\int_0^t\int_0^1\rho^6u_x^4dxd\eta\\ &\le C(T). \quad \Box\end{aligned} \tag{82}$$

Now we are in a position to prove the existence of weak solutions. Indeed based on Lemma 3.1–3.11 and Corollary 3.5, following the similar argument in

[19, 25, 34], we can construct a weak solution to the initial boundary value problem $(10)-(12)$ by using a finite difference method. Since the details are standard , so will be omited. This complete the proof of the existence part of Theorem 2.1.

4 Uniqueness of weak solution

In the above section, under the assumptions $(A_1)--(A_3)$, we obtained a global weak solution $(\rho(x,t), u(x,t))$ to (10) such that for any $T>0$, $(\rho(x,t), u(x,t))$ satisfies (10)– (12) and the estimate

$$C_1[(1-x)]^{1/2} \le \rho(x,t) \le C_2[(1-x)]^{1/2}, \|\rho u_x\|_{L^\infty([0,1]\times[0,T])} \le C(T). \tag{83}$$

In this section, we will use the energy method to prove the uniqueness of such a weak solution in Theorem 2.1.

Theorem 4.1 *Assume (A_1), (A_2) and (A_3). Let both $(\rho_1(x,t),\ u_1(x,t))$ and $(\rho_2(x,t),\ u_2(x,t))$ be solutions of* (10) *satisfying* (10), (11) *and* (12) *for any* $T>0$. *Then $\rho_1=\rho_2$, $u_1=u_2$, a.e. .*

Proof: In the following, we may assume that $(\rho_1,u_1)(x,t)$ and $(\rho_2,u_2)(x,t)$ are suitably smooth since the following estimates are valid for the solutions by using the Friedrichs mollifier.

From $(10)_2$, we have

$$(u_2-u_1)_t + ((\rho_2)^2-(\rho_1)^2)_x = (\rho_2^2 u_{2x} - \rho_1^2 u_{1x}) \tag{84}$$

Multiplying the above equation by u_2-u_1 and integrating it with respect to x, using integration by parts, we get

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \frac{d}{dt}(u_2-u_1)^2 dx \\
&= \int_0^1 (u_2-u_1)_x(\rho_2^2-\rho_1^2) - (u_2-u_1)_x(\rho_2^2-\rho_1^2)u_{2x} dx \\
&- \int_0^1 (u_2-u_1)_x^2 \rho_1^2 dx,
\end{aligned} \tag{85}$$

Set

$$u_2 - u_1 = w, \rho_2 - \rho_1 = \rho.$$

Then it follows from (37), (65) and (85) that

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \frac{d}{dt} w^2 dx + \int_0^1 w_x^2 \rho_1^2 dx \\
&= \int_0^1 \rho(\rho_2+\rho_1) w_x dx - \int_0^1 \rho(\rho_2+\rho_1) u_{2x} w_x dx \\
&\leq \frac{1}{8}\int_0^1 \rho_1^2 w_x^2 dx + 2\int_0^1 \rho^2(\rho_2+\rho_1)^2 \rho_1^{-2} dx \\
&+ 2\int_0^1 \rho^2(\rho_2+\rho_1)^2 u_{2x}^2 \rho_1^{-2} dx + \frac{1}{8}\int_0^1 \rho_1^2 w_x^2 dx \\
&\leq \frac{1}{4}\int_0^1 \rho_1^2 w_x^2 dx + C\int_0^1 \rho_2^{-2}\rho^2 dx \{\sup_x [(\rho_2+\rho_1)^2\rho_1^{-2}\rho_2^2 + (\rho_2+\rho_1)^2\rho_1^{-2}\rho_2^2 u_{2x}^2]\} \\
&\leq \frac{1}{4}\int_0^1 \rho_1^2 w_x^2 dx + C(1 + \|\rho_2^2 u_{2x}^2\|_{L^\infty(0,1)}) \int_0^1 \rho_2^{-2}\rho^2 dx \\
&\leq \frac{1}{4}\int_0^1 \rho_1^2 w_x^2 dx + C(T)\int_0^1 \rho_2^{-2}\rho^2 dx.
\end{aligned} \tag{86}$$

On the other hand,

$$\rho_{it} = \rho_i^2 u_{ix} \Rightarrow (\frac{1}{\rho_i})_t = u_{ix}, \quad i = 1, 2. \tag{87}$$

Thus,

$$(\frac{1}{\rho_1} - \frac{1}{\rho_2})_t = u_{1x} - u_{2x} = w_x, \tag{88}$$

and so

$$[\frac{\rho}{\rho_1\rho_2}]_t + w_x = 0. \tag{89}$$

Multiplying (89) by $\rho_1\rho_2^{-1}\rho$ yields

$$(\rho_1^{-1}\rho_2^{-1}\rho)_t \rho_1\rho_2^{-1}\rho + w_x\rho_1\rho_2^{-1}\rho = 0, \tag{90}$$

which can be rewritten as

$$(\rho_2^{-2}\rho^2)_t + 2\rho_1\rho_2^{-2}\rho^2 u_{1x} + 2\rho_1\rho_2^{-1}\rho w_x = 0. \tag{91}$$

Integrating (91) with respect to x over $[0,1]$ leads to

$$\frac{d}{dt}\int_0^1 \rho_2^{-2}\rho^2 dx + 2\int_0^1 \rho_1\rho_2^{-2}\rho^2 u_{1x} dx + 2\int_0^1 \rho_1\rho_2^{-1}\rho w_x dx = 0. \tag{92}$$

It follows from (92) and (37) that

$$\begin{aligned}\frac{d}{dt}\int_0^1 \rho_2^{-2}\rho^2 dx &\le C\int_0^1 \rho_2^{-2}\rho^2|\rho_1 u_{1x}| dx + \frac{1}{4}\int_0^1 \rho_1^2 w_x^2 dx + \int_0^1 \rho_2^{-2}\rho^2 dx \\ &\le C(T)\int_0^1 \rho_2^{-2}\rho^2 dx + \frac{1}{4}\int_0^1 \rho_1^2 w_x^2 dx.\end{aligned} \tag{93}$$

Combining (86) with (93) yields

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \rho_2^{-2}\rho^2 dx + \int_0^1 \frac{1}{2} w^2 dx + \int_0^1 \frac{1}{2}\rho_1^2 w_x^2 dx \\ &\le C(T)\int_0^1 \rho_2^{-2}\rho^2 dx \\ &\le C(T)\int_0^1 (\rho_2^{-2}\rho^2 + w^2) dx.\end{aligned}$$

By Gronwall's inequality, we conclude that

$$w = 0, \qquad \rho = 0. \tag{94}$$

Thus we finish the proof of the uniqueness. $\square$

5 Asymptotic behavior

In this section, we will consider the asymptotic behavior of the solution to the free boundary problem (10). We will show that the solution to the free boundary problem tends to a stationary solution as $t \to \infty$.

In order to obtain the result, we need a lemma.

Lemma 5.1 *Suppose that* $y \in W^{1,1}_{loc}(\mathbb{R}^+)$ *satisfies*

$$y = y_1 + y_2,$$

and

$$|y_2| \leq \sum_{i=1}^n \alpha_i, \ \ |y'| \leq \sum_{i=1}^n \beta_i, \ \ on \ \mathbb{R}^+$$

where $y_1 \in W^{1,1}_{loc}(\mathbb{R}^+)$, $\lim_{s\to+\infty} y_1(s) = 0$ *and* $\alpha_i, \beta_i \in L^{p_i}(\mathbb{R}^+)$ *for some* $p_i \in [1, \infty)$, $i = 1, \cdots n$. *Then*

$$\lim_{s\to+\infty} y(s) = 0.$$

Proof: Now by Sobolev's embedding theorem $W^{1,1} \hookrightarrow L^\infty$, and using the fact that

$$\|y\|_{W^{1,1}} \simeq |\int y(s)ds| + \int |y'(s)|ds,$$

we have

$$\begin{aligned} |y(t)| &\leq |\int_t^{t+1} yds| + \int_t^{t+1} |y'|ds \\ &\leq |y_1(t+1) - y_1(t)| + \sum_{i=1}^n \int_t^{t+1} (\alpha_i + \beta_i)ds \\ &\leq |y_1(t+1) - y_1(t)| + \sum_{i=1}^n (\|\alpha_i\|_{L^{p_i}(t,t+1))} + \|\beta_i\|_{L^{p_i}(t,t+1))} \to 0, \end{aligned}$$

as $t \to \infty$. $\square$

Proposition 5.2 *Under the conditions of Theorem 2.1, the total kinetic energy*

$$E(t) := \int_0^1 \frac{1}{2} u^2(x,t)dx \to 0, \ \ as \ \ t \to \infty.$$

Proof: First, $E(t) \geq 0$. It follows from (39) and (30) that

$$\int_0^\infty E(t)dt = \frac{1}{2} \int_0^\infty \int_0^1 u^2(x,t)dxdt \leq C, \tag{95}$$

and

$$\begin{aligned} |E'(t)| &= |-\int_0^1 \rho^2 u_x^2 dx + \int_0^1 \rho^2 u_x dx - \int_0^1 gudx| \\ &\leq \int_0^1 \rho^2 u_x^2 dx + (\int_0^1 \rho^2 u_x^2 dx)^{\frac{1}{2}} \cdot (\int_0^1 \rho^2)^{\frac{1}{2}} - g(\int_0^1 u^2(x,t)dx)^{\frac{1}{2}} \qquad (96) \\ &\leq \int_0^1 \rho^2 u_x^2 dx + C(\int_0^1 \rho^2 u_x^2 dx)^{\frac{1}{2}} + g(\int_0^1 u^2(x,t)dx)^{\frac{1}{2}}. \end{aligned}$$

Using Hölder inequality and Sobolev's embedding theorem $W^{1,1} \hookrightarrow L^\infty$, one has

$$
\begin{aligned}
E(\tau) \leq & \int_\tau^{\tau+1} E(t)dt + \int_\tau^{\tau+1} |E'(\tau)|dt \\
\leq & \frac{1}{2}\int_\tau^{\tau+1}\int_0^1 u^2(x,t)dxdt + \int_\tau^{\tau+1}\int_0^1 \rho^2 u_x^2 dxdt \\
& + C\int_\tau^{\tau+1}(\int_0^1 \rho^2 u_x^2 dx)^{\frac{1}{2}}dt + g\int_\tau^{\tau+1}(\int_0^1 u^2(x,t)dx)^{\frac{1}{2}}dt \\
\leq & \int_\tau^{\tau+1}\int_0^1 u^2(x,t)dxdt + \int_\tau^{\tau+1}\int_0^1 \rho^2 u_x^2 dxdt \\
& + C(\int_\tau^{\tau+1}\int_0^1 \rho^2 u_x^2 dxdt)^{\frac{1}{2}} + g(\int_\tau^{\tau+1}\int_0^1 u^2(x,t)dxdt)^{\frac{1}{2}},
\end{aligned}
$$

for all $\tau \geq 0$.

Hence, it follows from (30), (37), (39) and Lemma 5.1 that the right hand side of the last estimate converges to zero as $\tau \to \infty$, thus

$$\lim_{\tau\to\infty} E(\tau) = 0. \quad \square$$

In order to show that the density function ρ tends to the stationary state ρ_∞, we also need some uniform estimate with respect to time. The following lemma is essential to obtain the desired conclusion.

Lemma 5.3 *Under the condition of theorem 2.1, it holds that*

$$\int_0^t\int_0^1 (\rho^2 - \rho_\infty^2)^2 dxds \leq C, \tag{97}$$

where C is independent of $t \geq 0$.

Proof: Note that

$$\rho^2 = -\rho_t + \int_x^1 u_t dy + g(1-x),$$

then

$$\rho^2 - \rho_\infty^2 = -\rho_t + \int_x^1 u_t dy. \tag{98}$$

We have

$$
\begin{aligned}
&\int_0^t\int_0^1(\rho^2-\rho_\infty^2)^2dxds=\int_0^t\int_0^1(\rho^2-\rho_\infty^2)(\rho^2-\rho_\infty^2)dxds\\
&=\int_0^t\int_0^1(\rho^2-\rho_\infty^2)(-\rho_t+\int_x^1 u_tdy)dxds\\
&=\int_0^t\int_0^1(\rho^2-\rho_\infty^2)(\int_x^1 u_tdy)dxds+\int_0^t\int_0^1(-\rho^2\rho_t+\rho_t\rho_\infty^2)dxds\\
&=\int_0^t\int_0^1(\rho^2-\rho_\infty^2)(\int_x^1 u_tdy)dxds+\int_0^t\int_0^1(-\frac{1}{3}\rho^3+\rho_\infty^2\rho)_tdxds\\
&=I_1+I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1&=\int_0^t\int_0^1(\rho^2-\rho_\infty^2)(\int_x^1 u_tdy)dxds\\
&=\int_0^1\int_0^t(\rho^2-\rho_\infty^2)(\int_x^1 udy)_tdsdx,\\
I_2&=\int_0^t\int_0^1(-\frac{1}{3}\rho^3+\rho_\infty^2\rho)_tdxds.
\end{aligned}
\tag{99}
$$

Integrating by parts and using $(10)_1$ lead to

$$
\begin{aligned}
I_1=&-\int_0^t\int_0^1 2\rho\rho_t(\int_x^1 udy)dxds+\int_0^1(\rho^2-\rho_\infty^2)(\int_x^1 udy)dx\\
&-\int_0^1(\rho_0^2-\rho_{0\infty}^2)(\int_x^1 u_0dy)dx\\
=&\int_0^t\int_0^1 2\rho^3u_x(\int_x^1 udy)dxds+\int_0^1(\rho^2-\rho_\infty^2)(\int_x^1 udy)dx\\
&-\int_0^1(\rho_0^2-\rho_\infty^2)(\int_x^1 u_0dy)dx.
\end{aligned}
$$

It follows from Hölder inequality, (30), (37), (39) and the initial conditions that

$$
\begin{aligned}
I_1\le\ & C\int_0^t\int_0^1\rho^2u_x^2dxds\\
&+C\int_0^t\int_0^1 u^2(x,t)dxds+C(\int_0^1 u^2(x,t)dx)^{\frac{1}{2}}\cdot(\int_0^1(1-x)^{\frac{3}{2}}dx)+C\\
\le\ & C.
\end{aligned}
$$

Similarly, integrating by parts, from (37) and the initial condition, we have

$$
\begin{aligned}
I_2 &= \int_0^t \int_0^1 (-\frac{1}{3}\rho^3 + \rho_\infty \rho)_t dx ds \\
&= \int_0^1 (-\frac{1}{3}\rho^3 + \rho\rho_\infty^2) dx - \int_0^1 (-\frac{1}{3}\rho_0^3 + \rho_0\rho_\infty^2) dx) \\
&\leq C \int_0^1 (1-x)^{\frac{3}{2}} dx \\
&\leq C.
\end{aligned}
$$

Now (97) holds. $\square$

Proposition 5.4 *Under the conditions of Theorem 2.1, we have*

$$\int_0^1 (\rho^2 - \rho_\infty^2)^2 (x,t) dx \to 0, \tag{100}$$

and

$$\|(\rho - \rho_\infty)(\cdot, t)\|_{L^q} \to 0, q \in [1, \infty), \tag{101}$$

as $t \to \infty$.

Proof: Lemma 5.3 shows that

$$\int_0^1 (\rho^2 - \rho_\infty^2)^2 dx \in L^1(\mathbb{R}^+).$$

On the other hand, by (10), (37) and Hölder inequality, one has

$$
\begin{aligned}
&|\frac{d}{dt} \int_0^1 (\rho^2 - \rho_\infty^2)^2 dx| \\
&= |\int_0^1 2(\rho^2 - \rho_\infty^2) 2\rho\rho_t dx| \\
&\leq 4 \int_0^1 |\rho^2 - \rho_\infty^2| \rho^3 |u_x| dx \\
&\leq 4 (\int_0^1 \rho^2 u_x^2 dx)^{\frac{1}{2}} (\int_0^1 \rho^4 |\rho^2 - \rho_\infty^2|^2 dx)^{\frac{1}{2}} \\
&\leq C (\int_0^1 \rho^2 u_x^2 dx)^{\frac{1}{2}}.
\end{aligned}
$$

Taking into account the uniform energy estimates (30), and using the same method as in Proposition 5.2 and Lemma 5.1, we have (100).

To prove (101), we note that

$$
\begin{aligned}
&\int_0^1 (\rho-\rho_\infty)^4(x,t)dx \\
&= \int_0^1 \frac{(\rho-\rho_\infty)^4}{(\rho^2-\rho_\infty^2)^2}((\rho^2-\rho_\infty^2)^2)dx \\
&\le C\int_0^1 (\rho^2-\rho_\infty^2)^2 dx \to 0,
\end{aligned}
$$

as $t\to\infty$, where (37) and (100) have been used.

This and Hölder inequality imply that for $q\in[1,4)$,

$$
\begin{aligned}
&\int_0^1 (\rho-\rho_\infty)^q(x,t)dx \\
&\le C(\int_0^1 (\rho-\rho_\infty)^4 dx)^{\frac{q}{4}} \to 0.
\end{aligned}
$$

On the other hand, for $q\in(4,\infty)$, we have from (37) that

$$
\begin{aligned}
&\int_0^1 (\rho-\rho_\infty)^q(x,t)dx \\
&= \int_0^1 (\rho-\rho_\infty)^4(\rho-\rho_\infty)^{q-4}dx \\
&\le C\int_0^1 (\rho-\rho_\infty)^4(1-x)^{\frac{q-4}{2}}dx \\
&\le C\int_0^1 (\rho-\rho_\infty)^4 dx \to 0.
\end{aligned}
$$

So (101) follows. $\Box$

Thus, we finish the proof of Theorem 2.1.

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