A BLOW-UP CRITERION FOR CLASSICAL SOLUTIONS TO THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

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ABSTRACT. In this paper, we obtain a blow up criterion for classical solutions to the 3-D compressible Naiver-Stokes equations just in terms of the gradient of the velocity, similar to the Beal-Kato-Majda criterion for the ideal incompressible flow. In addition, initial vacuum is allowed in our case.

1. INTRODUCTION

Let $\Omega \subset \mathcal{R}^n$ be a n-dimensional domain. The time evolution of the density and the velocity of a general viscous compressible barotropic fluid occupying a domain Ω is governed by the Navier-Stokes system of equations

$$\begin{cases} \partial_t \rho + \text{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho u) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \triangle \mathbf{u} - (\mu + \lambda)\nabla(\text{div}\mathbf{u}) + \nabla \mathrm{P}(\rho) = 0 \end{cases} \tag{1.1}$$

Where ρ, u, P denotes the density, velocity and pressure respectively. The equation of state is given by

$$P(\rho) = a\rho^\gamma \quad (a > 0, \gamma > 1) \tag{1.2}$$

μ and λ are the shear viscosity and the bulk viscosity coefficients respectively satisfying the condition:

$$\mu > 0, \lambda + \frac{2}{3}\mu \geq 0 \tag{1.3}$$

Lions [?] [?], Feireisl [?][?] et. established the global existence of weak solutions to the problem $(1.1) - (1.3)$, where vacuum is allowed initially. The existence of global smooth solutions to the compressible Navier-Stokes equations is obtained by Matsumura[?]and Nishida under the condition that the initial data is a small perturbation of a non-vacuum constant. It is also shown by Xin[?] that there is no global in time regular solution in R^3 to the compressible Naiver-Stokes equations provided that the initial density is compactly supported.

There are many results concerning the existence of strong solutions to the Navier-Stokes equations, only local existence results have been established, see [?], [?],[?],[?]. V.A.Solonnikov proved in [?] that for C^2 pressure laws and initial data satisfies for some $q > N$,

$$0 < m \leq \rho_0(x) \leq M < \infty, \quad and \quad \rho_0 \in W^{1,q}(T^N) \tag{1.4}$$

$$u_0 \in W^{2-\frac{2}{q},q}(T^N)^N \tag{1.5}$$

there exists a local unique strong solution (ρ, u) to $(1.4) - (1.5)$ for periodic data, such that

$$\rho \in L^\infty(0,T;W^{1,q}(T^N)), \quad \rho_t \in L^q((0,T) \times T^N)$$
$$u \in L^q(0,T;W^{2,q}(T^N)), \quad u_t \in L^q((0,T) \times T^N)^N \tag{1.6}$$

Later, it was shown in [?] that if Ω is either a bounded domain or the whole space, the initial data ρ_0 and u_0 satisfy

$$0 \leq \rho_0 \in W^{1,\tilde{q}}(\Omega), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \tag{1.7}$$

for some $\tilde{q} \in (3, \infty)$ and the compatibility condition:

$$-\mu\triangle u_0 - (\lambda + \mu)\nabla \mathrm{div} u_0 + \nabla P(\rho_0) = \rho_0^{1/2}g \quad for \quad some \quad g \in L^2(\Omega) \tag{1.8}$$

then there exists a positive time $T_1 \in (0, \infty)$ and a unique strong solution (ρ, u) to the isentropic problem, such that

$$\rho \in C([0,T_1];W^{1,q_0}(\Omega))$$
$$u \in C([0,T_1];D_0^1 \cap D^2(\Omega)) \cap L^2(0,T_1;D^{2,q_0}(\Omega)) \tag{1.9}$$

Furthermore, one has the following blow-up criterion: if T^* is the maximal time of existence of the strong solution (ρ, u) and $T^* < \infty$, then

$$\sup_{t \to T^*} (\|\rho\|_{W^{1,q_0}} + \|u\|_{D_0^1}) = \infty \tag{1.10}$$

where $q_0 = \min(6, \tilde{q})$.

Here and throughout this paper, we use the following notations for the standard homogeneous and inhomogeneous Sobolev spaces.

$$D^{k,r}(\Omega) = \{u \in L^1_{loc}(\Omega) : \|\nabla^k u\|_{L^r} < \infty\},$$
$$W^{k,r} = L^r \cap D^{k,r}, H^k = W^{k,2}, \quad D^k = D^{k,2}$$

$$D_0^1 = \{u \in L^6(\Omega) : \|\nabla u\|_{L^2} < \infty \quad and \quad u = 0 \quad on \quad \partial\Omega\},$$

$$H_0^1 = L^2 \cap D_0^1, \quad \|u\|_{D^{k,r}} = \|\nabla^k u\|_{L^r}$$

Recently, it is established in [?] that if $[0, T^*)$ is the finite maximal interval for such strong solutions. and $7\mu > 9\lambda$, then

$$\lim_{T \to T^*} \{\sup_{0 \leq T < T^*} \|\rho\|_{L^\infty} + \int_0^T (\|\rho\|_{W^{1,q_0}} + \|\nabla\rho\|_{L^2}^4)dt\} = \infty \tag{1.11}$$

Here they only require a sufficient regularity of density ρ to admit the global existence of strong solutions, as (1.11) revealed.

It is shown in [?], we can obtain a blow up criterion for strong solutions similar to Beal-Kato-Majda for ideal incompressible fluid, i.e,

$$\int_0^{T_*} \|\nabla u\|_{L^\infty} dt = \infty$$

where we assume that

$$\mu + \lambda \geq 0, \quad N = 2, \quad \Omega = T^2$$
$$\mu + \lambda = 0, \quad N = 3, \quad \Omega \subset R^3 \tag{1.12}$$

Recently, it is shown in[?] that if the domain is either a bounded domain or the whole space R^3 and the initial data ρ_0, u_0 satisfy

$$(\rho_0, P_0) \in H^3, \rho_0 \geq 0$$
$$u_0 \in H_0^1 \cap H^3 \tag{1.13}$$

and the compatibility condition

$$- Lu_0 + \nabla P(\rho_0) = \rho_0 g \quad for \quad some \quad g \in H_0^1(\Omega) \quad with \quad \rho_0^{\frac{1}{2}} g \in L^2 \tag{1.14}$$

where

$$Lu = \mu\triangle u + (\lambda + \mu)\nabla \text{divu}, \quad P(\rho_0) = a\rho_0^\gamma$$

Then there exist a small time $T_* \in (0, T)$ and a unique classical solution (ρ, p, u) such that

$(\rho, P) \in C([0, T_*]; H^3(\Omega))$

$u \in C([0, T_*]; D_0^1 \cap D^3(\Omega)) \cap L^2(0, T_*; D^4(\Omega))$

$u_t \in L^\infty(0, T_*; D_0^1(\Omega)) \cap L^2(0, T_*; D^2(\Omega)) \quad and \quad \sqrt{\rho}u_t \in L^\infty(0, T_*; L^2(\Omega))$

$(u_t, \nabla^2 u) \in C((0, T_*] \times \bar\Omega)$

In this paper, under the assumption

$$\mu > \frac{1}{7}\lambda \tag{1.16}$$

we establish a blow up criterion for classical solutions.

Here and thereafter C always denotes a generic constant depending only on Ω, T and initial data.

For the initial boundary value problem, we have the following result:

Theorem 1.1. *Let $\Omega \subset R^3$ be a bounded domain. $Q_T = (0,T) \times \Omega$. Assume that the initial data satisfy $(1.13) - (1.14)$. Let (ρ, u) be a classical solution of the problem $(1.1)-(1.3)$ satisfying the regularity (1.15). If $T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T \to T^*} \int_0^T \|\nabla u\|_{L^\infty(\Omega)} dt = \infty \tag{1.17}$$

provided that (1.16) holds.

In case of the Cauchy problem, it holds that

Theorem 1.2. *Let $\Omega = R^3$. Assume that the initial data satisfy*

$$(\rho_0, P_0) \in H^3(R^3), \quad u_0 \in D_0^1(R^3) \cap D^3(R^3) \tag{1.18}$$

the compatibility condition (1.14). Let (ρ, u) be a classical solutions to the problem $(1.1) - (1.3)$ in the sense of $[?]$ satisfying

$(\rho, P) \in C([0, T_*], H^3(R^3))$

$u \in C([0, T_*], D_0^1(R^3) \cap D^3(R^3)) \cap L^2(0, T_*; D^4(R^3)) \tag{1.19}$

$u_t \in L^\infty(0, T_*; D_0^1(R^3)) \cap L^2(0, T_*; D^2(R^3)), \quad \sqrt{\rho}u_t \in L^\infty(0, T_*; L^2(R^3))$

If $T^* < \infty$ is the maximal time of existence, then

$$\lim_{T \to T^*} \int_0^T \|\nabla u\|_{L^\infty(\Omega)} dt = \infty \tag{1.20}$$

provided that (1.16) holds.

Remark 1.1 The blow up criterion (1.10) involves both the density and velocity. It may be natural to expect the higher regularity of velocity if the density is regular enough. (1.11) shows that sufficient regularity of the gradient of density indeed guarantees the global existence of strong solutions. The main difficulty in our case is to

control the gradient of density, which is not a priorily known and coupled with the second derivative of velocity.

In this paper, we establish a blow up criterion under condition (1.16) instead of (1.12). Obviously, (1.16) becomes physical condition (1.3) if $\lambda \leq 0$. We develop some new estimates under the condition that the integral on the left of (1.17) is finite. In fact, the key estimate in our analysis is $L^\infty H^1$ bound of $\nabla \rho$. To control the $L^\infty(0, T; L^2(\Omega))$ norm of $\nabla \rho$, we observe that that the space-time square mean of the convection term $F = \rho u_t + \rho u \cdot \nabla u$ is controlled by that of $\nabla \rho$ (see Lemma 2.3). This, in turn, gives the desired $L^\infty(0, T; L^2(\Omega))$ estimate on $\nabla \rho$, and thus the $L^2(0, T; H^2(\Omega))$ of u. To obtain a higher regularity of $\nabla \rho$, one need to improve the regularity of pressure P, as we can't deduce $P \in L^\infty H^3$ directly even ρ is sufficient regular unless $\gamma = 2$ or $\gamma \geq 3$ due to the presence of vacuum. Our proof relies on the observation that, the pressure P is solution of a transport equation $P_t + \text{div}(\text{Pu}) + (\gamma - 1)\text{Pdivu} = 0$. Hence we can deduce a high regularity of P provided that u and P_0 are regular enough. As a consequence, the high order regularity of the density follows from the mass equation and a sufficient regularity of pressure.

Remark 1.2 There are many results concerning blow-up criteria of the incompressible flows. In the well-known paper [**?**], Beal-Kato-Majda established a blow-up criterion for the incompressible Euler equations. One can get global smooth solution if $\int_0^T \|\omega\|_{L^\infty} dt$ is bounded. It's worth noting that only the vorticity ω plays an important role in the existence of global smooth solutions. Moreover, as pointed out by Constantin[**?**], the solution is smooth if and only if $\int_0^T \|((\nabla u)\xi) \cdot \xi\|_{L^\infty}$ is bounded, where ξ is the unit vector in the direction of ω. It turns out that the solution becomes smooth either the asymmetric or symmetric part of ∇u is controlled. Later, Constantin[**?**], Fefferman and Majda showed a sufficient geometric condition to control the breakdown of smooth solutions of incompressible Euler involving the Lipschitz regularity of the direction of the vorticity. It is also shown by Constantin[**?**] and Fefferman that the solution of incompressible Navier-Stokes equations is smooth if the direction of vorticity is well behaved.

Recently, in[**?**], assuming that the added stress tensor is given in a proper form, and using an idea of J.-Y. Chemin and N. Masmoudi [**?**], Constantin, P. and Fefferman,

C., Titi, E. S. and Zarnescu, A obtain a logarithmic bound for $\int_0^T \|\nabla u\|_{L^\infty} dt$. to conclude that the solution to Navier-Stokes-Fokker-Planck system exists for all time and is smooth.

In our paper, we establish a similar criterion to Beal-Kato-Majda. Our blow up criteria involve both the symmetric and asymmetric part of ∇u, as the compressibility and the vorticity of the compressible flow are two key issues in the formation of singularities of the compressible Navier-Stokes.

Remark 1.3 The paper is organized as follows. Section 2 is devoted to improve the regularity of the density and the velocity in strong sense. In section 3, we derive some high order regularity estimate for the density, pressure and velocity, which guarantee the extension of classical solutions.

2. Regularity of the density and the velocity

Let (ρ, u) be a classical solution to the problem $(1.1) - (1.3)$. We assume that the opposite holds, i.e

$$\lim_{T \to T^*} \int_0^T \|\nabla u\|_{L^\infty(\Omega)} dt \le C < \infty \tag{2.1}$$

First, the standard energy estimate yields

$$\sup_{0 \le t \le T} \|\rho^{1/2} u(t)\|_{L^2} + \int_0^T \|u\|_{H^1}^2 dt \le C, \quad 0 \le T < T^* \tag{2.2}$$

By assumption (2.1) and the conservation of mass, the L^∞ bounds of density follows immediately,

Lemma 2.1. *Assume that*

$$\int_0^T \|divu\|_{L^\infty} dt \le C, \quad 0 \le T < T^* \tag{2.3}$$

then

$$\|\rho\|_{L^\infty(Q_T)} \le C, \quad 0 \le T < T^* \tag{2.4}$$

Proof. It follows from the conservation of mass that for $\forall q > 1$,

$$\partial_t(\rho^q) + \text{div}(\rho^q u) + (q-1)\rho^q \text{divu} = 0 \tag{2.5}$$

Integrating (2.5) over Ω to obtain,

$$\partial_t \int_\Omega \rho^q dx \le (q-1)\|\nabla u\|_{L^\infty(\Omega)} \int_\Omega \rho^q dx \tag{2.6}$$

i.e

$$\partial_t \|\rho\|_{L^q} \leq \frac{q-1}{q} \|\nabla u\|_{L^\infty(\Omega)} \|\rho\|_{L^q} \tag{2.7}$$

which implies immediately

$$\|\rho\|_{L^q}(t) \leq C \tag{2.8}$$

with C independent of q, so our lemma follows.

$\square$

Next, we improve the energy estimate (2.2). It's worth noting that only here we require that the condition (1.16) holds.

Lemma 2.2. *Let* $\mu > \frac{1}{7}\lambda$, *then there exist a small* $\delta > 0$

$$\sup_{0 \leq t \leq T} \int_\Omega \rho |u|^{3+\delta} dx \leq C, \quad 0 < T < T_*, \tag{2.9}$$

where C *is a positive constant depending only on* $\|\rho\|_{L^\infty(Q_T)}$.

Proof. This follows from an argument due to Hoff[**?**].

Indeed, setting $q > 3$ and multiplying (1.2) by $q|u|^{q-2}u$, and integrating over Ω, we obtain by using lemma 2.1 that

$$\begin{aligned}
\frac{d}{dt} &\int_\Omega \rho |u|^q dx + \int_\Omega (q|u|^{q-2}[\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}u)^2 + \mu(q-2)|\nabla|u||^2] \\
&+ q(\lambda+\mu)(\nabla|u|^{q-2}) \cdot u\mathrm{div}u)\mathrm{dx} \\
&= q\int_\Omega \mathrm{div}(|u|^{q-2}u)\mathrm{pdx} \\
&\leq C\int_\Omega \rho^{\frac{1}{2}}|u|^{q-2}|\nabla u|dx \\
&\leq \epsilon\int_\Omega |u|^{q-2}|\nabla u|^2 dx + C(\epsilon)\int_\Omega \rho|u|^{q-2}dx \\
&\leq \epsilon\int_\Omega |u|^{q-2}|\nabla u|^2 dx + C(\epsilon)(\int_\Omega \rho|u|^q dx)^{\frac{q-2}{q}}
\end{aligned} \tag{2.10}$$

Note that $|\nabla|u|| \leq |\nabla u|$, one gets that

$$
\begin{aligned}
q|u|^{q-2}&[\mu|\nabla u|^2 + (\lambda + \mu)(\mathrm{divu})^2 + \mu(\mathrm{q}-2)|\nabla|\mathrm{u}||^2] + \mathrm{q}(\lambda + \mu)(\nabla|u|^{q-2}) \cdot u\mathrm{divu} \\
&\geq q|u|^{q-2}[\mu|\nabla u|^2 + (\lambda + \mu)(\mathrm{divu})^2 + \mu(\mathrm{q}-2)|\nabla|\mathrm{u}||^2 \\
&\quad - (\lambda + \mu)(q-2)|\nabla|u|| \cdot |\mathrm{divu}|] \\
&= q|u|^{q-2}[\mu|\nabla u|^2 + (\lambda + \mu)(\mathrm{divu} - \frac{1}{2}|\nabla|\mathrm{u}||)^2] \\
&\quad + q|u|^{q-2}[\mu(q-2) - \frac{1}{4}(\lambda + \mu)(q-2)^2]|\nabla|u||^2 \\
&\geq C|u|^{q-2}|\nabla u|^2
\end{aligned}
\tag{2.11}
$$

where we use the fact $\mu > \frac{1}{7}\lambda$, then there exist a small δ, such that

$$
\mu(q-2) - \frac{1}{4}(\lambda + \mu)(q-2)^2 > 0 \tag{2.12}
$$

where $q = 3 + \delta$.

Inserting (2.11) into (2.10), and taking ϵ small enough, we may apply Gronwall's inequality to conclude (2.9).

$\square$

The next lemma shows a connection between a convection term and the gradient of the density, which will play an important role in deriving the desired bounds on $\nabla\rho$.

Lemma 2.3. *Let* $F = \rho u_t + \rho u \cdot \nabla u$. *Then it holds that*

$$
\int_{Q_T} F^2 dxdt + \sup_{0 \leq t \leq T} \int_{\Omega} |\nabla u|^2 dx \leq C \int_{Q_T} |\nabla\rho|^2 dxdt + C, \quad 0 \leq T < T^*
$$

Proof. Note that

$$
\int_{Q_T} F^2 dxdt \leq C^*(\|\rho\|_{L^\infty(Q_T)}) \int_{Q_T} \rho u_t^2 dxdt + 2 \int_{Q_T} |\rho u \cdot \nabla u|^2 dxdt \tag{2.13}
$$

The last term of (2.13) can be estimated as follows

$$\int_{Q_T} \rho^2 |u|^2 |\nabla u|^2 dxdt$$

$$\leq C \int_0^T \|\rho^{\frac{1}{2}} u\|_{L^4}^2 \|\nabla u\|_{L^4}^2 dt$$

$$\leq C \int_0^T \|\rho^{\frac{1}{2}} u\|_{L^{3+\delta}}^{\alpha} \|\rho^{\frac{1}{2}} u\|_{L^6}^{2-\alpha} \|\nabla u\|_{L^4}^2 dt \tag{2.14}$$

$$\leq C \int_0^T \|\nabla u\|_{L^2}^{3-\alpha} \|\nabla u\|_{L^\infty} dt$$

$$\leq C \sup_{0 \leq T < T^*} \|\nabla u\|_{L^2}^{3-\alpha} \int_0^T \|\nabla u\|_{L^\infty} dt$$

$$\leq C \sup_{0 \leq T < T^*} \|\nabla u\|_{L^2}^{3-\alpha}$$

where $\frac{\alpha}{3+\delta} + \frac{2-\alpha}{6} = \frac{1}{2}$ and $1 < \alpha = \frac{3+\delta}{3-\delta} < 2$. It follows from (2.13) and (2.14) that

$$\int_{Q_T} F^2 dxdt \leq C^*(\|\rho\|_{L^\infty(Q_T)}) \int_{Q_T} \rho u_t^2 dxdt + C \sup_{0 \leq T < T^*} \|\nabla u\|_{L^2}^{3-\alpha} \tag{2.15}$$

Multiplying the momentum equation by u_t and integrating show that

$$\int_\Omega \rho u_t^2 dx + \int_\Omega \rho u \cdot \nabla u \cdot u_t dx + \frac{d}{dt} \int_\Omega \frac{\mu}{2} |\nabla u|^2 + \frac{\lambda+\mu}{2} (\mathrm{div} u)^2 dx = \int_\Omega P \mathrm{div} u_t dx \tag{2.16}$$

Note that

$$\int_\Omega P \mathrm{div} u_t dx = \frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega P \mathrm{div} u dx - \int_\Omega P_t \mathrm{div} u dx, \tag{2.17}$$

and

$$P_t + \mathrm{div}(Pu) + (\gamma - 1)P \mathrm{div} u = 0 \tag{2.18}$$

One gets

$$\int_\Omega P \mathrm{div} u_t dx = \frac{d}{dt} \int_\Omega P \mathrm{div} u dx + \int_\Omega \mathrm{div}(Pu) \mathrm{div} u dx + (\gamma - 1) \int_\Omega P(\mathrm{div} u)^2 dx$$

$$= \frac{d}{dt} \int_\Omega P \mathrm{div} u dx - \int_\Omega (Pu) \cdot \nabla \mathrm{div} u dx + (\gamma - 1) \int_\Omega P(\mathrm{div} u)^2 dx \tag{2.19}$$

This, together with (2.16), yields

$$\int_\Omega \frac{\mu}{2} |\nabla u|^2 + \frac{\lambda+\mu}{2} (\mathrm{div} u)^2 dx(T) + \int_{Q_T} \rho u_t^2 dxdt + \int_{Q_T} \rho u \cdot \nabla u \cdot u_t dxdt$$

$$= \int_\Omega \frac{\mu}{2} |\nabla u_0|^2 + \frac{\lambda+\mu}{2} (\mathrm{div} u_0)^2 dx(T) + \int_\Omega P \mathrm{div} u dx(T) - \int_\Omega P_0 \mathrm{div} u_0 dx \tag{2.20}$$

$$- \int_{Q_T} Pu \cdot \nabla \mathrm{div} u dxdt + (\gamma - 1) \int_{Q_T} P(\mathrm{div} u)^2 dxdt$$

Direct estimates show that

$$\int_\Omega P\mathrm{div}u\mathrm{d}x(T) \leq \frac{\mu}{4}\int_\Omega |\nabla u|^2 \mathrm{d}x(T) + C \tag{2.21}$$

$$\int_{Q_T} \rho u \cdot \nabla u \cdot u_t dxdt \leq \frac{1}{2}\int_{Q_T}\rho u_t^2 + \int_{Q_T}\rho|u\cdot\nabla u|^2 dxdt$$
$$\leq \frac{1}{2}\int_{Q_T}\rho u_t^2 + C\sup_{0\leq T<T^*}\|\nabla u\|_{L^2}^{3-\alpha} \tag{2.22}$$

On the other hand, it follows from $Lu = F + \nabla P$ and standard elliptic regularity that

$$\|u\|_{H^2} \leq C(\|F\|_{L^2} + \|\nabla P\|_{L^2}) \tag{2.23}$$

$$\int_{Q_T} Pu\cdot\nabla\mathrm{div}u\mathrm{d}x\mathrm{d}t \leq C\|\rho^{\frac{1}{2}}u\|_{L^2}\|\nabla\mathrm{div}u\|_{\mathrm{L}^2}$$
$$\leq C\int_{Q_T}|\nabla\rho|^2 dxdt + \epsilon\int_{Q_T}F^2 dxdt + C \tag{2.24}$$

Consequently,

$$\int_{Q_T}\rho u_t^2 dxdt + \frac{\mu}{2}\int_\Omega|\nabla u|^2 dx(T) \leq C\int_{Q_T}|\nabla\rho|^2 dxdt + 2\epsilon\int_{Q_T}F^2 dxdt$$
$$+ C\sup_{0\leq T<T^*}\|\nabla u\|_{L^2}^{3-\alpha} \tag{2.25}$$

Choosing ϵ as $2C^*\epsilon < 1$, one concludes that

$$\sup_{0\leq t\leq T}\int_\Omega|\nabla u|^2 dx + \int_{Q_T}F^2 dxdt \leq C\int_{Q_T}|\nabla\rho|^2 dxdt + C$$

This completes the proof of Lemma 2.2.

$\square$

We are now ready to obtain the desired $L^\infty(0,T;L^2(\Omega))$ estimate of $\nabla\rho$.

Proposition 2.4. *Under the assumption* (2.1), *it holds that*

$$\sup_{0\leq t\leq T}\int_\Omega|\nabla\rho|^2 dx \leq C \quad, 0\leq T < T^* \tag{2.26}$$

$$\int_{Q_T}\rho u_t^2 dxdt + \sup_{0\leq t\leq T}\int_\Omega|\nabla u|^2 dx \leq C \quad, 0\leq T < T^* \tag{2.27}$$

$$\int_0^T\|u\|_{H^2(\Omega)}^2 dt \leq C, \quad 0\leq T < T^* \tag{2.28}$$

Proof. Differentiating the mass equation in (1.1) with respect to x_i, and multiplying the resulting identity by $2\partial_i\rho$ yield

$$\partial_t|\partial_i\rho|^2 + \text{div}(|\partial_i\rho|^2\mathbf{u}) + |\partial_i\rho|^2\text{divu} + 2\partial_i\rho\rho\partial_i\text{divu} + 2\partial_i\rho\partial_i\mathbf{u}\cdot\nabla\rho = 0 \qquad (2.29)$$

Integrating (2.29) over Ω shows that

$$\partial_t\int_\Omega|\partial_i\rho|^2dx = -\int_\Omega|\partial_i\rho|^2\text{divudx} - 2\int_\Omega\rho\partial_i\rho\partial_i\text{divudx} - \int_\Omega 2\partial_i\rho\partial_i\mathbf{u}\cdot\nabla\rho dx$$
$$= -(A_1 + A_2 + A_3) \qquad (2.30)$$

Each term on the right hand side of (2.30) can be estimated as follows:

$$|A_1(t)| \le \|divu\|_{L^\infty}(t)\int_\Omega|\partial_i\rho|^2dx \le \|divu\|_{L^\infty}(t)\int_\Omega|\nabla\rho|^2dx \qquad (2.31)$$

It follows from (2.22) that,

$$|A_2(t)| \le C\|\nabla\rho\|_{L^2}(\|\nabla P\|_{L^2} + \|F\|_{L^2}) \le C(\int_\Omega|\nabla\rho|^2dx + \int_\Omega F^2dx) \qquad (2.32)$$

$$|A_3(t)| \le C\|\nabla u\|_{L^\infty}(t)\int_\Omega|\nabla\rho|^2dx \qquad (2.33)$$

Consequently,

$$\partial_t\int_\Omega|\nabla\rho|^2dx \le C(\|\nabla u\|_{L^\infty}(t) + 1)\int_\Omega|\nabla\rho|^2dx + C\int_\Omega F^2dx \qquad (2.34)$$

This, together with Gronwall's inequality, yields

$$\int_\Omega|\nabla\rho|^2dx(t)$$
$$\le Ce^{C\int_0^t(\|\nabla u\|_{L^\infty}(s)+1)ds}(\int_\Omega|\nabla\rho_0|^2dx + \int_0^t(\int_\Omega F^2(s)dx)e^{-C\int_0^s(\|\nabla u\|_{L^\infty}(\tau)+1)d\tau}ds)$$
$$\le C\int_0^t\int_\Omega F^2dxds + C$$
$$\le C\int_0^t\int_\Omega|\nabla\rho|^2dxds + C$$
$$\qquad (2.35)$$

Hence

$$\sup_{0\le t\le T}\int_\Omega|\nabla\rho|^2dx \le C \qquad (2.36)$$

Next, it follows from (2.15), (2.24) and (2.35) that

$$\int_{Q_T}\rho u_t^2dxdt + \sup_{0\le t\le T}\int_\Omega|\nabla u|^2dx \le C \qquad (2.37)$$

This, together with $Lu = \rho u_t + \rho u \cdot \nabla u + \nabla P$, shows that

$$\|u\|_{L^2(0,T;H^2(\Omega))} \le \|\rho u_t\|_{L^2(Q_T)} + \|\rho u \cdot \nabla u\|_{L^2(Q_T)} + \|\nabla P\|_{L^2(Q_T)}$$

$$\le C + C\|\nabla\rho\|_{L^2(Q_T)} \le C \tag{2.38}$$

$\square$

Next, we proceed to improve the regularity of ρ and u. To this end, we first derive some bounds on derivatives of u based on above estimates.

Proposition 2.5. *Under the condition* (2.1), *it holds that*

$$\sup_{0 \le t \le T} \|\rho^{1/2} u_t(t)\|_{L^2}^2 + \int_{Q_T} |\nabla u_t|^2 dx dt \le C, \quad 0 \le T < T^* \tag{2.39}$$

$$\sup_{0 \le t \le T} \|u\|_{H^2} \le C, \quad 0 \le T < T^* \tag{2.40}$$

Proof. Differentiating the momentum equations in (1.1) with respect to time t yields

$$\rho u_{tt} + \rho u \cdot \nabla u_t - \mu \triangle u_t - (\mu + \lambda)\nabla \text{div} u_t + \nabla p_t = -\rho_t(u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u \tag{2.41}$$

Taking the inner product of the above equation with u_t in $L^2(\Omega)$ and integrating by parts, one gets

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho u_t^2 dx + \int_\Omega (\mu|\nabla u_t|^2 + (\lambda + \mu)(\text{div} u_t)^2)dx - \int_\Omega P_t \text{div} u_t dx$$

$$= -\int_\Omega (\rho u \cdot \nabla[(u_t + u \cdot \nabla u)u_t] + \rho(u_t \cdot \nabla u) \cdot u_t)dx \tag{2.42}$$

Due to (2.18), the last term on the left-hand side of (2.42) can be rewritten as

$$-\int_\Omega P_t \text{div} u_t dx = \frac{d}{dt}\int_\Omega \frac{\gamma}{2}P(\text{div} u)^2 dx + \int_\Omega \nabla P \cdot (u \text{div} u_t)dx$$

$$+ \frac{\gamma}{2}\int_\Omega (-Pu \cdot \nabla(\text{div} u)^2 + (\gamma - 1)P(\text{div} u)^3)dx \tag{2.43}$$

It follows from (2.41) and (2.42) that

$$\frac{d}{dt}\int_\Omega (\frac{1}{2}\rho u_t^2 + \frac{\gamma}{2}P(\text{div} u)^2)dx + \int_\Omega \mu|\nabla u_t|^2 dx$$

$$\le \int_\Omega (2\rho|u||u_t||\nabla u_t| + \rho|u||u_t||\nabla u|^2 + \rho|u|^2|u_t||\nabla^2 u| + \rho|u|^2||\nabla u||\nabla u_t|$$

$$+ \rho|u_t|^2|\nabla u| + |\nabla P||u||\nabla u_t| + \gamma P|u||\nabla u||\nabla^2 u| + \gamma^2 P|\nabla u|^3)dx \tag{2.44}$$

$$\equiv \sum_{i=0}^8 F_i$$

Now, we estimate each F_i separately, where the Sobolev inequality and Hölder inequality will be used frequently.

$$
\begin{aligned}
|F_1| &= \int_\Omega 2\rho|u||u_t||\nabla u_t|dx \\
&\leq C\|u\|_{L^6}\|\rho^{1/2}u_t\|_{L^3}\|\nabla u_t\|_{L^2} \\
&\leq C\|\rho^{1/2}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}} \\
&\leq \epsilon\|\nabla u_t\|_{L^2}^2 + C\|\rho^{1/2}u_t\|_{L^2}^2 \quad,
\end{aligned}
\tag{2.45}
$$

where one has used (2.27) and the interpolation inequality.

Similarly, it follows from $Lemma\,2.1$ and $Proposition\,2.4$ that

$$
\begin{aligned}
|F_2| &= \int_\Omega \rho|u||u_t||\nabla u|^2 dx \\
&\leq C\|u\|_{L^6}\|u_t\|_{L^6}\|\nabla u\|_{L^3}^2 \\
&\leq C\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{L^6} \\
&\leq C\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^6} \\
&\leq \epsilon\|\nabla u_t\|_{L^2}^2 + C\|u\|_{H^2}^2 \quad,
\end{aligned}
\tag{2.46}
$$

$$
\begin{aligned}
|F_3| &= \int_\Omega \rho|u|^2|u_t||\nabla^2 u|dx \\
&\leq \|u^2\|_{L^3}\|u_t\|_{L^6}\|\nabla^2 u\|_{L^2} \\
&\leq \epsilon\|\nabla u_t\|_{L^2}^2 + C\|u\|_{H^2}^2 \quad,
\end{aligned}
\tag{2.47}
$$

$$
\begin{aligned}
|F_4| &= \int_\Omega \rho|u|^2|\nabla u||\nabla u_t|dx \\
&\leq C\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^6}\|u^2\|_{L^3} \\
&\leq C\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2} \\
&\leq \epsilon\|\nabla u_t\|_{L^2}^2 + C\|u\|_{H^2}^2 \quad,
\end{aligned}
\tag{2.48}
$$

$$
\begin{aligned}
|F_5| &= \int_\Omega \rho|u_t|^2|\nabla u|dx \\
&\leq C\|\rho u_t^2\|_{L^2}\|\nabla u\|_{L^2} \\
&\leq C\|\rho^{1/2}u_t\|_{L^4}^2 \\
&\leq \epsilon\|u_t\|_{L^6}^2 + C\|\rho^{1/2}u_t^2\|_{L^2} \quad,
\end{aligned}
\tag{2.49}
$$

$$|F_6| = \int_\Omega |\nabla P||u||\nabla u_t|dx$$

$$\leq C\|\nabla P\|_{L^2}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}$$

$$\leq C\|u\|_{H^2}\|\nabla u_t\|_{L^2}$$

$$\leq \epsilon\|\nabla u_t\|_{L^2}^2 + C\|u\|_{H^2}^2 \quad ,$$

(2.50)

$$|F_7| = \int_\Omega \gamma P|u||\nabla u||\nabla^2 u|dx$$

$$\leq C\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^2}\|u\|_{L^\infty}$$

$$\leq C\|\nabla^2 u\|_{L^2}\|u\|_{L^\infty}$$

$$\leq C\|u\|_{H^2}^2 \quad ,$$

(2.51)

and finally,

$$|F_8| = \int_\Omega \gamma^2 P|\nabla u|^3 dx$$

$$\leq C\int_\Omega |\nabla u|^3 dx$$

$$\leq C\|\nabla u\|_{L^\infty(\Omega)}\int_\Omega |\nabla u|^2 dx$$

$$\leq C\|\nabla u\|_{L^\infty(\Omega)} \quad .$$

(2.52)

Collecting all the estimates for F_i, we conclude

$$\frac{d}{dt}\int_\Omega (\frac{1}{2}\rho u_t^2 + \frac{\gamma}{2}P(\mathrm{div}u)^2)\mathrm{dx} + \int_\Omega \mu|\nabla u_t|^2\mathrm{dx}$$

$$\leq 6\epsilon\int_\Omega |\nabla u_t|^2 dx + C(\|\rho^{1/2}u_t\|_{L^2}^2 + \|u\|_{H^2}^2 + \|\nabla\rho\|_{L^2}^2 + \|\nabla u\|_{L^\infty})$$

(2.53)

Thanks to the compatibility condition:

$$\rho_0(x)^{\frac{1}{2}}(\rho_0(x)^{\frac{1}{2}}u_t(t=0,x) + \rho_0^{\frac{1}{2}}u_0\cdot\nabla u_0(x) - \rho_0^{\frac{1}{2}}g) = 0$$

(2.54)

it holds that

$$\rho_0(x)^{\frac{1}{2}}u_t(t=0,x) = \rho_0^{\frac{1}{2}}u_0\cdot\nabla u_0(x) - \rho_0^{\frac{1}{2}}g \in L^2(\Omega)$$

(2.55)

Therefore, for arbitrary small ϵ, (2.53) yields

$$\sup_{0\leq t\leq T}\|\rho^{1/2}u_t(t)\|_{L^2}^2 + \int_{Q_T}|\nabla u_t|^2 dxdt \leq C, \quad 0\leq T < T^*$$

(2.56)

Moreover,

$$Lu = \rho u_t + \rho u\cdot\nabla u + \nabla P$$

thus,

$$\|u\|_{H^2} \le C(\|\rho^{\frac{1}{2}} u_t\|_{L^2} + \|u\|_{L^6}\|\nabla u\|_{L^3} + \|\nabla P\|_{L^2}) \le C(\|\rho^{\frac{1}{2}} u_t\|_{L^2} + \|\nabla u\|_{L^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}} + \|\nabla P\|_{L^2})$$

Hence,

$$\sup_{0 \le T < T^*} \|u\|_{H^2}^2 \le C \tag{2.57}$$

Thus, *Proposition* 2.5 follows immediately.

$\square$

Finally, the following lemma gives bounds of the first order derivatives of the density and the second derivatives of the velocity.

Lemma 2.6. *Under the condition* (2.1), *it holds that*

$$\sup_{0 \le t \le T} (\|\rho_t(t)\|_{L^6} + \|\rho\|_{W^{1,6}}) \le C, \quad 0 \le T < T^*$$

$$\int_0^T \|u(t)\|_{W^{2,6}}^2 dt \le C, \quad 0 \le T < T_*$$

Proof. It follows from (2.55) and (2.56) that

$$u_t \in L^2(0, T; L^6(\Omega)), \nabla u \in L^6(Q_T)$$

$$F \in L^2(0, T; L^6(\Omega))$$

Differentiating the mass equation in (1.1) with respect to x_i, and multiplying the resulting identity by $6|\partial_i \rho|^4 \partial_i \rho$, one gets after integration that

$$\partial_t \int_\Omega |\partial_i \rho|^6 dx = -5 \int_\Omega |\partial_i \rho|^6 \mathrm{div} u dx - 6 \int_\Omega \rho |\partial_i \rho|^4 \partial_i \rho \partial_i \mathrm{div} u dx$$

$$- 6 \int_\Omega |\partial_i \rho|^4 \partial_i \rho \partial_i u \cdot \nabla \rho dx \tag{2.58}$$

$$= -(B_1 + B_2 + B_3)$$

Using $Lu = F + \nabla P$, one can estimate each term on the righthand side of (2.57) as follows:

$$|B_1(t)| \le 5\|\nabla u\|_{L^\infty}(t) \int_\Omega |\partial_i \rho|^6 dx \le C\|\nabla u\|_{L^\infty}(t) \int_\Omega |\nabla \rho|^6 dx \quad , \tag{2.59}$$

$$|B_2(t)| \le C\||\nabla \rho|^5\|_{L^{\frac{6}{5}}} (\|\nabla P\|_{L^6} + \|F\|_{L^6}) \quad , \tag{2.60}$$

$$|B_4(t)| \leq C\|\nabla u\|_{L^\infty}(t) \int_\Omega |\nabla \rho|^6 dx \quad . \tag{2.61}$$

It follows from $(2.57) - (2.60)$ that

$$\partial_t \|\nabla \rho\|_{L^6} \leq C(\|\nabla u\|_{L^\infty}(t) + 1)\|\nabla \rho\|_{L^6} + C\|F\|_{L^6} \tag{2.62}$$

Hence,

$$\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^6} \leq C \quad . $$

Therefore, due to this, (2.57) and interpolation inequality, one has

$$\rho_t = -(u \cdot \nabla \rho + \rho \mathrm{div} u) \in \mathrm{L}^\infty \mathrm{L}^6 \quad . \tag{2.63}$$

Finally, taking into account that

$$Lu = F + \nabla P \in L^2 L^6 \quad , $$

one has

$$\int_0^T \|u\|_{W^{2,6}(\Omega)}^2 dt \leq C \quad . \tag{2.64}$$

This finishes the proof of *Lemma*2.6. □

3. Improved regularity of the density and the velocity

In this section, we obtain some higher order regularity of the density and the velocity. However, we may not deduce the $L^\infty H^1$ estimate of $\nabla \rho$ directly just similar to *Lemma*2.4 or *Lemma*2.6, as the L^2 norm of $\nabla^2 P$ can't be controlled by that of $\nabla^2 \rho$ due to the presence of vacuum, unless γ is large is enough. In order to circumvent such difficulties, we first need to improve the regularity of the pressure by observing that P satisfies a linear transport equation.

In fact, we have the following lemma.

Lemma 3.1.

$$\|P\|_{L^\infty H^2} + \|P_t\|_{L^\infty H^1} + \|P_{tt}\|_{L^2 L^2} \leq C, \quad 0 \leq T < T^*$$

$$\|\rho\|_{L^\infty H^2} + \|\rho_t\|_{L^\infty H^1} + \|\rho_{tt}\|_{L^2 L^2} \leq C, \quad 0 \leq T < T^* \tag{3.1}$$

Proof. For the proof of (3.1), we will make use of the transport equation (2.18) for the pressure and the elliptic regularity of the system $Lu = F + \nabla P$ for the velocity u.

Indeed, it follows from the elliptic regularity that

$$\|u\|_{H^3} \leq C(\|F\|_{H^1} + \|\nabla P\|_{H^1}) \leq C(\|F\|_{H^1} + \|\nabla^2 P\|_{L^2} + C) \tag{3.2}$$

Apply D_{ij} to both side of (2.18) to yield

$$(D_{ij}P)_t + D_{ij}u \cdot \nabla P + u \cdot \nabla D_{ij}P + D_i u \cdot \nabla D_j P + D_j u \cdot \nabla D_i P$$
$$+ \gamma D_{ij}P\text{divu} + \gamma P D_{ij}\text{divu} + \gamma(D_i P D_j \text{divu} + D_j P D_i \text{divu}) = 0 \tag{3.3}$$

Multiplying (3.3) by $2D_{ij}P$, one gets

$$\partial_t (D_{ij}P)^2 + \text{div}(|D_{ij}P|^2 u) + (2\gamma - 1)|D_{ij}P|^2 \text{divu} + 2D_{ij}PD_i u \cdot \nabla D_j P + 2D_{ij}PD_j u \cdot \nabla D_i P$$
$$+ 2\gamma P D_{ij}PD_{ij}\text{divu} + 2\gamma D_i P D_{ij}PD_j\text{divu} + 2\gamma D_j P D_{ij}PD_i\text{divu} + +2D_{ij}PD_{ij}u \cdot \nabla P = 0 \tag{3.4}$$

Integrating ()3.4 over Ω, yields

$$\partial_t \int_\Omega |D_{ij}P|^2 dx = -(2\gamma - 1) \int_\Omega |D_{ij}P|^2 \text{divudx} - 2 \int_\Omega D_{ij}PD_i u \cdot \nabla D_j P dx$$
$$- 2 \int_\Omega D_{ij}PD_j u \cdot \nabla D_i P dx - 2\gamma \int_\Omega P D_{ij}PD_{ij}\text{divudx}$$
$$- 2\gamma \int_\Omega D_i P D_{ij}PD_j\text{divudx} - 2\gamma \int_\Omega D_j P D_{ij}PD_i\text{divudx} - 2 \int_\Omega D_{ij}PD_{ij}u \cdot \nabla P dx$$
$$= -\sum_{i=1}^{7} P_i \tag{3.5}$$

Each term of P_i can be estimated as follows

$$|P_1, P_2, P_3| \leq C\|\nabla u\|_{L^\infty} \int_\Omega |\nabla^2 P|^2 dx \quad , \tag{3.6}$$

$$|P_4| \leq C\|\nabla^2 P\|_{L^2}\|D_{ij}\text{divu}\|_{L^2}$$
$$\leq C\|\nabla^2 P\|_{L^2}(\|F\|_{H^1} + \|\nabla^2 P\|_{L^2} + C) \tag{3.7}$$
$$\leq C\|\nabla^2 P\|_{L^2}^2 + C\|F\|_{H^1}^2 + C \quad ,$$

$$|P_5, P_6, P_7| \leq C\|D_i P\|_{L^3}\|\nabla^2 P\|_{L^2}\|\nabla^2 u\|_{L^6}$$
$$\leq C\|\nabla^2 P\|_{L^2}\|\nabla^2 u\|_{L^6} \tag{3.8}$$
$$\leq C\|\nabla^2 P\|_{L^2}^2 + C\|\nabla^2 u\|_{L^6}^2 \quad .$$

where one has used $lemma 2.6$. Collecting $(3.5) - (3.8)$ yields

$$\partial_t \int_\Omega |D_{ij}P|^2 dx \leq C(\|\nabla u\|_{L^\infty} + 1) \int_\Omega |\nabla^2 P|^2 dx$$
$$+ C(\|F\|_{H^1}^2 + \|\nabla^2 u\|_{L^6}^2 + 1) \tag{3.9}$$

Using Gronwall's inequality and $P_0 \in H^3$, $F \in L^2 H^1$, $u \in L^2 W^{2,6}$, one has

$$\|P\|_{L^\infty H^2} \leq C \tag{3.10}$$

As a consequence of (2.18), (3.10), $Lemma 2.1$ and $Proposition 2.4, 2.5$, one has

$$\|P_t\|_{L^\infty H^1} \leq C \tag{3.11}$$

In view of $(3.10) - (3.11)$, we may apply the same technique to the mass equation to derive

$$\|\rho\|_{L^\infty H^2} + \|\rho_t\|_{L^\infty H^1} \leq C \tag{3.12}$$

Note that

$$\rho_{tt} + \rho_t \text{divu} + \rho \text{divu}_t + u_t \cdot \nabla\rho + u \cdot \nabla\rho_t = 0$$

$$P_{tt} + \gamma P_t \text{divu} + \gamma P \text{divu}_t + u_t \cdot \nabla P + u \cdot \nabla P_t = 0$$

then one has $\rho_{tt} \in L^2 L^2$ and $P_{tt} \in L^2 L^2$. Thus the lemma is proved due to $(3.10) - (3.12)$, $Lemma 2.4$ and $Proposition 2.5$. $\qquad\square$

In order to obtain high regularity of (ρ, u), we need the following improved estimate.

Lemma 3.2.

$$\int_{Q_T} \rho u_{tt}^2 dx dt + \sup_{0 \leq t \leq T} \int_\Omega |\nabla u_t|^2 dx \leq C, \quad 0 \leq T < T^* \tag{3.13}$$

Proof. Multiplying (2.41) by u_{tt}, and integrating by parts, one gets that

$$\int_\Omega \rho u_{tt}^2 dx + \int_\Omega \rho u \cdot \nabla u_t \cdot u_{tt} dx + \frac{d}{dt} \int_\Omega \frac{\mu}{2} |\nabla u_t|^2 + \frac{\lambda + \mu}{2} (\text{divu}_t)^2 dx$$
$$= \int_\Omega P_t \text{divu}_{tt} dx - \int_\Omega \rho_t (u_t + u \cdot \nabla u) u_{tt} dx - \int_\Omega \rho u_t \cdot \nabla u \cdot u_{tt} dx \tag{3.14}$$

Note that

$$|\int_\Omega \rho u \cdot \nabla u_t \cdot u_{tt} dx| \leq \epsilon \int_\Omega \rho u_{tt}^2 dx + C \int_\Omega \rho (u \cdot \nabla u_t)^2 dx \quad , \tag{3.15}$$

$$|\int_\Omega \rho u_t \cdot \nabla u \cdot u_{tt} dx| \leq \epsilon \|\rho^{\frac{1}{2}} u_{tt}\|_{L^2}^2 + C\|\rho^{\frac{1}{2}} u_t\|_{L^3}^2 \|\nabla u\|_{L^6}^2$$

$$\leq \epsilon \|\rho^{\frac{1}{2}} u_{tt}\|_{L^2}^2 + C\|\rho^{\frac{1}{2}} u_t\|_{L^2} \|u_t\|_{L^6} \tag{3.16}$$

$$\leq \epsilon \|\rho^{\frac{1}{2}} u_{tt}\|_{L^2}^2 + C\|\nabla u_t\|_{L^2} \quad,$$

The first term of the right hand side of (3.14) becomes

$$\int_\Omega P_t \mathrm{div} u_{tt} \mathrm{dx} = \frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \mathrm{P_t div u_t dx} - \int_\Omega \mathrm{P_{tt} div u_t dx} \tag{3.17}$$

which can be estimated by

$$|\int_\Omega P_t \mathrm{div} u_t(\mathrm{T})\mathrm{dx}| \leq \frac{\mu}{8} \int_\Omega |\nabla u_t|^2(\mathrm{T})\mathrm{dx} + C\|\mathrm{P_t}\|_{\mathrm{L^2}}^2(\mathrm{T}) \leq \frac{\mu}{8} \int_\Omega |\nabla u_t|^2(\mathrm{T})\mathrm{dx} + \mathrm{C}$$

$$|\int_\Omega P_{tt} \mathrm{div} u_t \mathrm{dx}| \leq \|\mathrm{P_{tt}}\|_{\mathrm{L^2}} \|\mathrm{div} u_t\|_{\mathrm{L^2}} \leq \|\mathrm{P_{tt}}\|_{\mathrm{L^2}}^2 + \|\nabla u_t\|_{\mathrm{L^2}}^2$$

$$\tag{3.18}$$

The second term of the righthand side of (3.14), can be rewritten as

$$\int_\Omega \rho_t(u_t + u \cdot \nabla u)u_{tt} dx$$

$$= \frac{d}{dt} \int_\Omega \rho_t(\frac{1}{2}|u_t|^2)dx - \int_\Omega \rho_{tt}(\frac{1}{2}|u_t|^2)dx + \int_\Omega \rho_t(u \cdot \nabla u)u_{tt} dx \tag{3.19}$$

Each term of the right hand side of (3.19) can be estimated as follows

$$|\int_\Omega \rho_t(\frac{1}{2}|u_t|^2)(T)dx| = |\int_\Omega \mathrm{div}(\rho u)(\frac{1}{2}|u_t|^2)(\mathrm{T})\mathrm{dx}|$$

$$= |\int_\Omega \rho u \cdot \nabla(\frac{1}{2}|u_t|^2)(T)dx|$$

$$\leq \frac{\mu}{8} \int_\Omega |\nabla u_t|^2(T)dx + C(\mu)\|\rho^{\frac{1}{2}}|u_t|(T)\|_{L^2}^2 \tag{3.20}$$

$$\leq \frac{\mu}{8} \int_\Omega |\nabla u_t|^2(T)dx + C$$

It follows from $Lu_t = F_t + \nabla P_t$ and the standard elliptic regularity theory that

$$\|u_t\|_{H^2} \leq C\|F_t\|_{L^2} + C\|\nabla P_t\|_{L^2} \tag{3.21}$$

A simple calculation based on the previous estimates shows that

$$\|F_t\|_{L^2} \leq \|\rho_t u_t\|_{L^2} + \|\rho u_{tt}\|_{L^2} + \|\rho_t u \cdot \nabla u\|_{L^2} + \|\rho u_t \cdot \nabla u\|_{L^2} + \|\rho u \cdot \nabla u_t\|_{L^2}$$

$$\leq C(\|\rho_t\|_{L^3}\|u_t\|_{L^6} + \|\rho u_{tt}\|_{L^2} + \|\rho_t\|_{L^3}\|\nabla u\|_{L^6} + \|u_t\|_{L^6}\|\nabla u\|_{L^3} + \|\nabla u_t\|_{L^2})$$

$$\leq C(\|\rho^{\frac{1}{2}} u_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} + 1)$$

$$\tag{3.22}$$

Accordingly, the second term of righthand side of (3.19) becomes

$$
\begin{aligned}
|\int_\Omega \rho_{tt}(\frac{1}{2}|u_t|^2)dx| &= |\int_\Omega \mathrm{div}(\rho_t \mathrm{u} + \rho \mathrm{u_t})(\frac{1}{2}|\mathrm{u_t}|^2)\mathrm{dx}| \\
&= |\int_\Omega (\rho_t u + \rho u_t)\nabla(\frac{1}{2}|u_t|^2)dx| \\
&\leq C\|\rho_t\|_{L^3}\|u\|_{L^\infty}\|u_t\|_{L^6}\|\nabla u_t\|_{L^2} + \int_\Omega \rho|u_t|^2|\nabla u_t|dx \\
&\leq C\|\nabla u_t\|_{L^2}^2 + C\|\rho^{\frac{1}{2}}u_t\|_{L^3}^2\|\nabla u_t\|_{L^3} \\
&\leq C\|\nabla u_t\|_{L^2}^2 + C\|\rho^{\frac{1}{2}}u_t\|_{L^2}\|\rho^{\frac{1}{2}}u_t\|_{L^6}\|\nabla u_t\|_{L^3} \\
&\leq C\|\nabla u_t\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}\|\nabla u_t\|_{L^3} \\
&\leq C\|\nabla u_t\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}\|u_t\|_{H^2} \\
&\leq C\|\nabla u_t\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}(\|\rho^{\frac{1}{2}}u_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} + 1) \\
&\leq C\|\nabla u_t\|_{L^2}^2 + \epsilon\|\rho^{\frac{1}{2}}u_{tt}\|_{L^2}^2 + C
\end{aligned}
\tag{3.23}
$$

where we use (3.23) and (3.24). We write the last term of righthand side of (3.21) as

$$
\int_\Omega \rho_t(u \cdot \nabla u)u_{tt}dx
$$
$$
= \frac{d}{dt}\int_\Omega \rho_t(u \cdot \nabla u)u_t dx - \int_\Omega \rho_{tt}(u \cdot \nabla u)u_t dx - \int_\Omega \rho_t(u_t \cdot \nabla u)u_t dx - \int_\Omega \rho_t(u \cdot \nabla u_t)u_t dx
\tag{3.24}
$$

Observe that

$$
\begin{aligned}
|\int_\Omega \rho_t(u \cdot \nabla u)u_t dx| &\leq \|\rho_t\|_{L^3}\|u \cdot \nabla u\|_{L^2}\|u_t\|_{L^6} \\
&\leq \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2 + C \quad,
\end{aligned}
\tag{3.25}
$$

and

$$
\begin{aligned}
|\int_\Omega \rho_{tt}(u \cdot \nabla u)u_t dx| &\leq \|\rho_{tt}\|_{L^2}\|u \cdot \nabla u\|_{L^3}\|u_t\|_{L^6} \\
&\leq C\|\rho_{tt}\|_{L^2}\|u_t\|_{L^6} \\
&\leq C\|\rho_{tt}\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2 \quad,
\end{aligned}
\tag{3.26}
$$

$$
\begin{aligned}
|\int_\Omega \rho_t(u_t \cdot \nabla u)u_t dx| &\leq \|\rho_t\|_{L^2}\||u_t|^2\|_{L^3}\|\nabla u\|_{L^6} \\
&\leq C\|\nabla u_t\|_{L^2}^2 \quad,
\end{aligned}
\tag{3.27}
$$

and

$$|\int_\Omega \rho_t(u\cdot\nabla u_t)u_t dx| \le \|\rho_t\|_{L^3}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|u_t\|_{L^6}$$

$$\le C\|\nabla u_t\|_{L^2}\|u_t\|_{L^6} \tag{3.28}$$

$$\le C\|\nabla u_t\|_{L^2}^2$$

It follows from $Lemma\, 3.1$ and $Proposition\, 2.5$ that

$$(\rho_t, P_t) \in L^\infty H^1, (\rho_{tt}, P_{tt}) \in L^2 L^2, \nabla u_t \in L^2 L^2$$

In view of regularity (1.15), there exist a sequence ϵ_i, such that $\epsilon_i \to 0, \epsilon_i > 0$, and

$$\|u_t(\epsilon_i)\|_{H^1} \le \|u_t\|_{L^\infty(0,T_*;H_0^1(\Omega))} \le C(\|u_0\|_{H^3}, \|\rho_0\|_{H^3}, \|P_0\|_{H^3}) \tag{3.29}$$

Collecting all the estimates $(3.14) - (3.29)$, integrating over (ϵ_i, T), accordingly

$$\int_{\epsilon_i}^T \int_\Omega \rho u_{tt}^2 dx dt + \int_\Omega \frac{\mu}{8}|\nabla u_t(T)|^2 + \frac{\lambda+\mu}{2}(\mathrm{div}u_t(T))^2 dx$$

$$\le 3\epsilon \int_{\epsilon_i}^T \int_\Omega \rho u_{tt}^2 dx dt + C\int_{\epsilon_i}^T (\|P_{tt}\|_{L^2}^2 + \|\rho_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|\rho^{\frac{1}{2}}u_t\|_{L^2}^2 + 1)dt$$

$$+ \int_\Omega \frac{\mu}{8}|\nabla u_t(\epsilon_i)|^2 + \frac{\lambda+\mu}{2}(\mathrm{div}u_t(\epsilon_i))^2 dx \tag{3.30}$$

$$\le 3\epsilon \int_0^T \int_\Omega \rho u_{tt}^2 dx dt + C\int_0^T (\|P_{tt}\|_{L^2}^2 + \|\rho_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|\rho^{\frac{1}{2}}u_t\|_{L^2}^2 + 1)dt$$

$$+ C(\|u_0\|_{H^3}, \|\rho_0\|_{H^3}, \|P_0\|_{H^3})$$

The righthand of (3.30) is independent of ϵ_i. Therefore, letting ϵ_i go to 0 and choosing ϵ small enough, we complete the proof of lemma 3.2.

$$\square$$

Finally, we have

Lemma 3.3.

$$\|\rho\|_{L^\infty H^3} + \|P\|_{L^\infty H^3} + \|u\|_{L^\infty H^3} \le C \tag{3.31}$$

Proof. It follows from (3.1) and (3.15) that

$$F = \rho u_t + \rho u \cdot \nabla u \in L^\infty H^1, \quad \nabla P \in L^\infty H^1 \tag{3.32}$$

which gives

$$Lu = F + \nabla P \in L^\infty H^1 \tag{3.33}$$

As a consequence,

$$\|u\|_{L^\infty H^3} \leq C \tag{3.34}$$

Therefore,

$$Lu_t = F_t + \nabla P_t \in L^2 L^2 \tag{3.35}$$

which implies

$$u_t \in L^2 H^2, F \in L^2 H^2 \tag{3.36}$$

By an estimate similar to lemma 3.1, one can derive the high regularity of pressure P, it holds that

$$\|P\|_{L^\infty H^3} \leq C \tag{3.37}$$

In view of the mass equation, one can show that

$$\|\rho\|_{L^\infty H^3} \leq C \tag{3.38}$$

$$\square$$

This will be enough to extend the classical solutions of (ρ, u) beyond $t \geq T^*$.

In fact, in view of Lemma $3.1 - 3.3$, the functions $(\rho, P, u)|_{t=T^*} = \lim_{t \to T^*}(\rho, P, u)$ satisfy the conditions imposed on the initial data $(1.13) - (1.14)$ at the time $t = T^*$ Furthermore,

$$\rho u_t + \rho u \cdot \nabla u \in L^\infty H_0^1$$

$$-Lu + \nabla P|_{t=T^*} = \lim_{t \to T^*}(\rho u_t + \rho u \cdot \nabla u) \triangleq \rho g|_{t=T^*} \quad, \tag{3.39}$$

where $g|_{t=T^*} \in H_0^1(\Omega)$ and $\rho^{\frac{1}{2}} g|_{T^*} \in L^2$. Therefore, we can take $(\rho, P, u)|_{t=T^*}$ as the initial data and apply the local existence theorem [?] to extend our local classical solution beyond T^*. This contradicts the assumption on T^*.

Note that a few modifications can be applied for both periodic case and $\Omega = R^3$, so theorem 1.2 holds.

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