Existence of solutions for three dimensional stationary incompressible Euler equations with nonvanishing vorticity*

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Dedicated to Professor Andrew Majda for his 60th Birthday

Abstract

In this paper, solutions with nonvanishing vorticity are established for the three dimensional stationary incompressible Euler equations on simply connected bounded three dimensional domains with smooth boundary. A class of additional boundary conditions for the vorticities are identified so that the solution is unique and stable.

Key words: three dimensional stationary incompressible Euler equations, boundary value condition, nonvanishing vorticity.

1 Introduction and main results

Consider the stationary incompressible Euler equations

$$(v \cdot \nabla)v + \nabla p = 0, \quad x \in \Omega, \tag{1}$$

$$\text{div } v = 0, \quad x \in \Omega, \tag{2}$$

with the boundary condition

$$n \cdot v = f, \quad x \in \partial\Omega, \tag{3}$$

where $\Omega(\subset R^3)$ is a bounded, simply connected domain, $v \in C^1(\overline{\Omega}, R^3)$ denotes the velocity and $p \in C^1(\overline{\Omega}, R)$ the pressure of the flow, n denotes the exterior unit vector field normal to the boundary $\partial\Omega$. The given function f is assumed to satisfy

$$\int_{\partial\Omega} f dS_x = 0. \tag{4}$$

It is well known that for simply connected domains Ω problem (1)-(3) has an irrotational solution (v, p), which is unique up to addition of constants

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to the pressure. Based on a solution (v_0, p_0) to the problem (1)-(3), H. D. Alber [1] constructs solutions with nonvanishing vorticity to problem (1)-(3). Under some assumptions, for suitable h and g, he proves that the problem (1)-(3) has a unique steady solution in a neighborhood of (v_0, p_0) satisfying the additional boundary conditions

$$n(x) \cdot \text{curl } v(x) = h(x) + n(x) \cdot \text{curl } v_0(x)$$

and

$$\frac{1}{2}|v(x)|^2 + p(x) = g(x) + \frac{1}{2}|v_0(x)|^2 + p_0(x)$$

for all $x \in \partial\Omega_-$, where

$$\partial\Omega_- = \{x \in \partial\Omega \mid f(x) < 0\}, \qquad \partial\Omega_+ = \{x \in \partial\Omega \mid f(x) > 0\}.$$

In this paper, we will establish the well-posendess of the solution to the problem (1)-(3) satisfying the following additional boundary conditions

$$\text{curl } v = av + b \qquad\qquad \text{for all } x \in \partial\Omega_-$$

with suitable given a and b.

Incompressible flows with nontrivial vorticity are important topics for fluid dynamics [16, 17]. There exist huge literatures dealing with the stationary incompressible Euler equations, such as, exact solutions (see [19, 30] and references therein), the existence of solutions (see [2, 3, 5, 6, 7, 11, 12, 14, 16, 20, 21, 22, 23, 24, 25, 27, 31, 32] and references therein), symmetry of solutions (see [13] and references therein), stability of solutions (see [15, 16] and references therein), topological properties of solutions ([10]) and numerical approximations of solutions (see [8, 9, 28, 35] and references therein). For proving the existence of solutions, there are various methods, such as the variational methods (see [2, 3, 5, 12, 14, 20, 31, 32] and references therein), the statistical mechanics methods ([6, 7]), the pseudo-advection method ([22, 24, 25]), the magnetohydrodynamic approach (see [21, 23]), the fixed points method (see [1]) and some other methods in [29, 34]. Most of them can only be used to the two-dimensional or the axisymmetric cases, except for [1, 4, 23, 36]. In [21], a measure-valued solution is found for three-dimensional steady Euler equations with nontrivial vorticity. While in [4, 34], the problem has been well studied in the special case that v and curl v are parallel.

Motivated by the results in [1], we establish the well-posedness of classical solutions for problem (1)-(3) without any reference solutions. The main result is the following theorem.

Theorem 1.1 Suppose that Ω is a bounded, simply connected domain of R^3 with C^2 boundary $\partial\Omega$. Assume that $f \in H^2(\partial\Omega, R)$ satisfying (4).

Let $v_0 \in H^3(\Omega, R^3)$ and $\alpha_0, \beta_0, \gamma_0, L_0 \in (0, +\infty)$ satisfying that

$$\begin{aligned} \text{div } v_0 = 0, \quad & x \in \Omega, \\ n \cdot v_0 = f, \quad & x \in \partial\Omega, \end{aligned}$$

$$|v_0(x)| \geq 2\alpha_0 \tag{5}$$

for all $x \in \Omega$,

$$\|v_0\|_{3,\Omega} \leq \frac{1}{2}\beta_0, \tag{6}$$

v_0 does not have closed stream lines, the length of all stream line of v_0 in Ω is less than L_0, and

$$\liminf_{t \to 0^+} \frac{\text{dist}(\partial\Omega_-, x + tv_0(x))}{t} > 0 \tag{7}$$

uniformly for all $x \in \partial\partial\Omega_-$ and

$$\liminf_{t \to 0^+} \frac{\text{dist}(\partial\Omega_+, x - tv_0(x))}{t} > 0$$

uniformly for all $x \in \partial\partial\Omega_+$, where

$$\partial\partial\Omega_\pm = \overline{\partial\Omega_\pm} \cap \overline{(\partial\Omega \setminus \partial\Omega_\pm)}$$

is the boundary of $\partial\Omega_\pm$ in $\partial\Omega$.

Then there exists a constant

$$\gamma_0 = \gamma_0(\alpha_0, \beta_0, L_0) > 0$$

and for every $0 < \gamma \leq \gamma_0$, there exist constants

$$K_i = K_i(\alpha_0, \beta_0, L_0, \gamma) > 0, \qquad i = 1, 2, 3$$

such that for all $a \in H^2(\partial\Omega_-, R)$, $b \in H^2(\partial\Omega_-, R^3)$ with

$$b \cdot n = 0, \quad \forall x \in \partial\Omega_-, \tag{8}$$

$$\text{div}(fb) = 0, \quad \forall x \in \partial\Omega_-, \tag{9}$$

(where div(fb) is the divergence of the vector-valued function fb on $\partial\Omega_-$ defined as

$$\text{div}(fb) = \lim \frac{1}{\Delta s} \int_l (fb) \cdot (n \times dl)$$

where s is a surface lying on $\partial\Omega_-$ with smooth boundary l) and v_0 with

$$\|a\| + \|b\| + \|\text{curl } v_0\|_{2,\Omega} \leq K_1, \tag{10}$$

the problem (1)-(3) has a solution $(v, p) \in H^3(\Omega, R^3 \times R)$ with

$$\text{curl } v(x) = a(x)v(x) + b(x) \tag{11}$$

for all $x \in \partial\Omega_-$, and

$$\frac{1}{|\Omega|}\int_\Omega p(x)dx = 1, \tag{12}$$

where

$$\begin{aligned}\|a\| &= \||f|^{-2}a\|_{L^\infty(\partial\Omega_-)} + \||f|^{-3}a\|_{L^2(\partial\Omega_-)} \\ &\quad + \||f|^{-3}\nabla_T a\|_{L^2(\partial\Omega_-)} + \||f|^{-2}\nabla_T^2 a\|_{L^2(\partial\Omega_-)},\end{aligned}$$

$$\begin{aligned}\|b\| &= \||f|^{-2}b\|_{L^\infty(\partial\Omega_-)} + \||f|^{-3}b\|_{L^2(\partial\Omega_-)} \\ &\quad + \||f|^{-3}\nabla_T b\|_{L^2(\partial\Omega_-)} + \||f|^{-2}\nabla_T^2 b\|_{L^2(\partial\Omega_-)},\end{aligned}$$

$|\Omega|$ is the Lebesgue measure of Ω, $\nabla_T a$ is the tangential gradient of the function a and $\nabla_T^2 a = \nabla_T(\nabla_T a)$.

Furthermore, v satisfies

$$\|v - v_0\|_{3,\Omega} \leq \gamma, \tag{13}$$

and (v, p) is the only solution to (1)-(3), (11), (12) in $H^3(\Omega, R^3 \times R)$ satisfying (13).

In addition, if $(a^{(1)}, b^{(1)})$ and (a, b) are two sets of boundary data on $\partial\Omega_-$ both satisfying (10), and $(v^{(1)}, p^{(1)})$, (v, p) are solutions of (1)-(3), (11), (12) to the boundary data $(a^{(1)}, b^{(1)})$ and (a, b), respectively, both satisfying (13), then it holds that

$$\|v^{(1)} - v\|_{1,\Omega} \leq K_2(\|a^{(1)} - a\|_{0,\partial\Omega_-} + \|b^{(1)} - b\|_{0,\partial\Omega_-}), \tag{14}$$

$$\|p^{(1)} - p\|_{1,\Omega} \leq K_3(\|a^{(1)} - a\|_{0,\partial\Omega_-} + \|b^{(1)} - b\|_{0,\partial\Omega_-}). \tag{15}$$

Remark 1.1 Compared with the main results in [1], Theorem 1.1 in this paper has several advantages. First, we do not require that v_0 be a velocity field of a solution to the problem (1)-(3) in contrast to [1]. Second, Theorem 1.1 requires less regularity on v_0 that the ones required in [1]. And finally, there is no requirement that $\partial\Omega_-$ is a manifold with Lipschitz boundary as in [1].

Remark 1.2 As motivated by the approach in [1], we prove Theorem 1.1 by a fixed point argument. The key in our analysis is to solve a boundary value problem for a nonlinear first order transport system satisfied by the vorticity field.

The rest of the paper is organized as follows. In §2, we give the proof of Theorem 1.1 by the contraction mapping principle provided that we can solve a boundary value problem for a linear first system. The solvability the necessary estimates, and properties of the solutions for this linearized problem are carried out in details in §2 - §6.

2 Proof of Theorem 1.1

Let $\Omega \subset R^m$ be an open set and k be any nonnegative integer. Denote by $H^k(\Omega) = H^k(\Omega, R^m)$ the usual Sobolev space of functions from Ω into R^m with the norm

$$\|u\|_{k,\Omega} = \left(\sum_{|\beta| \le k} \int_\Omega |D^\beta u(x)|^2 dx \right)^{\frac{1}{2}},$$

where $\beta = (\beta_1, \cdots, \beta_l)$ is a multi-index. Set

$$\|u\|_{k,r,\Omega} = \left(\sum_{|\beta| \le k} \int_\Omega |D^\beta u(x)|^r dx \right)^{\frac{1}{r}}, \quad r \le 1.$$

It follows from Sobolev imbedding theorem and Sobolev's trace theorem that there exists a positive constant M such that

$$\begin{array}{ll} \|v\|_{i,4,\Omega} \le M\|v\|_{i+1,\Omega}; & \|v\|_{i,\partial\Omega} \le M\|v\|_{i+1,\Omega}, \quad i = 0, 1, 2, \\ \|\hat{v}\|_{C^i_B(R^3,R^3)} \le M\|\hat{v}\|_{i+2,R^3}; & \|v\|_{C^i(\overline{\Omega},R^3)} \le M\|v\|_{i+2,\Omega} \quad i = 0, 1 \end{array} \tag{16}$$

for all $v \in H^3(\Omega, R^3)$ and $\hat{v} \in H^3(R^3, R^3)$. Define

$$L^2_\sigma(\Omega, R^3) \triangleq \{u \in L^2(\Omega, R^3) \mid \text{ div } u = 0, x \in \Omega; n \cdot u = 0, x \in \partial\Omega\},$$
$$V = L^2_\sigma(\Omega, R^3) \cap H^3(\Omega, R^3),$$

and

$$V_\gamma = \{u \in V \mid \|u\|_{3,\Omega} \le \gamma\}$$

for $\gamma > 0$.

For given $v \in v_0 + V_r$, $a \in H^2(\partial\Omega_-, R)$, and $b \in H^2(\partial\Omega_-, R^3)$ satisfying (8) and (9), we consider the following boundary value problem

$$(v \cdot \nabla)z = (z \cdot \nabla)v, \qquad x \in \Omega, \tag{17}$$
$$z = av + b, \qquad x \in \partial\Omega_-. \tag{18}$$

The keys in the proof of Theorem 1 are the following lemmas which yield the solvability of the problem (17) and (18) and necessary estimate.

Lemma 2.1 There exists $\gamma_0 > 0$ such that for every $0 < \gamma \le \gamma_0$ and every $v \in v_0 + V_\gamma$, problem (17) and (18) has a unique solution z denoted by $Av = A[a, b](v)$.

The proof of this lemma will be given in Section 3.

Lemma 2.2 For $0 < \gamma < \gamma_0$, there exists $K = K(\gamma) > 0$ such that

$$\|Av\|_{0,\Omega} \le K(\|a\|_{0,\partial\Omega_-} + \|b\|_{0,\partial\Omega_-}) \tag{19}$$
$$\|Av\|_{2,\Omega} \le K(\|a\| + \|b\|) \tag{20}$$
$$\|Av^{(1)} - Av\|_{0,\Omega} \le K(\|a\| + \|b\|)\|v^{(1)} - v\|_{1,\Omega} \tag{21}$$

for all $v, w \in v_0 + V_\gamma$.

The next lemma shows that the solution to (17)-(18) is divergence free.

Lemma 2.3 For every $v \in v_0 + V_\gamma$, one has

$$\text{div } Av = 0, \qquad x \in \Omega.$$

The proof of the two lemmas will be given in Section 6. We also need the following two lemmas.

Lemma 2.4 [26, 33] For every $z \in H^2(\Omega, R^3)$ with

$$\text{div } z = 0, \qquad x \in \Omega,$$

there exists a unique $w \in V$ such that

$$z = \text{curl } w.$$

Moreover, there exists a constant $M_1 > 0$, only depending on Ω, such that

$$\|w\|_{3,\Omega} \le M_1 \|z\|_{2,\Omega}.$$

Lemma 2.5 [36] There exists a constant $M_2 > 0$ such that

$$\|u\|_{1,\Omega} \le M_2 \|\text{curl } u\|_{0,\Omega}$$

for all $u \in L^2_\sigma(\Omega, R^3) \cap H^1(\Omega, R^3)$.

We now assume that Lemmas 2.1-2.3 hold and proceed to prove Theorem 1.1.

Proof of Theorem 1.1 Let

$$K_1 = \min\left\{\frac{\gamma}{M_1(K+1)}, \frac{1}{2M_2K}\right\}.$$

For $v \in v_0 + V_\gamma$, it follows from Lemma 2.3 that

$$\text{div } (Av - \text{curl } v_0) = 0. \tag{22}$$

Moreover, by Lemma 2.4, there exists a unique $w \in V$ such that

$$Av - \text{curl } v_0 = \text{curl } w. \tag{23}$$

Define

$$Bv = B[a, b](v) = v_0 + w. \tag{24}$$

We shall prove that $B : v_0 + V_\gamma (\subset H^1(\Omega, R^3)) \to v_0 + V_\gamma$, is a contraction. In fact, by (24), (23), (22), Lemma 2.4, (20) and (10), one may obtain

$$\begin{aligned}
\|Bv - v_0\|_{3,\Omega} &= \|w\|_{3,\Omega} \\
&\le M_1 \|\text{curl } w\|_{2,\Omega} \\
&= M_1 \|Av - \text{curl } v_0\|_{2,\Omega} \\
&\le M_1 (\|Av\|_{2,\Omega} + \|\text{curl } v_0\|_{2,\Omega}) \\
&\le KM_1(\|a\| + \|b\|) + M_1 \|\text{curl } v_0\|_{2,\Omega} \\
&\le \gamma,
\end{aligned}$$

which implies that B is into. Next, it follows from (24), (23), (22), Lemma 2.5, (21) and (10) that

$$\begin{aligned}
\|Bv^{(1)} - Bv\|_{1,\Omega} &= \|w^{(1)} - w\|_{1,\Omega} \\
&\leq M_2\|\text{curl } w^{(1)} - \text{curl } w\|_{0,\Omega} \\
&= M_2\|Av^{(1)} - Av\|_{0,\Omega} \\
&\leq M_2K(\|a\| + \|b\|)\|v^{(1)} - v\|_{1,\Omega} \\
&\leq \frac{1}{2}\|v^{(1)} - v\|_{1,\Omega}
\end{aligned}$$

Hence B is a contraction on $v_0+V_\gamma(\subset H^1(\Omega, R^3))$. It follows from Banach's fixed point theorem that B has a unique fixed point v in $v_0 + V_\gamma$. By (24), we have

$$v = Bv = v_0 + w,$$

for some $w \in V$ satisfying (23), which implies that

$$\text{curl } v = \text{curl } v_0 + \text{curl } w = Av.$$

Due to the definition of A,

$$\begin{aligned}
(\text{curl } v \cdot \nabla)v &= (v \cdot \nabla)\text{curl } v, && x \in \Omega, \\
\text{curl } v &= av + b, && x \in \partial\Omega_-.
\end{aligned} \tag{25}$$

Noting

$$\text{curl } (v \times z) = v \text{ div } z - z \text{ div } v + (z \cdot \nabla)v - (v \cdot \nabla)z \tag{26}$$

and (25), one obtains

$$\text{curl } (v \times \text{curl } v) = 0, \qquad x \in \Omega,$$

which implies that there exists a function $g \in C^1(\overline{\Omega}, R)$ such that

$$v \times \text{curl } v = \nabla g, \qquad x \in \Omega$$

since Ω is simply connected. Set

$$p(x) = g(x) - \frac{1}{|\Omega|}\int_\Omega g(x)dx - \frac{1}{2}|v(x)|^2 + \frac{1}{2|\Omega|}\int_\Omega |v(x)|^2dx + 1, \qquad x \in \Omega,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Then one has

$$\frac{1}{|\Omega|}\int_\Omega p(x)dx = 1,$$

and

$$v(x) \times \text{curl } v(x) = \nabla\left(p(x) + \frac{1}{2}|v(x)|^2\right), \qquad x \in \Omega,$$

which implies that

$$(v \cdot \nabla)v + \nabla p = 0, \qquad x \in \Omega$$

due to the relation that

$$(v \cdot \nabla)v = \nabla \left(\frac{1}{2}|v|^2\right) - v \times \text{curl } v, \qquad x \in \Omega. \tag{27}$$

Hence (v, p) is a solution of problem (1)-(3) with $v \in v_0 + V_\gamma$ satisfying conditions (11) and (12).

Next, we prove the uniqueness of the solution to the problem (1)-(3) with $v \in v_0 + V_\gamma$ satisfying conditions (11) and (12). Assume that $(\tilde{v}, \tilde{p})$ is another solution to the problem (1)-(3) with $\tilde{v} \in v_0 + V_\gamma$ satisfying conditions (11) and (12). Then it follows from (1) and (27) that

$$\tilde{v} \times \text{curl } \tilde{v} = \nabla(\frac{1}{2}|\tilde{v}|^2 + \tilde{p}),$$

which implies that

$$\text{curl } (\tilde{v} \times \text{curl } \tilde{v}) = 0.$$

Moreover, by (26) and (2), it holds that

$$(\text{curl } \tilde{v} \cdot \nabla)\tilde{v} = (\tilde{v} \cdot \nabla)\text{curl } \tilde{v}.$$

This, together with (11), shows that

$$A\tilde{v} = \text{curl } \tilde{v}.$$

By the definition of B, one has

$$B\tilde{v} = \tilde{v}.$$

It follows from the uniqueness of the fixed point of B in $v_0 + V_\gamma$ that

$$\tilde{v} = v.$$

Hence

$$\nabla \tilde{p} = \nabla p,$$

which implies that

$$\tilde{p} = p$$

by (12).

Finally, we prove the stability of the solutions. From (24), (23), Lemma 2.5, (21) and (19) we obtain

$$\begin{aligned} \|v^{(1)} - v\|_{1,\Omega} &= \|B[a^{(1)}, b^{(1)}]v^{(1)} - B[a, b]v\|_{1,\Omega} \\ &\le \|B[a^{(1)}, b^{(1)}]v^{(1)} - B[a^{(1)}, b^{(1)}]v\|_{1,\Omega} \\ &\quad + \|B[a^{(1)}, b^{(1)}]v - B[a, b]v\|_{1,\Omega} \end{aligned}$$

$$
\begin{aligned}
&\le M_2(\|A[a^{(1)}, b^{(1)}](v^{(1)} - v)\|_{0,\Omega} \\
&\quad +\|A[a^{(1)} - a, b^{(1)} - b]v\|_{0,\Omega}) \\
&\le M_2(K(\|a^{(1)}\| + \|b^{(1)}\|)\|v^{(1)} - v\|_{1,\Omega} \\
&\quad +K(\|a^{(1)} - a\|_{0,\partial\Omega_-} + \|b^{(1)} - b\|_{0,\partial\Omega_-})) \\
&\le M_2 K K_1 \|v^{(1)} - v\|_{1,\Omega} + M_2 K(\|a^{(1)} - a\|_{0,\partial\Omega_-} + \|b^{(1)} - b\|_{0,\partial\Omega_-}) \\
&\le \frac{1}{2}\|v^{(1)} - v\|_{1,\Omega} + M_2 K(\|a^{(1)} - a\|_{0,\partial\Omega_-} + \|b^{(1)} - b\|_{0,\partial\Omega_-})
\end{aligned}
$$

which implies that

$$
\|v^{(1)} - v\|_{1,\Omega} \le K_2(\|a^{(1)} - a\|_{0,\partial\Omega_-} + \|b^{(1)} - b\|_{0,\partial\Omega_-}),
$$

where $K_2 = 2M_2K$. Hence (14) holds. It follows from (1) that

$$
\begin{aligned}
|\nabla p^{(1)} - \nabla p| &\le |(v^{(1)} \cdot \nabla)v^{(1)} - (v \cdot \nabla)v| \\
&\le |((v^{(1)} - v) \cdot \nabla)v^{(1)}| + |(v \cdot \nabla)(v^{(1)} - v)| \\
&\le |v^{(1)} - v||v^{(1)}|_1 + |v||v^{(1)} - v|_1 \\
&\le (\beta_0 + \gamma)\,(|v^{(1)} - v| + |v^{(1)} - v|_1)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\nabla p^{(1)} - \nabla p\|_{0,\Omega} &\le (\beta_0 + \gamma)\,\|v^{(1)} - v\|_{1,\Omega} \\
&\le (\beta_0 + \gamma)\,K_2(\|a^{(1)} - a\|_{0,\partial\Omega_-} + \|b^{(1)} - b\|_{0,\partial\Omega_-}), \quad (28)
\end{aligned}
$$

where one has used the notation

$$
|v|_1 = |v|_1(x) = \left(\sum_{i=1}^{3} \sum_{|\beta|=1} |D^\beta v_i(x)|^2\right)^{\frac{1}{2}}.
$$

Due to

$$
\int_\Omega (p^{(1)} - p)dx = \int_\Omega p^{(1)}dx - \int_\Omega p dx = |\Omega| - |\Omega| = 0,
$$

one has

$$
\sqrt{\mu_2}\|p^{(1)} - p\|_{0,\Omega} \le \|\nabla p^{(1)} - \nabla p\|_{0,\Omega}, \tag{29}
$$

where $\mu_2 > 0$ is the first positive eigenvalue of the eigenvalue problem

$$
-\triangle u = \mu u \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.
$$

It follows from (28) and (29) that

$$
\|p^{(1)} - p\|_{1,\Omega} \le K_3(\|a^{(1)} - a\|_{0,\partial\Omega_-} + \|b^{(1)} - b\|_{0,\partial\Omega_-}),
$$

where $K_3 = \frac{1}{\sqrt{\mu_2}} K_2\,(\beta_0 + \gamma)$. Hence (15) holds. Thus we have completed the proof of Theorem 1.1.

3 Solvability of (17)-(18)

We now prove Lemma 2.1 in this section. First, we give the following lemma, which shows that the conditions (5) and (6) in Theorem 1.1 are invariant for small perturbations.

Lemma 3.1 Under assumptions of Theorem 1.1, there exists a constant $\gamma_1 > 0$ such that

$$|v(x)| \geq \alpha_0 \tag{30}$$

for all $x \in \Omega$, and

$$\|v\|_{3,\Omega} \leq \beta_0 \tag{31}$$

for all $v \in v_0 + V_{\gamma_1}$.

Proof Set

$$\gamma_1 = \min\left\{\frac{\alpha_0}{M}, \frac{\beta_0}{2}\right\}. \tag{32}$$

Then for $v \in v_0 + V_{\gamma_1}$, it holds that

$$\begin{aligned} |v(x)| &\geq |v_0(x)| - |v(x) - v_0(x)| \\ &\geq 2\alpha_0 - \|v - v_0\|_{C^1(\overline{\Omega}, R^3)} \\ &\geq 2\alpha_0 - M\|v - v_0\|_{3,\Omega} \\ &\geq 2\alpha_0 - M\gamma_1 \\ &\geq \alpha_0 \end{aligned}$$

for all $x \in \Omega$ by (5), (16) and (32), which proves (30). It follows from (6) and (32) that

$$\|v\|_{3,\Omega} \leq \|v_0\|_{3,\Omega} + \|v - v_0\|_{3,\Omega} \leq \frac{\beta_0}{2} + \gamma_1 \leq \beta_0$$

for all $v \in v_0 + V_{\gamma_1}$. Hence (31) holds. This complete the proof of this lemma. □

We will solve the boundary value problem (17)-(18) by the characteristic method. Thus we consider the following initial value problem for ordinary differential equations

$$\begin{aligned} \frac{d}{dt}\omega(t, x, v) &= v(\omega(t, x, v)), \\ \omega(0, x, v) &= x, \end{aligned}$$

where $x \in \overline{\Omega}$, $v \in C^1(\overline{\Omega}, R^3)$. By the theory of the ordinary differential equations, this equations has a unique solution $\omega(t, x, v)$ which is continuously differentiable in $(x, v) \in \overline{\Omega} \times C^1(\overline{\Omega}, R^3)$. Let $[0, T(x, v))$ be the maximal existence interval of $\omega(t, x, v)$ to right. Define

$$T(v) = \sup_{x \in \overline{\Omega}} T(x, v)$$

for $v \in C^1(\overline{\Omega}, R^3)$.

By Calderó's extension theorem there exists a constant $M_3 > 0$ such that, for every $w \in H^3(\Omega, R^3)$ there exists an extension to $\hat{w} \in H^3(R^3, R^3)$ satisfying

$$\|\hat{w}\|_{3,R^3} \leq M_3 \|w\|_{3,\Omega}. \tag{33}$$

Then $\omega(t, x, v)$ can be extended to $\hat{\omega}(t, x, \hat{v})$ which is defined on $[0, +\infty)$.

To show that each stream line going through a point in Ω must exit Ω is finite time, we need the following Lemma.

Lemma 3.2 Let L_γ be the least super bound of the length of all stream line of v in Ω with $v \in v_0 + V_\gamma$. Then there exists a constant $\gamma_2 \in (0, \gamma_1]$ such that

$$L_{\gamma_2} < +\infty.$$

Proof We first prove the continuity of the mapping $(x, v) \to \hat{\omega}(t, x, \hat{v})$ at v_0. By the mean value theorem, (16), (33) and (31), one can get

$$\begin{aligned}
\frac{d}{dt}|\hat{\omega}(t,x,\hat{v}) - \hat{\omega}(t,x_0,\hat{v}_0)| &\leq |\frac{d}{dt}(\hat{\omega}(t,x,\hat{v}) - \hat{\omega}(t,x_0,\hat{v}_0))| \\
&= |\hat{v}(\hat{\omega}(t,x,\hat{v})) - \hat{v}_0(\hat{\omega}(t,x_0,\hat{v}_0))| \\
&\leq |\hat{v}(\hat{\omega}(t,x,\hat{v})) - \hat{v}_0(\hat{\omega}(t,x,\hat{v}))| \\
&\quad + |\hat{v}_0(\hat{\omega}(t,x,\hat{v})) - \hat{v}_0(\hat{\omega}(t,x_0,\hat{v}_0))| \\
&\leq \|\hat{v}_0\|_{C^1_B(R^3,R^3)} |\hat{\omega}(t,x,\hat{v}) - \hat{\omega}(t,x_0,\hat{v}_0)| \\
&\quad + \|\hat{v} - \hat{v}_0\|_{C_B(R^3,R^3)} \\
&\leq M\|\hat{v}_0\|_{3,R^3} |\hat{\omega}(t,x,\hat{v}) - \hat{\omega}(t,x_0,\hat{v}_0)| \\
&\quad + M\|\hat{v} - \hat{v}_0\|_{3,R^3} \\
&\leq MM_3\|v_0\|_{3,\Omega} |\hat{\omega}(t,x,\hat{v}) - \hat{\omega}(t,x_0,\hat{v}_0)| \\
&\quad + MM_3\|v - v_0\|_{3,\Omega} \\
&\leq MM_3\beta_0 |\hat{\omega}(t,x,\hat{v}) - \hat{\omega}(t,x_0,\hat{v}_0)| \\
&\quad + MM_3\|v - v_0\|_{3,\Omega}
\end{aligned}$$

which implies that

$$\begin{aligned}
|\hat{\omega}(t,x,\hat{v}) - \hat{\omega}(t,x_0,\hat{v}_0)| &\leq e^{MM_3\beta_0 t}(|\hat{\omega}(0,x,\hat{v}) - \hat{\omega}(0,x_0,\hat{v}_0)| \\
&\quad + MM_3\|v - v_0\|_{3,\Omega} t) \qquad (34) \\
&\leq e^{MM_3\beta_0 t}(|x - x_0| + MM_3\|v - v_0\|_{3,\Omega} t). \qquad (35)
\end{aligned}$$

Let $l(\omega(\cdot, x, v_0))$ be the length of the stream line $\omega(\cdot, x, v_0)$ starting at x. Then (30) yields

$$L_0 \geq l(\omega(\cdot, x, v_0))$$

$$
\begin{aligned}
&= \int_0^{T(x,v_0)} \left| \frac{d}{dt}\omega(t,x,v_0) \right| dt \\
&= \int_0^{T(x,v_0)} |v_0(\omega(t,x,v_0))|\, dt \\
&\geq T(x,v_0)\alpha_0
\end{aligned}
$$

which leads to

$$
T(v_0) \leq \frac{L_0}{\alpha_0}. \tag{36}
$$

Then we claim that for every $\varepsilon > 0$, there exists a positive constant $\gamma_\varepsilon \leq \gamma_1$ such that

$$
T(v) \leq T(v_0) + \varepsilon \tag{37}
$$

for all $\|v - v_0\|_{3,\Omega} < \gamma_\varepsilon$. Indeed, it follows from the definition of $T(v_0)$ that there exists $t_0 = t_0(\varepsilon, x_0) \in (0, T(v_0) + \varepsilon)$ such that

$$
\hat{\omega}(t_0, x_0, \hat{v}_0) \notin \overline{\Omega}.
$$

By (35), there exists $\delta_{x_0} > 0$ such that

$$
\hat{\omega}(t_0, x, \hat{v}) \notin \overline{\Omega}
$$

for all $x \in \overline{\Omega}$ with $|x - x_0| < \delta_{x_0}$ and $\|v - v_0\|_{3,\Omega} < \delta_{x_0}$, which implies that

$$
T(x, v) \leq T(v_0) + \varepsilon
$$

for all $x \in \overline{\Omega}$ with $|x - x_0| < \delta_{x_0}$ and $\|v - v_0\|_{3,\Omega} < \delta_{x_0}$. It follows from the compactness of $\overline{\Omega}$ that there exist finite $x_1, x_2, \cdots, x_k$ and positive constants $\delta_1, \delta_2, \cdots, \delta_k$ such that

$$
T(x, v) \leq T(v_0) + \varepsilon
$$

for all $x \in \overline{\Omega}$ with $|x - x_j| < \delta_j$ and $\|v - v_0\|_{3,\Omega} < \delta_j$ for some $1 \leq j \leq k$, and

$$
\overline{\Omega} \subset \cup_{j=1}^{k} B(x_j; \delta_j),
$$

where $B(x_j; \delta_j)$ is the open ball in R^3 with center x_j and radius δ_j. Set

$$
\gamma_\varepsilon = \min\{\delta_1, \delta_2, \cdots, \delta_k\}.
$$

Then

$$
T(x, v) \leq T(v_0) + \varepsilon
$$

for all $x \in \overline{\Omega}$ and $\|v - v_0\|_{3,\Omega} < \gamma_\varepsilon$. Hence one has

$$
T(v) \leq T(v_0) + \varepsilon
$$

for all $\|v - v_0\|_{3,\Omega} < \gamma_\varepsilon$, which verifies (37).

It follows from (37) that there exists a positive constant $\gamma_2 \leq \gamma_1$ such that

$$
T(v) \leq T(v_0) + 2 \tag{38}
$$

for all $\|v - v_0\|_{3,\Omega} < \gamma_2$. Let $l(\omega(\cdot, x, v))$ be the length of the stream line $\omega(\cdot, x, v)$ starting at x. Then

$$\begin{aligned} l(\omega(\cdot, x, v)) &= \int_0^{T(x,v)} \left| \frac{d}{dt}\omega(t, x, v) \right| dt \\ &\leq \int_0^{T(x,v)} |v(\omega(t, x, v))|\, dt \\ &\leq T(x, v)\|v\|_{C^1(\overline{\Omega}, R^3)} \\ &\leq (T(v_0) + 2)M\|v\|_{3,\Omega} \\ &\leq \left(\frac{L_0}{\alpha_0} + 2\right) M\beta_0 < +\infty \end{aligned}$$

by (38), (16), (36) and (31). Hence the lemma holds. □

We are now ready to show

Lemma 3.3 There exists a positive constant $\gamma_3 \leq \gamma_2$ such that, for every $v \in v_0 + V_{\gamma_3}$, every integral curve of v that passes over a point in Ω meets the boundary in exactly two different points, one point in $\partial\Omega_-$, the starting point of the integral curve, and another point in $\partial\Omega_+$, the endpoint of this integral curve.

Proof Assume that $x_0 \in \partial\Omega$. Set $\hat{\omega}(t) = \hat{\omega}(t, x_0, \hat{v})$. It follows from the continuously differential property of $\hat{\omega}$ and the implicit function theorem that the equations

$$\hat{\omega}(t) - x = -\rho n(x)$$

has a unique continuously differentiable solution (x, ρ) from a suitable neighborhood of 0 to $\partial\Omega \times R$ such that

$$x(0) = x_0, \quad \rho(0) = 0.$$

Hence,

$$\hat{\omega}(t) - x(t) = -\rho(t)n(x(t)).$$

It follows that

$$\hat{v}(\hat{\omega}(t)) - \frac{d}{dt}x(t) = -\rho'(t)n(x(t)) - \rho(t)\frac{d}{dt}(n(x(t))).$$

Taking the inner product of the above equation with $n(x(t))$ yields

$$\rho'(t) = -\hat{v}(\hat{\omega}(t)) \cdot n(x(t)). \tag{39}$$

In the case that $x_0 \in \partial\Omega_-$, it holds that

$$\rho'(0) = -\hat{v}(\hat{\omega}(0)) \cdot n(x(0)) = -v(x_0) \cdot n(x_0) = -f(x_0) > 0.$$

Hence, there exists a constant $\delta > 0$ such that $\rho(t) > 0$ for all $0 < t < \delta$. And so $\hat{\omega}(t, x_0, \hat{v}) \in \Omega$ for all $0 < t < \delta$.

Consider now the case that $x_0 \in \partial\Omega \setminus \partial\Omega_-$. Due to (39), one may have

$$\begin{aligned}\rho'(t) &= -\hat{v}(\hat{\omega}(t)) \cdot n(x(t)) + v(x(t)) \cdot n(x(t)) - f(x(t)) \\ &= a(t)\rho(t) - f(x(t)),\end{aligned}$$

where

$$\begin{aligned}|a(t)| &= \left|\frac{1}{\rho(t)}(-\hat{v}(\hat{\omega}(t)) \cdot n(x(t)) + v(x(t)) \cdot n(x(t)))\right| \\ &\leq \|\hat{v}\|_{C^1_B(R^3,R^3)} \\ &\leq M\|\hat{v}\|_{3,R^3} \\ &\leq MM_3\|v\|_{3,\Omega} \\ &\leq MM_3\beta_0\end{aligned}$$

by the mean value theorem, (16), (33) and (31). Hence,

$$\rho(t) = -e^{\int_0^t a(\tau)d\tau} \int_0^t e^{-\int_0^\tau a(s)ds} f(x(\tau))d\tau.$$

In the case that $x_0 \in \partial\Omega \setminus \overline{\partial\Omega_-}$, by the continuity of $x(t)$, there exists a positive constant δ such that

$$x(t) \in \partial\Omega \setminus \overline{\partial\Omega_-}$$

for all $0 \leq t < \delta$, which implies that

$$\rho(t) = -e^{\int_0^t a(\tau)d\tau} \int_0^t e^{-\int_0^\tau a(s)ds} f(x(\tau))d\tau \leq 0$$

for all $0 \leq t < \delta$. Hence one has

$$\hat{\omega}(t, x_0, \hat{v}) \notin \Omega, \qquad \forall 0 \leq t < \delta.$$

In the case that $x_0 \in \partial\partial\Omega_-$, by the fact that $\dot{x}(0) = v(x_0)$ and (16), one has

$$\begin{aligned}\text{dist }(\partial\Omega_-, x(t)) &\geq \text{dist }(\partial\Omega_-, x_0 + tv(x_0)) - |x(t) - x_0 - tv(x_0)| \\ &\geq \text{dist }(\partial\Omega_-, x_0 + tv_0(x_0)) - t|v(x_0) - v_0(x_0)| \\ &\quad -|x(t) - x_0 - tv(x_0)| \\ &\geq \text{dist }(\partial\Omega_-, x_0 + tv_0(x_0)) - t\|v - v_0\|_{C(\overline{\Omega})} \\ &\quad -|x(t) - x_0 - t\dot{x}(0)| \\ &\geq \text{dist }(\partial\Omega_-, x_0 + tv_0(x_0)) - tM\|v - v_0\|_{3,\Omega} \\ &\quad -|x(t) - x_0 - t\dot{x}(0)|,\end{aligned}$$

which leads

$$\liminf_{t\to 0^+} \frac{1}{t}\text{dist }(\partial\Omega_-, x(t)) \geq \liminf_{t\to 0^+} \frac{1}{t}\text{dist }(\partial\Omega_-, x_0 + tv_0(x_0)) - M\|v - v_0\|_{3,\Omega}.$$

Hence there exists a positive constant $\gamma_3 \le \gamma_2$ such that, for every $v \in v_0 + V_{\gamma_3}$ and $x_0 \in \partial\partial\Omega_-$, one has

$$\liminf_{t\to 0^+} \frac{1}{t} \text{dist } (\partial\Omega_-, x(t)) \;>\; 0$$

by (7). Therefore, there exists a positive constant δ such that

$$x(t) \in \partial\Omega \setminus \partial\Omega_-$$

for all $0 \le t < \delta$, which implies that

$$\rho(t) = -e^{\int_0^t a(\tau)d\tau} \int_0^t e^{-\int_0^\tau a(s)ds} f(x(\tau))d\tau \le 0$$

for all $0 \le t < \delta$. Thus one has

$$\hat{\omega}(t, x_0, \hat{v}) \notin \Omega, \qquad \forall 0 \le t < \delta.$$

Hence every integral curve of v that passes through a point $x \in \Omega$ can only start exactly one point in $\partial\Omega_-$, the starting point of the integral curve. Similarly, every integral curve of v that passes through a point $x \in \Omega$ can only end in exactly one point in $\partial\Omega_+$, the endpoint of this integral curve. It follows from (38) that every integral curve of v that passes through a point $x \in \Omega$ must start one point in $\partial\Omega_-$, the starting point of the integral curve, and must end in one point in $\partial\Omega_+$, the endpoint of this integral curve. Therefore Ω is completely covered by integral curves of v starting at $\partial\Omega_-$. □

Let $\omega(s) = \omega(s, y) = \omega(s, y, v)$ be the solution of

$$\frac{d}{ds}\omega(s, y, v) = \frac{1}{|v(\omega(s, y, v))|} v(\omega(s, y, v)), \quad \omega(0, y, v) = y \in \partial\Omega_-.$$

We are now ready to prove Lemma 2.1.

Proof of Lemma 2.1 Let γ_0 be γ_3 in Lemma 3.3. On one hand, assume that z is a solution to (17) and (18). Set

$$z(s) = z(s, y) = z(s, y, v) = z(\omega(s, y, v)).$$

Then,

$$\begin{aligned} \frac{d}{ds} z(s) &= (\frac{d}{ds}\omega(s) \cdot \nabla) z(s) \\ &= \frac{1}{|v(s)|}(v(s) \cdot \nabla) z(s) \\ &= \frac{1}{|v(s)|}(z(s) \cdot \nabla) v(s) \end{aligned}$$

and

$$z(0,y) = av(0,y) + b.$$

That is, for every $y \in \partial\Omega_-$, $z(s) = z(\omega(s,y,v))$ is a solution of the initial problem for the first order linear homogeneous ordinary differential equations

$$\frac{d}{ds}z(s) = \frac{1}{|v(s)|}(z(s)\cdot\nabla)v(s), \tag{40}$$

$$z(0) = av(0,y) + b. \tag{41}$$

On the other hand, assume $z(s)$ is a solution of the initial problem for the first order linear homogeneous ordinary differential equations (40) and (41). Then $\forall x \in \Omega$, by Lemma 3.3, there exists unique $(t,y) = (s(x), y(x))$ such that $w(s,y,v) = x$. Set

$$z(x) = z(s(x), y(x)).$$

Then

$$z(s,y) = z(\omega(s,y,v)).$$

It follows that

$$\begin{aligned} \frac{d}{ds}z(s) &= (\frac{d}{ds}\omega(s)\cdot\nabla)z(s) \\ &= \frac{1}{|v(s)|}(v(s)\cdot\nabla)z(s). \end{aligned}$$

Moreover, by (40), it holds that

$$(v(s)\cdot\nabla)z(s) = (z(s)\cdot\nabla)v(s),$$

that is,

$$(v(x)\cdot\nabla)z(x) = (z(x)\cdot\nabla)v(x).$$

Hence $z(x) = z(s(x), y(x))$ is a solution to (17) and (18). Therefore $z(x)$ is a solution to (17) and (18) if and only if $z(s)$ is a solution to the initial problem for the first order linear homogeneous ordinary differential equations (40) and (41).

By the theory of the ordinary differential equations, the problem (40) and (41) has a unique solution. Hence the problem (17) and (18) has a unique solution. This completes the proof of Lemma 2.1. □

4 Estimates of solutions to (17) and (18)

For easy presentation, we use the following notations. For a function $q = (q_1, \cdots, q_m) : \Omega(\subset R^3) \to R^m$, set

$$|q|_k(x) = \left(\sum_{i=1}^{m} \sum_{|\beta|=k} |D^\beta q_i(x)|^2 \right)^{\frac{1}{2}},$$

$$q_{|i}(x) = \frac{\partial}{\partial x_i} q,$$

$$q_{|ij}(x) = \frac{\partial^2}{\partial x_i \partial x_j} q,$$

and

$$q(s) = q(s, y) = q(\omega(s, y, v)).$$

First we estimate solutions to (17) and (18).

Lemma 4.1 Suppose that $v \in v_0 + V_\gamma$ with $\gamma \le \gamma_0$. Assume that z is a solution to (17) and (18). Then it holds that

$$|z(s)| \le C_1 |z(0)|$$

for some positive constant $C_1 = C_1(\alpha_0, \beta_0, \gamma_0, L_0)$.

Proof It follows from (17), (30), (31) and (16) that

$$\begin{aligned}
\frac{d}{ds}|z(s)| &\le \left| \frac{d}{ds} z(s) \right| \\
&\le |v(s)|^{-1} |(z(s) \cdot \nabla) v(s)| \\
&\le \alpha_0^{-1} |z(s)| |v|_1(s) \\
&\le \alpha_0^{-1} |z(s)| \|v\|_{C^1(\overline{\Omega}, R^3)} \\
&\le \alpha_0^{-1} |z(s)| M \|v\|_{3,\Omega} \\
&\le \alpha_0^{-1} M \beta_0 |z(s)|,
\end{aligned}$$

which implies that

$$\begin{aligned}
|z(s)| &\le e^{\alpha_0^{-1} M \beta_0 s} |z(0)| \\
&\le e^{\alpha_0^{-1} M \beta_0 L_{\gamma_0}} |z(0)| \\
&\triangleq C_1 |z(0)|.
\end{aligned}$$

□

Next, we estimate the first derivatives of the solution to (17) and (18).

Lemma 4.2 Suppose that z is a solution of (17) and (18). Then

$$|z|_1(s) \le C_2 \left(|z|_1(0) + |z(0)| \int_0^s |v|_2(\tau)|d\tau \right)$$

for some positive constant $C_2 = C_2(\alpha_0, \beta_0, \gamma_0, L_0)$.

Proof Differentiating (17) yields

$$(v \cdot \nabla) z_{|i} + (v_{|i} \cdot \nabla) z = z_{|i} \cdot \nabla) v + (z \cdot \nabla) v_{|i}. \tag{42}$$

Hence,

$$\begin{aligned}
\frac{d}{ds}|z|_1 &\le \left(\sum_{i=1}^3 \left| \frac{d}{ds} z_{|i} \right|^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^3 \left| |v|^{-1} (v \cdot \nabla) z_{|i} \right|^2 \right)^{\frac{1}{2}} \\
&= |v|^{-1} \left(\sum_{i=1}^3 \left| (z_{|i} \cdot \nabla) v + (z \cdot \nabla) v_{|i} - (v_{|i} \cdot \nabla) z \right|^2 \right)^{\frac{1}{2}} \\
&\le |v|^{-1} (|z|_1 |v|_1 + |z||v|_2 + |z|_1 |v|_1) \\
&= |v|^{-1} (2|z|_1 |v|_1 + |z||v|_2) \\
&\le 2\alpha_0^{-1} M \beta_0 |z|_1 + \alpha_0^{-1} C_1 |z(0)||v|_2
\end{aligned}$$

which implies that

$$\begin{aligned}
|z|_1(s) &\le e^{2\alpha_0^{-1} M \beta_0 s} \left(|z|_1(0) + \alpha_0^{-1} C_1 |z(0)| \int_0^s |v|_2(\tau) d\tau \right) \\
&\le e^{2\alpha_0^{-1} M \beta_0 L_{\gamma_0}} \left(|z|_1(0) + \alpha_0^{-1} C_1 |z(0)| \int_0^s |v|_2(\tau) d\tau \right) \\
&\le C_2 \left(|z|_1(0) + |z(0)| \int_0^s |v|_2(\tau) d\tau \right).
\end{aligned}$$

□

Now we estimate the second derivatives of the solution to (17) and (18).

Lemma 4.3 Let z be a solution of (17) and (18). Then

$$|z|_2(s) \le C_3 (|z|_2(0) + |z|_1(0) + |z(0)| (\int_0^s |v|_2(\tau) d\tau)^2 + |z(0)| \int_0^s |v|_3(\tau) d\tau)$$

for some positive constant $C_3 = C_3(\alpha_0, \beta_0, \gamma_0, L_0)$.

Proof Due to (42), one has

$$\begin{aligned}
(v \cdot \nabla) z_{|ij} + (v_{|ij} \cdot \nabla) z + (v_{|i} \cdot \nabla) z_{|j} + (v_{|j} \cdot \nabla) z_{|i} &= (z_{|ij} \cdot \nabla) v + (z_{|i} \cdot \nabla) v_{|j} \\
&\quad + (z \cdot \nabla) v_{|ij} + (z_{|j} \cdot \nabla) v_{|i}.
\end{aligned}$$

It follows that

$$
\begin{aligned}
\frac{d}{ds}|z|_2 &\leq \left(\sum_{i,j=1}^{3}\left|\frac{d}{ds}z_{|ij}\right|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i,j=1}^{3}\left||v|^{-1}(v\cdot\nabla)z_{|ij}\right|^2\right)^{\frac{1}{2}} \\
&\leq |v|^{-1}(\sum_{i,j=1}^{3}|(z_{|ij}\cdot\nabla)v|^2)^{\frac{1}{2}} + (\sum_{i,j=1}^{3}|(z_{|i}\cdot\nabla)v_{|j}|^2)^{\frac{1}{2}} \\
&\quad +(\sum_{i,j=1}^{3}|(z\cdot\nabla)v_{|ij}|^2)^{\frac{1}{2}} + (\sum_{i,j=1}^{3}|(z_{|j}\cdot\nabla)v_{|i}|^2)^{\frac{1}{2}} \\
&\quad +(\sum_{i,j=1}^{3}|(v_{|ij}\cdot\nabla)z|^2)^{\frac{1}{2}} + (\sum_{i,j=1}^{3}|(v_{|i}\cdot\nabla)z_{|j}|^2)^{\frac{1}{2}} \\
&\quad +(\sum_{i,j=1}^{3}|(v_{|j}\cdot\nabla)z_{|i}|^2)^{\frac{1}{2}} \\
&\leq |v|^{-1}(|z|_2|v|_1 + |z|_1|v|_2 + |z||v|_3 \\
&\quad +|z|_1|v|_2 + |v|_2|z|_1 + |v|_1|z|_2 + |v|_1|z|_2) \\
&= |v|^{-1}(3|z|_2|v|_1 + 3|z|_1|v|_2 + |z||v|_3) \\
&\leq \alpha_0^{-1}(3M\beta_0|z|_2 + 3C_2\left(|z|_1(0) + |z(0)|\int_0^s|v|_2(\tau)d\tau\right)|v|_2 \\
&\quad +C_1|z(0)||v|_3) \\
&\leq 3M\alpha_0^{-1}\beta_0|z|_2 + 3C_2\alpha_0^{-1}|z|_1(0) + 3C_2\alpha_0^{-1}|z(0)|\int_0^s|v|_2(\tau)d\tau|v|_2 \\
&\quad +C_1\alpha_0^{-1}|z(0)||v|_3,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
|z|_2(s) &\leq e^{3M\alpha_0^{-1}\beta_0 s}(|z|_2(0) + 3C_2\alpha_0^{-1}|z|_1(0)s \\
&\quad +3C_2\alpha_0^{-1}|z(0)|\int_0^s\int_0^r|v|_2(\tau)d\tau|v|_2(r)dr + C_1\alpha_0^{-1}|z(0)|\int_0^s|v|_3(r)dr) \\
&\leq e^{3M\alpha_0^{-1}\beta_0 L_{\gamma_0}}(|z|_2(0) + 3C_2\alpha_0^{-1}|z|_1(0)L_{\gamma_0} \\
&\quad +3C_2\alpha_0^{-1}|z(0)|(\int_0^s|v|_2(\tau)d\tau)^2 + C_1\alpha_0^{-1}|z(0)|\int_0^s|v|_3(\tau)d\tau) \\
&\leq C_3\left(|z|_2(0) + |z|_1(0) + |z(0)|(\int_0^s|v|_2(\tau)d\tau)^2 + |z(0)|\int_0^s|v|_3(\tau)d\tau\right).
\end{aligned}
$$

□

In order to prove (21) we need the following lemma.

Lemma 4.4 Let

$$[z] = Av^{(1)} - Av, \qquad [v] = v^{(1)} - v,$$

where $v^{(1)}, v \in v_0 + V_\gamma$. Then one has

$$|[z](s)| \leq C_4(|[z](0)| + \int_0^s (|Av^{(1)}||[v]|_1 + |[v]||Av^{(1)}|_1)d\tau)$$

for some positive constant $C_4 = C_4(\alpha_0, \beta_0, \gamma_0, L_0)$.

Proof By (17), one has

$$\begin{aligned}(v\cdot\nabla)[z] &= (v^{(1)}\cdot\nabla)Av^{(1)} - ([v]\cdot\nabla)Av^{(1)} - (v\cdot\nabla)Av \\ &= (Av^{(1)}\cdot\nabla)v^{(1)} - ([v]\cdot\nabla)Av^{(1)} - (Av\cdot\nabla)v \\ &= (Av^{(1)}\cdot\nabla)[v] - ([v]\cdot\nabla)Av^{(1)} + ([z]\cdot\nabla)v.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{d}{ds}|[z]| &\leq \left|\frac{d}{ds}[z]\right| \\ &\leq |v|^{-1}|(v\cdot\nabla)[z]| \\ &\leq \alpha_0^{-1}(|Av^{(1)}||[v]|_1 + |[v]||Av^{(1)}|_1 + |[z]||v|_1) \\ &\leq \alpha_0^{-1}M\beta_0|[z]| + \alpha_0^{-1}|Av^{(1)}||[v]|_1 + \alpha_0^{-1}|[v]||Av^{(1)}|_1,\end{aligned}$$

which implies that

$$|[z](s)| \leq C_4(|[z](0)| + \int_0^s (|Av^{(1)}||[v]|_1 + |[v]||Av^{(1)}|_1)d\tau).$$

□

In order to obtain the L^2 estimate we need the following lemmas.

Lemma 4.5 [1] Assume that $q \in L^1(\Omega; R^m)$. Then it holds that

$$\int_\Omega q(x)dx = \int_{\partial\Omega_-}\int_0^{l(y)} q(s,y)\frac{|f(y)|}{|v(s,y)|}dsdS_y,$$

where $l(y)$ is the exit time of $w(s, y, v)$.

Lemma 4.6 Suppose that $v \in v_0 + V_\gamma$ with $\gamma \leq \gamma_0$. Then there exists a positive constant $C = C(\alpha_0, \beta_0, \gamma_0, L_0)$ such that

$$\left\|\int_0^{s(\cdot)} q(\tau, y(\cdot))d\tau\right\|_{0,\Omega} \leq C\|q\|_{0,\Omega}, \qquad \forall q \in L^2(\Omega; R^m), \tag{43}$$

$$\left\|\int_0^{s(\cdot)} q(\tau, y(\cdot))d\tau\right\|_{0,4,\Omega} \leq C\|q\|_{0,4,\Omega}, \qquad \forall q \in L^4(\Omega; R^m),$$

$$\|q(0, y(\cdot))\|_{0,\Omega} \leq C\|q\|_{0,\partial\Omega_-}, \qquad \forall q \in L^2(\partial\Omega_-; R^m), \tag{44}$$

and

$$\|q(0,y(\cdot))\|_{0,4,\Omega} \le C\|q\|_{0,4,\partial\Omega_-}, \qquad \forall q \in L^4(\partial\Omega_-; R^m).$$

Proof It follows from Lemma 4.5, (30), (16), (31) and Lemma 3.2 that

$$\begin{aligned}
\left\|\int_0^{s(\cdot)} q(\tau, y(\cdot))d\tau\right\|^2_{L^2(\Omega;R^m)} &= \int_\Omega \left|\int_0^{s(x)} q(\tau, y(x))d\tau\right|^2 dx \\
&= \int_{\partial\Omega_-}\int_0^{l(y)} \left|\int_0^s q(\tau,y)d\tau\right|^2 \frac{|f(y)|}{|v(s,y)|} ds dS_y \\
&\le \int_{\partial\Omega_-}\int_0^{l(y)} s\int_0^s |q(\tau,y)|^2 d\tau \frac{|f(y)|}{|v(s,y)|} ds dS_y \\
&\le \alpha_0^{-1} L^2_{\gamma_0} \int_{\partial\Omega_-}\int_0^{l(y)} |q(\tau,y)|^2 d\tau |f(y)| dS_y \\
&\le \alpha_0^{-1} L^2_{\gamma_0} M\beta_0 \int_{\partial\Omega_-}\int_0^{l(y)} |q(\tau,y)|^2 \frac{|f(y)|}{|v(\tau,y)|} d\tau dS_y \\
&\le C\|q\|^2_{L^2(\Omega;R^m)}
\end{aligned}$$

for all $q \in L^2(\Omega; R^m)$ and some

$$C = \max\{\alpha_0^{-1}M\beta_0 L^2_{\gamma_0}, \alpha_0^{-1}M\beta_0 L^4_{\gamma_0}, \alpha_0^{-1}M\beta_0 L_{\gamma_0}, \alpha_0^{-1}M\beta_0 L_{\gamma_0}\}.$$

Similarly,

$$\begin{aligned}
\left\|\int_0^{s(\cdot)} q(\tau, y(\cdot))d\tau\right\|^4_{L^4(\Omega;R^m)} &= \int_\Omega \left|\int_0^{s(x)} q(\tau, y(x))d\tau\right|^4 dx \\
&= \int_{\partial\Omega_-}\int_0^{l(y)} \left|\int_0^s q(\tau,y)d\tau\right|^4 \frac{|f(y)|}{|v(s,y)|} ds dS_y \\
&\le \int_{\partial\Omega_-}\int_0^{l(y)} s^3\int_0^s |q(\tau,y)|^4 d\tau \frac{|f(y)|}{|v(s,y)|} ds dS_y \\
&\le \alpha_0^{-1} L^4_{\gamma_0} \int_{\partial\Omega_-}\int_0^{l(y)} |q(\tau,y)|^4 d\tau |f(y)| dS_y \\
&\le \alpha_0^{-1} L^4_{\gamma_0} M\beta_0 \int_{\partial\Omega_-}\int_0^{l(y)} |q(\tau,y)|^4 \frac{|f(y)|}{|v(\tau,y)|} d\tau dS_y \\
&\le C\|q\|^4_{L^4(\Omega;R^m)}
\end{aligned}$$

for all $q \in L^4(\Omega; R^m)$. For $q \in L^2(\partial\Omega_-; R^m)$, one may get

$$\begin{aligned}
\|q(0,y(\cdot))\|^2_{L^2(\Omega;R^m)} &= \int_\Omega |q(0,y(x))|^2 dx \\
&= \int_{\partial\Omega_-}\int_0^{l(y)} |q(0,y)|^2 \frac{|f(y)|}{|v(s,y)|} ds dS_y \\
&\le \alpha_0^{-1} L_{\gamma_0} M\beta_0 \|q\|^2_{L^2(\partial\Omega_-;R^m)} \\
&\le C\|q\|^2_{L^2(\partial\Omega_-;R^m)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|q(0,y(\cdot))\|^4_{L^4(\Omega;R^m)} &= \int_\Omega |q(0,y(x))|^4 dx \\
&= \int_{\partial\Omega_-}\int_0^{l(y)} |q(0,y)|^4 \frac{|f(y)|}{|v(s,y)|} ds dS_y \\
&\leq \alpha_0^{-1} L_{\gamma_0} M\beta_0 \|q\|^4_{L^4(\partial\Omega_-;R^m)} \\
&\leq C\|q\|^4_{L^4(\partial\Omega_-;R^m)}
\end{aligned}$$

for $q \in L^4(\partial\Omega_-;R^m)$. $\square$

Next, we estimate the solution of (17) and (18) and its derivatives in terms of their boundary values.

Lemma 4.7 Suppose that $v \in v_0 + V_\gamma$ with $\gamma \leq \gamma_0$. Assume that z is a solution of (17) and (18). Then one has

$$\|z\|_{0,\Omega} \leq C_1 C\|z\|_{0,\partial\Omega_-},$$

$$\||z|_1\|_{0,\Omega} \leq K_4(\||z|_1\|_{0,\partial\Omega_-} + \|z\|_{0,\infty,\partial\Omega_-})$$

and

$$\||z|_2\|_{0,\Omega} \leq K_5(\||z|_2\|_{0,\partial\Omega_-} + \||z|_1\|_{0,\partial\Omega_-} + \|z\|_{0,\infty,\partial\Omega_-}).$$

Proof It follows from Lemma 4.2 and Lemma 4.6 that

$$\begin{aligned}
\||z|_1\|_{0,\Omega} &\leq C_2\left(\||z|_1(0)\|_{0,\Omega} + \|z\|_{0,\infty,\partial\Omega_-}\|\int_0^s |v|_2(\tau)|d\tau\|_{0,\Omega}\right) \\
&\leq C_2C\left(\||z|_1\|_{0,\partial\Omega_-} + \|z\|_{0,\infty,\partial\Omega_-}\||v|_2\|_{0,\Omega}\right) \\
&\leq C_2C\left(\||z|_1\|_{0,\partial\Omega_-} + \|z\|_{0,\infty,\partial\Omega_-}\|v\|_{2,\Omega}\right) \\
&\leq C_2C\left(\||z|_1\|_{0,\partial\Omega_-} + \|z\|_{0,\infty,\partial\Omega_-}\|v\|_{3,\Omega}\right) \\
&\leq C_2C\left(\||z|_1\|_{0,\partial\Omega_-} + \|z\|_{0,\infty,\partial\Omega_-}\beta_0\right) \\
&\leq K_4(\||z|_1\|_{0,\partial\Omega_-} + \|z\|_{0,\infty,\partial\Omega_-}).
\end{aligned}$$

Similarly, one deduces from Lemma 4.3 and Lemma 4.6 that

$$\begin{aligned}
\||z|_2\|_{0,\Omega} \leq\ & C_3(\||z|_2(0)\|_{0,\Omega} + \||z|_1(0)\|_{0,\Omega} \\
& +\|z\|_{0,\infty,\partial\Omega_-}(\|(\int_0^s |v|_2(\tau)d\tau)^2\|_{0,\Omega} + \|\int_0^s |v|_3(\tau)d\tau\|_{0,\Omega})) \\
\leq\ & C_3(\||z|_2(0)\|_{0,\Omega} + \||z|_1(0)\|_{0,\Omega} \\
& +\|z\|_{0,\infty,\partial\Omega_-}(\|\int_0^s |v|_2(\tau)d\tau\|^2_{0,4,\Omega} + \|\int_0^s |v|_3(\tau)d\tau\|_{0,\Omega}))) \\
\leq\ & C_3C(\||z|_2\|_{0,\partial\Omega_-} + \||z|_1\|_{0,\partial\Omega_-} \\
& +\|z\|_{0,\infty,\partial\Omega_-}(\||v|_2\|^2_{0,4,\Omega} + \||v|_3\|_{0,\Omega}))
\end{aligned}$$

$$
\begin{aligned}
&\leq C_3C(\||z|_2\|_{0,\partial\Omega_-} + \||z|_1\|_{0,\partial\Omega_-} \\
&\quad +\|z\|_{0,\infty,\partial\Omega_-}(\|v\|^2_{2,4,\Omega} + \|v\|_{3,\Omega})) \\
&\leq C_3C(\||z|_2\|_{0,\partial\Omega_-} + \||z|_1\|_{0,\partial\Omega_-} \\
&\quad +\|z\|_{0,\infty,\partial\Omega_-}(M^2\|v\|^2_{3,\Omega} + \|v\|_{3,\Omega})) \\
&\leq C_3C(\||z|_2\|_{0,\partial\Omega_-} + \||z|_1\|_{0,\partial\Omega_-} + \|z\|_{0,\infty,\partial\Omega_-}(M^2\beta_0^2 + \beta_0)) \\
&\leq K_5(\||z|_2\|_{0,\partial\Omega_-} + \||z|_1\|_{0,\partial\Omega_-} + \|z\|_{0,\infty,\partial\Omega_-}).
\end{aligned}
$$

Thus Lemma 4.7 is proved.

5 Boundary Estimates

In this section, we give the boundary estimates for solution to (17) and (18). For $q = (q_1, \cdots, q_m)$: $\partial\Omega_- \to R^m$, set

$$q_{l|Ti} = e_i \cdot (\nabla_T q_l)$$

for all $1 \leq l \leq m, 1 \leq i \leq 3$, where $\{e_1, e_2, e_3\}$ is the standard orthogonal basis of R^3. Moreover, we will use the following notations in this section:

$$q_{l|Tij} = e_j \cdot (\nabla_T q_{l|Ti})$$

for all $1 \leq l \leq m, 1 \leq i, j \leq 3$,

$$q_{|Ti} = (q_{1|Ti}, \cdots, q_{m|Ti})$$

for all $1 \leq i \leq 3$,

$$q_{|Tij} = (q_{1|Tij}, \cdots, q_{1|Tij})$$

for all $1 \leq l \leq m, 1 \leq i, j \leq 3$,

$$\nabla_T q = (\nabla_T q_1, \cdots, \nabla_T q_m),$$

$$\nabla_T^2 q_l = (\nabla_T q_{l|T1}, \cdots, \nabla_T q_{l|T3})$$

for all $1 \leq l \leq m$,

$$\nabla_T^2 q = (\nabla_T^2 q_1, \cdots, \nabla_T^2 q_m),$$

$$|\nabla_T q| = \left(\sum_{i=1}^{3} |q_{|Ti}|^2\right)^{\frac{1}{2}}$$

and

$$|\nabla_T^2 q| = \left(\sum_{i=1}^{3}\sum_{j=1}^{3} |q_{|Tij}|^2\right)^{\frac{1}{2}}.$$

First, we have the following elementary facts:

Lemma 5.1 The tangential gradient ∇_T has the following properties:

$$|\nabla_T(aq)| \le |\nabla_T a||q| + |a||\nabla_T q|, \quad \forall y \in \partial\Omega_-, \tag{45}$$

$$|\nabla_T(q \cdot r)| \le |\nabla_T q||r| + |q||\nabla_T r|, \quad \forall y \in \partial\Omega_-, \tag{46}$$

$$|\nabla_T^2(aq)| \le |\nabla_T^2 a||q| + 2|\nabla_T a||\nabla_T q| + |a||\nabla_T^2 q|, \quad \forall y \in \partial\Omega_-, \tag{47}$$

$$|\nabla_T((v_T \cdot \nabla)z)| \le 2(|\nabla v| + |v||\nabla_T n|)|\nabla_T z| + |v||\nabla_T^2 z|, \quad \forall y \in \partial\Omega_-, \tag{48}$$

and

$$|\nabla_T((z \cdot \nabla)v)| \le |v|_1|\nabla_T z| + |z||v|_2, \quad \forall y \in \partial\Omega_-. \tag{49}$$

Proof First we prove (45). By the multiplication formula of tangential gradient and Minkowski inequality, one has

$$\begin{aligned}
|\nabla_T(aq)| &= \left(\sum_{l=1}^m |\nabla_T(aq_l)|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{l=1}^m |q_l\nabla_T a|^2\right)^{\frac{1}{2}} + \left(\sum_{l=1}^m |a\nabla_T q_l|^2\right)^{\frac{1}{2}} \\
&= |\nabla_T a||q| + |a||\nabla_T q|
\end{aligned}$$

which shows (45).

Next we prove (46). Due to

$$\nabla_T(q \cdot r) = \sum_{l=1}^m \nabla_T(q_l r_l) = \sum_{l=1}^m q_l \nabla_T r_l + \sum_{l=1}^m r_l \nabla_T q_l, \tag{50}$$

and Cauchy inequality, one can obtain

$$\begin{aligned}
|\nabla_T(q \cdot r)| &\le \sum_{l=1}^m |q_l||\nabla_T r_l| + \sum_{l=1}^m |r_l||\nabla_T q_l| \\
&\le \left(\sum_{l=1}^m |q_l|^2\right)^{\frac{1}{2}} \left(\sum_{l=1}^m |\nabla_T r_l|^2\right)^{\frac{1}{2}} + \left(\sum_{l=1}^m |r_l|^2\right)^{\frac{1}{2}} \left(\sum_{l=1}^m |\nabla_T q_l|^2\right)^{\frac{1}{2}} \\
&= |\nabla_T q||r| + |q||\nabla_T r|
\end{aligned}$$

which is just (46).

To prove (47), we apply Minkowski inequality, (46) and Cauchy inequality to get

$$\begin{aligned}
|\nabla_T^2(aq)| &= \left(\sum_{l=1}^m |\nabla_T^2(aq_l)|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{l=1}^m |\nabla_T(q_l\nabla_T a)|^2\right)^{\frac{1}{2}} + \left(\sum_{l=1}^m |\nabla_T(a\nabla_T q_l)|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sum_{l=1}^{m}(|\nabla_T q_l||\nabla_T a|)^2\right)^{\frac{1}{2}} + \left(\sum_{l=1}^{m}(|q_l||\nabla_T^2 a)|)^2\right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{l=1}^{m}(|\nabla_T a||\nabla_T q_l|)^2\right)^{\frac{1}{2}} + \left(\sum_{l=1}^{m}(|a||\nabla_T^2 q_l)|)^2\right)^{\frac{1}{2}} \\
&= |\nabla_T^2 a||q| + 2|\nabla_T a||\nabla_T q| + |a||\nabla_T^2 q|.
\end{aligned}
$$

Thus (47) follows.

Next, it follows from (47) and Minkowski inequality that

$$
\begin{aligned}
|\nabla_T((v_T\cdot\nabla)z)| &= \left(\sum_{l=1}^{3}|\nabla_T((v_T\cdot\nabla)z_l)|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{l=1}^{m}|\nabla_T(v_T\cdot\nabla_T z_l)|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{l=1}^{3}|\nabla_T v_T|^2|\nabla_T z_l|^2\right)^{\frac{1}{2}} + \left(\sum_{l=1}^{3}|v_T|^2|\nabla_T^2 z_l|^2\right)^{\frac{1}{2}} \\
&= |\nabla_T v_T||\nabla_T z| + |v_T||\nabla_T^2 z|. \qquad (51)
\end{aligned}
$$

Moreover, (45) and (46) imply that

$$
\begin{aligned}
|\nabla_T v_T| &\leq |\nabla_T v| + |\nabla_T((v\cdot n)n)| \\
&\leq |\nabla v| + |\nabla_T(v\cdot n)| + |v\cdot n||\nabla_T n| \\
&\leq |\nabla v| + |\nabla_T v| + |v||\nabla_T n| + |v||\nabla_T n| \\
&\leq 2(|\nabla v| + |v||\nabla_T n|).
\end{aligned}
$$

Then (48) follows from (51) and (52).

Finally, we prove (49). From (46) and Minkowski inequality, one obtains

$$
\begin{aligned}
|\nabla_T((z\cdot\nabla)v)| &= \left(\sum_{l=1}^{3}|\nabla_T((z\cdot\nabla)v_l)|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{l=1}^{3}(|\nabla_T z||\nabla v_l| + |z||\nabla_T\nabla v_l|)^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{l=1}^{3}|\nabla_T z|^2|\nabla v_l|^2\right)^{\frac{1}{2}} + \left(\sum_{l=1}^{3}|z|^2|\nabla_T\nabla v_l|^2\right)^{\frac{1}{2}} \\
&= |\nabla_T z||\nabla v| + |z||\nabla_T\nabla v| \\
&\leq |\nabla_T z||v|_1 + |z||v|_2.
\end{aligned}
$$

Hence (49) holds. So the proof of this lemma is complete. □

Lemma 5.2 Suppose that z is a solution of (17)-(18). Then it holds that

$$|z|_1(0,y) \leq C_5|1/f|(|z| + |\nabla_T z|), \quad \forall y \in \partial\Omega_- \tag{52}$$

for some positive constant C_5.

Proof Note that for any $y \in \partial\Omega_-$,

$$(z \cdot \nabla)v = (v \cdot \nabla)z = ((v_T + (n \cdot v)n) \cdot \nabla)z = (v_T \cdot \nabla)z + f\partial_n z.$$

Thus

$$\begin{aligned} f\partial_n z &= ((v \cdot n)n \cdot \nabla)z \\ &= (v \cdot \nabla)z - (v_T \cdot \nabla)z \\ &= (z \cdot \nabla)v - (v_T \cdot \nabla)z. \end{aligned}$$

Hence one has

$$\begin{aligned} |f||\partial_n z| &\leq |z||v|_1 + |v_T||\nabla_T z| \\ &\leq |z||v|_1 + |v||\nabla_T z|, \end{aligned}$$

which implies that

$$|\partial_n z| \leq |1/f|(|z||v|_1 + |v||\nabla_T z|). \tag{53}$$

It follows that

$$\begin{aligned} |z|_1 &= \left(\sum_{l=1}^{3} |\nabla z_l)|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{l=1}^{3} |\nabla_T z_l)|^2\right)^{\frac{1}{2}} + \left(\sum_{l=1}^{3} |\partial_n z_l)|^2\right)^{\frac{1}{2}} \\ &= |\nabla_T z| + |\partial_n z| \\ &\leq |\nabla_T z| + |1/f|(|z||v|_1 + |v||\nabla_T z|) \\ &\leq |\nabla_T z| + |1/f|(|z|M\beta_0 + M\beta_0|\nabla_T z|) \\ &\leq C_5|1/f|(|z| + |\nabla_T z|), \end{aligned}$$

which leads to (52) with $C_5 = M\beta_0 + \|f\|_\infty$. □

Lemma 5.3 Assume that that z is a solution to the problem (17)-(18). Then

$$|z|_2(0,y) \leq C_6|1/f|^3(|z| + |\nabla_T z|) + C_6|1/f|^2(|z||v|_2 + |\nabla_T^2 z|), \quad \forall y \in \partial\Omega_- \tag{54}$$

for some positive constant C_6.

Proof It follows from (48), (49), the proofs of (51) and (53), and (53) that

$$\begin{array}{rcl} |f\nabla_T\partial_n z| & \le & |\nabla_T(f\partial_n z)| + |\nabla_T f||\partial_n z| \\ & \le & |\nabla_T(z\cdot\nabla)v| + |\nabla_T(v_T\cdot\nabla)z| + M_4|\partial_n z| \\ & \le & |\nabla_T z||\nabla v| + |z||\nabla_T\nabla v| + |\nabla_T v_T||\nabla_T z| \\ & & +|v_T||\nabla_T^2 z| + M_4|\partial_n z| \\ & \le & |\nabla_T z||v|_1 + |z||v|_2 + M_5|\nabla_T z| \\ & & +|v||\nabla_T^2 z| + M_4|\partial_n z| \\ & \le & |z||v|_2 + M_6|\nabla_T^2 z| + M_6|1/f|(|z| + |\nabla_T z|), \end{array}$$

where one has used the estimates

$$\begin{array}{rcl} |\nabla_T f| & = & |\nabla_T n\cdot v_0| \\ & \le & |\nabla_T n||v_0| + |\nabla_T v_0| \\ & \le & |\nabla_T n||v_0| + |v_0|_1 \\ & \le & M_4 \end{array}$$

and

$$\begin{array}{rcl} |\nabla_T v_T| & = & |\nabla_T v| + |\nabla_T(fn)| \\ & \le & |v|_1 + |\nabla_T f| + |f||\nabla_T n| \\ & \le & |v|_1 + M_4 + |f||\nabla_T n| \\ & \le & M_5. \end{array}$$

Hence,

$$|\nabla_T\partial_n z| \le M_7|1/f|(|z||v|_2 + +|\nabla_T^2 z|) + M_7|1/f|^2(|z| + |\nabla_T z|). \qquad (55)$$

Note that

$$\begin{array}{rcl} \nabla_T z_{|i} & = & \nabla_T((e_i\cdot\nabla)z) \\ & = & \nabla_T((e_{iT}\cdot\nabla)z) + \nabla_T(n_i\partial_n z). \end{array}$$

We obtain

$$\begin{array}{rcl} |\nabla_T z_{|i}| & \le & |\nabla_T(e_{iT}\cdot\nabla z)| + |\nabla_T(n_i\partial_n z)| \\ & \le & |\nabla_T e_{iT}|\nabla_T z| + |e_{iT}||\nabla_T^2 z| + |\nabla_T n_i||\partial_n z| + |n_i||\nabla_T\partial_n z|. \end{array}$$

Hence,

$$\begin{array}{rcl} \left(\sum_{i=1}^3 |\nabla_T z_{|i}|^2\right)^{\frac{1}{2}} & \le & \left(\sum_{i=1}^3 |\nabla_T e_{iT}|^2|\nabla_T z|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^3 |e_{iT}|^2|\nabla_T^2 z|^2\right)^{\frac{1}{2}} \\ & & + \left(\sum_{i=1}^3 |\nabla_T n_i|^2|\partial_n z|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^3 |n_i|^2|\nabla_T\partial_n z|^2\right)^{\frac{1}{2}} \end{array}$$

$$
\begin{aligned}
&= |\nabla_T z| \left(\sum_{i=1}^{3} |\nabla_T e_{iT}|^2\right)^{\frac{1}{2}} + |\nabla_T^2 z| \left(\sum_{i=1}^{3} |e_{iT}|^2\right)^{\frac{1}{2}} \\
&\quad + |\partial_n z| \left(\sum_{i=1}^{3} |\nabla_T n_i|^2\right)^{\frac{1}{2}} + |\nabla_T \partial_n z| \left(\sum_{i=1}^{3} |n_i|^2\right)^{\frac{1}{2}} \\
&\le M_8(|\nabla_T z| + |\nabla_T^2 z| + |\partial_n z| + |\nabla_T \partial_n z|). \qquad (56)
\end{aligned}
$$

By (42), it holds that

$$
\begin{aligned}
f\partial_n z_{|i} &= f(n\cdot\nabla) z_{|i} \\
&= ((v\cdot n)n\cdot\nabla) z_{|i} \\
&= (v\cdot\nabla) z_{|i} - (v_T\cdot\nabla) z_{|i} \\
&= (z\cdot\nabla) v_{|i} + (z_{|i}\cdot\nabla) v - (v_{|i}\cdot\nabla) z - (v_T\cdot\nabla) z_{|i}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|f||\partial_n z_{|i}| &\le |(z\cdot\nabla) v_{|i}| + |(z_{|i}\cdot\nabla) v| + |(v_{|i}\cdot\nabla) z| + |(v_T\cdot\nabla) z_{|i}| \\
&\le |z||v_{|i}|_1 + |z_{|i}||v|_1 + |v_{|i}||z|_1 + |v_T||\nabla_T z_{|i}|.
\end{aligned}
$$

Moreover, it follows from Minkowski inequality that

$$
\begin{aligned}
|f| \left(\sum_{i=1}^{3} |\partial_n z_{|i}|^2\right)^{\frac{1}{2}} &\le \left(\sum_{i=1}^{3} |z|^2 |v_{|i}|_1^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{3} |z_{|i}|^2 |v|_1^2\right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{i=1}^{3} |v_{|i}|^2 |z|_1^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{3} |v_T|^2 |\nabla_T z_{|i}|^2\right)^{\frac{1}{2}} \\
&= |z||v|_2 + |v|_1|z|_1 + |z|_1|v|_1 + |v_T| \left(\sum_{i=1}^{3} |\nabla_T z_{|i}|^2\right)^{\frac{1}{2}} \\
&\le |z||v|_2 + M_9(|z|_1 + \left(\sum_{i=1}^{3} |\nabla_T z_{|i}|^2\right)^{\frac{1}{2}}). \qquad (57)
\end{aligned}
$$

Therefore by Minkowski inequality, (56), (57), (55), (53) and (52), we have

$$
\begin{aligned}
|z|_2(0,y) &= \left(\sum_{l=1}^{3}\sum_{i=1}^{3} |\nabla z_{l|i}|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{l=1}^{3}\sum_{i=1}^{3} |\nabla_T z_{l|i}|^2\right)^{\frac{1}{2}} + \left(\sum_{l=1}^{3}\sum_{i=1}^{3} |\partial_n z_{l|i}|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{3} |\nabla_T z_{|i}|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{3} |\partial_n z_{|i}|^2\right)^{\frac{1}{2}} \\
&\le M_{10}|1/f|(|\nabla_T z| + |\nabla_T^2 z| + |\partial_n z| + |\nabla_T \partial_n z| + |z||v|_2 + |z|_1) \\
&\le C_6|1/f|^3(|z| + |\nabla_T z|) + C_6|1/f|^2(|z||v|_2 + |\nabla_T^2 z|).
\end{aligned}
$$

□

Lemma 5.4 Let z be a solution to (17)-(18). Then the following estimates hold:

$$|z|(0,y) \le C_7|a| + |b|, \quad \forall y \in \partial\Omega_-, \tag{58}$$

$$|z|_1(0,y) \le C_7|1/f|(|a| + |b| + |\nabla_T a| + |\nabla_T b|), \quad \forall y \in \partial\Omega_-, \tag{59}$$

$$\begin{aligned} |z|_2(0,y) &\le C_7|f|^{-3}(|a| + |b| + |\nabla_T a| + |\nabla_T b|) \\ &\quad + C_7|f|^{-2}(|\nabla_T^2 a| + |\nabla_T^2 b| + (|a| + |b|)|v|_2), \quad \forall y \in \partial\Omega_-. \end{aligned} \tag{60}$$

Proof Set

$$C_7 = 2(C_5 + C_6 + 1)(M^2\beta_0^2 + M\beta_0 + 1).$$

By (18), (16) and (31), one has

$$|z|(0,y) \le |a||v| + |b| \le M\beta_0|a| + |b| \le C_7|a| + |b|, \quad \forall y \in \partial\Omega_-.$$

Then (58) holds. It follows from (18), (45), (16) and (31) that

$$\begin{aligned} |\nabla_T z| &\le |\nabla_T(av)| + |\nabla_T b| \\ &\le |\nabla_T a||v| + |a||v|_1 + |\nabla_T b| \\ &\le (M\beta_0 + 1)(|\nabla_T a| + |a| + |\nabla_T b|). \end{aligned}$$

Hence by (52), one has

$$\begin{aligned} |z|_1(0,y) &\le C_5|1/f|(|z| + |\nabla_T z|) \\ &\le C_5|1/f|(M\beta_0|a| + |b| + (M\beta_0 + 1)(|\nabla_T a| + |a| + |\nabla_T b|)) \\ &\le C_7(|a| + |b| + |\nabla_T a| + |\nabla_T b|) \end{aligned}$$

for all $y \in \partial\Omega_-$, which shows (59). Due to (18), (47), (16) and (31), one can obtain

$$\begin{aligned} |\nabla_T^2 z| &\le |\nabla_T^2(av)| + |\nabla_T^2 b| \\ &\le |\nabla_T^2 a||v| + 2|\nabla_T a||\nabla_T v| + |a||\nabla_T^2 v| + |\nabla_T^2 b| \\ &\le |\nabla_T^2 a||v| + 2|\nabla_T a||v|_1 + |a||\nabla_T^2 v| + |\nabla_T^2 b| \\ &\le M\beta_0(|\nabla_T^2 a| + 2|\nabla_T a|) + |a||v|_2 + |\nabla_T^2 b|. \end{aligned}$$

Then by (54) we have

$$\begin{aligned} |z|_2(0,y) &\le C_6|1/f|^3(|z| + |\nabla_T z|) + C_6|1/f|^2(|z||v|_2 + |\nabla_T^2 z|) \\ &\le C_6|1/f|^3(M\beta_0|a| + |b| + (M\beta_0 + 1)(|\nabla_T a| + |a| + |\nabla_T b|)) \\ &\quad + C_6|1/f|^2((M\beta_0|a| + |b|)|v|_2 + M\beta_0(|\nabla_T^2 a| \\ &\quad + 2|\nabla_T a|) + |a||v|_2 + |\nabla_T^2 b|) \\ &\le C_7|1/f|^3(|a| + |b| + |\nabla_T a| + |\nabla_T b|) \\ &\quad + C_7|1/f|^2(|\nabla_T^2 a| + |\nabla_T^2 b| + (|a| + |b|)|v|_2), \end{aligned}$$

where we have used the fact that

$$\|f\|_\infty \le \|v_0\| \le M\|v_0\| \le M\beta_0.$$

So the proof of this lemma is complete. $\square$

Based on these estimates, we have the following desired boundary estimates.

Lemma 5.5 Suppose that $v \in v_0 + V_\gamma$ with $\gamma \le \gamma_0$. Let z be a solution to (17)-(18). Then the following estimates hold:

$$\|z\|_{0,\partial\Omega_-} \le C_8(\|a\|_{0,\partial\Omega_-} + \|b\|_{0,\partial\Omega_-}), \tag{61}$$

$$\|z\|_{0,\infty,\partial\Omega_-} \le C_8(\|a\| + \|b\|), \tag{62}$$

$$\||z|_1\|_{0,\partial\Omega_-} \le C_8(\|a\| + \|b\|), \tag{63}$$

$$\||z|_2\|_{0,\partial\Omega_-} \le C_8(\|a\| + \|b\|). \tag{64}$$

Proof Set

$$C_8 = C_7(M^2\beta_0^2 + M\beta_0 + 1).$$

Then (61) and (63) are obtained easily from (58) and (59). It follows from (58) that

$$\begin{aligned}\|z\|_{0,\infty,\partial\Omega_-} &\le C_7\|a\|_{0,\infty,\partial\Omega_-} + \|b\|_{0,\infty,\partial\Omega_-} \\ &\le C_7\|f\|_\infty^2\|a\| + \|f\|_\infty^2\|b\| \\ &\le C_7 M^2\beta_0^2\|a\| + M^2\beta_0^2\|b\| \\ &\le C_8(\|a\| + \|b\|).\end{aligned}$$

Then (62) holds. Finally, by (60), (16) and (31), one can obtain

$$\begin{aligned}\||z|_2\|_{0,\partial\Omega_-} &\le C_7(\|a\| + \|b\|)(1 + \||v|_2\|_{0,\partial\Omega_-}) \\ &\le C_7(\|a\| + \|b\|)(1 + \|v\|_{2,\partial\Omega}) \\ &\le C_7(\|a\| + \|b\|)(1 + M\|v\|_{3,\Omega}) \\ &\le C_7(1 + M\beta_0)(\|a\| + \|b\|) \\ &\le C_8(\|a\| + \|b\|),\end{aligned}$$

which proves (64). Thus Lemma 5.5 is proved.

6 Proof of Lemmas 2.2 and 2.3

Based on the preparations in previous two sections, we are now ready to prove Lemmas 2.2 and 2.3. We start with the proof of Lemma 2.2.

Proof of Lemma 2.2 It follows from Lemmas 4.7 and 5.5 that

$$\begin{aligned}\|z\|_{0,\Omega} &\le C_1C\|z\|_{0,\partial\Omega_-}\\ &\le C_1CC_8(\|a\|_{0,\partial\Omega_-}+\|b\|_{0,\partial\Omega_-}).\end{aligned}$$

Hence (19) holds.

Applying Lemmas 4.7 and 5.5 again shows that

$$\begin{aligned}\|z\|_{2,\Omega} &\le C_1C\|z\|_{0,\partial\Omega_-}+K_4(\||z|_1\|_{0,\partial\Omega_-}+\|z\|_{0,\infty,\partial\Omega_-})\\ &\quad +K_5(\||z|_2\|_{0,\partial\Omega_-}+\||z|_1\|_{0,\partial\Omega_-}+\|z\|_{0,\infty,\partial\Omega_-})\\ &\le C_1C(C_8(\|a\|_{0,\partial\Omega_-}+\|b\|_{0,\partial\Omega_-}))+2K_4C_8(\|a\|+\|b\|)\\ &\quad +3K_5C_8(\|a\|+\|b\|)\\ &\le C_8(CC_1M^3\beta_0^3+2K_4+3K_5)(\|a\|+\|b\|),\end{aligned}$$

which implies (20).

By (44), (11), Sobolev's embedding theorem and Sobolev's trace theorem (see (16)), we have

$$\begin{aligned}\|[z](0,y(\cdot))\|_{0,\Omega} &\le C\|[z](0,y)\|_{0,\partial\Omega_-}\\ &\le C\||a||[v]|\|_{0,\partial\Omega_-}\\ &\le C\|a\|_\infty\|[v]\|_{0,\partial\Omega_-}\\ &\le CM^3\beta_0^2\|a\|\|[v]\|_{1,\Omega}.\end{aligned}$$

It follows from (43) and Sobolev's embedding theorem that

$$\begin{aligned}\|\int_0^{s(\cdot)}(|Av^{(1)}||[v]|_1)(\tau,y(\cdot)))d\tau\|_{0,\Omega} &\le C\||Av^{(1)}||[v]|_1\|_{0,\Omega}\\ &\le C\|Av^{(1)}\|_\infty\||[v]|_1\|_{0,\Omega}\\ &\le CM\|Av^{(1)}\|_{2,\Omega}\|[v]\|_{1,\Omega}\end{aligned}$$

and

$$\begin{aligned}\|\int_0^{s(\cdot)}((|[v]||Av^{(1)}|_1)(\tau,y(\cdot)))d\tau\|_{0,\Omega} &\le C\||[v]||Av^{(1)}|_1\|_{0,\Omega}\\ &\le C\|[v]\|_{0,4,\Omega}\||Av^{(1)}|_1\|_{0,4,\Omega}\\ &\le CM^2\|[v]\|_{1,\Omega}\|Av^{(1)}\|_{2,\Omega}.\end{aligned}$$

Combining these estimates with Lemma 4.4 and (20) leads to

$$\begin{aligned}\|(Av^{(1)}-Av)(s)\|_{0,\Omega} &\le C_4(\|[z](0)\|_{0,\Omega}+\|\int_0^s(|Av^{(1)}||[v]|_1d\tau\|_{0,\Omega}\\ &\quad +\|\int_0^s|[v]||Av^{(1)}|_1)d\tau\|_{0,\Omega})\\ &\le C_4(CM^3\beta_0^2\|a\|\|[v]\|_{1,\Omega}+CM\|Av^{(1)}\|_{2,\Omega}\|[v]\|_{1,\Omega}\\ &\quad +CM^2\|[v]\|_{1,\Omega}\|Av^{(1)}\|_{2,\Omega})\\ &\le C_4(CM^3\beta_0^2+CM(M+1)C_8(CC_1M^3\beta_0^3\\ &\quad +2K_4+3K_5))(\|a\|+\|b\|)\|[v]\|_{1,\Omega},\end{aligned}$$

which implies (21). $\square$

We now turn to the proof of Lemma 2.3.

Proof Lemma 2.3 Due to (8), it holds that

$$n \times (b \times n) = (n \cdot n)b - (n \cdot b)n = b.$$

This, together with (18), yields

$$z = av + n \times (b \times n).$$

Hence,

$$\begin{aligned} v \times z &= v \times (n \times (b \times n)) \\ &= (v \cdot (b \times n))n - (v \cdot n)(b \times n) \\ &= (v \cdot (b \times n))n - (fb) \times n. \end{aligned} \tag{65}$$

It follows from (17), (26), and div $v = 0$, that

$$\text{curl } (v \times z) = v \text{ div } z.$$

This, together with (9), implies

$$\begin{aligned} f \text{ div } z &= (n \cdot v) \text{ div } z \\ &= n \cdot \text{curl } (v \times z) \\ &= \lim_{\triangle S \to 0} \frac{1}{\triangle S} \int_l (v \times z) \cdot dl \\ &= \lim_{\triangle S \to 0} \frac{1}{\triangle S} \int_l ((fb) \times n) \cdot dl \\ &= \lim_{\triangle S \to 0} \frac{1}{\triangle S} \int_l (fb) \cdot dr \\ &= \text{div } (fb) \\ &= 0 \end{aligned}$$

where S is a smooth surface lying in $\partial\Omega_-$ with smooth boundary l. Thus we have

$$\text{div } z = 0 \text{ on } \partial\Omega_-. \tag{66}$$

Note that

$$\begin{aligned} \text{div } ((v \cdot \nabla)z) &= \sum_{i=1}^{3} D_i(\sum_{j=1}^{3} v_j D_j z_i) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} D_i v_j D_j z_i + (v \cdot \nabla) \text{ div } z. \end{aligned}$$

This, together with (17), implies

$$(v \cdot \nabla)\ \text{div}\ z = (z \cdot \nabla)\ \text{div}\ v = 0, \quad x \in \Omega.$$

Hence div z is a constant on the stream line of v. It follows from this and (66) that div $z = 0$, $x \in \Omega$. So the proof of Lemma 2.3 is completed. $\square$

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