Multiple nontrivial solutions for some fourth order quasilinear elliptic problems with local superlinearity and sublinearity

Zhang Jihui
Institute of Mathematics, School of Mathematics Sciences,
Nanjing Normal University, Nanjing, People's Republic of China
The Institute of Mathematical Sciences,
The Chinese University of Hong Kong, Shatin, N.T., Hong Kong

Abstract

In this paper, we consider the existence of multiple nontrivial solutions for some fourth order quasilinear elliptic boundary value problems. The weak solutions are sought by using variational methods.

Keywords: Quasilinear; Superlinear; Sublinear; Fourth order elliptic problem; Variational method

1. Introduction

Let Ω be a bounded smooth open set in R^n. In this paper, we are concerned with the existence of multiple nontrivial solutions to the fourth order quasilinear elliptic boundary value problem

$$\begin{cases} \Delta\left(g_1\left((\Delta u)^2\right)\Delta u\right) + c\,\mathrm{div}\left(g_2\left(|\nabla u|^2\right)\nabla u\right) = f(x,u) & \text{in } \Omega, \\ u = 0,\ \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $c \in R$, $g_1,\ g_2 \in C(R,R)$ and $f : \Omega \times R \to R$ is a Carathéodory's function and satisfies the local superlinearity and sublinearity condition.

[0]E-mail address: zhangjihui@njnu.edu.cn(J. Zhang)

Papers [16-18] considered the fourth order semilinear elliptic boundary value problem

$$\begin{cases} \Delta^2 u + c\ \Delta u = f(x,u) & \text{in } \Omega, \\ \\ u = 0,\ \ \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Δ^2 denotes the biharmonic operator, and the fourth order quasilinear elliptic boundary value problem

$$\begin{cases} \Delta\left(g_1\left((\Delta u)^2\right)\Delta u\right) + c\ \mathrm{div}\left(g_2\left(|\nabla u|^2\right)\nabla u\right) = f(x,u) & \text{in } \Omega, \\ \\ u = 0,\ \ \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where h_1 and $h_2 \in C(R,R)$.

In problem (1.2), let $f(x,u) = b[(u+1)^+ - 1]$, then we get the following Dirichlet problem

$$\begin{cases} \Delta^2 u + c\ \Delta u = b[(u+1)^+ - 1] & \text{in } \Omega, \\ \\ u = 0,\ \ \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $u^+ = \max\ \{u, 0\}$.

There are many results about problems (1.1) -(1.4) (cf. [1, 2, 5-7, 16-19]). For problem (1.4), Lazer and McKenna in [2] proved the existence of $2k-1$ solutions when $\Omega \subset R$ is an interval and $b > \lambda_k(\lambda_k - c)$ by the global bifurcation method. In [5] Tarantello obtained a negative solution when $b \geq \lambda_1(\lambda_1 - c)$ by the degree theory. For problem (1.2) when $f(x,u) = bg(x,u)$, Micheletti and Pistoia in [6, 7] proved that there exist two solutions or three solutions for a more general nonlinearity g by variational method. Papers [16-19] proved the existence of weak solutions of problems (1.2) and (1.3) for a more general nonlinearity f by means of variational method, Morse theory and local linking.

In paper [19], it was studied that the existence of positive solutions for the fourth order semilinear elliptic boundary value problem

$$\begin{cases} \Delta^2 u + c\ \Delta u = f(x,u) & \text{in } \Omega, \\ \\ u \geq 0,\ u \not\equiv 0 & \text{in } \Omega, \\ \\ u = 0,\ \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where f satisfies the local superlinearity and sublinearity.

In the problem (1.1), let $f(x,u) = \lambda a(x)(u^+)^q + b(x)(u^+)^p$, then we get the following fourth order quasilinear elliptic boundary value problem

$$\begin{cases} \Delta\left(g_1\left((\Delta u)^2\right)\Delta u\right) + c\ \mathrm{div}\left(g_2\left(|\nabla u|^2\right)\nabla u\right) = a(x)(u^+)^q + b(x)(u^+)^p & \text{in } \Omega, \\ \\ u = 0,\ \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $u^+ = \max\{u, 0\}$, $\lambda > 0$ is a parameter and the exponents p and q satisfy $0 \leq q < 1 < p$ with $p < 2^* - 1$ if $N \geq 3$, $p < +\infty$ if $N = 1$ or 2, here $2^* = 2N/(N-2)$.

In the problem (1.1), let $f(x,u) = \lambda a(x)|u|^q + b(x)|u|^p$, then we get the following fourth order quasilinear elliptic boundary value problem

$$\begin{cases} \Delta\left(g_1\left((\Delta u)^2\right)\Delta u\right) + c\ \mathrm{div}\left(g_2\left(|\nabla u|^2\right)\nabla u\right) = a(x)|u|^q + b(x)|u|^p & \text{in } \Omega, \\ \\ u = 0,\ \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\lambda > 0$ is a parameter and the exponents p and q satisfy $0 \leq q < 1 < p$ with $p < 2^* - 1$ if $N \geq 3$, $p < +\infty$ if $N = 1$ or 2, here $2^* = 2N/(N-2)$.

Motivated by the above works, it is the purpose of this paper to use variational methods for the fourth order quasilinear problem (1.1) when nonlinearity f satisfies the local superlinearity and sublinearity condition and the fourth order quasilinear problems (1.6) and (1.7).

The plan of the following sections are as follows. In Section 2 we give some notations and main results. The main results are proved in Section 3.

2. Notations and main results

In this paper we use the following definitions.

Definition 2.1. Problem (1.1) is said to be sublinear (superlinear) at 0 if there exist $\alpha > 0$ and $t_0 > 0$ such that

$$f(x,t) \geq (\leq) \alpha |t| \tag{2.1}$$

for a.e. $x \in \Omega$ and all $0 \leq |t| \leq t_0$.
Problem (1.1) is said to be superlinear (sublinear) at ∞ if there exist $\beta > 0$ and $t_1 > 0$ such that

$$f(x,t) \geq (\leq) \beta |t| \tag{2.2}$$

for a.e. $x \in \Omega$ and all $|t| \geq t_1$. Let V be a real Banach space and let $E \in C^1(V,R)$ be a functional.

Definition 2.2. We say that E satisfies the (PS) condition if for every sequence $\{u_n\}$ in V with $E(u_n)$ bounded and $\lim_{n\to\infty} E'(u_n) = 0$, there exists a convergent subsequence.

Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_k < \cdots$ be the sequence of distinct eigenvalues of the eigenvalue problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

The eigenvalue problem

$$\begin{cases} \Delta^2 u + c\,\Delta u = \mu u & \text{in } \Omega, \\ u = 0,\ \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

has infinitely many eigenvalues $\mu_k = \lambda_k(\lambda_k - c), \quad k = 1, 2, \cdots$.

We will always assume $N \geq 3$, $c < \lambda_1(\Omega)$, denote by σ' the Holder conjugate of σ , by $\lambda_1(\Omega)$ the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ and by $\lambda_1(\Omega_1)$ the first eigenvalue of

$-\Delta$ on $H_0^1(\Omega_1)$. Let V denote the Hilbert space $H^2(\Omega)\cap H_0^1(\Omega)$ is equipped with the inner product

$$\langle u,v\rangle_V=\int_\Omega[\Delta u\Delta v-c\ \nabla u\nabla v]dx. \tag{2.5}$$

We denote by $\| u \|_p$ the norm in $L^p(\Omega)$, by $\| u \|_{0,1}$ the norm in $H_0^1(\Omega)$, and by $\| u \|$ the norm in V is given by

$$\| u \|^2=\langle u,u\rangle_V. \tag{2.6}$$

Let V' denote the dual of V and let $\langle\ ,\ \rangle$ be the duality pairing between V' and V.

Now we give the following assumptions:

Let $g_1,\ g_2: R\to R$ be two functions and satisfy the following conditions:

(i_g) g_1 is a continuous and nondecreasing function;

(ii_g) cg_2 is a continuous and nonincreasing function, where $c\in R$;

(iii_g) There exist $\alpha_1,\ \alpha_2,\ \beta_1$ and $\beta_2\in R$ such that

$$\begin{aligned}0<\alpha_1\le g_1(t)\le\beta_1,\\ c\alpha_2\le cg_2(t)\le c\beta_2\end{aligned}$$

for all $t\in R$

Let $f:\Omega\times R\to R$ be a Carathéodory's function and satisfy the following conditions:

(i_f) There exist $1\le\sigma<2^*,\ d_1\in L^{\sigma'}(\Omega),\ d_2>0$ such that

$$|f(x,t)|\le d_1(x)+d_2|t|^{\sigma-1}$$

for a.e. $x\in\Omega$ and all $t\in R$;

(ii_f) There exist $\theta>2, 1\le r<2,\ d\in L^{(\frac{2^*}{r})'}(\Omega)$, with $d\ge 0$ a.e. in Ω , $t_0\ge 0$, such that

$$\theta F(x,t)\le tf(x,t)+d(x)|t|^r$$

for a.e. $x\in\Omega$ and all $|t|>t_0$, where $F(x,t)=\int_0^t f(x,s)ds$;

(iii_f) There exist $0\le q<1<p<2^*-1,\ a_0\in L^{\sigma_q}(\Omega)$, with $\sigma_q=(\frac{2^*}{q+1})'$ and $a_0\ge 0$ a.e. in Ω, $b_0\in L^{\sigma_p}(\Omega)$, with $\sigma_p=(\frac{2^*}{p+1})'$ and $b_0\ge 0$ a.e. in Ω such that

$$f(x,t)\le a_0|t|^q+b_0|t|^p$$

for a.e. $x \in \Omega$ and all $t \in R$;

(iv_f) There exist a nonempty subdomain $\Omega_1 \subset \Omega$, $\theta_1 > \lambda_1(\Omega_1)(\lambda_1(\Omega_1)\beta_1 - c\alpha_2)/2$, $\mathrm{t}_1 > 0$, such that

$$F(x,t) \geq \theta_1 t^2$$

for a.e. $x \in \Omega_1$ and all $0 \leq t \leq t_1$;

(v_f) There exist a nonempty open subset $\Omega_2 \subset \Omega$, $\theta_2 > 0$, $t_2 \geq 0$, such that

$$F(x,t) \geq \theta_2 t^2$$

for a.e. $x \in \Omega_2$ and all $t \geq t_2$, with the additional requirement that the function $d(x)$ appearing in (iii_f) is bounded on Ω_2.

Remark 2.1. The hypothesis (iv_f) implies that f satisfies the local sublinearity condition at 0, the hypothesis (v_f) implies that f satisfies the local superlinearity condition at ∞.

Let us define the mapping $B_g : V \to V'$ by

$$\langle B_g u, v\rangle = \int_\Omega \left[g_1\left((\Delta u)^2\right) \Delta u \Delta v - c\ g_2\left(|\nabla u|^2\right) \nabla u \nabla v \right] dx, \tag{2.7}$$

foy any $u,\ v \in V$.

Definition 2.3. We say that $u \in V$ is the weak solution of problem (1.1) if the identity

$$\langle B_g u, v\rangle = \int_\Omega f(x,u) v dx \tag{2.8}$$

holds for any $v \in V$. Let E denote the associated energy:

$$Eu = \int_\Omega \left[G_1\left((\Delta u)^2\right) - c\ G_2\left(|\nabla u|^2\right) - F(x,u) \right] dx, \tag{2.9}$$

where $G_1(t) = \frac{1}{2}\int_0^t g_1(s)ds, G_2(t) = \frac{1}{2}\int_0^t g_2(s)ds,\ F(x,t) = \int_0^t f(x,s)ds.$

Under the above assumptions, we shall give the existence of weak solutions, by means of Mountain Pass theorem and local minimization, for the quasilinear problem (1.1), (1.6) and (1.7). The main results of this paper are the following theorems.

Theorem 2.1. *Assume that (i_g)-(iii_g) and (i_f-(v_f) hold. Suppose in addition that*

$\theta\alpha_1 - 2\beta_1 > 0$, if $0 \le c < \lambda_1(\Omega)$, $\alpha_1 \ge \beta_2$ and $\theta\alpha_1 - 2\beta_1 \ge \theta\beta_2 - 2\alpha_2$, or if $c \le 0$, $\alpha_1 \le \beta_2$ and $\theta\alpha_1 - 2\beta_1 \le \theta\beta_2 - 2\alpha_2$. Then there exists $\eta = \eta(p, q, n) > 0$ such that for a_0 and b_0 :

$$\|a_0\|_{\sigma_q}^{p-1}\|b_0\|_{\sigma_p}^{1-q} < \eta,$$

problem (1.1) has at least two solutions v and w which satisfy $E(v) > 0$ and $E(w) < 0$, where E denotes the associated energy in (2.9). In addition if f varies in such a way that the coefficients in (iii$_f$) satisfy

$$a_0 \to 0 \text{ in } L^{\sigma_q}(\Omega) \text{ and } b_0 \text{ bounded in } L^{\sigma_p}(\Omega),$$

then the solution $w = w_f$ can be constructed such that $w_f \to 0$ in V.

Theorem 2.2. *Assume that (i$_g$)-(iii$_g$) and (i$_f$)-(v$_f$) hold. Suppose in addition that $\theta\beta_2 - 2\alpha_2 > 0$, $\alpha_1 \ge \beta_2$ and $\theta\alpha_1 - 2\beta_1 \ge \theta\beta_2 - 2\alpha_2$. Then there exists $\eta = \eta(p, q, n) > 0$ such that for a_0 and b_0 :*

$$\|a_0\|_{\sigma_q}^{p-1}\|b_0\|_{\sigma_p}^{1-q} < \eta,$$

problem (1.1) has at least two solutions v and w which satisfy $E(v) > 0$ and $E(w) < 0$, where E denotes the associated energy in (2.9). In addition if f varies in such a way that the coefficients in (iii$_f$) satisfy

$$a_0 \to 0 \text{ in } L^{\sigma_q}(\Omega) \text{ and } b_0 \text{ bounded in } L^{\sigma_p}(\Omega),$$

then the solution $w = w_f$ can be constructed such that $w_f \to 0$ in V.

We apply these theorems to the quasilinear problem (1.6), we have

Theorem 2.3. *Assume that (i$_g$)-(iii$_g$) hold. Suppose that $\theta\alpha_1 - 2\beta_1 > 0$, if $0 \le c < \lambda_1(\Omega)$, $\alpha_1 \ge \beta_2$ and $\theta\alpha_1 - 2\beta_1 \ge \theta\beta_2 - 2\alpha_2$, or if $c \le 0$, $\alpha_1 \le \beta_2$ and $\theta\alpha_1 - 2\beta_1 \le \theta\beta_2 - 2\alpha_2$ and that in (1.6) $\lambda > 0$, $0 \le q < 1 < p < 2^* - 1$, $a \in L^{\tau_q}(\Omega)$ with $\tau_q > \sigma_q$, $b \in L^{\tau_p}(\Omega)$ with $\tau_p > \sigma_p$, with in addition $a(x) \ge 0$ a.e. in Ω in case $q = 0$. Suppose in addition that*

(vi$_f$) there exists a nonempty open subset $\Omega_1 \subset \Omega$ such that, on Ω_1, $a(x) \ge \epsilon_1$ for some $\epsilon_1 > 0$ and $b(x)$ is bounded from below;

(vii$_f$) there exists a nonempty open subset $\Omega_2 \subset \Omega$ such that, on Ω_2, $b(x) \geq \epsilon_2$ for some $\epsilon_2 > 0$ and $a(x)$ is bounded from above and from below.
Then there exists $\widetilde{\eta} = \widetilde{\eta}(p, q, n) > 0$ for

$$\lambda < \frac{\widetilde{\eta}}{\|a^+\|_{\sigma_q} \|b^+\|_{\sigma_p}^{(1-q)(p-1)}},$$

problem (1.6) has at least two solutions v and w which satisfy $J(v) > 0$ and $J(w) < 0$, where J denotes the energy functional associated to (1.6). Moreover, if $\lambda \to 0$, the solution $w = w_f$ can be constructed such that $w_f \to 0$ in V.

Theorem 2.4. *Assume that (i$_g$)-(iii$_g$) hold. Suppose that $\theta\beta_2 - 2\alpha_2 > 0$, $\alpha_1 \geq \beta_2$ and $\theta\alpha_1 - 2\beta_1 \geq \theta\beta_2 - 2\alpha_2$ and that in (1.6) $\lambda > 0$, $0 \leq q < 1 < p < 2^* - 1$, $a \in L^{\tau_q}(\Omega)$ with $\tau_q > \sigma_q$, $b \in L^{\tau_p}(\Omega)$ with $\tau_p > \sigma_p$, with in addition $a(x) \geq 0$ a.e. in Ω in case $q = 0$. Suppose in addition that (vi$_f$) and (vii$_f$) hold. Then there exists $\widetilde{\eta} = \widetilde{\eta}(p, q, n) > 0$ for*

$$\lambda < \frac{\widetilde{\eta}}{\|a^+\|_{\sigma_q} \|b^+\|_{\sigma_p}^{(1-q)(p-1)}},$$

problem (1.6) has at least two solutions v and w which satisfy $J(v) > 0$ and $J(w) < 0$, where J denotes the energy functional associated to (1.6). Moreover, if $\lambda \to 0$, the solution $w = w_f$ can be constructed such that $w_f \to 0$ in V.

We apply Theorem 2.1 and 2.2 to the quasilinear problem (1.7), we have

Theorem 2.5. *Assume that hypotheses in Theorem 2.3 hold. Then there exists $\widetilde{\eta} = \widetilde{\eta}(p, q, n) > 0$ for*

$$\lambda < \frac{\widetilde{\eta}}{\|a^+\|_{\sigma_q} \|b^+\|_{\sigma_p}^{(1-q)(p-1)}},$$

problem (1.7) has at least two solutions v and w which satisfy $J(v) > 0$ and $J(w) < 0$, where J denotes the energy functional associated to (1.7). Moreover, if $\lambda \to 0$, the solution $w = w_f$ can be constructed such that $w_f \to 0$ in V.

Theorem 2.6. *Assume that hypotheses in Theorem 2.4 hold. Then there exists $\widetilde{\eta} = \widetilde{\eta}(p, q, n) > 0$ for*

$$\lambda < \frac{\widetilde{\eta}}{\|a^+\|_{\sigma_q} \|b^+\|_{\sigma_p}^{(1-q)(p-1)}},$$

problem (1.7) has at least two solutions v and w which satisfy $J(v) > 0$ and $J(w) < 0$, where J denotes the energy functional associated to (1.7). Moreover, if $\lambda \to 0$, the solution $w = w_f$ can be constructed such that $w_f \to 0$ in V.

Remark 2.2. Theorem 2.1-2.6 are includes an interesting interplay between the behavior of the functions g_1, g_2, f and the potential function E and J. On the other hand, Theorem 2.1-2.6 deal with some fourth order quasilinear elliptic problems. Figueiredo, Gossez and Ubilla in [12] studied second semilinear elliptic problems, as well as of another paper by Xu and Zhang [19] considered some fourth order semilinear elliptic problems of the local superlinearity and sublinearity conditions.

3. Some lemmas and proofs of main results

Let Ω be a bounded smooth open subset of R^n and let $V = H^2(\Omega) \cap H_0^1(\Omega)$. In this section we give the proofs of Theorem 2.1-2.6. For this we need the following lemmas.

Let $G_1(t) = \frac{1}{2}\int_0^t g_1(s)ds, G_2(t) = \frac{1}{2}\int_0^t g_2(s)ds$ and let the energy $E : V \to R$ be given by

$$Eu = \int_\Omega \left[G_1\left((\Delta u)^2\right) - c\ G_2\left(|\nabla u|^2\right) - F(x,u)\right] dx, \tag{3.1}$$

for any $u \in V$.

Lemma 3.1. *Assume that* (i_f) *holds, let $F' : V \to V'$ be given by*

$$\langle F'u, v\rangle = \int_\Omega f(x,u)v dx, \tag{3.2}$$

for any $u,\ v \in V$. Then F' is completely continuous.

Proof. Let $I : V \to L^\sigma(\Omega)$ be given by $Iu = u$, since for $\sigma < 2^*$ by Rellich's theorem the space $H_0^1(\Omega)$ embeds into $L^\sigma(\Omega)$ compactly, then I is a compact mapping. From [19]. Let $K : L^\sigma(\Omega) \to L^{\sigma'}(\Omega)$ $(1/\sigma + 1/\sigma' = 1)$ be given by $K(u) = f(x,u)$ and let $I' : L^{\sigma'} \to V'$ be given by $I'v = v$, then K *and* I' are continuous. Hence $F' = I'KI$ is a completely continuous mapping from V to V'.

Lemma 3.2. *Let that* (i_g)-(iii_g) *and* (i_f) *hold. Then* $E \in C^1(V, R)$.

Proof. Let $Gu = \int_\Omega [G_1((\Delta u)^2) - c\, G_2(|\nabla u|^2)]\, dx$, by (i_g) and (ii_g), it is easy to check that G is a continuous. From Lemma 3.1 we know that F' is completely continuous.

Thus by combining assumptions (i_g)-(iii_g) and (i_f), it follows that $E \in C^1(V, R)$ and

$$\begin{aligned} \langle E'u, v\rangle &= \int_\Omega \left[g_1\left((\Delta u)^2\right)\Delta u \Delta v - c\, g_2\left(|\nabla u|^2\right)\nabla u \nabla v - f(x,u)v\right] dx \\ &= \langle B_g u, v\rangle - \langle F'u, v\rangle. \end{aligned} \tag{3.3}$$

Remark 3.1. By Lemma 3.1 and 3.2, $E'u = 0$ implies that

$$\langle B_g u, v\rangle = \int_\Omega f(x,u) v dx.$$

Therefore critical points of E are weak solutions of problem (1.1).

Lemma 3.3. *Assume that* (i_g)-(iii_g) *and* (i_f)-(ii_f) hold. *Let* $\theta\alpha_1 - 2\beta_1 > 0$, *if* $0 \le c < \lambda_1$, $\theta\alpha_1 - 2\beta_1 \ge \theta\beta_2 - 2\alpha_2$, *or if* $c \le 0$, $\theta\alpha_1 - 2\beta_1 \le \theta\beta_2 - 2\alpha_2$. *Then* E *satisfies the* (PS) *condition on* V.

Proof. By Lemma 3.1 and 3.2, E is a C^1 functional on V. To see that E satisfies (PS) condition, at first we show that any (PS)-sequence $\{u_n\}$ for E is bounded in V. Let $\{u_n\}$ be a (PS) sequence, i.e. $E(u_n)$ bounded and $E'(u_n) \to 0$. So, for θ as in (iii_f) and for some $\epsilon_n \to 0$ and some constant C,

$$\theta E(u_n) - \langle E'(u_n), u_n\rangle \le C + \varepsilon_n \|u_n\|. \tag{3.4}$$

When $0 \le c < \lambda_1$, by combining (iii_g), (3.1), (3.2) and (3.3) we have

$$\begin{aligned} \theta E(u_n) - \langle E'(u_n), u_n\rangle &= \theta \int_\Omega \left[G_1\left((\Delta u_n)^2\right) - c\, G_2\left(|\nabla u_n|^2\right)\right] dx \\ &\quad - \int_\Omega \left[g_1\left((\Delta u_n)^2\right)(\Delta u_n)^2 - c\, g_2\left(|\nabla u_n|^2\right)|\nabla u_n|^2\right] dx \\ &\quad - \int_\Omega (\theta F(x,u_n) - u_n f(x,u_n)) dx \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\theta}{2}\int_\Omega \Big[\alpha_1(\Delta u_n)^2 - c\ \beta_2|\nabla u_n|^2\Big]\,dx \\
&\quad -\int_\Omega \Big[\beta_1(\Delta u_n)^2 - c\ \alpha_2|\nabla u_n|^2\Big]\,dx \\
&\quad -\int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx \\
&= \int_\Omega \Big[(\frac{\theta}{2}\alpha_1 - \beta_1)(\Delta u_n)^2 - c\ (\frac{\theta}{2}\beta_2 - \alpha_2)|\nabla u_n|^2\Big]\,dx \\
&\quad -\int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx \\
&= (\frac{\theta}{2}\alpha_1 - \beta_1)\int_\Omega \Big[(\Delta u_n)^2 - \frac{\theta\beta_2 - 2\alpha_2}{\theta\alpha_1 - 2\beta_1}c\ |\nabla u_n|^2\Big]\,dx \\
&\quad -\int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx \\
&\geq (\frac{\theta}{2}\alpha_1 - \beta_1)\int_\Omega \Big[(\Delta u_n)^2 - c\ |\nabla u_n|^2\Big]\,dx \\
&\quad -\int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx \\
&\geq (\frac{\theta}{2}\alpha_1 - \beta_1)\parallel u_n \parallel^2 \\
&\quad -\int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx. \qquad (3.5)
\end{aligned}
$$

When $c < 0$, by combining (iii$_g$), (3.1), (3.2), (3.3) and (3.5), similarly, from (3.5) we have

$$
\begin{aligned}
\theta E(u_n) - \langle E^{'}(u_n), u_n\rangle &\geq \frac{\theta}{2}\int_\Omega \Big[\alpha_1(\Delta u_n)^2 - c\ \beta_2|\nabla u_n|^2\Big]\,dx \\
&\quad -\int_\Omega \Big[\beta_1(\Delta u_n)^2 - c\ \alpha_2|\nabla u_n|^2\Big]\,dx \\
&\quad -\int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx \\
&= (\frac{\theta}{2}\alpha_1 - \beta_1)\int_\Omega \Big[(\Delta u_n)^2 - \frac{\theta\beta_2 - 2\alpha_2}{\theta\alpha_1 - 2\beta_1}c\ |\nabla u_n|^2\Big]\,dx \\
&\quad -\int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx \\
&\geq (\frac{\theta}{2}\alpha_1 - \beta_1)\int_\Omega \Big[(\Delta u_n)^2 - c\ |\nabla u_n|^2\Big]\,dx \\
&\quad -\int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx \\
&\geq (\frac{\theta}{2}\alpha_1 - \beta_1)\parallel u_n \parallel^2
\end{aligned}
$$

$$-\int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx. \tag{3.6}$$

by combining (3.4), (3.5) and (3.6), it implies that

$$(\frac{\theta}{2}\alpha_1 - \beta_1)\|u_n\|^2 - \int_\Omega (\theta F(x,u_n) - u_n f(x,u_n)) \leq C + \varepsilon_n \|u_n\|. \tag{3.7}$$

By (3.7) and assumption (ii$_f$), we get

$$(\frac{\theta}{2}\alpha_1 - \beta_1)\|u_n\|^2 \leq C' + \int_\Omega d(x)|u_n|^r dx + \varepsilon_n \|u_n\|, \tag{3.8}$$

where C' is another constant. Since $r < 2$, we deduces that $\|u_n\|$ remains bounded.

Next, we show that E satisfies the (PS) condition. From assumptions (i$_g$), (ii$_g$) and (iii$_g$), it follows that

$$\begin{aligned}
\langle B_g(u-v),(u-v)\rangle &= \int_\Omega \Big[\Big(g_1\left((\Delta u)^2\right)\Delta u - g_1\left((\Delta v)^2\right)\Delta v\Big)\Delta(u-v) \\
&\quad + \Big(-c\ g_2\left(|\nabla u|^2\right)\nabla u + c\ g_2\left(|\nabla v|^2\right)\nabla v\Big)\nabla(u-v)\Big]dx \\
&= \frac{1}{2}\int_\Omega \Big[\Big(g_1\left((\Delta u)^2\right) + g_1\left((\Delta v)^2\right)\Big)(\Delta(u-v))^2 \\
&\quad + \Big(g_1\left((\Delta u)^2\right) - g_1\left((\Delta v)^2\right)\Big)((\Delta u)^2 - (\Delta v)^2)\Big]dx \\
&\quad + \frac{1}{2}\int_\Omega \Big[\Big(-cg_2\left(|\nabla u|^2\right) - cg_2\left(|\nabla v|^2\right)\Big)|\nabla(u-v)|^2 \\
&\quad + \Big(-cg_2\left(|\nabla u|^2\right) + cg_2\left(|\nabla v|^2\right)\Big)(|\nabla u|^2 - |\nabla v|^2)\Big]dx \\
&\geq \alpha_1 \parallel u - v \parallel^2
\end{aligned} \tag{3.9}$$

for any $u, v \in V$. Thus, by combining assumptions (i$_g$)-(iii$_g$), (3.9), Lemma 3.1 and 3.2, this implies that $B_g : V \to V'$ is an homeomorphism.

Let $\{u_n\}$ in V be a (PS)-sequence for E, from (3.8) we know that $\{u_n\}$ is bounded in V, assume after passing to a subsequence that $u_n \to u_0$ weakly in V, strongly in $L^2(\Omega)$. Since

$$u_n = B_g^{-1}(E'(u_n) - F'(u_n)), \tag{3.10}$$

by Lemma 3.1, we know that $F' : V \to V'$ is is a completely continuous mapping. Thus there exists a convergent subsequence, hence E satisfies the (PS) condition. This completes the proof of Lemma 3.3.

Lemma 3.4. *Assume that* (i_g)-(iii_g) *and* (i_f)-(ii_f) hold. *Let* $\theta\beta_2 - 2\alpha_2 > 0$, $\theta\alpha_1 - 2\beta_1 \geq \theta\beta_2 - 2\alpha_2$, *Then* E *satisfies the* (PS) *condition on* V.

Proof. As in the proof of Lemma 3.3, we know that E is a C^1 functional on V. At first we show that any (PS)-sequence $\{u_n\}$ for E is bounded in V. Let $\{u_n\}$ be a (PS) sequence, i.e. $E(u_n)$ bounded and $E^{'}(u_n) \to 0$. Thus, for θ as in (iii_f) and for some $\epsilon_n \to 0$ and some constant C,

$$\theta E(u_n) - \langle E^{'}(u_n), u_n \rangle \leq C + \varepsilon_n \|u_n\|. \tag{3.11}$$

From (3.1), (3.2) and (3.3) we have

$$\begin{aligned}
\theta E(u_n) - \langle E^{'}(u_n), u_n \rangle &= \theta \int_\Omega \left[G_1\left((\Delta u_n)^2\right) - c\, G_2\left(|\nabla u_n|^2\right)\right] dx \\
&\quad - \int_\Omega \left[g_1\left((\Delta u_n)^2\right)(\Delta u_n)^2 - c\, g_2\left(|\nabla u_n|^2\right)|\nabla u_n|^2\right] dx \\
&\quad - \int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx \\
&\geq \frac{\theta}{2} \int_\Omega \left[\alpha_1(\Delta u_n)^2 - c\, \beta_2 |\nabla u_n|^2\right] dx \\
&\quad - \int_\Omega \left[\beta_1(\Delta u_n)^2 - c\, \alpha_2 |\nabla u_n|^2\right] dx \\
&\quad - \int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx \\
&= \int_\Omega \left[(\frac{\theta}{2}\alpha_1 - \beta_1)(\Delta u_n)^2 - c\, (\frac{\theta}{2}\beta_2 - \alpha_2)|\nabla u_n|^2\right] dx \\
&\quad - \int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx \\
&= (\frac{\theta}{2}\beta_2 - \alpha_2) \int_\Omega \left[\frac{\theta\alpha_1 - 2\beta_1}{\theta\beta_2 - 2\alpha_2}(\Delta u_n)^2 - c\, |\nabla u_n|^2\right] dx \\
&\quad - \int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx \\
&\geq (\frac{\theta}{2}\beta_2 - \alpha_2) \int_\Omega \left[(\Delta u_n)^2 - c\, |\nabla u_n|^2\right] dx \\
&\quad - \int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx \\
&\geq (\frac{\theta}{2}\beta_2 - \alpha_2) \parallel u_n \parallel^2 \\
&\quad - \int_\Omega (\theta F(x,u_n) - u_n f(x,u_n))dx.
\end{aligned} \tag{3.12}$$

By (3.11) and (3.12), it follows that

$$(\frac{\theta}{2}\beta_2 - \alpha_2)\|u_n\|^2 - \int_\Omega (\theta F(x,u_n) - u_n f(x,u_n)) \le C + \varepsilon_n \|u_n\|. \tag{3.13}$$

From (3.13) and assumption (ii$_f$), we have

$$(\frac{\theta}{2}\beta_2 - \alpha_2)\|u_n\|^2 \le C^{''} + \int_\Omega d(x)|u_n|^r dx + \varepsilon_n \|u_n\|, \tag{3.14}$$

where $C^{''}$ is another constant. Since $r < 2$, we deduces that $\|u_n\|$ remains bounded.

Next, we show that E satisfies the (PS) condition. By combining (3.9), (3.15), assumptions (i$_g$)-(iii$_g$), Lemma 3.1 and 3.2, , we know that

$$\langle B_g(u-v),(u-v)\rangle \quad \ge \quad \alpha_1 \parallel u - v \parallel^2 \tag{3.15}$$

for any $u, v \in V$ and that $B_g : V \to V'$ is an homeomorphism.

Let $\{u_n\}$ in V be a (PS)-sequence for E, from (3.14) we know that $\{u_n\}$ is bounded in V, assume after passing to a subsequence that $u_n \to u_0$ weakly in V, strongly in $L^2(\Omega)$. Since

$$u_n = B_g^{-1}(E'(u_n) - F'(u_n)), \tag{3.16}$$

by Lemma 3.1, we know that $F' : V \to V'$ is is a completely continuous mapping. Thus there exists a convergent subsequence, hence E satisfies the (PS) condition. This completes the proof of Lemma 3.4.

Lemma 3.5.[12,19] *Let $0 \le q < 1 < p,\ A > 0,\ B > 0$, and consider the function*

$$\Psi_{A,B}(t) = t^2 - At^{q+1} - Bt^{p+1}$$

for $t \ge 0$. Then max $\{\Psi_{A,B}(t) : t \ge 0\}$ is > 0 if and only if

$$A^{p-1}B^{q-1} < \frac{(p-1)^{p-1}(1-q)^{1-q}}{(p-q)^{p-q}} = \eta_1(p,q). \tag{3.17}$$

Moreover, for $t = t_B := [\frac{1-q}{B(p-q)}]^{\frac{1}{p-1}}$, one has

$$\Psi_{A,B}(t_B) = t_B^2[\frac{p-1}{p-q} - AB^{\frac{1-q}{p-1}}(\frac{p-q}{1-q})^{\frac{1-q}{p-1}}]. \tag{3.18}$$

Proof of Theorem 2.1. By means of Lemma 3.3, we know that the associated energy E in (2.9) satisfies the (PS) condition. At first, we will show the existence of the first solution by the Mountain Pass theorem. Thus, recalling (iii$_f$), using Hölder inequality and Sobolev inequality, when $0 \leq c < \lambda_1$, by combining (iii$_g$) and (3.1) we have

$$\begin{aligned}
E(u) &= \int_\Omega \left[G_1\left((\Delta u)^2\right) - c\, G_2\left(|\nabla u|^2\right)\right] dx - \int_\Omega F(x,u)dx \\
&\geq \frac{1}{2}\int_\Omega \left[\alpha_1(\Delta u)^2 - c\,\beta_2|\nabla u|^2\right] dx - \int_\Omega \left[\frac{a_0|u|^{q+1}}{q+1} + \frac{b_0|u|^{p+1}}{p+1}\right] dx \\
&= \frac{\alpha_1}{2}\int_\Omega \left[(\Delta u)^2 - \frac{\beta_2}{\alpha_1}c\,|\nabla u|^2\right] dx - \int_\Omega \left[\frac{a_0|u|^{q+1}}{q+1} + \frac{b_0|u|^{p+1}}{p+1}\right] dx \\
&\geq \frac{\alpha_1}{2}\int_\Omega \left[(\Delta u)^2 - c\,|\nabla u|^2\right] dx - m_1\|a_0\|_{\sigma_q}\|u\|_{0,1}^{q+1} - m_2\|b_0\|_{\sigma_p}\|u\|_{0,1}^{p+1} \\
&\geq \frac{\alpha_1}{2}\parallel u\parallel^2 - c_1\|a_0\|_{\sigma_q}\|u\|^{q+1} - c_2\|b_0\|_{\sigma_p}\|u\|^{p+1} \qquad (3.19)
\end{aligned}$$

for all $u \in V$, where $m_1 = (q+1)^{-1}S^{-\frac{q+1}{2}},\ m_2 = (p+1)^{-1}S^{-\frac{p+1}{2}},\ c_1 = m_1(\lambda_1(\Omega) - c)^{-\frac{q+1}{2}},\ c_2 = m_2(\lambda_1(\Omega) - c)^{-\frac{p+1}{2}}$, and

$$S := \inf\{\int_\Omega |\nabla u|^2 dx : u \in H_0^1(\Omega) \text{ and } \int_\Omega |u|^{2^*} dx = 1\}. \qquad (3.20)$$

When $c < 0$, by combining (iii$_g$), (3.1) and (3.19), similarly, from (3.19) we have

$$\begin{aligned}
E(u) &= \int_\Omega \left[G_1\left((\Delta u)^2\right) - c\, G_2\left(|\nabla u|^2\right)\right] dx - \int_\Omega F(x,u)dx \\
&\geq \frac{\alpha_1}{2}\int_\Omega \left[(\Delta u)^2 - \frac{\beta_2}{\alpha_1}c\,|\nabla u|^2\right] dx - \int_\Omega \left[\frac{a_0|u|^{q+1}}{q+1} + \frac{b_0|u|^{p+1}}{p+1}\right] dx \\
&\geq \frac{\alpha_1}{2}\|u\|^2 - c_1\|a_0\|_{\sigma_q}\|u\|^{q+1} - c_2\|b_0\|_{\sigma_p}\|u\|^{p+1} \qquad (3.21)
\end{aligned}$$

for all $u \in V$, where c_1 and c_2 as in (3.19). Let $A = (2c_1/\alpha_1)\|a_0\|_{\sigma_q}$ and $B = (2c_2/\alpha_1)\|b_0\|_{\sigma_p}$, it follows from (iv$_f$) and (v$_f$) that $A > 0$ and $B > 0$. By Lemma 3.5, (3.19) and (3.21), we get

$$E(u) \geq \frac{\alpha_1}{2}\Psi_{A,B}(t_B) > 0 \qquad (3.22)$$

for all $u \in V$ with $\|u\| = t_B$ and $A^{p-1}B^{q-1} < \eta_1(p,q)$, this implies that

$$\|a_0\|_{\sigma_q}^{p-1}\|b_0\|_{\sigma_p}^{1-q} < \frac{\alpha_1^{p-q}\eta_1(p,q)}{2^{p-q}(c_1)^{p-1}(c_2)^{1-q}} = \eta(p,q,N). \qquad (3.23)$$

Noting that $E(0) = 0$, thus by (3.22) and (3.23) we have obtained a range of mountains around 0.

Now we will show that for some $u_2 \in V$ such that $E(tu_2) \to -\infty$ as $t \to +\infty$. By assumption (v_f), there exist $t_3 > t_0$ and $\theta_3 > 0$, such that $F(x,t) \geq \theta_3 t^2 + 1$ for a.e. $x \in \Omega_2$ and all $t \geq t_3$. For $x \in \Omega_2$ and $t \geq t_3$, we then divide the inequality of (ii_f) by $tF(x,t)$, integrate from t_3 to t and take the exponential to get

$$F(x,t) \geq F(x,t_3)(\frac{t}{t_3})^\theta \exp(-d(x)\int_{t_3}^t \frac{s^{r-1}}{F(x,s)}ds).$$

From (v_f), it implies that

$$F(x,t) \geq Ct^\theta \tag{3.24}$$

for a.e. $x \in \Omega_2$ and all $t \geq t_3$, where $C > 0$ is a constant. Choosing a function $u_2 \in V \bigcap H^\theta(\Omega)$ with support in Ω_2 and $u_2 \geq 0$, $u_2 \not\equiv 0$, and taking t_2 such that measure of $\{x \in \Omega_2 : tu_2(x) \geq t_3\}$ is > 0 for $t \geq t_2$. By (iii_g) and (3.1) we have

$$\begin{aligned} E(tu_2) &= \int_\Omega \left[G_1\left((\Delta tu_2)^2\right) - c\, G_2\left(|\nabla tu_2|^2\right)\right] dx - \int_\Omega F(x,tu_2)dx \\ &\leq \frac{t^2}{2}\int_{\Omega_2}\left[\beta_1(\Delta u_2)^2 - c\alpha_2|\nabla u_2|^2\right] dx - \int_{\Omega_2} F(x,tu_2)dx, \quad (3.25) \end{aligned}$$

by splitting the integral over Ω_2 into an integral over $\{x \in \Omega_2 : tu_2(x) < t_3\}$ and an integral over $\{x \in \Omega_2 : tu_2(x) \geq t_3\}$, and by using (3.24) and (3.25), this follows that, for some constants $C_1,\ C_2$ and C_3, with $C_3 > 0$,

$$\begin{aligned} E(tu_2) &\leq \frac{t^2}{2}\int_{\Omega_2}\left[\beta_1(\Delta u_2)^2 - c\alpha_2|\nabla u_2|^2\right] dx + C^2 \\ &\quad -Ct^\theta \int_{\{x\in\Omega_2 : tu_2(x)\geq t_3\}} (u_2)^\theta dx \\ &\leq C_1 t^2 + C_2 - C_3 t^\theta. \end{aligned} \tag{3.26}$$

Since $\theta > 2$, this implies that $E(tu_2) \to -\infty$ as $t \to +\infty$.

By combining (3.22), (3.23) and (3.26), the Mountain Pass theorem yields a critical point v of E with

$$E(v) \geq \frac{\alpha_1}{2}\Psi_{A,B}(t_B) > 0. \tag{3.27}$$

This v is the first nontrivial solution of problem (1.1).

Next, we will show the existence of the second solution by local minimization. Let e_1 be the positive eigenfunction associated to the principle eigenvalue of $-\Delta$ on

$H_0^1(\Omega_1)$, it is known that $e_1 \in L^\infty(\Omega_1)$. Set $t_1 = t_1/\|e_1\|_\infty$, using the positivity of e_1 and the hypothesis (iv$_f$), it is easy to see that for all $0 < t \leq t_1$, $0 \leq te_1 \leq t_1$. By (iii$_g$) and (3.1), this implies that

$$
\begin{aligned}
E(te_1) &= \int_\Omega \left[G_1\left((\Delta te_1)^2\right) - c\ G_2\left(|\nabla te_1|^2\right)\right]dx - \int_\Omega F(x, te_1)dx \\
&\leq \frac{t^2}{2}\int_{\Omega_1}\left[\beta_1(\Delta e_1)^2 - c\alpha_2|\nabla e_1|^2\right]dx - \theta_1 t^2 \int_{\Omega_1} e_1^2 dx \\
&\leq \lambda_1(\Omega_1)(\lambda_1(\Omega_1)\beta_1 - c\alpha_2)\frac{t^2}{2}\int_{\Omega_1} e_1^2 dx - \theta_1 t^2 \int_{\Omega_1} e_1^2 dx \\
&= (\lambda_1(\Omega_1)(\lambda_1(\Omega_1)\beta_1 - c\alpha_2)/2 - \theta_1)t^2 \int_{\Omega_1} e_1^2 dx \\
&< 0, \qquad (3.28)
\end{aligned}
$$

since $\theta_1 > \lambda_1(\Omega_1)(\lambda_1(\Omega_1)\beta_1 - c\alpha_2)/2$. Thus, there exists $t > 0$ sufficiently small such that

$$E(te_1) < 0. \qquad (3.29)$$

Let $Gu = \int_\Omega \left[G_1\left((\Delta u)^2\right) - c\ G_2\left(|\nabla u|^2\right)\right]dx$, and $Wu = \int_\Omega F(x, u)dx$. By (i$_g$)-(iii$_g$), it is easy to check that G is a convex and lower semicontinuous, hence G is weakly lower semicontinuous. As in the proof of Lemma 3.1 from assumption (i$_f$), we know that W is weakly continuous. Thus $E = G - W$ is weakly lower semicontinuous. It follows from (3.29) that the minimum of the functional E on the closed ball in V with center 0 and radius t_B is achieved in the corresponding open ball and thus yields a nontrivial solution w of problem (1.1) with

$$E(w) < 0 \text{ and } \|w\| < t_B. \qquad (3.30)$$

Thus, we obtain the second solution. This completes the proof of the existence of at least two solutions in Theorem 2.1.

We now turn to the study of the asymptotic behavior of one of these two solutions. When f varies in such a way that $a_0 \to 0$ in $L^{\sigma_q}(\Omega)$ and b_0 remains bounded in $L^{\sigma_p}(\Omega)$, let $\gamma \in (0, 1/1-q)$ and $t_B = \|a_0\|_{\sigma_q}^\gamma$. By (3.19), we have

$$
\begin{aligned}
E(u) &\geq \frac{\alpha_1}{2}\parallel u \parallel^2 - c_1\|a_0\|_{\sigma_q}\|u\|^{q+1} - c_2\|b_0\|_{\sigma_p}\|u\|^{p+1} \\
&\geq \frac{\alpha_1}{2}\|a_0\|_{\sigma_q}^{2\gamma} - c_1\|a_0\|_{\sigma_q}^{1+\gamma(q+1)} - c_2\|b_0\|_{\sigma_p}\|a_0\|_{\sigma_q}^{\gamma(p+1)} \\
&= \|a_0\|_{\sigma_q}^{2\gamma}\left(\frac{\alpha_1}{2} - c_1\|a_0\|_{\sigma_q}^{1-\gamma(1-q)} - c_2\|b_0\|_{\sigma_p}\|a_0\|_{\sigma_q}^{\gamma(p-1)}\right) \qquad (3.31)
\end{aligned}
$$

for all u with $\|u\| = t_B$. Since $\gamma \in (0, 1/1-q)$, so $1-\gamma(1-q) > 0$, thus it follows from (3.31) that

$$E(u) \geq \frac{\alpha_1}{2}\Psi_{A,B}(t_B) > 0 \tag{3.32}$$

for all u with $\|u\| = \|a_0\|_{\sigma_q}^{\gamma} = t_B$ sufficiently small. By combining (3.30), and (3.31) and (3.32), this implies that the corresponding solution w will converges to 0 in V as $a_0 \to 0$. This complete the proof of Theorem 2.1.

Proof of Theorem 2.2. By combining assumptions (i_g)-(iii_g), (i_f)-(v_f), Lemma 3.1, 3.2 and 3.4, we can give the proof of Theorem 2.2 similar to the proof of Theorem 2.1.

Proof of Theorem 2.3. Let

$$f(x,t) = \begin{cases} a(x)t^q + b(x)t^p & t \geq 0, \\ \\ 0 & t < 0. \end{cases}$$

By combining assumptions (i_g)-(iii_g), (vi_f)-(vii_f), Lemma 3.1, 3.2 and 3.3 and the proof of Theorem 2.1, we can show that $J \in C^1(V, R)$, it is enough to verify that for each $\lambda > 0$, hypotheses (i_f)-(v_f) hold. Noting the strict inequalities $\tau_q > \sigma_q$ and $\tau_p > \sigma_p$, by using Young's inequality, we know that (i_f) holds. Set $\theta = p+1$, $r = q+1$, $d(x) = \lambda(\theta/(q+1)-1)a^+(x)$ and $t_0 = 0$, it follows that (ii_f) holds. Hypothesis (iii_f) is clear. By assumption (vi_f), we have

$$\begin{aligned} \lim_{t\to 0+} \frac{f(x,t)}{t} &= \lim_{t\to 0+} a(x)t^{q-1} + b(x)t^{p-1} \\ &= +\infty \end{aligned} \tag{3.33}$$

uniformly for a.e. $x \in \Omega_1$. So, there exist $\theta_1 > \lambda_1(\Omega_1)(\lambda_1(\Omega_1)\beta_1 - c\alpha_2)/2$ and $t_1 > 0$, such that

$$F(x,t) \geq \theta_1 t^2 \tag{3.34}$$

for a.e. $x \in \Omega_1$ and all $0 \leq t \leq t_1$, where $F(x,t) = \int_0^t f(x,s)ds$. It implies from (3.34) that hypothesis (iv_f) holds on Ω_1. By assumption (vii_f), set $d(x) = \lambda(\theta/(q+1)-1)a^+(x)$, which is bounded on Ω_2, we have

$$\begin{aligned} \lim_{t\to +\infty} \frac{f(x,t)}{t} &= \lim_{t\to +\infty} a(x)t^{q-1} + b(x)t^{p-1} \\ &= +\infty \end{aligned} \tag{3.35}$$

uniformly for a.e. $x \in \Omega_2$. So, there exist $\theta_2 > 0$ and $t_2 \geq 0$, such that

$$F(x,t) \geq \theta_2 t^2 \tag{3.36}$$

for a.e. $x \in \Omega_2$ and all $t \geq t_2$, where $F(x,t) = \int_0^t f(x,s)ds$, with the additional requirement that the function $d(x)$ appearing in (iii$_f$) is bounded on Ω_2. It implies from (3.36) that hypothesis (v$_f$) holds on Ω_2. This complete the proof of Theorem 2.3.

Proof of Theorem 2.4. By combining assumptions (i$_g$)-(iii$_g$), (vi$_f$),(vii$_f$), Lemma 3.1, 3.2 and 3.4, the proof of Theorem 2.4 can be completed similar to the proof of Theorem 2.3.

Remark 3.2. The proofs of Theorem 2.5 and 2.6 are similar to the proofs of Theorem 2.3 and 2.4. So we omit them.

Acknowledgments

This paper is completed while the author is visiting The Institute of Mathematical Sciences, The Chinese University of Hong Kong. He would like to thank Prof. Z. Xin and the members of IMS very much for their invitation and hospitality. The author also would like to thank Prof. D. Cao, Prof. J. Su and Prof. Z. Xin for their help and many valuable discussions.

This research was supported by the NSFC (10871096), Foundation of Major Project of Science and Technology of Chinese Education Ministry and The Institute of Mathematical Sciences, The Chinese University of Hong Kong.

References

[1] A.C. Lazer and P.J. McKenna, Large amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis, SIAM Review 32 (1990) 537-578.

[2] A.C. Lazer and P.J. McKenna, Global bifurcation and a theorem of Tarantello, J. Math. Anal. Appl. 181 (1994) 648-655.

[3] P.J. McKenna and W. Walter, Nonlinear oscillations in a suspension bridge, Arch. Rational Mech. Anal. 98 (1987) 167-177.

[4] P.J. McKenna and W. Walter, Travelling waves in a suspension bridge, SIAM J. Appl. Math. 50 (1990) 703-715.

[5] G. Tarantello, A note on a semilinear elliptic problem, Diff. Integral Eqns. 5 (1992) 561-566.

[6] A.M. Micheletti and A. Pistoia, Multiplicity results for a fourth-order semilinear elliptic problem, Nonlinear Anal. 31 (1998) 895-908.

[7] A.M. Micheletti and A. Pistoia, Nontrivial solutions for some fourth order semilinear elliptic problems, Nonlinear Anal. 34 (1998) 509-523.

[8] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical points theory and applications, J. Funct. Anal. 14 (1973) 349-381.

[9] T. Bartsch, S. Li, Critical point theory for asymptotically quadratic functionals and applications to problems with resonance, Nonlinear Anal. 28 (1997) 419-441.

[10] S. Li and J. Liu, Nontrivial critical point for asymptotically quadratic functions, J. Math. Anal. Appl. 165 (1992) 333-345.

[11] J. Su, Existence and multiplicity results for classes of elliptic resonant problems, J. Math. Anal. Appl. 273 (2002) 565-579.

[12] D.G. De Figueiredo, J.P. Gossez, P. Ubilla, Local superlinearity and sublinearity for indefinite semilinear elliptic problem, J. Func. Anal. 199(2003) 452-467

[13] P. Rabinowitz, Minimax Methods in Critical Point Theory With Application to Differential Equations, AMS, Providence, 1986.

[14] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Springer, Berlin, 1989.

[15] P. Drabek and A. Kufner and F. Nicolosi, On the solvability of degenerated quasilinear elliptic equations of higher order, J. Diff. Eqns. 109 (1994) 325-347.

[16] J. Zhang, Exitence results for some fourth-order nonlinear elliptic problems, Nonlinear Anal. 45 (2001) 29-36.

[17] J. Zhang, S. Li Multiple nontrivial solutions for some fourth-order semilinear elliptic problems, Nonlinear Anal. 60 (2005) 221-230.

[18] J. Zhang, Multiple nontrivial solutions for some fourth-order quasilinear elliptic problems, Nonlinear Anal. 63 (2005) e2533-e2544.

[19] G. Xu and J. Zhang, Existence results for some fourth-order nonlinear elliptic problems of Local superlinearity and sublinearity, J. Math. Anal. Appl. 281(2003) 633-640.

[20] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin New York, 1983.

[21] M. Struwe, Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer (1990).

[22] A. Ambrosetti, H. Brêzis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122(1994) 519-543.