Convergence to rarefaction waves for the nonlinear Boltzmann equation and compressible Navier-Stokes equations*

Zhouping XIN[†], Huihui ZENG

The Institute of Mathematical Sciences
The Chinese University of Hong Kong

Abstract

The main purpose of this paper is to study the asymptotic equivalence of the Boltzmann equation for the hard-sphere collision model to its corresponding Euler equations of compressible gas dynamics in the limit of small mean free path. When the fluid flow is a smooth rarefaction (or centered-rarefaction) wave with finite strength, the corresponding Boltzmann solution exists globally in time, and the solution converges to the rarefaction wave uniformly for all time (or away from $t = 0$) as $\epsilon \to 0$. A decomposition of a Boltzmann solution into its macroscopic (fluid) part and microscopic (kinetic) part is adopted to rewrite the Boltzmann equation in a form of compressible Navier-Stokes equations with source terms. In this setting, the same asymptotic equivalence of the full compressible Navier-Stokes equations to its corresponding Euler equations in the limit of small viscosity and heat-conductivity (depending on the viscosity) is also obtained.

1 Introduction

The Boltzmann equation of kinetic theory gives a statistical description of a gas of interacting particles. An important property of this equation is its asymptotic equivalence to the Euler or Navier-Stokes

*This work is supported in part by Zheng Ge Ru Foundation, Hong Kong RGC Earmarked Research Grants CUHK4040/06P, CUHK4042/08P, and RGC Central Allocation Grant CA05/06.SC01.

[†]Xin's email address is: zpxin@ims.cuhk.edu.hk

equations of the compressible gas dynamics, in the limit of small mean free path. It is expect that, away from initial layers, boundary layers and shock layers, the Boltzmann solutions should relax to its equilibrium state (local Maxwellian) in the small mean free path, and the gas should be governed by the macroscopic equations- the fluid equations. This asymptotic relationship can be seen formally by using the Hilbert expansion and Chapman-Enskog expansions. The rigorous mathematical justification of this asymptotic equivalence poses challenging problems. For a simplified model, Broadwell model of Boltzmann equations, the asymptotic equivalence between which and corresponding Euler equations in the small mean free path was justified when the Euler flow is either smooth (by Caflish and Papanicolou [6]), or contains finitely many non-interacting shocks (by Xin [31]) or is a rarefaction wave (by Xin [32] and Wang, Xin [30]). For the Boltzmann equation, some progress has been made for smooth flows (cf. [5]) or in the case when the fluid flow has finitely many non-interacting shocks ([34]). In [5], Caflisch showed that, if the compressible Euler equation has a smooth solution which exists up to time $T < \infty$, then there exists corresponding solutions to the Boltzmann equation in the same time interval such that the Boltzmann solutions converge to the local Maxwellian state determined by the fluid solutions as the mean free path goes to zero. This result was generalized by Yu ([34]) to the case when the Euler flow contains finitely many non-interacting shocks. Besides the shock wave, it is well-known that the rarefaction wave is another important elementary nonlinear wave for compressible Euler equations. One of main purpose is to extend Caflish's finite time convergence result mentioned above, which is not uniform for all time, to the uniform in time case when the Euler flow is a rarefaction wave. More precisely, we can obtain the following results: First, when the fluid flow is a centered-rarefaction wave for the Euler equations, then we can construct a sequence of global in time solutions which converges to the local Maxwellian determined by the centered-rarefaction wave, uniformly away from $t = 0$. Next, we obtain a rate of convergence in the mean free path which is valid uniformly for all time, when we specialize to smooth rarefaction waves of the Euler equations. Our analysis is strongly motivated by the idea of Xin in [32] for the Broadwell model equations, where a decomposition by using the Chapman-Enskog ansatz is introduced and an entropy energy estimate is used. The main difficulty in extending the analysis in [32] is the treatment of the high velocity tail of the distribution. To overcome this obstacle, we adopt here a decomposition of a Boltzmann solution into its macroscopic (fluid) part and microscopic (kinetic) part to rewrite the Boltzmann equation in a form of compressible Navier-Stokes equations with source terms. This decomposition was first used in [22] and further elaborated in [20]. In this setting, we can handle the zero dissipation problems, when the mean free path goes to zero for Boltzmann equation and the viscosity and heat conductivity tend to zero for compressible Navier-Stokes equations, in a uniform way. Therefore, besides the results for Boltzmann equations aforementioned, we also obtain the convergence results for compressible full Navier-Stokes equations when the corresponding Euler flow is a rarefaction wave. It should be mentioned here that the same convergence result was obtained by Xin ([33]) for the isentropic flow. We extend here the result in [33] to the general non-isentropic case. Related result for the non-isentropic Navier-Stokes equations with the Riemann data being center rarefaction wave can be found in [15], requiring that the strength of the rarefaction wave is suitably small. Here, for our problem, we can allow the rarefaction wave arbitrarily strong, but for the initial data depending on the viscosity and heat conductivity, in the same spirit as in [33] for the isentropic flow.

2 Botltzmann Equation, Navier-Stokes Equations, Euler Equations and Rarefaction Waves

The one space dimensional Boltzmann equation is

$$f_t + \xi_1 f_x = \frac{1}{\epsilon} Q(f,f), \quad (f,t,x,\xi) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^3, \qquad (2.1)$$

in which $f(t,x,\xi)$ represents the distributional density of particles at time-space (t,x) with velocity ξ , and $Q(f,f)$ is a bilinear collision operator (cf. [4]), and ϵ is the mean free path. We consider the hard sphere model, for which $Q(f,g)$ takes the form:

$$Q(f,g)(\xi) \equiv \frac{1}{2} \int_{\mathbb{R}^3} \int_{S_+^2} (f(\xi')g(\xi_*') + f(\xi_*')g(\xi)' - f(\xi)g(\xi_*) \\ - f(\xi_*)g(\xi)) |(\xi - \xi_*) \cdot \Omega| d\xi_* d\Omega.$$

Here $S_+^2 = \{\Omega \in S^2 : (\xi - \xi_*) \cdot \Omega \geq 0\}$ and

$$\xi' = \xi - [(\xi - \xi_*) \cdot \Omega]\Omega, \quad \xi_*' = \xi_* - [(\xi - \xi_*) \cdot \Omega]\Omega.$$

The important property of the asymptotic relation between the Boltzmann equation and the macroscopic fluid-dynamical equation, i.e., Euler system and Navier-Stokes equations has been investigated intensively in the literature for either small mean free path or large time, see [1, 2, 3, 5, 7, 9, 14, 13, 16, 17, 22, 21, 25, 26, 29, 34] and references therein. In this paper, we study the macroscopic limit to rarefaction waves of the fluid equations for small mean free path.

For a given solution $f(t,x,\xi)$ of the Boltzmann equation, there are five conserved macroscopic quantities: the mass density $\rho(t,x)$, momentum $m(t,x)=\rho(t,x)u(t,x)$, and energy density $e(t,x)+|u(t,x)|^2/2$:

$$\begin{cases} \rho(t,x) \equiv \int_{\mathbb{R}^3} f(t,x,\xi)d\xi, \\ m(t,x) \equiv (m_1,m_2,m_3)^t \equiv \int_{\mathbb{R}^3} \xi f(t,x,\xi)d\xi, \\ \rho(e+\frac{1}{2}|u|^2)(t,x) \equiv \int_{\mathbb{R}^3} \frac{1}{2}|\xi|^2 f(t,x,\xi)d\xi. \end{cases} \tag{2.2}$$

The local Maxwellian M associated to the Boltzmann solution $f(t,x,\xi)$ is defined in terms of the conserved fluid variables:

$$M \equiv M_{[\rho,u,\theta]}(t,x,\xi) \equiv \frac{\rho(t,x)}{\sqrt{(2\pi R\theta(t,x))^3}} \exp(-\frac{|\xi-u(t,x)|^2}{2R\theta(t,x)}). \tag{2.3}$$

Here $\theta(t,x)$ is the temperature which is related to the internal energy e by $e=3R\theta/2=\theta$ with the gas constant R taken to be $2/3$, and $u(x,t)=(u_1(t,x),u_2(t,x),u_3(t,x))^t$ is the fluid velocity. Following [22, 20], we decompose the solution into the macroscopic (fluid) part and microscopic (kinetic) part as:

$$f(x,t,\xi)=M(x,t,\xi)+G(x,t,\xi),$$

where the local Maxwellian M and G represent the fluid and non-fluid components in the solution, respectively. Since our problem is in one-dimensional space $x\in\mathbb{R}^1$, in the macroscopic level, it is more convenient to rewrite the system by using the Lagrangian coordinates. That is, consider the coordinate transformation:

$$x \Rightarrow \int_0^x \rho(t,y)dy, \quad t \Rightarrow t. \tag{2.4}$$

We will still denote the Lagrangian coordinates by (t,x) for simplicity of notation. System (2.1) in the Lagrangian coordinates becomes,

$$f_t + \frac{\xi_1-u_1}{v} f_x = \frac{1}{\epsilon} Q(f,f), \tag{2.5}$$

where $v=1/\rho$ represents the specific volume. Under the macro-micro decomposition, the Boltzmann equation (2.5) is equivalent to the following fluid-type system for the fluid components (cf. [22, 20]),

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = \epsilon(\frac{4\mu(\theta)}{3v}u_{1x})_x - \int_{\mathbb{R}^3} \xi_1^2 \Xi_x d\xi, \\ u_{it} = \epsilon(\frac{\mu(\theta)}{v}u_{ix})_x - \int_{\mathbb{R}^3} \xi_1\xi_i\Xi_x d\xi, \quad i = 2, 3 \\ (e + \frac{1}{2}|u|^2)_t + (pu_1)_x = \epsilon(\frac{\kappa(\theta)}{v}\theta_x)_x + \epsilon(\frac{4\mu(\theta)}{3v}u_1u_{1x})_x \\ \qquad\qquad + \epsilon\sum_{i=2}^{3}(\frac{\mu(\theta)}{v}u_iu_{ix})_x - \int_{\mathbb{R}^3}\frac{1}{2}\xi_1|\xi|^2\Xi_x d\xi, \end{cases} \tag{2.6}$$

together with the equation for the non-fluid component G

$$G_t + \frac{1}{v}P_1(\xi_1 G_x) - \frac{u_1}{v}G_x + \frac{1}{v}P_1(\xi_1 M_x) = \frac{2}{\epsilon}Q(M, G) + \frac{1}{\epsilon}Q(G, G), \tag{2.7}$$

where the pressure $p = R\theta/v$, the viscosity coefficient $\mu(\theta)(> 0)$ and the heat conductivity coefficient $\kappa(\theta)(> 0)$ are smooth functions of the temperature θ, and Ξ is higher order term of the non-fluid part (see section 7.1).

When Ξ is set to be zero in (2.6), this system becomes the compressible Navier-Stokes equations. So we neglect the effect of Ξ in (2.6) and study the following compressible Navier-Stokes equations under the assumptions that the coefficients of viscosity ϵ and heat-conductivity κ, are positive constants and satisfy

$$(H1) \quad \kappa = O(\epsilon) \text{ as } \epsilon \to 0; \ \kappa/\epsilon \geq c > 0, \ \forall\epsilon > 0$$

for some positive constant c,

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \epsilon(\frac{u_x}{v})_x, & t > 0, x \in \mathbb{R}^1 \\ (e + \frac{u^2}{2})_t + (up)_x = (\kappa\frac{\theta_x}{v} + \epsilon\frac{uu_x}{v})_x. \end{cases} \tag{2.8}$$

Here the unknowns $v(> 0)$, u, p, and e represent the specific volume, the velocity, the pressure and the internal energy, respectively. We assume, as usual in thermodynamics, that by any given two of the five thermodynamical variables, v, p, e, the temperature $\theta(> 0)$, and the entropy s, the remaining three variables can be expressed. Choosing (v, θ) as independent variables, we can deduce that

$$s_v(v,\theta) = p_\theta(v,\theta), \ s_\theta(v,\theta) = \frac{e_\theta(v,\theta)}{\theta}, \ e_v(v,\theta) = \theta p_\theta(v,\theta) - p(v,\theta)$$

by the second law of thermodynamics

$$\theta ds = de + pdv.$$

Then a straightforward calculation gives

$$s_t = \kappa(\frac{\theta_x}{v\theta})_x + \kappa\frac{\theta_x^2}{v\theta^2} + \epsilon\frac{u_x^2}{v\theta} \tag{2.9}$$

and

$$\theta_t + \frac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}u_x = \frac{\kappa}{e_\theta(v,\theta)}(\frac{\theta_x}{v})_x + \frac{\epsilon}{e_\theta(v,\theta)}\frac{u_x^2}{v}. \tag{2.10}$$

One may also choose (v, s) as independent variables and write $p = p(v, s)$, $\theta = \theta(v, s)$. In fact, for smooth solutions, (2.8) are equivalent to $(2.8)_{1,2}, (2.9)$ or $(2.8)_{1,2}, (2.10)$.

When the mean free path ϵ and Ξ are set zero in (2.6), the system becomes the following compressible Euler equations:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \quad t > 0, x \in \mathbb{R}^1 \\ s_t = 0. \end{cases} \tag{2.11}$$

We assume that the pressure p and the internal energy satisfy, for $v > 0$,

$$(H2)\begin{cases} p_v(v,\theta) < 0,\ e_\theta(v,\theta) > 0, \\ p_{vv}(v,s) > 0,\ p(v,s) \text{ is convex with respect to } (v,s), \end{cases}$$

which, together with the second law of thermodynamics, assures that

$$p_v(v,s) < 0 \text{ and } e(v,s) \text{ is convex with respect to } (v,s). \tag{2.12}$$

For $v > 0$, system (2.11) has characteristic speeds

$$\lambda_1 = -\sqrt{-p_v(v,s)},\ \lambda_2 = 0,\ \lambda_3 = \sqrt{-p_v(v,s)}$$

and the 1st and 3rd characteristic fields are genuinely nonlinear and can give rise to rarefaction waves (cf. [18]). For illustration, we consider only the 1-rarefaction wave, the case for the 3-rarefaction wave can be handled similarly.

Suppose that two states $(v_\pm, u_\pm, s_\pm)$ satisfy

$$s_- = s_+ \equiv \bar{s},\ u_+ = u_- + \int_{v_+}^{v_-} \lambda_1(z,\bar{s})dz,\ v_+ > v_- > 0, \tag{2.13}$$

then the Riemann problem of (2.11) with the initial data

$$(v_0, u_0, s_0)(x) = (v_0^r, u_0^r, s_0^r)(0, x) \equiv \begin{cases} (v_-, u_-, s_-), & x < 0 \\ (v_+, u_+, s_+), & x > 0 \end{cases} \tag{2.14}$$

admits a self-similar solution, the centered 1-rarefaction wave $(v^r, u^r, s^r)(x/t)$, which is defined by (cf. [8, 28])

$$\begin{cases} s^r(x/t) = \bar{s},\ u^r(x/t) = u_- + \displaystyle\int_{v^r(x/t)}^{v_-} \lambda_1(z, \bar{s}) dz, \\ \lambda_1(v^r(x/t), \bar{s}) = \begin{cases} \lambda_1(v_-, \bar{s}), & x/t < \lambda_1(v_-, \bar{s}), \\ x/t, & \lambda_1(v_-, \bar{s}) \le x/t \le \lambda_1(v_+, \bar{s}), \\ \lambda_1(v_+, \bar{s}), & x/t > \lambda_1(v_+, \bar{s}). \end{cases} \end{cases} \tag{2.15}$$

3 Convergence to Rarefaction Waves for The Compressible Navier-Stokes Equations

We consider the compressible Navier-Stokes equations $(2.8)_{1,2}, (2.9)$ with the initial data:

$$(v, u, s)(0, x) = (v_0, u_0, s_0)(x) \to (v_\pm, u_\pm, s_\pm), \text{ as } x \to \pm\infty, \tag{3.1}$$

where $v_\pm (> 0)$, $u_\pm$, $s_\pm$ are given constants.
We ignore the effects of initial layers by allowing the initial data for the Navier-Stokes equations to depend on the viscosity, and show that for a given 1-centered rarefaction wave $(v^r, u^r, s^r)(x/t)$ of arbitrary strength, there exists global smooth solutions $(v, u, s)(t, x)$ to the Navier-Stokes equations such that the viscous solution converges to the centered rarefaction wave as viscosity $\epsilon \to 0$, uniformly away from $t = 0$. More precisely, we have

Theorem 3.1 *Assume that* $(H1)$ *and* $(H2)$ *hold. Let* $(v^r, u^r, s^r)(x/t)$ *be the centered* 1*-rarefaction wave defined by* (2.15)*, which connects two constant states* $(v_\pm, u_\pm, s_\pm)$ *satisfying* (2.13) *with* $v_- > 0$*. Set* $\theta^r(x/t) = \theta(v^r, s^r)$*, then there exists a small positive constant* ϵ_0 *such that for each* $\epsilon \in (0, \epsilon_0]$*, we can construct a global smooth solution* $(v, u, \theta)(t, x)$ *to the Navier-Stokes equations* $(2.8)_{1,2}, (2.10)$ *with the following properties:*

$$(i) \begin{cases} (v - v^r, u - u^r, \theta - \theta^r) \in C^0(0, +\infty; L^2), \\ (v, u, \theta)_x \in C^0(0, +\infty; L^2), (u, \theta)_{xx} \in L^2(0, +\infty; L^2). \end{cases}$$

(ii) As viscosity $\epsilon \to 0$, $(v,u,\theta)(t,x)$ converges to $(v^r,u^r,\theta^r)(x/t)$ pointwise except at $(0,0)$. Furthermore, for any given positive constant h, there is a constant $c_h > 0$, independent of ϵ, so that

$$\sup_{t\geq h} \|(v-v^r, u-u^r, \theta-\theta^r)(t,\cdot)\|_{L^\infty} \leq c_h \epsilon^{1/4} |\ln \epsilon|. \tag{3.2}$$

To prove Theorem 3.1, we first approximate the centered 1-rarefaction wave $(v^r,u^r,\theta^r)(x/t)$ by a smooth rarefaction wave $(v^r_{\delta(\epsilon)}, u^r_{\delta(\epsilon)}, \theta^r_{\delta(\epsilon)})(t,x)$, which converges to (v^r,u^r,θ^r) at a rate as $\epsilon \to 0$ (see section 6). And then, we prove that the smooth rarefaction wave dominates the behavior of the solution $(v,u,\theta)(t,x)$ to the Navier-Stokes equations with the same initial data as that of $(v^r_{\delta(\epsilon)}, u^r_{\delta(\epsilon)}, \theta^r_{\delta(\epsilon)})$. This is done by decomposing the solution (v,u,θ) as a small perturbation of the smooth rarefaction wave and using an elementary energy method on two time scales (see section 6.1), which is first estimated by the entropy method, motivated by the ideas in [33]. However, since our aim is to solve the Boltzamann equation, the scaling argument in [33] does not apply directly to the analysis of Boltzmann equation, so we do not use the scaling argument here.

Next, we prove that in the case that the inviscid flow is a smooth rarefaction wave, the global smooth solution to the Navier-Stokes equations with the same initial data as the inviscid flow, exists and converges to the smooth rarefaction wave at a rate as $\epsilon \to 0$. By a smooth 1-rarefaction wave $(v^R,u^R,s^R)(t,x)$, we mean the unique solution to the Euler equations (2.11) with sufficiently smooth initial data $(v^R,u^R,s^R)(0,x)$ satisfying

$$\begin{cases} (v^R,u^R)(0,x) = (v_\pm,u_\pm), \ \text{as } x\to\pm\infty, \ \text{with } v_\pm>0; \\ s^R(0,x) = \bar{s}, \ \forall x\in\mathbb{R}^1; \ \partial_x u^R(0,x) > 0, \\ u_- = u^R(0,x) + \displaystyle\int_{v_-}^{v^R(0,x)} \lambda_1(z,\bar{s})dz, \ \forall x\in\mathbb{R}^1; \\ \dfrac{\partial^l u^R(0,x)}{\partial x^l} \to 0, \ \text{as } x\to\pm\infty, \ \ l=1,2,3; \ \dfrac{\partial}{\partial x}u^R(0,x)\in H^4; \\ |\dfrac{\partial^l u^R(0,x)}{\partial x^l}| \leq c_l |\dfrac{\partial}{\partial x}u^R(0,x)|, \ \text{when } |x| \text{ large and } l=2,3, \end{cases} \tag{3.3}$$

for some constants c_l $(l=2,3)$. The smooth 1-rarefaction wave $(v^R,u^R,s^R)(t,x)$ can be defined by

$$\begin{cases} s^R(t,x) = \bar{s}, \lambda_1(v^R(t,x),\bar{s}) = w(t,x), \\ u^R(t,x) = u_- + \int_{v^R(t,x)}^{v_-} \lambda_1(z,\bar{s})dz, \\ \begin{cases} w_t + ww_x = 0, \\ w(0,x) = \lambda_1(v^R(0,x),\bar{s}). \end{cases} \end{cases} \tag{3.4}$$

The second theorem in this paper is stated as follows:

Theorem 3.2 *Assume that* $(H1)$ *and* $(H2)$ *hold. Let* $(v^R, u^R, s^R)(t,x)$ *be a smooth* 1*-rarefaction wave defined by* (3.4) *with the initial data* $(v^R, u^R, s^R)(0,x)$ *satisfying* (3.3)*. Set* $\theta^R(t,x) = \theta(v^R, s^R)$*, then there exists a small positive constant* ϵ_0 *such that for each* $\epsilon \in (0, \epsilon_0]$*, the Navier-Stokes equations* $(2.8)_{1,2}, (2.10)$ *with initial data*

$$(v, u, \theta)(0,x) = (v^R, u^R, \theta^R)(0,x)$$

have a unique smooth solution $(v, u, \theta)(t,x)$ *satisfying*

$$\begin{cases} (v - v^R, u - u^R, \theta - \theta^R) \in C^0(0, +\infty; H^1), \\ (v - v^R)_x \in L^2(0, +\infty; L^2), (u - u^R, \theta - \theta^R)_x \in L^2(0, +\infty; H^1) \end{cases}$$

and

$$\sup_{t \geq 0} \|(v - v^R, u - u^R, \theta - \theta^R)(t, \cdot)\|_{L^\infty} \leq c\epsilon^{1/2}, \tag{3.5}$$

where c *is a positive constant independent of* ϵ*.*

For the proof of Theorem 3.2, we will combine a finite time estimate, which applies for arbitrary smooth solution of (2.11), with a large time estimate which can be obtained by an energy method making use of the expansion nature of the rarefaction wave (see section 6.2). The proof of the above two theorems can be found in Section 6.

4 Convergence to Rarefaction Waves for The Boltzmann equation

Now, we turn to the Boltzmann equation. Similar to the compressible Navier-Stokes equations, we can get the following two theorems. First we ignore the effects of the initial layers by allowing the initial data for the Boltzmann equation to depend on the mean free path ϵ, and show that the global Boltzmann solution exists and converges to the local Maxwellian determined by the given centered rarefaction wave as $\epsilon \to 0$, uniformly away from $t = 0$. In order to state our theorem, we introduce some notations first.
Let $M_* = M_{[v_*, u_*, \theta_*]}$ be a global Maxwellian satisfying

$$\begin{cases} \frac{1}{2}\theta(t,x) < \theta_* < \theta(t,x), \quad \text{for} \quad t \geq 0, \ x \in \mathbb{R}^1, \\ |v - v_*| + |u - u_*| + |\theta - \theta_*| < \vartheta, \end{cases} \tag{4.1}$$

where $\vartheta = \vartheta(v, u, \theta; v_*, u_*, \theta_*)$ is the constant to be given in Lemma 7.2. We say that $f(\xi) \in L^2_\xi(\frac{1}{\sqrt{M_*}})$ if $\frac{f}{\sqrt{M_*}} \in L^2_\xi$.

With these notations, we have the following

Theorem 4.1 *Let $(v^r, u_1^r, s^r)(x/t)$ be the centered 1-rarefaction wave defined by (2.15), which connects two constant states $(v_\pm, u_{1\pm}, s_\pm)$ satisfying (2.13) with $v_- > 0$. Set $\theta^r(x/t) = 3(v^r)^{-2/3}\exp(s^r)/(4e\pi)$ and $u^r(x/t) = (u_1^r, 0, 0)^t$, if*

$$\alpha \equiv |v_+ - v_-| + |u_+ - u_-| < \vartheta, \quad \sup_{t\geq 0, x\in\mathbb{R}^1} \theta^r(x/t) < 2 \inf_{t\geq 0, x\in\mathbb{R}^1} \theta^r(x/t),$$

then there exists a small positive constant ϵ_0 such that for each $\epsilon \in (0, \epsilon_0]$, we can construct a global solution $f(t, x, \xi)$ to the Boltzmann equation (2.5). Furthermore, for any given positive constant h, there is a constant $c_h > 0$, independent of ϵ, so that

$$\sup_{t\geq h} \|f(t, x, \xi) - M_{[v^r, u^r, \theta^r]}(t, x, \xi)\|_{L_x^\infty L_\xi^2(\frac{1}{\sqrt{M_*}})} \leq c_h \epsilon^{1/5} |\ln \epsilon|. \quad (4.2)$$

To prove Theorem 4.1, we first approximate (v^r, u^r, θ^r) by a smooth rarefaction wave $(v^r_{\delta(\epsilon)}, u^r_{\delta(\epsilon)}, \theta^r_{\delta(\epsilon)})$ as in the proof of Theorem 3.1. Then we prove that the local Maxwellian determined by the smooth rarefaction wave governs the asymptotic behavior of the solution to the Boltzmann equation with the same initial data as that of this Maxwellian. This is done by combing the techniques for the compressible Navier-Stokes equations as in Section 6 and the weighted energy method, based on the macro-micro decomposition of the Boltzmann solution. The treatment of the macroscopic components (v, u, θ) is almost the same as the one for Navier-Stokes equations except estimating the relevant part of the non-fluid component Ξ with respect to the weight $M_{[v,u,\theta]}$. As in [21], we use the microscopic H-theorem (7.7) to estimate the non-fluid component. Since the energy estimate with respect to the weight M has an error term with a polynomial of ξ of order greater than 1 because of the derivatives on M, but the order of growth in ξ of the dissipation on the microscopic component in only 1. Hence, another set of energy estimates based on a global suitably chosen Maxwellian M_* (ensuring the H-theorem hold in Lemma 7.2) is needed to complete the analysis (similar to the argument given in [5]). Notice that the error term in the energy estimate with respect to M is an integral with the weight M_* and a small factor. Although an additional term in the form of integrals of the fluid components and their derivatives appear because of the invalid orthogonality of M and G with respect to the weight M_*, the small factor helps to yield the desired estimates.

Finally, we prove that in the case that the fluid is a smooth rarefaction wave, the global solution to the Boltzmann equation with the same initial data as the fluid, exists and converges to the relevant Maxwellian at a rate as $\epsilon \to 0$. We state our final theorem as follows:

Theorem 4.2 *Let* $(v^R, u_1^R, s^R)(t,x)$ *be a smooth* 1*-rarefaction wave defined by* (3.4) *with the initial data* $(v^R, u_1^R, s^R)(0,x)$ *satisfying* (3.3)*. Set* $\theta^R(t,x) = 3(v^R)^{-2/3}\exp(s^R)/(4e\pi)$ *and* $u^R(t,x) = (u_1^R, 0, 0)^t$*, if*

$$\alpha \equiv |v_+ - v_-| + |u_+ - u_-| < \vartheta, \quad \sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}} \theta^R(t,x) < 2 \inf_{(t,x)\in\mathbb{R}^+\times\mathbb{R}} \theta^R(t,x),$$

then there exists a small positive constant ϵ_0 *such that for each* $\epsilon \in (0, \epsilon_0]$*, the Boltzmann equation* (2.5) *with initial data*

$$f(0,x,\xi) = M_{[v^R,u^R,\theta^R]}(0,x,\xi)$$

has a unique global solution $f(t,x,\xi)$ *satisfying*

$$\sup_{t\geq 0} \|f(t,x,\xi) - M_{[v^R,u^R,\theta^R]}(t,x,\xi)\|_{L^\infty_x L^2_\xi(\frac{1}{\sqrt{M_*}})} \leq c\epsilon^{1/4}, \tag{4.3}$$

where c *is a positive constant independent of* ϵ*.*

The proof of Theorem 4.2 will be based on a combination of a finite time estimate, which applies for arbitrary smooth flows, with a large time estimate, which can be obtained by the same argument as in the proof of Theorem 4.1.

Remark 1 The restriction $\alpha < \vartheta$ on the strength α of the rarefaction wave is mainly to ensure that the microscopic version of H-theorem holds for some $\sigma(v,u,\theta; v_*, u_*, \theta_*) > 0$, and ϑ depends on the first non-zero eigenvalue of the linearized operator L_M, which is not necessary to be small. So α need not be small.

Remark 2 By a analogous analysis, one can check that the aforementioned four theorems also hold for the superposition of 1-rarefaction wave and 3-rarefaction wave.

Remark 3 On any finite time interval $[0,T]$ with $T < +\infty$, the rates of convergence in Theorem 3.2, Theorem 4.2 are ϵ, $\epsilon^{1/2}$ respectively, i.e.

$$\sup_{0\leq t\leq T} \|(v - v^R, u - u^R, \theta - \theta^R)(t,\cdot)\|_{L^\infty} \leq c(T)\epsilon,$$

$$\sup_{0\leq t\leq T} \|f(t,x,\xi) - M_{[v^R,u^R,\theta^R]}(t,x,\xi)\|_{L^\infty_x L^2_\xi(\frac{1}{\sqrt{M_*}})} \leq c(T)\epsilon^{1/2}.$$

This is so even in the case that (v^R, u^R, θ^R) is replaced by an arbitrary smooth solution of the Euler equations over $[0,T]\times\mathbb{R}^1$. Note that Caflisch (cf. [5]) has proved that for smooth, spatially periodic Euler solution $(V,U,\Theta)(t,x)$ over $[0,T]\times\mathbb{R}^1$, there exists a smooth Boltzmann solution f to (2.1) with initial values depending on ϵ, such that

$$\sup_{0\leq t\leq T} \|f(t,x,\xi) - M_{[V,U,\Theta]}(t,x,\xi)\|_{L^\infty_x L^2_\xi} \leq c(T)\epsilon.$$

Although our rate is only $\epsilon^{1/2}$, the initial value is independent of ϵ.
Notations. Throughout this paper, c denotes a generic positive constant, ϵ represents the mean free path, viscosity in the study of Boltzamann equation, Navier-Stokes equations, respectively, and γ is taken as a small positive constant to be determined later. For functional space, $H^l(\mathbb{R})$ denotes the l-th order sobolev space with its norm

$$\|u\|_l = \sum_{j=0}^{l} \|\partial_x^j u\|, \qquad \text{when } \|\cdot\| = \|\cdot\|_{L^2}(\mathbb{R}).$$

For the integral, $\int(\cdots)d\xi$ means $\int_{\mathbb{R}^3}(\cdots)d\xi$.

5 Approximate rarefaction waves

In this section, we construct smooth rarefaction waves which approximate centered rarefaction waves. Consider

$$\begin{cases} w_t + ww_x = 0, \\ w(0,x) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0. \end{cases} \end{cases} \tag{5.1}$$

If $w_- < w_+$, then (5.1) has the centered rarefaction wave solution $w^r(t,x) = w^r(x/t)$ given by

$$w^r(t,x) = \begin{cases} w_-, & x/t < w_-, \\ x/t, & w_- \le x/t \le w_+, \\ w_+, & x/t > w_+. \end{cases}$$

To construct a smooth rarefaction wave solution of the Burgers equation which approximates the centered rarefaction wave, we set for each $\delta > 0$,

$$w_\delta(x) = w(x/\delta) \equiv (w_+ + w_-)/2 + tanh(x/\delta)(w_+ - w_-)/2 \tag{5.2}$$

and solve the following initial value problem

$$\begin{cases} w_t + ww_x = 0, \\ w(0,x) = w_\delta(x). \end{cases} \tag{5.3}$$

Next, we state certain properties for the smooth rarefaction wave (see [23, 33] for the proof).

Lemma 5.1 *For each $\delta > 0$, (5.3) has a unique global smooth solution $w_\delta^r(t,x)$, such that the following hold:*

(1) $w_- < w^r_\delta(t,x) < w_+$, $\partial_x w^r_\delta(t,x) > 0$, *for* $x \in \mathbb{R}^1,\ t \geq 0,\ \delta > 0$.
(2) For any $1 \leq p \leq \infty$, *there is a constant* c_p *depending only on* p, *such that the following estimates hold for all* $t > 0,\ \delta > 0$,

$$\begin{aligned}
&\|\partial_x w^r_\delta(t,\cdot)\|_{L^p} \leq c_p \min\{(w_+ - w_-)\delta^{-1+1/p}, (w_+ - w_-)^{1/p} t^{-1+1/p}\},\\
&\|\partial_x^2 w^r_\delta(t,\cdot)\|_{L^p} \leq c_p \min\{(w_+ - w_-)\delta^{-2+1/p}, \delta^{-1+1/p} t^{-1}\},\\
&\|\partial_x^3 w^r_\delta(t,\cdot)\|_{L^p} \leq c_p \min\{(w_+ - w_-)\delta^{-3+1/p}, \delta^{-2+1/p} t^{-1}\}.
\end{aligned}$$

(3) There exist constants $\delta_0 \in (0,1)$, c *such that for* $\delta \in (0,\delta_0], t > 0$,

$$\|w^r_\delta(t,\cdot) - w^r(t,\cdot)\|_{L^\infty} \leq c t^{-1} \delta(\ln(1+t) + |\ln \delta|).$$

Set $w_\pm = \lambda_1(v_\pm, \bar{s})$ in (5.1)-(5.3), and define the smooth approximation $(v^r_\delta, u^r_\delta, s^r_\delta, \theta^r_\delta)(t,x)$ of the centered-rarefaction wave $(v^r, u^r, s^r)(x/t)$ in (2.15) by

$$\begin{cases}
s^r_\delta(t,x) = \bar{s},\ \lambda_1(v^r_\delta(t,x), \bar{s}) = w^r_\delta(t,x),\\
u^r_\delta(t,x) = u_- + \displaystyle\int_{v^r_\delta(t,x)}^{v_-} \lambda_1(z,\bar{s})dz,\\
\theta^r_\delta(t,x) = \theta(v^r_\delta(t,x), \bar{s}).
\end{cases} \tag{5.4}$$

Then, it is easy to check that $(v^r_\delta, u^r_\delta, s^r_\delta, \theta^r_\delta)(t,x)$ satisfy the Euler equations

$$\begin{cases}
v^r_{\delta t} - u^r_{\delta x} = 0,\\
u^r_{\delta t} + p(v^r_\delta, \theta^r_\delta)_x = 0,\\
(e(v^r_\delta, \theta^r_\delta) + \dfrac{(u^r_\delta)^2}{2})_t + (u^r_\delta p(v^r_\delta, \theta^r_\delta))_x = 0,\\
\theta^r_{\delta t} + \dfrac{\theta p_\theta(v^r_\delta, \theta^r_\delta)}{e_\theta(v^r_\delta, \theta^r_\delta)} u^r_{\delta x} = 0,\\
s^r_{\delta t} = 0.
\end{cases} \tag{5.5}$$

And due to Lemma 5.1, the following lemma holds.

Lemma 5.2 *The functions* $(v^r_\delta, u^r_\delta, s^r_\delta, \theta^r_\delta)(t,x)$ *constructed by* (5.4) *have the following properties:*
(1) $\partial_x u^r_\delta(t,x) > 0$ *for* $x \in \mathbb{R}^1,\ t \geq 0$ *and* $\delta > 0$.
(2) For any $1 \leq p \leq \infty$, *there is a constant* c_p *depending only on* p, *such that the following estimates hold for all* $t > 0,\ \delta > 0$,

$$\begin{aligned}
&\|\partial_x(v^r_\delta, u^r_\delta, \theta^r_\delta)(t,\cdot)\|_{L^p} \leq c_p \min\{\alpha\delta^{-1+1/p}, \alpha^{1/p} t^{-1+1/p}\},\\
&\|\partial_{xx}(v^r_\delta, u^r_\delta, \theta^r_\delta)(t,\cdot)\|_{L^p} \leq c_p \min\{\alpha\delta^{-2+1/p}, \delta^{-1+1/p} t^{-1}\},\\
&\|\partial_{xxx}(v^r_\delta, u^r_\delta, , \theta^r_\delta)(t,\cdot)\|_{L^p} \leq c_p \min\{\alpha\delta^{-3+1/p}, \delta^{-2+1/p} t^{-1}\},
\end{aligned}$$

where $\alpha \equiv |v_+ - v_-| + |u_+ - u_-|$.
(3) There exist constants $\delta_0 \in (0,1)$ *and* c *such that for* $\delta \in (0, \delta_0]$ *and* $t > 0$,

$$\|(v^r_\delta, u^r_\delta, \theta^r_\delta)(t,x) - (v^r, u^r, \theta^r)(x/t)\|_{L^\infty_x} \le ct^{-1}\delta(\ln(1+t) + |\ln\delta|).$$

The smooth rarefaction wave $(v^R, u^R, s^R)(t,x)$, defined by (3.4) with the initial data satisfying (3.3), has the following properties

Lemma 5.3 *Let* $(v^R, u^R, \theta^R)(t,x)$ *be the smooth rarefaction wave given in Theorem 3.2, then it holds that:*
(1) $u^R_x(t,x) > 0$ *for* $x \in \mathbb{R}^1,\ t \ge 0$.
(2) For any $1 \le p \le \infty$, *there is a constant* c_p *depending only on* p, *such that the following estimates hold for all* $t > 0$,

$$\|(v^R_x, u^R_x, \theta^R_x, v^R_{xx}, u^R_{xx}, \theta^R_{xx}, v^R_{xxx}, u^R_{xxx}, \theta^R_{xxx})(t,\cdot)\|_{L^p} \le c_p(1+t)^{-1}.$$

6 Proof of Theorems 3.1 and 3.2 for Compressible Navier-Stokes Equations

This section is devoted to proving Theorem 3.1 and Theorem 3.2. To this end, set

$$(\phi, \psi, \zeta, \omega)(\tau, y) \equiv (v - V, u - U, \theta - \Theta, s - S)(t, x),$$

in which

$$y = \epsilon^{-1}x,\ \tau = \epsilon^{-1}t,$$

where $(v, u, \theta, s)(t,x)$ and $(V, U, \Theta, S)(t,x)$ are assumed to be the solutions of the Navier-Stokes equations $(2.8)_{1,2}, (2.9), (2.10)$ and the corresponding Euler Equations, respectively. Then $(\phi, \psi, \zeta)(\tau, y)$ solves the following initial value problem:

$$\begin{cases} \phi_\tau - \psi_y = 0, \\ \psi_\tau + (p(v,\theta) - p(V,\Theta))_y = (\dfrac{u_y}{v})_y, \\ \zeta_\tau + \dfrac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)}\psi_y + \epsilon(\dfrac{\theta p_\theta(v,\theta)}{e_\theta(v,\theta)} - \dfrac{\Theta p_\theta(V,\Theta)}{e_\theta(V,\Theta)})U_y \\ \qquad = \dfrac{\kappa}{\epsilon e_\theta(v,\theta)}(\dfrac{\theta_y}{v})_y + \dfrac{u_y^2}{v e_\theta(v,\theta)}, \end{cases} \tag{6.1}$$

$$(\phi, \psi, \zeta)(0, y) \equiv (\phi_0, \psi_0, \zeta_0)(y) = (v - V, u - U, \theta - \Theta)(0, x). \tag{6.2}$$

The entropy difference $\omega(\tau,y)$ satisfies the following equation

$$\omega_\tau = \frac{\kappa}{\epsilon}(\frac{\theta_y}{v\theta})_y + \frac{\kappa}{\epsilon}\frac{\theta_y^2}{v\theta^2} + \frac{u_y^2}{v\theta}.$$

For convenience of presentation, in what follows, we will choose (v,θ) as independent variables in the equation of state. We seek a global (in time) solution (ϕ,ψ,ζ) to the problem (6.1) and (6.2) in the space defined as

$$\begin{aligned} X(0,\tau_1) \equiv &\{(\phi,\psi,\zeta)(\tau,y)|(\phi,\psi,\zeta)\in C^0(0,\tau_1;H^1),\\ &\phi_y\in L^2(0,\tau_1;L^2),(\psi_y,\zeta_y)\in L^2(0,\tau_1;H^1)\}. \end{aligned}$$

Define a normalized entropy $\eta(v,u,\theta;V,U,\Theta)$ around (V,U,Θ) as

$$\begin{aligned} \eta(\tau,y) \equiv \{&(e(v,\theta)+u^2/2)-(e(V,\Theta)+U^2/2)\\ &+p(V,\Theta)(v-V)-U(u-U)-\Theta(s-S)\}. \end{aligned} \tag{6.3}$$

Since $e(v,s)+u^2/2$ is a strictly convex function of (v,u,s) by (2.12), then $\eta\geq 0$. A simple computation shows that

$$\begin{aligned} &\eta_\tau + \epsilon U_x q_1 + \frac{\Theta}{v\theta}\psi_y^2 + \frac{\kappa\Theta}{\epsilon v\theta^2}\zeta_y^2 + (\cdots)_y\\ =&\epsilon\{(\frac{2\zeta\psi_y}{v\theta} - \frac{\psi\phi_y}{v^2})U_x + \frac{\kappa}{\epsilon}(\frac{\zeta\zeta_y}{v\theta^2} - \frac{\zeta\phi_y}{v^2\theta})\Theta_x\}\\ &+\epsilon^2(\frac{\zeta U_x^2}{v\theta} - \frac{\psi V_x U_x}{v^2} - \frac{\kappa}{\epsilon}\frac{\zeta V_x\Theta_x}{v^2\theta}) + \epsilon^2(\frac{\psi U_{xx}}{v} + \frac{\kappa}{\epsilon}\frac{\zeta\Theta xx}{v\theta}), \end{aligned} \tag{6.4}$$

where $q_1 = p(v,s)-p(V,S)-p_v(V,S)(v-V)-p_s(V,S)(s-S)\geq 0$ by the convexity of $p(v,s)$.

6.1 Proof of Theorem 3.1

In this subsection, we prove Theorem 3.1. Let $(V,U,\Theta)(t,x)\equiv(v_\delta^r,u_\delta^r,\theta_\delta^r)(t,x)$ be defined by (5.4) and $(v,u,\theta)(t,x)$ be the solution of the initial value problem of the Navier-Stokes equation $(2.8)_{1,2}$, (2.10) with the same initial data as that of the smooth rarefaction wave, that is, $(v_0,u_0,\theta_0)(x)=(v_\delta^r,u_\delta^r,\theta_\delta^r)(0,x)$. Consider Cauchy problem (6.1) and (6.2), in which

$$(\phi,\psi,\zeta)(\tau,y)\equiv(v-v_\delta^r,u-u_\delta^r,\ \theta-\theta_\delta^r)(t,x),(\phi_0,\psi_0,\zeta_0)(y)=0. \tag{6.5}$$

Due to the smoothness of $(v_\delta^r,u_\delta^r,\theta_\delta^r)$, the local existence of smooth solutions to (6.1) and (6.2) is standard (cf. [19, 24]), while the global existence and the estimate (3.2) will follow from the following a priori estimate.

Proposition 6.1 *(A priori estimate) Suppose that* (6.5) *holds and the problem* (6.1) *and* (6.2) *has a solution* $(\phi, \psi, \zeta) \in X(0, \tau_1)$ *for some* $\tau_1 > 0$. *There exist positive constants* $\epsilon_1, \delta_1, k_1, \nu_1$ *and* c, *independent of* ϵ, δ *and* τ_1, *such that if*

$$0 < \epsilon \leq \epsilon_1, \ 0 < \delta \leq \delta_1, \ \epsilon^{1/4} \leq k_1 \delta, \ \sup_{0 \leq \tau \leq \tau_1} \|(\phi, \psi, \zeta)(\tau)\|_1 \leq \nu_1 \quad (6.6)$$

for small ϵ_1 *and* ν_1, *then*

$$\begin{aligned} \sup_{0 \leq \tau \leq \tau_1} \|(\phi, \psi, \zeta)(\tau)\|^2 + \epsilon \int_0^{\tau_1} \|\sqrt{u^r_{\delta x}}(\phi, \zeta)(\tau)\|^2 d\tau \\ + \int_0^{\tau_1} \|(\phi_y, \psi_y, \zeta_y)(\tau)\|^2 d\tau \leq c\epsilon^{1/4}, \end{aligned} \quad (6.7)$$

$$\sup_{0 \leq \tau \leq \tau_1} \|(\phi_y, \psi_y, \zeta_y)(\tau)\|^2 + \int_0^{\tau_1} \|(\psi_{yy}, \zeta_{yy})(\tau)\|^2 d\tau \leq c\epsilon^{3/4}. \quad (6.8)$$

Once Proposition 6.1 is proved, one can take $\delta = k_1^{-1}\epsilon^{1/4}$, so that (6.7) and (6.8) imply that there exists a positive constant c independent of ϵ such that

$$\sup_{0 \leq \tau \leq +\infty} \|(\phi, \psi, \zeta)(\tau, \cdot)\|_{L^\infty} \leq c\epsilon^{1/4}.$$

Therefore, the Navier-Stokes equations $(2.8)_{1,2}, (2.10)$ have a global smooth solution $(v, u, \theta)(t, x)$ satisfying (i) of Theorem 3.1 and for all t,

$$\|(v, u, \theta)(t, \cdot) - (v^r_\delta, u^r_\delta, \theta^r_\delta)(t, \cdot)\|_{L^\infty} \leq c\epsilon^{1/4}.$$

By (3) in Lemma 5.2, it follows that for $t > 0$,

$$\|(v^r_\delta, u^r_\delta, \theta^r_\delta)(t, \cdot) - (v^R, u^R, \theta^R)(t, \cdot)\|_{L^\infty} \leq ct^{-1}\epsilon^{1/4}(\ln(1+t) + |\ln \epsilon|).$$

It is clear now that the desired estimate (3.2) and Theorem 3.1 follow from the above two estimates. Thus, the main task is to prove Proposition 6.1. Notice that the smallness of ν_1 in (6.6) guarantees that

$$v^r_\delta + \phi \geq v_-/2, \ \theta^r_\delta + \zeta \geq \hat{\theta}/2, \text{ for } \hat{\theta} = \inf_{t \geq 0, x \in \mathbb{R}^1} \theta^r_\delta(t, x).$$

We will derive the lower order estimate (6.7) and the higher order estimate (6.8) on $(\phi, \psi, \zeta)(\epsilon^{-1}t, \epsilon^{-1}y)$ in two time scales, $0 \leq t \leq T \leq 1$ and $1 \leq t \leq +\infty$. Set

$$\tau_0 = \epsilon^{-3/4}T, \tau^0 = \epsilon^{-3/4}.$$

Then $\tau_0 \leq \tau^0$. Moreover $\tau_0 = \tau^0$ when $T = 1$. Proposition 6.1 can be proved by the following two lemmas and choosing $\tau^0 = \tau_0$ in (6.10).

Lemma 6.2 *(Basic energy estimate) Suppose that the assumptions in Proposition 6.1 hold. Then for* $\tau_0 \le \tau_1$,

$$
\begin{aligned}
&\sup_{0\le\tau\le\tau_0} \|(\phi,\psi,\zeta)(\tau)\|^2 + \epsilon\int_0^{\tau_0} \|\sqrt{u^r_{\delta x}}(\phi,\zeta)(\tau)\|^2 d\tau \\
&\quad + \int_0^{\tau_0} \|(\phi_y,\psi_y,\zeta_y)(\tau)\|^2 d\tau \le c\frac{\epsilon}{\delta^3},
\end{aligned}
\tag{6.9}
$$

while for $\tau^0 \le \tau_1$,

$$
\begin{aligned}
&\sup_{\tau^0\le\tau\le\tau_1} \|(\phi,\psi,\zeta)(\tau)\|^2 + \epsilon\int_{\tau^0}^{\tau_1} \|\sqrt{u^r_{\delta x}}(\phi,\zeta)(\tau)\|^2 d\tau \\
&\quad + \int_{\tau^0}^{\tau_1} \|(\phi_y,\psi_y,\zeta_y)(\tau)\|^2 d\tau \le c(\|(\phi,\psi,\zeta)(\tau^0)\|_1^2 + \epsilon^{1/2}).
\end{aligned}
\tag{6.10}
$$

Proof We prove the basic energy estimate in finite time, (6.9), first. Integrating (6.4) with respect to τ and y over $[0,\tau]\times\mathbb{R}^1$ ($\tau \le \tau_0$) shows that

$$
\begin{aligned}
&\int_{\mathbb{R}^1} \eta(\tau)dy + \int_0^\tau\int_{\mathbb{R}^1} (\epsilon u^r_{\delta x} q_1 + \psi_y^2 + \zeta_y^2)dyd\tau \\
\le& c\int_0^\tau\int_{\mathbb{R}^1} (R_1+R_2+R_3)dyd\tau,
\end{aligned}
$$

where

$$
\begin{cases}
R_1 = \epsilon\{(|\zeta\psi_y| + |\psi\phi_y|)|u^r_{\delta x}| + (|\zeta\zeta_y| + |\zeta\phi_y|)|\theta^r_{\delta x}|\},\\
R_2 = \epsilon^2\{|\zeta|(u^r_{\delta x})^2 + |\zeta v^r_{\delta x}\theta^r_{\delta x}| + |\psi v^r_{\delta x} u^r_{\delta x}|\},\\
R_3 = \epsilon^2\{|\psi u^r_{\delta xx}| + |\zeta\theta^r_{\delta xx}|\}.
\end{cases}
$$

We only estimate some typical terms in the $R_i (i = 1,2,3)$, the others can be handled similarly. For R_1,

$$
\begin{aligned}
&\epsilon\int_0^\tau\int_{\mathbb{R}^1} |\zeta\psi_y u^r_{\delta x}|dyd\tau \\
\le& \int_0^\tau\int_{\mathbb{R}^1} (\epsilon^{1/2}\psi_y^2 + \epsilon^{3/2}\|u^r_{\delta x}\|^2_{L^\infty_{t,x}}\zeta^2)dyd\tau \\
\le& \int_0^\tau\int_{\mathbb{R}^1} (\epsilon^{1/2}\psi_y^2 + c\frac{\epsilon^{3/2}}{\delta^2}\zeta^2)dyd\tau \\
\le& \int_0^\tau\int_{\mathbb{R}^1} (\epsilon^{1/2}\psi_y^2 + c\epsilon\zeta^2)dyd\tau,
\end{aligned}
\tag{6.11}
$$

provided that $\delta \ge k_1^{-1}\epsilon^{1/4}$. Similarly, for R_2 and R_3,

$$
\begin{aligned}
&\epsilon^2\int_0^\tau\int_{\mathbb{R}^1} |\psi|((u^r_{\delta x})^2 + |u^r_{\delta xx}|)dyd\tau \\
\le&\int_0^\tau\int_{\mathbb{R}^1} \{\epsilon\psi^2 + \epsilon^3((u^r_{\delta x})^4 + (u^r_{\delta xx})^2)\}dyd\tau \\
\le&\epsilon\int_0^\tau\int_{\mathbb{R}^1} \psi^2 dyd\tau + c\frac{\epsilon}{\delta^3}.
\end{aligned}
\tag{6.12}
$$

So,

$$
\begin{aligned}
&\int_{\mathbb{R}^1} \eta(\tau) dy + \int_0^\tau \int_{\mathbb{R}^1} (\epsilon u_{\delta x}^r q_1 + \psi_y^2 + \zeta_y^2) dy d\tau \\
\leq & c \int_0^\tau \int_{\mathbb{R}^1} \{\epsilon^{1/2} \phi_y^2 + \epsilon(\phi^2 + \psi^2 + \zeta^2)\} dy d\tau + c\frac{\epsilon}{\delta^3}.
\end{aligned} \tag{6.13}
$$

Now we deal with the double integral of ϕ_y^2. Multiply $(6.1)_2$ by $v^{-1}\phi_y$ and use $(6.1)_1$ to get

$$
\begin{aligned}
&(\frac{\phi_y^2}{2v^2} - \frac{\psi\phi_y}{v})_\tau + \frac{|p_v|}{v}\phi_y^2 + (\cdots)_y = \frac{\psi_y^2 + p_\theta \phi_y \zeta_y}{v} \\
&+ \frac{\epsilon}{v^2}\psi(\phi_y u_{\delta x}^r - \psi_y v_{\delta x}^r) + \frac{\epsilon}{v^3}\phi_y \psi_y v_{\delta x}^r + \frac{\epsilon \phi_y}{v}\{(p_v - \bar{p}_v) v_{\delta x}^r \\
&+ (p_\theta - \bar{p}_\theta)\theta_{\delta x}^r\} + \frac{\epsilon^2 \phi_y}{v^3}(v_{\delta x}^r u_{\delta x}^r - v u_{\delta xx}^r),
\end{aligned} \tag{6.14}
$$

where $p = p(v, \theta), \bar{p} = p(v_\delta^r, \theta_\delta^r)$. Integrating (6.14) with respect to τ and y over $[0, \tau] \times \mathbb{R}^1$ $(\tau \leq \tau_0)$ yields

$$
\begin{aligned}
&\int_{\mathbb{R}^1} (\frac{\phi_y^2}{2v^2} - \frac{\psi\phi_y}{v})(\tau) dy + \int_0^\tau \int_{\mathbb{R}^1} \phi_y^2 dy d\tau \\
\leq & c \int_0^\tau \int_{\mathbb{R}^1} (R_4 + R_5 + R_6) dy d\tau,
\end{aligned}
$$

where

$$
\begin{cases}
R_4 = \psi_y^2 + |\phi_y \zeta_y| + \epsilon |\phi_y \psi_y v_{\delta x}^r|, \\
R_5 = \epsilon\{(|\phi\phi_y| + |\zeta\phi_y|)(|v_{\delta x}^r| + |\theta_{\delta x}^r|) + |\psi\phi_y u_{\delta x}^r| + |\psi\psi_y v_{\delta x}^r|\}, \\
R_6 = \epsilon^2 |\phi_y|(|v_{\delta x}^r u_{\delta x}^r| + |u_{\delta xx}^r|).
\end{cases}
$$

Since the estimate for R_5 is similar to that for R_1, it suffices to estimate R_4 and R_6. For R_4,

$$
\int_0^\tau \int_{\mathbb{R}^1} \phi_y \zeta_y dy d\tau \leq \gamma \int_0^\tau \int_{\mathbb{R}^1} \phi_y^2 dy d\tau + c(\gamma) \int_0^\tau \int_{\mathbb{R}^1} \zeta_y^2 dy d\tau,
$$

and

$$
\begin{aligned}
\int_0^\tau \int_{\mathbb{R}^1} \epsilon |\phi_y \psi_y v_{\delta x}^r| dy d\tau &\leq \epsilon \|v_{\delta x}^r\|_{L_{t,x}^\infty} \int_0^\tau \int_{\mathbb{R}^1} (\phi_y^2 + \psi_y^2) dy d\tau \\
&\leq c\frac{\epsilon}{\delta} \int_0^\tau \int_{\mathbb{R}^1} (\phi_y^2 + \psi_y^2) dy d\tau.
\end{aligned}
$$

For R_6,

$$
\epsilon^2 \int_0^\tau \int_{\mathbb{R}^1} |\phi_y|(|v_{\delta x}^r u_{\delta x}^r| + |u_{\delta xx}^r|) dy d\tau \leq \epsilon \int_0^\tau \int_{\mathbb{R}^1} \phi_y^2 dy d\tau + c\frac{\epsilon}{\delta^3}.
$$

The above inequalities yield

$$\begin{aligned}&\int_{\mathbb{R}^1}(\frac{\phi_y^2}{2v^2}-\frac{\psi\phi_y}{v}(\tau))dy+\int_0^\tau\int_{\mathbb{R}^1}\phi_y^2dyd\tau\\ \le& c\int_0^\tau\int_{\mathbb{R}^1}\{\psi_y^2+\zeta_y^2+\epsilon(\phi^2+\psi^2+\zeta^2)\}dyd\tau+c\frac{\epsilon}{\delta^3}.\end{aligned}\tag{6.15}$$

Since $\eta(v,u,\theta;v_\delta^r,u_\delta^r,\theta_\delta^r)\ge c(\phi^2+\psi^2+\zeta^2)$ for some positive constant c, then a suitable linear combination of (6.13) and (6.15) implies that

$$\begin{aligned}&\|(\phi,\psi,\zeta)(\tau,\cdot)\|^2+\int_0^\tau\|(\phi_y,\psi_y,\zeta_y,\sqrt{\epsilon u_{\delta x}^r q_1})(s)\|^2ds\\ \le& c\epsilon\int_0^\tau\int_{\mathbb{R}^1}(\phi^2+\psi^2+\zeta^2)dyd\tau+c\frac{\epsilon}{\delta^3},\end{aligned}\tag{6.16}$$

We conclude from (6.16) by using the classical Gronwall inequality that for $\tau\in[0,\tau_0]$,

$$\|(\phi,\psi,\zeta)(\tau,\cdot)\|^2+\int_0^\tau\|(\phi_y,\psi_y,\zeta_y,\sqrt{\epsilon u_{\delta x}^r q_1})(s)\|^2ds\le c\frac{\epsilon}{\delta^3}.\tag{6.17}$$

Hence, the estimate (6.9) holds due to the definition of q_1, which gives $q_1\ge c(\phi^2+\zeta^2)$ for some constant $c(>0)$. Next, we derive the basic energy estimate in large time, (6.10), by modifying the procedure for the finite time. Note that for R_1,

$$\begin{aligned}&\epsilon\int_{\tau^0}^\tau\int_{\mathbb{R}^1}|\zeta\psi_y u_{\delta x}^r|dyd\tau\\ \le&\epsilon^{1/4}\int_{\tau^0}^\tau\int_{\mathbb{R}^1}\psi_y^2dyd\tau+\epsilon^{7/4}\int_{\tau^0}^\tau\int_{\mathbb{R}^1}\zeta^2(u_{\delta x}^r)^2dyd\tau\\ \le&\epsilon^{1/4}\int_{\tau^0}^\tau\|\psi_y\|^2d\tau+c\epsilon^{3/4}\int_{\tau^0}^\tau\|\zeta\|\|\zeta_y\|\|u_{\delta x}^r\|_{L_x^2}^2d\tau\\ \le&c\epsilon^{1/4}\int_{\tau^0}^\tau\|\phi_y,\zeta_y\|^2d\tau+c\epsilon^{1/4}\sup_{\tau^0\le s\le\tau}\|\zeta(s)\|^2.\end{aligned}\tag{6.18}$$

For R_2 and R_3,

$$\begin{aligned}&\epsilon^2\int_{\tau^0}^\tau\int_{\mathbb{R}^1}|\psi|((u_{\delta x}^r)^2+|u_{\delta xx}^r|)dyd\tau\\ \le&c\epsilon\int_{\tau^0}^\tau\|\psi\|^{1/2}\|\psi_y\|^{1/2}(\|u_{\delta x}^r\|_{L_x^2}^2+\|u_{\delta xx}^r\|_{L_x^1})d\tau\\ \le&\gamma\int_{\tau^0}^\tau\int_{\mathbb{R}^1}\psi_y^2dyd\tau+c(\gamma)\epsilon^{4/3}\int_{\tau_0}^\tau\|\psi\|^{2/3}t^{-4/3}d\tau\\ \le&\gamma\int_{\tau^0}^\tau\int_{\mathbb{R}^1}\psi_y^2dyd\tau+\gamma\sup_{\tau^0\le s\le\tau}\|\psi(s)\|^2+c(\gamma)\epsilon^{1/2}.\end{aligned}\tag{6.19}$$

Finally, for R_6,

$$
\begin{aligned}
&\epsilon^2\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}|\phi_y|(|v^r_{\delta x}u^r_{\delta x}|+|u^r_{\delta xx}|)dyd\tau\\
\le&\epsilon\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\phi_y^2dyd\tau+\epsilon^3\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}(|v^r_{\delta x}u^r_{\delta x}|+|u^r_{\delta xx}|)^2dyd\tau\\
\le&\epsilon\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\phi_y^2dyd\tau+c\frac{\epsilon}{\delta}.
\end{aligned}
$$

Then we have obtained the basic energy estimate for the large time, (6.10). This completes the proof of the lemma. □

Lemma 6.3 *(Derivative estimate) Suppose that the assumptions in Proposition 6.1 hold, then*

$$
\sup_{0\le\tau\le\tau_1}\|(\phi_y,\psi_y,\zeta_y)(\tau)\|^2+\int_0^{\tau_1}\|(\psi_{yy},\zeta_{yy})(\tau)\|^2d\tau\le c\epsilon^{3/4}. \tag{6.20}
$$

Proof The derivative estimate, (6.20), will be derived in two time intervals. We start with the estimate in finite time ($0\le\tau\le\tau_0\le\tau_1$). Multiplying $(6.1)_2$ by $-\psi_{yy}$, $\partial_y(6.1)_3$ by $e_\theta(v^r_\delta,\theta^r_\delta)\zeta_y/\theta^r_\delta$, respectively, adding the resulting equations and using $(6.1)_1$, we have

$$
\begin{aligned}
&\frac{1}{2}\{|\bar p_v|\phi_y^2+\psi_y^2+\frac{\bar e_\theta}{\theta^r_\delta}\zeta_y^2\}_\tau+\frac{\psi_{yy}^2}{v}+\frac{\kappa\bar e_\theta}{\epsilon v\theta^r_\delta e_\theta}\zeta_{yy}^2\\
=&q_{2y}\psi_{yy}-\epsilon[\phi(\bar p_v)_x+\zeta(\bar p_\theta)_x]_y\psi_y-\epsilon[(\bar p_v)_t\frac{\phi_y^2}{2}-(\frac{\bar e_\theta}{\theta^r_\delta})_t\frac{\zeta_y^2}{2}\\
&+\frac{\bar e_\theta}{\theta^r_\delta}(\frac{\theta^r_\delta\bar p_\theta}{\bar e_\theta})_x\psi_y\zeta_y]+\epsilon^2(\frac{vu^r_{\delta xx}-v^r_{\delta x}u^r_{\delta x}}{v^2})_y\psi_y+\frac{\epsilon}{v^2}(v^r_{\delta x}\psi_y\\
&+u^r_{\delta x}\phi_y)\psi_{yy}+\frac{\phi_y\psi_y\psi_{yy}}{v^2}-\frac{\kappa}{v^2e_\theta}(\frac{\bar e_\theta}{\theta^r_\delta})_x\zeta_y[v\zeta_{yy}-\phi_y\zeta_y\\
&-\epsilon(v^r_{\delta x}\zeta_y+\theta^r_{\delta x}\phi_y)+\epsilon^2(v\theta^r_{\delta xx}-v^r_{\delta x}\theta^r_{\delta x})]+\epsilon^2[\frac{\kappa\bar e_\theta}{\epsilon v^2\theta^r_\delta e_\theta}(v\theta^r_{\delta xx}\\
&-v^r_{\delta x}\theta^r_{\delta x})]_y\zeta_y+\frac{\kappa\bar e_\theta}{\epsilon v^2\theta^r_\delta e_\theta}\zeta_{yy}[\epsilon(\theta^r_{\delta x}\phi_y+v^r_{\delta x}\zeta_y)+\phi_y\zeta_y]\\
&+\frac{\bar e_\theta}{\theta^r_\delta}(\psi_y\zeta_{yy}-\epsilon^2u^r_{\delta xx}\zeta_y)q_3-\epsilon\frac{\bar e_\theta}{\theta^r_\delta}u^r_{\delta x}\zeta_yq_{3y}+\epsilon(\frac{\bar e_\theta}{\theta^r_\delta})_x\psi_y\zeta_yq_3\\
&-\frac{\bar e_\theta}{ve_\theta\theta^r_\delta}\psi_y\zeta_{yy}(\psi_y+2\epsilon u^r_{\delta x})+\epsilon^2[\frac{\bar e_\theta}{ve_\theta\theta^r_\delta}(u^r_{\delta x})^2]_y\zeta_y\\
&-\frac{\epsilon}{ve_\theta}(\frac{\bar e_\theta}{\theta^r_\delta})_x\zeta_y[\psi_y^2+2\epsilon u^r_{\delta x}\psi_y+\epsilon^2(u^r_{\delta x})^2]+(\cdots)_y,
\end{aligned}
\tag{6.21}
$$

where

$$\begin{cases} e \equiv e(v,\theta), p \equiv p(v,\theta), \bar{e} \equiv e(v_\delta^r,\theta_\delta^r), \bar{p} \equiv p(v_\delta^r,\theta_\delta^r) \\ q_2 \equiv p - \bar{p} - \bar{p}_v\phi - \bar{p}_\theta\zeta = O(1)(\phi^2+\zeta^2), \\ q_3 \equiv \dfrac{\theta p_\theta}{e_\theta} - \dfrac{\theta_\delta^r \bar{p}_\theta}{\bar{e}_\theta} = O(1)(|\phi|+|\zeta|). \end{cases}$$

Then integrating (6.21) with respect to τ and y over $[0,\tau]\times\mathbb{R}^1$ $(\tau \le \tau_0)$ leads to

$$\|(\phi_y,\psi_y,\zeta_y)(\tau)\|^2 + \int_0^\tau \|(\psi_{yy},\zeta_{yy})(s)\|^2 ds \le c\int_0^\tau \int_{\mathbb{R}^1} \sum_{i=0}^3 h_i dy d\tau,$$

where

$$\begin{aligned} h_0 = &(|\phi|+|\zeta|)[|\psi_y\zeta_{yy}| + (|\phi_y|+|\zeta_y|)|\psi_{yy}|] + |\phi_y\psi_y\psi_{yy}| \\ &+ (|\phi_y\zeta_y| + \psi_y^2)|\zeta_{yy}| + (\phi^2+\zeta^2)(|\phi_y|+|\zeta_y|)|\psi_{yy}|, \end{aligned}$$

$$\begin{aligned} h_1 = &\epsilon\{(|v_{\delta t}^r| + |\theta_{\delta t}^r|)(\phi_y^2+\zeta_y^2) + (|v_{\delta x}^r| + |\theta_{\delta x}^r|)(|\psi_y\zeta_y| + |\phi_y\psi_y|) \\ &+ u_{\delta x}^r(\zeta_y^2 + |\zeta_y\phi_y|) + (|u_{\delta x}^r\phi_y| + |v_{\delta x}^r\psi_y|)|\psi_{yy}| + [|\theta_{\delta x}^r\phi_y| + |u_{\delta x}^r\psi_y| \\ &+ (|v_{\delta x}^r| + |\theta_{\delta x}^r|)|\zeta_y|]|\zeta_{yy}| + (|\phi|+|\zeta|)[(|v_{\delta x}^r| + |\theta_{\delta x}^r|)|\psi_y\zeta_y| + |u_{\delta x}^r|(|\phi_y\zeta_y| \\ &+ \zeta_y^2)] + (\phi^2+\zeta^2)(|v_{\delta x}^r| + |\theta_{\delta x}^r|)|\psi_{yy}| + (|v_{\delta x}^r| + |\theta_{\delta x}^r|)(|\phi_y|\zeta_y^2 + \psi_y^2|\zeta_y|)\}, \end{aligned}$$

$$\begin{aligned} h_2 = &\epsilon^2\{(|\phi|+|\zeta|)[((|v_{\delta x}^r| + |\theta_{\delta x}^r|)|u_{\delta x}^r| + |u_{\delta xx}^r|)|\zeta_y| + (|v_{\delta xx}^r| + |v_{\delta x}^r\theta_{\delta x}^r| \\ &+ |\theta_{\delta xx}^r|)|\psi_y|] + (|v_{\delta x}^r| + |\theta_{\delta x}^r|)(|v_{\delta x}^r|\zeta_y^2 + |\theta_{\delta x}^r\phi_y\zeta_y| + |u_{\delta x}^r\psi_y\zeta_y|) \\ &+ |u_{\delta xx}^r\phi_y\psi_y| + (|\theta_{\delta xx}^r| + |v_{\delta x}^r\theta_{\delta x}^r| + (u_{\delta x}^r)^2)(|\phi_y\zeta_y| + \zeta_y^2)\}, \end{aligned}$$

$$\begin{aligned} h_3 = &\epsilon^3\{[(|v_{\delta x}^r| + |\theta_{\delta x}^r|)(|\theta_{\delta xx}^r| + |v_{\delta x}^r\theta_{\delta x}^r| + (u_{\delta x}^r)^2) + |\theta_{\delta xxx}^r| + |(v_{\delta x}^r\theta_{\delta x}^r)_x| \\ &+ |u_{\delta x}^r u_{\delta xx}^r|]|\zeta_y| + (|u_{\delta xx}^r v_{\delta x}^r| + |u_{\delta xxx}^r| + |(v_{\delta x}^r u_{\delta x}^r)_x|)|\psi_y|\}. \end{aligned}$$

Since many terms in $h_i (i = 0,\cdots,3)$ can be estimated similarly, so we only treat the typical terms. For h_0,

$$\begin{aligned} &\int_0^\tau \int_{\mathbb{R}^1} |\phi\phi_y\psi_{yy}| dy d\tau \\ \le &\int_0^\tau \int_{\mathbb{R}^1} (\gamma\psi_{yy}^2 + c(\gamma)\phi^2\phi_y^2) dy d\tau \\ \le &\gamma\int_0^\tau \int_{\mathbb{R}^1} \psi_{yy}^2 dy d\tau + c(\gamma)\int_0^\tau \|\phi\|\|\phi_y\|^3 d\tau \\ \le &\gamma\int_0^\tau \int_{\mathbb{R}^1} \psi_{yy}^2 dy d\tau + c(\gamma)\sup_{0\le s\le\tau}\|\phi(s)\|\|\phi_y(s)\| \int_0^\tau \int_{\mathbb{R}^1} \phi_y^2 dy \\ \le &\gamma\int_0^\tau \int_{\mathbb{R}^1} \psi_{yy}^2 dy d\tau + \gamma\sup_{0\le s\le\tau}\|\phi_y(s)\|^2 + c(\gamma)(\int_0^\tau \int_{\mathbb{R}^1} \phi_y^2 dy)^2 \sup_{0\le s\le\tau}\|\phi(s)\|^2, \end{aligned}$$

and

$$
\begin{aligned}
&\int_0^\tau\int_{\mathbb{R}^1}(|\phi_y\psi_y\psi_{yy}|+|\phi^2\phi_y\psi_{yy}|)dyd\tau\\
\le&\int_0^\tau\int_{\mathbb{R}^1}(\gamma\psi_{yy}^2+c(\gamma)(\phi_y^2\psi_y^2+\phi^4\phi_y^2))dyd\tau\\
\le&\gamma\int_0^\tau\int_{\mathbb{R}^1}\psi_{yy}^2dyd\tau+c(\gamma)\int_0^\tau\|\phi_y\|^2(\|\psi_y\|\|\psi_{yy}\|+\|\phi\|^2\|\phi_y\|^2)d\tau\\
\le&\ 2\gamma\int_0^\tau\int_{\mathbb{R}^1}\psi_{yy}^2dyd\tau+c(\gamma)\nu_1^2\int_0^\tau\|(\phi_y,\psi_y)(\tau)\|^2d\tau\sup_{0\le s\le\tau}\|\phi_y(s)\|^2.
\end{aligned}
$$

For h_1,

$$
\begin{aligned}
&\epsilon\int_0^\tau\int_{\mathbb{R}^1}(|v_{\delta x}^r|\phi_y^2+|u_{\delta x}^r\phi_y\psi_{yy}|+|v_{\delta x}^r\phi\psi_y\zeta_y|)dyd\tau\\
\le&c\epsilon(1+\nu_1)\|(v_\delta^r,u_\delta^r)\|_{L^\infty_{t,x}}\int_0^\tau\int_{\mathbb{R}^1}(\psi_{yy}^2+\phi_y^2+\psi_y^2+\zeta_y^2)dyd\tau\\
\le&c(1+\nu_1)\frac{\epsilon}{\delta}\int_0^\tau\int_{\mathbb{R}^1}(\psi_{yy}^2+\phi_y^2+\psi_y^2+\zeta_y^2)dyd\tau,
\end{aligned}
$$

$$
\begin{aligned}
&\epsilon\int_0^\tau\int_{\mathbb{R}^1}|v_{\delta x}^r\phi^2\psi_{yy}|dyd\tau\\
&\le\gamma\int_0^\tau\int_{\mathbb{R}^1}\psi_{yy}^2dyd\tau+c(\gamma)\epsilon^2\int_0^\tau\|\phi\|^2\|\phi_y\|^2(\int_{\mathbb{R}^1}(v_{\delta x}^r)^2dy)d\tau\\
&\le\gamma\int_0^\tau\int_{\mathbb{R}^1}\psi_{yy}^2dyd\tau+c(\gamma)\nu_1^2\frac{\epsilon}{\delta}\int_0^\tau\|\phi_y\|^2d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
\epsilon\int_0^\tau\int_{\mathbb{R}^1}|v_{\delta x}^r\phi_y\zeta_y^2|dyd\tau&\le\epsilon\int_0^\tau\|\zeta_y\|\|\zeta_{yy}\|\int_{\mathbb{R}^1}|v_{\delta x}^r\phi_y|dyd\tau\\
&\le c\epsilon^{1/2}\|v_{\delta x}^r\|_{L_t^\infty L_x^2}\int_0^\tau\|\phi_y\|\|\zeta_y\|\|\zeta_{yy}\|d\tau\\
&\le\nu_1\int_0^\tau\|\zeta_{yy}\|^2d\tau+c\nu_1\frac{\epsilon}{\delta}\int_0^\tau\|\phi_y\|^2.
\end{aligned}
$$

For h_2,

$$
\begin{aligned}
&\epsilon^2\int_0^\tau\int_{\mathbb{R}^1}\{(|v_{\delta x}^r u_{\delta x}^r|+|u_{\delta xx}^r|)|\phi\zeta_y|+(v_{\delta x}^r)^2\zeta_y^2+|\theta_{\delta xx}^r\phi_y\zeta_y|\}dyd\tau\\
\le&(\epsilon+\frac{\epsilon^2}{\delta^2})\int_0^\tau\int_{\mathbb{R}^1}(\phi_y^2+\zeta_y^2)dyd\tau+\epsilon^3\int_0^\tau\int_{\mathbb{R}^1}\phi^2(|v_{\delta x}^r u_{\delta x}^r|+|u_{\delta xx}^r|)^2dyd\tau\\
\le&(\epsilon+\frac{\epsilon^2}{\delta^2})\int_0^\tau\int_{\mathbb{R}^1}(\phi_y^2+\zeta_y^2)dyd\tau+\epsilon^3\|(v_{\delta x}^r u_{\delta x}^r,u_{\delta xx}^r)\|_{L^\infty_{t,x}}^2\int_0^\tau\|\phi\|^2d\tau\\
\le&(\epsilon+\frac{\epsilon^2}{\delta^2})\int_0^\tau\int_{\mathbb{R}^1}(\phi_y^2+\zeta_y^2)dyd\tau+c\frac{\epsilon^2}{\delta^4}\sup_{0\le s\le\tau}\|\phi(s)\|^2.
\end{aligned}
$$

Finally, for h_3,

$$\begin{aligned}
&\epsilon^3\int_0^\tau\int_{\mathbb{R}^1}(|v^r_{\delta x}|^2|\theta^r_{\delta x}|+|v^r_{\delta x}\theta^r_{\delta xx}|+|\theta^r_{\delta xxx}|)|\zeta_y|dyd\tau\\
\le&\epsilon\int_0^\tau\int_{\mathbb{R}^1}\zeta_y^2dyd\tau+\epsilon^5\int_0^\tau\int_{\mathbb{R}^1}(|v^r_{\delta x}|^2|\theta^r_{\delta x}|+|v^r_{\delta x}\theta^r_{\delta xx}|+|v^r_{\delta xxx}|)^2dyd\tau\\
\le&\epsilon\int_0^\tau\int_{\mathbb{R}^1}\zeta_y^2dyd\tau+c\frac{\epsilon^3}{\delta^5}.
\end{aligned}$$

These estimates, together with the basic energy estimate in finite time (6.9) and the a priori assumption $\delta\ge k_1^{-1}\epsilon^{1/4}$, give

$$\sup_{0\le\tau\le\tau_0}\|(\phi_y,\psi_y,\zeta_y)(\tau)\|^2+\int_0^{\tau_0}\|(\psi_{yy},\zeta_{yy})(\tau)\|^2d\tau\le c\epsilon^{3/4}. \qquad (6.22)$$

Next, we derive the derivative estimate in large time ($\tau^0\le\tau\le\tau_1$) by following the same procedure as for finite time. Note that for h_2,

$$\begin{aligned}
&\epsilon^2\int_{\tau^0}^\tau\int_{\mathbb{R}^1}(|v^r_{\delta x}u^r_{\delta x}|+|u^r_{\delta xx}|)|\phi\zeta_y|dyd\tau\\
\le&\epsilon\int_{\tau^0}^\tau\int_{\mathbb{R}^1}\zeta_y^2dyd\tau+\epsilon^2\int_{\tau^0}^\tau(\|(v^r_{\delta x},u^r_{\delta x})\|^4_{L^4}+\|v^r_{\delta xx}\|^2)\|\phi\|\|\phi_y\|d\tau\\
\le&\epsilon\int_{\tau^0}^\tau\int_{\mathbb{R}^1}\zeta_y^2dyd\tau+\epsilon\sup_{\tau^0\le s\le\tau}\|\phi(s)\|^2\int_{\tau^0}^\tau\|\phi_y(\tau)\|^2d\tau+c\frac{\epsilon^2}{\delta}
\end{aligned}$$

and for h_3,

$$\begin{aligned}
&\epsilon^3\int_{\tau^0}^\tau\int_{\mathbb{R}^1}(|v^r_{\delta x}|^2|\theta^r_{\delta x}|+|v^r_{\delta x}\theta^r_{\delta xx}|+|\theta^r_{\delta xxx}|)|\zeta_y|dyd\tau\\
\le&\epsilon\int_{\tau^0}^\tau\int_{\mathbb{R}^1}\zeta_y^2dyd\tau+\epsilon^5\int_{\tau^0}^\tau\int_{\mathbb{R}^1}(|v^r_{\delta x}|^2|\theta^r_{\delta x}|+|v^r_{\delta x}\theta^r_{\delta xx}|+|v^r_{\delta xxx}|)^2dyd\tau\\
\le&\epsilon\int_{\tau^0}^\tau\int_{\mathbb{R}^1}\zeta_y^2dyd\tau+c\frac{\epsilon^3}{\delta^3}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\sup_{\tau^0\le\tau\le\tau_1}\|(\phi_y,\psi_y,\zeta_y)(\tau)\|^2+\int_{\tau^0}^{\tau_1}\|(\psi_{yy},\zeta_{yy})(\tau)\|^2d\tau\\
\le&c(\|(\phi_y,\psi_y,\zeta_y)(\tau^0)\|^2+\epsilon^{3/4})
\end{aligned} \qquad (6.23)$$

The desired estimate (6.20) follows by choosing $\tau_0=\tau^0$. □

Remark In fact, (6.21) can be computed much easier to estimate (6.20). The reason that we do such a complicated calculation here is to derive the convergence rate in the following subsection.

6.2 Proof of Theorem 3.2

The problem of inviscid limit to smooth flow is well understood. The main interest here is to obtain a rate of convergence in the viscosity which is valid uniformly for all time, when we specialize to smooth rarefaction waves of the Euler equations. We first prove a finite time result which justifies the vanishing viscosity method for a fairly large class of smooth flows on any given fixed time interval $[0,T]$ with $T < +\infty$, which, in particular, yields Theorem 3.2 on $[0,T]$. We then complete the proof of Theorem 3.2 by deriving a large time a priori estimate.

6.2.1 Smooth Flows in Finite Time

Let $0 < T < +\infty$, and $(V,U,\Theta)(t,x)$ be a smooth solution to the Euler equation $(5.5)_{1,2,4}$ on $[0,T] \times \mathbb{R}^1$ with initial data

$$(V,U,\Theta)(0,x) = (V_0,U_0,\Theta_0)(x)$$

satisfying

$$\sup_{0\le t\le T} \sum_{1\le l\le 3} \int_{\mathbb{R}^1} |\partial_x^l (V,U,\Theta)|^2 dx \le c_0 < +\infty, \tag{6.24}$$

and

$$\inf_{0\le t\le T, x\in\mathbb{R}^1} V(t,x) \ge \underline{v} > 0, \quad \inf_{0\le t\le T, x\in\mathbb{R}^1} \Theta(t,x) \ge \underline{\theta} > 0,$$

for some positive constants c_0, $\underline{v}$ and $\underline{\theta}$. Our following theorem asserts that $(V,U,\Theta)(t,x)$ is a strong limits as $\epsilon \to 0$, of the viscous solution $(v,u,\theta)(t,x)$ to the Navier-Stokes equations $(2.8)_{1,2}$, (2.10) with the same initial data

$$(v,u,\theta)(0,x) = (V_0,U_0,\Theta_0)(x). \tag{6.25}$$

Theorem 6.4 *Let $(V,U,\Theta)(t,x)$ be a smooth Euler solution as described above. Then there exist positive constants ϵ_2 and $c(T)$ such that for each $\epsilon \in (0,\epsilon_2]$, the Cauchy problem for the Navier-Stokes equations* $(2.8)_{1,2}$, (2.10) *and* (6.25) *has a unique smooth solution $(v,u,\theta)(t,x)$ for $t \in [0,T]$ such that*

$$\sup_{0\le t\le T} \|(v-V, u-U, \theta-\Theta)(t,\cdot)\|_{L^\infty} \le c(T)\epsilon. \tag{6.26}$$

To prove this theorem, we consider the initial value problem (6.1) with the initial data

$$(\phi_0,\psi_0,\zeta_0)(y) = 0. \tag{6.27}$$

It is easy to see that the above theorem follows immediately from the following a priori estimate.

Proposition 6.5 *Suppose that the initial value problem* (6.1) *and* (6.27) *has a solution* $(\phi, \psi, \zeta)(\tau, y)$ *in* $X(0, \tau_1)$ *for some* $(0 < \tau_1 \leq \epsilon^{-1}T$*. There exist positive constants* ϵ_2, ν_2 *and* $c(T)$*, independent of* ϵ *and* τ_1*, such that if*

$$\epsilon \in (0, \epsilon_2], \sup_{0 \leq s \leq \tau_1} \|(\phi, \psi, \zeta)(s)\|_1^2 \leq \nu_2 \tag{6.28}$$

for small ϵ_2 *and* ν_2*, then for* $\tau \in (0, \tau_1]$*,*

$$\sup_{0 \leq s \leq \tau} \|(\phi, \psi, \zeta)(s)\|^2 + \int_0^\tau \|(\phi_y, \psi_y, \zeta_y)(s)\|^2 ds \leq c(T)\epsilon \tag{6.29}$$

and

$$\sup_{0 \leq s \leq \tau} \|(\phi_y, \psi_y, \zeta_y)(s)\|^2 + \int_0^\tau \|(\psi_{yy}, \zeta_{yy})(s)\|^2 ds \leq c(T)\epsilon^3. \tag{6.30}$$

Proof First, note that the smallness of ν_2 in (6.28) guarantees that $V+\phi \geq \underline{v}/2, \Theta+\zeta \geq \underline{\theta}/2$. (6.29) can be derived in a similar way to (6.9) by taking into account the fact that $|U_x| \leq c_0$. Although U_x has no sign here, the term $\epsilon \int\int U_x q_1 dy d\tau$ is bounded by $c\epsilon \int\int (\phi^2 + \zeta^2) dy d\tau$, which can be put into the right hand side of (6.16). (6.30) can be derived similarly as for (6.22). Following the derivation step by step, then we get

$$\begin{aligned}&\sup_{0 \leq s \leq \tau} \|(\phi_y, \psi_y, \zeta_y)(s)\|^2 + \int_0^\tau \|(\psi_{yy}, \zeta_{yy})(s)\|^2 ds \\ \leq & c\epsilon \int_0^\tau \|(\phi_y, \psi_y, \zeta_y)(s)\|^2 ds + c(T)\epsilon^3\end{aligned}$$

based on (6.29). This together with a classical Gronwall inequality yields the desired estimate (6.30).□

6.2.2 Smooth Rarefaction Waves in Large Time

Let us now turn to smooth rarefaction waves. Applying Theorem 6.4 to be the smooth rarefaction wave $(v^R, u^R, \theta^R)(t, x)$ implies Theorem 3.2 on any finite time interval. To complete the proof of Theorem 3.2, we need only show the following large time a priori estimate. In the rest of this section, $(\phi, \psi, \zeta)(\tau, y) = (v - v^R, u - u^R, \theta - \theta^R)(t, x)$.

Proposition 6.6 *Suppose that the Cauchy problem of the Navier-Stokes equations has a solution* $(v, u, \theta)(t, x)$ *as in Theorem 3.2, which is defined on* $[0, T_1] \times \mathbb{R}^1$ $(1 \leq T_1 \leq +\infty)$ *and with the regularity as stated in Theorem 3.2. There exist small positive constants* ϵ_3, ν_3 *and* c*, which are independent of* ϵ *and* τ_1*, such that if*

$$\epsilon \in (0, \epsilon_3], \sup_{\tau^0 \leq \tau \leq \tau_1} \|(\phi, \psi, \zeta)(\tau)\|_1^2 \leq \nu_3$$

with $\tau^0 = \epsilon^{-1}$ *and* $\tau_1 = \epsilon^{-1}T_1$ *for small* ϵ_3 *and* ν_3*, then the following estimates hold*

$$\begin{aligned}&\sup_{\tau^0\le\tau\le\tau_1}\|(\phi,\psi,\zeta)(\tau)\|^2+\epsilon\int_{\tau^0}^{\tau_1}\int_{\mathbb{R}^1}u_x^R(\phi^2+\zeta^2)dyd\tau\\&\quad+\int_{\tau^0}^{\tau_1}\|(\phi_y,\psi_y,\zeta_y)(\tau)\|^2d\tau\le c(\|(\phi,\psi,\zeta)(\tau^0)\|_1^2+\epsilon^{1/2}),\end{aligned}\tag{6.31}$$

$$\begin{aligned}&\sup_{\tau^0\le\tau\le\tau_1}\|(\phi_y,\psi_y,\zeta_y)(\tau)\|^2+\int_{\tau^0}^{\tau_1}\|(\psi_{yy},\zeta_{yy})(\tau)\|^2d\tau\\&\qquad\le c\|(\phi_y,\psi_y,\zeta_y)(\tau^0)\|^2+\epsilon^{3/2}).\end{aligned}\tag{6.32}$$

Proof By virtue of Lemma 5.3, the proposition can be proved by modifying the proof of lemmas in Section 6.1. First, (6.31) follows from the derivation of (6.10) and the properties of $(v^R,u^R,\theta^R)(t,x)$ in Lemma 5.3. Next, applying the argument for the estimate (6.23) and using the estimates (6.29), (6.30) and (6.31), we obtain the desired estimate (6.32).

7 Proof of Theorems 4.1 and 4.2 for Boltzmann Equation

In order to prove Theorems 4.1 and 4.2 on the zero mean free path limit to rarefaction waves for Boltzmann Equation, we need the decomposition, which decomposes a Boltzmann solution into its macroscopic (fluid) part and microscopic (kinetic) part, and the celebrated H-theorem.

7.1 Macro-micro decomposition

The collision operator has five collision invariants (cf. [4]),

$$\begin{cases}\psi_0(\xi)\equiv 1,\\ \psi_i(\xi)\equiv \xi, \qquad \text{for} \quad i=1,2,3,\\ \psi_4(\xi)\equiv |\xi|^2/2\end{cases}$$

satisfying

$$\int\psi_\alpha(\xi)Q(f,f)d\xi=0,\ \text{for any function } f \text{ and } \alpha=0,\cdots,4.$$

The inner product in $\xi\in\mathbb{R}^3$ with respect to the local Maxwellian M is defined by

$$<h,g>\equiv\int_{\mathbb{R}^3}\frac{1}{M}h(\xi)g(\xi)d\xi$$

for functions h, g of ξ such that the above integral is well defined. With respect to this inner product, the following functions spanning the space of macroscopic, i.e. fluid components of the solution, are pairwise orthogonal:

$$
\begin{cases}
\chi_0(\xi;\rho,u,\theta) \equiv \dfrac{1}{\sqrt{\rho}} M, \\
\chi_i(\xi;\rho,u,\theta) \equiv \dfrac{\xi_i - u_i}{\sqrt{R\rho\theta}} M, \text{ for } i = 1,2,3, \\
\chi_4(\xi;\rho,u,\theta) \equiv \dfrac{1}{\sqrt{6\rho}} (\dfrac{|\xi-u|^2}{R\theta} - 3) M, \\
< \chi_i, \chi_j >= \delta_{ij}, \text{ for } i,j = 0, \cdots, 4.
\end{cases}
$$

The macroscopic projection P_0 and microscopic projection P_1 can be defined as:

$$
P_0 h \equiv \sum_{j=0}^{4} < h, \chi_j > \chi_j, \quad P_1 h \equiv h - P_0 h.
$$

The operators P_0 and P_1 are orthogonal (and thus self-adjoint) projections for the inner product$< \cdot, \cdot >$. A function $h(\xi)$ is called microscopic, or non-fluid, if it has no fluid components, i.e.,

$$
\int_{\mathbb{R}^3} h(\xi)\psi_j(\xi) d\xi = 0, \quad \text{for } j = 0, \cdots, 4.
$$

The solution of the Boltzmann equation $f(t,x,\xi)$ is decomposed into the macroscopic (fluid) part, i.e. the local Mawellian $M = M(t,x,\xi) = M_{[\rho,u,\theta]}$ and the microscopic (non-fluid) part, i.e. $G = G(t,x,\xi)$:

$$
f(t,x,\xi) = M(t,x,\xi) + G(t,x,\xi), \quad P_0 f = M, \ P_1 f = G.
$$

The Boltzmann equation (2.1) hence becomes

$$
(M+G)_t + \xi_1 (M+G)_x = \frac{2}{\epsilon} Q(M,G) + \frac{1}{\epsilon} Q(G,G), \tag{7.1}
$$

which is equivalent to the following fluid-type system (see [20, 22] for details):

$$
\begin{cases}
\rho_t + (\rho u_1)_x = 0, \\
(\rho u_1)_t + (\rho u_1^2 + p)_x = -\displaystyle\int \xi_1^2 G_x d\xi, \\
(\rho u_i)_t + (\rho u_1 u_i)_x = -\displaystyle\int \xi_1 \xi_i G_x d\xi, \quad i = 2,3, \\
[\rho(e + \frac{1}{2}|u|^2)]_t + [u_1(\rho(e + \frac{1}{2}|u|^2) + p)]_x = -\displaystyle\int \frac{1}{2} \xi_1 |\xi|^2 G_x d\xi,
\end{cases} \tag{7.2}
$$

together with the equation for the non-fluid part G:

$$G_t + P_1(\xi_1 M_x) + P_1(\xi_1 G_x) = \frac{1}{\epsilon}(2Q(M,G) + Q(G,G)). \qquad (7.3)$$

Consider the coordinate transformation (2.4). We will still denote the Lagrangian coordinates by (t,x) for simplicity of notations. The system (7.2) in the Lagrangian coordinates becomes

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = -\displaystyle\int_{\mathbb{R}^3} \xi_1^2 G_x d\xi, \\ u_{it} = -\displaystyle\int_{\mathbb{R}^3} \xi_1 \xi_i G_x d\xi, \quad i = 2,3, \\ (e + \dfrac{1}{2}|u|^2)_t + (pu_1)_x = -\displaystyle\int_{\mathbb{R}^3} \frac{1}{2}\xi_1 |\xi|^2 G_x d\xi \end{cases} \qquad (7.4)$$

or more precisely the form of (2.6) (cf. [1, 2] for details), and the non-fluid equation (7.3) in the Lagrangian coordinates becomes (2.7). It follows from (2.7) that

$$G = \frac{\epsilon}{v} L_M^{-1}(P_1(\xi_1 M_x)) + \Xi \qquad (7.5)$$

with

$$\Xi = L_M^{-1}[\epsilon G_t + \frac{\epsilon}{v} P_1(\xi_1 G_x) - \frac{\epsilon u_1}{v} G_x - Q(G,G)] \qquad (7.6)$$

Here L_M is the linearized collision operator around the local Maxwellian M:

$$L_M h = L_{[\rho,u,\theta]} h = 2Q(h, M)$$

and the null space $\mathcal{N}$ of L_M is spanned by the macroscopic variables: $\chi_j, \quad j = 0, \cdots, 4$. Furthermore, there exists a positive constant $\sigma_0(\rho,u,\theta) > 0$ such that for any function $h(\xi) \in \mathcal{N}^\perp$, cf. [11],

$$< h, L_M h > \leq -\sigma_0(\rho,u,\theta) < (1 + |\xi|)h, h > \qquad (7.7)$$

In the above presentation, we have normalized the gas constant R to be $2/3$ for simplicity so that $e = \theta$ and $p = 2\theta/(3v)$. Notice also that the viscosity coefficient $\mu(\theta) > 0$ and the heat conductivity coefficient $\kappa(\theta) > 0$ are smooth functions of the temperature θ. The entropy $s = 2(\ln v)/3 + \ln(4\pi\theta/3) + 1$ is constant across the Euler rarefaction waves. And the temperature $\theta(v,s) = 3v^{-2/3}\exp(s)/(4e\pi)$ satisfies the following equations

$$\theta_t + pu_{1x} = -\int (\frac{|\xi|^2}{2} - u \cdot \xi)\xi_1 G_x d\xi \qquad (7.8)$$

or

$$
\begin{aligned}
\theta_t + pu_{1x} =& \epsilon(\frac{\kappa(\theta)}{v}\theta_x)_x + \epsilon\frac{4\mu(\theta)}{3v}u_{1x}^2 + \epsilon\sum_{i=2}^{3}\frac{\mu(\theta)}{v}u_{ix}^2 \\
& - \int(\frac{|\xi|^2}{2} - u\cdot\xi)\xi_1\Xi_x d\xi.
\end{aligned}
\tag{7.9}
$$

Since the approximate rarefaction waves for the Boltzmann equation $M_{[V,U,\Theta]}$ are not sufficiently accurate for the energy method, we have to subtract from $G(t,x,\xi)$ the term $\overline{G}(t,x,\xi)$:

$$
\overline{G}(t,x,\xi) = \epsilon\frac{L_M^{-1}\{P_1[\xi_1(\frac{|\xi-u|^2}{2\theta}\Theta_x + \xi U_x)M]\}}{Rv\theta}, \tag{7.10}
$$

which is the first term in the Chapman-Enskog expansion, cf. (7.5).

For later use, we list some basic properties of the projections P_0, P_1 and the linearized collision operator L_M as follows (cf. [22]):

$$
\begin{cases}
P_0(\chi_j M) = \chi_j M,\ P_1(\chi_j M) = 0,\ j = 0,1,2,3,4, \\
L_M P_1 = P_1 L_M = L_M,\ P_1(Q(h,h)) = Q(h,h), \\
L_M(P_0) = P_0 L_M = 0,\ P_0(Q(h,h)) = 0, \\
<\chi_j M, h> = <\chi_j M, P_0 h>,\ j = 0,1,2,3,4, \\
<h, L_M g> = <P_1 h, L_M(P_1 g)>, \\
<h, L_M^{-1}(P_1 g)> = <L_M^{-1}(P_1 h), P_1 g> = <P_1 h, L_M^{-1}(P_1 g)>.
\end{cases}
$$

7.2 H-theorem

We list the following basic lemmas based on the celebrated H-theorem for later use. The first lemma is from [10].

Lemma 7.1 *There exists a positive constant $C > 0$ such that*

$$
\begin{aligned}
\int_{\mathbb{R}^3}\frac{(1+|\xi|)^{-1}Q(f,g)^2}{M}d\xi \leq& C\{\int_{\mathbb{R}^3}\frac{(1+|\xi|)f^2}{M}d\xi\int_{\mathbb{R}^3}\frac{g^2}{M}d\xi \\
& + \int_{\mathbb{R}^3}\frac{f^2}{M}d\xi\int_{\mathbb{R}^3}\frac{(1+|\xi|)g^2}{M}d\xi\}
\end{aligned}
$$

where M can be any Maxwellian so that the above integrals are well defined.

Based on Lemma 7.1, the following three lemmas are proved in [21].

Lemma 7.2 *If $\theta/2 < \theta_* < \theta$, then there exist two positive constants $\sigma = \sigma(\rho, u, \theta; \rho_*, u_*, \theta_*) > 0$ and $\vartheta = \vartheta(\rho, u, \theta; \rho_*, u_*, \theta_*) > 0$ such that if $|\rho - \rho_*| + |u - u_*| + |\theta - \theta_*| < \vartheta$, it holds that for $h(\xi) \in \mathcal{N}^\perp$*

$$-\int_{\mathbb{R}^3} \frac{hL_M h}{M_*} d\xi \geq \sigma \int_{\mathbb{R}^3} \frac{(1+|\xi|)h^2}{M_*} d\xi,$$

where $M_* = M_{[\rho_*, u_*, \theta_*]}$ and the definition of $M_{[\rho,u,\theta]}$ can be found in (2.3).

Corollary 7.3 *Under the assumptions in Lemma 7.2, the following estimates hold*

$$\int_{\mathbb{R}^3} \frac{1+|\xi|}{M} |L_M^{-1} h|^2 d\xi \leq \sigma^{-2} \int_{\mathbb{R}^3} \frac{(1+|\xi|)^{-1} h^2}{M} d\xi,$$
$$\int_{\mathbb{R}^3} \frac{1+|\xi|}{M_*} |L_M^{-1} h|^2 d\xi \leq \sigma^{-2} \int_{\mathbb{R}^3} \frac{(1+|\xi|)^{-1} h^2}{M_*} d\xi$$

for each $h(\xi) \in \mathcal{N}^\perp$.

Lemma 7.4 *Under the conditions in Lemma 7.2, there exists a constant $C > 0$ such that for positive constants k and β,*

$$\Big|\int_{\mathbb{R}^3} \frac{g_1 P_1(|\xi|^k g_2)}{M_*} d\xi - \int_{\mathbb{R}^3} \frac{g_1 |\xi|^k g_2}{M_*} d\xi\Big| \leq C \int_{\mathbb{R}^3} \frac{\beta |g_1|^2 + \beta^{-1} |g_2^2|}{M_*} d\xi.$$

7.3 Reformulation of the problem

In order to prove Theorem 4.1 and Theorem 4.2, we set

$$(\phi, \psi, \zeta, \widetilde{G})(\tau, y) \equiv (v - V, u - U, \theta - \Theta, G - \overline{G})(t, x)$$

with

$$\tau = \epsilon^{-1} t, y = \epsilon^{-1} x,$$

where the macroscopic parts (v, u, θ) and the microscopic component G of the Boltzmann solution f satisfy the fluid-type system $(7.4)_{1,2,3,4}$, (7.8) or $(2.6)_{1,2,3,4}$, (7.9) and the non-fluid equation (2.7), respectively. The macroscopic ansatz (V, U, Θ) satisfy the Euler equations $(5.5)_{1,2,4}$ and the microscopic ansatz $\overline{G}$ is defined by (7.10). Then $(\phi, \psi, \zeta)(\tau, y)$ and $\widetilde{G}(\tau, y)$ satisfy the following equations

$$\begin{cases} \phi_\tau - \psi_{1y} = 0, \\ \psi_{1\tau} + (p(v,\theta) - p(V,\Theta))_y = -\displaystyle\int \xi_1^2 G_y d\xi, \\ \psi_{i\tau} = -\displaystyle\int \xi_1 \xi_i G_y d\xi, \quad i = 2, 3, \\ \zeta_\tau + p(v,\theta)\psi_{1y} + (p(v,\theta) - p(V,\Theta))U_{1y} \\ \qquad = -\displaystyle\int (\frac{|\xi|^2}{2} - u \cdot \xi) \xi_1 G_y d\xi \end{cases} \tag{7.11}$$

or

$$
\begin{cases}
\phi_\tau - \psi_{1y} = 0, \\
\psi_{1\tau} + (p(v,\theta) - p(V,\Theta))_y = (\frac{4\mu(\theta)}{3v} u_{1y})_y - \int \xi_1^2 \Xi_y d\xi, \\
\psi_{i\tau} = (\frac{\mu(\theta)}{v} u_{iy})_y - \int \xi_1 \xi_i \Xi_y d\xi, \qquad i = 2,3, \\
\zeta_\tau + p(v,\theta)\psi_{1y} + (p(v,\theta) - p(V,\Theta))U_{1y} = (\frac{\kappa(\theta)}{v}\theta_y)_y \\
\quad + \frac{4\mu(\theta)}{3v} u_{1y}^2 + \sum_{i=2}^3 \frac{\mu(\theta)}{v} u_{iy}^2 - \int (\frac{|\xi|^2}{2} - u\cdot\xi)\xi_1 \Xi_y d\xi,
\end{cases}
\tag{7.12}
$$

and

$$
\begin{aligned}
\widetilde{G}_\tau - L_M \widetilde{G} = & -\frac{1}{R\theta v} P_1[\xi_1(\frac{|\xi - u|^2}{2\theta}\zeta_y + \xi\psi_y)M] + \frac{u_1}{v} G_y \\
& - \frac{1}{v} P_1(\xi_1 G_y) + Q(G,G) - \epsilon \overline{G}_t.
\end{aligned}
\tag{7.13}
$$

We seek a global (in time) solution f to the Cauchy problem of the Boltzmann equation (2.5) with the initial data $f(0,x,\xi) = M_{[V,U,\Theta]}(0,x,\xi)$. To do so, we define the function space for the difference

$$
g(\tau, y, \xi) = f(t,x,\xi) - M_{[V,U,\Theta]}(t,x,\xi)
$$

to be

$$
\begin{aligned}
\widehat{X}(0,\tau_1) \equiv \{ g(\tau,y,\xi) | & \frac{\partial^\alpha g}{\sqrt{M_*}} \in C^0(0,\tau_1; L^2_{y,\xi}(\mathbb{R}^1 \times \mathbb{R}^3), \\
& \frac{\sqrt{1+|\xi|}\partial^\alpha g}{\sqrt{M_*}} \in L^2(0,\tau_1; L^2_{y,\xi}(\mathbb{R}^1 \times \mathbb{R}^3), |\alpha| \le 2 \}
\end{aligned}
$$

with the differential operator $\partial^\alpha = \partial^{(\alpha_0,\alpha_1)} = \partial_\tau^{\alpha_0}\partial_y^{\alpha_1}$, $|\alpha| = \alpha_0 + \alpha_1$, where α_0 and α_1 are non-negative integers. We set also for $\tau_0 < \tau_1$

$$
\begin{aligned}
N^2(\tau_0,\tau_1) \equiv & \sup_{\tau_0 \le \tau \le \tau_1} \{ \|(\phi,\psi,\zeta)(\tau)\|^2 + \sum_{|\alpha|=1} \|\partial^\alpha (v,u,\theta)(\tau)\|^2 \\
& + \int_{\mathbb{R}^1} \int [\frac{\widetilde{G}^2(\tau)}{M_*} + \sum_{|\alpha|=1} \frac{(\partial^\alpha G)^2(\tau)}{M_*} + \sum_{|\alpha|=2} \frac{(\partial^\alpha f)^2(\tau)}{M_*}] d\xi dy \}.
\end{aligned}
$$

Note that for the monatomic gas, the normalized entropy $\eta(v,u,\theta;V,U,\Theta)$ defined by (6.3) in Section 6 can be written as

$$
\eta(\tau,y) \equiv \frac{2}{3}\Theta\Phi(\frac{v}{V}) + \frac{1}{2}(u-U)^2 + \Theta\Phi(\frac{\theta}{\Theta})
$$

with $\Phi(s) = s - \ln s - 1$. Direct computations give that

$$
\begin{aligned}
&\eta_\tau + \epsilon q_1 U_{1x} + \frac{\mu(\theta)\Theta}{v\theta}(|\psi_y|^2 + \frac{1}{3}\psi_{1y}^2) + \frac{\kappa(\theta)\Theta}{v\theta^2}\zeta_y^2 \\
=&\epsilon[\zeta(\frac{\kappa'(\theta)}{v\theta}\Theta_x\zeta_y - \frac{\kappa(\theta)}{v^2\theta}\Theta_x\phi_y + \frac{\kappa(\theta)}{v^2\theta}\Theta_x\zeta_y + \frac{8\mu(\theta)}{3v\theta}U_{1x}\psi_{1y}) \\
&+\psi_1(\frac{4\mu'(\theta)}{3v}U_{1x}\zeta_y - \frac{4\mu(\theta)}{3v^2}U_{1x}\phi_y)] + \epsilon^2[\zeta\frac{\kappa(\theta)}{v\theta}\Theta_{xx} \\
&+\psi_1\frac{4\mu(\theta)}{3v}U_{1xx}] + \epsilon^2[\zeta(\frac{\kappa'(\theta)\Theta_x^2}{v\theta} - \frac{\kappa(\theta)\Theta_x V_x}{v^2\theta} + \frac{4\mu(\theta)}{3v\theta}U_{1x}^2) \\
&+\psi_1(\frac{4\mu'(\theta)}{3v}U_{1x}\Theta_x - \frac{4\mu(\theta)}{3v^2}U_{1x}V_x)] + \int[(\frac{|\xi|^2}{2} - \xi\cdot u)\frac{\Theta}{\theta^2}\zeta_y \\
&+\frac{\Theta}{\theta}\xi\cdot\psi_y]\xi_1\Xi d\xi - \epsilon\zeta\int[\frac{\xi\cdot U_x}{\theta} + (\frac{|\xi|^2}{2} - \xi\cdot u)\frac{\Theta_x}{\theta^2}]\xi_1\Xi d\xi \\
&+(\cdots)_y
\end{aligned}
\tag{7.14}
$$

with $q_1 = p(v,s) - p(V,S) - p_v(V,S)(v-V) - p_s(V,S)(s-S) \geq 0$ by the convexity of $p(v,s) = v^{-5/3}\exp(s)/(2e\pi)$.

7.4 Proof of Theorem 4.1

In this subsection, we prove Theorem 4.1. Thus, we consider the Cauchy problem of the Boltzmann equation (2.5) with the initial data

$$
f(0,x,\xi) \equiv f_0(x,\xi) = M_{[v_\delta^r, u_\delta^r, \theta_\delta^r]}(0,x,\xi), \tag{7.15}
$$

where $(v_\delta^r, u_{1\delta}^r, \theta_\delta^r)$ is the approximation rarefaction wave given by (5.4) and $u_\delta^r = (u_{1\delta}^r, 0, 0)^t$. Due to the smoothness of $(v_\delta^r, u_\delta^r, \theta_\delta^r)$, the local existence of (2.5) and (7.15) is standard (cf. [12]), the global existence and the estimate (4.2) will follow from the following a priori estimate. Throughout this subsection, $(V, U, \Theta)(t,x) = (v_\delta^r, u_\delta^r, \theta_\delta^r)(t,x)$.

Proposition 7.5 *(A priori estimate) Suppose that the Cauchy problem (2.5) and (7.15) has a solution f with $(f - M_{[v_\delta^r, u_\delta^r, \theta_\delta^r]}) \in \widehat{X}(0,\tau_1)$ for some $\tau_1 > 0$. There exist positive constants $\epsilon_1, \delta_1, k_1, \nu_1$ and c, independent of ϵ, δ and τ_1, such that if*

$$
0 < \epsilon \leq \epsilon_1,\ 0 < \delta \leq \delta_1,\ \epsilon^{1/4} \leq k_1\delta,\ N^2(0,\tau_1) \leq \nu_1^2 \tag{7.16}
$$

for small ϵ_1 *and* ν_1, *then*

$$
\begin{aligned}
&N^2(0,\tau_1)+\int_0^{\tau_1}\{\epsilon\|\sqrt{u^r_{\delta 1x}}(\phi,\zeta)\|^2+\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2\\
&+\sum_{|\alpha|=2}\|\partial^\alpha(v,u,\theta)\|^2+\int_{\mathbb{R}^1}\int[\frac{(1+|\xi|)\widetilde{G}^2}{M_*}\\
&+\sum_{1\le|\alpha|\le 2}\frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*}]d\xi dy\}d\tau\le c(\frac{\epsilon}{\delta^3}+\epsilon^{1/2}).
\end{aligned}
\tag{7.17}
$$

Once Proposition 7.5 is proved, we can take $\delta=k_1^{-1}\epsilon^{1/5}$, then (7.17) implies that there exists a positive constant c independent of ϵ such that

$$
\sup_{0\le\tau\le+\infty,y\in\mathbb{R}^1}\{|(\phi,\psi,\zeta)(\tau,y)|+\|G(\tau,y,\xi)\|_{L^2_\xi(\frac{1}{\sqrt{M_*}})}\}\le c\epsilon^{1/5}.
$$

Therefore, the Boltzmann equation (2.5) has a global solution $f(t,x,\xi)$ satisfying

$$
\|f(t,x,\xi)-M_{[v^r_\delta,u^r_\delta,\theta^r_\delta]}(t,x,\xi)\|_{L^\infty_x L^2_\xi(\frac{1}{\sqrt{M_*}})}\le c\epsilon^{1/5},
$$

for all $t\in(0,+\infty)$. By (3) in Lemma 5.2, we have for $t>0$,

$$
\|(v^r_\delta,u^r_\delta,\theta^r_\delta)(t,\cdot)-(v^r,u^r,\theta^r)(t,\cdot)\|_{L^\infty}\le ct^{-1}\epsilon^{1/5}(\ln(1+t)+|\ln\epsilon|).
$$

Thus, combining the above two estimates yields the desired estimate (4.2), and the proof of theorem 4.1 is complete.

Now, we turn to the proof of Proposition 7.5. The bound for $N^2(0,\tau_1)$ yields the following $L^\infty_\tau L^2_y$ and $L^\infty_{(\tau,y)}$ estimates by the sobolev imbedding theorem:

$$
\begin{aligned}
&\sum_{|\alpha|=2}\sup_{\tau\in[0,\tau_1]}\int_{\mathbb{R}^1}\int\frac{(\partial^\alpha M)^2(\tau)+(\partial^\alpha G)^2(\tau)}{M_*}d\xi dy\\
&+\sup_{\tau\in[0,\tau_1],y\in\mathbb{R}^1}\{|(\phi,\psi,\zeta)|+\int\frac{\widetilde{G}^2}{M_*}d\xi+\sum_{|\alpha|=1}(|\partial^\alpha(v,u,\theta)|\\
&+\int\frac{(\partial^\alpha G)^2}{M_*}d\xi)\}(\tau,y)\le c(\nu_1+\sqrt{\epsilon/\delta})\equiv c\bar{\nu},
\end{aligned}
\tag{7.18}
$$

for some small constant $\bar{\nu}$, independent of ϵ,τ_1. Note that

$$
\int\frac{(1+|\xi|)G^2}{M}d\xi\le c\int\frac{G^2}{M_*}d\xi,\ \int\frac{(1+|\xi|)G_y^2}{M}d\xi\le c\int\frac{G_y^2}{M_*}d\xi \tag{7.19}
$$

for some constant $c(>0)$, independent of ϵ and τ_1. We should keep in mind that the above two estimates (7.18) and (7.19) give the smallness in the energy estimate. Notice also that the smallness of ν_1 and ϵ_1 guarantees that

$$v^r_\delta + \phi \geq v_-/2,\ \theta^r_\delta + \zeta \geq \hat{\theta}/2,\ \text{for } \hat{\theta} = \inf_{t\geq 0, x\in\mathbb{R}^1} \theta^r_\delta(t,x)$$

and the existence of M_* satisfying (4.1). Similar to the proof of Proposition 6.1, we will derive the energy estimate (7.17) in two time levels, $0 \leq t \leq T \leq 1$ and $1 \leq t \leq +\infty$. Set

$$\tau_0 = \epsilon^{-3/4}T,\ \tau^0 = \epsilon^{-3/4},$$

then $\tau_0 \leq \tau^0$. Moreover, $\tau_0 = \tau^0$ when $T = 1$. Proposition 7.5 can be proved by the following two lemmas with $\tau^0 = \tau_0$ in lemma 7.7.

Lemma 7.6 *(Finite time estimate) Suppose that the assumptions in Proposition 7.5 hold. Then for $\tau_0 \leq \tau_1$,*

$$\begin{aligned} &N^2(0,\tau_0) + \int_0^{\tau_0} \{\epsilon\|\sqrt{u^r_{\delta 1x}}(\phi,\zeta)\|^2 + \sum_{|\alpha|=1} \|\partial^\alpha(\phi,\psi,\zeta)\|^2 \\ &+ \sum_{|\alpha|=2} \|\partial^\alpha(v,u,\theta)\|^2 + \int_{\mathbb{R}^1}\int [\frac{(1+|\xi|)\widetilde{G}^2}{M_*} \\ &+ \sum_{1\leq|\alpha|\leq 2} \frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*}] d\xi dy\} d\tau \leq c\frac{\epsilon}{\delta^3}. \end{aligned} \tag{7.20}$$

Proof This lemma is proved in the following three steps.

Step 1. Basic Energy Estimate

First, integrating (7.14) with respect to τ and y over $[0,\tau]\times\mathbb{R}^1$ $(\tau \leq \tau_0)$ yields that

$$\begin{aligned} &\int_{\mathbb{R}^1} \eta(\tau) dy + \int_0^\tau \int_{\mathbb{R}^1} (\epsilon u^r_{1\delta x} q_1 + |\psi_y|^2 + \zeta_y^2) dy d\tau \\ \leq& c\int_0^\tau \int_{\mathbb{R}^1} (R_1 + R_2 + R_3 + R_4 + R_5) dy d\tau, \end{aligned} \tag{7.21}$$

where

$$\begin{cases} R_1 = \epsilon\{(|\zeta\psi_{1y}| + |\psi_1\phi_y| + |\psi_1\zeta_y|)|u^r_{\delta x}| + (|\zeta\zeta_y| + |\zeta\phi_y|)|\theta^r_{\delta x}|\}, \\ R_2 = \epsilon^2\{|\zeta|[|u^r_{\delta x}|^2 + |v^r_{\delta x}\theta^r_{\delta x}| + (\theta^r_{\delta x})^2] + |\psi_1 u^r_{\delta x}|(|v^r_{\delta x}| + |\theta^r_{\delta x}|)\}, \\ R_3 = \epsilon^2\{|\psi u^r_{\delta xx}| + |\zeta\theta^r_{\delta xx}|\}, \\ R_4 = |\psi_y \int \xi\xi_1 \Xi d\xi| + |\zeta_y \int (|\xi|^2 - 2\xi\cdot u)\xi_1 \Xi d\xi|, \\ R_5 = \epsilon|\zeta|\{|u^r_{\delta x}\int \xi\xi_1\Xi d\xi| + |\theta^r_{\delta x}\int (|\xi|^2 - 2\xi\cdot u)\xi_1\Xi d\xi|\}. \end{cases}$$

Since R_1, R_2 and R_3 can be estimated by (6.11) and (6.12) as in Section 6, then it suffices to treat R_4 and R_5. Note that for any polynomial $g(\xi)$,

$$|\int_{\mathbb{R}^3} g(\xi)\Xi d\xi|^2 \le (\int_{\mathbb{R}^3} g^2(\xi)M d\xi)(\int_{\mathbb{R}^3} \frac{\Xi^2}{M} d\xi) \le c\int_{\mathbb{R}^3} \frac{\Xi^2}{M} d\xi. \quad (7.22)$$

Then

$$\int_0^\tau \int_{\mathbb{R}^1} R_4 dy d\tau \le \int_0^\tau \int_{\mathbb{R}^1} \{\gamma(|\psi_y|^2+\zeta_y^2)+c(\gamma)\int \frac{\Xi^2}{M}d\xi\}dyd\tau \quad (7.23)$$

and

$$\int_{\tau_0}^\tau \int_{\mathbb{R}^1} R_5 dy d\tau \le \int_0^\tau \int_{\mathbb{R}^1} \{\epsilon^2(|u^r_{\delta x}|+|\theta^r_{\delta x}|)^2\zeta^2+c\int \frac{\Xi^2}{M}d\xi\}dyd\tau. \quad (7.24)$$

It follows from (7.6) and Corollary 7.3 that

$$\int \frac{\Xi^2}{M}d\xi \le c\int \frac{G_\tau^2+(1+|\xi|)G_y^2+(1+|\xi|)^{-1}Q^2(G,G)}{M}d\xi. \quad (7.25)$$

Thus the following basic energy estimate holds,

$$\begin{aligned}
&\int_{\mathbb{R}^1} \eta(\tau)dy + \int_0^\tau \int_{\mathbb{R}^1} (\epsilon u^r_{1\delta x} q_1 + |\psi_y|^2+\zeta_y^2)dyd\tau \\
\le& c\frac{\epsilon}{\delta^3} + c\int_0^\tau \int_{\mathbb{R}^1} \{\epsilon^{1/2}\phi_y^2 + \epsilon(\phi^2+\psi^2+\zeta^2)\}dyd\tau \\
&+c\int_0^\tau \int_{\mathbb{R}^1} \int \frac{G_\tau^2+(1+|\xi|)G_y^2+(1+|\xi|)^{-1}Q(G,G)^2}{M}d\xi dyd\tau.
\end{aligned} \quad (7.26)$$

It remains to estimate the microscopic component G and the double integral for $|\phi_y|^2$. Multiplying (7.13) by $\frac{\widetilde{G}}{M}$ and $\frac{\widetilde{G}}{M_*}$, and integrating the products over $[0,\tau]\times\mathbb{R}^1\times\mathbb{R}^3$ ($\tau\le\tau_0$), respectively, we obtain from (7.7) and Lemma 7.2 that

$$\begin{aligned}
&\int_{\mathbb{R}^1}\int \frac{\widetilde{G}^2(\tau)}{2M}dyd\tau + \sigma_0\int_0^\tau \int_{\mathbb{R}^1}\int \frac{(1+|\xi|)\widetilde{G}^2}{M}d\xi dyd\tau \\
\le& \int_0^\tau \int_{\mathbb{R}^1}\int \{\frac{\widetilde{G}}{M}[-\frac{1}{R\theta v}P_1[\xi_1(\frac{|\xi-u|^2}{2\theta}\zeta_y+\xi\psi_y)M]+\frac{u_1}{v}G_y \\
&-\frac{1}{v}P_1(\xi_1 G_y)+Q(G,G)-\epsilon\overline{G}_t] - \frac{\widetilde{G}^2}{2M^2}M_\tau\}d\xi dyd\tau
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^1}\int \frac{\widetilde{G}(\tau)^2}{2M_*}dyd\tau + \sigma\int_0^\tau \int_{\mathbb{R}^1}\int \frac{(1+|\xi|)\widetilde{G}^2}{M_*}d\xi dyd\tau \\
\le& \int_0^\tau \int_{\mathbb{R}^1}\int \frac{\widetilde{G}}{M_*}\{-\frac{1}{R\theta v}P_1[\xi_1(\frac{|\xi-u|^2}{2\theta}\zeta_y+\xi\psi_y)M]+\frac{u_1}{v}G_y \\
&-\frac{1}{v}P_1(\xi_1 G_y)+Q(G,G)-\epsilon\overline{G}_t\}d\xi dyd\tau.
\end{aligned}$$

From (7.10), (7.18), (7.19), Lemma 5.2 and Lemma 7.1, one may obtain

$$
\begin{aligned}
&\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)^{-1}Q^2(G,G)}{M_i}d\xi dyd\tau\\
\le& c\int_0^\tau\int_{\mathbb{R}^1}(\int\frac{(1+|\xi|)G^2}{M_i}d\xi)(\int\frac{G^2}{M_i}d\xi)dyd\tau\\
\le& c\int_0^\tau\int_{\mathbb{R}^1}(\int\frac{(1+|\xi|)(\overline{G}^2+\widetilde{G}^2)}{M_i}d\xi)(\int\frac{\overline{G}^2+\widetilde{G}^2}{M_i}d\xi)dyd\tau \qquad (7.27)\\
\le& c\bar{\nu}\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)\widetilde{G}^2}{M_i}d\xi dyd\tau+c\epsilon^3\int_0^\tau\|(u^r_{1\delta x},\theta^r_{\delta x})\|^4_{L^4_x}d\tau\\
\le& c\bar{\nu}\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)\widetilde{G}^2}{M_i}d\xi dyd\tau+c\frac{\epsilon^2}{\delta^3},
\end{aligned}
$$

with $M_i=M($ or $M_*)$. Note that for any h and polynomial $g(\xi)$,

$$
\int\frac{hg(\xi)}{M}d\xi=\int\frac{h}{M_*}\frac{M_*}{M}g(\xi)d\xi\le c\int\frac{h}{M_*}d\xi.
$$

Thus, together with Lemma 5.2, Lemma 7.4 and (7.27), shows

$$
\begin{aligned}
&\int_{\mathbb{R}^1}\int\frac{\widetilde{G}^2(\tau)}{M}dyd\tau+\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)\widetilde{G}^2}{M}d\xi dyd\tau\\
\le& c\int_0^\tau\int_{\mathbb{R}^1}(\frac{\epsilon^2}{\delta^2}\phi_y^2+|\psi_y|^2+\zeta_y^2+\int\frac{(1+|\xi|)G_y^2}{M}d\xi)dyd\tau \qquad (7.28)\\
&+c\bar{\nu}\int_0^\tau\int_{\mathbb{R}^1}\int\frac{\widetilde{G}^2}{M_*}d\xi dyd\tau+c\frac{\epsilon^2}{\delta^3}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^1}\int\frac{\widetilde{G}(\tau)^2}{M_*}dyd\tau+\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)\widetilde{G}^2}{M_*}d\xi dyd\tau\\
\le& c\int_0^\tau\int_{\mathbb{R}^1}(\frac{\epsilon^2}{\delta^2}\phi_y^2+|\psi_y|^2+\zeta_y^2+\int\frac{(1+|\xi|)G_y^2}{M_*}d\xi)dyd\tau+c\frac{\epsilon^2}{\delta^3}. \qquad (7.29)
\end{aligned}
$$

The estimate on the double integral of ϕ_y^2 is obtained from the coupling through the conservation laws. Indeed, multiplying $(7.11)_2$ by $-\phi_y$ and integrating the product over $[0,\tau]\times\mathbb{R}^1$ $(\tau\le\tau_0)$ lead to

$$
\begin{aligned}
&-\int_{\mathbb{R}^1}(\psi_1\phi_y)(\tau)dy+\int_0^\tau\int_{\mathbb{R}^1}\frac{p}{v}\phi_y^2dyd\tau=\int_0^\tau\int_{\mathbb{R}^1}\{\phi_y[\epsilon(p_v-\bar{p}_v)v^r_{\delta x}\\
&+\epsilon(p_\theta-\bar{p}_\theta)\theta^r_{\delta x}\ +p_\theta\zeta_y+\int\xi_1^2G_yd\xi]+\psi_{1y}^2\}dyd\tau.
\end{aligned}
$$

Direct calculations give

$$\begin{aligned}\int_0^\tau\int_{\mathbb{R}^1}\phi_y^2dyd\tau\le c\|(\psi_1\phi_y)(\tau)\|_{L^1}+c\int_0^\tau\int_{\mathbb{R}^1}\{\zeta_y^2+\psi_{1y}^2\\ +\int\frac{G_y^2}{M}d\xi+\epsilon^2[(v_{\delta x}^r)^2+(\theta_{\delta x}^r)^2](\phi^2+\zeta^2)\}dyd\tau.\end{aligned}\tag{7.30}$$

Similarly, the double integral of $|(\phi_\tau,\psi_\tau,\zeta_\tau)|^2$ can be estimated by (7.11),

$$\begin{aligned}\int_0^\tau\int_{\mathbb{R}^1}|(\phi_\tau,\psi_\tau,\zeta_\tau)|^2dyd\tau\le c\int_0^\tau\int_{\mathbb{R}^1}\{|(\phi_y,\psi_y,\zeta_y)|^2\\ +\int\frac{G_y^2}{M}d\xi+\epsilon^2[(v_{\delta x}^r)^2+(\theta_{\delta x}^r)^2](\phi^2+\zeta^2)\}dyd\tau.\end{aligned}\tag{7.31}$$

Due to (7.25) and (7.27), suitable linear combinations of (7.26), (7.28), (7.30) and (7.31) yield the following basic energy estimate

$$\begin{aligned}&\|(\phi,\psi,\zeta)(\tau)\|^2+\int_{\mathbb{R}^1}\int\frac{\widetilde{G}^2(\tau)}{M}dyd\tau+\int_0^\tau\int_{\mathbb{R}^1}\{\epsilon u_{\delta x}^r q_1\\ &+\int\frac{(1+|\xi|)\widetilde{G}^2}{M}d\xi+\sum_{|\alpha|=1}|\partial^\alpha(\phi,\psi,\zeta)|^2\}dyd\tau\\ \le& c\int_0^\tau\int_{\mathbb{R}^1}\{\epsilon(\phi^2+\psi^2+\zeta^2)+\sum_{|\alpha|=1}\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}d\xi\}dyd\tau\\ &+c\bar{\nu}\int_0^\tau\int_{\mathbb{R}^1}\int\frac{\widetilde{G}^2}{M_*}d\xi dyd\tau+c\|\phi_y(\tau)\|^2+c\frac{\epsilon}{\delta^3}.\end{aligned}\tag{7.32}$$

Step 2. Derivative estimate.

For $|\alpha|=1$, multiplying $\partial^\alpha(2.6)_i$ by $\partial^\alpha u_{i-1}(i=2,3,4)$, $\partial^\alpha(7.9)$ by $\partial^\alpha\theta/\theta$ respectively, adding up all the resulting equations and using $(2.6)_1$, one has

$$\begin{aligned}&\frac{1}{2}[\frac{p}{v}(\partial^\alpha v)^2+|\partial^\alpha u|^2+\frac{1}{\theta}(\partial^\alpha\theta)^2]_\tau+\frac{\mu(\theta)}{v}[|\partial^\alpha u_y|^2+\frac{1}{3}(\partial^\alpha u_{1y})^2]\\ &+\frac{\kappa(\theta)}{v\theta}(\partial^\alpha\theta_y)^2=(\frac{p}{2v})_\tau(\partial^\alpha v)^2+(\frac{1}{2\theta})_\tau(\partial^\alpha\theta)^2-\frac{\partial^\alpha p}{\theta}u_{1y}\partial^\alpha\theta\\ &-\partial^\alpha(\frac{\mu(\theta)}{v})(u_y\partial^\alpha u_y+\frac{1}{3}u_{1y}\partial^\alpha u_{1y})-[\partial^\alpha(\frac{\kappa(\theta)}{v}\theta_y)(\frac{\partial^\alpha\theta}{\theta})_y\\ &-\frac{\kappa(\theta)}{v\theta}(\partial^\alpha\theta_y)^2]+\frac{1}{\theta}\partial^\alpha[\frac{\mu(\theta)}{v}|u_y|^2+\frac{\mu(\theta)}{3v}(u_{1y})^2]\partial^\alpha\theta\\ &+(\frac{\partial^\alpha\theta}{\theta})_y\partial^\alpha(\int(\frac{|\xi|^2}{2}-u\cdot\xi)\xi_1\Xi d\xi)+(\partial^\alpha u_y)\int\xi\xi_1(\partial^\alpha\Xi)d\xi\\ &-(\frac{\partial^\alpha\theta}{\theta})\int\xi\xi_1\partial^\alpha(u_y\Xi)d\xi+(\cdots)_y\end{aligned}\tag{7.33}$$

Integrating (7.33) over $[0,\tau]\times\mathbb{R}^1$ ($\tau\le\tau_0$) yields

$$\begin{aligned}&\|\partial^\alpha(v,u,\theta)(\tau)\|^2+\int_0^\tau\|(\partial^\alpha u_y,\partial^\alpha\theta_y)(\tau)\|^2d\tau\\ \le&c\int_0^\tau\int_{\mathbb{R}^1}\{|\partial_\tau(v,\theta)|(\partial^\alpha v)^2+|\theta_\tau|(\partial^\alpha\theta)^2+|\partial^\alpha(v,\theta)|^2(|u_{1y}|\\ &+|u_y|^2+\theta_y^2)+(|\partial^\alpha u|^2+|\partial^\alpha\theta|^2)\int\frac{\Xi^2}{M}d\xi+\int\frac{(\partial^\alpha\Xi)^2}{M}d\xi\}dyd\tau.\end{aligned}$$

The macroscopic components (v,u,θ) can be estimated as follows

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}^1}|\theta_\tau|(\partial^\alpha v)^2dyd\tau\\ \le&c\int_0^\tau\int_{\mathbb{R}^1}\{\bar\nu(\partial^\alpha\phi)^2+|\partial^\alpha v_\delta^r|\zeta_\tau^2+|\partial^\alpha v_\delta^r|^2|\theta_{\delta y}^r|\}dyd\tau\\ \le&c(\bar\nu+\frac{\epsilon}{\delta})\int_0^\tau\int_{\mathbb{R}^1}\sum_{|\alpha|=1}|\partial^\alpha(\phi,\zeta)|^2dyd\tau+c\frac{\epsilon}{\delta^2}.\end{aligned}\tag{7.34}$$

To estimate the microscopic part Ξ, one first notes that the linearized operator L_M^{-1} satisfies, for any $h\in\mathcal{N}^\perp$,

$$\partial^\alpha(L_M^{-1}h)=L_M^{-1}(\partial^\alpha h)-2L_M^{-1}\{Q(L_M^{-1}h,\partial^\alpha M)\},\ |\alpha|=1$$

and the projection P_1 satisfies, for any h,

$$\partial^\alpha(P_1(\xi_1h))=P_1(\xi_1\partial^\alpha h)-\sum_{j=0}^4<\xi_1h,\chi_j>P_1(\partial^\alpha\chi_j).$$

Thus, it follows from (7.18), (7.19) and lemma 7.1 that

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)^{-1}Q^2(\partial^\alpha G,G)}{M}d\xi dyd\tau\\ \le&c\int_0^\tau\int_{\mathbb{R}^1}(\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*}d\xi)(\int\frac{(1+|\xi|)G^2}{M}d\xi)dyd\tau\\ \le&c\bar\nu\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*}d\xi dyd\tau.\end{aligned}\tag{7.35}$$

Then the following estimate holds

$$\begin{aligned}\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(\partial^\alpha\Xi)^2}{M}d\xi dyd\tau\le c\int_0^\tau\int_{\mathbb{R}^1}\{\sum_{|\alpha|=2}\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}d\xi\\ +\bar\nu[\sum_{|\alpha|=1}\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}d\xi+\int\frac{(1+|\xi|)\widetilde{G}^2}{M}d\xi]\}dyd\tau+c\frac{\epsilon^2}{\delta^3}.\end{aligned}$$

This analysis leads to the following basic estimate on derivatives

$$
\begin{aligned}
&\|\partial^\alpha(v,u,\theta)(\tau)\|^2+\int_0^\tau\|(\partial^\alpha u_y,\partial^\alpha\theta_y)\|^2d\tau\\
\le& c\int_0^\tau\int_{\mathbb{R}^1}\{\sum_{|\alpha|=2}\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}d\xi+\bar{\nu}\sum_{|\alpha|=1}[|\partial^\alpha(\phi,\psi,\zeta)|^2\\
&+\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}d\xi]+\bar{\nu}\int\frac{(1+|\xi|)\widetilde{G}^2}{M}d\xi\}dyd\tau+c\frac{\epsilon}{\delta^2}.
\end{aligned}
\tag{7.36}
$$

We now turn to estimate the derivatives of G. For $|\alpha|=1$, multiplying $\partial^\alpha(2.7)$ by $\partial^\alpha G/M$, $\partial^\alpha G/M_*$ and integrating the products over $[0,\tau]\times\mathbb{R}^1\times\mathbb{R}^3$ $(\tau\le\tau_0)$ respectively, one obtains from (7.7) and Lemma 7.2 that

$$
\begin{aligned}
&\int_{\mathbb{R}^1}\int\frac{(\partial^\alpha G)^2(\tau)}{2M}dyd\tau+\sigma_0\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}d\xi dyd\tau\\
&\le\int_0^\tau\int_{\mathbb{R}^1}\int\{\frac{\partial^\alpha G}{M}\{[\partial^\alpha(L_MG)-L_M\partial^\alpha G]+\partial^\alpha Q(G,G)\\
&-\partial^\alpha[\frac{1}{v}P_1(\xi_1G_y)-\frac{u_1}{v}G_y+\frac{1}{v}P_1(\xi_1M_y)]\}-\frac{(\partial^\alpha G)^2}{2M^2}M_\tau\}d\xi dyd\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^1}\int\frac{(\partial^\alpha G)^2(\tau)}{2M_*}dyd\tau+\sigma\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*}d\xi dyd\tau\\
&\le\int_0^\tau\int_{\mathbb{R}^1}\int\frac{\partial^\alpha G}{M_*}\{[\partial^\alpha(L_MG)-L_M\partial^\alpha G]+\partial^\alpha Q(G,G)\\
&-\partial^\alpha[\frac{1}{v}P_1(\xi_1G_y)-\frac{u_1}{v}G_y+\frac{1}{v}P_1(\xi_1M_y)]\}d\xi dyd\tau.
\end{aligned}
$$

Note that

$$
\partial^\alpha(L_MG)=L_M(\partial^\alpha G)+2Q(\partial^\alpha M,G),\ |\alpha|=1.
$$

Similar to the estimates of $\widetilde{G}$, it holds that

$$
\begin{aligned}
&\sum_{|\alpha|=1}\{\|\partial^\alpha G(\tau)\|^2_{L^2_yL^2_\xi(\frac{1}{\sqrt{M}})}+\int_0^\tau\|\sqrt{1+|\xi|}\partial^\alpha G\|^2_{L^2_yL^2_\xi(\frac{1}{\sqrt{M}})}d\tau\}\\
\le& c\int_0^\tau\int_{\mathbb{R}^1}\{\sum_{|\alpha|=2}[|\partial^\alpha(v,u,\theta)|^2+\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}d\xi]\\
&+\bar{\nu}\sum_{|\alpha|=1}[|\partial^\alpha(\phi,\psi,\zeta)|^2+\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*}d\xi]\\
&+\bar{\nu}\int\frac{(1+|\xi|)\widetilde{G}^2}{M}d\xi\}dyd\tau+c\frac{\epsilon^2}{\delta^3}
\end{aligned}
\tag{7.37}
$$

and

$$\sum_{|\alpha|=1}\{\|\partial^\alpha G(\tau)\|^2_{L^2_y L^2_\xi(\frac{1}{\sqrt{M_*}})}+\int_0^\tau \|\sqrt{1+|\xi|}\partial^\alpha G\|^2_{L^2_y L^2_\xi(\frac{1}{\sqrt{M_*}})}d\tau\}$$

$$\leq c\int_0^\tau\int_{\mathbb{R}^1}\{\sum_{|\alpha|=2}[|\partial^\alpha(v,u,\theta)|^2+\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*}d\xi] \tag{7.38}$$

$$+\bar{\nu}[\sum_{|\alpha|=1}|\partial^\alpha(\phi,\psi,\zeta)|^2+\int\frac{(1+|\xi|)\widetilde{G}^2}{M_*}d\xi]\}dyd\tau+c\frac{\epsilon^2}{\delta^3}.$$

Here one has used the following fact

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)^{-1}Q^2(\partial^\alpha G,G)}{M_*}d\xi dyd\tau\\ \leq& c\int_0^\tau\int_{\mathbb{R}^1}\{(\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*}d\xi)(\int\frac{G^2}{M_*}d\xi)\\ &+(\int\frac{(\partial^\alpha G)^2}{M_*}d\xi)(\int\frac{(1+|\xi|)G^2}{M_*}d\xi)\}dyd\tau\\ \leq& c\bar{\nu}\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)[\widetilde{G}^2+(\partial^\alpha G)^2]}{M_*}d\xi dyd\tau.\end{aligned}$$

The estimate for the double integral of $(\partial^\alpha v_y)^2$ is also obtained from the coupling through the conservation laws. Multiplying $\partial^\alpha(7.4)_2$ by $-\partial^\alpha v_y$ and integrating the product over $[0,\tau]\times\mathbb{R}^1$ ($\tau\leq\tau_0$), we obtain

$$\begin{aligned}&-\int_{\mathbb{R}^1}(\partial^\alpha u_1\cdot\partial^\alpha v_y)(s)dy\Big|_{s=0}^{s=\tau}+\int_0^\tau\int_{\mathbb{R}^1}\frac{p}{v}(\partial^\alpha v_y)^2dyd\tau\\ =&\int_0^\tau\int_{\mathbb{R}^1}\{(\partial^\alpha u_{1y})^2+\partial^\alpha v_y[(\partial^\alpha p_y-p_v\cdot\partial^\alpha v_y)+\int\xi_1^2\partial^\alpha G_y d\xi]\}dyd\tau\end{aligned}$$

This can be estimated directly to get

$$\begin{aligned}\int_0^\tau\int_{\mathbb{R}^1}(\partial^\alpha v_y)^2dyd\tau\leq& c\int_0^\tau\int_{\mathbb{R}^1}\{(\partial^\alpha u_{1y})^2+(\partial^\alpha\theta_y)^2\\ &+\sum_{|\alpha|=2}\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}d\xi+\bar{\nu}\sum_{|\alpha|=1}|\partial^\alpha(\phi,\psi,\zeta)|^2\}dyd\tau\\ &+c\|(\partial^\alpha u_1\cdot\partial^\alpha v_y)(\tau)\|_{L^1}+c\frac{\epsilon^2}{\delta^3}.\end{aligned}\tag{7.39}$$

Similarly, one can estimate the double integral of $|\partial^\alpha(v_\tau,u_\tau,\theta_\tau)|^2$ by $(7.4)_{1,2,3,4}$ and (7.8),

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}^1}|\partial^\alpha(v_\tau,u_\tau,\theta_\tau)|^2dyd\tau\leq c\int_0^\tau\int_{\mathbb{R}^1}\{|\partial^\alpha(v_y,u_y,\theta_y)|^2\\ &+\int\frac{(\partial^\alpha G_y)^2}{M}d\xi+\bar{\nu}[\sum_{|\alpha|=1}|\partial^\alpha(\phi,\psi,\zeta)|^2+\int\frac{G_y^2}{M}]\}dyd\tau+c\frac{\epsilon^2}{\delta^3}.\end{aligned}\tag{7.40}$$

The following estimate holds as a consequence of (7.36), (7.37), (7.39) and (7.40):

$$
\begin{aligned}
&\sum_{|\alpha|=1}\{\|\partial^\alpha(v,u,\theta)(\tau)\|^2+\int_{\mathbb{R}^1}\int\frac{(\partial^\alpha G)^2(\tau)}{M}d\xi dy\}\\
&+\sum_{|\alpha|=1}\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}d\xi dy d\tau\\
&+\sum_{|\alpha|=2}\int_0^\tau\int_{\mathbb{R}^1}|\partial^\alpha(v,u,\theta)|^2 dy d\tau\le c\sum_{|\alpha|=2}\|\partial^\alpha v(\tau)\|^2 \qquad (7.41)\\
&+c\int_0^\tau\int_{\mathbb{R}^1}\{\sum_{|\alpha|=2}\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}d\xi+\bar\nu\sum_{|\alpha|=1}[|\partial^\alpha(\phi,\psi,\zeta)|^2\\
&+\int\frac{(\partial^\alpha G)^2}{M_*}d\xi]+\bar\nu\int\frac{(1+|\xi|)\widetilde{G}^2}{M}d\xi\}dy d\tau+c\frac{\epsilon}{\delta^2}.
\end{aligned}
$$

Step 3. Higher order estimates.

For $|\alpha|=2$, multiplying $\partial^\alpha(2.5)$ by $\partial^\alpha f/M$, $\partial^\alpha f/M_*$ and integrating the products over $[0,\tau]\times\mathbb{R}^1\times\mathbb{R}^3$ ($\tau\le\tau_0$), respectively, we obtain from (7.7) and Lemma 7.2 that

$$
\begin{aligned}
&\int_{\mathbb{R}^1}\int\frac{|\partial^\alpha f(\tau)|^2}{2M}dy d\tau+\sigma_0\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)|\partial^\alpha G|^2}{M}d\xi dy d\tau\\
&\le\int_0^\tau\int_{\mathbb{R}^1}\int\{\frac{\partial^\alpha M}{M}L_M(\partial^\alpha G)-\frac{(\partial^\alpha f)^2}{2M}[\frac{M_\tau}{M}-(\frac{\xi_1-u_1}{v})_y\\
&+\frac{\xi_1-u_1}{v}\frac{M_y}{M}]\}d\xi dy d\tau+\int_0^\tau\int_{\mathbb{R}^1}\int\frac{\partial^\alpha f}{M}\{[\partial^\alpha(L_M G)-L_M(\partial^\alpha G)]\\
&+\partial^\alpha Q(G,G)-[\partial^\alpha(\frac{\xi_1-u_1}{v}f_y)-\frac{\xi_1-u_1}{v}\partial^\alpha f_y]\}d\xi dy d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^1}\int\frac{|\partial^\alpha f(\tau)|^2}{2M_*}dy d\tau+\sigma\int_0^\tau\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)|\partial^\alpha G|^2}{M_*}d\xi dy d\tau\\
&\le\int_0^\tau\int_{\mathbb{R}^1}\int\{\frac{\partial^\alpha M}{M_*}L_M(\partial^\alpha G)+\frac{(\partial^\alpha f)^2}{2M_*}(\frac{\xi_1-u_1}{v})_y\}d\xi dy d\tau\\
&+\int_0^\tau\int_{\mathbb{R}^1}\int\frac{\partial^\alpha f}{M_*}\{[\partial^\alpha(L_M G)-L_M\partial^\alpha G]+\partial^\alpha Q(G,G)\\
&-[\partial^\alpha(\frac{\xi_1-u_1}{v}f_y)-\frac{\xi_1-u_1}{v}\partial^\alpha f_y]\}d\xi dy d\tau.
\end{aligned}
$$

Note that

$$
\int_0^\tau\int_{\mathbb{R}^1}\int\frac{\partial^\alpha M}{M}L_M(\partial^\alpha G)d\xi dy d\tau=\int_0^\tau\int_{\mathbb{R}^1}\int\frac{P_1(\partial^\alpha M)}{M}L_M(\partial^\alpha G)d\xi dy d\tau
$$

and

$$\partial^\alpha(L_M G) = L_M(\partial^\alpha G) + \sum_{|\beta|=1} 2Q(\partial^{\alpha-\beta} M, \partial^\beta G) + 2Q(\partial^\alpha M, G), \ |\alpha| = 2.$$

Since $P_1(\partial^\alpha M)$ does not contain $\partial^\alpha(v,u,\theta)$, then we may get

$$\begin{aligned}
&\sum_{|\alpha|=2}\{\|\partial^\alpha f(\tau)\|^2_{L^2_y L^2_\xi(\frac{1}{\sqrt{M}})} + \int_0^\tau \|\sqrt{1+|\xi|}\partial^\alpha G\|^2_{L^2_y L^2_\xi(\frac{1}{\sqrt{M}})} d\tau\} \\
\leq & c\bar{\nu}\int_0^\tau \int_{\mathbb{R}^1} \{\sum_{|\alpha|=1} |\partial^\alpha(\phi,\psi,\zeta)|^2 + \sum_{|\alpha|=2} |\partial^\alpha(v,u,\theta)|^2 \\
&+ \sum_{1\leq|\alpha|\leq 2}\int \frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*} d\xi\} dy d\tau + c\frac{\epsilon^2}{\delta^3},
\end{aligned} \tag{7.42}$$

where (7.19) has been used. Notice that for $|\alpha| = 2$,

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{R}^1} (\int \frac{(\partial^\alpha M)^2}{M_*} d\xi \int \frac{(1+|\xi|)G^2}{M_*} d\xi) dy d\tau \\
\leq & \int_0^\tau \{\int_{\mathbb{R}^1}\int \frac{(\partial^\alpha M)^2}{M_*} d\xi dy (\int \frac{(1+|\xi|)G^2}{M_*} d\xi dy)^{1/2} \\
&\cdot (\int \frac{(1+|\xi|)G_y^2}{M_*} d\xi dy)^{1/2}\} d\tau \\
\leq & c\bar{\nu}\int_0^\tau \int_{\mathbb{R}^1}\int [\frac{(\partial^\alpha M)^2}{M_*} + \frac{(1+|\xi|)\widetilde{G}^2 + G_y^2}{M_*}] d\xi dy d\tau,
\end{aligned}$$

due to a priori assumption in (7.18). Then it holds that

$$\begin{aligned}
&\sum_{|\alpha|=2}\{\|\partial^\alpha f(\tau)\|^2_{L^2_y L^2_\xi(\frac{1}{\sqrt{M_*}})} + \int_0^\tau \|\sqrt{1+|\xi|}\partial^\alpha G\|^2_{L^2_y L^2_\xi(\frac{1}{\sqrt{M_*}})} d\tau\} \\
\leq & c\int_0^\tau \int_{\mathbb{R}^1} \{\sum_{|\alpha|=2} |\partial^\alpha(v,u,\theta)|^2 dy d\tau + \bar{\nu}\sum_{|\alpha|=1} [|\partial^\alpha(\phi,\psi,\zeta)|^2 \\
&+ \int \frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*} d\xi] + \bar{\nu}\int \frac{(1+|\xi|)\widetilde{G}^2}{M_*} d\xi\} dy d\tau + c\frac{\epsilon^2}{\delta^3}.
\end{aligned} \tag{7.43}$$

Suitable linear combinations of (7.32), (7.41) and (7.42) give

$$\begin{aligned}
&\|(\phi,\psi,\zeta)(\tau)\|^2+\sum_{|\alpha|=1}\|\partial^\alpha(v,u,\theta)(\tau)\|^2+\int_{\mathbb{R}^1}\int\{\frac{\widetilde{G}^2(\tau)}{M}\\
&+\sum_{|\alpha|=1}\frac{(\partial^\alpha G)^2(\tau)}{M}+\sum_{|\alpha|=2}\frac{(\partial^\alpha f)^2(\tau)}{M}\}d\xi dy+\int_0^\tau\int_{\mathbb{R}^1}\{\epsilon u^r_{\delta x}q_1\\
&+\sum_{|\alpha|=1}|\partial^\alpha(\phi,\psi,\zeta)|^2+\sum_{|\alpha|=2}|\partial^\alpha(v,u,\theta)|^2+\int[\frac{(1+|\xi|)\widetilde{G}^2}{M}\\
&+\sum_{1\le|\alpha|\le2}\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}]d\xi\}dyd\tau\le c\int_0^\tau\int_{\mathbb{R}^1}\{\epsilon(\phi^2+\psi^2\\
&+\zeta^2)+\bar{\nu}\int\frac{1+|\xi|}{M_*}[\widetilde{G}^2+\sum_{1\le|\alpha|\le2}(\partial^\alpha G)^2]d\xi\}dyd\tau+c\frac{\epsilon}{\delta^3},
\end{aligned}\tag{7.44}$$

where the following fact has been used:

$$\begin{aligned}
&\int_{\mathbb{R}^1}\int\frac{(\partial^\alpha f)^2}{M}d\xi dy=\int_{\mathbb{R}^1}\int\frac{(\partial^\alpha M)^2+2(\partial^\alpha M)(\partial^\alpha G)+(\partial^\alpha G)^2}{M}d\xi dy\\
=&\int_{\mathbb{R}^1}\int\frac{(P_0(\partial^\alpha M))^2+(P_1(\partial^\alpha M))^2+2P_1(\partial^\alpha M)(\partial^\alpha G)+(\partial^\alpha G)^2}{M}d\xi dy\\
\ge&\int_{\mathbb{R}^1}\int\frac{(P_0(\partial^\alpha M))^2}{M}d\xi dy.
\end{aligned}$$

Finally, we get the following estimate by appropriate linear combinations of the estimates (7.29), (7.38), (7.43) and (7.44)

$$\begin{aligned}
&N^2(0,\tau)+\int_0^\tau\int_{\mathbb{R}^1}\{\sum_{|\alpha|=1}|\partial^\alpha(\phi,\psi,\zeta)|^2+\sum_{|\alpha|=2}|\partial^\alpha(v,u,\theta)|^2\\
&+\int[\frac{(1+|\xi|)\widetilde{G}^2}{M_*}+\sum_{1\le|\alpha|\le2}\frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*}]d\xi+\epsilon u^r_{\delta x}q_1\}dyd\tau\\
&\le c\frac{\epsilon}{\delta^3}+c\epsilon\int_0^\tau\int_{\mathbb{R}^1}(\phi^2+\psi^2+\zeta^2)dyd\tau.
\end{aligned}\tag{7.45}$$

This, together with a classic Grownwall inequality, yields the desired estimate (7.20). □

Lemma 7.7 *(Large time estimate) Suppose that the assumptions in*

Proposition 7.5 hold, then

$$
\begin{aligned}
&N^2(\tau^0,\tau_1)+\int_{\tau^0}^{\tau_1}\{\epsilon\|\sqrt{u^r_{\delta 1x}}(\phi,\zeta)\|^2+\sum_{|\alpha|=1}\|\partial^\alpha(\phi,\psi,\zeta)\|^2\\
&+\sum_{|\alpha|=2}\|\partial^\alpha(v,u,\theta)\|^2+\int_{\mathbb{R}^1}\int[\frac{(1+|\xi|)\widetilde{G}^2}{M_*}\\
&+\sum_{1\le|\alpha|\le 2}\frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*}]d\xi dy\}d\tau\le c(N^2(\tau^0,\tau^0)+\epsilon^{1/2}).
\end{aligned}
\tag{7.46}
$$

Proof The procedure is similar to the one for the finite time estimate in Lemma 7.6. Thus we only point out the differences between them here. For the basic energy estimate, $R_i (i=1,2,3)$ in (7.21) are estimated by (6.18) and (6.19). Noting that

$$
\begin{aligned}
&\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)^{-1}Q^2(G,G)}{M_i}d\xi dy d\tau\\
\le& c\bar{\nu}\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)\widetilde{G}^2}{M_i}d\xi dy d\tau+c\epsilon^3\int_{\tau^0}^{\tau}\|(u^r_{1\delta x},\theta^r_{\delta x})\|^4_{L^4_x}d\tau\\
\le& c\bar{\nu}\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)\widetilde{G}^2}{M_i}d\xi dy d\tau+c\epsilon^2,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\tau^0}^{\tau}\int\int\frac{\epsilon^2\overline{G}_t^2}{M_i}d\xi dy d\tau\\
\le& c\bar{\nu}\int_{\tau^0}^{\tau}\int(|(\phi_y,\psi_y,\zeta_y)|^2+\int\frac{G_y^2}{M_i}d\xi)dy d\tau+c\frac{\epsilon^2}{\delta},
\end{aligned}
$$

one then can get the basic energy estimate in large time

$$
\begin{aligned}
&\sup_{\tau^0\le s\le\tau}\|(\phi,\psi,\zeta)(s)\|^2+\int_{\mathbb{R}^1}\int\frac{\widetilde{G}^2(\tau)}{M}dy d\tau+\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\{\epsilon u^r_{\delta x}q_1\\
&+\sum_{|\alpha|=1}|\partial^\alpha(\phi,\psi,\zeta)|^2+\int\frac{(1+|\xi|)\widetilde{G}^2}{M}d\xi\}dy d\tau\\
\le& cN^2(\tau^0,\tau^0)+c\|\phi_y(\tau)\|^2+c\epsilon^{1/2}+c\bar{\nu}\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\int\frac{\widetilde{G}^2}{M_*}d\xi dy d\tau\\
&+c\sum_{|\alpha|=1}\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\int\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}d\xi dy d\tau.
\end{aligned}
$$

To estimate the derivatives ($|\alpha|=1$), noting that

$$
\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}|\theta_\tau|(\partial^\alpha v)^2 dy d\tau\le c\bar{\nu}\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\sum_{|\alpha|=1}|\partial^\alpha(\phi,\zeta)|^2 dy d\tau+c\epsilon
$$

and

$$\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\int \frac{(\partial^\alpha \Xi)^2}{M} d\xi dy d\tau \le c\epsilon^2 + c\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\int \{\bar{\nu}\int \frac{(1+|\xi|)\widetilde{G}^2}{M}$$
$$+\sum_{|\alpha|=2}\frac{(1+|\xi|)(\partial^\alpha G)^2}{M} + \bar{\nu}\sum_{|\alpha|=1}\frac{(1+|\xi|)(\partial^\alpha G)^2}{M}\} d\xi dy d\tau,$$

then we have the following derivative estimate

$$\sum_{|\alpha|=1}\{\|\partial^\alpha(v,u,\theta)(\tau)\|^2 + \int_{\mathbb{R}^1}\int \frac{(\partial^\alpha G)^2(\tau)}{M} d\xi dy\}$$
$$+\sum_{|\alpha|=1}\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\int \frac{(1+|\xi|)(\partial^\alpha G)^2}{M} d\xi dy d\tau$$
$$+\sum_{|\alpha|=2}\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1} |\partial^\alpha(v,u,\theta)|^2 dy d\tau \le cN^2(\tau^0,\tau^0) + c\epsilon$$
$$+c\int_{\tau^0}^{\tau}\int_{\mathbb{R}^1}\{\sum_{|\alpha|=2}\int \frac{(1+|\xi|)(\partial^\alpha G)^2}{M} d\xi + \bar{\nu}\sum_{|\alpha|=1}[|\partial^\alpha(\phi,\psi,\zeta)|^2$$
$$+\int \frac{(\partial^\alpha G)^2}{M_*} d\xi] + \bar{\nu}\int \frac{(1+|\xi|)\widetilde{G}^2}{M} d\xi\} dy d\tau + c\sum_{|\alpha|=2}\|\partial^\alpha v(\tau)\|^2.$$

Since the higher order estimates of f are similar to those for the finite time estimate in Lemma 7.6, then we obtain the desired estimate (7.46) by an appropriate linear combination as in the last lemma. □

7.5 Proof of Theorem 4.2

The main interest here is to obtain a rate of convergence in the mean free path ϵ which is valid uniformly for all time, when we specialize to smooth rarefaction waves of the Euler equations. We first prove a finite time result which justifies the fluid-dynamical limit for a fairly large class of smooth flows on any given fixed time interval $[0,T]$ with $T<+\infty$, which, in particular, yields Theorem 4.2 on $[0,T]$. We then complete the proof of Theorem 4.2 by deriving a large time a priori estimate as in the previous subsection.

7.5.1 Smooth Flows in Finite Time

Let $0<T<+\infty$, and $(V,U,\Theta)(t,x)$ be a smooth solution to the Euler equations as stated in Section 6.2.1. Our following theorem asserts that $M_{[V,U,\Theta]}(t,x,\xi)$ is a limits as $\epsilon\to 0$, of the Boltzmann solution $f(t,x,\xi)$ to Boltzmann equation (2.5) with the same initial data

$$f(0,x,\xi) = M_{[V,U,\Theta]}(0,x,\xi). \tag{7.47}$$

Theorem 7.8 *Let* $(V,U,\Theta)(t,x)$ *be a smooth Euler solution as described above. Then there exist positive constants* ϵ_2 *and* $c(T)$ *such that for each* $\epsilon \in (0,\epsilon_2]$*, the Cauchy problem for the Boltzmann equation* (2.5)*,* (7.47) *has a unique solution* $f(t,x,\xi)$ *such that*

$$\sup_{0\le t\le T} \|f(t,x,\xi) - M_{[V,U,\Theta]}(t,x,\xi)\|_{L^\infty_x L^2_\xi(\frac{1}{\sqrt{M_*}})} \le c(T)\epsilon^{1/2}. \tag{7.48}$$

It is easy to see that the above theorem follows immediately from the following a priori estimate.

Proposition 7.9 *Suppose that the Cauchy problem* (2.5) *and* (7.47) *has a solution* f *with* $(f - M_{[V,U,\Theta]}) \in \widehat{X}(0,\tau_1)$ *for some positive* $\tau_1 \le \epsilon^{-1}T$*. There exist positive constants* ϵ_2, ν_2 *and* $c(T)$*, independent of* ϵ *and* τ_1*, such that if*

$$\epsilon \in (0,\epsilon_2],\; N^2(0,\tau_1) \le \nu_2 \tag{7.49}$$

for small ϵ_2 *and* ν_2*, then for* $\tau \in (0,\tau_1]$*,*

$$\begin{aligned} &N^2(0,\tau) + \int_0^\tau \{\sum_{|\alpha|=1} \|\partial^\alpha(\phi,\psi,\zeta)\|^2 + \sum_{|\alpha|=2} \|\partial^\alpha(v,u,\theta)\|^2 \\ &+ \int_{\mathbb{R}^1}\int \frac{(1+|\xi|)}{M_*}[\widetilde{G}^2 + \sum_{1\le|\alpha|\le 2} (\partial^\alpha G)^2] d\xi dy\} d\tau \le c(T)\epsilon. \end{aligned} \tag{7.50}$$

Proof First, note that the smallness of ϵ_2 and ν_2 in (7.49) guarantees that $V+\phi \ge \underline{v}/2$, $\Theta+\zeta \ge \underline{\theta}/2$ and the existence of M_*. The estimate (7.50) can be derived in a similar way as for (7.17) by taking into account the fact that $|U_x| \le c_0$. For the basic energy estimate, although U_x has no sign here, the term $\epsilon\int\int U_x q_1 dy d\tau$ is bounded by $c\epsilon\int\int(\phi^2+\zeta^2)dyd\tau$, which can be put into the right hand side of (7.32). Then we have

$$\begin{aligned} &\|(\phi,\psi,\zeta)(\tau)\|^2 + \int_{\mathbb{R}^1}\int \frac{\widetilde{G}(\tau)^2}{M} dy d\tau + \int_0^\tau \int_{\mathbb{R}^1} \{\sum_{|\alpha|=1} |\partial^\alpha(\phi,\psi,\zeta)|^2 \\ &+ \int \frac{(1+|\xi|)\widetilde{G}^2}{M} d\xi\} dy d\tau \le c\|\phi_y(\tau)\|^2 + c\int_0^\tau \int_{\mathbb{R}^1} \{\epsilon(\phi^2+\psi^2+\zeta^2) \\ &+ \sum_{|\alpha|=1} \int \frac{(1+|\xi|)(\partial^\alpha G)^2}{M} d\xi\} dy d\tau + c\bar{\nu}\int_0^\tau \int_{\mathbb{R}^1}\int \frac{\widetilde{G}^2}{M_*} d\xi dy d\tau + c\epsilon, \end{aligned}$$

where $\bar{\nu} = \bar{\nu}(\epsilon_2, \nu_2)$ are small. The estimates for the derivatives are the same as in Lemma 7.6. It then holds that

$$N^2(0,\tau) + \int_0^\tau \int_{\mathbb{R}^1} \{ \sum_{|\alpha|=1} |\partial^\alpha(\phi,\psi,\zeta)|^2 + \sum_{|\alpha|=2} |\partial^\alpha(v,u,\theta)|^2$$

$$+ \int [\frac{(1+|\xi|)\widetilde{G}^2}{M_*} + \sum_{1\leq|\alpha|\leq 2} \frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*}] d\xi \} dy d\tau$$

$$\leq c\epsilon + c\epsilon \int_0^\tau \int_{\mathbb{R}^1} (\phi^2 + \psi^2 + \zeta^2) dy d\tau.$$

This, together with a classic Grownwall inequality, yields the desired estimate (7.50). □

7.5.2 Smooth Rarefaction Waves in Large Time

Let us now turn to the smooth rarefaction waves. Take $(V, U, \Theta)(t, x)$ as in Theorem 7.8 to be the smooth rarefaction wave $(v^R, u^R, \theta^R)(t, x)$ given in Theorem 4.2. Then Theorem 7.8 implies immediately Theorem 4.2 on any finite time interval. To complete the proof of Theorem 4.2, we need only show the following large time a priori estimate. In what follows, $(V, U, \Theta)(t, x) = (v^R, u^R, \theta^R)(t, x)$

Proposition 7.10 *Suppose that the Cauchy problem of the Boltzmann equation has a solution $f(t, x, \xi)$ as in Theorem 4.2, which is defined on $[0, T_1] \times \mathbb{R}^1$ $(1 \leq T_1 \leq +\infty)$ and with $(f - M_{[v^R,u^R,\theta^R]}) \in \widehat{X}(0, \tau_1)$ for some $\tau_1 > \tau^0 (= \epsilon^{-1})$. There exist positive constants ϵ_3, ν_3 and c, independent of ϵ and τ_1, such that if*

$$\epsilon \in (0, \epsilon_3], \; N^2(\tau^0, \tau_1) \leq \nu_3$$

for small ϵ_3 and ν_3, then it holds that

$$N^2(\tau^0, \tau_1) + \int_{\tau^0}^{\tau_1} \{ \epsilon \| \sqrt{u^R_{1x}} (\phi, \zeta) \|^2 + \sum_{|\alpha|=1} \| \partial^\alpha (\phi, \psi, \zeta) \|^2$$

$$+ \sum_{|\alpha|=2} \| \partial^\alpha (v, u, \theta) \|^2 + \int_{\mathbb{R}^1} \int [\frac{(1+|\xi|)\widetilde{G}^2}{M_*}$$

$$+ \sum_{1\leq|\alpha|\leq 2} \frac{(1+|\xi|)(\partial^\alpha G)^2}{M_*}] d\xi dy \} d\tau \leq c(N^2(\tau^0, \tau^0) + \epsilon^{1/2}).$$

Proof By virtue of Lemma 5.3, the proposition can be proved in the same way as for Lemma 7.7. □

References

[1] C. Bardos, F. Golse, and D. Levermore. Fluid dynamic limits of kinetic equations. i. formal derivations. *J. Statist. Phys.*, 63(1-2):323–344, 1991.

[2] C. Bardos, F. Golse, and D. Levermore. Fluid dynamic limits of kinetic equations. ii. convergence proofs for the boltzmann equation. *Comm. Pure Appl. Math.*, 46(5):667–753, 1993.

[3] C. Bardos and S. Ukai. The classical incompressible navier-stokes limit of the boltzmann equation. *Math. Models Methods Appl. Sci.*, 1:235–257, 1991.

[4] L. Boltzmann. *Lectures on Gas Theory. Dover Publications.* Inc. NewYork, 1964.

[5] R. Caflisch. The fluid dynamic limit of the nonlinear boltzmann equation. *Comm. Pure Appl. Math.*, 33:651–666, 1980.

[6] R. Caflisch and G. Papanicolaou. The fluid-dynamical limit of a nonlinear model boltzmann equation. *Comm. Pure Appl. Math.*, 32:589–616, 1979.

[7] R. Caflish and B. Nicolaenko. Shock profile solutions of the boltzmann equation. *Comm. Math. Phys.*, 86:161–194, 1982.

[8] R. Courant and K. Friedrichs. *Supersonic Flows and Shock Waves.* New York: Wiley-Interscience, 1948.

[9] R. Esposito and M. Pulvirenti. From particles to fluid. *Handbooks of Mathematical Fluid Dynamics*, III, 2004.

[10] F. Golse, B. Perthame, and C. Sulem. On a boundary layer problem for the nonlinear boltzmann equation. *Arch. Ration. Mech. Anal.*, 103:81–96, 1986.

[11] H. Grad. Asymptotic theory of the boltzmann equation ii. *Rarefied Gas Dynamics*, 1:26–59, 1963.

[12] Y. Guo. The boltzmann equation in the whole space. *Indiana University Mathematics Journal*, 53:1081–1094, 2004.

[13] F. Huang and T. Yang. Stability of contact discontinuity for the boltzmann equation. *J. Differential Equations*, 229(2):698–742, 2006.

[14] F. Huang, T. Yang, and Z. Xin. Contact discontinuity with general perturbation for gas motion. *to appear in Advances in Math.*

[15] S. Jiang, G. Ni, and W. Sun. Vanishing viscosity limit to rarefaction waves for the navier-stokes equations of one-dimensional

compressible heat-conducting fluids. *SIAM J. Math. Anal.*, 38(2):368–384, 2006.

[16] S. Kawashima, A. Matsumura, and T. Nishida. On the fluid-dynamical approximation to the boltzmann equation at the level of the navier-stokes equation. *Comm. Math. Phys.*, 70:97–124, 1979.

[17] M. Lachowicz. On the initial layer and the existence theorem for the nonlinear boltzmann equation. *Math. Methods Appl. Sci.*, 9:342–366, 1987.

[18] P. Lax. Hyperbolic system of conservation laws, ii. *Comm. Pure Appl. Math.*, 10:537–566, 1957.

[19] T. Liu and Z. Xin. Nonlinear stability of rarefaction waves for compressible navier-stokes equations. *Comm. Math. Phys.*, 118(3):451–465, 1988.

[20] T.-P. Liu, T. Yang, and S.-H. Yu. Energy method for the boltzmann equation. *Physics D.*, 188:178–192, 2004.

[21] T.-P. Liu, T. Yang, S.-H. Yu, and H.-J. Zhao. Nonlinear stability of rarefaction waves for the boltzmann equation. *Arch. Ration. Mech. Anal.*, 181(2):333–371, 2006.

[22] T.-P. Liu and S.-H. Yu. Boltzmann equation: Micro-macro decompositions and positivity of shock profiles. *Comm. Math. Phys.*, 246:133–179, 2004.

[23] A. Matsumura and K. Nishihara. Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas. *Japan J. Appl. Math.*, 3(1):1–13, 1986.

[24] A. Matsumura and K. Nishihara. Global stability of the rarefaction wave of a one-dimensional model system for compressible viscous gas. *Comm. Math. Phys.*, 144(2):325–335, 1992.

[25] B. Nicolaenko. Shock wave solutions of the boltzmann equation as a nonlinear bifurcation problem from the essential spectrum. *Theories cinetiques classiques et relativistes*, pages 127–150, 1975.

[26] T. Nishida. Fluid dynamical limit of the nonlinear boltzmann equation to the level of the compressible euler equation. *Comm. Math. Phys.*, 61:119–148, 1978.

[27] K. Nishihara, T. Yang, and H. Zhao. Nonlinear stability of strong rarefaction waves for compressible navier-stokes equations. *SIAM J. Math. Anal.*, 35(6):1561–1597, 2004.

[28] J. Smoller. *Shock waves and reaction-diffusion equations*. New York: Springer-Verlag, 1994.

[29] Y. Sone. *Kinetic Theory and Fluid Dynamics*. Boston: Birkhauser, 2002.

[30] W. C. Wang and Z. Xin. Fluid-dynamic limit for the center rarefaction wave of the broadwell equation. *J. Diff. Eqns.*, 150:438–461, 1998.

[31] Z. Xin. The fluid dynamic limit for the broadwell model of the nonlinear boltzmann equation in the presence of shocks. *Comm. Pure Appl. Math.*, 44:679–741, 1991.

[32] Z. P. Xin. The fluid-dynamic limit of the broadwell equation. *SIAM, Philadelphia, PA*, pages 237–252, 1991.

[33] Z. P. Xin. Zero dissipation limit to rarefaction waves for the one-dimensional naviercstokes equations of compressible isentropic gases. *Comm. Pure Appl. Math.*, 46:621–665, 1993.

[34] S.-H. Yu. Hydrodynamic limits with shock waves of the boltzmann equations. *Commun. Pure Appl. Math.*, 58:397–431, 2005.