Transonic Shock Solutions for a System of Euler-Poisson Equations

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Abstract

A boundary value problem for a system of Euler-Poisson equations modelling semiconductor devices or plasma is considered. The boundary conditions are supersonic inflow and subsonic outflow. The purpose of this paper is to elucidate the role played by the electric filed to the structure of solutions with transonic shocks. The existence, non-existence, uniqueness and multiplicity of solutions with transonic shocks are obtained according to the different cases of the boundary data and physical interval length. Detailed structures of solutions are given. Shock locations are determined by the boundary data. Different phenomena are shown for the different situations when the density of fixed, positively charged background ions is in supersonic and subsonic regimes.

1 Introduction

The following system of 1-dimensional Euler-Poisson equations:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (p(\rho) + \rho u^2)_x = \rho E, \\ E_x = \rho - b, \end{cases} \tag{1.1}$$

models several physical flows including the propagation of electrons in submicron semiconductor devices and plasma (cf. [17])(hydrodynamic model), and the biological transport of ions for channel proteins (cf [3]). In the hydrodynamical model of semiconductor devices or plasma, u, ρ and p represent the average particle velocity, electron density and pressure, respectively, E is the electric filed, which is generated by the Coulomb force of particles. $b > 0$ stands for the density of fixed, positively charged background ions. The biological model describes the transport of ions between the extracellular side and the cytoplasmic side of the membranes([3]). In this case, ρ, ρu and E are the ion concentration, the ions translational mass, and the electric field, respectively.

In this paper, we consider the transonic shock solutions for following time-independent problem

$$\begin{cases} (\rho u)_x = 0, \\ (p(\rho) + \rho u^2)_x = \rho E, \\ E_x = \rho - b. \end{cases} \tag{1.2}$$

Assuming that p satisfies:

$$p(0) = 0, p'(\rho) > 0, p''(\rho) > 0, \ for \ \rho > 0, p(+\infty) = +\infty, \tag{1.3}$$

we consider boundary value problem for (1.2) in an interval $0 \leq x \leq L$ with the boundary condition:

$$(\rho, u, E)(0) = (\rho_l, u_l, \alpha), \quad (\rho, u)(L) = (\rho_r, u_r). \tag{1.4}$$

We assume $u_l > 0$ and $u_r > 0$. By the first equation in (1.2), we know that $\rho u(x) = constant(0 \leq x \leq L)$ so the boundary data should satisfy

$$\rho_l u_l = \rho_r u_r \tag{1.5}$$

We denote

$$\rho_l u_l = \rho_r u_r = J. \tag{1.6}$$

Then $\rho u(x) = J(0 \leq x \leq L)$ and the velocity is given by

$$u = J/\rho. \tag{1.7}$$

The boundary value problem for system (1.2) reduces to

$$\begin{cases} (p(\rho) + \frac{J^2}{\rho})_x = \rho E, \\ E_x = \rho - b, \end{cases} \tag{1.8}$$

with the boundary conditions:

$$(\rho, E)(0) = (\rho_l, \alpha), \quad \rho(L) = \rho_r. \tag{1.9}$$

We use the terminology from gas dynamics to call $c = \sqrt{p'(\rho)}$ the sound speed. There is a unique solution $\rho = \rho_s$ for the equation

$$p'(\rho)\rho^2 = J^2, \tag{1.10}$$

which is the sonic state (recall that $J = \rho u$).
In this case, the flow is called supersonic if

$$p'(\rho)\rho^2 < J^2, \ i.e. \ \rho < \rho_s. \tag{1.11}$$

If

$$p'(\rho)\rho^2 > J^2, \ i.e., \ \rho > \rho_s, \tag{1.12}$$

then the flow is called subsonic.

We notice that $(1.8)_1$ is singular at sonic state ($p'(\rho_s) - \frac{J^2}{\rho_s^2} = 0$) and the coefficient of ρ_x changes the sign for the supersonic flow and subsonic flow. This makes the problem of determining which kind of boundary conditions should be posed to make the boundary value problem well-posed a subtle one. In the previous works, some pure subsonic or supersonic solutions are obtained for both 1-dimensional and multidimensional cases (cf. [10] and [17]). For a viscous approximation of transonic solutions in 2-d case for the equations of semiconductors, see [12]. However, there have been only a few results for the transonic flow. In the following, we list several results which are closely related to the present paper. First, a boundary value problem for (1.8) was discussed in [1] for a linear pressure function of the form $p(\rho) = k\rho$ with the special boundary condition $\rho(0) = \rho(L) = \bar{\rho}$ with $\bar{\rho}$ being a subsonic state for the case when $0 < b < \rho_s$. The solution obtained in [1] may contain transonic shock. On the other hand, since the boundary conditions and the pressure function are special in [1], it is desired to consider the more general boundary conditions with more general pressure function. Moreover, only the case when $0 < b < \rho_s$ (i.e., when b is in the supersonic regime) is considered. As we will show later, the cases when $0 < b < \rho_s$ and $b > \rho_s$ are completely different. Actually, $(b, 0)$ is a center when $0 < b < \rho_s$ and a saddle point when $b > \rho_s$ for system (1.8). We will construct solutions with transonic shocks for both cases. In [18], the local-in-time stability of transonic shock solutions for the Cauchy problem of (1.1) is considered by assuming the existence of steady transonic shocks. In [19], a phase plane analysis is given for system (1.8). However, no transonic shock solutions are constructed in [19]. A transonic solution which may contain transonic shocks was constructed by I. Gamba (cf. [13]) by using a vanishing viscosity limit method. However, the solutions as the limit of vanishing viscosity may contain boundary layers. Therefore, the question of well-posedness of the boundary value problem for the inviscid problem can not be answered by the vanishing viscosity method. Moreover, the structure of the solutions constructed by the vanishing viscosity method in [13] is shown to be of bounded total variation and possibly contain more than one transonic shock. One of the main purposes of the present paper is to obtain more detailed structure of the solutions for the boundary value problem (1.8) and (1.9) and answer the question of well-posedness of solutions for this boundary value problem. We give a throughout study of the structure of the solutions to the boundary value problem for the different situations of boundary data and the interval length L. The existence, non-existence, uniqueness and non-uniqueness of solutions with transonic shocks are obtained according to the different cases of boundary data and physical interval length. The solution (when it exists) that we construct contains exactly one transonic shock in the interval $[0, L]$. On the left of this transonic shock, the flow is supersonic, it is subsonic on the right of this shock. Moreover, we can determine the shock location by the boundary data and L. It is interesting to compare this result with the transonic solutions of a quasi-one-dimensional gas flow through a nozzle studied by Embid, Goodman and Majda ([9]). The time-dependent equations for the one dimensional isentropic nozzle flow are

$$\begin{cases} \rho_t + (\rho u)_x = -\frac{A'(x)}{A(x)}\rho u, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = -\frac{A'(x)}{A(x)}\rho u^2, \end{cases} \tag{1.13}$$

where ρ, u and p denote respectively the density, velocity and pressure, $A(x)$ is the cross-sectional area of the nozzle. In [9], steady state solutions containing transonic shocks are constructed for the boundary value problem in the interval $[0, 1]$ with the boundary conditions $(\rho, u)(0) = (\rho_l, u_l)$ and $(\rho, u)(1) = (\rho_r, u_r)$ satisfying $\rho_l u_l = \rho_r u_r$ with (ρ_l, u_l) being supersonic and (ρ_r, u_r) being subsonic. It is shown in [9] that, if $A(x)$ is not strictly monotone, then there exist multiple steady state transonic shock solutions, and the shock locations are not unique. Particularly, when $A'(x) \equiv 0$ (this means the duct is uniform), the transonic shock can be anywhere in the duct. Therefore, the structure of solutions depend on the structure of the geometry of the nozzle. The electric field E plays a similar role as we will show later. The difference is that the geometry of the nozzle is given, while the electric field E is unknown and is a part of solutions.

There have been many studies on the stability of transonic shocks for system (1.13)(cf. [15], [16] and [14]). It would be interesting to investigate the stability of steady transonic solutions obtained in the paper. It would be interesting to extend the results of this paper to the multi-dimensional case, as those for gas dynamics (cf. [2], [4], [5], [6], [7], [8], [20], [21] and [22]). An effort in this direction was made in [12] for a viscous approximation of transonic solutions in 2-d case for the equations of semiconductors. However, passing limit when the viscosity tends to zero for the viscosity approximation in [12] is still an open problem. Some progress has been made in this direction [6] for the potential flow equations of gas dynamics.

2 Initial Value Problem For System (1.8)

In this section, we study the initial value problem for (1.8), i.e., we consider the initial value problem:

$$\begin{cases} (p(\rho) + \frac{J^2}{\rho})_x = \rho E, \quad E_x = \rho - b, \quad \text{for } x > x_0, \\ (\rho, E)(x_0) = (\rho_0, E_0). \end{cases} \tag{2.1}$$

which will be used when we construct the transonic shock solutions for the boundary value problem.

The solution of (1.8) can be analyzed in (ρ, E)-phase plane. Any trajectory in (ρ, E)-plane satisfies the following equation,

$$d\left(\frac{1}{2}E^2 - H(\rho)\right) = 0, \quad \text{where } H'(\rho) = \frac{\rho - b}{\rho}(p'(\rho) - \frac{J^2}{\rho^2}). \tag{2.2}$$

The trajectory passing through the point (ρ_0, E_0) with $\rho_0 > 0$ is given by

$$\frac{1}{2}E^2 - \int_{\rho_0}^{\rho} H'(s)ds = \frac{1}{2}E_0^2. \tag{2.3}$$

Since the cases when $0 < b < \rho_s$ (b is in supersonic region) and $b > \rho_s$ (b is in subsonic region) are completely different, we discuss these two cases separately. The phase portraits of those two different cases are in Figure 1 and Figure 2, respectively (all figures are at the end of this paper).

2.1 The Case when $0 < b < \rho_s$.

The following facts will be useful:

$$H'(\rho_s) = H'(b) = 0, H'(\rho) > 0 \text{ for } 0 < \rho < b \text{ and } \rho > \rho_s,\ H'(\rho) > 0 \text{ for } b < \rho < \rho_s, \tag{2.4}$$

$$\lim_{\rho \to 0^+} \int_{\rho_0}^{\rho} H'(s)ds = -\infty, \text{ for any } \rho_0 > 0. \tag{2.5}$$

For the different situations of the initial value (ρ_0, E_0) on the (ρ, E)-plane, we give the following classification of solutions. First, we define the **Critical Trajectory for the case when** $0 < b < \rho_s$.

Definition: The critical trajectory is the trajectory passing through the point $(\rho_s, 0)$ with the equation:

$$\frac{1}{2}E^2 - \int_{\rho_s}^{\rho} H'(s)ds = 0. \tag{2.6}$$

There are two branches of the critical trajectory, a supersonic branch and a subsonic branch. The supersonic branch is for $\rho_{min}^c \le \rho \le \rho_s$ where ρ_{min}^c is determined by

$$\int_{\rho_s}^{\rho_{min}^c} H'(s)ds = 0, \quad 0 < \rho_{min}^c < b. \tag{2.7}$$

The subsonic branch is for $\rho > \rho_s$. The supersonic branch is a loop with the center $(b, 0)$ (we call this the supersonic loop of the critical trajectory). The supersonic branch and subsonic branch intersect at the sonic point $(\rho_s, 0)$.

Solutions for IVP (2.1) for the case $0 < b < \rho_s$.

Case 1 $(\rho_0,\ E_0)$ is inside the critical supersonic loop, i. e., $(\frac{1}{2}E_0^2 - \int_{\rho_s}^{\rho_0} H'(s)ds < 0$ and $0 < \rho_0 < \rho_s$ $(\rho_0, E_0) \ne (b, 0))$.

In this case, initial value problem (2.1) admits a unique solution (ρ, E) for all $x \ge x_0$. In (ρ, E)-plane, the trajectory of the solution is given by equation (2.3). In this case, the trajectory is a loop with the center $(b, 0)$. The direction of the trajectory is counter clockwise. The solution is periodic and always supersonic.

Case 2 $(\rho_0,\ E_0)$ is inside the critical subsonic branch of the critical trajectory, i. e.,($\frac{1}{2}E_0^2 - \int_{\rho_s}^{\rho_0} H'(s)ds < 0$ and $\rho_0 > \rho_s$).

In this case, initial value problem (2.1) admits a unique solution (ρ, E) for all $x \geq x_0$. E is strictly increasing. The solution is always subsonic. Moreover,

$$\lim_{x\to\infty}(\rho, E) = (\infty, \infty). \tag{2.8}$$

Case 3 ((ρ_0, E_0) is on the critical supersonic trajectory , i.e., $\frac{1}{2}E_0^2 - \int_{\rho_s}^{\rho_0} H'(s)ds = 0$ and $0 < \rho_0 \leq \rho_s$. In this case, there are infinitely many smooth solutions for IVP (2.1) for all $x \geq x_0$. These solutions are of the following types:

i)(Type I)(Periodic) The solution (ρ, E) is always on the supersonic loop of the critical trajectory.

ii) (Type II) The solution travels along the supersonic loop of the critical trajectory n times $(n = 0, 1, 2, \cdots)$, and then travels to the sonic point $(\rho_s, 0)$. From this sonic point, it travels along the upper subsonic branch of the critical trajectory $\frac{1}{2}E^2 - \int_{\rho_s}^{\rho} H'(s)ds = 0,\ E > 0,\ \rho > \rho_s$. In this case, we have

$$\lim_{x\to\infty}(\rho, E) = (\infty, \infty). \tag{2.9}$$

Case 4 ((ρ_0, E_0) is on the critical trajectory, and $\rho_0 > \rho_s$ (subsonic) and $E_0 > 0$.)

In this case, there exists a unique solution $(\rho, E)(x)$ of the initial value problem (2.1) for all $x \geq x_0$, which travels along the upper subsonic branch of the critical trajectory $\frac{1}{2}E^2 - \int_{\rho_s}^{\rho} H'(s)ds = 0,\ E > E_0,\ \rho > \rho_0$. In this case, we have

$$\rho_x > 0, E_x > 0, \lim_{x\to\infty}(\rho, E) = (\infty, \infty). \tag{2.10}$$

Case 5((ρ_0, E_0) is on the critical trajectory, and $\rho_0 > \rho_s$ (subsonic) and $E_0 < 0$.) In this case, there are infinitely many solutions. In (ρ, E) plane, the solutions start from (ρ_0, E_0), travel along the lower subsonic branch of the critical trajectory $\frac{1}{2}E^2 - \int_{\rho_s}^{\rho} H'(s)ds = 0,\ 0 > E > E_0,\ \rho < \rho_0$ in the direction ρ decreases and E increases. The solutions reaches the sonic point $(\rho_s, 0)$ at some $x_1 > x_0$. After then $(x > x_1)$, this case reduces to case 3).

Case 6($\frac{1}{2}E_0^2 - \int_{\rho_s}^{\rho_0} H'(s)ds > 0$ and $0 < \rho_0 < \rho_s$)

In this case, the solution for initial value problem (2.1) exists only on a finite interval $[x_0, x_2)$ for some $x_2 > x_0$. Moreover,

$$\lim_{x\to x_2^-}(\rho, E) = (\rho_s, E_1), \tag{2.11}$$

where E_1 is determined by

$$\frac{1}{2}E_1^2 - \int_{\rho_0}^{\rho_s} H'(s)ds = \frac{1}{2}E_0^2, E_1 < 0.$$

Furthermore,

$$\lim_{x\to x_2^-}\rho_x(x) = +\infty. \tag{2.12}$$

Case 7$(\frac{1}{2}E_0^2 - \int_{\rho_s}^{\rho_0} H'(s)ds > 0$ and $\rho_0 > \rho_s, E_0 > 0$)
In this case, the solution for initial value problem (2.1) exists for all $x \geq x_0$. Along the trajectory of the solution, both ρ and E are strictly increasing. Moreover,

$$\lim_{x\to\infty}(\rho, E)(x) = (+\infty, +\infty). \tag{2.13}$$

Case 8$(\frac{1}{2}E_0^2 - \int_{\rho_s}^{\rho_0} H'(s)ds > 0$ and $\rho_0 > \rho_s, E_0 < 0$)
In this case, the solution for initial value problem (2.1) exists only on a finite interval $[x_0, x_3)$ for some $x_3 > x_0$. Moreover,

$$\lim_{x\to x_3^-}(\rho, E) = (\rho_s, E_2), \tag{2.14}$$

where E_2 is determined by

$$\frac{1}{2}E_2^2 - \int_{\rho_0}^{\rho_s} H'(s)ds = \frac{1}{2}E_0^2, \quad E_2 < 0.$$

Furthermore,

$$\lim_{x\to x_3^-} \rho_x(x) = -\infty. \tag{2.15}$$

2.2 The case when $b > \rho_s$.

In this subsection, we solve the initial value problem (2.1) for the different situations of the initial values (ρ_0, E_0). In this case, the equilibrium point $(b, 0)$ is a saddle point on the phase plane (see Figure 2).
We define the **Critical Trajectory for the case** $b > \rho_s$.
Definition: The critical trajectory (for the case $b > \rho_s$) is the trajectory passing through the point $(b, 0)$ with the equation:

$$\frac{1}{2}E^2 - \int_b^{\rho} H'(s)ds = 0. \tag{2.16}$$

We solve the initial value problem (2.1) for the differnet cases of the initial data (ρ_0, E_0).
Case 1 $(\rho_0 < \rho_s)$, i.e., ρ_0 is supersonic. In the case, the solution of (2.1) only exists in a finite interval $[x_0, x_4)$. Moreover,

$$\lim_{x\to x_4-}(\rho, E) = (\rho_s, -\sqrt{E_0^2 + 2\int_{\rho_0}^{\rho_s} H'(s)ds}), \quad \lim_{x\to x_4-} \rho_x = +\infty.$$

Case 2 $\rho_0 > \rho_s$.
a) (ρ_0, E_0) is inside the critical trajectory, i.e.,

$$\frac{1}{2}E_0^2 + \int_{\rho_0}^{b} H'(s)ds < 0, \rho_0 > \rho_s.$$

There are two subcases.

a1) $\rho_s < \rho_0 < b$.

In this case, initial value problem (2.1) admits a unique solution (ρ, E) in a finite interval $[x_0,\ x_5)$. Moreover,

$$
\begin{aligned}
& b > \rho(x) > \rho_s,\ \ x \in [x_0, x_5), \\
& \lim_{x \to x_5-} (\rho, E)(x) = (\rho_s, -\sqrt{E_0^2 + 2\int_{\rho_0}^{\rho_s} H'(s)ds}), \lim_{x \to x_5-} \rho_x(x) = -\infty.
\end{aligned} \tag{2.17}
$$

a2) $\rho_0 > b$.

In this case, initial value problem (2.1) admits a unique solution (ρ, E) for all $x \geq x_0$. Moreover,

$$
\begin{aligned}
& \rho(x) > b > \rho_s, E_x > 0,\ \ x \in [x_0, \infty), \\
& \lim_{x \to \infty} (\rho, E)(x) = (+\infty, +\infty).
\end{aligned} \tag{2.18}
$$

b) (ρ_0, E_0) is outside the critical trajectory, i.e.,

$$
\frac{1}{2}E_0^2 + \int_{\rho_0}^{b} H'(s)ds > 0, \rho_0 > \rho_s.
$$

There are two subcases.

b1) $E_0 > 0$.

In this case, initial value problem (2.1) admits a unique solution (ρ, E) for all $x \geq x_0$. Moreover,

$$
\begin{aligned}
& \rho(x) > \rho_s,\ \ x \in [x_0, \infty), \\
& \lim_{x \to \infty} (\rho, E)(x) = (+\infty, +\infty).
\end{aligned} \tag{2.19}
$$

b2) $E_0 < 0$.

In this case, the initial value problem (2.1) admits a unique solution (ρ, E) in a finite interval $[x_0,\ x_6)$. Moreover,

$$
\begin{aligned}
& \rho(x) > \rho_s,\ \ x \in [x_0, x_6), \\
& \lim_{x \to x_6-} (\rho, E)(x) = (\rho_s, -\sqrt{E_0^2 + 2\int_{\rho_0}^{\rho_s} H'(s)ds}), \lim_{x \to x_6-} \rho_x(x) = -\infty.
\end{aligned} \tag{2.20}
$$

c) (ρ_0, E_0) is on the critical supersonic trajectory , i.e.,

$$
\frac{1}{2}E_0^2 + \int_{\rho_0}^{b} H'(s)ds = 0.
$$

c1) $\rho_s < \rho_0 < b, E_0 > 0$.
In this case, initial value problem (2.1) admits a unique solution (ρ, E) for all $x \geq x_0$. Moreover,

$$\begin{aligned} &\rho_x > 0, E_x < 0, \;\; x > x_0, \\ &\lim_{x\to\infty} (\rho, E)(x) = (b, \;0). \end{aligned} \tag{2.21}$$

c2) $\rho_s < \rho_0 < b, E_0 < 0$.
In this case, , the initial value problem (2.1) admits a unique solution (ρ, E) in a finite interval $[x_0, \; x_7)$. Moreover,

$$\begin{aligned} &\rho_x(x) < 0, E_x(x) < 0, \;\; x \in [x_0, x_7), \\ &\lim_{x\to x_7-} (\rho, E)(x) = (\rho_s, -\sqrt{2\int_b^{\rho_s} H'(s)ds}), \lim_{x\to x_7-} \rho_x(x) = -\infty. \end{aligned} \tag{2.22}$$

c3) $\rho_0 > b, E_0 > 0$.
In this case, initial value problem (2.1) admits a unique solution (ρ, E) for all $x \geq x_0$. Moreover,

$$\begin{aligned} &\rho_x > 0, E_x > 0, \;\; x > x_0, \\ &\lim_{x\to\infty} (\rho, E)(x) = (\infty, \;\infty). \end{aligned} \tag{2.23}$$

c4) $\rho_0 > b, E_0 < 0$.
In this case, the initial value problem (2.1) admits a unique solution (ρ, E) for all $x \geq x_0$. Moreover,

$$\begin{aligned} &\rho_x(x) > 0, E_x(x) > 0, \;\; x > x_0, \\ &\lim_{x\to\infty} (\rho, E)(x) = (b, \;0). \end{aligned} \tag{2.24}$$

3 Transonic Shocks

We use $(\rho, E)(x, \rho_0, E_0)$ $(x \geq x_0)$ to denote the solution of the initial value problem (2.1) and use $T(\rho_0, E_0)$ to denote the trajectory passing through the state (ρ_0, E_0) in the direction as x increases. Precisely, we define

Definition 3.1 *We say that a state $(\rho_1, E_1) \in T(\rho_0, E_0)$ if there exist $x_0 \in \mathrm{R}^1$ and $x_1 \in \mathrm{R}^1$ satisfying $x_1 \geq x_0$ such that $(\rho_1, E_1) = (\rho, E)(x_1, \rho_0, E_0)$.*

Therefore, if $(\rho_1, E_1) \in T(\rho_0, E_0)$, then

$$\frac{1}{2}E^2 - \int_{\rho_0}^{\rho} H'(s)ds = \frac{1}{2}E_0^2.$$

For boundary value problem (1.8) and (1.9), we assume $\rho_l < \rho_s$ and $\rho_r > \rho_s$. This means the flow is supersonic at $x = 0$ and subsonic at $x = L$. By the results in section 2, we know that this boundary value problem does not have a smooth solution in general. The solution is expected to have a transonic shock in the interval $[0, L]$. A transonic shock solution is a discontinuous solution of the boundary value problem (1.8) and (1.9). Suppose the shock location is at a point $a \in [0, L]$, then we require the following Rankine-Hugoniot condition and entropy condition:

Rankine-Hugoniot Condition

$$\left(p(\rho) + \frac{J^2}{\rho}\right)(a+) = \left(p(\rho) + \frac{J^2}{\rho}\right)(a-), E(a+) = E(a-), \tag{3.1}$$

Entropy Condition

$$\rho(a+) > \rho(a-). \tag{3.2}$$

The shock is transonic means

$$\rho(a+) > \rho_s > \rho(a-). \tag{3.3}$$

For any $\rho \in (0, \rho_s)$, there exists one and only one $F(\rho)$ satisfying

$$p(F(\rho)) + \frac{J^2}{F(\rho)} = p(\rho) + \frac{J^2}{\rho}, \quad F(\rho) > \rho_s. \tag{3.4}$$

Also it is easy to verify that

$$F'(\rho) = \frac{p'(\rho) - \frac{J^2}{\rho^2}}{p'(F(\rho)) - \frac{J^2}{F(\rho)^2}} < 0, \ for\ 0 < \rho < \rho_s, \tag{3.5}$$

$$H'(F(\rho))F'(\rho) = \frac{F(\rho) - b}{F(\rho)}\left(p'(\rho) - \frac{J^2}{\rho^2}\right), \ for\ 0 < \rho < \rho_s. \tag{3.6}$$

For the trajectory passing through (ρ_l, α), we define the shock curve by T_{shock}

$$T_{shock} = \{(F(\rho), E) : (\rho, E) \in T((\rho_l, \alpha)\}.$$

We denote $\ell((\rho_1, E_1); (\rho_2, E_2))$ the length in x for the trajectory of (1.8) traveling from the state (ρ_1, E_1) to the state $(\rho_2, E_2))$ when (ρ_1, E_1) and $(\rho_2, E_2))$ are on the same trajectory. In order to show the existence and uniqueness of transonic shocks, we need the following lemmas.

Lemma 3.1. *If the two states (ρ_1, E_1) and (ρ_2, E_2) are on the same trajectory of system (1.8), i.e., $(\rho_2, E_2) \in T(\rho_1, E_1)$ and on the trajectory connecting these two states, E does not change sign (then E is a function of ρ, denoted by $E(\rho)$) ,*

$$\ell((\rho_1, E_1); (\rho_2, E_2)) = \int_{\rho_1}^{\rho_2} \frac{p'(\rho) - \frac{J^2}{\rho^2}}{\rho E(\rho)} d\rho. \tag{3.7}$$

Proof. From $(1.8)_1$ we have,

$$\frac{p'(\rho) - \frac{J^2}{\rho^2}}{\rho E} d\rho = dx, \tag{3.8}$$

when E does not change sign. This proves (3.7).□

Lemma 3.2. *If two states (ρ_1, E_1) and (ρ_2, E_2) are on the same trajectory of system (1.8), i.e., $(\rho_2, E_2) \in T(\rho_1, E_1)$, and on the trajectory connecting these two states, E is strictly increasing or decreasing (then ρ is a function of E, denoted by $\rho(E, \rho_1)$), then*

$$\ell\left((\rho_1, E_1); (\rho_2, E_2)\right) = \int_{E_1}^{E_2} \frac{dE}{\rho(E, \rho_1) - b}, \tag{3.9}$$

as long as $\rho(E, \rho_1) \neq b$ for E between E_1 and E_2.

Proof. By the second equation in (1.8), we have $\frac{dE}{\rho - b} = dx$. (3.9) follows then. □

Lemma 3.3. *For the fixed (ρ_0, E_0) and ρ_r, let*

$$X(\bar\rho) = \ell((\rho_0, E_0); (\bar\rho, E(\bar\rho))) + \ell((F(\bar\rho), E(\bar\rho)); (\rho_r, E_r(\bar\rho))), \tag{3.10}$$

where $\rho_0 < \rho_s$, $\bar\rho < \rho_s$, $\rho_r > \rho_s$, $(\bar\rho, E(\bar\rho)) \in T(\rho_0, E_0)$, $(\rho_r, E_r(\bar\rho)) \in T(F(\bar\rho), E(\bar\rho))$. If E does not change the sign along the trajectories from (ρ_0, E_0) to $(\bar\rho, E(\bar\rho))$ and from $(F(\bar\rho), E(\bar\rho))$ to $(\rho_r, E_r(\bar\rho))$, then

$$X'(\bar\rho) = (p'(\bar\rho) - \frac{J^2}{\bar\rho^2})(\frac{1}{\bar\rho} - \frac{1}{F(\bar\rho)})Q(\bar\rho), \tag{3.11}$$

where

$$Q(\bar\rho) = \frac{1}{E(\bar\rho)} + b \int_{F(\bar\rho)}^{\rho_r} \frac{p'(t) - \frac{J^2}{t^2}}{tE^3(\bar\rho, t)} dt, \tag{3.12}$$

provided $E(\bar\rho) \neq 0$ and $F(\bar\rho) \neq 0$, where

$$E(\bar\rho, t) = sgn(E(\bar\rho)) \sqrt{E^2(\bar\rho) + 2 \int_{F(\bar\rho)}^{t} H'(s) ds}, \tag{3.13}$$

for t between $F(\bar\rho)$ and ρ_r. Moreover,

$$Q'(\bar\rho) = (p'(\bar\rho) - \frac{J^2}{\bar\rho^2}) \left(\frac{1}{E^3(\bar\rho)} [\frac{b}{\bar\rho} - \frac{b}{F(\bar\rho)} - 1] + 3b^2 [\frac{1}{\bar\rho} - \frac{1}{F(\bar\rho)}] \int_{F(\bar\rho)}^{\rho_r} \frac{p'(t) - \frac{J^2}{t^2}}{tE^5(\bar\rho, t)} dt \right). \tag{3.14}$$

Proof. Let $X_1(\bar\rho) = \ell((\rho_0, E_0); (\bar\rho, E(\bar\rho)))$ and $X_2(\bar\rho) = \ell((F(\bar\rho), E(\bar\rho)); (\rho_r, E_r(\bar\rho)))$. Then we have, by Lemma 3.1,

$$X_1(\bar\rho) = \int_{\rho_0}^{\bar\rho} \frac{p'(t) - \frac{J^2}{t^2}}{tE(t)} dt, \tag{3.15}$$

where $(t, E(t)) \in T(\rho_0, E_0)$. So

$$\frac{1}{2}E^2(t) - H(t) = \frac{1}{2}E_0^2 - H(\rho_0). \tag{3.16}$$

Especially,

$$\frac{1}{2}E^2(\bar\rho) - H(\bar\rho) = \frac{1}{2}E_0^2 - H(\rho_0). \tag{3.17}$$

Therefore

$$E(\bar\rho)E'(\bar\rho) = H'(\bar\rho). \tag{3.18}$$

Moreover,

$$X_2(\bar\rho) = \int_{F(\bar\rho)}^{\rho_r} \frac{p'(t) - \frac{J^2}{t^2}}{tE(\bar\rho, t)} dt,$$

where $E(\bar\rho, t)$ is given by

$$\frac{1}{2}E^2(\bar\rho, t) - H(t) = \frac{1}{2}E^2(\bar\rho) - H(F(\bar\rho)), \tag{3.19}$$

for t between $F(\bar\rho)$ and ρ_r. By (3.15), we have

$$X_1'(\bar\rho) = \frac{p'(\bar\rho) - \frac{J^2}{\bar\rho^2}}{\bar\rho E(\bar\rho)}, \tag{3.20}$$

and

$$\begin{aligned} X_2'(\bar\rho) = & -\frac{p'(F(\bar\rho) - \frac{J^2}{(F(\bar\rho))^2}}{F(\bar\rho)E(\bar\rho)} F'(\bar\rho), \\ & - \int_{F(\bar\rho)}^{\rho_r} \frac{(}{p'(t) - \frac{J^2}{t^2})\partial E(\bar\rho, t)/\partial \bar\rho} tE^2(\bar\rho, t) dt. \end{aligned} \tag{3.21}$$

From (3.5), we have

$$\left(p'(F(\bar\rho)) - \frac{J^2}{(F(\bar\rho))^2}\right) F'(\bar\rho) = (p'(\bar\rho) - \frac{J^2}{\bar\rho^2}). \tag{3.22}$$

By virtue of (3.6) and (3.18), we obtain,

$$\begin{aligned} E(\bar\rho, t)\frac{\partial E(\bar\rho, t)}{\partial \bar\rho} &= E(\bar\rho)E'(\bar\rho) - H'(F(\bar\rho)F'(\bar\rho) \\ &= H'(\bar\rho) - H'(F(\bar\rho))F'(\bar\rho) \\ &= (p'(\bar\rho) - \frac{J^2}{\bar\rho^2})(\frac{b}{F(\bar\rho)} - \frac{b}{\bar\rho}). \end{aligned} \tag{3.23}$$

Since $X(\bar\rho) = X_1(\bar\rho) + X_2(\bar\rho)$. Therefore, (3.11) follows from (3.20)-(3.23). (3.14) can be obtained by the same method. □.

We give an alternative lemma on how to calculate $X'(\bar\rho)$ which will be used later.

Lemma 3.4. *For the fixed* (ρ_0, E_0) *and* ρ_r*, let*

$$X(\bar\rho) = \ell((\rho_0, E_0); (\bar\rho, E(\bar\rho))) + \ell((F(\bar\rho), E(\bar\rho)); (\rho_r, E_r(\bar\rho))), \tag{3.24}$$

where $\rho_0 < \rho_s$*,* $\bar\rho < \rho_s$*,* $\rho_r > \rho_s$*,* $(\bar\rho, E(\bar\rho) \in T(\rho_0, E_0)$*,* $(\rho_r, E_r(\bar\rho)) \in T(F(\bar\rho), E(\bar\rho))$*. If* E *does not change the sign along the trajectories from* (ρ_0, E_0) *to* $(\bar\rho, E(\bar\rho))$ *and from* $(F(\bar\rho), E(\bar\rho))$ *to* $(\rho_r, E_r(\bar\rho))$*, and* $\rho \neq b$ *along the trajectory from* $(F(\bar\rho), E(\bar\rho))$ *to* $(\rho_r, E_r(\bar\rho))$*, then*

$$\frac{dX(\bar\rho)}{d\bar\rho} = \left(p'(\bar\rho) - \frac{J^2}{\bar\rho^2}\right)\frac{F(\bar\rho) - \bar\rho}{\bar\rho} \cdot \left\{\frac{1}{F(\bar\rho)E(\bar\rho)} + \frac{b}{F(\bar\rho)}\int_{F(\bar\rho)}^{\rho_r} \frac{H'(\hat\rho)}{(\hat\rho - b)sgn(E(\bar\rho))[E^2(\bar\rho) + 2\int_{F(\bar\rho)}^{\hat\rho} H'(t)dt]^{3/2}} d\hat\rho.\right\}, \tag{3.25}$$

provided $E(\bar\rho) \neq 0$ *,* $E_r(\bar\rho) \neq 0$ *and* $F(\bar\rho) \neq 0$ *.*

Proof. Let

$$L_1(\bar\rho) = \ell((\rho_0, E_0); (\bar\rho, E(\bar\rho))),$$

and

$$L_2(\bar\rho) = \ell((F(\bar\rho), E(\bar\rho)); (\rho_r, E_r(\bar\rho))). \tag{3.26}$$

First, we have from Lemma 3.1 that

$$L_1(\bar\rho) = \int_{\rho_0}^{\bar\rho} \frac{p'(s) - \frac{J^2}{s^2}}{sE(s)} ds. \tag{3.27}$$

So,

$$L_1'(\bar\rho) = \frac{p'(\bar\rho) - \frac{J^2}{\bar\rho^2}}{\bar\rho E(\bar\rho)}, \tag{3.28}$$

as long as $E(\bar\rho) \neq 0$. Next, since E does not change sign along the trajectory from the state $(F(\bar\rho), E(\bar\rho))$ to $(\rho_r, E_r(\bar\rho))$, it follows from $(p'(\rho) - \frac{J^2}{\rho^2})\rho_x = \rho E$ that ρ is a function of E on the trajectory from the state $(F(\bar\rho), E(\bar\rho))$ to $(\rho_r, E_r(\bar\rho))$ $(sgn(\rho_x) = sgnE)$. We denote this function by $\rho(E, \bar\rho)$. It follows from Lemma 3.2 that

$$L_2(\bar\rho) = \int_{E(\bar\rho)}^{E_r(\bar\rho)} \frac{dE}{\rho(E, \bar\rho) - b}. \tag{3.29}$$

Notice that $\rho\,(E(\bar\rho), \bar\rho)) = F(\bar\rho)$ and $\rho(E_r(\bar\rho), \bar\rho) = \rho_r$, we have

$$L_2'(\bar\rho) = \frac{E_r'(\bar\rho)}{\rho_r - b} - \frac{E'(\bar\rho)}{F(\bar\rho) - b} - \int_{E(\bar\rho)}^{E_r(\bar\rho)} \frac{\frac{\partial\rho(E,\bar\rho)}{\partial\bar\rho}}{(\rho(E, \bar\rho) - b)^2} dE. \tag{3.30}$$

Since

$$\frac{1}{2}E^2(\bar\rho) - H(\bar\rho) = \frac{1}{2}E_0^2 - H(\rho_0),$$

$$E'(\bar\rho) = \frac{H'(\bar\rho)}{E(\bar\rho)}, \tag{3.31}$$

provided $E(\bar\rho) \neq 0$. Moreover, in view of the fact $E_r^2(\bar\rho) = E^2(\bar\rho) + 2\int_{F(\bar\rho)}^{\rho_r} H'(t)dt$, we obtain

$$\begin{aligned} E_r(\bar\rho)E_r'(\bar\rho) &= E(\bar\rho)E'(\bar\rho) - H'(F(\bar\rho))F'(\bar\rho) \\ &= H'(\bar\rho) - H'(F(\bar\rho))F'(\bar\rho). \end{aligned} \tag{3.32}$$

We calculate $\frac{\partial\rho(E,\bar\rho)}{\partial\bar\rho}$ as follows. First, along the trajectory form $(F(\bar\rho), E(\bar\rho))$ to $(\rho_r, E_r(\bar\rho))$,

$$\frac{1}{2}E^2 - H(\rho(E,\bar\rho)) = \frac{1}{2}E^2(\bar\rho) - H(F(\bar\rho)). \tag{3.33}$$

It follows from (3.6) and (3.33) that

$$\begin{aligned} \frac{\partial\rho(E,\bar\rho)}{\partial\bar\rho} &= \frac{H'(F(\bar\rho)F'(\bar\rho) - E(\bar\rho)E'(\bar\rho)}{H'(\rho(E,\bar\rho))} \\ &= \frac{H'(F(\bar\rho))F'(\bar\rho) - H'(\bar\rho)}{H'(\rho(E,\bar\rho))}. \end{aligned} \tag{3.34}$$

Therefore, (3.30)-(3.34) imply

$$\begin{aligned} L_2'(\bar\rho) = &\big(H'(\bar\rho) - H'(F(\bar\rho))F'(\bar\rho)\big)\left(\frac{1}{(\rho_r - b)E_r(\bar\rho)} + \int_{E(\bar\rho)}^{E_r(\bar\rho)} \frac{dE}{(\rho(E,\bar\rho)-b)^2H'(\rho(E,\bar\rho))}ds\right) \\ &- \frac{H'(\bar\rho)}{(F(\bar\rho)-b)E(\bar\rho)}. \end{aligned} \tag{3.35}$$

By (2.1), (3.5) and (3.6), we have

$$H'(\bar\rho) - H'(F(\bar\rho))F'(\bar\rho) = (p'(\bar\rho) - \frac{J^2}{\bar\rho^2})\frac{b(\bar\rho - F(\bar\rho)}{\bar\rho F(\bar\rho)}. \tag{3.36}$$

Therefore, by virtue of (3.28), (3.35) and (3.36), we have

$$\begin{aligned} L_1'(\bar\rho) + L_2'(\bar\rho) = &\left(p'(\bar\rho) - \frac{J^2}{\bar\rho^2}\right)(F(\bar\rho) - \bar\rho) \\ &\cdot\left\{-\frac{b}{\bar\rho F(\bar\rho)}\left(\frac{1}{(\rho_r - b)E_r(\bar\rho)} + \int_{E(\bar\rho)}^{E_r(\bar\rho)} \frac{dE}{(\rho(E,\bar\rho)-b)^2H'(\rho(E,\bar\rho))}\right) + \frac{1}{\bar\rho E(\bar\rho)(F(\bar\rho)-b)}\right\}. \end{aligned} \tag{3.37}$$

Next, we calculate the term $\int_{E(\bar\rho)}^{E_r(\bar\rho)} \frac{dE}{(\rho(E,\bar\rho)-b)^2H'(\rho(E,\bar\rho))}$. We make a substitution $\hat\rho = \rho(E,\bar\rho)$. By the definition of $\rho(E,\bar\rho)$, we have

$$\frac{1}{2}E^2 - H(\hat\rho) = \frac{1}{2}E^2(\bar\rho) - H(F(\bar\rho)). \tag{3.38}$$

Thus

$$EdE = H'(\hat\rho)d\hat\rho. \tag{3.39}$$

Therefore, noticing that $\rho(E(\bar\rho),\bar\rho)=F(\bar\rho)$ and $\rho(E_r(\bar\rho),\bar\rho)=\rho_r$, we have

$$\int_{E(\bar\rho)}^{E_r(\bar\rho)}\frac{dE}{(\rho(E,\bar\rho)-b)^2H'(\rho(E,\bar\rho))}=\int_{F(\bar\rho)}^{\rho_r}\frac{d\hat\rho}{(\hat\rho-b)^2sgn(E(\bar\rho))\sqrt{E^2(\bar\rho)+2\int_{F(\bar\rho)}^{\hat\rho}H'(t)dt}}. \tag{3.40}$$

Here we have used the fact $E=sgn(E(\bar\rho))\sqrt{E^2(\bar\rho)+2\int_{F(\bar\rho)}^{\hat\rho}H'(t)dt}$ along the trajectory from $(F(\bar\rho),E(\bar\rho))$ to $(\rho_r,E_r(\bar\rho))$ and (3.39). Next, integration by parts gives

$$\begin{aligned}&\int_{E(\bar\rho)}^{E_r(\bar\rho)}\frac{dE}{(\rho(E,\bar\rho)-b)^2H'(\rho(E,\bar\rho))}\\&=\int_{F(\bar\rho)}^{\rho_r}\frac{d\hat\rho}{(\hat\rho-b)^2sgn(E(\bar\rho))\sqrt{E^2(\bar\rho)+2\int_{F(\bar\rho)}^{\hat\rho}H'(t)dt}}\\&=-\frac{1}{(\rho_r-b)E_r(\bar\rho)}+\frac{1}{(F(\bar\rho)-b)E(\bar\rho)}-\int_{F(\bar\rho)}^{\rho_r}\frac{H'(\hat\rho)}{(\hat\rho-b)sgn(E(\bar\rho))[E^2(\bar\rho)+2\int_{F(\bar\rho)}^{\hat\rho}H'(t)dt]^{3/2}}d\hat\rho.\end{aligned} \tag{3.41}$$

It follows from (3.37) and (3.41) that

$$\begin{aligned}L_1'(\bar\rho)+L_2'(\bar\rho)=&\left(p'(\bar\rho)-\frac{J^2}{\bar\rho^2}\right)\frac{F(\bar\rho)-\bar\rho}{\bar\rho}\\&\cdot\left\{\frac{1-\frac{b}{F(\bar\rho)}}{(F(\bar\rho)-b)E(\bar\rho)}+\frac{b}{F(\bar\rho)}\int_{F(\bar\rho)}^{\rho_r}\frac{H'(\hat\rho)}{(\hat\rho-b)sgn(E(\bar\rho))[E^2(\bar\rho)+2\int_{F(\bar\rho)}^{\hat\rho}H'(t)dt]^{3/2}}d\hat\rho.\right\}\end{aligned} \tag{3.42}$$

This proves (3.42). □

Lemma 3.5. *For the fixed (ρ_0,E_0) and ρ_r satisfying $\rho_0<\rho_s$, $\rho_r>\rho_s$, let $(\bar\rho,E(\bar\rho))\in T(\rho_0,E_0)$ be a state satisfying $0<\bar\rho<\rho_s$ and E does not change sign along the trajectory from (ρ_0,E_0) to $(\bar\rho,E(\bar\rho))$. Moreover, the trajectory starting from $(F(\bar\rho),E(\bar\rho))$ crosses the ρ-axis at the point $(q(\bar\rho),0)$ and then intersects the line $\rho=\rho_r$ at $(\rho_r,E_r(\bar\rho))$ (i.e. $(q(\bar\rho),0)\in T(F(\bar\rho),E(\bar\rho))$ and $(\rho_r,E_r(\bar\rho))\in T(q(\bar\rho),0)$). Furthermore, we assume that $\rho\neq b$ on the trajectory $T(F(\bar\rho),E(\bar\rho))$. Let*

$$X(\bar\rho)=\ell((\rho_0,E_0);(\bar\rho,E(\bar\rho)))+\ell((F(\bar\rho),E(\bar\rho));(\rho_r,E_r(\bar\rho))), \tag{3.43}$$

then

$$\begin{aligned}\frac{dX(\bar\rho)}{d\bar\rho}=&\left(p'(\bar\rho)-\frac{J^2}{\bar\rho^2}\right)\frac{F(\bar\rho)-\bar\rho}{\bar\rho}\\&\cdot\left\{\frac{1}{(F(\bar\rho)-b)E(\bar\rho)}-\frac{b}{F(\bar\rho)}[\frac{1}{(\rho_r-b)E_r(\bar\rho)}+\int_{F(\bar\rho)}^{q(\bar\rho)}\frac{d\rho}{(\rho-b)^2E_1(\rho,\bar\rho)}+\int_{q(\bar\rho)}^{\rho_r}\frac{d\rho}{(\rho-b)^2E_2(\rho,\bar\rho)}]\right\},\end{aligned} \tag{3.44}$$

provided $E(\bar{\rho}) \neq 0$ *and* $E_r(\bar{\rho}) \neq 0$, *where*

$$E_1(\rho, \bar{\rho}) = sgn(E(\bar{\rho}))\sqrt{E^2(\bar{\rho}) + 2\int_{F(\bar{\rho})}^{\rho} H'(t)dt}, \tag{3.45}$$

$$E_2(\rho, \bar{\rho}) = -sgn(E(\bar{\rho}))\sqrt{E^2(\bar{\rho}) + 2\int_{F(\bar{\rho})}^{\rho} H'(t)dt}. \tag{3.46}$$

Remark 1. By the definition of $E_1(\rho, \bar{\rho})$, $E_2(\rho, \bar{\rho})$ and $q(\bar{\rho})$, it is clear that

$$E_1(q(\bar{\rho}), \bar{\rho}) = E_2(q(\bar{\rho}), \bar{\rho}) = 0, \tag{3.47}$$

$$E_1(F(\bar{\rho}), \bar{\rho}) = E(\bar{\rho}), \ E_2(\rho_r, \bar{\rho}) = E_r(\bar{\rho}). \tag{3.48}$$

Proof of Lemma 3.5.

Let

$$X_1(\bar{\rho}) = \ell((\rho_0, E_0); (\bar{\rho}, E(\bar{\rho}))),$$

and

$$X_2(\bar{\rho}) = \ell((F(\bar{\rho}), E(\bar{\rho})); (\rho_r, E_r(\bar{\rho}))). \tag{3.49}$$

Similar to (3.28), we have,

$$X_1'(\bar{\rho}) = \frac{p'(\bar{\rho}) - \frac{J^2}{\bar{\rho}^2}}{\bar{\rho}E(\bar{\rho})}, \tag{3.50}$$

as long as $E(\bar{\rho}) \neq 0$. Since

$$\frac{E_1(\rho, \bar{\rho})}{dx} = \rho - b,$$

thus

$$\frac{\partial E_1(\rho, \bar{\rho})/\partial \rho}{\rho - b} = dx.$$

Therefore

$$L_2(\bar{\rho}) =: \ell\left((F(\bar{\rho}), E(\bar{\rho}); (q(\bar{\rho}), 0)\right) = \int_{F(\bar{\rho})}^{q(\bar{\rho})} \frac{\partial E_1(\rho, \bar{\rho})/\partial \rho}{\rho - b} d\rho. \tag{3.51}$$

Noticing (3.47) and (3.48), integration by parts gives,

$$L_2(\bar{\rho}) = -\frac{E(\bar{\rho})}{F(\bar{\rho}) - b} + \int_{F(\bar{\rho})}^{q(\bar{\rho})} \frac{E_1(\rho, \bar{\rho})}{(\rho - b)^2} d\rho. \tag{3.52}$$

Similarly,

$$L_3(\bar{\rho}) =: \ell\left((q(\bar{\rho}), 0); (\rho_r, E_r(\bar{\rho})\right) = \frac{E_r(\bar{\rho})}{\rho_r - b} + \int_{q(\bar{\rho})}^{\rho_r} \frac{E_2(\rho, \bar{\rho})}{(\rho - b)^2} d\rho. \tag{3.53}$$

It should be noted that

$$\ell((F(\bar{\rho}), E(\bar{\rho})); (\rho_r, E_r(\bar{\rho}))) = L_2(\bar{\rho}) + L_3(\bar{\rho}). \tag{3.54}$$

By (3.52), (3.47) and (3.48), we have,

$$L_2'(\bar\rho) = -\frac{E'(\bar\rho)}{F(\bar\rho)-b} + \int_{F(\bar\rho)}^{q(\bar\rho)} \frac{\partial E_1(\rho,\bar\rho)/\bar\rho}{(\rho-b)^2} d\rho. \tag{3.55}$$

Similar to (3.31), we have

$$E'(\bar\rho)E'(\bar\rho) = H'(\bar\rho). \tag{3.56}$$

It follows from (3.45) that

$$\frac{1}{2}E_1^2(\rho,\bar\rho) = \frac{1}{2}E^2(\bar\rho) + \int_{F(\bar\rho)}^{\rho} H'(t)dt.$$

Therefore, in view of (3.56),

$$E_1(\rho,\bar\rho)\frac{\partial E_1}{\partial\bar\rho} = H'(\bar\rho) - H'(F(\bar\rho))F'(\bar\rho). \tag{3.57}$$

So

$$L_2'(\bar\rho) = -\frac{H'(\bar\rho)}{(F(\bar\rho)-b)E(\bar\rho)} + (H'(\bar\rho) - H'(F(\bar\rho))F'(\bar\rho)) \int_{F(\bar\rho)}^{q(\bar\rho)} \frac{1}{(\rho-b)^2 E_1(\rho,\bar\rho)} d\rho. \tag{3.58}$$

Now we show that $\int_{F(\bar\rho)}^{q(\bar\rho)} \frac{1}{(\rho-b)^2 E_1(\rho,\bar\rho)} d\rho$ is finite. This is necessary because $E_1(q(\bar\rho),\bar\rho)) = 0$. Let

$$g(\rho) = E^2(\bar\rho) + 2\int_{F(\bar\rho)}^{\rho} H'(t)dt.$$

By (3.45)

$$E_1(\rho,\bar\rho) = sgn(E(\rho))\sqrt{g(\rho)}. \tag{3.59}$$

It is clear that $g((q(\bar\rho)) = 0$ and

$$g'(q(\bar\rho)) = H'(q(\bar\rho)) \neq 0, \tag{3.60}$$

because $q(\bar\rho) \neq b$ and $q(\bar\rho) \neq \rho_s$. So

$$g(\rho) = g'(q(\bar\rho)(\rho - q(\bar\rho)) + O((\rho - q(\bar\rho))^2),$$

as $|\rho - q(\bar\rho))|$ is small. This, together with (3.59) and (3.60), implies that $\int_{F(\bar\rho)}^{q(\bar\rho)} \frac{1}{(\rho-b)^2 E_1(\rho,\bar\rho)} d\rho$ is finite. By a similar method as above, we can show that

$$L_3'(\bar\rho) = (H'(\bar\rho) - H'(F(\bar\rho))F'(\bar\rho)) \left(\frac{1}{(\rho_r - b)E_r(\bar\rho)} + \int_{q(\bar\rho)}^{\rho_r} \frac{1}{(\rho-b)^2 E_2(\rho,\bar\rho)} d\rho \right). \tag{3.61}$$

(3.44) follows from (3.50), (3.58) and (3.61), in view of (3.22). □

In the following, since the cases of $0 < b < \rho_s$ and $b > \rho_s$ are completely different, we consider them separately.

4 Transonic shock solutions for the case of $0 < b < \rho_s$.

We consider this problem for the following different cases.
Case 3.1 (ρ_l, α) is inside the critical trajectory, i.e.,

$(\frac{1}{2}\alpha^2 - \int_{\rho_s}^{\rho_l} H'(s)ds < 0$ and $0 < \rho_l < \rho_s$ $(\rho_l, \alpha) \neq (b, 0))$ (see Figure 3).

By case 1 discussed in section 2, the initial value problem

$$\begin{cases} (p(\rho) + \frac{J^2}{\rho})_x = \rho E, \quad E_x = \rho - b, \quad \text{for } x > 0, \\ (\rho, E)(0) = (\rho_l, \alpha) \end{cases} \tag{4.1}$$

has a unique periodic supersonic solution. We denote the period of the solution by P. We assume there exists a positive integer k such that

$$kP + 2B < L < (k+1)P + 2B, \tag{4.2}$$

where B is the length of x for the solution of (1.8) to travel from the state $(\rho_r, -E_c^r)$ to the state $(\rho_s, 0)$. Here E_c^r is defined as follows: there two intersection points of the line $\rho = \rho_r$ with the critical trajectory in (ρ, E) plane, we denote those two intersection points by (ρ_r, E_c^r) and $(\rho_r, -E_c^r)$ $(E_c^r > 0)$. The length in x for the solution of system (1.8) to travel from $(\rho_r, -E_c^r)$ to the state $(\rho_s, 0)$ is the same as that for the solution to travel from $(\rho_s, 0)$ to (ρ_r, E_c^r). In the case of (4.2), the solution starts from (ρ_l, α) and travels k times along the periodic trajectory and come back to the state (ρ_l, α) at $x = kP$. In this case, we expect the shock location is in the interval (kP, L). Due to this, for simplicity, we may assume $k = 0$. Let $\rho_L(x)$ be the solution of initial value problem (4.1). Let ρ_{min} and ρ_{max} be the minimum and maximum values of $\rho_L(x)$, i. e.,

$$\rho_{min} = \min_{0 \le x \le P} \rho_L(x), \;\; \rho_{max} = \max_{0 \le x \le P} \rho_L(x), \tag{4.3}$$

where P is the period of the solution.

In order to construct the transonic shock solution, we assume $\alpha > 0$, the case for $\alpha < 0$ can be handled similarly. Suppose $\rho_r > F(\rho_{min})$, we define E_{max} by the value of E such that the states $(F(\rho_l), \alpha))$ and (ρ_r, E_{max}) are on the same trajectory of system (1.8), i.e.,

$$E_{max} = \sqrt{\alpha^2 + 2\int_{F(\rho_l)}^{\rho_r} H'(s)ds}, \tag{4.4}$$

and E_{min} by the value of E such that the states $(F(\rho_{min}), 0))$ and (ρ_r, E_{min}) are on the same trajectory of system (1.8), i.e.,

$$E_{min} = \sqrt{2\int_{F(\rho_{min})}^{\rho_r} H'(s)ds}. \tag{4.5}$$

We have the following theorem

Theorem 4.1. *For $\alpha > 0$, suppose that (ρ_l, α) is inside the supersonic loop of critical trajectory, i.e.,*

$$\frac{1}{2}\alpha^2 - \int_{\rho_s}^{\rho_l} H'(s)ds < 0, \text{ and } 0 < \rho_l < \rho_s, \tag{4.6}$$

and

$$(\rho_l, \alpha) \neq (b, 0). \tag{4.7}$$

If $\rho_r > F(\rho_{min})$ and

$$\ell((F(\rho_l), \alpha); (\rho_r, E_{max})) \leq L \leq \ell((\rho_l, \alpha),\ (\rho_{min}, 0)) + \ell((F(\rho_{min}), 0); (\rho_r, E_{min})), \tag{4.8}$$

then there exists a unique state (ρ^*, E^*) on the trajectory of system (1.8) passing through (ρ_l, α) satisfying $\rho_{min} \leq \rho^* \leq \rho_l$ and $E^* \geq 0$ and a unique number β satisfying $E_{min} \leq \beta \leq E_{max}$ such that the following equality holds true:

$$L = \ell\left((\rho_l, \alpha); (\rho^*, E^*)\right) + \ell\left((F(\rho^*), E^*); (\rho_r, \beta)\right). \tag{4.9}$$

So the transonic shock location is $a = \ell((\rho_l,\ \alpha); (\rho^*, E^*))$ (see Fig. 3).

Proof. For any $\bar{\rho} \in [\rho_{min}, \rho_l]$, let

$$E(\bar{\rho}) = \sqrt{\alpha^2 + 2\int_{\rho_l}^{\bar{\rho}} H'(s)ds}. \tag{4.10}$$

$$E_r(\bar{\rho}) = \sqrt{E^2(\bar{\rho}) + 2\int_{F(\bar{\rho})}^{\rho_r} H'(s)ds}, \tag{4.11}$$

i.e., (ρ_l, α) and $(\bar{\rho}, E(\bar{\rho}))$ are on the same supersonic trajectory of system (1.8) and $(\rho_r, E_r(\bar{\rho}))$ and $(F(\bar{\rho}), E(\bar{\rho}))$ are on the same subsonic trajectory of (1.8). Let

$$X(\bar{\rho}) = \ell((\rho_l, \alpha); (\bar{\rho}, E(\bar{\rho}))) + \ell((F(\bar{\rho}), E(\bar{\rho})); (\rho_r, E_r(\bar{\rho}))), \text{ for } \rho_{min} < \rho \leq \rho_l. \tag{4.12}$$

By (3.11), we have

$$X'(\bar{\rho}) < 0, \text{ for } \rho_{min} < \rho \leq \rho_l. \tag{4.13}$$

This is because $0 < \bar{\rho} < \rho_s < F(\bar{\rho}) < \rho_r$, $E(\bar{\rho}) > 0$ and $0 < b < \rho_s$. (4.8) follows from (4.13) then. □

We still assume $\rho_r > F(\rho_{min})$. We denote

$$L_3 = \ell\left((\rho_l, \alpha); (\rho_{min}, 0)\right), \tag{4.14}$$

$$L_4 = \ell\left((F(\rho_{min}), 0); (\rho_r, E_{min})\right), \tag{4.15}$$

where $E_{min} = \sqrt{2\int_{F(\rho_{min})}^{\rho_r} H'(s)ds}$,

$$L_5 = \ell\left((\rho_{min}, 0); (\rho_{max}, 0)\right) \tag{4.16}$$

$$L_6 = \ell\left(F(\rho_{max}), 0); (\rho_r, \tilde{E})\right), \tag{4.17}$$

where $\tilde{E} = \sqrt{2\int_{F(\rho_{max})}^{\rho_r} H'(s)ds}$. At first, we have the following lemma.

Lemma 4.1. *For $\alpha > 0$, suppose that (ρ_l, α) is inside the supersonic loop of critical trajectory, i.e. ,*

$$\frac{1}{2}\alpha^2 - \int_{\rho_s}^{\rho_l} H'(s)ds < 0, \text{ and } 0 < \rho_l < \rho_s, \tag{4.18}$$

and

$$(\rho_l, \alpha) \neq (b, 0). \tag{4.19}$$

If $\rho_r > F(\rho_{min})$, then

$$L_4 < L_6. \tag{4.20}$$

Proof. We define

$$x(\rho) = \ell\left((\rho, 0); (\rho_r, \bar{E}_s(\rho)\right), \ for \ F(\rho_{max}) \leq \rho \leq F(\rho_{min}), \tag{4.21}$$

where $\bar{E}_s(\rho) = \sqrt{2\int_\rho^{\rho_r} H'(t)dt}$. By Lemma 3.2, we get

$$x(\rho) = \int_0^{\bar{E}_s(\rho)} \frac{dz}{\bar{\rho}(z, \rho) - b}, \tag{4.22}$$

where the function $\bar{\rho}(z, \rho)$ is given by

$$\frac{1}{2}z^2 = \int_\rho^{\bar{\rho}} H'(t)dt, \ for \ 0 \leq z \leq \bar{E}_s(\rho) \ and \ \rho \leq \bar{\rho} \leq \rho_r. \tag{4.23}$$

Notice that $\bar{\rho}(\bar{E}_s(\rho), \rho) = \rho_r$, we have

$$x'(\rho) = \frac{\bar{E}'_s(\rho)}{\rho_r - b} - \int_0^{\bar{E}_s(\rho)} \frac{\frac{\partial \bar{\rho}(z,\rho)}{\partial \rho}}{(\bar{\rho}(z, \rho) - b)^2} dz, \ F(\rho_{max}) \leq \rho \leq F(\rho_{min}). \tag{4.24}$$

By the definition of $\bar{E}_s(\rho)$ ($\bar{E}_s(\rho) = \sqrt{2\int_\rho^{\rho_r} H'(t)dt}$), we have

$$\bar{E}_s(\rho)\bar{E}'_s(\rho) = -H'(\rho). \tag{4.25}$$

It follows from (4.23) that

$$\frac{\partial \bar{\rho}(z, \rho)}{\partial \rho} = \frac{H'(\rho)}{H'(\bar{\rho})}. \tag{4.26}$$

Therefore, (4.24)-(4.26) imply

$$x'(\rho) = -\frac{H'(\rho)}{(\rho_r - b)\bar{E}_s(\rho)} - \int_0^{\bar{E}_s(\rho)} \frac{H'(\rho)}{(\bar{\rho}(z, \rho) - b)^2 H'(\bar{\rho}(z, \rho))} dz, \ F(\rho_{max}) \leq \rho \leq F(\rho_{min}). \tag{4.27}$$

For $\rho_{sonic} < F(\rho_{max}) \leq \rho \leq F(\rho_{min})$, we have $H'(z) > 0$, $\bar{E}_s(\rho) > 0$ and $H'(\bar{\rho}(z, \rho)) > 0$. By (4.27), we have

$$x'(\rho) < 0 \ for \ F(\rho_{max}) \leq \rho \leq F(\rho_{min}) \tag{4.28}$$

Because $L_4 = x(F(\rho_{min}))$ and $L_6 = x(F(\rho_{max}))$ and $F(\rho_{min}) > F(\rho_{max})$, (4.20) follows from (4.28).

□

Theorem 4.2. *For $\alpha > 0$, suppose that (ρ_l, α) is inside the supersonic loop of critical trajectory, i.e.,*

$$\frac{1}{2}\alpha^2 - \int_{\rho_s}^{\rho_l} H'(s)ds < 0, \text{ and } 0 < \rho_l < \rho_s, \tag{4.29}$$

and

$$(\rho_l, \alpha) \neq (b, 0). \tag{4.30}$$

If $\rho_r > F(\rho_{min})$ and

$$L_3 + L_4 < L < L_3 + L_5 + L_6 \tag{4.31}$$

there exist a unique state (ρ_*, E_*) on the trajectory of system (1.8) passing through (ρ_l, α) satisfying $\rho_{min} < \rho_* < \rho_{max}$ and $E_* < 0$ and a unique number β_1 such that the following equality holds true:

$$L = \ell\left((\rho_l, \alpha); (\rho_*, E_*)\right) + \ell\left((F(\rho_*), E_*); (\rho_r, \beta_1)\right). \tag{4.32}$$

So the transonic shock location is $a = \ell((\rho_l), \alpha); (\rho_*, E_*))$ (see Fig. 4).

Proof. By Lemma 4.1, we know that

$$L_3 + L_4 < L_3 + L_5 + L_6. \tag{4.33}$$

The existence of (ρ_*, E_*) follows from intermediate value theorem. So the remaining task is to prove the uniqueness. This is done as follows. For $\bar{\rho} \in [\rho_{min}, \rho_{max}]$, we define

$$E(\bar{\rho}) = -\sqrt{2\int_{\rho_{min}}^{\bar{\rho}} H'(s)ds}. \tag{4.34}$$

$$E_r(\bar{\rho}) = \sqrt{E(\bar{\rho})^2 + 2\int_{F(\bar{\rho})}^{\rho_r} H'(s)ds}, \tag{4.35}$$

i.e., $(\rho_{min}, 0)$ and $(\bar{\rho}, E(\bar{\rho}))$ are on the same supersonic trajectory of system (1.8) and $(\rho_r, E_r(\bar{\rho})))$ and $(F(\bar{\rho}), E(\bar{\rho}))$ are on the same subsonic trajectory of (1.8).Let

$$X(\bar{\rho}) = \ell\left((\rho_{min}), 0); (\bar{\rho}, E(\bar{\rho}))\right) + \ell((F(\bar{\rho}), E(\bar{\rho})); (\rho_r, E_r(\bar{\rho}))), \quad \text{for} \quad \rho_{min} < \bar{\rho} \leq \rho_{max}. \tag{4.36}$$

Then we can apply Lemma 3.4 to show that

$$X'(\bar{\rho}) > 0, \quad \text{for } \bar{\rho}_{min} < \rho < \rho_{max}. \tag{4.37}$$

This is because $0 < b < \rho_s$, and for $\rho_{min} < \bar{\rho} < \rho_{max}$, $0 < \bar{\rho} < \rho_s$, $E(\bar{\rho}) < 0$, $E_r(\bar{\rho}) > 0$, $\rho_r > \rho_s > b$, $q(\bar{\rho}) < F(\bar{\rho})$ and $q(\bar{\rho}) < F(\bar{\rho})$. Moreover,

$$E_1(\rho, \bar{\rho}) < 0, \quad \text{for } \ q(\bar{\rho}) < \rho \leq F(\bar{\rho}),$$

and

$$E_1(\rho,\bar\rho) > 0, \text{ for } q(\bar\rho) < \rho \le \rho_r.$$

These quantities are defined in Lemma 3.4. Theorem 4.2 follows from (4.37). □

We define L_7 by

$$L_7 = \ell\left((\rho_{max}, 0); (\rho_l, \alpha)\right), \tag{4.38}$$

$$L_8 = \ell\left(F(\rho_l, \alpha); (\rho_r, E_\alpha)\right), \tag{4.39}$$

where $E_\alpha = \sqrt{\alpha^2 + 2\int_{F(\rho_l)}^{\rho^r} H'(s)ds}$. Then we have

Theorem 4.3. *For $\alpha > 0$, suppose that (ρ_l, α) is inside the supersonic loop of critical trajectory, i.e.,*

$$\frac{1}{2}\alpha^2 - \int_{\rho_s}^{\rho_l} H'(s)ds < 0, \text{ and } 0 < \rho_l < \rho_s, \tag{4.40}$$

and

$$(\rho_l, \alpha) \neq (b, 0). \tag{4.41}$$

If $\rho_r > F(\rho_{min})$ and

$$L_3 + L_5 + L_6 < L < L_3 + L_5 + L_7 + L_8, \tag{4.42}$$

there exist a unique state (ρ^{**}, E^{**}) on the trajectory of system (1.8) passing through (ρ_l, α) satisfying $\rho_l < \rho^{**} < \rho_{max}$ and $E^{**} > 0$ and a unique number β_2 such that the following equality holds true:

$$L = \ell\left((\rho_l, \alpha); (\rho^{**}, E^{**})\right) + \ell\left((F(\rho^{**}), E{*}{*}); (\rho_r, \beta_2)\right). \tag{4.43}$$

So the transonic shock location is $\ell((\rho_l, \alpha); (\rho^{**}, E^{**}))$.

Proof. For $\bar\rho \in [\rho_l, \rho_{max}]$, we define

$$E(\bar\rho) = \sqrt{2\int_{\rho_{max}}^{\bar\rho} H'(s)ds.} \tag{4.44}$$

$$E_r(\rho) = \sqrt{E(\bar\rho)^2 + 2\int_{F(\bar\rho)}^{\rho_r} H'(s)ds}, \tag{4.45}$$

i.e., $(\rho_{max}, 0)$ and $(\bar\rho, E(\bar\rho))$ are on the same supersonic trajectory of system (1.8) and $(\rho_r, E_r(\bar\rho)))$ and $(F(\bar\rho), E(\bar\rho))$ are on the same subsonic trajectory of (1.8). Let

$$X(\bar\rho) = \ell\left((\rho_{max}), 0); (\bar\rho, E(\bar\rho))\right) + \ell((F(\bar\rho), E(\bar\rho)); (\rho_r, E_r(\bar\rho))), \text{ for } \rho_l < \bar\rho < \rho_{max}. \tag{4.46}$$

By a similar approach as in the proof of Theorem 4.1, we can show that

$$X_3'(\bar\rho) < 0, \text{ for } \rho_l < \rho < \rho_{max}. \tag{4.47}$$

This shows that $L_6 < L_7 + L_8$. So the assumption (4.42) makes sense. Also Theorem 4.3 follows from (4.47). $\square$

Theorems 4.1-4.3 complete all the possible cases for the interval length L for the case when (ρ_l, α) is inside the supersonic loop of the critical trajectory. We turn to the case when (ρ_l, α) is on the supersonic loop of the critical trajectory, i. e.,

$$\frac{1}{2}\alpha^2 - \int_{\rho_s}^{\rho_l} H'(s)ds = 0.$$

and $\rho_l < \rho_s$. We still assume $\alpha > 0$. (The case when $\alpha < 0$ can be handled similarly). There are two intersection points of the supersonic loop of the critical trajectory and the line $E = 0$. One is $(\rho_s, 0)$, another one is $(\rho^c_{\min}, 0)$ $(\int_{\rho_s}^{\rho^c_{min}} H'(t)dt = 0, 0 < \rho^c_{min} < b)$. We define the following quantities:

$$\bar{E}(\rho_l) = \sqrt{\alpha^2 + 2\int_{F(\rho_l)}^{\rho_r} H'(t)dt}, \bar{E}(\rho^c_{min}) = \sqrt{2\int_{F(\rho^c_{min})}^{\rho_r} H'(t)dt}, \tag{4.48}$$

$$L^c_1 = \ell((\rho_l, \alpha);\ (\rho^c_{min}, 0)), \tag{4.49}$$

$$L^c_2 = \ell((\rho^c_{min}, 0);\ (\rho_s, 0)), \tag{4.50}$$

$$L^c_3 = \ell((\rho^c_{min}, 0);\ (\rho_l, \alpha)), \tag{4.51}$$

$$L^c_4 = \ell((F(\rho_l), \alpha);\ (\rho_r, \bar{E}(\rho_l))), \tag{4.52}$$

$$L^c_5 = \ell((F(\rho^c_{min}), 0);\ (\rho_r, \bar{E}(\rho^c_{min}))), \tag{4.53}$$

$$L^c_6 = \ell((\rho_s, 0);\ (\rho_r, E_c)), \tag{4.54}$$

where E_c is defined by $E_c = \sqrt{2\int_{\rho_s}^{\rho_r} H'(t)dt}$. We have the following theorem.

Theorem 4.4. *For $\alpha > 0$, suppose that (ρ_l, α) is on the supersonic loop of critical trajectory, i.e.,*

$$\frac{1}{2}\alpha^2 - \int_{\rho_s}^{\rho_l} H'(s)ds = 0, \text{ and } 0 < \rho_l < \rho_s, \tag{4.55}$$

If $\rho_r > F(\rho^c_{min})$, then
a) If

$$L^c_4 \le L \le L^c_1 + L^c_5, \tag{4.56}$$

where $\bar{E}(\rho_l) = \sqrt{\alpha^2 + 2\int_{F(\rho_l}^{\rho_r} H'(t)dt}$, $\bar{E}(\rho^c_{min}) = \sqrt{2\int_{F(\rho^c_{min}}^{\rho_r} H'(t)dt}$, then there exists a unique state (ρ^*_c, E^*_c) on supersonic loop of the critical trajectory satisfying $\rho^c_{min} \le \rho^*_c \le \rho_l$ and $E^*_c \ge 0$ and a unique number β^c satisfying $\bar{E}(\rho^c_{min})) \le \beta^c \le \bar{E}(\rho_l)$ such that the following equality holds true:

$$L = \ell\left((\rho_l, \alpha); (\rho^*_c, E^*_c)\right) + \ell\left((F(\rho^*_c), E^*_c); (\rho_r, \beta^c)\right). \tag{4.57}$$

So the transonic shock location is $a = \ell((\rho_l,\ \alpha); (\rho_c^*, E_c^*))$;
b)If $L_1^c + L_5^c \le L < L_1^c + L_2^c + L_6^c$, then there exists a unique state (ρ_c^{**}, E_c^{**}) on supersonic loop of the critical trajectory satisfying $\rho_{min}^c \le \rho_c^* < \rho_s$ and $E_c^* \le 0$ and a unique number β_1^c such that the following equality holds true:

$$L = L_1^c + \ell\left((\rho_{min}^c, 0); (\rho_c^{**} E_c^{**})\right) + \ell\left((F(\rho_c^{**}), E_c^{**}); (\rho_r, \beta_1^c)\right). \tag{4.58}$$

So the transonic shock location is $a = L_1^c + \ell\left((\rho_{min}^c, 0); (\rho_c^{**} E_c^{**})\right)$;
c)If $L = L_1^c + L_2^c + L_6^c$, then the solution of the boundary value problem of (1.8) and (1.9) is smooth (no transonic shock). In (ρ, E)-phase plane, the solution starts from (ρ_l, α), travels along the supersonic loop of the critical trajectory to the sonic state $(\rho_s, 0)$, then travels along the subsonic branch of the critical trajectory to the state (ρ_r, E_c);
(d) If $L_1^c + L_2^c + L_6^c < L \le L_1^c + L_2^c + L_3^c + L_4^c$, then there exists a unique state (ρ_c^0, E_c^0) on supersonic loop of the critical trajectory satisfying $\rho_l \ge \rho_c^0 < \rho_s$ and $E_c^0 > 0$ and a unique number β_2^c such that the following equality holds true:

$$L = L_1^c + L_2^c + \ell\left((\rho_s, 0); ((\rho_c^0, E_c^0))\right) + \ell\left((F(\rho_c^0), E_c^0); (\rho_r, \beta_2^c)\right). \tag{4.59}$$

So the transonic shock location is $a = L_1^c + L_2^c + \ell\left((\rho_s, 0); ((\rho_c^0, E_c^0))\right)$;

The proof of this theorem is similar to those of Theorems 4.2 and 4.3. So we omit it.

When the state (ρ_l, α) is outside the supersonic loop of the critical trajectory, i.e.

$$\frac{1}{2}\alpha^2 - \int_{\rho_s}^{\rho_l} H'(s)ds > 0, \text{ and } 0 < \rho_l < \rho_s, \tag{4.60}$$

the situation is more complicated. We consider the following cases. We use T_o to denote the supersonic trajectory passing through the point (ρ_l, α) on (ρ, E) phase plane, i.e.,

$$T_o = \{(\rho, E) : \frac{1}{2}E^2 = \frac{1}{2}\alpha^2 + \int_{\rho_l}^{\rho} H'(t)dt, 0 < \rho < \rho_s\}. \tag{4.61}$$

We use the T_o^{shock} to denote the shock conjugate of T_o,

$$T_o^{shock} = \{(F(\rho), E) : (\rho, E) \in T_o\}. \tag{4.62}$$

It is easy to verify that T_o^{shock} intersects the subsonic branch of the critical trajectory at two points, denoted by $(\check{\rho}, \check{E})$ and $(\check{\rho}, -\check{E})$, where $\check{E} > 0$. Also, we denote the intersection point of T_o with ρ-axis by $(\rho_{min}^o, 0)$. We assume that $\rho_r > F(\rho_{min}^o)$. We assume $\alpha > 0$. We define the following quantity:

$$\bar{L}(\rho) = \ell\left((\rho_l, \alpha); (\rho, E(\rho))\right) + \ell\left((F(\rho), E(\rho)); (\rho_r, \bar{E}(\rho))\right), \text{ for } (\rho, E(\rho)) \in T_o, E(\rho) \ge 0, \tag{4.63}$$

where $\bar{E}(\rho) = \sqrt{E^2(\rho) + 2\int_{F(\rho)}^{\rho_r} H'(t)dt}$. By the same argument as in the proof of theorem 4.2, we can show that $\bar{L}(\rho)$ is a strictly decreasing function of ρ for $\rho_{min}^o \leq \rho \leq \rho_l$. We also define the quantity

$$\tilde{L}(\rho) = \ell\left((\rho_{min}^o, 0); (\rho, E(\rho))\right) + \ell\left((F(\rho), E(\rho)); (\rho_r, \bar{E}(\rho))\right), \tag{4.64}$$

for $(\rho, E(\rho)) \in T_o, \rho_{min}^o \leq \rho < F^{-1}(\check{\rho}), E(\rho) < 0$, where $\bar{E}(\rho) = \sqrt{E^2(\rho) + 2\int_{F(\rho)}^{\rho_r} H'(t)dt}$. By the same argument as in the proof of theorem 3.2, we can show that $\bar{L}(\rho)$ is a strictly increasing function of ρ for $\rho_{min}^o \leq \rho < F^{-1}(\check{\rho})$.

Then we have the following Theorem.

Theorem 4.5. *Assume $\alpha > 0$, then*
1. If $\bar{L}(\rho_l) \leq L \leq \bar{L}(\rho_{min}^o)$, then there exits a unique state $(\rho_1, E_1) \in T_o$ with $E_1 \geq 0$ such that $L = \bar{L}(\rho_1)$. In this case, the transonic shock location is at $a = \ell\left((\rho_l, \alpha); (\rho_1, E_1)\right)$.
2. If $\ell\left((\rho_l, \alpha); (\rho_{min}^o, 0)\right) + \tilde{L}(\rho_{min}^o) \leq L < \ell\left((\rho_l, \alpha); (\rho_{min}^o), 0)\right) + \lim_{\rho \to F^{-1}(\check{\rho})} \tilde{L}(\rho)$,then there exits a unique state $(\rho_2, E_2) \in T_o$ with $-\check{E} < E_2 \leq 0$ such that $L = \tilde{L}(\rho_1)$. In this case, the transonic shock location is at $a = \ell\left((\rho_l, \alpha); (\rho_2, E_2)\right)$.

The proof of this theorem is similar to those for Theorems 4.2 and 4.3. So we omit it. The case when $L \geq \ell\left((\rho_l, \alpha); (\rho_{min}^o), 0)\right) + \lim_{\rho \to F^{-1}(\check{\rho})} \tilde{L}(\rho)$ is more complicated. We have the following theorem for this case. In this case, we do not assume $\rho_r > F(\rho_{min}^o)$.

Theorem 4.6. *Assume $\alpha > 0$. If $L \geq \ell\left((\rho_l, \alpha); (\rho_{min}^o), 0)\right) + \lim_{\rho \to F^{-1}(\check{\rho})} \tilde{L}(\rho)$, the only possible solution of the boundary value problem is described as follows: In (ρ, E)-phase plane, the solution starts from (ρ_l, α), travels along the T_o in the counterclockwise direction and reaches the point $(F^{-1}(\check{\rho}), -\check{E})$, then jumps to the point $(\check{\rho}, -\check{E})$ by a transonic shock. Starting from $(\check{\rho}, -\check{E})$, the solution travels along the lower portion of the subsonic branch of the critical trajectory $\{(\rho, E) : E = -\sqrt{2\int_{\rho_s}^{\rho} H'(t)dt}, \rho > \rho_s\}$ and reaches the sonic point $(\rho_s, 0)$. Starting from the sonic point $(\rho_s, 0)$, the solution travels along the supersonic loop $\{(\rho, E) : \frac{1}{2}E^2 = \int_{\rho_s}^{\rho} H'(t)dt, \rho < \rho_s\}$ k times $k = 0, 1, 2, \cdots$ and comes back to the sonic point. Starting form the sonic point, the solution travels along the upper portion of the subsonic branch of the critical trajectory $\{(\rho, E) : E = \sqrt{2\int_{\rho_s}^{\rho} H'(t)dt}, \rho > \rho_s\}$ in the direction that ρ increases and reaches the state (ρ_r, E_c) where $E_c = \sqrt{2\int_{\rho_s}^{\rho_r} H'(t)dt}$.*

Proof. In (ρ, E)-phase plane, starting from (ρ_l, α), the solution travels along the T_o in the counterclockwise direction. The solution can not jump by a transonic shock before it reaches the point $(F^{-1}(\check{\rho}), -\check{E})$, otherwise it reduces to the case that $L < \ell\left((\rho_l, \alpha); (\rho_{min}^o), 0)\right) + \lim_{\rho \to F^{-1}(\check{\rho})} \tilde{L}(\rho)$ discussed in Theorem 4.5. Also , it can not travel beyond the point $(F^{-1}(\check{\rho}), -\check{E})$. This is because if it travels beyond the point $(F^{-1}(\check{\rho}), -\check{E})$, it can never reach the state ρ_r. This can be shown clearly by a phase plane analysis. So the only possibility is as described in the theorem. □

5 Transonic shock solutions for the case when $b > \rho_s$.

In this section, we study the case when $b > \rho_s$, i. e., b is in subsonic region. It is easy to see that

$$H'(\rho) > 0 \text{ for } 0 < \rho < \rho_s \text{ and } \rho > b, H'(\rho) < 0, \text{ for } \rho_s < \rho < b. \tag{5.1}$$

In order to solve the boundary value problem (1.8) and (1.9), we need several lemmas. First, we have the following Lemma.

Lemma 5.1. *Suppose the pressure function p satisfies (1.3) and $b > \rho_s$. Let ρ_b be the density satisfying*

$$0 < \rho_b < \rho_s, \qquad H(\rho_b) = H(b). \tag{5.2}$$

Then

$$\rho_s < F(\rho_b) < b. \tag{5.3}$$

Proof. Since $H(\rho_b) = H(b)$,

$$\int_{\rho_b}^{b} \frac{t-b}{t}\left(p'(t) - \frac{J^2}{t^2}\right) dt = 0.$$

So

$$\begin{aligned} &(p(b) + \frac{J^2}{b}) - (p(\rho_b) + \frac{J^2}{\rho_b}) \\ &- \int_{\rho_b}^{b}\left(p'(t) - \frac{J^2}{t^2}\right)\frac{b}{t}dt = 0. \end{aligned} \tag{5.4}$$

Let

$$\begin{aligned} f(z) =:&(p(z) + \frac{J^2}{z}) - (p(g(z)) + \frac{J^2}{(g(z)}) \\ &- \int_{g(z)}^{z}\left(p'(t) - \frac{J^2}{t^2}\right)\frac{z}{t}dt, \end{aligned} \tag{5.5}$$

for $z \geq \rho_s$, where $g(z) = F^{-1}(z)$. Since $g(\rho_s) = \rho_s$, we have

$$f(\rho_s) = 0. \tag{5.6}$$

On the other hand,

$$p(g(z)) + \frac{J^2}{g(z)} = p(z) + \frac{J^2}{z}, \quad z \geq \rho_s.$$

Hence,

$$\left(p'(g(z)) - \frac{J^2}{(g(z))^2}\right)g'(z) = p'(z) - \frac{J^2}{z^2}, \quad z \geq \rho_s. \tag{5.7}$$

(5.5) and (5.7) yield,

$$f'(z) = \left(p'(z) - \frac{J^2}{z^2}\right)\frac{1}{g(z)}(z - g(z)) - \int_{g(z)}^{z}\left(p'(t) - \frac{J^2}{t^2}\right)\frac{1}{t}dt$$
$$= \int_{g(z)}^{z}\left(p'(z) - \frac{J^2}{z^2}\right)\frac{1}{g(z)}dt - \int_{g(z)}^{z}\left(p'(t) - \frac{J^2}{t^2}\right)\frac{1}{t}dt. \tag{5.8}$$

Since $p''(\rho) > 0$ for $\rho > 0$ (see (1.3)), we have

$$p'(z) - \frac{J^2}{z^2} > p'(t) - \frac{J^2}{t^2}, \text{ for } g(z) \le t < z.$$

This, together with (5.8), implies

$$f'(z) > 0, \text{ for } z > \rho_s.$$

Therefore, in view of (5.6), we have

$$f(b) > 0, \tag{5.9}$$

since $b > \rho_s$. This means, in view of (5.5),

$$(p(b) + \frac{J^2}{b}) - (p(g(b)) + \frac{J^2}{(g(b)})$$
$$- \int_{g(b)}^{b}\left(p'(t) - \frac{J^2}{t^2}\right)\frac{b}{t}dt > 0. \tag{5.10}$$

Next, we define

$$q(\rho) =:(p(b) + \frac{J^2}{b}) - (p(\rho) + \frac{J^2}{\rho})$$
$$- \int_{\rho}^{b}\left(p'(t) - \frac{J^2}{t^2}\right)\frac{b}{t}dt, \text{ for } 0 < \rho < \rho_s. \tag{5.11}$$

It is easy to verify that

$$q'(\rho) = (p'(\rho) - \frac{J^2}{\rho^2})(\frac{b}{\rho} - 1) < 0, \text{for } 0 < \rho < \rho_s, \tag{5.12}$$

since $b > \rho_s > \rho$. This, together with (5.4) and (5.10), implies

$$\rho_b < g(b) = F^{-1}(b). \tag{5.13}$$

Since $F'(\rho) < 0$ for $0 < \rho < \rho_s$ (cf. (3.5)), (5.3) follows. □

Let

$$T_b =: \{(\rho, E) : \frac{1}{2}E^2 - H(\rho) = -H(F^{-1}(b)), \rho \le \rho_s\}, \tag{5.14}$$

and S_b be the set of states which can be connected to the states of T_b by transonic shocks, i.e.,

$$S_b := \{(F(\rho), E) : (\rho, E) \in T_b.\}$$

Then S_b is a curve in (ρ, E)-plane satisfying the following equation

$$\frac{1}{2}E^2 - H(F^{-1}(\rho)) = -H(F^{-1}(b), \qquad \rho_s \le \rho \le b. \tag{5.15}$$

Clearly $(b, 0) \in S_b$. Let

$$C_b^{sub} =: \{(\rho, E) : \frac{1}{2}E^2 - H(\rho) = -H(b), \rho_s \le \rho \le b\},$$

the subsonic branch of the critical trajectory passing through $(b, 0)$. In the next lemma, we will show that curve S_b is outside the curve C_b^{sub} . Precisely, we have

Lemma 5.2.

$$H(F^{-1}(\rho)) - H(F^{-1}(b)) > H(\rho) - H(b), \qquad \rho_s \le \rho < b. \tag{5.16}$$

Proof. Let

$$h(\rho) = H(F^{-1}(\rho)) - H(\rho) + H(b) - H(F^{-1}(b)), \qquad \rho_s \le \rho < b.$$

Since $F^{-1}(\rho_s) = \rho_s$, we have

$$h(\rho_s) = H(b) - H(F^{-1}(b)) = H(\rho_b) - H(F^{-1}(b)), \tag{5.17}$$

where $\rho_b < \rho_s$ is the constant defined in (5.2). Since $\rho_s < b$ we have $H'(\rho) > 0$ for $0 < \rho < \rho_s$. Thus, (5.13) and (5.17) imply

$$h(\rho_s) > 0. \tag{5.18}$$

On the other hand, just as (5.7), we have

$$\left(p'(g(\rho)) - \frac{J^2}{(g(\rho))^2}\right) g'(\rho) = p'(\rho) - \frac{J^2}{\rho^2}, \quad \rho \ge \rho_s, \tag{5.19}$$

where and in the following

$$g(\rho) = F^{-1}(\rho).$$

This gives

$$H'(g(\rho))g'(\rho) = \left(p'(\rho) - \frac{J^2}{\rho^2}\right) \left(\frac{g(\rho) - b}{g(\rho)}\right) \quad \rho \ge \rho_s. \tag{5.20}$$

Therefore,

$$h'(\rho) = \left(p'(\rho) - \frac{J^2}{\rho^2}\right) \left(\frac{b}{\rho} - \frac{b}{g(\rho)}\right), \quad \rho \ge \rho_s. \tag{5.21}$$

Since $g(\rho) = F^{-1}(\rho) < \rho_s$ for $\rho > \rho_s$, we have

$$h'(\rho) < 0, \quad \rho > \rho_s. \tag{5.22}$$

On the other hand

$$h(b) = 0. \tag{5.23}$$

This, together with (5.18) and (5.22), implies (5.16). □

For (ρ_l, α) satisfying $0 < \rho_l < \rho_s$, let

$$\bar{T}(\rho_l, \alpha) = \{(\rho, E) : \frac{1}{2}E^2 - H(\rho) = \frac{1}{2}\alpha^2 - H(\rho_l), \rho > 0\}, \tag{5.24}$$

and

$$S(\rho_l, \alpha) = \{(\rho, E) : \frac{1}{2}E^2 - H(F^{-1}(\rho)) = \frac{1}{2}\alpha^2 - H(\rho_l), \rho \geq \rho_s\}. \tag{5.25}$$

So $S(\rho_l, \alpha)$ is the set of states which can be connected to the set $\{(\rho, E) \in \bar{T}(\rho_l, \alpha) : 0 < \rho \leq \rho_s\}$ by a transonic shock. For the set $\{(\rho, E) \in \bar{T}(\rho_l, \alpha), \rho \geq \rho_s\}$, E^2 is a function of ρ, we denote this function by $E_1^2(\rho)$, i.e.,

$$E_1^2(\rho) = \alpha^2 + 2(H(\rho) - H(\rho_l)). \tag{5.26}$$

For the set $S(\rho_l, \alpha)$, E^2 is also a function of ρ, we denote this function by $E_2^2(\rho)$, i.e.,

$$E_2^2(\rho) = \alpha^2 + 2(H(F^{-1}(\rho)) - H(\rho_l)), \rho_s \leq \rho \leq \rho^\alpha, \tag{5.27}$$

where ρ^α is determined by

$$H(F^{-1}(\rho^\alpha)) = H(\rho_l) - \frac{1}{2}\alpha^2, \rho^\alpha > \rho_s. \tag{5.28}$$

Obviously

$$E_2(\rho^\alpha) = 0. \tag{5.29}$$

Then we have following lemma.

Lemma 5.3.

$$E_1^2(\rho) > E_2^2(\rho), \qquad \text{for } \rho_s < \rho \leq \rho^\alpha. \tag{5.30}$$

Proof. Obviously

$$E_1^2(\rho_s) = E_2^2(\rho_s). \tag{5.31}$$

Let $g(\rho) = F^{-1}(\rho)$ for $\rho \geq \rho_s$. By (3.5), we have

$$(p'(g(\rho)) - \frac{J^2}{(g(\rho))^2}g'(\rho) = p'(\rho) - \frac{J^2}{\rho^2},\ \rho \geq \rho_s. \tag{5.32}$$

It follows from (5.26), (5.27) and (5.32 that

$$\begin{aligned}
&\frac{d(E_1^2(\rho) - E_2^2(\rho))}{d\rho}\\
&= 2(H'(\rho) - H'(g(\rho))g'(\rho))\\
&= 2(p'(\rho) - \frac{J^2}{\rho^2})(\frac{b}{g(\rho)} - \frac{b}{\rho}),
\end{aligned} \tag{5.33}$$

for $\rho_s \leq \rho \leq \rho^\alpha$. For $\rho > \rho_s$, $g(\rho) = F^{-1}(\rho) < \rho_s$, $p'(\rho) - \frac{J^2}{\rho^2} > 0$. Therefore, $\frac{d(E_1^2(\rho) - E_2^2(\rho))}{d\rho} > 0$ for $\rho_s < \rho \leq \rho^\alpha$. This, together with (5.31), implies (5.30). □

We construct transonic shock solutions according to the different situations of (ρ_l, α), ρ_r and L.

5.1 The case for (ρ_l, α) is outside the trajectory though $(F^{-1}(b), 0)$.

In this case,

$$\frac{1}{2}\alpha^2 - H(\rho_l) > -H(F^{-1}(b)),\ 0 < \rho_l < \rho_s. \tag{5.34}$$

We define ρ_{min}^{out} by

$$H(\rho_{min}^{out}) = H(\rho_l) - \frac{1}{2}\alpha^2, 0 < \rho_{min}^{out} < \rho_s$$

(see Figure 6). We construct the solution for the different situations of ρ_r.

Subcase 1. $\rho_r \geq F(\rho_{min}^{out})$**.**

We define

$$\mathfrak{T}(\rho_l, \alpha) =: \{(\rho, E) : \frac{1}{2}E^2 - H(\rho) = \frac{1}{2}\alpha^2 - H(\rho_l),\ \rho_{min}^{out} \leq \rho < \rho_s\} \tag{5.35}$$

the supersonic trajectory passing through (ρ_l, α), and

$$S(\rho_l, \alpha) =: \{(F(\rho, E) : (\rho, E) \in \mathfrak{T}(\rho_l, \alpha)\} \tag{5.36}$$

the curve on (ρ, E)-plane consisting of the states which can be connected to those on $\mathfrak{T}(\rho_l, \alpha)$ by a transonic shock. Then $S(\rho_l, \alpha)$ intersects the critical trajectory passing through $(b, 0)$ at two points (ρ_c, E_c) and $(\rho_c, -E_c)$ with $\rho_c > b$ and $E_c > 0$ (see Figure 6).

In this case, we have the following result.

Theorem 5.1. *Suppose that (ρ_l, α) satisfies (5.34) and $\rho_r \geq F(\rho_{min}^{our})$.*
1) If $\alpha > -E_c$, then we have
1a) the boundary value problem (1.8) and (1.9) does not have a solution with a single transonic shock, if

$$L < \ell((F(\rho_l),\ \alpha), (\rho_r, \beta)), \tag{5.37}$$

where β is determined by

$$\beta = \sqrt{\alpha^2 + 2(H(\rho_r) - H(F(\rho_l))}, \tag{5.38}$$

such that $(\rho_r, \beta) \in T(F(\rho_l),\ \alpha)$,
1b) if

$$\ell((F(\rho_l),\ \alpha); (\rho_r, \beta)) \leq L < +\infty, \tag{5.39}$$

Then there exists a unique state $(\rho^*, E^*) \in T(\rho_l, \alpha)$ satisfying $\rho_{min}^{out} \leq \rho^* \leq \rho_l$ and $-E_c < E^* \leq \alpha$ and a constant E_r such that

$$(\rho_r, E_r) \in T(F(\rho^*), E^*),\ L = \ell\left((\rho_l, \alpha); (\rho^*, E^*)\right) + \ell\left((F(\rho^*), E^*); (\rho_r, E_r)\right), \tag{5.40}$$

so the transonic shock location is $a = \ell((\rho_l,\ \alpha); (\rho^*, E^*))$.
2) If $\alpha < -E_c$, then the boundary value problem (1.8) and (1.9) does not have a solution with a single transonic shock. (See Figure 6).

Proof. 1a) and 2) are clear by looking at the phase portrait (see Figure 6). Therefore, the task is to prove 1b). We prove this for the different cases of α.
Case 1.

$$\alpha \geq 0. \tag{5.41}$$

In this case, we claim:
i) if

$$\ell((F(\rho_l),\ \alpha);(\rho_r,\beta)) \leq L \leq \ell((\rho_l,\alpha);\ (\rho_{min}^{out},0)) + \ell((F(\rho_{min}^{out}),0);(\rho_r,E_{r1}) \tag{5.42}$$

where β is given by (5.38), E_{r1} is determined by

$$E_{r1} = \sqrt{2(H(\rho_r) - H(F(\rho_{min}^{out})),} \tag{5.43}$$

such that $(\rho_r, E_{r1}) \in T(F(\rho_{min}^{out}),\ 0)$, then there exists a unique state $(\rho^*, E^*) \in T(\rho_l, \alpha)$ satisfying $\rho_{min}^{out} \leq \rho^* \leq \rho_l$ and $0 \leq E^* \leq \alpha$ and a constant E_r such that

$$(\rho_r, E_r) \in T(F(\rho^*, E^*),\ L = \ell\left((\rho_l,\alpha);(\rho^*,E^*)\right) + \ell\left((F(\rho^*),E^*);(\rho_r,E_r)\right), \tag{5.44}$$

ii) if

$$\ell((\rho_l,\alpha);(\rho_{min}^{out},0)) + \ell((F(\rho_{min}^{out}),0);(\rho_r,E_{r1})) \leq L < +\infty, \tag{5.45}$$

then there exists a unique state $(\rho^*, E^*) \in T(\rho_{min}^{out}, 0)$ satisfying $\rho_{min}^{out} \leq \rho^* \leq F^{-1}(\rho_c)$ and -$E_c < E^* \leq 0$ and a constant $E_r^* > 0$ such that

$$(\rho_r, E_r^*) \in T(F(\rho^*), E^*),\ L = \ell\left((\rho_l,\alpha);(\rho^*,E^*)\right) + \ell\left((F(\rho^*),E^*);(\rho_r,E_r^*)\right). \tag{5.46}$$

We prove i) and ii) by using Lemmas 3.3 and 3.4. First, if (5.41) and (5.42) hold, we define

$$X(\bar\rho) = \ell\left((\rho_l,\alpha);(\bar\rho,E(\bar\rho)\right) + \ell\left((F(\bar\rho),E(\bar\rho));(\rho_r,E_r(\bar\rho))\right),$$

for $(\bar\rho, E(\bar\rho)) \in T(\rho_l,\alpha),\ \rho_{min}^{out} \leq \rho < \rho_l$ and $0 < E(\bar\rho) < \alpha$. Here the meaning of $E_r(\bar\rho)$ is the same as that in Lemma 3.3, i.e.

$$E_r(\bar\rho) = \sqrt{E^2(\bar\rho) + 2(H(F(\bar\rho)) - H(\rho_r)).} \tag{5.47}$$

Then we can apply (3.11) in Lemma 3.3 to obtain

$$X'(\bar\rho) < 0,\ \text{for}\ \rho_{min}^{out} \leq \rho < \rho_l. \tag{5.48}$$

This is because $0 < \bar\rho < \rho_s$, $\rho_r > F(\bar\rho) > \rho_s > \bar\rho$, $E(\bar\rho) > 0$ and $E(\bar\rho, t) > 0$ for $\rho_r > t > F(\bar\rho)$ (The definition of $E(\bar\rho, t)$ can be found in Lemma 3.3). This proves i). In order to prove ii), we let

$$\phi(\bar\rho) = \ell\left((\rho_{min}^{out},0);(\bar\rho,E(\bar\rho)\right) + \ell\left((F(\bar\rho),E(\bar\rho));(\rho_r,E_r(\bar\rho))\right),$$

for $(\bar\rho, E(\bar\rho)) \in T(\rho_{min}^{out}, 0)$, $\rho_{min}^{out} < \bar\rho < F^{-1}(\rho_c)$ and $-E_c < E(\bar\rho) < 0$. Here the meaning of $E_r(\bar\rho)$ is the same as that in Lemma 3.5, i.e.,

$$E_r(\bar\rho) = \sqrt{E^2(\bar\rho) + 2(H(F(\bar\rho)) - H(\rho_r))}. \tag{5.49}$$

By using Lemma 3.5, we obtain

$$\begin{aligned}\frac{d\phi(\bar\rho)}{d\bar\rho} = &= \left(p'(\bar\rho) - \frac{J^2}{\bar\rho^2}\right)\frac{F(\bar\rho)-\bar\rho}{\bar\rho} \\ &\cdot \left\{\frac{1}{(F(\bar\rho)-b)E(\bar\rho)} - \frac{b}{F(\bar\rho)}\left[\frac{1}{(\rho_r - b)E_r(\bar\rho)} + \int_{F(\bar\rho)}^{q(\bar\rho)} \frac{d\rho}{(\rho-b)^2E_1(\rho,\bar\rho)} + \int_{q(\bar\rho)}^{\rho_r}\frac{d\rho}{(\rho-b)^2E_2(\rho,\bar\rho)}\right]\right\},\end{aligned} \tag{5.50}$$

where

$$E_1(\rho,\bar\rho) = -\sqrt{E^2(\bar\rho) + 2\int_{F(\bar\rho)}^{\rho} H'(t)dt}, \tag{5.51}$$

$$E_2(\rho,\bar\rho) = \sqrt{E^2(\bar\rho) + 2\int_{F(\bar\rho)}^{\rho} H'(t)dt}, \tag{5.52}$$

where $q(\bar\rho)$ is determined by

$$E_1(q(\bar\rho),\bar\rho) = E_2(q(\bar\rho),\bar\rho) = 0, \tag{5.53}$$

It is clear that $q(\bar\rho) < F(\bar\rho)$ and $q(\bar\rho) < \rho_r$ for $\rho_{min}^{out} < \bar\rho < F^{-1}(\rho_c)$. Moreover, $E_1(\rho,\rho) < 0$ as $q(\bar\rho) < \rho \le F(\bar\rho)$, so

$$\int_{F(\bar\rho)}^{q(\bar\rho)} \frac{1}{(\rho-b)^2E_1(\rho,\bar\rho)}d\rho > 0. \tag{5.54}$$

On the other hand, $q(\rho) < \rho_r$ and $E_2(\rho,\bar\rho) > 0$ as $q(\bar\rho) < \rho \le \rho_r$, so

$$\int_{q(\bar\rho)}^{\rho_r} \frac{1}{(\rho-b)^2E_2(\rho,\bar\rho)}d\rho > 0. \tag{5.55}$$

Therefore,

$$\int_{F(\rho)}^{\rho_r} \frac{1}{(t-b)^2E(t,\rho)}dt > 0. \tag{5.56}$$

Due to the fact that $p'(\bar\rho) - \frac{J^2}{\bar\rho^2} < 0$ and $F(\bar\rho) > b > \bar\rho$ and $E(\bar\rho) < 0$ for $\rho_{min}^{out} < \bar\rho < F^{-1}(\rho_c)$. In view of (5.55) and (5.54), we have

$$\phi'(\bar\rho) > 0, \text{ for } \rho_{min}^{out} < \bar\rho < F^{-1}(\rho_c). \tag{5.57}$$

Finally, we show that

$$\lim_{\bar\rho \to F^{-1}(\rho_c)-} \ell((F(\bar\rho), E(\bar\rho)); (\rho_r, E_r(\bar\rho)) = +\infty, \tag{5.58}$$

where $E_r(\bar\rho)$ is determined by (5.47). This can be shown as follows. The trajectory $T(F(\bar\rho), E(\bar\rho))$ intersects the ρ-axis at $(q(\bar\rho), 0)$ In order to show (5.58), it suffices to show that

$$\ell(F(\bar\rho), E(\bar\rho)); (q(\bar\rho), 0)) = \int_{F(\rho)}^{q(\bar\rho)} \frac{p'(t) - \frac{J^2}{t^2}}{E_1(t, \bar\rho)t} dt \to +\infty, \tag{5.59}$$

as $\bar\rho \to F^{-1}(\rho_c)-$, where

$$E_1(t, \bar\rho) = -\sqrt{2\int_{q(\bar\rho)}^{t} \frac{(s-b)(p'(s) - \frac{J^2}{s^2}}{s} ds}, \; q(\bar\rho) \le t \le F(\bar\rho). \tag{5.60}$$

In fact, as $\bar\rho < F^{-1}(\rho_c)$, $F(\bar\rho) \ge q(\bar\rho) > b$. Therefore

$$|E_1(t, \bar\rho)| \le \; C\sqrt{\int_{q(\rho)}^{t} (s-b)ds} = C\sqrt{\frac{1}{2}((t-b)^2 - (q(\bar\rho) - b)^2)}, \; q(\bar\rho) \le t \le F(\bar\rho). \tag{5.61}$$

By (5.59) and (5.60), we have

$$\ell(F(\bar\rho), E(\bar\rho)); (q(\bar\rho), 0) \ge C \int_{q(\bar\rho)}^{F(\bar\rho)} \frac{1}{|E_1(t, \bar\rho)|} dt \ge C \int_{q(\bar\rho)}^{F(\bar\rho)} \frac{1}{\sqrt{\frac{1}{2}((t-b)^2 - (q(\bar\rho) - b)^2)}}.$$

As $\bar\rho \to F^{-1}(\rho_c)-$, $F(\bar\rho) \to \rho_c > b$, $q(\bar\rho) \to b$. Thus, (5.59) follows from (5.60)and (5.61). □

Case when $b < \rho_r < F(\rho_{min}^{out})$

Next, we consider the case when $b < \rho_r \le F(\rho_{min}^{out})$. We still denote (ρ_c, E_c) and $(\rho_c, -E_c)$ the intersection points of the shock curve $S(\rho_l, \alpha)$ and the trajectory though $(b, 0)$ (see Figure 7). There are two subcases needed to be handled separately.

Subcase 1:

$$\rho_c \le \rho_r \le F(\rho_{min}^{out}). \tag{5.62}$$

In this case, the line $\rho = \rho_r$ intersects the shock curve $S(\rho_l, \alpha)$ at two points (ρ_r, E_r^0) and $(\rho_r, -E_r^0)$ with $E_r^0 > 0$, the trajectory passing through $(\rho_r, 0)$ satisfying $\frac{1}{2}E^2 - H(\rho) = -H(\rho_r)$ intersects the shock curve $S(\rho_l, \alpha)$ at two points $(\rho_r^1, \; E_r^1)$ and $(\rho_r^1, \; -E_r^1)$ with $E_r^1 > 0$ (see Figure 7). Clearly, $\rho_r^1 > \rho_r$ and $E_r^0 > E_r^1$.

For $\bar\rho \in [F^{-1}(\rho_r^1), F^{-1}(\rho_c)]$, we let

$$E_r(\bar\rho) = \sqrt{E^2(\bar\rho) + 2((H(\rho_r) - H(F(\bar\rho))}, \tag{5.63}$$

where

$$E(\bar\rho) = -\sqrt{\alpha^2 + 2(H(\bar\rho) - H(\rho_l))}$$

satisfying $(\bar\rho, E(\bar\rho) \in T(\rho_{min}^{out}, 0) \subset T(\rho_l, \alpha)$.

In this case, for any state $(\bar\rho, E(\bar\rho) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho_r^1) < \bar\rho < F^{-1}(\rho_c)$, $-E_c < E(\bar\rho) < -E_r^1$, the trajectory $T(F(\bar\rho), E(\bar\rho))$ starting from $(F(\bar\rho), E(\bar\rho))$ intersects the line $\rho = \rho_r$ twice at $(\rho_r, -E_r(\bar\rho)$ and $(\rho_r, E_r(\bar\rho)$. Obviously,

$$E_r\left(F^{-1}(\rho_r^1)\right) = 0, E_r\left(F^{-1}(\rho_r)\right) = E_r^0. \tag{5.64}$$

For $(\bar{\rho}, E(\bar{\rho}) \in T(\rho_{min}^{out}, 0) \subset T(\rho_l, \alpha)$, we define

$$\begin{aligned} Y(\bar{\rho}) =& \ell((\rho_l, \alpha); (\bar{\rho}, E(\bar{\rho}))) + \ell((F(\bar{\rho}), E(\bar{\rho})); (\rho_r, -E_r(\bar{\rho}))), \\ & \text{for } \bar{\rho} \in [F^{-1}(\rho_r^1), F^{-1}(\rho_r)],\ -E_r^0 \leq E(\bar{\rho}) \leq -E_r^1, \\ Z(\bar{\rho}) =& \ell((\rho_l, \alpha); (\bar{\rho}, E(\bar{\rho}))) + \ell((F(\bar{\rho}), E(\bar{\rho})); (\rho_r, E_r(\bar{\rho}))), \\ & \text{for } \bar{\rho} \in [F^{-1}(\rho_r^1), F^{-1}(\rho_c)), -E_c < E(\bar{\rho}) \leq -E_r^1. \end{aligned} \tag{5.65}$$

It should be noted that $Z(\bar{\rho}) = Y(\bar{\rho}) + \ell((\rho_r, -E_r(\bar{\rho})); (\rho_r, E_r(\bar{\rho})))$ for $\bar{\rho} \in [F^{-1}(\rho_r^1), F^{-1}(\rho_r)]$ and $Y(F^{-1}(\rho_r^1)) = Z(F^{-1}(\rho_r^1))$. With those notations, we have the following Lemma.

Lemma 5.4. *Suppose that (ρ_l, α) satisfies (5.34), ρ_r satisfies (5.62) and $\alpha > E_r^0$. Then there exists a unique state $(\hat{\rho}, E(\hat{\rho}) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho_r^1) < \hat{\rho} < F^{-1}(\rho_r)$ and $-E_r^0 < E(\hat{\rho}) < -E_r^1$ such that*

$$Y'(F^{-1}(\rho_r^1)) = -\infty, \begin{cases} Y'(\bar{\rho}) < 0, \text{for } F^{-1}(\rho_r^1) < \bar{\rho} < \hat{\rho}, \\ Y'(\bar{\rho}) > 0, \text{for } \hat{\rho} < \bar{\rho} \leq F^{-1}(\rho_r). \end{cases} \tag{5.66}$$

So

$$Y(\hat{\rho}) = \min_{F^{-1}(\rho_r^1) \leq \bar{\rho} \leq F^{-1}(\rho_r)} Y(\bar{\rho}). \tag{5.67}$$

Proof. We prove (5.66) first. Notice that

$$Y(\bar{\rho}) = \ell((\rho_l, \alpha); (\rho_{min}^{out}, 0)) + X(\bar{\rho}), \tag{5.68}$$

where

$$X(\bar{\rho})) = \ell((\rho_{min}^{out}, 0); (\bar{\rho}, E(\bar{\rho}))) + \ell((F(\bar{\rho}), E(\bar{\rho})); (\rho_r, -E_r(\bar{\rho}))),$$

for $\bar{\rho} \in [F^{-1}(\rho_r^1), F^{-1}(\rho_r))$. So

$$Y'(\bar{\rho})) = X'(\bar{\rho}).$$

Applying (3.11) in Lemma 3.3, we get

$$Y'(F^{-1}(\rho_r)) = X'(F^{-1}(\rho_r)) = \left(p'(F^{-1}(\rho_r)) - \frac{J^2}{(F^{-1}(\rho_r))^2}\right)\left(\frac{1}{F^{-1}(\rho_r)} - \frac{1}{\rho_r}\right)\frac{1}{E(F^{-1}(\rho_r))}.$$

Since $E(F^{-1}(\rho_r)) < 0$, $p'(F^{-1}(\rho_r)) - \frac{J^2}{(F^{-1}(\rho_r))^2} < 0$ and $F^{-1}(\rho_r) < \rho_r$,

$$Y'(F^{-1}(\rho_r)) > 0. \tag{5.69}$$

Again, by (3.11), we have

$$Y'(F^{-1}(\rho_r^1)) = X'(F^{-1}(\rho_r^1)) = \left(p'(F^{-1}(\rho_r^1)) - \frac{J^2}{(F^{-1}(\rho_r^1))^2}\right)\left(\frac{1}{F^{-1}(\rho_r^1)} - \frac{1}{\rho_r^1}\right)Q(F^{-1}(\rho_r^1)), \tag{5.70}$$

where

$$Q(F^{-1}(\rho_r^1)) = \frac{1}{E(F^{-1}(\rho_r^1))} + b\int_{\rho_r^1}^{\rho_r} \frac{p'(t)-\frac{J^2}{t^2}}{tE^3(F^{-1}(\rho_r^1),t)}dt, \tag{5.71}$$

$$E(F^{-1}(\rho_r^1),t) = -\sqrt{E^2(F^{-1}(\rho_r^1)) + 2(H(t)-H(\rho_r^1))},\ \rho_r \le t \le \rho_r^1.$$

We know that

$$-\infty < E(F^{-1}(\rho_r^1)) < 0. \tag{5.72}$$

We now show that

$$\int_{\rho_r^1}^{\rho_r} \frac{p'(t)-\frac{J^2}{t^2}}{tE^3(F^{-1}(\rho_r^1),t)}dt = +\infty. \tag{5.73}$$

This can be shown as follows. Let

$$g(t) = E^2(F^{-1}(\rho_r^1),t),\ \rho_r \le t \le \rho_r^1.$$

Then

$$\frac{1}{2}g(t) - H(t) = \frac{1}{2}E^2(F^{-1}(\rho_r^1)) - H(\rho_r^1),\ \rho_r \le t \le \rho_r^1.$$

Therefore,

$$g'(t) = 2H'(t) = 2(1-\frac{b}{t})(p'(t)-\frac{J^2}{t^2}),\ \rho_r \le t \le \rho_r^1.$$

Since $\rho_r^1 > \rho_r > b > \rho_s$, there exist positive constants C_1 and C_2 such that

$$C_1 \le g'(t) \le C_2,\ \rho_r \le t \le \rho_r^1. \tag{5.74}$$

Since $g(\rho_r) = 0$, we have

$$g(t) = O(|t-\rho_r|),$$

as $|t-\rho_r|$ is small. This means

$$E(F^{-1}(\rho_r^1),t) = O(|t-\rho_r|^{1/2}),$$

as $|t-\rho_r|$ is small. (5.73) follows since $\rho_r^1 > \rho_r$ and $E(F^{-1}(\rho_r^1),t) < 0$ for $\rho_r \le t \le \rho_r^1$. By (5.70)-(5.73), we have

$$Y'(F^{-1}(\rho_r^1)) = -\infty. \tag{5.75}$$

In view of (5.69) and (5.75), we know that $Y'(\bar\rho)$ changes the sign in the interval $[F^{-1}(\rho_r^1), F^{-1}(\rho_r)]$. Since

$$signQ(\bar\rho) = -signX'(\bar\rho) = -signY'(\bar\rho)$$

for $\bar\rho \in [F^{-1}(\rho_r^1), F^{-1}(\rho_r)]$ (where $Q(\bar\rho)$ is defined in (3.12)), $Q(\bar\rho)$ changes the sign in the interval $[F^{-1}(\rho_r^1), F^{-1}(\rho_r)]$. Suppose

$$Q(\hat\rho) = X'(\hat\rho) = Y'(\hat\rho) = 0, \tag{5.76}$$

for $\hat{\rho} \in [F^{-1}(\rho_r^1), F^{-1}(\rho_r)]$. By (3.11) and (3.12), we have

$$Q(\hat{\rho}) = \frac{1}{E(\hat{\rho})} + b\int_{F(\hat{\rho})}^{\rho_r} \frac{p'(t) - \frac{J^2}{t^2}}{tE^3(\hat{\rho}, t)} dt = 0. \tag{5.77}$$

This, together with (3.14), gives

$$\begin{aligned}
&\frac{Q'(\hat{\rho})}{p'(\hat{\rho}) - \frac{J^2}{\hat{\rho}^2}} \\
&= \frac{1}{E^3(\hat{\rho})}\left(\frac{b}{\hat{\rho}} - \frac{b}{F(\hat{\rho})} - 1\right) + 3b^2\left(\frac{1}{\hat{\rho}} - \frac{1}{F(\hat{\rho})}\right)\int_{F(\hat{\rho})}^{\rho_r} \frac{p'(t) - \frac{J^2}{t^2}}{tE^5(\hat{\rho}, t)} dt \\
&= b^2\left(\frac{1}{\hat{\rho}} - \frac{1}{F(\hat{\rho})}\right)\int_{F(\hat{\rho})}^{\rho_r} \frac{p'(t) - \frac{J^2}{t^2}}{tE^3(\hat{\rho}, t)}\left(\frac{3}{E^2(\hat{\rho}, t)} - \frac{1}{E^2(\hat{\rho})}\right) dt - \frac{1}{E^3(\hat{\rho})}.
\end{aligned} \tag{5.78}$$

Since $\rho_r < F(\hat{\rho})$, $E^2(\hat{\rho}, t) < E^2(\hat{\rho})$ for $\rho_r \le t < F(\hat{\rho})$, $E(\hat{\rho}) < 0$, $F(\hat{\rho}) > \hat{\rho}$ and $p'(\hat{\rho}) - \frac{J^2}{\hat{\rho}^2} < 0$ ($\hat{\rho} < \rho_s$), (5.78) implies

$$Q'(\hat{\rho}) < 0.$$

Therefore, by (5.69), (5.75) and (3.11), we have

$$Q(F^{-1}(\rho_r)) < 0,\ Q(F^{-1}(\rho_r^1)) = +\infty,\ Q'(\hat{\rho}) < 0 \text{ as } Q(\hat{\rho}) = 0 \text{ for } \hat{\rho} \in [F^{-1}(\rho_r^1), F^{-1}(\rho_r)].$$

Therefore, $Q(\bar{\rho})$ only changes the sign once for $\bar{\rho} \in [F^{-1}(\rho_r^1), F^{-1}(\rho_r)]$ at $\bar{\rho} = \hat{\rho}$ where $Q(\hat{\rho}) = 0$. Therefore, we can claim that

$$Q(F^{-1}(\rho_r^1)) = +\infty, \begin{cases} Q(\bar{\rho}) > 0 \text{ as } F^{-1}(\rho_r^1) < \bar{\rho} < \hat{\rho}, \\ Q(\hat{\rho}) = 0, \\ Q(\bar{\rho}) < 0 \text{ as } \hat{\rho} < \bar{\rho} \le \hat{\rho}, F^{-1}(\rho_r). \end{cases}$$

This proves (5.66) and (5.67) in view of (3.11). □

With this lemma, we have the following theorem.

Theorem 5.2. *Suppose that (ρ_l, α) satisfies (5.34) and ρ_r satisfies(5.62). Then 1) If $\alpha > E_r^0$, 1a) the boundary value problem (1.8) and (1.9) does not have a solution with a single transonic shock, if*

$$L < \ell((F(\rho_l),\ \alpha), (\rho_r, \beta)), \tag{5.79}$$

where β is determined by

$$\beta = \sqrt{\alpha^2 + 2(H(\rho_r) - H(F(\rho_l))}, \tag{5.80}$$

such that $(\rho_r, \beta) \in T(F(\rho_l),\ \alpha)$.
1b) if

$$\ell((F(\rho_l), \alpha); (\rho_r, \beta) \le L \le \ell((\rho_l, \alpha); (F^{-1}(\rho_r), E_r^0)), \tag{5.81}$$

where β is determined by in (5.109), then there exists a unique state $(\rho^*, E^*) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho_r) \le \rho^* \le \rho_l$ and $E_r^0 \le E^* \le \alpha$ and a constant E_r such that

$$(\rho_r, E_r) \in T(F(\rho^*), E^*),\ \ L = \ell\left((\rho_l, \alpha); (\rho^*, E^*)\right) + \ell\left((F(\rho^*), E^*); (\rho_r, E_r)\right). \tag{5.82}$$

So the transonic shock location is $a = \ell((\rho_l,\ \alpha); (\rho^*, E^*))$;
2) If $\alpha > E_r^0$ and

$$\ell((\rho_l, \alpha); (F^{-1}(\rho_r), E_r^0)) < L < Y(\hat{\rho})), \tag{5.83}$$

(where and in the following $\hat{\rho}$ is given in (5.66) and (5.67)), then the boundary value problem (1.8) and (1.9) has no solution with single transonic shock;
3) If $\alpha > E_r^0$ and

$$Y(\hat{\rho}) < L \le \min\{Y(F^{-1}(\rho_r^1)), Y(F^{-1}(\rho_r))\}, \tag{5.84}$$

then there exist two and only two states $(\rho_1^*, E(\rho_1^*)) \in T(\rho_l, \alpha)$ and $(\rho_2^*, E(\rho_2^*)) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho_r^1) < \rho_1^* < \hat{\rho}$ and $\hat{\rho} < \rho_2^* < F^{-1}(\rho_r)$, $E(\hat{\rho}) < E(\rho_1^*) < -E_r^1$ and $-E_r^0 < E(\rho_2^*) < E(\hat{\rho})$ such that

$$L = Y(\rho_1^*) = Y(\rho_2^*). \tag{5.85}$$

In this case, there are two shock locations, i.e., $\ell((\rho_l, \alpha); (\rho_1^*, E(\rho_1^*))$ and $\ell((\rho_l, \alpha); (\rho_2^*, E(\rho_2^*)))$.
4) Suppose $\alpha > E_r^0$ and $Y(F^{-1}(\rho_r^1)) \ne Y(F^{-1}(\rho_r))$ (the case $Y(F^{-1}(\rho_r^1)) = Y(F^{-1}(\rho_r))$ can be handled similarly).
If

$$\min\{Y(F^{-1}(\rho_r^1)), Y(F^{-1}(\rho_r))\} < L < \max\{Y(F^{-1}(\rho_r^1)), Y(F^{-1}(\rho_r))\}, \tag{5.86}$$

then we have the following results:
4a) if

$$Y(F^{-1}(\rho_r^1)) < Y(F^{-1}(\rho_r)), \tag{5.87}$$

then there exist two states $(\bar{\rho}_1^*, E(\bar{\rho}_1^*)) \in T(\rho_l, \alpha)$ and $(\bar{\rho}_2^*, E(\bar{\rho}_2^*)) \in T(\rho_l, \alpha)$ satisfying $(F^{-1}(\rho_r^1)) < \bar{\rho}_1^*, \bar{\rho}_2^* < F^{-1}(\rho_r)$ such that

$$L = Z(\rho_1^*) = Y(\rho_2^*), \tag{5.88}$$

4b) if

$$Y(F^{-1}(\rho_r^1)) > Y(F^{-1}(\rho_r)), \tag{5.89}$$

then there exists a unique state $(\bar{\rho}*, E(\bar{\rho}^*)) \in T(\rho_l, \alpha)$ satisfying $(F^{-1}(\rho_r^1)) < \bar{\rho}^* < F^{-1}(\rho_r)$ such that

$$L = Z(\bar{\rho}*). \tag{5.90}$$

So the shock location is $a = \ell((\rho_l, \alpha); (\bar{\rho}^*, E(\bar{\rho}^*))$.
5) if

$$\max\{Y(F^{-1}(\rho_r^1)), Y(F^{-1}(\rho_r))\} \le L < +\infty, \tag{5.91}$$

then there exists a unique state $(\tilde{\rho}^*, E(\tilde{\rho}^*)) \in T(\rho_l, \alpha)$ *satisfying* $F^{-1}(\rho_r^1) \le \tilde{\rho}^* < F^{-1}(\rho_c)$ *and* $-E_c < E(\tilde{\rho}^*) < -E_r^0$ *such that*

$$L = \ell((\rho_l, \alpha); (\tilde{\rho}^*, E(\tilde{\rho}^*))) + \ell(\tilde{\rho}^*, E(\tilde{\rho}^*)); (\rho_r, E_r^*)) = Z(\tilde{\rho}^*), \tag{5.92}$$

where $E_r^* = \sqrt{(E^2(\tilde{\rho}^*) + 2(H(\rho_r) - H(F(\tilde{\rho}^*))}$ *so that* $(\rho_r, E_r^*) \in T(F(\tilde{\rho}^*), E(\tilde{\rho}^*)$. *So the transonic shock location is* $a = \ell((\rho_l,\ \alpha); (\rho^*, E^*))$.
6) If $-E_r^1 < \alpha < E_r^0$ *or* $\alpha < -E_c$*, then the boundary value problem (1.8) and (1.9) has no solution with single transonic shock.*

Proof. We prove 1a)and 1b) as follows, we define

$$x(\bar{\rho}) = \ell((\rho_l, \alpha);\ (\bar{\rho}, E(\bar{\rho}))) + \ell((F(\bar{\rho}), E(\bar{\rho})); (\rho_r, E_r(\bar{\rho}))),$$

for $(\bar{\rho}, E(\bar{\rho})) \in T(\rho_l, \alpha),\ F^{-1}(\rho_r) \le \bar{\rho} \le \rho_l$ and $E_r^0 \le E(\bar{\rho}) \le \alpha$, where $E_r(\bar{\rho})$ is determined by $(\rho_r, E_r(\bar{\rho})) \in T(F(\rho),\ E(\bar{\rho})$ satisfying $E_r^0 \le E_r(\bar{\rho}) \le \beta$. By (3.11) and (3.12), we have

$$x'(\bar{\rho}) = (p'(\bar{\rho}) - \frac{J^2}{\bar{\rho}^2})(\frac{1}{\bar{\rho}} - \frac{1}{F(\bar{\rho})})Q(\bar{\rho}),$$

where

$$Q(\bar{\rho}) = \frac{1}{E(\bar{\rho})} + b\int_{F(\bar{\rho})}^{\rho_r} \frac{p'(t) - \frac{J^2}{t^2}}{tE^3(\bar{\rho}, t)} dt.$$

Therefore,

$$x'(\bar{\rho}) < 0,\ F^{-1}(\rho_r) \le \bar{\rho} \le \rho_l,$$

since $E(\bar{\rho}) \ge E_r^0 > 0$, $F(\bar{\rho}) \le \rho_r$ and $E(\bar{\rho}, t) > 0$ as $F(\bar{\rho}) \le t \le \rho_r$. Thus 1a) and 1b) are proved.
2) can be proved as follows. For any state $(\rho, E) \in T(\rho_l, \alpha)$ on the portion between two states $(F^{-1}(\rho_r), E_r^0)$ and $(F^{-1}(\rho_r^1), -E_r^1)$, i.e. ,

$$\frac{1}{2}E^2 - H(\rho) = \frac{1}{2}\alpha^2 - H(\rho_l),\ -E_r^1 < E < E_r^0,$$

the trajectory passing through $(F(\rho), E)$ is on the right of the trajectory passing through $(\rho_r, 0)$ and thus can not intersect the line $\rho = \rho_r$. This, together with (5.69), proves 2).
3) can be proved by using (5.66) and (5.67).
In order to prove 4) and 5), we first show that

$$Z'(\bar{\rho}) > 0,\ for\ F^{-1}(\rho_r^1) < \bar{\rho} < F^{-1}(\rho_c), \tag{5.93}$$

where $Z(\bar{\rho})$ is defined in (5.65).
In fact, we may write $Z(\bar{\rho})$ as

$$Z(\bar{\rho}) = \ell((\rho_l, \alpha); (\rho_{min}^{out}, 0)) + z(\bar{\rho}), \tag{5.94}$$

for $F^{-1}(\rho_r^1) < \bar\rho < F^{-1}(\rho_c)$. Then

$$Z'(\bar\rho) = z'(\bar\rho), \tag{5.95}$$

for $F^{-1}(\rho_r^1) < \bar\rho < F^{-1}(\rho_c)$. It follows (3.44) in Lemma 3.5 that

$$\frac{dz(\bar\rho)}{d\bar\rho} = \left(p'(\bar\rho) - \frac{J^2}{\bar\rho^2}\right)\frac{F(\bar\rho)-\bar\rho}{\bar\rho} \cdot \left\{\frac{1}{(F(\bar\rho)-b)E(\bar\rho)} - \frac{b}{F(\bar\rho)}\left[\frac{1}{(\rho_r-b)E_r(\bar\rho)} + \int_{F(\bar\rho)}^{q(\bar\rho)}\frac{d\rho}{(\rho-b)^2E_1(\rho,\bar\rho)} + \int_{q(\bar\rho)}^{\rho_r}\frac{d\rho}{(\rho-b)^2E_2(\rho,\bar\rho)}\right]\right\}, \tag{5.96}$$

for $F^{-1}(\rho_r^1) < \bar\rho < F^{-1}(\rho_c)$, where the definitions of quantities in (5.96) are the same as those in Lemma 3.5. Since $p'(\bar\rho) - \frac{J^2}{\bar\rho^2} < 0$ $(\bar\rho < \rho_s)$, $F(\bar\rho) > b > \bar\rho$, $E(\bar\rho) < -E_r^1 < 0$, $\rho_r > b$, $E_r(\bar\rho) > 0$, $\rho_r > q(\bar\rho)$, $F(\bar\rho) > q(\rho)$, $E_1(\rho,\bar\rho) < 0$ and $E_2(\rho,\bar\rho) > 0$, we conclude,

$$z'(\bar\rho) > 0, \ \ for \ F^{-1}(\rho_r^1) < \bar\rho < F^{-1}(\rho_c). \tag{5.97}$$

(5.93) follows from (5.94) and (5.97).
Proof of 4a).
If (5.76) holds, then (5.75) implies

$$Y(F^{-1}(\rho_r^1)) < L < Y(F^{-1}(\rho_r)). \tag{5.98}$$

Since $Y(F^{-1}(\rho_r^1)) = Z(F^{-1}(\rho_r^1))$, 4a) is proved by using (5.98) and (5.93).
Proof of 4b).
If (5.78) holds, then (5.75) implies

$$Y(F^{-1}(\rho_r)) < L < Y(F^{-1}(\rho_r^1)). \tag{5.99}$$

Since $Y(F^{-1}(\rho_r^1)) = Z(F^{-1}(\rho_r^1))$ and $Z'(\bar\rho) > 0$ for $F^{-1}(\rho_r^1) < \bar\rho < F^{-1}(\rho_c)$, 4b) is proved.
Proof of 5).
5) is proved by using (5.93) and the following fact

$$\lim_{\bar\rho \to F^{-1}(\rho_c)^-} Z(\bar\rho) = +\infty. \tag{5.100}$$

The proof of (5.100) is similar to that for (5.58). We thus omit it.
6) can be easily seen by looking at the phase portrait (see Figure 7). □

Next, we consider
Subcase 2:

$$b < \rho_r < \rho_c. \tag{5.101}$$

In this case, the line $\rho = \rho_r$ intersects the shock curve $S(\rho_l, \alpha)$ at two points (ρ_r, E_r^0) and $(\rho_r, -E_r^0)$ with $E_r^0 > 0$, the trajectory passing through $(\rho_r, 0)$ satisfying $\frac{1}{2}E^2 - H(\rho) = -H(\rho_r)$

intersects the shock curve $S(\rho_l, \alpha)$ at two points $(\rho_r^1,\ E_r^1)$ and $(\rho_r^1,\ -E_r^1)$ with $E_r^1 > 0$ (see Figure 8). Clearly, $\rho_r^1 > \rho_r$ and $E_r^0 > E_r^1$.
For $\bar\rho \in [F^{-1}(\bar\rho_r), F^{-1}(\rho_r)]$, we let

$$E_r(\bar\rho) = \sqrt{E^2(\bar\rho) + 2((H(\rho_r) - H(F(\bar\rho))}, \tag{5.102}$$

where

$$E(\bar\rho) = -\sqrt{\alpha^2 + 2(H(\bar\rho) - H(\rho_l))}$$

satisfying $(\bar\rho, E(\bar\rho)) \in T(\rho_{min}^{out}, 0) \subset T(\rho_l, \alpha)$.
In this case, for any state $(\bar\rho, E(\bar\rho) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\bar\rho_r) < \bar\rho < F^{-1}(\rho_c)$, $-E_c < E(\bar\rho) < -\bar E_r$, the trajectory $T(F(\bar\rho), E(\bar\rho))$ starting from $(F(\bar\rho), E(\bar\rho))$ intersects the line $\rho = \rho_r$ twice at $(\rho_r, -E_r(\bar\rho)$ and $(\rho_r, E_r(\bar\rho)$. Obviously,

$$E_r\left(F^{-1}(\bar\rho_r)\right) = 0, E_r\left(F^{-1}(\rho_c)\right) = E_c. \tag{5.103}$$

For $(\bar\rho, E(\bar\rho) \in T(\rho_{min}^{out}, 0) \subset T(\rho_l, \alpha)$, we define

$$\begin{aligned} Y(\bar\rho) =& \ell((\rho_l, \alpha); (\bar\rho, E(\bar\rho))) + \ell((F(\bar\rho), E(\bar\rho)); (\rho_r, -E_r(\bar\rho))), \\ & \text{for } \bar\rho \in [F^{-1}(\bar\rho_r), F^{-1}(\rho_r)],\ -E_r \le E(\bar\rho) \le -\bar E_r, \end{aligned} \tag{5.104}$$

$$\begin{aligned} Z(\bar\rho) =& \ell((\rho_l, \alpha); (\bar\rho, E(\bar\rho))) + \ell((F(\bar\rho), E(\bar\rho)); (\rho_r, E_r(\bar\rho))), \\ & \text{for } \bar\rho \in [F^{-1}(\bar\rho_r), F^{-1}(\rho_c)), -E_c < E(\bar\rho) \le -\bar E_r. \end{aligned} \tag{5.105}$$

It should be noted that $Z(\bar\rho) = Y(\bar\rho) + \ell((\rho_r, -E_r(\bar\rho)); (\rho_r, E_r(\bar\rho)))$ for $\bar\rho \in [F^{-1}(\bar\rho_r), F^{-1}(\rho_c)]$ and $Y(F^{-1}(\bar\rho_r)) = Z(F^{-1}(\bar\rho_r))$. With those notations, we have the following Lemma.

Lemma 5.5. *Suppose that (ρ_l, α) satisfies (5.34), (5.101) holds and $\alpha > E_r^0$. Then there exists a unique state $(\hat\rho, E(\hat\rho) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\bar\rho_r) < \hat\rho < F^{-1}(\rho_c)$ and $-E_r < E(\hat\rho) < -E_c$ such that*

$$Y'(F^{-1}(\bar\rho_r)) = -\infty, \begin{cases} Y'(\bar\rho) < 0, \text{for } F^{-1}(\bar\rho_r) < \bar\rho < \hat\rho, \\ Y'(\bar\rho) > 0, \text{for } \hat\rho < \bar\rho \le F^{-1}(\rho_r). \end{cases} \tag{5.106}$$

So

$$Y(\hat\rho) = \min_{F^{-1}(\bar\rho_r) \le \bar\rho \le F^{-1}(\rho_c)} Y(\bar\rho). \tag{5.107}$$

The proof of this lemma is almost the same as that for Lemma 5.4. So we omit it.
With this lemma, we have the following theorem.

Theorem 5.3. *Suppose that (ρ_l, α) satisfies (5.34) and (5.101) holds. Then 1) If $\alpha > E_r^0$, 1a) the boundary value problem (1.8) and (1.9) does not have a solution with a single transonic shock, if*

$$L < \ell((F(\rho_l),\ \alpha), (\rho_r, \beta)), \tag{5.108}$$

where β is determined by

$$\beta = \sqrt{\alpha^2 + 2(H(\rho_r) - H(F(\rho_l))}, \tag{5.109}$$

such that $(\rho_r, \beta) \in T(F(\rho_l),\ \alpha)$.
1b) if

$$\ell((F(\rho_l), \alpha); (\rho_r, \beta) \le L \le \ell((\rho_l, \alpha); (F^{-1}(\rho_r), E_r)), \tag{5.110}$$

where β is determined by in (5.109), then there exists a unique state $(\rho^*, E^*) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho_r) \le \rho^* \le \rho_l$ and $E_r^0 \le E^* \le \alpha$ and a constant E_r such that

$$(\rho_r, E_r) \in T(F(\rho^*), E^*),\ \ L = \ell\left((\rho_l, \alpha); (\rho^*, E^*)\right) + \ell\left((F(\rho^*), E^*); (\rho_r, E_r)\right). \tag{5.111}$$

So the transonic shock location is $a = \ell((\rho_l,\ \alpha); (\rho^*, E^*))$;
2) If $\alpha > E_r^0$ and

$$\ell((\rho_l, \alpha); (F^{-1}(\rho_r), E_r)) < L < Y(\hat{\rho})), \tag{5.112}$$

(where and in the following $\hat{\rho}$ is given in (5.107)), then the boundary value problem (1.8) and (1.9) has no solution with single transonic shock;
3) If $\alpha > E_r^0$ and

$$Y(\hat{\rho}) < L \le \min\{Y(F^{-1}(\bar{\rho}_r)), Y(F^{-1}(\rho_r))\}, \tag{5.113}$$

then there exist two and only two states $(\rho_1^*, E(\rho_1^*)) \in T(\rho_l, \alpha)$ and $(\rho_2^*, E(\rho_2^*)) \in T(\rho_l, \alpha)$ satisfying $(F^{-1}(\bar{\rho}_r)) < \rho_1^* < \hat{\rho}$ and $\hat{\rho} < \rho_2^* < F^{-1}(\rho_r)$, $E(\hat{\rho}) < E(\rho_1^*) < -\bar{E}_r$ and $-E_r < E(\rho_2^*) < E(\hat{\rho})$ such that

$$L = Y(\rho_1^*) = Y(\rho_2^*). \tag{5.114}$$

In this case, there are two shock locations, i.e., $\ell((\rho_l, \alpha); (\rho_1^*, E(\rho_1^*))$ and $\ell((\rho_l, \alpha); (\rho_2^*, E(\rho_2^*)))$.
4) Suppose $\alpha > E_r^0$ and $Y(F^{-1}(\bar{\rho}_r)) \ne Y(F^{-1}(\rho_r))$ (the case $Y(F^{-1}(\bar{\rho}_r)) = Y(F^{-1}(\rho_r))$ can be handled similarly).
if

$$\min\{Y(F^{-1}(\bar{\rho}_r)), Y(F^{-1}(\rho_r))\} < L < \max\{Y(F^{-1}(\bar{\rho}_r)), Y(F^{-1}(\rho_r))\}, \tag{5.115}$$

then we have the following results:
4a) if

$$Y(F^{-1}(\bar{\rho}_r)) < Y(F^{-1}(\rho_r)), \tag{5.116}$$

then there exist two states $(\bar{\rho}_1^*, E(\bar{\rho}_1^*)) \in T(\rho_l, \alpha)$ and $(\bar{\rho}_2^*, E(\bar{\rho}_2^*)) \in T(\rho_l, \alpha)$ satisfying $(F^{-1}(\bar{\rho}_r)) < \bar{\rho}_1^* < F^{-1}(\rho_c) < \bar{\rho}_2^* < F^{-1}(\rho_r)$ such that

$$L = Z(\rho_1^*) = Y(\rho_2^*),$$

4b) if

$$Y(F^{-1}(\bar{\rho}_r)) > Y(F^{-1}(\rho_r)), \tag{5.117}$$

then there exists a unique state $(\bar{\rho}*, E(\bar{\rho}^*)) \in T(\rho_l, \alpha)$ *satisfying* $(F^{-1}(\bar{\rho}_r)) < \bar{\rho}^* < F^{-1}(\rho_c)$ *such that*

$$L = Z(\bar{\rho}*).$$

So the shock location is $a = \ell((\rho_l, \alpha); (\bar{\rho}^*, E(\bar{\rho}^*))$.
5) if

$$\max\{Y(F^{-1}(\bar{\rho}_r)), Y(F^{-1}(\rho_r))\} \leq L < +\infty,$$

then there exists a unique state $(\tilde{\rho}^*, E(\tilde{\rho}^*)) \in T(\rho_l, \alpha)$ *satisfying* $F^{-1}(\bar{\rho}_r) \leq \tilde{\rho}^* < F^{-1}(\rho_c)$ *and* $-E_c < E(\tilde{\rho}^*) < -E_r$ *such that*

$$L = \ell((\rho_l, \alpha); (\tilde{\rho}^*, E(\tilde{\rho}^*))) + \ell(\tilde{\rho}^*, E(\tilde{\rho}^*)); (\rho_r, E_r^*)) = Z(\tilde{\rho}^*), \tag{5.118}$$

where $E_r^* = \sqrt{(E^2(\tilde{\rho}^*) + 2(H(\rho_r) - H(F(\tilde{\rho}^*))}$ *so that* $(\rho_r, E_r^*) \in T(F(\tilde{\rho}^*), E(\tilde{\rho}^*)$. *So the transonic shock location is* $a = \ell((\rho_l,\ \alpha); (\rho^*, E^*))$.
6) If $-\bar{E}_r < \alpha < E_r$ *or* $\alpha < -E_c$, *then the boundary value problem (1.8) and (1.9) has no solution with single transonic shock.*

The proof of this theorem is similar to that for Theorem 5.2. So we omit it.

Next, we consider the following subcase:

Subcase 3:

$$b < \rho_r < \rho_c. \tag{5.119}$$

In this case, the line $\rho = \rho_r$ intersects the shock curve $S(\rho_l, \alpha)$ at two points (ρ_r, E_r^0) and $(\rho_r, -E_r^0)$ with $E_r^0 > 0$. The trajectory passing through $(b, 0)$ satisfying $\frac{1}{2}E^2 - H(\rho) = -H(b)$ intersects the shock curve $S(\rho_l, \alpha)$ at two points $(\rho_c,\ E_c)$ and $(\rho_c,\ -E_c)$ with $E_c > 0$ (see Figure 9).
For $\bar{\rho} \in [F^{-1}(\rho_c), F^{-1}(\rho_r)]$, we let

$$e_r(\bar{\rho}) = -\sqrt{E^2(\bar{\rho}) + 2((H(\rho_r) - H(F(\bar{\rho})),} \tag{5.120}$$

where

$$E(\bar{\rho}) = -\sqrt{\alpha^2 + 2(H(\bar{\rho}) - H(\rho_l))}$$

satisfying $(\bar{\rho}, E(\bar{\rho}) \in T(\rho_{min}^{out}, 0) \subset T(\rho_l, \alpha)$.
For $(\bar{\rho}, E(\bar{\rho}) \in T(\rho_{min}^{out}, 0) \subset T(\rho_l, \alpha)$, we define

$$\begin{aligned} Y(\bar{\rho}) =& \ell((\rho_l, \alpha); (\bar{\rho}, E(\bar{\rho}))) + \ell((F(\bar{\rho}), E(\bar{\rho})); (\rho_r, -e_r(\bar{\rho}))), \\ & \text{for } \bar{\rho} \in [F^{-1}(\rho_c), F^{-1}(\rho_r)],\ -E_r^0 \leq E(\bar{\rho}) \leq -E_c, \end{aligned} \tag{5.121}$$

Then we have,

Lemma 5.6. *Suppose that (ρ_l, α) satisfies (5.34) and (5.119) holds and $\alpha > E_r^0$. Then there exists a unique state $(\hat{\rho}, E(\hat{\rho}) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho_c) < \hat{\rho} < F^{-1}(\rho_r)$ and $-E_r^0 < E(\hat{\rho}) < -E_c$ such that*

$$\lim_{\bar{\rho}\to\rho_c+} Y'(\bar{\rho}) = -\infty, \begin{cases} Y'(\bar{\rho}) < 0, \text{for } F^{-1}(\rho_c) < \bar{\rho} < \hat{\rho}, \\ Y'(\bar{\rho}) > 0, \text{for } \hat{\rho} < \bar{\rho} \le F^{-1}(\rho_r). \end{cases} \tag{5.122}$$

So

$$Y(\hat{\rho}) = \min_{F^{-1}(\rho_c)\le\bar{\rho}\le F^{-1}(\rho_r)} Y(\bar{\rho}). \tag{5.123}$$

Proof. We only prove

$$\lim_{\bar{\rho}\to\rho_c+} Y'(\bar{\rho}) = -\infty$$

in (5.122). The proof of the rest is almost the same as that for Lemma 5.4.
For any $\bar{\rho} \in (F^{-1}(\rho_c), F^{-1}(\rho_r))$, we apply (3.11) and (3.15) to get

$$Y'(\bar{\rho}) = (p'(\bar{\rho}) - \frac{J^2}{\bar{\rho}^2})(\frac{1}{\bar{\rho}} - \frac{1}{F(\bar{\rho})})Q(\bar{\rho},$$

where

$$Q(\bar{\rho}) = \frac{1}{E(\bar{\rho})} + b\int_{F(\bar{\rho})}^{\rho_r} \frac{p'(t) - \frac{J^2}{t^2}}{tE^3(\bar{\rho}, t)} dt,$$

with $-E_r^0 < E(\bar{\rho}) < -E_c$. The meaning of $E(\bar{\rho}, t)$ is given in (3.13). Now we want to show that

$$\lim_{\bar{\rho}\to F^{-1}(\rho_c)+} Q(\bar{\rho}) = +\infty.$$

This is equivalent to

$$\int_{F(\bar{\rho})}^{\rho_r} \frac{p'(t) - \frac{J^2}{t^2}}{tE^3(\bar{\rho}, t)} dt = +\infty. \tag{5.124}$$

Since $E(\bar{\rho}, t) < 0$, $p'(t) - \frac{J^2}{t^2} > 0$ and $F(\bar{\rho}) > b > \rho_r$ for $F^{-1}(\rho_c) < \bar{\rho} < F^{-1}(\rho_r)$, $\rho_r \le t \le F(\bar{\rho})$, we have

$$\int_{F(\bar{\rho})}^{\rho_r} \frac{p'(t) - \frac{J^2}{t^2}}{tE^3(\bar{\rho}, t)} dt \ge \int_{F(\bar{\rho})}^{b} \frac{p'(t) - \frac{J^2}{t^2}}{tE^3(\bar{\rho}, t)} dt. \tag{5.125}$$

Let

$$g(\bar{\rho}, t) = E^2(\bar{\rho}, t), \text{ for } b \le t \le F(\bar{\rho}).$$

Then we have

$$\frac{1}{2} g(\bar{\rho}, t) - H(t) = C(\bar{\rho}),$$

where $C(\bar{\rho})$ is a quantity only depending on $\bar{\rho}$ but not on t. Therefore,

$$\frac{\partial g(\bar{\rho}, t)}{\partial t} = 2H'(t) = 2(1 - \frac{b}{t})(p'(t) - \frac{J^2}{t^2}).$$

Thus

$$\frac{\partial g(\bar\rho, t)}{\partial t}|_{t=b} = 0. \tag{5.126}$$

On the other hand,

$$\lim_{\bar\rho \to F^{-1}(\rho_c)+, t\to b+} g(\bar\rho, t) = 0. \tag{5.127}$$

It follows from (5.126) and (5.127) that

$$g(\bar\rho, t) = o(|t-b|), \tag{5.128}$$

as $|t-b|$ is mall. Therefore, (5.124) follows from (5.125) and (5.128). □

With this lemma, we have the following theorem.

Theorem 5.4. *Suppose that (ρ_l, α) satisfies (5.34) and (5.119) holds. Then 1) If $\alpha > E_r^0$, 1a) the boundary value problem (1.8) and (1.9) does not have a solution with a single transonic shock, if*

$$L < \ell((F(\rho_l),\ \alpha), (\rho_r, \beta)),$$

where β is determined by

$$\beta = \sqrt{\alpha^2 + 2(H(\rho_r) - H(F(\rho_l))}, \tag{5.129}$$

such that $(\rho_r, \beta) \in T(F(\rho_l),\ \alpha)$.
1b) if

$$\ell((F(\rho_l), \alpha); (\rho_r, \beta) \le L \le \ell((\rho_l, \alpha); (F^{-1}(\rho_r), E_r^0)),$$

where β is determined by in (5.129), then there exists a unique state $(\rho^*, E^*) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho_r) \le \rho^* \le \rho_l$ and $E_r^0 \le E^* \le \alpha$ and a constant E_r such that

$$(\rho_r, E_r) \in T(F(\rho^*), E^*),\ \ L = \ell\left((\rho_l, \alpha); (\rho^*, E^*)\right) + \ell\left((F(\rho^*), E^*); (\rho_r, E_r)\right).$$

So the transonic shock location is $a = \ell((\rho_l,\ \alpha); (\rho^*, E^*))$;
2) If $\alpha > E_r^0$ and

$$\ell((\rho_l, \alpha); (F^{-1}(\rho_r), E_r^0)) < L < Y(\hat\rho)), \tag{5.130}$$

(where and in the following $\hat\rho$ is given in (5.122)), then the boundary value problem (1.8) and (1.9) has no solution with single transonic shock;
3) If $\alpha > E_r^0$ and

$$Y(\hat\rho) < L \le Y(F^{-1}(\rho_r)),$$

then there exist two and only two states $(\rho_1^*, E(\rho_1^*)) \in T(\rho_l, \alpha)$ and $(\rho_2^*, E(\rho_2^*)) \in T(\rho_l, \alpha)$ satisfying $(F^{-1}(\rho_c)) < \rho_1^* < \hat\rho$ and $\hat\rho < \rho_2^* < F^{-1}(\rho_r)$, $E(\hat\rho) < E(\rho_1^*) < -E_c$ and $-E_r^0 < E(\rho_2^*) < E(\hat\rho)$ such that

$$L = Y(\rho_1^*) = Y(\rho_2^*).$$

In this case, there are two shock locations, i.e., $\ell((\rho_l,\alpha);(\rho_1^*,E(\rho_1^*))$ *and* $\ell((\rho_l,\alpha);(\rho_2^*,E(\rho_2^*)))$.
4) if

$$Y(F^{-1}(\rho_r)) \leq L < +\infty,$$

then there exists a unique state $(\tilde{\rho}^*,E(\tilde{\rho}^*)) \in T(\rho_l,\alpha)$ *satisfying* $F^{-1}(\rho_c) \leq \tilde{\rho}^* < F^{-1}(\hat{\rho})$ *and* $E(\hat{\rho}) < E(\tilde{\rho}^*) < -E_c$ *such that*

$$L = \ell((\rho_l,\alpha);(\tilde{\rho}^*,E(\tilde{\rho}^*))) + \ell(\tilde{\rho}^*,E(\tilde{\rho}^*));(\rho_r,E_r^*)) = Z(\tilde{\rho}^*),$$

where $E_r^* = \sqrt{(E^2(\tilde{\rho}^*) + 2(H(\rho_r) - H(F(\tilde{\rho}^*))}$ *so that* $(\rho_r,E_r^*) \in T(F(\tilde{\rho}^*),E(\tilde{\rho}^*))$. *So the transonic shock location is* $a = \ell((\rho_l,\ \alpha);(\rho^*,E^*))$.
6) If $-E_c \leq \alpha < E_r^0$ *or* $\alpha < -E_r^0$, *then the boundary value problem (1.8) and (1.9) has no solution with single transonic shock.*

Proof. The proof of this theorem is similar to that for Theorem 5.2 by noticing that

$$\lim_{\bar{\rho}\to F^{-1}(\rho_c)+} Y(\bar{\rho}) = +\infty,$$

which can be shown by a similar argument to that for (5.59). □

Next, we consider the following case

5.2 The case when (ρ_l,α) is between the trajectory passing through $(F^{-1}(b),0)$ and the subsonic part of the trajectory passing through $(b,0)$.

In this case, (ρ_l,α) satisfies:

$$-H(b) < \frac{1}{2}\alpha^2 - H(\rho_l) < -H(F^{-1}(b)),\ 0 < \rho_l < \rho_s. \tag{5.131}$$

The supersonic part of the trajectory passing through (ρ,α) intersects the line $E = 0$ at $(\rho_{min}^{bw},0)$, the shock curve $S(\rho_l,\alpha)$ intersects the subsonic part of the critical trajectory passing through $(b,0)$ at two points, denoted by (ρ^c,E^c) and $(\rho^c,-E^c)$.
We first consider the case when $\rho_r > b$. We have the following result (See Figure 10).

Theorem 5.5. ***Case for*** $\rho_r > b$
Suppose that (ρ_l,α) *satisfies (5.131),*
1) If $\alpha > E^c$, *then*
1a) the boundary value problem (1.8) and (1.9) does not have solutions with a single transonic shock if

$$L < \ell(F(\rho_l),\alpha);(\rho_r,E_r^\alpha), \tag{5.132}$$

(where E_r^α *is determined by* $(\rho_r,E_r^\alpha) \in T(F(\rho_l),\alpha)$*),*
1b) if

$$\ell(F(\rho_l),\alpha);(\rho_r,E_r^\alpha) \leq L < \infty, \tag{5.133}$$

then there exists a unique state $(\rho*, E(\rho^*) \in T(\rho_l, \alpha)$ *satisfying* $F^{-1}(\rho^c) < \rho^* \leq \rho_l$ *and* $E^c < E(\rho^*) \leq \alpha$ *such that*

$$L = \ell((\rho_l, \alpha); (\rho^*, E(\rho^*)) + \ell((F(\rho^*), E(\rho^*), (\rho_r, E_r^*)), \tag{5.134}$$

where E_r^* *satisfies* $(\rho_r, E_r^*) \in T(F(\rho^*), E(\rho^*))$.
2) If $\alpha \leq E^c$, *the boundary value problem (1.8) and (1.9) does not have solutions with a single transonic shock .*

Proof. 1a) and 2) are easily seen on phase plane (see Figure 10). We prove 1b) as follows: If $\rho_r > b$ and $\alpha > E^c$, for $(\rho, E(\rho) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho^c) < \rho \leq \rho_l$, we define

$$X(\rho) = \ell((\rho_l, \alpha); (\rho, E(\rho)) + \ell((F(\rho), E(\rho), (\rho_r, E_r(\rho))),$$

where $E_r(\rho)$ satisfies $(\rho_r, E_r(\rho)) \in T(F(\rho), E(\rho))$. By (3.11), we can show that

$$X'(\rho) < 0, \tag{5.135}$$

for $(\rho, E(\rho) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho^c) < \rho \leq \rho_l$ and $E^c < E(\rho) \leq \alpha$. Moreover, just as (5.58),

$$\lim_{\rho \to F^{-1}(\rho^c)+} \ell((F(\rho), E(\rho), (\rho_r, E_r(\rho))) \to +\infty. \tag{5.136}$$

This implies

$$\lim_{\rho \to F^{-1}(\rho^c)+} X(\rho) \to +\infty. \tag{5.137}$$

(5.134) follows from (5.135) and (5.137). □

Next, we consider the case when $F(\rho^{bw}_{min}) < \rho_r < b$. In this case, the trajectory passing through $(\rho_r, 0)$ satisfying $\frac{1}{2}E^2 - H(\rho) = -H(b)$ intersects the shock curve $S(\rho_l, \alpha)$ at two points, denoted by (ρ^K, E^K) and $(\rho^K, -E^K)$. In this case, for any state (ρ_0, E_0) in between the trajectory through $(\rho_r, 0)$ and the critical trajectory T_b through $(b, 0)$, i. e.,

$$-H(\rho_r) < \frac{1}{2}E_0^2 - H(\rho_0) < -H(b) \tag{5.138}$$

the trajectory through (ρ_0, E_0) is also in between the trajectory through $(\rho_r, 0)$ and the critical trajectory T_b through $(b, 0)$, and thus intersects the line $\rho = \rho_r$ at two points, denoted by $(\rho_r, E_r(\rho_0, E_0))$ and $(\rho_r, -E_r(\rho_0, E_0))$ (see Figure 11). With these notations, we have the following theorem.

Theorem 5.6. ***Case when*** $F(\rho^{bw}_{min}) < \rho_r < b$.
Suppose that (ρ_l, α) *satisfies (5.131) , then*

1) If $\alpha > E^c$, *then the boundary value problem (1.8) and (1.9) does not have solutions with a single transonic shock if*

$$L < \ell(F(\rho_l), \alpha); (\rho_r, E_r^\alpha)), \tag{5.139}$$

(where E_r^α is determined by $(\rho_r, E_r^\alpha) \in T(F(\rho_l), \alpha)$).
2) If $\alpha > E^K$, then
2a) if

$$\ell(F(\rho_l), \alpha); (\rho_r, E_r^\alpha)) \leq L \leq \ell((\rho_l, \alpha), (F^{-1}(\rho^K), E^K)) + \ell((\rho^K, E^K), (\rho_r, 0)), \tag{5.140}$$

then there exists a unique state $(\rho^*, E(\rho^*)) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho^K) \leq \rho^* \leq \rho_l$ and $E^K \leq E(\rho^*) \leq \alpha$ such that

$$L = \ell((\rho_l, \alpha); (\rho^*, E(\rho^*)) + \ell((F(\rho^*), E(\rho^*), (\rho_r, E_r^*)), \tag{5.141}$$

where E_r^* satisfies $(\rho_r, E_r^*) \in T(F(\rho^*), E(\rho^*))$,
2b) if

$$\ell((\rho_l, \alpha), (F^{-1}(\rho^K), E^K)) + \ell((\rho^K, E^K), (\rho_r, 0)) \leq L < +\infty, \tag{5.142}$$

then there exists a unique state $(\rho^*, E(\rho^*) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho^K) \leq \rho^* \leq \rho_l$ and $E^K \leq E(\rho^*) < E^c$ such that

$$L = \ell((\rho_l, \alpha); (\rho^*, E(\rho^*)) + \ell((F(\rho^*), E(\rho^*), (\rho_r, -E_r((F(\rho^*), E^*)). \tag{5.143}$$

3) If $\alpha < E^K$, the boundary value problem (1.8) and (1. 9) does not have solutions with a single transonic shock .

Proof. 1) and 3) are easily seen on phase plane (see Figure 11). We prove 2a) and 2b)as follows: $F(\rho_{min}^{bw}) < \rho_r < b$ and $\alpha > E^K$, for $(\rho, E(\rho) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho^K) < \rho \leq \rho_l$ and $E^K < E(\rho) \leq \alpha$, we define

$$X(\rho) = \ell((\rho_l, \alpha); (\rho, E(\rho)) + \ell((F(\rho), E(\rho), (\rho_r, E_r(\rho))),$$

where $E_r(\rho)$ satisfies $(\rho_r, E_r(\rho)) \in T(F(\rho), E(\rho))$. By (3.11), it can be readily shown that

$$X'(\rho) < 0, \tag{5.144}$$

for $(\rho, E(\rho) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho^K) < \rho \leq \rho_l$. This proves 2a).
For $(\bar\rho, E(\bar\rho) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho^K) < \bar\rho \leq \rho^c$ and $E^K < E(\rho) \leq \alpha$, we define

$$Y(\bar\rho) = \ell((\rho_l, \alpha); (\bar\rho, E(\bar\rho)) + \ell((F(\bar\rho), E(\bar\rho), (\rho_r, -E_r(F(\bar\rho), E(\bar\rho))).$$

By using (3.41), we obtain

$$\begin{aligned}\frac{dY(\bar\rho)}{d\bar\rho} &= \left(p'(\bar\rho) - \frac{J^2}{\bar\rho^2}\right)\frac{F(\bar\rho) - \bar\rho}{\bar\rho} \cdot \{\frac{1}{(F(\bar\rho) - b)E(\bar\rho)} \\ &- \frac{b}{F(\bar\rho)}[\frac{1}{(\rho_r - b)(-E_r(F(\bar\rho), E(\bar\rho))} + \int_{F(\bar\rho)}^{q(\bar\rho)} \frac{d\rho}{(\rho - b)^2 E_1(\rho, \bar\rho)} + \int_{q(\bar\rho)}^{\rho_r} \frac{d\rho}{(\rho - b)^2 E_2(\rho, \bar\rho)}]\}.\end{aligned} \tag{5.145}$$

The meanings of $q(\bar\rho)$, $E_1(\rho,\bar\rho)$, $E_2(\rho,\bar\rho)$ are the same as those in Lemma 3.5. Moreover, it should be noted that $\bar E_r(\bar\rho)$ in (3.41) is the same as $-E_r(F(\bar\rho),E(\bar\rho))$ here. Since $F(\bar\rho)<b$, $E(\bar\rho)>0$, $E_r(F(\bar\rho),E(\bar\rho))>0$, $\rho_r<b$, $q(\bar\rho)>F(\bar\rho)$, $E_1(\rho,\bar\rho)>0$, $q(\bar\rho)>\rho_r$ and $E_2(\rho,\bar\rho)<0$, by the same method we have already used, we can show that

$$Y'(\bar\rho)>0, \tag{5.146}$$

for $(\bar\rho,E(\bar\rho))\in T(\rho_l,\alpha)$ satisfying $F^{-1}(\rho^K)<\bar\rho\le\rho^c$. Moreover,

$$\lim_{\bar\rho\to F^{-1}(\rho^c)+}\ell((F(\bar\rho),E(\bar\rho),(\rho_r,-E_r(F(\bar\rho),E(\bar\rho)))\to+\infty. \tag{5.147}$$

This implies

$$\lim_{\rho\to F^{-1}(\rho^c)+}Y(\bar\rho)\to+\infty. \tag{5.148}$$

(5.143) follows from (5.146) and (5.148). This proves 2b). □

Next, we consider the case when $\rho^c<\rho_r<F(\rho^{bw}_{min})$ (see Figure 12).

$$\rho^c<\rho_r<F(\rho^{bw}_{min}) \tag{5.149}$$

In this case, the line $\rho=\rho_r$ intersects the shock curve $S(\rho_l,\alpha)$ at two points (ρ_r,E_r^0) and $(\rho_r,-E_r^0)$ with $E_r^0>0$. The trajectory passing through $(b,0)$ satisfying $\frac12E^2-H(\rho)=-H(b)$ intersects the shock curve $S(\rho_l,\alpha)$ at two points $(\rho^c,\ E^c)$ and $(\rho^c,\ -E^c)$ with $E^c>0$ (see Figure 12).

For $\bar\rho\in[\rho^{bw}_{min},F^{-1}(\rho_r)]$, we let

$$E_r^-(\bar\rho)=-\sqrt{E_-{}^2(\bar\rho)+2((H(\rho_r)-H(F(\bar\rho))}, \tag{5.150}$$

where

$$E_-(\bar\rho)=-\sqrt{\alpha^2+2(H(\bar\rho)-H(\rho_l))}$$

satisfying $(\bar\rho,E_-(\bar\rho)\in T(\rho^{bw}_{min},0)\subset T(\rho_l,\alpha)$. and $-E_r^0\le E_-(\bar\rho)\le0$. In this case, we define

$$\begin{aligned}\mathfrak{Y}(\bar\rho)=&\ell((\rho_l,\alpha);(\bar\rho,E_-(\bar\rho)))+\ell((F(\bar\rho),E_-(\bar\rho));(\rho_r,E_r(\bar\rho))),\\ &\text{for }\bar\rho\in[F^{-1}(\rho^{bw}_{min}),F^{-1}(\rho_r)],\ -E_r^0\le E_-(\bar\rho)\le0.\end{aligned} \tag{5.151}$$

For $\bar\rho\in[\rho^{bw}_{min},F^{-1}(\rho_c)]$, we let

$$E_r^-(\bar\rho)=-\sqrt{E_+{}^2(\bar\rho)+2((H(\rho_r)-H(F(\bar\rho))}, \tag{5.152}$$

where

$$E_+(\bar\rho)=\sqrt{\alpha^2+2(H(\bar\rho)-H(\rho_l))}$$

satisfying $(\bar{\rho}, E_+(\bar{\rho})) \in T(\rho_l, \alpha)$. and $0 \le E_+(\bar{\rho}) \le E^c$. In this case, we define

$$\begin{aligned}
&\mathfrak{Z}(\bar{\rho}) = \ell((\rho_l, \alpha); (\bar{\rho}, E_+(\bar{\rho}))) + \ell((F(\bar{\rho}), E_+(\bar{\rho})); (\rho_r, E_r^-(\bar{\rho}))),\\
&\text{for } \bar{\rho} \in [\rho_{min}^{bw}, F^{-1}(\rho_c)),\ 0 \le E_+(\bar{\rho}) < E^c.
\end{aligned} \tag{5.153}$$

It is easy to see that

$$\mathfrak{Y}(\rho_{min}^{bw}) = \mathfrak{Z}(\rho_{min}^{bw}). \tag{5.154}$$

Then we have,

Lemma 5.7. *Suppose that (ρ_l, α) satisfies (5.131), ρ_r satisfies (5.149) and $\alpha > E^c$. Then there exists a unique state $(\hat{\rho}, E(\hat{\rho}) \in T(\rho_{min}^{min}, 0) \subset T(\rho_l, \alpha)$ satisfying $\rho_{min}^c < \hat{\rho} < F^{-1}(\rho_r)$ and $-E_r^0 < E(\hat{\rho}) < 0$ such that*

$$\mathfrak{Y}'(\rho_{min}^{bw}) = -\infty, \begin{cases} \mathfrak{Y}'(\bar{\rho}) < 0, \text{for } \rho_{min}^{bw} < \bar{\rho} < \hat{\rho}, \\ \mathfrak{Y}'(\bar{\rho}) > 0, \text{for } \hat{\rho} < \bar{\rho} \le F^{-1}(\rho_r). \end{cases} \tag{5.155}$$

So

$$\mathfrak{Y}(\hat{\rho}) = \min_{\rho_{min}^{bw} \le \bar{\rho} \le F^{-1}(\rho_r)} \mathfrak{Y}(\bar{\rho}). \tag{5.156}$$

Also

$$\begin{aligned}
&\mathfrak{Z}'(\bar{\rho}) > 0, \text{ for } \rho_{min}^{bw} \le \bar{\rho} < F^{-1}(\rho^c),\\
&\lim_{\bar{\rho} \to F^{-1}(\rho^c)} \mathfrak{Z}(\bar{\rho}) = +\infty.
\end{aligned} \tag{5.157}$$

Proof. The proof of (5.155) is almost the same as that for Lemma 5.4. The proof of (5.157) follows a similar argument as that for (5.93) and (5.100). □

With this lemma, we have the following theorem.

Theorem 5.7. *Suppose that (ρ_l, α) satisfies (5.131) and ρ_r satisfies (5.149).*
1) If $\alpha > E_c$,
1a) the boundary value problem (1.8) and (1.9) does not have a solution with a single transonic shock, if

$$L < \ell((F(\rho_l),\ \alpha), (\rho_r, \beta)),$$

where β is determined by

$$\beta = \sqrt{\alpha^2 + 2(H(\rho_r) - H(F(\rho_l))}, \tag{5.158}$$

such that $(\rho_r, \beta) \in T(F(\rho_l),\ \alpha)$.
1b) if

$$\ell((F(\rho_l), \alpha); (\rho_r, \beta)) \le L \le \ell((\rho_l, \alpha); (F^{-1}(\rho_r), E_r^0)),$$

where β is determined by in (5.158), then there exists a unique state $(\rho^*, E^*) \in T(\rho_l, \alpha)$ satisfying $F^{-1}(\rho_r) \le \rho^* \le \rho_l$ and $E_r^0 \le E^* \le \alpha$ and a constant E_r such that

$$(\rho_r, E_r) \in T(F(\rho^*), E^*), \ \ L = \ell\left((\rho_l, \alpha); (\rho^*, E^*)\right) + \ell\left((F(\rho^*), E^*); (\rho_r, E_r)\right).$$

So the transonic shock location is $a = \ell((\rho_l, \ \alpha); (\rho^*, E^*))$;
2) If $\alpha > E^c$ and

$$\ell((\rho_l, \alpha); (F^{-1}(\rho_r), E_r^0)) < L < \mathfrak{Y}(\hat{\rho})), \tag{5.159}$$

(where and in the following $\hat{\rho}$ is given in (5.156)), then the boundary value problem (1.8) and (1.9) has no solution with single transonic shock;
3) If $\alpha > E^c$,

$$\mathfrak{Y}(\hat{\rho}) < L \le \min\{\mathfrak{Y}((\rho_{min}^{bw}), \mathfrak{Y}(F^{-1}(\rho_r))\}, \tag{5.160}$$

then there exist two and only two states $(\rho_1^*, E(\rho_1^*)) \in T(\rho_l, \alpha)$ and $(\rho_2^*, E(\rho_2^*)) \in T(\rho_l, \alpha)$ satisfying $\rho_{min}^{bw} < \rho_1^* < \hat{\rho} < \rho_2^* < F^{-1}(\rho_r)$, $E(\hat{\rho}) < E(\rho_1^*) < 0$ and $-E_r^0 < E(\rho_2^*) < E(\hat{\rho})$ such that

$$L = \mathfrak{Y}(\rho_1^*) = \mathfrak{Y}(\rho_2^*). \tag{5.161}$$

In this case, there are two shock locations, i.e., $\ell((\rho_l, \alpha); (\rho_1^*, E(\rho_1^*))$ and $\ell((\rho_l, \alpha); (\rho_2^*, E(\rho_2^*)))$.
4) Suppose $\alpha > E^c$ and $\mathfrak{Y}(F^{-1}(\rho_{min}^{bw}) \neq \mathfrak{Y}(F^{-1}(\rho_r))$ (the case $\mathfrak{Y}(F^{-1}(\rho_{min}^{bw}) = \mathfrak{Y}(F^{-1}(\rho_r))$ can be handled similarly).
If

$$\min\{\mathfrak{Y}(\rho_{min}^{bw}), \mathfrak{Y}(F^{-1}(\rho_r))\} < L < \max\{\mathfrak{Y}(F^{-1}(\rho_{min}^{bw})), \mathfrak{Y}(F^{-1}(\rho_r))\},$$

then we have the following results:
4a) if

$$\mathfrak{Y}(\rho_{min}^{bw}) < \mathfrak{Y}(F^{-1}(\rho_r)),$$

then there exist two states $(\bar{\rho}_1^*, E(\bar{\rho}_1^*)) \in T(\rho_l, \alpha)$ and $(\bar{\rho}_2^*, E(\bar{\rho}_2^*)) \in T(\rho_l, \alpha)$ satisfying $\rho_{min}^{bw} < \bar{\rho}_1^* < F^{-1}(\rho^c) < \bar{\rho}_2^* < F^{-1}(\rho_r)$, $0 \le E(\bar{\rho}_1^*)) < E^c$ and $-E_r^0 \le E(\bar{\rho}_2^*) \le 0$, such that

$$L = \mathfrak{Z}(\rho_1^*) = \mathfrak{Y}(\rho_2^*),$$

4b) if

$$\mathfrak{Y}(\rho_{min}^{bw}) > \mathfrak{Y}(F^{-1}(\rho_r)),$$

then there exists a unique state $(\bar{\rho}*, E(\bar{\rho}^*)) \in T(\rho_l, \alpha)$ satisfying $\rho_{min}^{bw} < \bar{\rho}^* < \hat{\rho}$ and $-E_c < E(\bar{\rho}^*)) \le 0$ such that

$$L = \mathfrak{Y}(\bar{\rho}*).$$

So the shock location is $a = \ell((\rho_l, \alpha); (\bar{\rho}^*, E(\bar{\rho}^*))$.
5) if

$$\max\{\mathfrak{Y}(F^{-1}(\rho_r^1)), \mathfrak{Y}(F^{-1}(\rho_r))\} \le L < +\infty,$$

then there exists a unique state $(\tilde{\rho}^*, E(\tilde{\rho}^*)) \in T(\rho_l, \alpha)$ *satisfying* $\rho^{bw}_{min} \le \tilde{\rho}^* < F^{-1}(\rho_c)$ *and* $0 < E(\tilde{\rho}^*) < E_c$ *such that*

$$L = \ell((\rho_l, \alpha); (\tilde{\rho}^*, E(\tilde{\rho}^*))) + \ell(\tilde{\rho}^*, E(\tilde{\rho}^*)); (\rho_r, E_r^*)) = \mathfrak{Z}(\tilde{\rho}^*),$$

where $E_r^* = -\sqrt{(E^2(\tilde{\rho}^*) + 2(H(\rho_r) - H(F(\tilde{\rho}^*))}$ *so that* $(\rho_r, E_r^*) \in T(F(\tilde{\rho}^*), E(\tilde{\rho}^*))$. *So the transonic shock location is* $a = \ell((\rho_l,\ \alpha); (\rho^*, E^*))$.
6) If $\alpha < -E_r^0$*, then the boundary value problem (1.8) and (1.9) has no solution with single transonic shock.*

The proof of this theorem is similar to that for Theorem 5.2 with the help of Lemma 5.7. So we omit it.

The case when

$$\rho_s < \rho_r < \rho^c. \tag{5.162}$$

can be handled in a similar manner to the case $\rho^c < \rho_r < F(\rho^{bw}_{min})$. A phase portrait of this case is given by Figure 13. We omit the details for this case.

5.3 The case when (ρ_l, α) is inside subsonic part of the trajectory passing through $(b, 0)$.

In this case, (ρ_l, α) satisfies:

$$\frac{1}{2}\alpha^2 - H(\rho_l) < -H(b),\ 0 < \rho_l < \rho_s. \tag{5.163}$$

The curve

$$\frac{1}{2}E^2 - H(\rho) = \frac{1}{2}\alpha^2 - H(\rho_l), \tag{5.164}$$

which is the trajectory passing through (ρ_l, α), intersects the line $E = 0$ at $(\rho^{in}_{min}, 0)$ and $(\rho_{max}, 0)$ satisfying

$$H((\rho^{in}_{min}) = H(\rho_{max}) = H(\rho_l) - \frac{1}{2}\alpha^2,\ \rho^{in}_{min} < \rho_s < \rho_{max}. \tag{5.165}$$

The curve (5.164) is a closed curve, lying inside the critical trajectory through $(b, 0)$. The shock curve $S(\rho_l, \alpha)$ lying inside the subsonic part of the curve (5.164), by Lemma 5.3 (see Figure 14).
The proofs of theorems in this subsection are similar to those in section 5.2, so we omit them.

By looking at the portrait, it is easy to see

Theorem 5.8. ***Case for*** $\rho_r > \rho_{max}$
Suppose that (ρ_l, α) *satisfies (5.163), if* $\rho_r > \rho_{max}$*, then the boundary value problem (1.8) and (1.9) does not have a solution for any* L *(see Figure 14).*

Next, we turn to the case when $F(\rho_{min}^{in}) < \rho_r < \rho_{max}$. In this case, the trajectory though the point $(F(\rho_{min}^{in}), 0)$ satisfying $\frac{1}{2}E^2 - H(\rho) = -H(\rho_{min}^{in})$ intersects the shock curve $S(\rho_l, \alpha)$ at two pints, denoted by (ρ_K, E_K) and $(\rho_K, -E_K)$ with $E_K > 0$ (See Figure 15). In this case, we have the following theorem.

Theorem 5.9. ***Case for*** $F(\rho_{min}^{in}) < \rho_r < \rho_{max}$
Suppose that (ρ_l, α) *satisfies (5.163), then*
1) If $\alpha > E_K$*, then*
1a) if

$$L < \ell((F(\rho_l), \alpha); (\rho_r, E_r^{\alpha}),$$

where E_r^{α} *is determined by* $(\rho_r, E_r^{\alpha}) \in T(F(\rho_l), \alpha)$ *satisfying* $E_r^{\alpha} > 0$*, then the boundary value problem (1.8) and (1.9) does not have a solution with a single transonic shock;*
1b) if

$$\ell((F(\rho_l), \alpha); (\rho_r, E_r^{\alpha}) \leq L \leq \ell((\rho_l), \alpha); (F^{-1}(\rho_K), E_K)) + \ell((\rho_K, E_K); (\rho_r, 0)), \qquad (5.166)$$

then there exists a unique state $(\rho*, E(\rho^*) \in T(\rho_l, \alpha)$ *satisfying* $F^{-1}(\rho_K) \leq \rho^* \leq \rho_l$ *and* $E_K \leq E(\rho^*) \leq \alpha$ *such that*

$$L = \ell((\rho_l, \alpha); (\rho^*, E(\rho^*)) + \ell((F(\rho^*), E(\rho^*), (\rho_r, E_r^*)), \qquad (5.167)$$

where E_r^* *satisfies* $(\rho_r, E_r^*) \in T(F(\rho^*), E(\rho^*))$ *and* $E_r^* \geq 0$*;*
1c) if

$$\ell((\rho_l), \alpha); (F^{-1}(\rho_K), E_K)) + \ell((\rho_K, E_K); (\rho_r, 0)) \leq L \leq \ell((F(\rho_l), \alpha); (\rho_r, -E_r^{\alpha}), \qquad (5.168)$$

where E_r^{α} *is determined as in 1a), then there exists a unique state* $(\rho*, E(\rho^*)) \in T(\rho_l, \alpha)$ *satisfying* $F^{-1}(\rho_K) \leq \rho^* \leq \rho_l$ *and* $E_K \leq E(\rho^*) \leq \alpha$ *such that*

$$L = \ell((\rho_l, \alpha); (\rho^*, E(\rho^*)) + \ell((F(\rho^*), E(\rho^*), (\rho_r, -E_r^*)), \qquad (5.169)$$

where E_r^* *satisfies* $(\rho_r, E_r^*) \in T(F(\rho^*), E(\rho^*))$ *and* $E_r^* \geq 0$*.*
2) If $\alpha < E_K$*, the boundary value problem (1.8) and (1.9) does not have solutions with a single transonic shock .*

The case $\rho_s < \rho_r < F(\rho_{min}^{in})$ can be handled in a similar way to the case of $F(\rho_{min}^{in}) < \rho_r < \rho_{max}$. So we omit it. A phase portrait is given by Figure 16, which illustrates how to handle this case.

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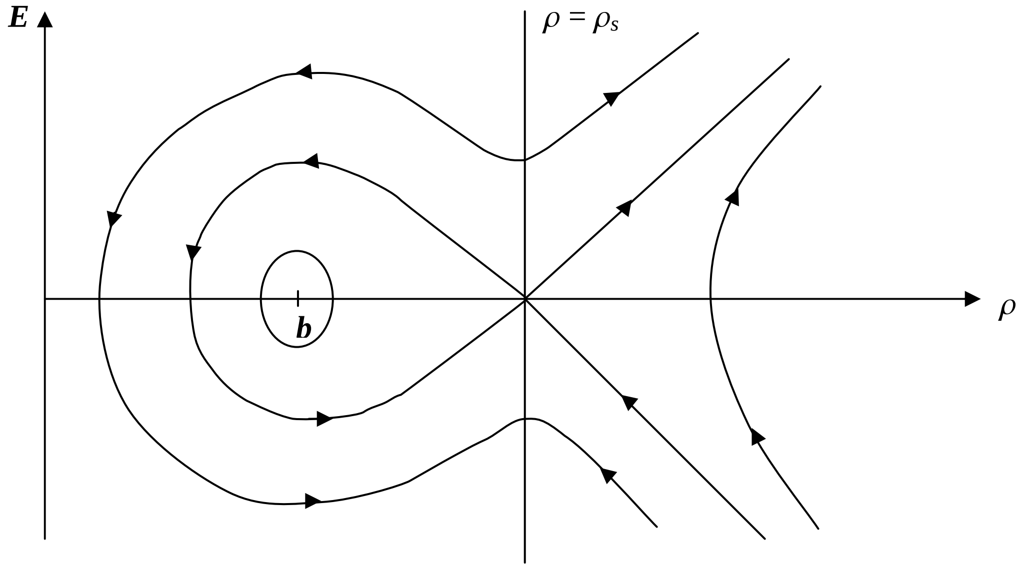


Figure 1
Phase Portrait for $0 < b < \rho_s$

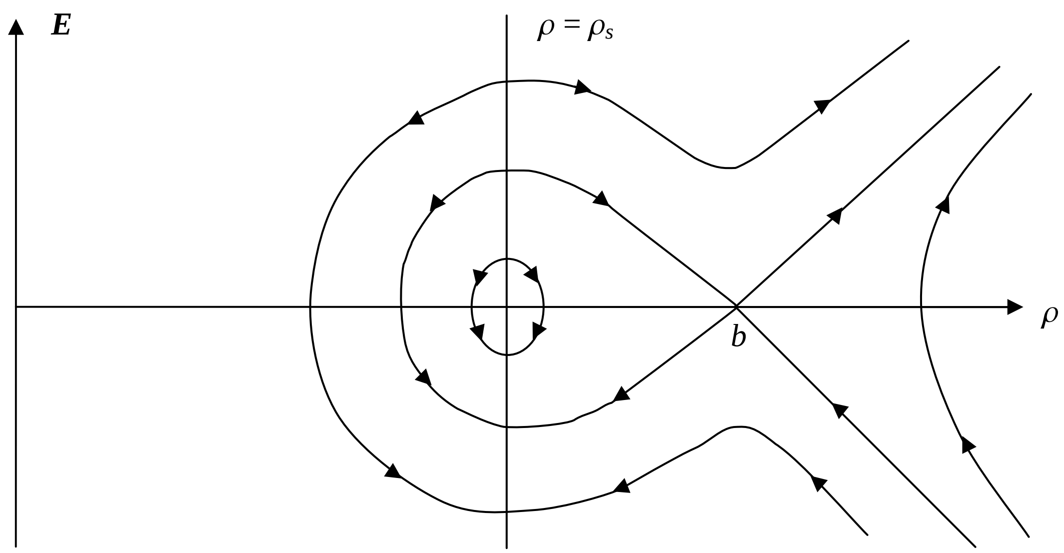
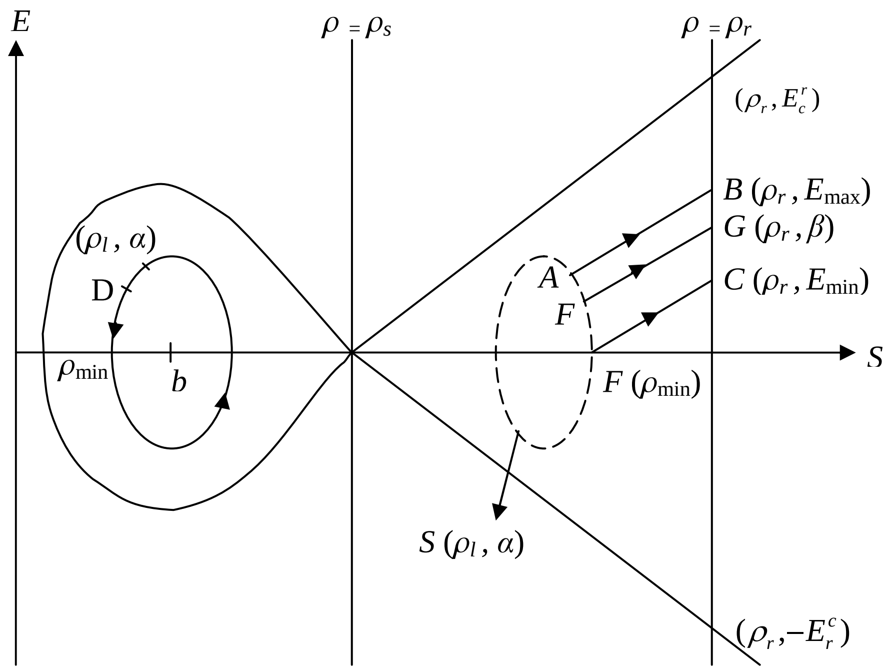
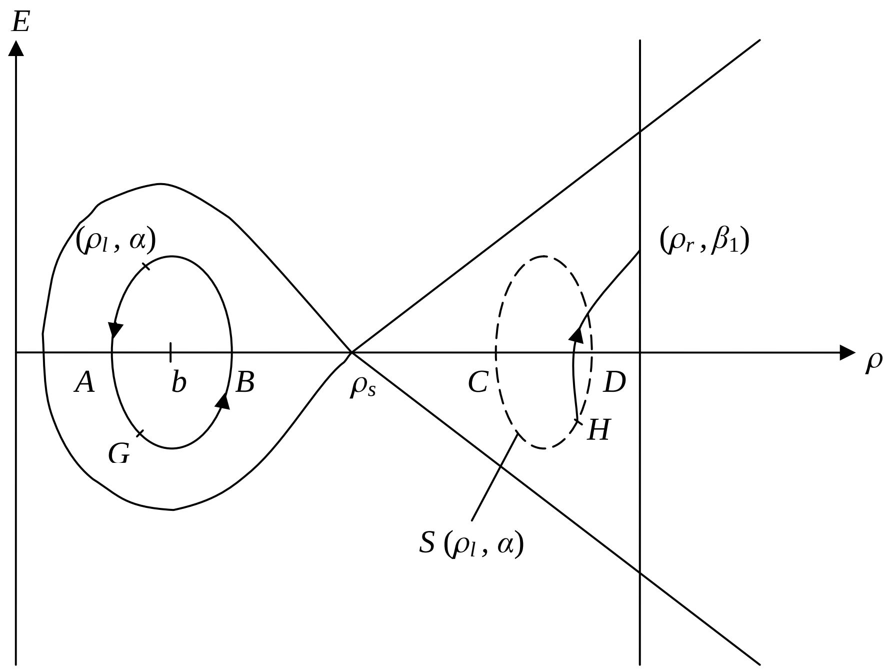


Figure 2
Phase Portrait of $b > \rho_s$



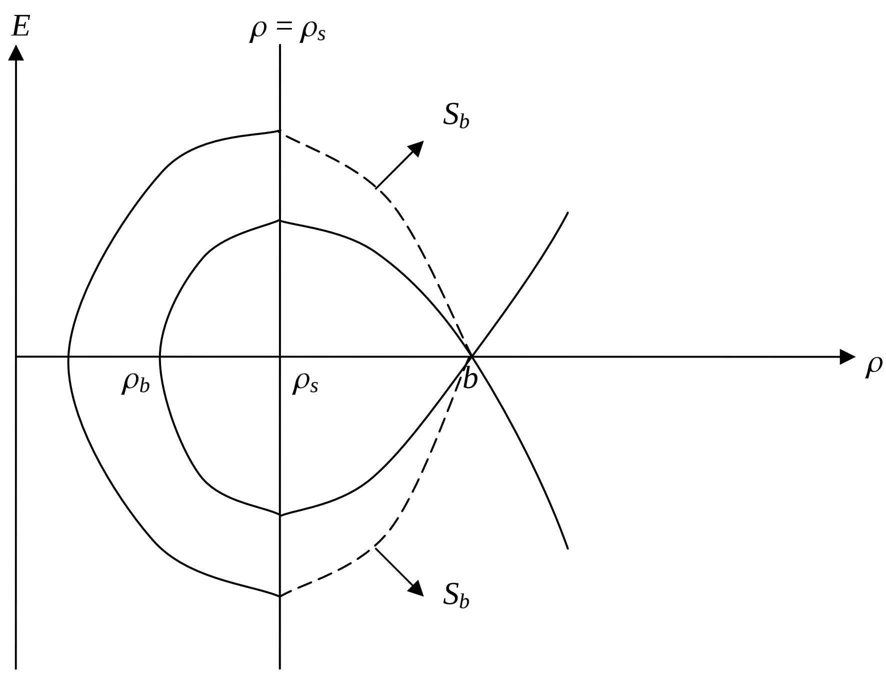
A: $(F(\rho_l),\alpha)$, B: $(\rho_r, E_{\max})$, C: $(\rho_r, E_{\min})$
D: (ρ^*, E^*), F: $(F(\rho^*), E^*)$.

Figure 3



A: $(\rho_{\min}, 0)$, *B*: $(\rho_{\max}, 0)$, *C*: $(F(\rho_{\max}), 0)$,
D: $(F(\rho_{\min}), 0)$, *G*: (ρ^*, E^*), *H*: $(F(\rho^*), E^*)$.

Figure 4



The dotted curve is S_b.

Figure 5

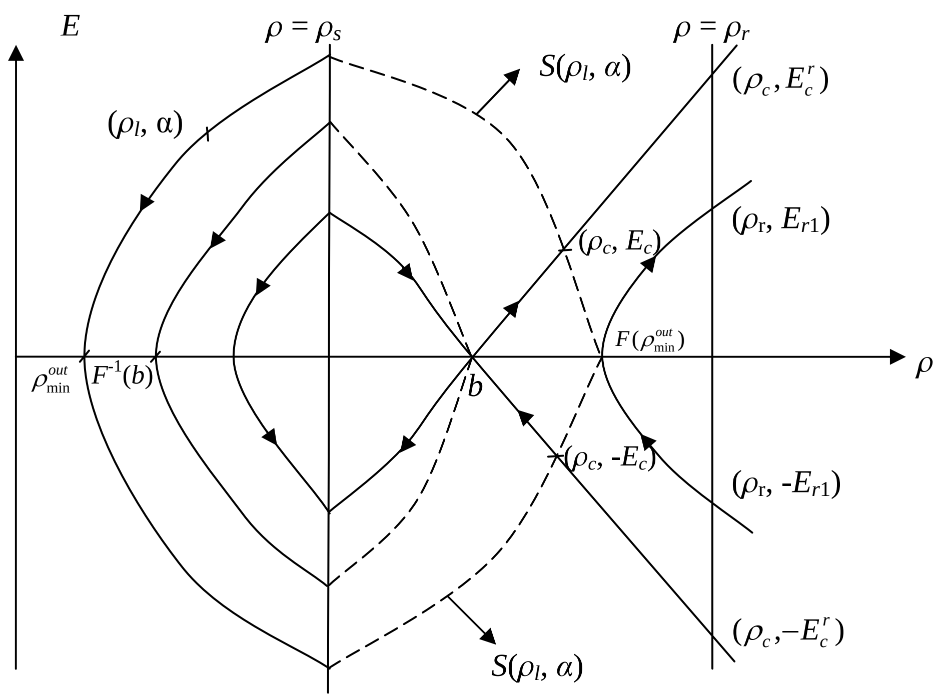
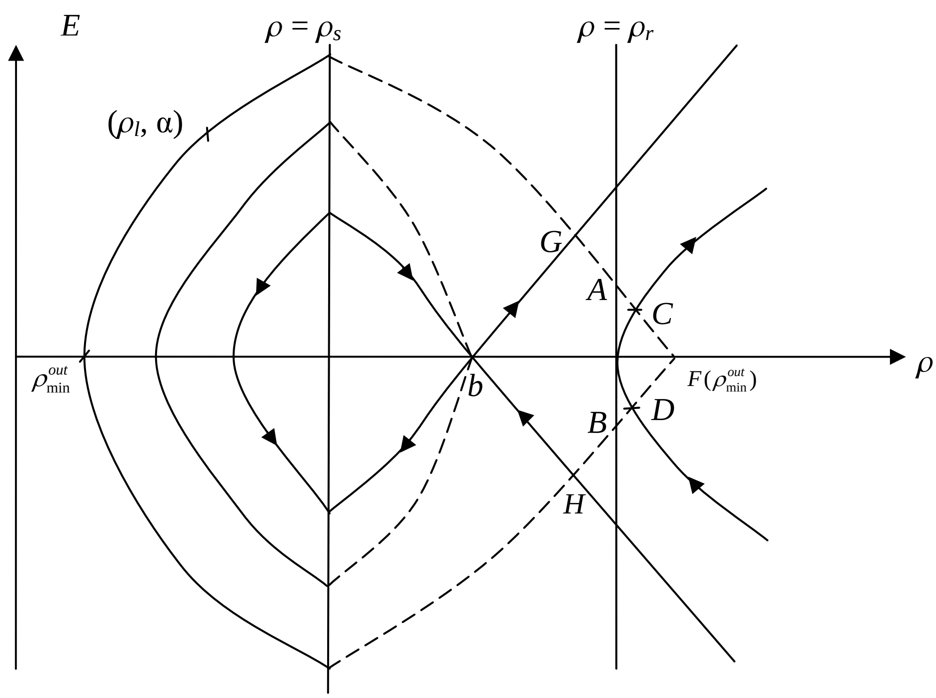
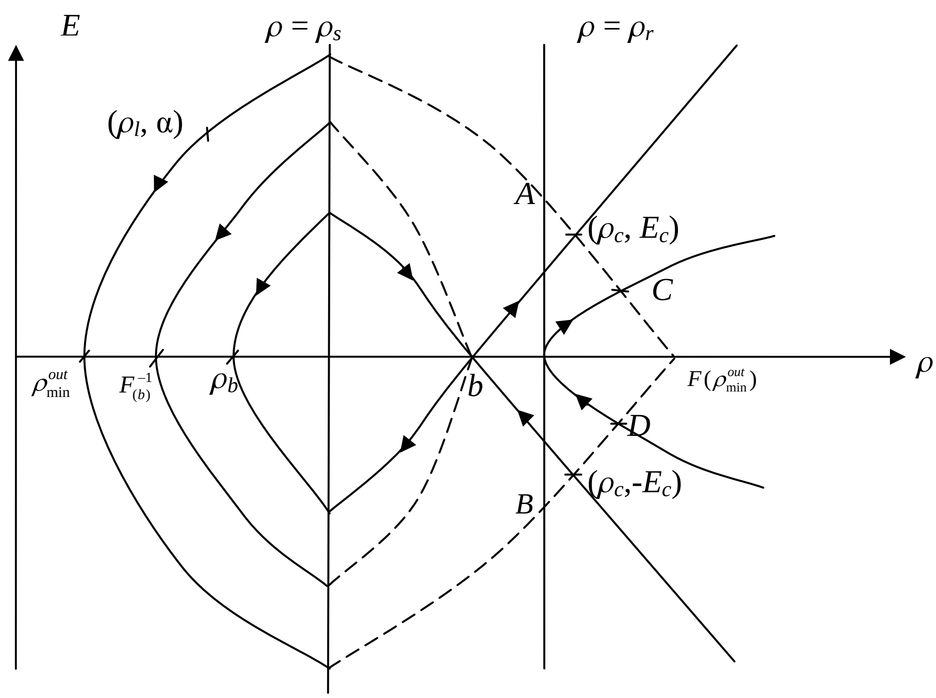

E
ρ = ρs
ρ = ρr
S(ρl, α)
(ρc, Ec^r)
(ρl, α)
(ρr, Er1)
(ρc, Ec)
F(ρmin^out)
ρ
ρmin^out
F^-1(b)
b
(ρc, -Ec)
(ρr, -Er1)
(ρc, −Ec^r)
S(ρl, α)

Figure 6



A: (ρ_r, E_r^0), B: $(\rho_r, -E_r^0)$, C: (ρ_r^1, E_r^1),
D: $(\rho_r^1, -E_r^1)$, G: (ρ_c, E_c), H: $(\rho_c, -E_c)$.

Figure 7



A: (ρ_{r}, E_r), B: $(\rho_{\mathrm{r}}, -E_r)$, C: $(\bar{\rho}_r, \bar{E}_r)$, D: $(\bar{\rho}_r, -\bar{E}_r)$.

Figure 8

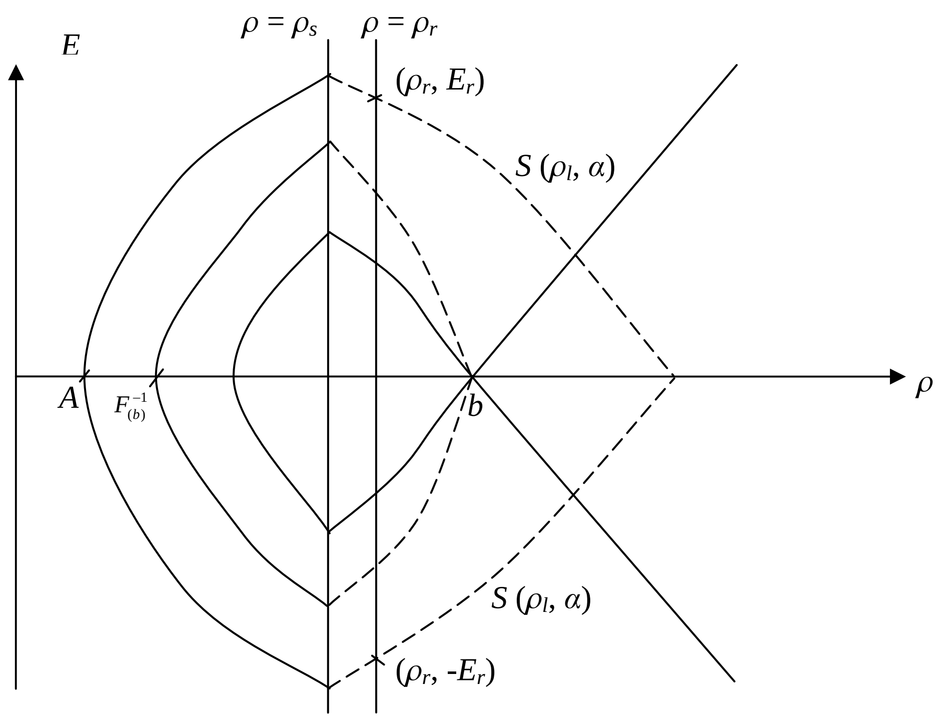
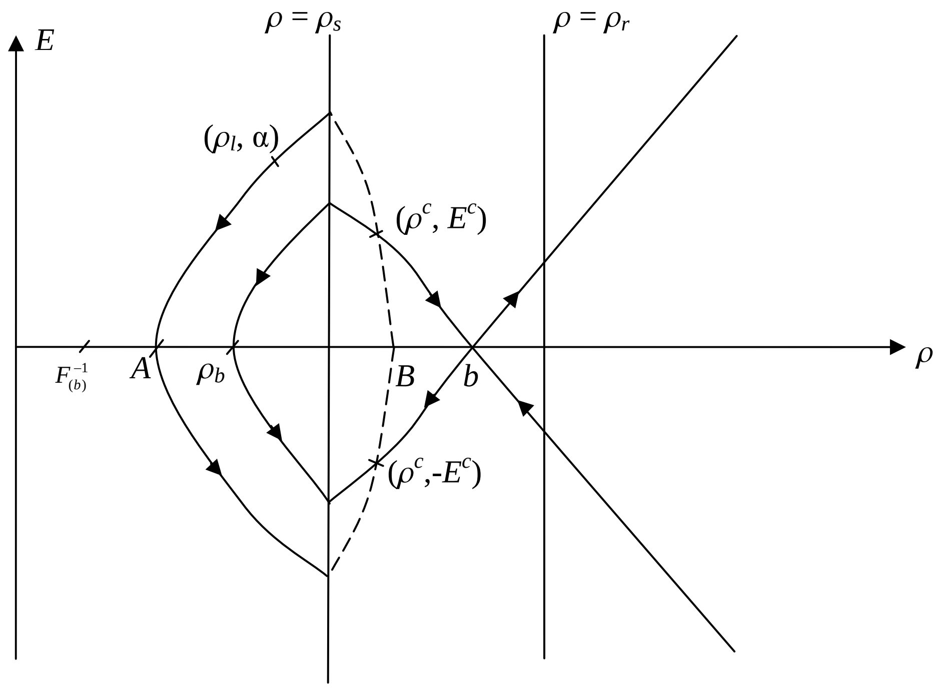
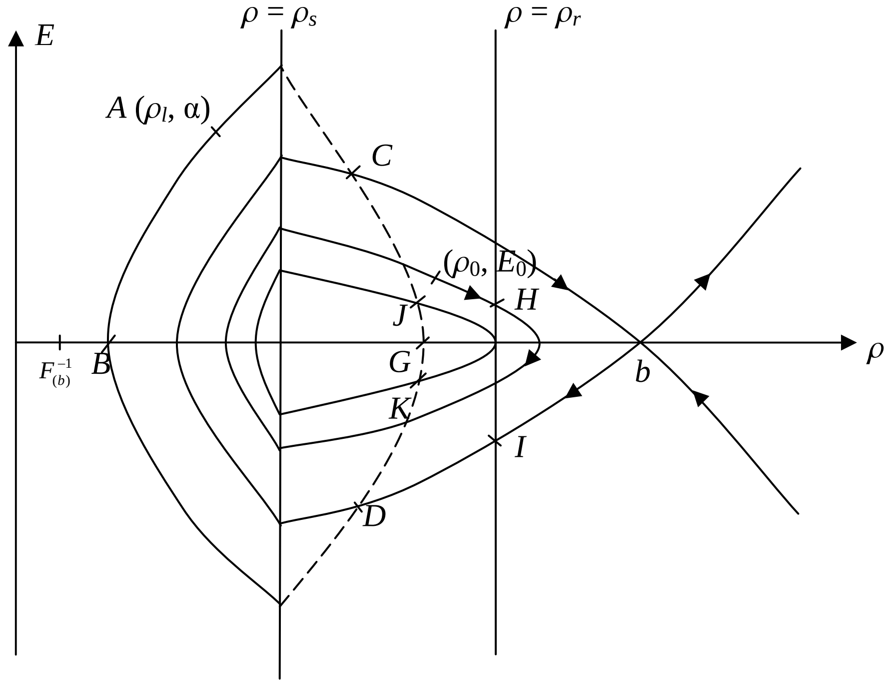


Figure 9



$A:(\rho_{\min}^{bw},0), \qquad B:(F(\rho_{\min}^{bw}),0)$

Figure 10



$B:(\rho^{bw}_{\min},0)$, $C:(\rho^{c}, E^{c})$, $D: (\rho^{c}, -E^{c})$, $G:(F(\rho^{bw}_{\min}),0)$,
$H: (\rho_r, E_r(\rho_0, E_0))$, $I: (\rho_r, -E_r(\rho_0, E_0))$, $J: (\rho^{k}, E^{k})$, $K: (\rho^{k}, -E^{k})$

Figure 11

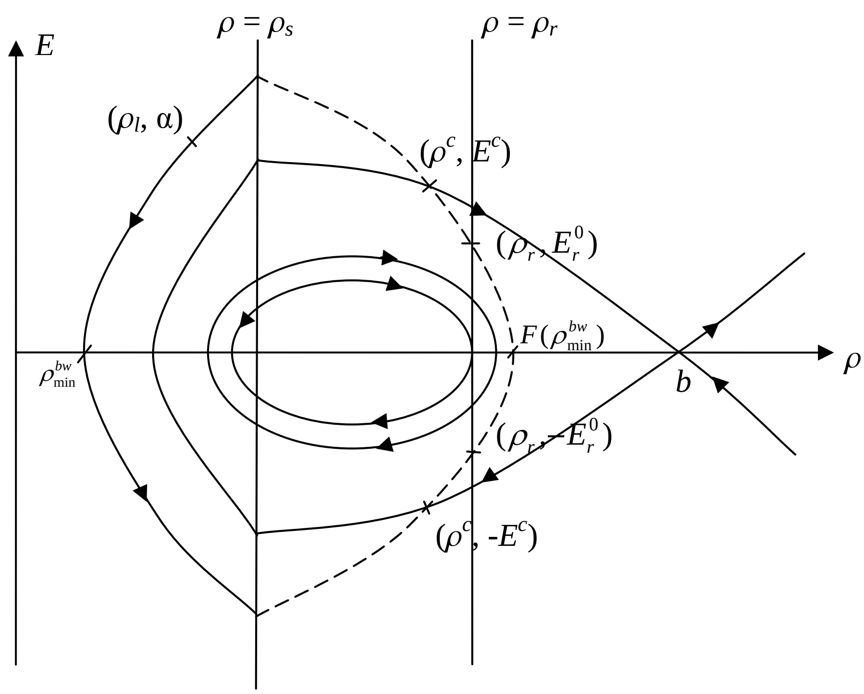
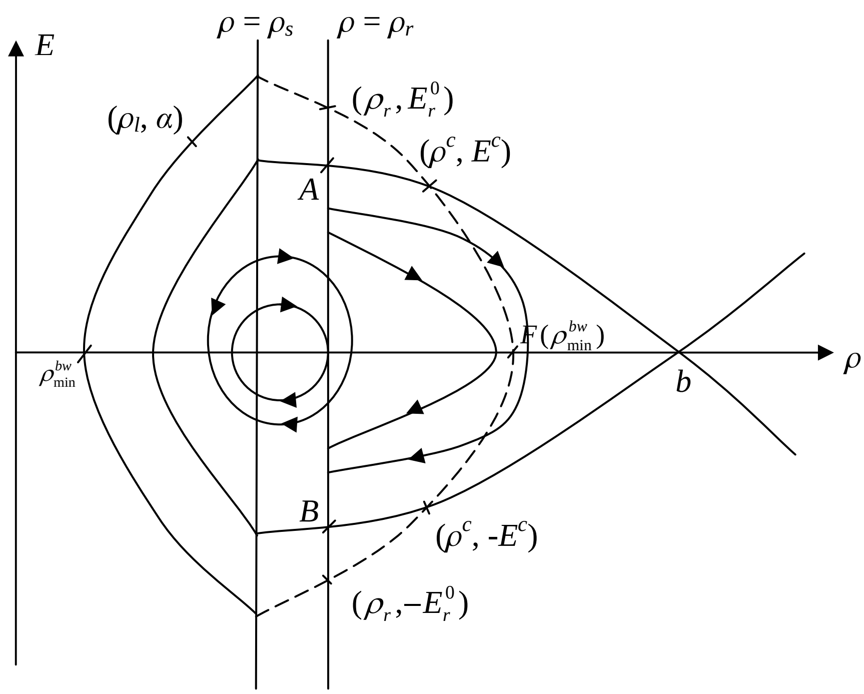
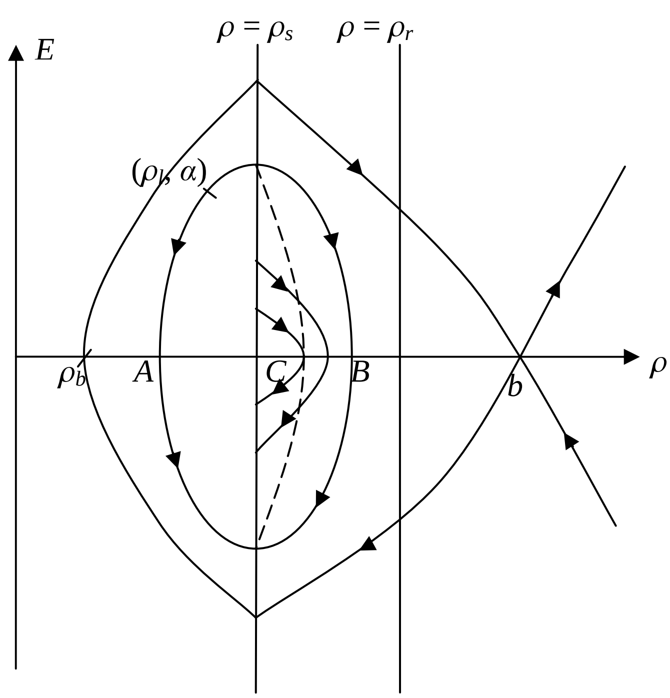


Figure 12



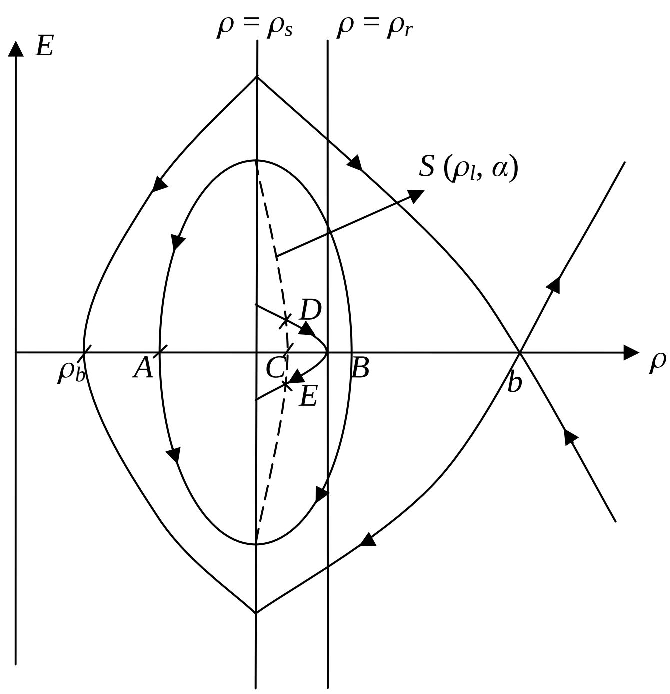
$A:(\rho_r,-E_r^1),\qquad B:(\rho_r,-E_r^1)$

Figure 13



$A:(\rho_{\min}^{in},0),\qquad B:(\rho_{\max},0),\qquad C:(F(\rho_{\min}^{in}),0)$

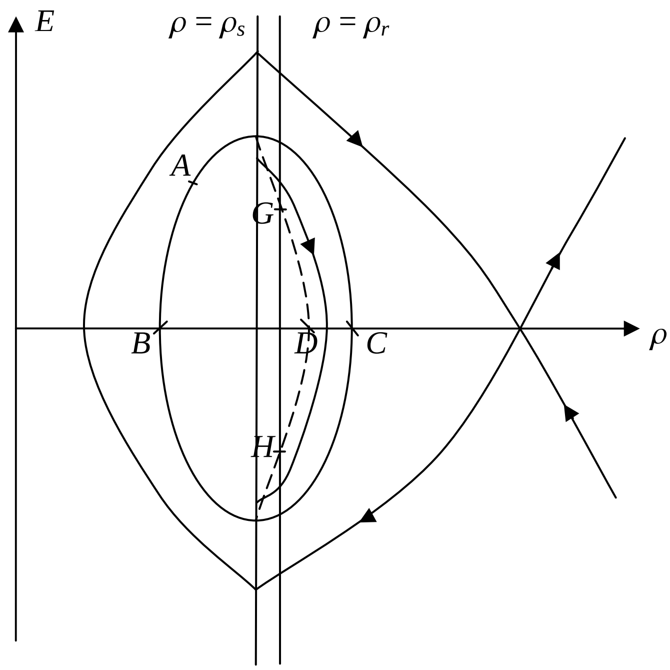
Figure 14



$A:(\rho_{\min}^{in},0), \quad B:(\rho_{\max},0), \quad C:(F(\rho_{\min}^{in}),0)$

$D:(\rho_k,E_k), \quad E:(\rho_k,-E_k)$

Figure 15



$A:(\rho_l,\alpha)$, $B:(\rho_{\min}^{in},0)$, $C:(\rho_{\max},0)$

$D:(F(\rho_{\min}^{in}),0)$, $G:(\rho_r,E_r^1)$, $H:(\rho_r,-E_r^1)$

Figure 16