ON GROUND STATE OF SPINOR BOSE-EINSTEIN CONDENSATES

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Abstract. We prove the existence of the ground state for the spinor Bose-Einstein condensates in the one-dimensional case.

1. Introduction

In 1925, Einstein predicted that massive non-interacting bosons at low temperature could occupy the same lowest-energy single-particle state and form so-called the Bosen-Einstein condensates (BEC). This was realized experimentally in 1995 by laser cooling technique for several alkali atomic dilute gases, such as ^{87}Rb [1], ^{23}Na [7], and ^{7}Li [5]. For bosonic atoms, the total spin number F corresponding to the lowest energy state has to be an integer with $2F+1$ hyperfine states ($m_F = -F, -F+1, ..., F-1, F$). They are called spin-$F$ BEC. For the above alkali atoms, $F = 1$.

In early experiments, the atoms are confined by magnetic trap, the spin direction follows the magnetic field and thereby the spin degree of freedom are frozen. The atomic gas is then described by a scalar wave function. Through the mean-field approximation, this wave function satisfies the Gross-Pitaevskii equation (GPE) [6, 11, 21]:

$$i\hbar\partial_t\psi = \frac{\delta H}{\delta\psi^*} = \Big[-\frac{\hbar^2}{2m}\nabla^2 + V(x) + U_0|\psi|^2\Big]\psi. \tag{1.1}$$

Here, $\hbar$ is the Planck constant, m the mass, ψ^* the complex conjugate of ψ, V the trapping potential

$$H(\psi) = \int_{\mathbb{R}^d} dx \Big[\frac{\hbar^2}{2m}|\nabla\psi|^2 + V(x)|\psi|^2 + \frac{U_0}{2}|\psi|^4\Big]$$

the Hamiltonian, and $d = 1, 2, 3$ the underlying space dimensions. The parameter $U_0 = 4\pi^2 a_0/m$ is the effective pairwise interaction energy between atoms, and a_0 is the s-wave scattering length.

Recently, optical dipole trap is used to confine alkali atoms. Unlike the magnetic trap, all hyperfine states are active [22, 23, 18, 3, 10]. The theory for these spinor BEC was developed independently by several researchers [20, 12, 13]. In the case of $F = 1$, the spin-1 BEC are described by a vector wave function $\mathbf{\Psi} = (\psi_1, \psi_0, \psi_{-1})^T$, where each component corresponds to the $m_F = 1, 0, -1$ hyperfine states, respectively. The governing equation is a generalized Gross-Pitaevskii equation (GGPE):

$$i\hbar\partial_t\mathbf{\Psi} = \frac{\delta H}{\delta\mathbf{\Psi}^\dagger}, \tag{1.2}$$

where the Hamiltonian is given by

$$H(\mathbf{\Psi}) = \int_{\mathbb{R}^d} dx \Big[\frac{\hbar^2}{2m} |\nabla \mathbf{\Psi}|^2 + V(x)|\mathbf{\Psi}|^2 + \frac{1}{2} g_n |\mathbf{\Psi}|^4 + \frac{1}{2} g_s \left(\mathbf{\Psi}^\dagger \mathbf{F} \mathbf{\Psi} \right)^2 \Big].$$

Here, the notation $\mathbf{\Psi}^\dagger$ stands for $(\psi_1^*, \psi_0^*, \psi_{-1}^*)^T$, $\mathbf{F}$ is the spin operator

$$\mathbf{F} = F_x \mathbf{e}_x + F_y \mathbf{e}_y + F_z \mathbf{e}_z,$$

$$F_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ F_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The parameters

$$g_n = \frac{4\pi\hbar^2}{m} \frac{a_0 + 2a_2}{3}, \ g_s = \frac{4\pi\hbar^2}{m} \frac{-a_0 + a_2}{3},$$

and a_0, a_2 are respectively the scattering lengths corresponding to the total spin zero channel and two channel. The parameter g_n characterizes the hyperfine-state independent interaction, while the parameter g_s characterizes the spin-exchange interaction.

Due to Feshbach resonance, the s-wave scattering length a_0 can be tuned over a large range by adjusting the applied magnetic field. Therefore, the parameters g_n and g_s can be positive or negative. For $g_n < 0$ (resp. $g_n > 0$), the spin-independent interaction is attractive (resp. repulsive). For $g_s < 0$ (resp. $g_s > 0$), the spin-exchange interaction is ferromagnetic (resp. anti-ferromagnetic).

This generalized GPE in component form is

$$\begin{cases} i\hbar\partial_t \psi_1 = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x) + g_n n \right) \psi_1 + g_s \left(n_1 + n_0 - n_{-1} \right) \psi_1 + g_s \psi_{-1}^* \psi_0^2, \\ i\hbar\partial_t \psi_0 = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x) + g_n n \right) \psi_0 + g_s \left(n_1 + n_{-1} \right) \psi_0 + g_s \psi_1 \psi_{-1} \psi_0^*, \\ i\hbar\partial_t \psi_{-1} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x) + g_n n \right) \psi_{-1} + g_s \left(n_{-1} + n_0 - n_1 \right) \psi_{-1} + g_s \psi_1^* \psi_0^2. \end{cases} \tag{1.3}$$

Here,

$$n_j = |\psi_j|^2, j = -1, 0, 1, \text{ and } n = n_1 + n_0 + n_{-1}.$$

In this paper, we give a first mathematical study on the ground states associated with (1.3). We consider the simplest case when $V(x) \equiv 0$ and all $\psi_i (i = -1, 0, 1)$ are real. We rename ψ_i by u_i. By rescaling, we may assume that $\frac{\hbar^2}{2m} = 1$. The energy functional associated with (1.3) is

$$\begin{aligned} \mathbb{E}(u_1, u_2, u_3) = \tfrac{1}{2} \textstyle\int_{\mathbb{R}^d} \sum_{j=1}^3 |\nabla u_j|^2 + \tfrac{1}{4} g_n \int_{\mathbb{R}^d} \left(\sum_{j=1}^3 |u_j|^2 \right)^2 \\ + \tfrac{1}{4} g_s \textstyle\int_{\mathbb{R}^d} \left[(u_1^2 - u_3^2)^2 + 2u_2^2(u_1^2 + u_3^2) + 4u_1 u_3 u_2^2 \right]. \end{aligned} \tag{1.4}$$

From (1.3), the following two integrals are conserved:

$$\int_{\mathbb{R}^d} (|u_1|^2 + |u_2|^2 + |u_3|^2) = N, \tag{1.5}$$

$$\int_{\mathbb{R}^d}(|u_1|^2-|u_3|^2)=M. \tag{1.6}$$

It is natural to assume that

$$N>0, \quad |M|<N. \tag{1.7}$$

For given real numbers (N,M), we define

$$\mathbb{H}_{N,M}=\left\{u=(u_1,u_2,u_3)\,\middle|\, u_j\in H^1(\mathbb{R}^d), j=1,2,3, \int_{\mathbb{R}^d}\sum_{j=1}^3|u_j|^2=N, \int_{\mathbb{R}^d}(|u_1|^2-|u_3|^2)=M\right\}.$$

We consider the minimization problem

$$E_0=inf\{\mathbb{E}(u)\mid u\in\mathbb{H}_{N,M}\}. \tag{1.8}$$

A solution to (1.8) is called a *ground state* since it has the smallest energy. A ground state (u_1,u_2,u_3) is nontrivial if $u_j\not\equiv 0$, for $j=1,2,3$.

Our main result in this paper is the following

Theorem 1.1. *Let $d=1$ and*

$$g_n<g_s<0. \tag{1.9}$$

Then a nontrivial ground state exists. Moreover, the ground state (u_1,u_2,u_3) is positive and strictly decreasing.

Theorem 1.1 is proved via approximation. Namely, we consider a related minimization problem in a bounded interval $I_k:=[-k,k]$ and then let $k\to+\infty$. More precisely, let us define an energy functional on I_k:

$$\begin{aligned}\mathbb{E}^k(u_1,u_2,u_3)&=\tfrac{1}{2}\int_{I_k}\textstyle\sum_{j=1}^3|\nabla u_j|^2+\tfrac{1}{4}g_n\int_{I_k}\left(\sum_{j=1}^3|u_j|^2\right)^2\\&+\tfrac{1}{4}g_s\int_{I_k}\left[(u_1^2-u_3^2)^2+2u_2^2(u_1^2+u_3^2)+4u_1u_3u_2^2\right].\end{aligned} \tag{1.10}$$

As before, for given real numbers (N,M), we define

$$\mathbb{H}^k_{N,M}=\left\{u=(u_1,u_2,u_3)\,\middle|\, u_j\in H^1_0(I_k), j=1,2,3, \int_{I_k}\sum_{j=1}^3|u_j|^2=N, \int_{I_k}(|u_1|^2-|u_3|^2)=M\right\}.$$

We consider the approximate minimization problem

$$E^k_0=inf\{\mathbb{E}^k(u)\mid u\in\mathbb{H}^k_{N,M}\}. \tag{1.11}$$

It is easy to see that

$$E^k_0\to E_0 \text{ as } k\to+\infty. \tag{1.12}$$

We will prove

Theorem 1.2. *Assume that (1.9) holds. Then the minimization problem (1.11) can be attained by some $\mathbf{u}_k=(u_{1,k},u_{2,k},u_{3,k})$ where $u_{j,k}>0$ and strictly decreasing.*

The assumption (1.9) is almost necessary. In fact we have

Theorem 1.3. *Suppose that $g_n \geq 0, g_s \geq 0$. Then (1.8) can not be achieved and hence there is no ground state.*

Remarks.

(1) Theorem 1.2 says that when $g_n \geq 0$ and $g_s \geq 0$, both spin-independent interaction and spin-exchange interaction are repulsive, and because there is no trapping potential, the atoms can not be confined, there is no nontrivial ground state.

(2) When $g_n < 0$, each hyperfine state is confined to form a spike due to self-attractive interaction. Theorem 1.1 characterize its normalized shape (symmetric about origin and decreasing in r). When $g_s < 0$, the spin-exchange interaction is also attractive, the three hyperfine states overlap to each other with peaks at origin. The condition $g_n < g_s$ should only be a technical condition.

(3) The fact that we can only prove the existence of ground state in the one-dimensional case is a serious restriction due to the Gagliardo-Nirenberg's inequality (3.4). This is also related to the fact that the critical exponent for nonlinear Schrödinger equation is $1 + \frac{4}{N}$ which equals 3 when $N = 2$.

There are two difficulties in proving Theorem 1.1. First, we need to show that all components of the minimizers are strictly positive. Second, in the one-dimensional case, the set of even and strictly decreasing functions in $H^1(\mathbb{R})$ is not compact in $L^2(\mathbb{R})$. To deal with the first difficulty, we have to use the special structure of the energy functional. To overcome the second difficulty, we solve the minimization problem (1.11) first and then show the compactness.

2. Proof of Theorem 1.3

From the definition of $\mathbb{E}(u_1, u_2, u_3)$ we have

$$\begin{aligned}\mathbb{E}(u_1, u_2, u_3) &= \tfrac{1}{2}\int_{\mathbb{R}}(|u_1'|^2 + |u_2'|^2 + |u_3'|^2) + \tfrac{1}{4}g_n\int_{\mathbb{R}}(u_1^2 + u_2^2 + u_3^2)^2 \\ &\quad + \tfrac{1}{4}g_s\int_{\mathbb{R}}\left[2u_2^2(u_1 + u_3)^2 + (u_1^2 - u_3^2)^2\right].\end{aligned} \tag{2.1}$$

If $g_n \geq 0, g_s \geq 0$ we always have $E_0 \geq 0$. We will show that $E_0 = 0$ if $g_n \geq 0,\ g_s \geq 0$. Suppose that (u_1, u_2, u_3) such that $u_j \in H^1(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}}\sum_{j=1}^{3}|u_j|^2 = N, \int_{\mathbb{R}}(|u_1|^2 - |u_3|^2) = M.$$

Set $v_j(x) = \rho^{\frac{1}{2}}u_j(\rho x)$ for $j = 1, 2, 3$. Then for any $\rho > 0$, (v_1, v_2, v_3) also satisfies

$$\int_{\mathbb{R}}\sum_{j=1}^{3}|v_j|^2 = N, \int_{\mathbb{R}}(|v_1|^2 - |v_3|^2) = M.$$

On the other hand, we have

$$\mathbb{E}(v_1,v_2,v_3) = \tfrac{1}{2}\rho^2 \int_{\mathbb{R}}(|u_1'|^2+|u_2'|^2+|u_3'|^2) + \tfrac{1}{4}\rho g_n \int_{\mathbb{R}}(u_1^2+u_2^2+u_3^2)^2 + \tfrac{1}{4}\rho g_s \int_{\mathbb{R}} \left[2u_2^2(u_1+u_3)^2 + (u_1^2-u_3^2)^2\right]. \tag{2.2}$$

Taking $\rho \to 0$ we have $E_0 = 0$. Therefore E_0 can be achieved only by $(0,0,0)$. This proves Theorem 1.3.

A corollary of the above proof is the following

Corollary 2.1. *If $g_n < 0, g_s < 0$ then $E_0 < 0$.*

In fact, this follows directly from (2.2), by taking ρ small enough.

3. Proof of Theorem 1.2

We now consider the minimization problem (1.11) and prove Theorem 1.2.

From the definition of $\mathbb{E}^k(u_1,u_2,u_3)$, we have

$$\begin{aligned}\mathbb{E}^k&(u_1,u_2,u_3)\\ &= \tfrac{1}{2}\int_{I_k}(|u_1'|^2+|u_2'|^2+|u_3'|^2) + \tfrac{1}{4}(g_n+g_s)\int_{I_k}(u_1^4+u_2^4+u_3^4)\\ &\quad + \tfrac{1}{4}(g_n+g_s)\int_{I_k}(2u_2^2(u_1^2+u_3^2)) + \tfrac{1}{4}(g_n-g_s)\int_{I_k} 2u_1^2u_3^2 + \tfrac{1}{4}g_s\int_{I_k}(4u_1u_3u_2^2).\end{aligned} \tag{3.1}$$

Let $\mathbf{u}^\ell = (u_1^\ell, u_2^\ell, u_3^\ell)$ be a minimizing sequence of (1.11). We can always assume that each component u_j^ℓ is non-negative, since it is easy to see that

$$\mathbb{E}^k(|u_1^\ell|,|u_2^\ell|,|u_3^\ell|) \le \mathbb{E}^k(u_1^\ell,u_2^\ell,u_3^\ell) \tag{3.2}$$

and $(|u_1^\ell|,|u_2^\ell|,|u_3^\ell|) \in \mathbb{H}^k_{N,M}$. Hence we can replace $(u_1^\ell,u_2^\ell,u_3^\ell)$ by $(|u_1^\ell|,|u_2^\ell|,|u_3^\ell|)$.

For $v \in H^1(\mathbb{R}), v \ge 0$, let us denote its Schwartz symmetrization by v^*. Then we have (see Theorem 3.2 of Lieb and Loss [14])

$$\begin{cases} \int_{\mathbb{R}} |u_j'|^2 \ge \int_{\mathbb{R}} |(u_j^*)'|^2, \quad j=1,2,3,\\ \int_{\mathbb{R}} u_j^4 = \int_{\mathbb{R}} (u_j^*)^4, \quad j=1,2,3,\\ \int_{\mathbb{R}} u_j^2u_k^2 \le \int_{\mathbb{R}} (u_j^*)^2(u_k^*)^2, \quad j,k=1,2,3,\\ \int_{\mathbb{R}} u_1u_3u_2^2 \le \int_{\mathbb{R}} u_1^*u_3^*(u_2^*)^2 \end{cases}$$

which implies (by (3.1)) that

$$\mathbb{E}^k((u_1^\ell)^*,(u_2^\ell)^*,(u_3^\ell)^*) \le \mathbb{E}^k(u_1^\ell,u_2^\ell,u_3^\ell) \tag{3.3}$$

and $((u_1^\ell)^*,(u_2^\ell)^*,(u_3^\ell)^*) \in \mathbb{H}^k_{N,M}$.

As a consequence, we can now assume that u_j^ℓ are nonnegative, even and non-increasing in I_k.

Next we show that minimizing sequence is uniformly bounded in $H_0^1(I_k)$ for $k \geq 1$. Recall that

$$\mathbb{E}^k(u_1^\ell, u_2^\ell, u_3^\ell) \to E_0^k \ (l \to \infty), \quad \int_{I_k} \big((u_1^\ell)^2 + (u_2^\ell)^2 + (u_3^\ell)^2\big) = N.$$

By Gagliardo-Nirenberg inequality ([25])

$$\int_{\mathbb{R}} |u|^4 \leq C \left(\int_{\mathbb{R}} |u'|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |u|^2\right)^{\frac{3}{2}}, \text{ for all } u \in H^1(\mathbb{R}), \tag{3.4}$$

we have for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$\int_{I_k} |u|^4 \leq C\epsilon \int_{I_k} |u'|^2 + C(\epsilon)\Big(\int_{I_k} |u|^2\Big)^3,$$

$$\begin{aligned}
\int_{I_k} u_k^2 u_j^2 &\leq C\Big(\int_{I_k} u_k^4\Big)^{\frac{1}{2}}\Big(\int_{I_k} u_j^4\Big)^{\frac{1}{2}} \leq \int_{I_k} u_k^4 + \int_{I_k} u_j^4 \\
&\leq C\epsilon\Big(\int_{I_k} |u_k'|^2 + \int_{I_k} |u_j'|^2\Big) + C(\epsilon)\Big(\Big(\int_{I_k} |u_k|^2\Big)^3 + \Big(\int_{I_k} |u_j|^2\Big)^3\Big),
\end{aligned}$$

and

$$\begin{aligned}
\Big|\int_{I_k} u_l^2 u_j u_k\Big| &\leq C\Big(\int_{I_k} u_l^4\Big)^{\frac{1}{2}}\Big(\int_{I_k} u_j^2 u_k^2\Big)^{\frac{1}{2}} \leq \int_{I_k} u_l^4 + \int_{I_k} u_j^2 u_k^2 \\
&\leq C\epsilon\Big(\int_{I_k} |u_l'|^2 + |u_j'|^2 + |u_k'|^2\Big) \\
&\quad + C(\epsilon)\Big(\Big(\int_{I_k} |u_l|^2\Big)^3 + \Big(\int_{I_k} |u_j|^2\Big)^3 + \Big(\int_{I_k} u_k^2\Big)^3\Big).
\end{aligned}$$

Hence, making use of the above inequalities, we get

$$\begin{aligned}
\tfrac{1}{2}\int_{I_k} \textstyle\sum_{j=1}^3 |(u_j^\ell)'|^2 &= E_0 - \tfrac{1}{4} g_n \int_{I_k} ((u_1^\ell)^2 + (u_2^\ell)^2 + (u_3^\ell)^2)^2 \\
&\quad - \tfrac{1}{4} g_s \int_{I_k} \left[2(u_2^\ell)^2(u_1^\ell + u_3^\ell)^2 + ((u_1^\ell)^2 - (u_3^\ell)^2)^2\right] + o_\ell(1) \\
&\leq E_0 - \tfrac{C}{4}(g_n + g_s)\epsilon\Big(\int_{I_k} \left[|(u_1^\ell)'|^2 + |(u_2^\ell)'|^2 + |(u_3^\ell)'|^2\right]\Big) + C(\epsilon)N^3.
\end{aligned}$$

where $o_\ell(1) \to 0$ as $\ell \to +\infty$.

Choosing ϵ sufficiently small, we see

$$\int_{I_k} \left[\sum_{j=1}^3 (|(u_j^\ell)'|^2 + (u_j^\ell)^2)\right] < C \tag{3.5}$$

which implies that by Sobolev embedding

$$|u_j^\ell|_{L^\infty} < C, \ \ j = 1, 2, 3, \ \ l = 1, 2, \cdots. \tag{3.6}$$

(Let us remark that the constant C here is independent of $k \geq 1$.)

Thus, $\{u_j^\ell\}$ can not blow up in $\mathbb{R}$, $j = 1, 2, 3$.

Therefore from $\mathbf{u}^\ell(x)$ we can obtain the existence of minimizer $(u_{1,k}, u_{2,k}, u_{3,k})$ by applying compactness of embedding of subspace of $H_0^1(I_k)$ consisting of even functions into $L^4(I_k)$. We can also assume that $u_{j,k} \geq 0$ for $j = 1, 2, 3$ and at least one component is not identically 0. Furthermore, there are Lagrange multipliers λ^k, μ^k such that $(u_{1,k}, u_{2,k}, u_{3,k})$ satisfies the Euler-Lagrange equation

$$\begin{cases} u_1'' - (g_n + g_s)u_1^3 - [(g_n + g_s)u_2^2 + (g_n - g_s)u_3^2]u_1 - g_s u_2^2 u_3 = (\lambda^k + \mu^k)u_1, & \text{in } I_k \\ u_2'' - g_n u_2^3 - [(g_n + g_s)u_1^2 + (g_n + g_s)u_3^2]u_2 - 2g_s u_1 u_2 u_3 = \lambda^k u_2, & \text{in } I_k \\ u_3'' - (g_n + g_s)u_3^3 - [(g_n + g_s)u_2^2 + (g_n - g_s)u_1^2]u_3 - g_s u_1 u_2^2 = (\lambda^k - \mu^k)u_3, & \text{in } I_k \\ u_j > 0 \text{ in } (-k, k),\ u_j(\pm k) = 0, j = 1, 2, 3. \end{cases} \tag{3.7}$$

Suppose that $(u_{1,k}, u_{2,k}, u_{3,k})$ is a minimizer of (1.11). It remains to show that $u_{j,k} \not\equiv 0$, for $j = 1, 2, 3$. This will be done by two claims.

Claim 1: $u_{2,k} > 0$.

We argue by contradiction. Suppose that $u_{2,k} \geq 0$ and $u_{2,k}(x_0) = 0$. Then by the Maximum Principle, $u_{2,k} \equiv 0$, and hence $(u_{1,k}, u_{3,k})$ is a solution of

$$\begin{cases} u_1'' - (g_n + g_s)u_1^3 - (g_n - g_s)u_3^2 u_1 = (\lambda^k + \mu^k)u_1, & \text{in } I_k, \\ u_3'' - (g_n + g_s)u_3^3 - (g_n - g_s)u_1^2 u_3 = (\lambda^k - \mu^k)u_3, & \text{in } I_k \end{cases} \tag{3.8}$$

satisfying the constraint

$$\int_{I_k} u_{1,k}^2 = \frac{N + M}{2}, \quad \int_{I_k} u_{3,k}^2 = \frac{N - M}{2}. \tag{3.9}$$

Set $(u_1, u_2, u_3) = (u_{1,k} + \varepsilon_1\varphi_1, \varepsilon_2\varphi_2, u_{3,k} + \varepsilon_3\varphi_3)$, with $\varepsilon_j > 0$, $\varphi_j \in H_0^1(I_k)$ for $j = 1, 2, 3$ such that

$$\int_{I_k} (|u_{1,k} + \varepsilon_1\varphi_1|^2 + |\varepsilon_2\varphi_2|^2 + |u_{3,k} + \varepsilon_3\varphi_3|^2) = N,$$

$$\int_{I_k} (|u_{1,k} + \varepsilon_1\varphi_1|^2 - |u_{3,k} + \varepsilon_3\varphi_3|^2) = M.$$

This can be done by choosing $\varepsilon_j > 0, \varphi_j \in H_0^1(I_k)(j = 1, 2, 3)$ such that $\int_{I_k} u_{1,k}\varphi_1 < 0, \int_{I_k} u_{3,k}\varphi_3 < 0$ and

$$2\varepsilon_1 \int_{I_k} u_{1,k}\varphi_1 + \varepsilon_1^2 \int_{I_k} |\varphi_1|^2 + 2\varepsilon_3 \int_{I_k} u_{3,k}\varphi_3 + \varepsilon_3^2 \int_{I_k} |\varphi_3|^2 + \varepsilon_2^2 \int_{I_k} |\varphi_2|^2 = 0, \tag{3.10}$$

$$2\varepsilon_1 \int_{I_k} u_{1,k}\varphi_1 + \varepsilon_1^2 \int_{I_k} |\varphi_1|^2 = 2\varepsilon_3 \int_{I_k} u_{3,k}\varphi_3 + \varepsilon_3^2 \int_{I_k} |\varphi_3|^2. \tag{3.11}$$

From (3.10) and (3.11)we obtain

$$\varepsilon_1 \int_{I_k} u_{1,k}\varphi_1 = \varepsilon_3 \int_{I_k} u_{3,k}\varphi_3 + O(\varepsilon_1^2+\varepsilon_3^2),$$

$$\varepsilon_1 \int_{I_k} u_{1,k}\varphi_1 + \varepsilon_3 \int_{I_k} u_{3,k}\varphi_3 = -\frac{1}{2}\varepsilon_2^2 \int_{I_k} |\varphi_2|^2 + O(\varepsilon_1^2+\varepsilon_3^2).$$

Applying the above equality to the expression in $\mathbb{E}^k(u_1,u_2,u_3)$, we obtain

$$\begin{aligned}
&\mathbb{E}^k(u_1,u_2,u_3)\\
&\quad = \varepsilon_1 \textstyle\int_{I_k} ((u_{1,k})'\varphi_1' + (g_n+g_s)(u_{1,k})^3\varphi_1 + (g_n-g_s)(u_{3,k})^2 u_{1,k}\varphi_1)\\
&\qquad + \varepsilon_3 \textstyle\int_{I_k} ((u_{3,k})'\varphi_3' + (g_n+g_s)(u_{3,k})^3\varphi_3 + (g_n-g_s)(u_{1,k})^2 u_{3,k}\varphi_3)\\
&\qquad + \varepsilon_2^2 \left[\tfrac{1}{2}\textstyle\int_{I_k} (|\varphi_2'|^2 + (g_n+g_s)((u_{1,k})^2+(u_{3,k})^2)\varphi_2^2) + g_s \int_{I_k} u_{1,k}u_{3,k}|\varphi_2|^2\right]\\
&\qquad + \mathbb{E}^k(u_{1,k},0,u_{3,k}) + o(\varepsilon_1^2+\varepsilon_3^2)\\
&\quad = -\varepsilon_1(\lambda^k+\mu^k)\textstyle\int_{I_k} u_{1,k}\varphi_1 - \varepsilon_3(\lambda^k-\mu^k)\int_{I_k} u_{3,k}\varphi_3\\
&\qquad + \varepsilon_2^2 \left[\tfrac{1}{2}\textstyle\int_{I_k} (|\varphi_2'|^2 + (g_n+g_s)((u_{1,k})^2+(u_{3,k})^2)\varphi_2^2) + g_s \int_{I_k} u_{1,k}u_{3,k}|\varphi_2|^2\right]\\
&\qquad + \mathbb{E}^k(u_{1,k},0,u_{3,k}) + o(\varepsilon_1^2+\varepsilon_3^2)\\
&\quad = \tfrac{1}{2}\varepsilon_2^2 \textstyle\int_{I_k} \left[|\varphi_2'|^2 + \lambda^k|\varphi_2|^2 + (g_n+g_s)((u_{1,k})^2+(u_{3,k})^2)\varphi_2^2 + 2g_s u_{1,k}u_{3,k}|\varphi_2|^2\right]\\
&\qquad + \mathbb{E}^k(u_{1,k},0,u_{3,k}) + o(\varepsilon_1^2+\varepsilon_3^2).
\end{aligned} \tag{3.12}$$

Let $\eta = \left(\frac{N+M}{N-M}\right)^{\frac{1}{2}}$, $\varphi_2 = u_{1,k}+\eta u_{3,k}$. We claim that if $g_s<0$ then

$$\int_{I_k} \left[|\varphi_2'|^2 + \lambda^k\varphi_2^2 + (g_n+g_s)((u_{1,k})^2+(u_{3,k})^2)\varphi_2^2\right] + 2g_s\int_{I_k} u_{1,k}u_{3,k}\varphi_2^2 < 0. \tag{3.13}$$

Indeed, from (3.8) and the choice of η we have

$$\begin{aligned}
&\textstyle\int_{I_k} (|(u_{1,k})'|^2 + \eta^2|(u_{3,k})'|^2 + (g_n+g_s)((u_{1,k})^4+\eta^2(u_{3,k})^4) + 2(g_n-g_s)(1+\eta^2)(u_{1,k})^2(u_{3,k})^2)\\
&\qquad + \lambda^k \textstyle\int_{I_k}((u_{1,k})^2+\eta^2(u_{3,k})^2) = 0,
\end{aligned} \tag{3.14}$$

$$\int_{I_k} \left[(u_{1,k})'(u_{3,k})' + \lambda^k u_{1,k}u_{3,k} + g_n((u_{1,k})^3u_{3,k} + (u_{3,k})^3u_{3,k})\right] = 0. \tag{3.15}$$

Using (3.14),(3.15) we have

$$\begin{aligned}
&\textstyle\int_{I_k} \left[|(u_{1,k}+\eta u_{3,k})'|^2 + \lambda^k(u_{1,k}+\eta u_{3,k})^2 + (g_n+g_s)((u_{1,k})^4+\eta^2(u_{3,k})^4)\right]\\
&\qquad + 2\eta g_n \textstyle\int_{I_k}((u_{1,k})^3u_{3,k} + (u_{3,k})^3u_{1,k}) + (g_n-g_s)(1+\eta^2)\int_{I_k}(u_{1,k})^2(u_{3,k})^2 = 0.
\end{aligned} \tag{3.16}$$

So, since $g_s < 0$, we get

$$\begin{aligned}&\textstyle\int_{I_k} \left[|(u_{1,k}+\eta u_{3,k})'|^2 + \lambda^k (u_{1,k}+\eta u_{3,k})^2 + (g_n+g_s)((u_{1,k})^2+(u_{3,k})^2)(u_{1,k}+\eta u_{3,k})^2\right] \\ &\qquad +2g_s \textstyle\int_{I_k} u_{1,k}u_{3,k}(u_{1,k}+\eta u_{3,k})^2 \\ &\quad = 2(1+\eta)g_s \textstyle\int_{I_k} u_{1,k}u_{3,k}\left((1+\eta)u_{1,k}u_{3,k} + (u_{1,k})^2 + \eta (u_{3,k})^2\right) < 0,\end{aligned} \tag{3.17}$$

which is a contradiction since in this case we can choose an element $\mathbf{u} \in \mathbb{H}^k_{N,M}$ such that $\mathbb{E}^k(\mathbf{u}) < E_0$.

Therefore $u_{2,k}$ is a positive function.

Claim 2: $u_{1,k} > 0, u_{3,k} > 0$.

Suppose that $u_{1,k} \geq 0$ and $u_{1,k}(x_0) = 0$. By the Maximum Principle, $u_{1,k}(x) \equiv 0$. By the equation satisfied by $u_{1,k}$ we obtain $g_s(u_{2,k})^2 u_{3,k} \equiv 0$. Therefore either $u_{2,k} \equiv 0$ or $u_{3,k} \equiv 0$. By Claim 1, $u_{2,k} > 0$. Therefore the only possibility is that $u_{3,k} \equiv 0$. In this case, $M = 0$ and $\int_{I_k} (u_{2,k})^2 = N$ and $u_{2,k}$ is a positive solution of

$$u'' - g_n u^3 = \lambda^k u \text{ in } I_k. \tag{3.18}$$

We will assume $M = 0$ from now on. Let $(u_1, u_2, u_3) = (\varepsilon_1\varphi_1, u_{2,k} + \varepsilon_2\varphi_2, \varepsilon_3\varphi_3)$. To ensure that (u_1, u_2, u_3) satisfies the constraint, we choose $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0$ such that

$$\varepsilon_1^2 \int_{I_k} |\varphi_1|^2 = \varepsilon_3^2 \int_{I_k} |\varphi_3|^2, \tag{3.19}$$

$$\varepsilon_1^2 \int_{I_k} |\varphi_1|^2 + \varepsilon_3^2 \int_{I_k} |\varphi_3|^2 + \varepsilon_2^2 \int_{I_k} |\varphi_2|^2 + 2\varepsilon_2 \int_{I_k} u_{2,k}\varphi_2 = 0. \tag{3.20}$$

Since $u_{2,k}$ satisfies (3.18) we have

$$\int_{I_k} \left((u_{2,k})'\varphi_2' + \lambda^k u_{2,k}\varphi_2 + g_n (u_{2,k})^3 \varphi_2\right) = 0. \tag{3.21}$$

From (3.19),(3.20) and (3.21) we have

$$\varepsilon_2 \int_{I_k} \left((u_{2,k})'\varphi_2' + g_n(u_{2,k})^3\varphi_2\right) = \frac{1}{2}\lambda^k \left(\varepsilon_1^2 \int_{I_k} |\varphi_1|^2 + \varepsilon_2^2 \int_{I_k} |\varphi_2|^2 + \varepsilon_3^2 \int_{I_k} |\varphi_3|^2\right). \tag{3.22}$$

Using (3.22) we get

$$\begin{aligned}
\mathbb{E}^k(u_1,u_2,u_3) &= \mathbb{E}^k(\varepsilon_1\varphi_1, u_{2,k}+\varepsilon_2\varphi_2, \varepsilon_3\varphi_3)\\
&= \tfrac{1}{2}\varepsilon_1^2\int_{I_k}|\varphi_1'|^2 + \tfrac{1}{4}(g_n+g_s)\varepsilon_1^4\int_{I_k}|\varphi_1|^4\\
&\quad +\tfrac{1}{2}\varepsilon_3^2\int_{I_k}(|\varphi_3'|^2 + \tfrac{1}{4}(g_n+g_s)\varepsilon_3^4\int_{I_k}|\varphi_3|^4\\
&\quad +\tfrac{1}{2}\int_{I_k}|(u_{2,k}+\varepsilon_2\varphi_2)'|^2 + \tfrac{1}{4}g_n\int_{I_k}|u_{2,k}+\varepsilon_2\varphi_2|^4\\
&\quad +\tfrac{1}{2}(g_n+g_s)\int_{I_k}[(\varepsilon_1^2\varphi_1^2+\varepsilon_3^2\varphi_3^2)(u_{2,k}+\varepsilon_2\varphi_2)^2]\\
&\quad +\tfrac{1}{2}(g_n-g_s)\int_{I_k}\varepsilon_1^2\varepsilon_3^2\varphi_1^2\varphi_3^2 + g_s\int_R \varepsilon_1\varepsilon_3\varphi_1\varphi_3(u_{2,k}+\varepsilon_2\varphi_2)^2\\
&= \varepsilon_1^2\left(\tfrac{1}{2}\int_{I_k}|\varphi_1'|^2 + \tfrac{1}{2}(g_n+g_s)\int_R|u_{2,k}\varphi_1|^2\right) + \tfrac{1}{4}\varepsilon_1^4(g_n+g_s)\int_{I_k}\varphi_1^4\\
&\quad +\varepsilon_3^2\left(\tfrac{1}{2}\int_{I_k}|\varphi_3'|^2 + \tfrac{1}{2}(g_n+g_s)\int_{I_k}|u_{2,k}\varphi_3|^2\right) + \tfrac{1}{4}\varepsilon_3^4(g_n+g_s)\int_{I_k}|\varphi_3|^4\\
&\quad +g_s\varepsilon_1\varepsilon_3\int_{I_k}|u_{2,k}+\varepsilon_2\varphi_2|^2\varphi_1\varphi_3 \qquad (3.23)\\
&\quad +\varepsilon_2\left[\int_{I_k}(u_{2,k})'\varphi_2' + g_n\int_{I_k}(u_{2,k})^3\varphi_2\right] + \varepsilon_2^2\left[\tfrac{1}{2}\int_{I_k}(|\varphi_2'|^2 + \tfrac{3}{2}g_n(u_{2,k})^2\varphi_2^2\right]\\
&\quad +\mathbb{E}^k(0,u_{2,k},0) + o(\varepsilon_1^2+\varepsilon_3^2)\\
&= \varepsilon_1^2\left(\tfrac{1}{2}\int_{I_k}(|\varphi_1'|^2+\lambda^k|\varphi_1|^2) + \tfrac{1}{2}(g_n+g_s)\int_{I_k}|u_{2,k}\varphi_1|^2\right)\\
&\quad +\varepsilon_3^2\left(\tfrac{1}{2}\int_{I_k}(|\varphi_3'|^2+\lambda^k|\varphi_2|^2) + \tfrac{1}{2}(g_n+g_s)\int_{I_k}|u_{2,k}\varphi_3|^2\right)\\
&\quad +g_s\varepsilon_1\varepsilon_3\int_{I_k}|u_{2,k}|^2\varphi_1\varphi_3\\
&\quad +\tfrac{1}{2}\varepsilon_2^2\left[\int_{I_k}\left(|\varphi_2'|^2+\lambda^k|\varphi_2|^2\right) + 3g_n(u_{2,k})^2\varphi_2^2\right]\\
&\quad +\mathbb{E}^k(0,u_{2,k},0) + o(\varepsilon_1^2+\varepsilon_3^2).
\end{aligned}$$

Let $\varepsilon_1=\varepsilon_3=\varepsilon, \varphi_1=\varphi_3=\varphi=u_{2,k}$. Then we obtain

$$\left\{\begin{array}{l}
\varepsilon_1^2\left(\frac{1}{2}\int_{I_k}(|\varphi_1'|^2+\lambda^k|\varphi_1|^2) + \frac{1}{2}(g_n+g_s)\int_{I_k}|u_{2,k}\varphi_1|^2\right)\\
+\varepsilon_3^2\left(\frac{1}{2}\int_{I_k}(|\varphi_3'|^2+\lambda^k|\varphi_3|^2) + \frac{1}{2}(g_n+g_s)\int_{I_k}|u_{2,k}\varphi_3|^2\right) + g_s\varepsilon_1\varepsilon_3\int_{I_k}|u_{2,k}|^2\varphi_1\varphi_3\\
=\varepsilon^2\left(\int_{I_k}(|\varphi'|^2+\lambda^k|\varphi|^2) + (g_n+2g_s)\int_{I_k}|u_{2,k}\varphi|^2\right) < 0.
\end{array}\right. \qquad (3.24)$$

On the other hand, if we take $\varphi_2 = u_{2,k}$, then

$$\int_{I_k}\left(|\varphi_2'|^2+\lambda^k|\varphi_2|^2 + 3g_n(u_{2,k})^2\varphi_2^2\right) < 0. \qquad (3.25)$$

From (3.23), (3.24) and (3.25) we have $\varepsilon_1>0, \varepsilon_2>0$ such that

$$\mathbb{E}^k(\varepsilon_1 u_{2,k}, (1+\varepsilon_2)u_{2,k}, \varepsilon_1 u_{2,k}) < \mathbb{E}^k(0,u_{2,k},0),$$

which is a contradiction.

Therefore, $u_{1,k} \not\equiv 0$. Similarly we can show that $u_{3,k} \not\equiv 0$. Thus we have completed the proof of Theorem 1.2.

4. Proof of Theorem 1.1

From Section 3, for each $k \geq 1$, we obtain a minimizer to the minimization problem (1.11) which satisfies the following Euler-Lagrange equation problem is

$$\begin{cases} u''_{1,k} - (g_n + g_s)u^3_{1,k} - [(g_n + g_s)u^2_{2,k} + (g_n - g_s)u^2_{3,k}]u_{1,k} - g_s u^2_{2,k}u_{3,k} = (\lambda^k + \mu^k)u_{1,k} & \text{in } (-k,k), \\ u''_{2,k} - g_n u^3_{2,k} - [(g_n + g_s)u^2_{1,k} + (g_n + g_s)u^2_{3,k}]u_{2,k} - 2g_s u_{1,k}u_{2,k}u_{3,k} = \lambda^k u_{2,k} & \text{in } (-k,k), \\ u''_{3,k} - (g_n + g_s)u^3_{3,k} - [(g_n + g_s)u^2_{2,k} + (g_n - g_s)u^2_{1,k}]u_{3,k} - g_s u_{1,k}u^2_{2,k} = (\lambda^k - \mu^k)u_{3,k} & \text{in } (-k,k), \\ u_{j,k}(\pm k) = 0, u_{j,k}(x) > 0 \text{ for } x \in (-k,k), \ \ j = 1,2,3, \end{cases} \tag{4.1}$$

where λ^k, μ^k are two Lagrange multipliers.

Let us collect some properties of $u_{j,k}$: we have $u_{j,k}(x) > 0$ in $(-k,k), u_{j,k}$ is even and decreasing. For $k \geq k_0, E^k_0 \leq -c_0 < 0$ by Corollary 2.1, and $\|u_{j,k}\|_{H^1(I_k)} \leq C$ for some C (independent of $k \geq k_0$).

Thus we can take a subsequence of $k \to +\infty$ such that $u_{j,k} \to u_j$ uniformly in $\mathbb{R}$ where $u_j \in H^1(\mathbb{R}), u_j \geq 0$ and u_j is decreasing. Note that we can conclude that $u_{j,k} \to u_j$ in $L^p(\mathbb{R})$ for $p > 2$ but not for $p = 2$.

Since $u_j \in H^1(\mathbb{R})$ and u_j is decreasing, we see that $u_j(x) \to 0$ as $|x| \to +\infty$. Thus, for any $\delta > 0$, we can find $R = R(\delta) > 0$ such that for $|x| \geq R$, $u_j(x) < \delta/2$. As a consequence of decreasing property of $u_{j,k}$, we can find k_0 such that

$$u_{j,k}(x) \leq u_{j,k}(R) < \delta, \text{ for } |x| > R, \ k \geq k_0. \tag{4.2}$$

The main difficulty is to show that $u_{j,k} \to u_j$ strongly in $L^2(\mathbb{R})$. If so, then (u_1, u_2, u_3) satisfies the constraint (1.6)-(1.7) and is a minimizer of the minimization problem (1.8). By the same arguments as those of Claim 1 and Claim 2, we can show that $u_j > 0$. So it remains to prove the strong convergence in $L^2(\mathbb{R})$. We proceed in a few claims.

Claim 3 $\lim_{k\to+\infty}(\lambda^k - \mu^k) \geq 0, \ \lim_{k\to+\infty} \lambda^k \geq 0, \ \lim_{k\to+\infty}(\lambda^k + \mu^k) \geq 0.$

In fact, suppose $\lim_{k\to+\infty}(\lambda^k + \mu^k) < -c_0 < 0$. Then from the equation for $u_{1,k}$, we see that $u_{1,k}$ satisfies

$$u''_{1,k} + \frac{c_0}{4}u_{1,k} \leq 0, \ u_{1,k}(x) > 0 \text{ in } (-k,k). \tag{4.3}$$

But for k large, by Sturm-Liouville Comparison theorem, $u_{1,k}$ must change signs in $(-\sqrt{c_0}\pi, \sqrt{c_0}\pi)$, which is a contradiction to the fact that $u_{1,k}(x) > 0$ in $(-k,k)$.

The other cases can be dealt with similarly.

Claim 4. There exists a positive constant $c_0 > 0$ such that

$$\lambda^k N + \mu^k M \geq c_0 > 0 \tag{4.4}$$

In fact, by integrating by parts, we obtain that

$$\lambda^k \int_{I_k} (\sum_{i=1}^{3} u_{i,k}^2) + \mu^k \int_{I_k} (u_{1,k}^2 - u_{3,k}^2) \geq -4\mathbb{E}^k(u_{1,k}, u_{2,k}, u_{3,k}) \geq c_0 > 0 \tag{4.5}$$

for k large.

Claim 5. There exists $c_0 > 0$ such that $\lambda^k \geq c_0 > 0$ and $\int_{I_k} u_{2,k}^2 \to \int_{I_k} u_2^2$ as $k \to +\infty$.

From Claim 3, we deduce that $\lim_{k\to+\infty}(\lambda^k - |\mu^k|) \geq 0$. By Claim 4, we have $\lambda^k \geq c_0 > 0$. From the equation for $u_{2,k}$ and using (4.2), we have that $u_{2,k}$ satisfies

$$u_{2,k}'' - \frac{c_0}{4} u_{2,k} \geq 0, \text{ for } |x| > R, k \geq k_0 \tag{4.6}$$

where R is a fixed large number. By comparison principle, we deduce that

$$u_{2,k}(x) \leq u_{2,k}(R) e^{-\frac{\sqrt{c_0}}{4}|x-R|} \leq C e^{-\frac{\sqrt{c_0}}{4}|x|}. \tag{4.7}$$

Note that R only depends on c_0. Thus we conclude that $u_{2,k}$ has exponential decay. So $\int_{I_k} u_{2,k}^2 \to \int_{\mathbb{R}} u_2^2$.

Since $\lambda^k \geq c_0 > 0$, we see that either $\lambda^k + \mu^k \geq c_0/2$ or $\lambda^k - \mu^k \geq c_0/2$. Let us assume that $\lambda^k + \mu^k \geq c_0/2$. Then by the same proof as Claim 5, we have

Claim 6. Assuming that $\lambda^k + \mu^k \geq \frac{c_0}{2} > 0$, then $\int_{I_k} u_{1,k}^2 \to \int_{\mathbb{R}} u_1^2$ as $k \to +\infty$.

Now it remains to show that $\int_{I_k} u_{3,k}^2 \to \int_{\mathbb{R}} u_3^2$. Suppose not. Then we can assume that

$$\int_{I_k} u_{3,k}^2 \geq c_0 > 0, \quad \lim_{k\to+\infty} (\lambda^k - \mu^k) = 0. \tag{4.8}$$

Claim 7. $u_1 u_2^2 \equiv 0$ and $u_3 \equiv 0$.

In fact, multiplying the equation for $u_{3,k}$ by $(1-x^2)^2$ and integrating over $(-1,1)$, we see that

$$\int_{-1}^{1} u_{1,k} u_{2,k}^2 (1-x^2)^2 = o(1)$$

which shows that $u_1 u_2^2 \equiv 0$.

Using (4.8), we see that the limit u_3 satisfies

$$u_3'' - (g_n + g_s) u_3^3 - [(g_n + g_s) u_2^2 + (g_n - g_s) u_1^2] u_3 = 0 \text{ in } \mathbb{R}. \tag{4.9}$$

Thus $u_3 \equiv 0$. So $u_{3,k} \to 0$ in $C^0(\mathbb{R})$.

If both $u_1 \equiv 0, u_2 \equiv 0$, we then derive that $N = -M$ (since $u_{1,k} \to u_1, u_{2,k} \to u_2$ strongly in $L^2(\mathbb{R})$), which is impossible.

There are two cases to be considered.

Case 1: $u_2 > 0, u_1 \equiv 0$.

In this case, using (4.2), the equation for $u_{2,k}$ and Claim 7, we see that $u_{2,k}$ satisfies

$$u_{2,k}'' - g_n u_{2,k}^3 \geq \frac{c_0}{2} u_{2,k}, \ u_{2,k} > 0 \text{ in } (-k,k), \ u_{2,k}(\pm k) = 0. \tag{4.10}$$

Using the equation for $u_{3,k}$ and (4.8), we see that $u_{3,k}$ satisfies

$$u''_{3,k} - g_n u^2_{2,k} u_{3,k} \le \frac{c_0}{4} u_{3,k},\ \ u_{3,k} > 0 \text{ in } (-k,k),\ \ u_{3,k}(\pm k) = 0. \tag{4.11}$$

Multiplying (4.10) by $u_{3,k}$ and (4.11) by $u_{2,k}$ and then integrating over $(-k, k)$, we obtain a contradiction.

Case 2: $u_1 > 0, u_2 \equiv 0$.

In this case, we observe that u_1 satisfies

$$u''_1 - (g_n + g_s)u^3_1 = 2\lambda^0 u_1 \text{ in } \mathbb{R},\ \ u_1 \in H^1(\mathbb{R}) \tag{4.12}$$

where $\lim_{k\to+\infty} \lambda^k = \lim_{k\to+\infty} \mu^k = \lambda^0 > 0$.

On the other hand, $\frac{u_{2,k}(x)}{u_{2,k}(0)} \to \hat{u}_2(x)$ which satisfies

$$\hat{u}''_2 - (g_n + g_s)u^2_1 \hat{u}_2 = \lambda^0 \hat{u}_2. \tag{4.13}$$

It is easy to see that $0 < \hat{u}_2 \le 1$ since $\hat{u}_2(0) = 1$. From (4.12) and (4.13), we obtain then

$$\lambda^0 \int_{\mathbb{R}} u_1 \hat{u}_2 = 0 \tag{4.14}$$

which is impossible.

In conclusion, we have proved that as $k \to +\infty$, $\int_{I_k} u^2_{j,k} \to \int_{\mathbb{R}} u^2_j$ for $j = 1, 2, 3$.

This completes the proof of Theorem 1.1.

□

5. Applications to 2×2 BEC system

Our proof of Theorem 1.1 can also be applied to obtain ground states for the two-component Bose-Einstein system. Such a system appears in a mixture of BEC of different hyperfine states confined under different kinds of traps [19]. A theoretical study was proposed by the following coupled nonlinear Schrödinger equation [8].

Let

$$\mathbb{E}_\beta(u, v) = \frac{1}{2}\int_{\mathbb{R}} (|u'|^2 + |v'|^2) - \frac{1}{4}\int_{\mathbb{R}} (\mu_1|u|^4 + \mu_2|v|^4 + 2\beta u^2 v^2), \tag{5.1}$$

where $\beta > 0$.

Let $N > 0$. Consider the constrained minimization problem

$$inf\{\, E_\beta(u, v) \mid \int_{\mathbb{R}} (|u|^2 + |v|^2) = N \,\}. \tag{5.2}$$

A nontrivial solution to (5.2) is called a *gound state* which satisfies the following Euler-Lagrange equation

$$\begin{cases} u'' + \mu_1 u^3 + \beta u v^2 = \lambda u, \\ v'' + \mu_2 v^3 + \beta u^2 v = \lambda v, \\ u, v \in H^1(\mathbb{R}). \end{cases} \tag{5.3}$$

Then we have

Theorem 5.1. *If $\beta > max\{\mu_1, \mu_2\}$, then (5.2) can be achieved by (u_0, v_0) such that $u_0 > 0, v_0 > 0$. Furthermore, we have*

$$u_0 = \sqrt{\frac{\beta - \mu_2}{\beta^2 - \mu_1\mu_2}}w, \;\; v_0 = \sqrt{\frac{\beta - \mu_1}{\beta^2 - \mu_1\mu_2}}w, \tag{5.4}$$

where w is the unique homoclinic solution of

$$w'' - \lambda w + w^3 = 0. \tag{5.5}$$

On the other hand, if we consider the following minimization problem

$$inf\{\, E_\beta(u, v) \mid \int_{\mathbb{R}} |u|^2 = N_1, \int_{\mathbb{R}} |v|^2 = N_2 \,\}, \tag{5.6}$$

where $N_1 > 0, N_2 > 0$, then we have

Theorem 5.2. *If $\beta > 0$ then (5.6) can be achieved by (u_0, v_0) such that $u_0 > 0, v_0 > 0$ and satisfy*

$$\begin{cases} u'' + \mu_1 u^3 + \beta u v^2 = \lambda_1 u \quad in \;\; \mathbb{R}, \\ v'' + \mu_2 v^3 + \beta u^2 v = \lambda_2 v \quad in \;\; \mathbb{R}, \\ u, v \in H^1(\mathbb{R}) \end{cases} \tag{5.7}$$

for some $\lambda_1 > 0, \lambda_2 > 0$.

We remark that the minimzation problems (5.2),(5.6) arise naturally in the study of standing waves of the coupled Gross-Pitaevskii equations, i.e., the coupled nonlinear Schrödinger equations,

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1, \\ -i\frac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2, \\ \Phi_j = \Phi_j(y, t) \in \mathbb{C}, \quad j = \; 1, 2, \end{cases} \tag{5.8}$$

where μ_1, μ_2 are positive constants, and β is a coupling constant.

For any solutions to (5.8), the following two integrals are conserved

$$\int_{\mathbb{R}^d} |\Phi_1(x, t)|^2 = \int_{\mathbb{R}^d} |\Phi(x, 0)|^2, \; \int_{\mathbb{R}^d} |\Phi_2(x, t)|^2 = \int_{\mathbb{R}^d} |\Phi_2(x, 0)|^2. \tag{5.9}$$

Thus it is natural to consider problems (5.2)-(5.6).

System (5.8) arises in the Hartree-Fock theory for a double condensate, i.e. a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ ([8]).

To obtain solitary wave solutions of the system (5.8), we set $\Phi_1(x, t) = e^{i\lambda_j t}u(x)$, $\Phi_2(x, t) = e^{i\lambda_j t}v(x)$, and the system (5.8) is transformed to an elliptic system given by

$$\begin{cases} \Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 \quad \text{in} \;\; \mathbb{R}^d, \\ \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \quad \text{in} \;\; \mathbb{R}^d, \\ u, v \in H^1(\mathbb{R}^d) \end{cases} \tag{5.10}$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ are positive constants and $\beta \neq 0$ is a coupling constant.

The existence of least energy solution to (5.10) is studied in [2, 15, 24]. In [2, 4, 17, 16, 24], the existence of bound states (i.e., solutions to (5.10)) when $\beta > 0$ is proved. Note that the minimization problems (5.2) and (5.6) are different from the minimization problems in [2] and [15].

Proof of Theorem 5.1: Following the proof of Theorem 1.1, and using Schwartz symmetrization, we can show the existence of minimizer (u_0, v_0) such that $u_0 \in H^1(\mathbb{R}), v_0 \in H^1(\mathbb{R})$, $u_0 \geq 0, v_0 \geq 0$ and satisfies (5.3). (Here as in the proof of Theorem 1.1, we have to first work on a bounded domain $(-k, k)$ for k large. Then we show the L^2-strong convergence of the sequence.)

The difficulty is to show that $u_0 \not\equiv 0, v_0 \not\equiv 0$.

We argue by way of contradiction. Suppose that $v_0 \equiv 0$. Then $u_0 \not\equiv 0$ and u_0 is a positive solution of

$$u'' + \mu_1 u^3 + \beta v^2 u = \lambda u \quad \text{in } \mathbb{R}.$$

Set $u = u_0 + \varepsilon_1 \varphi_1, v = \varepsilon_2 \varphi_2,$, with $\varepsilon_1 > 0, \varepsilon_2 > 0, \varphi_1 \in H^1(\mathbb{R}), \varphi_2 \in H^1(\mathbb{R})$ such that

$$\int_{\mathbb{R}} (|u|^2 + |v|^2) = \int_{\mathbb{R}} |u|^2 = N,$$

namely

$$2\varepsilon_1 \int_{\mathbb{R}} u_0 \varphi_1 + \varepsilon_1^2 \int_{\mathbb{R}} |\varphi_1|^2 + \varepsilon_2^2 \int_{\mathbb{R}} |\varphi_2|^2 = 0. \tag{5.11}$$

We can choose φ_1 such that $\int_{\mathbb{R}} u_0 \varphi_1 < 0$, then for fixed $\varepsilon_2 > 0$ small, $\varphi_2 \not\equiv 0$ there exists $\varepsilon_1 > 0$ so that (5.11) holds and $\varepsilon_1 = O(\varepsilon_2^2)$.

Using (5.11) and $\varepsilon_1 = O(\varepsilon_2^2)$ we have

$$\begin{aligned}
\mathbb{E}_\beta(u, v) &= \tfrac{1}{2} \textstyle\int_{\mathbb{R}} (|u_0' + \varepsilon_1 \varphi_1'|^2 + \varepsilon_2^2 |\varphi_2'|^2) \\
&\quad - \tfrac{1}{4} \textstyle\int_{\mathbb{R}} (\mu_1 |u_0 + \varepsilon_1 \varphi_1|^4 + \mu_2 \varepsilon_2^4 |\varphi_2|^4 + 2\beta \varepsilon_2^2 (u_0 + \varepsilon_1 \varphi_1)^2 \varphi_2^2) \\
&= \mathbb{E}_\beta(u_0, 0) + \varepsilon_1 \textstyle\int_{\mathbb{R}} (u_0' \varphi_1' - \mu_1 u_0^3 \varphi_1) + O(\varepsilon_1^2) \\
&\quad + \varepsilon_2^2 \left(\tfrac{1}{2} \textstyle\int_{\mathbb{R}} |\varphi_2'|^2 - \tfrac{\beta}{2} \textstyle\int_{\mathbb{R}} u_0^2 \varphi_2^2 \right) + O(\varepsilon_2^3) \\
&= \mathbb{E}_\beta(u_0, 0) - \varepsilon_1 \lambda \textstyle\int_{\mathbb{R}} u_0 \varphi_1 + O(\varepsilon_1^2) + \varepsilon_2^2 \left(\tfrac{1}{2} \textstyle\int_{\mathbb{R}} |\varphi_2'|^2 - \tfrac{\beta}{2} \textstyle\int_{\mathbb{R}} u_0^2 \varphi_2^2 \right) + O(\varepsilon_2^3) \\
&= \mathbb{E}_\beta(u_0, 0) + \varepsilon_2^2 \left(\tfrac{1}{2} \textstyle\int_{\mathbb{R}} (|\varphi_2'|^2 + \lambda \varphi_2^2) - \tfrac{\beta}{2} \textstyle\int_{\mathbb{R}} u_0^2 \varphi_2^2 \right) + O(\varepsilon_2^3).
\end{aligned} \tag{5.12}$$

Note that the first eigenvalue of

$$-\phi'' + \lambda \phi = \nu u_0^2 \phi$$

is μ_1. Thus if $\beta > \mu_1$, then we can choose φ_2 such that

$$\int_{\mathbb{R}} (|\varphi_2'|^2 + \lambda \varphi_2^2) - \beta \int_{\mathbb{R}} u_0^2 \varphi_2^2 < 0$$

and therefore $\mathbb{E}_\beta(u, v) < \mathbb{E}_\beta(u_0, 0)$, which deduces that $v_0 \not\equiv 0$.

Similarly, if $\beta > \mu_2$, then $u_0 > 0$. Therefore we have proved the first part of Theorem 5.1.

To prove the last part, we consider the set $I = \{u_0 > cv_0\}$, where $c = \sqrt{\frac{\beta-\mu_2}{\beta-\mu_1}}$. From the equation (5.3), we have

$$(u_0'v_0 - u_0v_0')|_{\partial I} + \int_I [(\mu_1 u_0^3 + \beta v_0^2 u_0)v_0 - (\mu_2 v_0^3 + \beta u_0^2 v_0)u_0] = 0.$$

Note that on I,

$$\int_I [(\mu_1 u_0^3 + \beta v_0^2 u_0)v_0 - (\mu_2 v_0^3 + \beta u_0^2 v_0)u_0] = (\mu_2 - \beta)\int_I u_0 v_0 (u_0^2 - c^2 v_0^2) \leq 0.$$

On the other hand,

$$(u_0'v_0 - u_0v_0')|_{\partial I} \leq 0.$$

Thus I must be an empty set. So $u_0 \leq cv_0$. Similarly we can prove that $u_0 \geq cv$.

So $u_0 = cv_0$. This proves (5.4).

□

Proof of Theorem 5.2. This follows easily from the proof of Theorem 1.1. Note that because of the constraints, we obtain $u_0 > 0, v_0 > 0$. □

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