Existence of Global Steady Subsonic Euler Flows through Infinitely Long Nozzles

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Abstract: In this paper, we study the global existence of steady subsonic Euler flows through infinitely long nozzles without the assumption of irrotationality. It is shown that when the variation of Bernoulli's function in the upstream is sufficiently small and mass flux is in a suitable regime with an upper critical value, then there exists a unique global subsonic solution in a suitable class for a general variable nozzle. One of the main difficulties for the general steady Euler flows, the governing equations are a mixed elliptic-hyperbolic system even for uniformly subsonic flows. A key point in our theory is to use a stream function formulation for compressible Euler equations. By this formulation, Euler equations are equivalent to a quasilinear second order equation for a stream function so that the hyperbolicity of the particle path is already involved. The existence of solution to the boundary value problem for stream function is obtained with the help of the estimate for elliptic equation of two variables. The asymptotic behavior for the stream function is obtained via a blow up argument and energy estimates. This asymptotic behavior, together with some refined estimates on the stream function, yields the consistency of the stream function formulation and thus the original Euler equations.

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1 Introduction and Main Results

Multidimensional gas flows give rise many outstanding challenging problems. Since the solutions for the unsteady compressible Euler equations develop singularities in general[29], it is not yet known which function space is suitable to study their wellposedness[27]. It is natural to start from the steady Euler equations to understand some important true multidimensional flow patterns. However, the steady Euler equations themselves are not easy to tackle, since the equations may not only be hyperbolic or hyperbolic-elliptic coupled system, but also have discontinuous solutions such as shock waves and vortex sheets. Therefore, a lot of approximate models were proposed to study fluid flows. An important approximate model is the potential flow, which originates from the study for flows without vorticity. Since 1950's, tremendous progress has been made on the study for potential flows. Subsonic potential flows around a body were studied extensively by Shiffman[28], Bers[2, 3], Finn, Gilbarg[13, 14], and Dong[10], et al. Subsonic-sonic flows around a body were established recently by Chen, et al [5] via compensated compactness method. Significant progress on transonic flows was made by Morawetz. She first showed the nonexistence of smooth transonic flows in general[21–24], and later worked on existence of weak solutions to transonic flows by the theory of compensated compactness[25, 26]. Existence and stability of transonic shocks in a nozzle for potential flows were achieved recently with prescribed potential at downstream in [6, 7]. Xin and Yin obtained existence and nonexistence of transonic shocks in a bounded nozzle with prescribed pressure at downstream was obtained in [34, 35]. Recently, well-posedness for subsonic and subsonic-sonic potential flows through infinitely long 2-D and 3-D axially symmetric nozzles, was established in [31, 32]. For the study on other aspects on subsonic potential flows, please refer to [12, 15, 16].

Besides the potential flow, there is another important approximate model to compressible Euler equations, incompressible Euler equations, which approximate to the compressible Euler equations for flows with small Mach numbers. For the study on the existence of steady incompressible Euler flows in a bounded domain, please refer to [1, 18, 30], etc, and references therein.

For the full compressible Euler equations, well-posedness and nonexistence of a transonic shock in bounded nozzles with prescribed pressure at downstream has been obtained in [19, 20, 33, 36]. Existence and stability of transonic shocks in nozzles with prescribed velocity at downstream was shown in [4].

In this paper, we study the existence of global steady subsonic Euler flows through general infinitely long nozzles.

Consider the 2-D steady isentropic Euler equations

$$(\rho u)_{x_1} + (\rho v)_{x_2} = 0, \tag{1}$$

$$(\rho u^2)_{x_1} + (\rho uv)_{x_2} + p_{x_1} = 0, \tag{2}$$

$$(\rho uv)_{x_1} + (\rho v^2)_{x_2} + p_{x_2} = 0, \tag{3}$$

where ρ, (u, v), and $p = p(\rho)$ denote the density, velocity and pressure respectively. In general, it is assumed that $p'(\rho) > 0$ for $\rho > 0$ and $p''(\rho) \geq 0$, where $c(\rho) = \sqrt{p'(\rho)}$ is called the sound speed. The most important examples include polytropic gases and isothermal gases. For polytropic gases, $p = A\rho^\gamma$ where A is a constant and γ is the adiabatic constant with $\gamma > 1$; and for isothermal gases, $p = c^2\rho$ with constant sound speed c [9].

We consider flows through an infinitely long nozzle given by

$$\Omega = \{(x_1, x_2) | f_1(x_1) < x_2 < f_2(x_1), -\infty < x_1 < \infty\},$$

which is bounded by $S_i = \{(x_1, x_2) | x_2 = f_i(x_1), -\infty < x_1 < \infty\}$, $(i = 1, 2)$. Suppose that $S_i (i = 1, 2)$ satisfy

$$f_2(x_1) > f_1(x_1) \text{ for } x_1 \in (-\infty, \infty), \tag{4}$$

$$f_1(x_1) \to 0, \quad f_2(x_1) \to 1, \qquad \text{as } x_1 \to -\infty, \tag{5}$$

$$f_1(x_1) \to a, \quad f_2(x_1) \to b > a, \qquad \text{as } x_1 \to +\infty, \tag{6}$$

and

$$\|f_i\|_{C^{2,\alpha}(\mathbb{R})} \leq C \text{ for some } \alpha > 0 \text{ and } C > 0. \tag{7}$$

It follows that Ω satisfies the uniform exterior sphere condition with some uniform radius $r > 0$.

Suppose that the nozzle walls are impermeable solid walls so that the flow satisfies the no flow boundary condition

$$(u, v) \cdot \vec{n} = 0 \text{ on } \partial\Omega, \tag{8}$$

where $\vec{n}$ is the unit outward normal to the nozzle wall. It follows from (1) and (8) that

$$\int_l (\rho u, \rho v) \cdot \vec{n} dl \equiv m \tag{9}$$

holds for some constant m, which is called the mass flux, where l is any curve transversal to the x_1-direction, and $\vec{n}$ is the normal of l in the positive x_1-axis direction.

Due to the continuity equation, when the flow is away from the vacuum, the momentum equations are equivalent to

$$u u_{x_1} + v u_{x_2} + h(\rho)_{x_1} = 0, \tag{10}$$

$$u v_{x_1} + v v_{x_2} + h(\rho)_{x_2} = 0, \tag{11}$$

where $h(\rho)$ is the enthalpy of the flow satisfying $h'(\rho) = p'(\rho)/\rho$. So $h(\rho)$ is determined up to a constant. In this paper, for example, we always choose $h(0) = 0$ for polytropic gases and $h(1) = 0$ for isothermal gases. After determining this integral constant, we denote $B_0 = \inf_{\rho>0} h(\rho)$.

It follows from (10) and (11) that

$$(u, v) \cdot \nabla(h(\rho) + \frac{1}{2}(u^2 + v^2)) = 0. \tag{12}$$

This implies that $\frac{u^2+v^2}{2} + h(\rho)$, which will be called Bernoulli's function, is a constant along each streamline. For Euler flows in the nozzle, we assume that in the upstream, Bernoulli's function is given, i.e.,

$$\frac{u^2 + v^2}{2} + h(\rho) \to B(x_2) \text{ as } x_1 \to -\infty, \tag{13}$$

where $B(x_2)$ is a function defined on $[0, 1]$.

Now let us state our main results in the paper

Theorem 1 *Let the nozzle satisfy (4)-(7) and* $\underline{B} > B_0$. *There exists a* $\delta_0 > 0$ *such that if*

$$\inf_{x_2\in[0,1]} B(x_2) = \underline{B},\ B'(0) \le 0,\ B'(1) \ge 0\ \text{ and } \|B'(x_2)\|_{C^{0,1}([0,1])} = \delta \le \delta_0, \tag{14}$$

then there exists $\hat{m} \ge 2\delta_0^{1/8}$ *such that for any* $m \in (\delta^{1/4}, \hat{m})$,

1. *(Existence) there exists a flow satisfying the Euler equations (1)-(3), the boundary condition (8), mass flux condition (9), and the asymptotic condition (13);*

2. *(Subsonic flows and positivity of horizontal velocity) the flow is globally uniformly subsonic and has positive horizontal velocity in the whole nozzle, i.e.,*

$$\sup_{\bar{\Omega}}(u^2 + v^2 - c^2(\rho)) < 0 \text{ and } u > 0 \text{ in } \bar{\Omega}; \tag{15}$$

3. *(Regularity and far fields behavior) Furthermore, the flow satisfies*

$$\|\rho\|_{C^{1,\alpha}(\Omega)}, \|u\|_{C^{1,\alpha}(\Omega)}, \|v\|_{C^{1,\alpha}(\Omega)} \le C \tag{16}$$

for some constant $C > 0$, *and the following asymptotic behavior in far fields*

$$\rho \to \rho_0 > 0,\ u \to u_0(x_2) > 0,\ v \to 0 \text{ as } x_1 \to -\infty, \tag{17}$$

$$\nabla\rho \to 0,\ \nabla u \to (0, u_0'(x_2)),\ \nabla v \to 0 \text{ as } x_1 \to -\infty, \tag{18}$$

uniformly for $x_2 \in K_1 \Subset (0,1)$, *and*

$$\rho \to \rho_1 > 0,\ u \to u_1(x_2) > 0,\ v \to 0 \text{ as } x_1 \to +\infty, \tag{19}$$

$$\nabla\rho \to 0,\ \nabla u \to (0, u_1'(x_2)),\ \nabla v \to 0 \text{ as } x_1 \to +\infty, \tag{20}$$

uniformly for $x_2 \in K_2 \Subset (a,b)$, *where* ρ_0 *and* ρ_1 *are both positive constants, and* ρ_0, ρ_1, u_0, *and* u_1 *can be determined by* m, $B(x_2)$ *and* $b-a$ *uniquely;*

4. *(Uniqueness) the Euler flow which satisfies (1)-(3), boundary condition (8), asymptotic condition (13), mass flux condition (9), (15), and asymptotic behavior (17)-(18) is unique;*

5. *(Critical mass flux) If, besides (14), B also satisfies*

$$B'(0) = B'(1) = 0, \tag{21}$$

then $\hat{m}$ is the upper critical mass flux for the existence of subsonic flow in the following sense: either

$$\sup_{\bar{\Omega}}(u^2 + v^2 - c^2(\rho)) \to 0 \text{ as } m \to \hat{m}, \tag{22}$$

or there is no $\sigma > 0$ such that for all $m \in (\hat{m}, \hat{m} + \sigma)$, there are Euler flows satisfying (1)-(3), boundary condition (8), asymptotic condition (13), mass flux condition (9), (15), and asymptotic behavior (17)-(18) and

$$\sup_{m \in (\hat{m}, \hat{m}+\sigma)} \sup_{\bar{\Omega}}(c^2(\rho) - (u^2 + v^2)) > 0. \tag{23}$$

There are a few remarks in order:

Remark 1 Here we obtained only the existence of the Euler flows in the nozzle, and the uniqueness in a special class of flows, but not the uniqueness for general Euler flows. For the issue on the uniqueness for steady incompressible Euler flows in a bounded domain, please refer to [30].

Remark 2 It can be shown by modifying the analysis in this paper slightly without further difficulties that there exists a subsonic full compressible Euler flow in the nozzle, if the entropy is prescribed in the upstream.

Remark 3 The subsonic Euler flows in half plane was studied in [8] recently. Although stream function formulation is also introduced in [8], however, the far fields conditions are different from ours. Furthermore, we obtain critical upper bound of mass flux for existence of subsonic flows in nozzles.

The rest of the paper is arranged as follows: in Section 3, we reformulate the problem by deriving the governing equation and boundary conditions for Euler flows in terms of a stream function, provided that the Euler flow has simple topological structure and satisfies the asymptotic behavior (17)-(18). In Section 3, existence of solutions to a modified elliptic problem is established. Subsequently, in Section 4, we will study asymptotic behavior of solutions in a larger class and show uniqueness of the solution to the boundary value problem. The existence of boundary value problem for the stream functions will be a direct consequence of these asymptotic behavior and uniqueness. In Section 5, some refined estimates for the stream function will be derived. Combining these estimates with the asymptotic behavior obtained in Section 4 will yield the existence of Euler flows which satisfy all properties in Theorem 1. Finally, in Section 6, we will show the existence of the critical mass flux.

2 Stream-Function Formulation of the Problem

We start with some basic structures of the steady Euler system. The steady Euler system (1)-(3) can be written in the following form,

$$AU_{x_1} + BU_{x_2} = 0,$$

where

$$A = \begin{pmatrix} \frac{uc^2(\rho)}{\rho} & c^2(\rho) & 0 \\ c^2(\rho) & \rho u & 0 \\ 0 & 0 & \rho u \end{pmatrix}, \; B = \begin{pmatrix} \frac{vc^2(\rho)}{\rho} & 0 & c^2(\rho) \\ 0 & \rho v & 0 \\ c^2(\rho) & 0 & \rho v \end{pmatrix}, \; U = \begin{pmatrix} \rho \\ u \\ v \end{pmatrix}.$$

Let λ be the solution of

$$\det(\lambda A - B) = 0. \tag{24}$$

It follows from straightforward computations that (24) has three eigenvalues

$$\lambda_1 = \frac{v}{u}, \; \lambda_\pm = \frac{uv \pm c(\rho)\sqrt{u^2 + v^2 - c^2(\rho)}}{u^2 - c^2}.$$

Therefore, at the points where $u^2 + v^2 - c^2(\rho) > 0$, i.e., the flow is supersonic, (24) has 3 real eigenvalues, the Euler system is hyperbolic. When $u^2 + v^2 - c^2(\rho) < 0$, i,e., the flow is subsonic, (24) has a real eigenvalue and two complex eigenvalues, the Euler system is a hyperbolic-elliptic coupled system. Therefore, even for globally subsonic flows, one has to resolve a hyperbolic mode. Moreover, for flows in infinitely long nozzles with both ends at infinity, it seems difficult to get uniform estimates for hyperbolic mode.

To overcome the difficulties mentioned above, we introduce the stream functions for the 2-D steady compressible Euler flows, and derive an equivalent formulation for Euler flows in terms of the stream functions when the flow satisfies certain asymptotic behavior.

It follows from (10) and (11) that

$$\partial_{x_2}(uu_{x_1} + vu_{x_2} + h(\rho)_{x_1}) - \partial_{x_1}(uv_{x_1} + vv_{x_2} + h(\rho)_{x_2}) = 0, \tag{25}$$

which yields that

$$(u, v) \cdot \nabla\omega + \omega \mathrm{div}(u, v) = 0, \tag{26}$$

where $\omega = v_{x_1} - u_{x_2}$ is the vorticity of the flow. By the continuity equation (1), one has

$$(u, v) \cdot \nabla\omega + \omega \mathrm{div}(u, v) = (u, v) \cdot (\nabla\omega - \frac{\omega\nabla\rho}{\rho}) = \rho(u, v) \cdot \nabla \left(\frac{\omega}{\rho}\right).$$

Therefore, away from vacuum, (26) is equivalent to

$$(u, v) \cdot \nabla \left(\frac{\omega}{\rho}\right) = 0. \tag{27}$$

We have the following proposition.

Proposition 2 *For a smooth flow away from vacuum in the nozzle Ω satisfying (4) and (5), the system consisting of (1), (12) and (27) is equivalent to the original Euler equations (1)-(3), if the given flow satisfies no flow boundary condition,*

$$u > 0 \text{ in } \Omega, \tag{28}$$

and the following asymptotic behavior

$$u,\ \rho \text{ and } v_{x_2} \text{ are bounded, while } v,\ v_{x_1} \text{ and } \rho_{x_2} \to 0,\ \text{ as } x_1 \to -\infty. \tag{29}$$

Proof: From previous analysis, it is easy to see that smooth solutions to the Euler equations (1)-(3) satisfy (1), (12) and (27). On the other hand, it follows from (1), (27) and the above derivation that (25) holds. Therefore, there exists a function Φ such that

$$\Phi_{x_1} = uu_{x_1} + vu_{x_2} + h(\rho)_{x_1},\ \Phi_{x_2} = uv_{x_1} + vv_{x_2} + h(\rho)_{x_2}.$$

So, (12) is equivalent to

$$(u, v) \cdot \nabla \Phi = 0. \tag{30}$$

Due to the no flow boundary condition (8), Φ is a constant along each component of the nozzle boundary. If, in addition,

$$\Phi_{x_2} \to 0 \text{ as } x_1 \to -\infty, \tag{31}$$

then $\Phi \to C$ as $x_1 \to -\infty$. On the other hand, it follows from (28) that through each point in Ω, there is one and only one streamline satisfying

$$\begin{cases} \frac{dx_1}{ds} = u(x_1(s), x_2(s)), \\ \frac{dx_2}{ds} = v(x_1(s), x_2(s)), \end{cases}$$

which can be defined globally in the nozzle (i.e., from the entry to the exit). Furthermore, it follows from (1) that any streamline through some point in Ω can not touch the nozzle wall. Suppose not, let the streamline through (x_1^0, x_2^0) pass through $(x_1, f_1(x_1))$. Due to (1) and no flow boundary condition, one has

$$\int_{f_1(x_1^0)}^{x_2^0} (\rho u)(x_1^0, x_2) dx_2 = 0.$$

This contradicts (28).

Thus, one can always solve (30) in the whole domain Ω, which yields

$$\Phi \equiv C \text{ in } \Omega,$$

if (31) holds. Therefore, $\Phi_{x_1} = \Phi_{x_2} \equiv 0$ in Ω, i.e., (10) and (11) hold globally in the nozzle. Thus, both (2) and (3) are true. It is obvious that (31) holds if (29) is valid. $\square$

It suffices to prove the existence of solutions to the system (1), (12) and (27) satisfying (28) and (29).

However, system (1), (12) and (27) is not easy to study directly either, since for infinitely long nozzles with both ends at infinity, it seems difficult to estimate the solutions for transport equations (12) and (27). Instead, we will use an equivalent formulation for (1), (12) and (27).

It follows from (1) that there exists a stream function ψ such that

$$\psi_{x_1} = -\rho v,\ \psi_{x_2} = \rho u.$$

Thus for the flows away from the vacuum, (27) is equivalent to

$$\nabla^{\perp}\psi \cdot \nabla \left(\frac{\omega}{\rho}\right) = 0, \tag{32}$$

where $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$. Note that (32) means that $\frac{\omega}{\rho}$ and ψ are functionally dependent, therefore, one may regard $\frac{\omega}{\rho}$ as a function of ψ. Set

$$\frac{\omega}{\rho} = W(\psi). \tag{33}$$

Similarly, (12) is equivalent to

$$\nabla^{\perp}\psi \cdot \nabla(h(\rho) + \frac{1}{2}(u^2 + v^2)) = 0,$$

therefore, $h(\rho) + \frac{1}{2}(u^2 + v^2)$ is also a function of ψ. We define this function by

$$h(\rho) + \frac{1}{2}(u^2 + v^2) = \mathcal{B}(\psi). \tag{34}$$

Furthermore, it follows from (8) that the nozzle walls are streamlines, so ψ is constant on each nozzle wall. Due to (9), one can assume that

$$\psi = 0 \text{ on } S_1, \text{ and } \psi = m \text{ on } S_2. \tag{35}$$

In order to get an explicit form of $\mathcal{B}$, first we study the density-speed relation via Bernoulli's law (34) carefully.

Note that $p'(\rho) > 0$ for $\rho > 0$ and $p''(\rho) \geq 0$, therefore $h'(\rho) = p'(\rho)/\rho > 0$ for $\rho > 0$ and for some fixed $\bar{\rho} > 0$,

$$h(\rho) = h(\bar{\rho}) + \int_{\bar{\rho}}^{\rho} \frac{p'(s)}{s} ds \geq h(\bar{\rho}) + \int_{\bar{\rho}}^{\rho} \frac{p'(\bar{\rho})}{s} ds \text{ for } \rho > \bar{\rho}.$$

This yields that $h(\rho) \to \infty$ as $\rho \to \infty$. On the other hand, since $\inf_{\rho>0} h(\rho) = B_0$, $h(\rho) \to B_0$ as $\rho \to 0$. Thus for any $s > B_0$, there exists a unique $\bar{\varrho} = \bar{\varrho}(s) > 0$ such that

$$h(\bar{\varrho}(s)) = s.$$

Moreover, for the state with given Bernoulli's constant s, the density and speed satisfy the relation,

$$h(\rho) + \frac{q^2}{2} = s.$$

Therefore, the speed q satisfies

$$q = \sqrt{2(s - h(\rho))}.$$

Hence, for fixed s, q is a strictly decreasing function of ρ on $[0, \bar{\varrho}(s)]$. By the definition of $\bar{\varrho}(s)$, one has $q(\bar{\varrho}(s)) = 0 < c(\bar{\varrho}(s))$. Now we claim that $q(0) > c(0)$. Indeed, one can prove this claim in two cases. First, if $c(0) > 0$, then

$$h(\rho) = h(\bar{\rho}) + \int_{\bar{\rho}}^{\rho} \frac{p'(s)}{s} ds \le h(\bar{\rho}) + \int_{\bar{\rho}}^{\rho} \frac{c^2(0)}{s} ds \to -\infty \text{ as } \rho \to 0.$$

Therefore, $q(0) \to \infty$. Thus $q(0) > c(0)$. Second, if $c(0) = 0$, $q(\rho) \to \sqrt{2(s - B_0)}$ as $\rho \to 0$, therefore, $q(0) > 0 = c(0)$. This completes the proof of the claim. Since $c^2(\rho) = p'(\rho)$ is an increasing function of ρ, there exists a unique $\varrho(s) \in [0, \bar{\varrho}(s)]$ such that

$$c^2(\varrho(s)) = q^2(\varrho(s)).$$

In summary, for any given $s > B_0$, there exist $\bar{\varrho} = \bar{\varrho}(s)$, $\varrho = \varrho(s)$ and $\Gamma = \Gamma(s)$ such that

$$h(\bar{\varrho}(s)) = s, \ h(\varrho(s)) + \frac{\Gamma^2(s)}{2} = s, \text{ and } c^2(\varrho(s)) = \Gamma^2(s), \tag{36}$$

where $\bar{\varrho}(s)$, $\varrho(s)$, and $\Gamma(s)$ are the maximum density, the critical density, and the critical speed, respectively for the states with given Bernoulli's constant s. Set

$$\Sigma(s) = \varrho(s)\sqrt{2(s - h(\varrho(s)))}. \tag{37}$$

Then direct calculations show that

$$\frac{d\bar{\varrho}}{ds} = \frac{\bar{\varrho}}{p'(\bar{\varrho})}, \ \frac{d\varrho}{ds} = \frac{1}{\frac{p'(\varrho)}{\varrho} + \frac{p''(\varrho)}{2}},$$

and

$$\frac{d\Sigma}{ds} = \frac{\sqrt{2(s-h(\varrho(s)))}}{\frac{p'(\varrho)}{\varrho}+\frac{p''(\varrho)}{2}} + \varrho\frac{1-\frac{2p'(\varrho)}{2p'(\varrho)+\varrho p''(\varrho)}}{\sqrt{2(s-h(\varrho(s)))}}.$$

Thus

$$\frac{d\bar{\varrho}}{ds} > 0,\ \frac{d\varrho}{ds} > 0,\ \text{and}\ \frac{d\Sigma}{ds} > 0.$$

Obviously, $\varrho(s) < \bar{\varrho}(s)$, if $s > B_0$. By the continuity and monotonicity of $\varrho(s)$ and $\bar{\varrho}(s)$, there exists a unique $\underline{\delta} > 0$ such that

$$\varrho(\underline{B}+\underline{\delta}) = \bar{\varrho}(\underline{B}). \tag{38}$$

Moreover, it follows from (36) that there exists a uniform constant $C > 0$ such that

$$\begin{cases} C^{-1} \le \varrho(\underline{B}) < \bar{\varrho}(\underline{B}) = \varrho(\underline{B}+\underline{\delta}) < \bar{\varrho}(\underline{B}+\underline{\delta}) \le C, \\ C^{-1} \le \varrho'(s) \le C,\ C^{-1} \le \bar{\varrho}'(s) \le C,\ \text{if } s \in (\underline{B}, \underline{B}+\underline{\delta}), \\ C^{-1} \le h'(\rho) \le C,\ \text{if } \rho \in (\varrho(\underline{B}), \bar{\varrho}(\underline{B}+\underline{\delta})), \\ C^{-1} \le \Sigma(s) \le C,\ \text{if } s \in (\underline{B}, \underline{B}+\underline{\delta}). \end{cases} \tag{39}$$

Later on, C will to denote generic constants which depend only on $\underline{B}$ and $\underline{\delta}$, and thus essentially on $\underline{B}$.

In order to study the relationship between density and mass flux with given Bernoulli's constant, let us investigate the function defined by

$$I(\rho) = 2\rho^2(s-h(\rho)).$$

Direct calculations show

$$\frac{dI}{d\rho} = 4\rho(s-h(\rho)-p'(\rho)/2) = 2\rho(q^2(\rho)-c^2(\rho)).$$

Therefore, for $\rho \in (0, \varrho(s))$, $\frac{dI}{d\rho} > 0$; and $\frac{dI}{d\rho} < 0$ for $\rho \in (\varrho(s), \bar{\varrho}(s))$. Moreover, $I(0) = I(\bar{\varrho}(s)) = 0$. Thus $I(\rho) > 0$ if $\rho \in (0, \bar{\varrho}(s))$ and I achieves its maximum at $\rho = \varrho(s)$. So, for fixed s, the relation

$$h(\rho) + \frac{\mathcal{M}}{2\rho^2} = s \tag{40}$$

defines a function $\mathcal{M} = I(\rho)$ which attains its maximum $\mathcal{M} = \Sigma^2(s)$ at $\rho = \varrho(s)$. Thus ρ is a two-valued function of $\mathcal{M}$ for $\mathcal{M} \in [0, \Sigma^2(s))$. Denote the subsonic branch by

$$\rho = J(\mathcal{M}) \text{ for } \mathcal{M} \in (0, \Sigma^2(s)),$$

which satisfies $J(\mathcal{M}) > \varrho(s)$. When s varies, this branch will be denoted by

$$\rho = J(\mathcal{M}, s) \text{ for } (\mathcal{M}, s) \in \{(\mathcal{M}, s) | \mathcal{M} \in (0, \Sigma^2(s)), s > B_0\}. \tag{41}$$

To determine the explicit form of W and $\mathcal{B}$, one may study W and $\mathcal{B}$ in the far fields of the nozzle where the flow may have certain simple asymptotic structure. Indeed, for flows satisfying the asymptotic behavior (17)-(20), one can determine ρ_0, ρ_1, $u_0(x_2)$ and $u_1(x_2)$ first. Suppose that the flow satisfies (17). Then

$$h(\rho_0) + \frac{u_0^2(x_2)}{2} = B(x_2),\ u_0(x_2) > 0, \tag{42}$$

and

$$\int_0^1 \rho_0 u_0(x_2) dx_2 = m \tag{43}$$

hold, which shows that

$$u_0(x_2) = \sqrt{2(B(x_2) - h(\rho_0))}, \tag{44}$$

and

$$m = \int_0^1 \rho_0 \sqrt{2(B(x_2) - h(\rho_0))} dx_2. \tag{45}$$

If $B(x_2)$ satisfies

$$\inf_{x_2 \in [0,1]} B(x_2) = \underline{B},\ \|B'(x_2)\|_{C^{0,1}([0,1])} \leq \delta, \tag{46}$$

then

$$\bar{B} = \sup_{x_2 \in [0,1]} B(x_2) \leq \underline{B} + \delta. \tag{47}$$

Let $\delta \leq \underline{\delta}/2$. Then it follows from (38) that $\varrho(B(x_2)) \leq \varrho(\bar{B}) < \bar{\varrho}(\underline{B})$. To obtain a global subsonic flow in the nozzle, it is necessary to show that for given $B_2(x_2)$ and m, (45) has a solution satisfying $\rho_0 \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B}))$. Direct calculations yield that

$$\frac{d}{d\rho_0} \int_0^1 \rho_0 \sqrt{2(B(x_2) - h(\rho_0))} dx_2 < 0, \text{ for } \rho_0 \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B})).$$

It follows from (39) and (46) that

$$\int_0^1 \bar{\varrho}(\underline{B})\sqrt{2(B(x_2)-h(\bar{\varrho}(\underline{B})))}dx_2 = \int_0^1 \bar{\varrho}(\underline{B})\sqrt{2(B(x_2)-\underline{B})}dx_2 \le C\delta^{1/2}.$$

In addition,

$$\begin{aligned}
&\int_0^1 \varrho(\bar{B})\sqrt{2(B(x_2)-h(\varrho(\bar{B})))}dx_2\\
\ge\ &\int_0^1 \varrho(\bar{B})\sqrt{2(\underline{B}-h(\varrho(\bar{B})))}dx_2\\
=\ &\int_0^1 \varrho(\bar{B})\sqrt{2(h(\bar{\varrho}(\underline{B}))-h(\varrho(\bar{B})))}dx_2\\
=\ &\int_0^1 \varrho(\bar{B})\sqrt{2(h(\varrho(\underline{B}+\underline{\delta}))-h(\varrho(\bar{B})))}dx_2\\
\ge\ &\int_0^1 \varrho(\bar{B})\sqrt{2\left(h(\varrho(\underline{B}+\underline{\delta}))-h(\varrho(\underline{B}+\underline{\delta}/2))\right)}dx_2\\
\ge\ &C^{-1}\underline{\delta}^{1/2}.
\end{aligned}$$

Therefore, for any $\gamma \in (0,1/3)$, there exists $\tilde{\delta}_0 \in (0,\underline{\delta}/2)$ such that (45) admits a unique solution $\rho_0 \in (\varrho(\bar{B}),\bar{\varrho}(\underline{B}))$, provided that $0 \le \delta \le \tilde{\delta}_0$ and $m \in (\delta^\gamma, m_1)$, where m_1 satisfies $m_1 \ge C^{-1}\underline{\delta}^{1/2} \ge 2\tilde{\delta}_0^{\gamma/2}$. Later on, for definiteness, we will choose $\gamma = 1/4$. However, all results hold for $\gamma \in (0,1/3)$.

By virtue of (45), one has

$$\begin{aligned}
m &= \int_0^1 \rho_0\sqrt{2(B(x_2)-h(\rho_0))}dx_2\\
&= \int_0^1 \rho_0\sqrt{2(B(x_2)-\underline{B}+\underline{B}-h(\rho_0))}dx_2\\
&\le C\int_0^1 \sqrt{2(\delta+h(\bar{\varrho}(\underline{B}))-h(\rho_0))}dx_2.
\end{aligned}$$

Thus

$$\delta + h(\bar{\varrho}(\underline{B})) - h(\rho_0) \ge C^{-1}m^2 \ge C^{-1}\delta^{2\gamma}.$$

Note that $\gamma < 1/3$, therefore, there exists $\tilde{\tilde{\delta}}_0 \in (0,\tilde{\delta}_0)$ such that if $0 < \delta \le \tilde{\tilde{\delta}}_0$, then

$$h(\bar{\varrho}(\underline{B})) - h(\rho_0) \ge C^{-1}\delta^{2\gamma}.$$

Consequently, if $\|B'(x_2)\|_{C^{0,1}([0,1])} = \delta \leq \tilde{\tilde{\delta}}_0$, by virtue of (39), there is a positive constant C such that

$$\begin{cases} C^{-1}\delta^{2\gamma} \leq \bar{\varrho}(\underline{B}) - \rho_0 \leq C, \\ C^{-1}\delta^{\gamma} \leq u_0 \leq C, \\ |u_0'(x_2)| = \left|\frac{B'(x_2)}{\sqrt{2(B(x_2)-h(\rho_0))}}\right| \leq C\delta^{1-\gamma}, \\ [u_0'(x_2)]_{C^{0,1}([0,1])} \leq C(\delta^{1-\gamma} + \delta^{2-3\gamma}). \end{cases} \tag{48}$$

To determine the states in the downstream, we parametrize the streamlines in the downstream by their positions in the upstream. Due to (17), (19), and (28), we can define

$$y = y(s) \text{ for } s \in [0,1] \tag{49}$$

such that

$$h(\rho_0) + \frac{u_0^2(s)}{2} = h(\rho_1) + \frac{u_1^2(y(s))}{2}, \ u_1(y(s)) > 0, \tag{50}$$

$$\int_0^s \rho_0 u_0(t)dt = \int_a^{y(s)} \rho_1 u_1(t)dt, \tag{51}$$

$$y(0) = a, \ y(1) = b. \tag{52}$$

The meaning of $y(s)$ is that the streamline which starts from $(-\infty, s)$ will flow to $(\infty, y(s))$. The map (49) is well-defined since (28) ensures a simple topological structure of streamlines. It follows from (51) that

$$\rho_0 u_0(s) = \rho_1 u_1(y(s))y'(s). \tag{53}$$

Hence,

$$\begin{cases} \frac{dy}{ds} = \frac{\rho_0 u_0(s)}{\rho_1\sqrt{2(h(\rho_0)+\frac{u_0^2(s)}{2}-h(\rho_1))}}, \\ y(0) = a, \end{cases} \tag{54}$$

where the parameter ρ_1 satisfies

$$\int_0^1 \frac{\rho_0 u_0(s)}{\rho_1\sqrt{2(h(\rho_0)+\frac{u_0^2(s)}{2}-h(\rho_1))}}ds = b - a. \tag{55}$$

It remains to show that there exists a $\rho_1 \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B}))$ satisfying (55). By direct calculations, one has

$$\frac{d}{d\rho_1}\int_0^1 \frac{\rho_0 u_0(s)}{\rho_1\sqrt{2(h(\rho_0)+\frac{u_0^2(s)}{2}-h(\rho_1))}}ds > 0, \text{ for } \rho_1 \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B})).$$

First, there exists $\bar{\delta}_0 \in (0, \bar{\bar{\delta}}_0)$ such that

$$\begin{aligned}
& \int_0^1 \frac{\rho_0 u_0(s)}{\bar{\varrho}(\underline{B}-\delta)\sqrt{2(h(\rho_0)+\frac{u_0^2(s)}{2}-h(\bar{\varrho}(\underline{B}-\delta)))}} ds \\
= & \int_0^1 \frac{\rho_0 u_0(s)}{\bar{\varrho}(\underline{B}-\delta)\sqrt{2(B(s)-\underline{B}+h(\bar{\varrho}(\underline{B}))-h(\bar{\varrho}(\underline{B}-\delta)))}} ds \\
\geq & \; C\delta^{(2\gamma-1)/2} > b-a,
\end{aligned}$$

provided $\delta \leq \bar{\delta}_0$. On the other hand,

$$\begin{aligned}
& \int_0^1 \frac{\rho_0 u_0(s)}{\varrho(\bar{B})\sqrt{2(h(\rho_0)+\frac{u_0^2(s)}{2}-h(\varrho(\bar{B})))}} ds \\
\leq & \; \frac{m}{\varrho(\bar{B})\sqrt{2(\underline{B}-h(\varrho(\bar{B})))}} \\
= & \; \frac{m}{\varrho(\bar{B})\sqrt{2(h(\bar{\varrho}(\underline{B}))-h(\varrho(\bar{B})))}} \\
= & \; \frac{m}{\varrho(\bar{B})\sqrt{2\left(h(\varrho(\underline{B}+\underline{\delta}))-h(\varrho(\bar{B}))\right)}} \\
\leq & \; \frac{m}{\varrho(\bar{B})\sqrt{2\left(h(\varrho(\underline{B}+\underline{\delta}))-h(\varrho(\underline{B}+\underline{\delta}/2))\right)}} \\
\leq & \; \frac{m}{C^{-1}\underline{\delta}^{1/2}}.
\end{aligned}$$

So there exists a unique $\rho_1 \in (\varrho(\underline{B}), \bar{\varrho}(\underline{B}))$ such that (55) holds, if $0 \leq \delta \leq \bar{\delta}_0$ and $m \in (\delta^\gamma, m_2)$ for some $m_2 \geq \min\{m_1, C^{-1}(b-a)\underline{\delta}^{1/2}\}$. Furthermore, one can choose $\bar{\delta}_0$ smaller if necessary such that $m_2 \geq 2\bar{\delta}_0^{\gamma/2}$. As soon as ρ_1 is determined, $y(s)$ and u_1 can be obtained from (54) and (50).

Let us summarize the above calculations in the following proposition:

Proposition 3 *Let $\underline{B} > B_0$. There exists $\bar{\delta}_0 > 0$ such that for any $B \in C^{1,1}([0,1])$ satisfying (46) with $\delta \leq \bar{\delta}_0$, there exists $\bar{m} \geq 2\bar{\delta}_0^{1/8}$ such that*

1. *there exist solutions (ρ_0, u_0) to (42)-(43) and (ρ_1, u_1) solving (50)-(52) if $m \in (\delta^{1/4}, \bar{m})$;*

2. *ρ_0, $\rho_1 \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B}))$;*

3. *(ρ_0, u_0) satisfies (48);*

4. *either $\rho_0 \to \varrho(\bar{B})$ or $\rho_1 \to \varrho(\bar{B})$ as $m \to \bar{m}$;*

where $\bar{B} = \max_{x_2 \in [0,1]} B(x_2)$.

Proof: We need only to verify the last statement.

First, if $m \in (\delta^{1/4}, m_2)$, both ρ_0 and ρ_1 belong to $(\varrho(\bar{B}), \bar{\varrho}(\underline{B}))$. For a given $B(x_2)$, as m increases, ρ_0 decreases. If $m \to \tilde{m} = \int_0^1 \varrho(\bar{B})\sqrt{2(B(x_2) - h(\varrho(\bar{B})))}dx_2$, then $\rho_0 \to \varrho(\bar{B})$. Therefore there exists an upper bound for m to ensure the existence of ρ_0, $\rho_1 \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B}))$. Define

$$\bar{m} = \sup\{s | m \in (\delta^\gamma, s), \text{ there exist } \rho_0, \rho_1 \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B}))\} \tag{56}$$

Obviously, $\bar{m} \in [m_2, \tilde{m}]$. Note that ρ_0 and ρ_1 are uniformly away from $\bar{\varrho}(\underline{B})$. If neither ρ_0 nor ρ_1 approaches to $\varrho(\bar{B})$ as $m \to \bar{m}$, then there always exist ρ_0, $\rho_1 \in (\varrho(\bar{B}), \bar{\varrho}(\underline{B}))$ for $m \in (\delta^\gamma, \bar{m} + \epsilon)$ with some small $\epsilon > 0$. This contradicts with the definition of $\bar{m}$. So the proof of the Proposition is finished. □

We now can determine W and $\mathcal{B}$ in the upstream. Suppose that the flow satisfies the asymptotic behavior (17). Then in the upstream, a stream function can be chosen so that

$$\psi = \int_0^{X_2} \rho_0 u_0(s) ds, \tag{57}$$

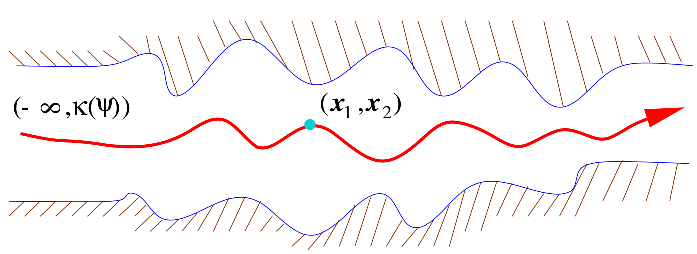
and $0 \le \psi \le m$. Since $\rho_0 u_0(s) > 0$ for $s \in [0,1]$, ψ is an increasing function of X_2. Thus one can represent X_2 as a function of ψ,

$$X_2 = \kappa(\psi), \quad 0 \le \psi \le m.$$

Define

$$f(\psi) = u_0'(\kappa(\psi)), \text{ and } F(\psi) = u_0(\kappa(\psi)). \tag{58}$$

Then f and F are well-defined on $[0, m]$.



The parametrization of flows by a stream function

It follows from the proof of Proposition 2 that through each point $(x_1, x_2) \in \Omega$, there is one and only one streamline which starts from the entry, provided that (28) holds in Ω. By the definition of streamlines, along each streamline, the stream function is a constant, therefore, through any (x_1, x_2) in the nozzle, there exists a unique streamline originated from $(-\infty, \kappa(\psi))$ with $\psi = \psi(x_1, x_2)$. Since Bernoulli's function is also invariant along a streamline,

$$(h(\rho) + \frac{|\nabla\psi|^2}{2\rho^2})(x_1, x_2) = (h(\rho) + \frac{u^2 + v^2}{2})(-\infty, \kappa(\psi)) = h(\rho_0) + \frac{F^2(\psi(x_1, x_2))}{2}.$$

Thus in the nozzle, one has

$$\mathcal{H}(\rho, |\nabla\psi|^2, \psi) = h(\rho) + \frac{|\nabla\psi|^2}{2\rho^2} - h(\rho_0) - \frac{F^2(\psi)}{2} = 0. \tag{59}$$

Similarly, by virtue of (33), one has

$$\frac{\omega}{\rho}(x_1, x_2) = -\frac{f(\psi(x_1, x_2))}{\rho_0}, \tag{60}$$

provided that (28) holds. Furthermore, note that (28) implies

$$0 \leq \psi \leq m. \tag{61}$$

Thus both (59) and (60) do make sense.

Next, we study the relationship between F and f. In the upstream,

$$\psi = \int_0^{\kappa(\psi)} \rho_0 u_0(s) ds,$$

which yields

$$\kappa'(\psi) = \frac{1}{\rho_0 u_0(\kappa(\psi))} = \frac{1}{\rho_0 F(\psi)}.$$

So, (58) shows

$$F'(\psi) = u_0'(\kappa(\psi))\kappa'(\psi) = f(\psi)\frac{1}{\rho_0 F(\psi)},$$

this implies

$$f(\psi) = \rho_0 F(\psi) F'(\psi). \tag{62}$$

Furthermore, if B satisfies (14) with $0 \le \delta \le \bar{\delta}_0$, and $m \in (\delta^\gamma, \bar{m})$, then

$$\begin{cases} C^{-1}\delta^\gamma \le F \le C, \\ F'(m) \ge 0 \text{ and } F'(0) \le 0, \\ |F'(\psi)| = \left|\frac{u_0'(\kappa(\psi))}{\rho_0 u_0(\kappa(\psi))}\right| \le C\delta^{1-2\gamma}, \\ [F'(\psi)]_{C^{0,1}([0,m])} \le C\delta^{1-3\gamma}. \end{cases} \tag{63}$$

It follows from (41) and (59) that the subsonic flows in the nozzle satisfy

$$\rho = J(|\nabla\psi|^2, h(\rho_0) + \frac{F^2(\psi)}{2}), \tag{64}$$

if they have asymptotic behavior (17). Furthermore, by the definitions of vorticity and stream function, one has

$$\omega = -\text{div}\left(\frac{\nabla\psi}{\rho}\right).$$

Thus, the stream function satisfies

$$\text{div}\left(\frac{\nabla\psi}{H(|\nabla\psi|^2,\psi)}\right) = F(\psi)F'(\psi)H(|\nabla\psi|^2,\psi), \tag{65}$$

where

$$H(|\nabla\psi|^2,\psi) = J(|\nabla\psi|^2, h(\rho_0) + \frac{F^2(\psi)}{2}). \tag{66}$$

Our major task in the rest of the paper is to show the existence of solutions to the following boundary value problem

$$\begin{cases} \text{div}\left(\frac{\nabla\psi}{H(|\nabla\psi|^2,\psi)}\right) = F(\psi)F'(\psi)H(|\nabla\psi|^2,\psi) \text{ in } \Omega, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1)-f_1(x_1)} m \text{ on } \partial\Omega, \end{cases} \tag{67}$$

and show that the flow field induced by

$$\rho = H(|\nabla\psi|^2,\psi),\ u = \frac{\psi_{x_2}}{\rho},\ v = -\frac{\psi_{x_1}}{\rho}$$

satisfies (17)-(20). We will obtain further the estimates (61) and (28) for the solution to (67) in order to get the existence of Euler flows.

Remark 4 It is easy to see that if $B = \underline{B}$, i.e., the flow has uniform Bernoulli's constant, then, F is a constant and the equation (65) reduces to

$$\text{div}\left(\frac{\nabla\psi}{H(|\nabla\psi|^2)}\right) = 0.$$

This is nothing but the potential equation. Therefore, it is reasonable to use (67) to formulate the problem for the Euler flows through the nozzles as a perturbation of the potential flows. The condition $m > \delta^\gamma(\gamma < 1/3)$ ensures

$$|F| \geq C^{-1}\delta^\gamma \text{ and } |FF'| \leq \delta^{1-2\gamma},$$

which guarantees that the magnitude of vorticity $|FF'|$ is sufficiently small, thus one can regard the potential flow as a leading ansatz for the Euler flow.

3 Existence of a Modified Boundary Value Problem for Stream Function

There are two main difficulties to solve the problem (67). The first difficulty is that the equation in (67) may become degenerate elliptic at sonic states. In addition, H is not well-defined for arbitrary ψ and $|\nabla\psi|$. The second difficulty is that this is a problem in an unbounded domain. Our basic strategy is that we extend the definition of F appropriately, truncate $|\nabla\psi|$ appeared in H in a suitable way, and use a sequence of problems on bounded domains to approximate the orginal problem. In this section we first get the existence of a modified problem on the unbounded domain, which indeed solves the original problem together with the asymptotic behavior established in the next section.

Set

$$\tilde{g}(s) = \begin{cases} F'(s), & \text{if } 0 \leq s \leq m, \\ F'(m)(2m-s)/m, & \text{if } m \leq s \leq 2m, \\ F'(0)(s+m)/m, & \text{if } -m \leq s \leq 0, \\ 0, & \text{if } s \geq 2m, \text{ or } s \leq -m. \end{cases}$$

It is obvious that $\tilde{g} \in C^{0,1}(\mathbb{R})$ and

$$\|\tilde{g}(s)\|_{C^0(\mathbb{R}^1)} \leq \|F'(s)\|_{C^0([0,m])}.$$

Moreover, it follows from (63) that

$$\tilde{g}(s) \geq 0 \text{ if } s \geq m \text{ and } \tilde{g}(s) \leq 0 \text{ if } s \leq 0. \tag{68}$$

Furthermore, it follows from $\|F'(s)\|_{C^0([0,m])} \leq C\delta^{1-2\gamma}$, $\|F'(s)\|_{C^{0,1}([0,m])} \leq C\delta^{1-3\gamma}$ and $m > \delta^\gamma$, that

$$\|\tilde{g}(s)\|_{C^{0,1}(\mathbb{R}^1)} \leq C\delta^{1-3\gamma}. \tag{69}$$

Define

$$\tilde{F}(s) = F(0) + \int_0^s \tilde{g}(t)dt.$$

Then $\tilde{F}' = \tilde{g}$ and $\tilde{F} \in C^{1,1}(\mathbb{R})$. Moreover, because $m > \delta^\gamma$, there exists a suitably small $\bar{\delta}_1$ such that when $\delta < \bar{\delta}_1$,

$$B_0 < \underline{B} - \varepsilon_0 \leq h(\rho_0) + \frac{\tilde{F}^2(s)}{2} \leq \bar{B} + \varepsilon_0 \tag{70}$$

holds for some $\varepsilon_0 > 0$, where $\bar{B} = \sup_{x_2 \in [0,1]} B(x_2)$. Moreover, (63) and (69) imply

$$\|\tilde{F}'\|_{C^0(\mathbb{R}^1)} \leq C\delta^{1-2\gamma} \quad \text{and} \quad \|\tilde{F}'\|_{C^{0,1}(\mathbb{R}^1)} \leq C\delta^{1-3\gamma}. \tag{71}$$

In the rest of the paper, we will always use the following notations

$$H_1(|\nabla\psi|^2, \psi) = \frac{\partial H}{\partial |\nabla\psi|^2}(|\nabla\psi|^2, \psi), \; H_2(|\nabla\psi|^2, \psi) = \frac{\partial H}{\partial \psi}(|\nabla\psi|^2, \psi).$$

It follows from direct calculations that

$$H_1(|\nabla\psi|^2, \psi) = -\frac{1}{2\rho(c^2 - \frac{|\nabla\psi|^2}{\rho^2})}$$

may go to negative infinity when the flow approaches sonic from subsonic.

Choose a smooth increasing function ζ_0 such that

$$\zeta_0(s) = \begin{cases} s, & \text{if } s < -2\varepsilon_0, \\ -\varepsilon_0, & \text{if } s \geq -\varepsilon_0. \end{cases}$$

Then define

$$\tilde{\Delta}(|\nabla\psi|^2,\psi) = \zeta_0(|\nabla\psi|^2 - \Sigma^2(\tilde{\mathcal{B}}(\psi))) + \Sigma^2(\tilde{\mathcal{B}}(\psi)), \tag{72}$$

where Σ is the function defined in (37) and

$$\tilde{\mathcal{B}}(\psi) = h(\rho_0) + \frac{\tilde{F}^2(\psi)}{2}. \tag{73}$$

Set

$$\tilde{H}(|\nabla\psi|^2,\psi) = J(\tilde{\Delta}(|\nabla\psi|^2,\psi), h(\rho_0) + \frac{\tilde{F}^2(\psi)}{2}), \tag{74}$$

where J is the function defined in (41). A direct calculation shows

$$\tilde{H}_1(|\nabla\psi|^2,\psi) = -\frac{\zeta_0'\tilde{H}}{2(\tilde{H}^2c^2 - \tilde{\Delta})}.$$

Obviously, there exist two positive constants $\lambda(\varepsilon_0)$ and $\Lambda(\varepsilon_0)$ such that

$$\lambda|\xi|^2 \le \tilde{A}_{ij}(q,z)\xi_i\xi_j \le \Lambda|\xi|^2 \tag{75}$$

holds for any $z \in \mathbb{R}^1$, $q \in \mathbb{R}^2$ and $\xi \in \mathbb{R}^2$, where

$$\tilde{A}_{ij}(q,z) = \tilde{H}(|q|^2,z)\delta_{ij} - 2\tilde{H}_1(|q|^2,z)q_iq_j. \tag{76}$$

Instead of (67), we first solve the following problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla\psi}{\tilde{H}(|\nabla\psi|^2,\psi)}\right) = \tilde{F}(\psi)\tilde{F}'(\psi)\tilde{H}(|\nabla\psi|^2,\psi) \text{ in } \Omega, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)} m \text{ on } \partial\Omega. \end{cases} \tag{77}$$

Proposition 4 *Let the boundary of Ω satisfy (4)-(7). Then there exists $0 < \delta_1 \le \min\{\bar{\delta}_0, \bar{\delta}_1\}$, where $\bar{\delta}_0$ is defined in Section 2, such that if $\|B'\|_{C^{0,1}([0,1])} = \delta \le \delta_1$ and $m \in (\delta^\gamma, m_1)$ with $m_1 = 2\delta_1^{\gamma/2} \le \bar{m}$, where $\bar{m}$ is defined in (56) in Section 2, then the problem (77) has a solution $\psi \in C^{2,\alpha}(\bar{\Omega})$ satisfying*

$$|\psi| \le C(\varepsilon_0,\delta),\ |\nabla\psi|^2 \le \Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0 \textit{ for some } \varepsilon_0 > 0. \tag{78}$$

Proof: Note that the equation (77) is uniformly elliptic and the domain is unbounded, one can use a sequence of boundary value problems on bounded domains to approximate it. The key point is to obtain the estimate (78). Therefore, we first solve the following boundary value problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla\psi}{\tilde{H}(|\nabla\psi|^2,\psi)}\right) = \tilde{F}(\psi)\tilde{F}'(\psi)\tilde{H}(|\nabla\psi|^2,\psi) \text{ in } \Omega_L, \\ \psi = \frac{x_2-f_1(x_1)}{f_2(x_1)-f_1(x_1)}m \text{ on } \partial\Omega_L, \end{cases} \tag{79}$$

where Ω_L satisfies $\{(x_1,x_2)|(x_1,x_2)\in\Omega, -L<x_1<L\}\subset\Omega_L\subset\{(x_1,x_2)|(x_1,x_2)\in\Omega, -4L<x_1<4L\}$ for $\forall L\in\mathbb{N}$. Furthermore, one may choose Ω_L so that $\Omega_L\in C^{2,\alpha_1}(0<\alpha_1\leq\alpha)$ satisfies the uniform exterior sphere condition with uniform radius r_0, $0<r_0<r$, for all $L>L_0$ with some L_0 sufficiently large. For the explicit construction of such Ω_L, please refer to Appendix in [31].

The equation in (77) can be rewritten as

$$\tilde{A}_{ij}(D\psi,\psi)\partial_{ij}\psi - \tilde{H}_2(|\nabla\psi|^2,\psi)|\nabla\psi|^2 = \tilde{F}(\psi)\tilde{F}'(\psi)\tilde{H}^3(|\nabla\psi|^2,\psi), \tag{80}$$

where

$$\tilde{H}_2(|\nabla\psi|^2,\psi) = \frac{\tilde{F}(\psi)\tilde{F}'(\psi)\tilde{H}(|\nabla\psi|^2,\psi)(\tilde{H}^2+\Sigma\Sigma'(\zeta_0'-1))}{\tilde{H}^2(|\nabla\psi|^2,\psi)c^2-\tilde{\Delta}(|\nabla\psi|^2,\psi)}.$$

Therefore, (80) becomes

$$\tilde{A}_{ij}(D\psi,\psi)\partial_{ij}\psi = \mathcal{F}(\psi,\nabla\psi), \tag{81}$$

where

$$\mathcal{F}(\psi,\nabla\psi) = \tilde{F}(\psi)\tilde{F}'(\psi)\tilde{H}(|\nabla\psi|^2,\psi)\left(\frac{(\tilde{H}^2+\Sigma\Sigma'(\zeta_0'-1))|\nabla\psi|^2}{\tilde{H}^2(|\nabla\psi|^2,\psi)c^2-\tilde{\Delta}(|\nabla\psi|^2,\psi)}+\tilde{H}^2\right).$$

Note that $\mathcal{F}$ has quadratic growth in $|\nabla\psi|$, so it is not easy to get a prior estimate and the existence for (81) directly. The strategy here is that, instead of (79), we first solve the problem

$$\begin{cases} \tilde{A}_{ij}(D\psi,\psi)\partial_{ij}\psi = \tilde{\mathcal{F}}(\psi,\nabla\psi) \text{ in } \Omega_L, \\ \psi = \frac{x_2-f_1(x_1)}{f_2(x_1)-f_1(x_1)}m \text{ on } \partial\Omega_L, \end{cases} \tag{82}$$

where

$$\tilde{\mathcal{F}}(\psi,\nabla\psi) = \tilde{F}(\psi)\tilde{F}'(\psi)\tilde{H}(|\nabla\psi|^2,\psi)\left(\frac{(\tilde{H}^2+\Sigma\Sigma'(\zeta_0'-1))\tilde{\Delta}(\nabla\psi,\psi)}{\tilde{H}^2(|\nabla\psi|^2,\psi)c^2-\tilde{\Delta}(|\nabla\psi|^2,\psi)}+\tilde{H}^2\right).$$

Thanks to (71), one has

$$|\tilde{\mathcal{F}}| \le C\delta^{1-2\gamma}. \tag{83}$$

It follows from Theorem 12.5 and Remark in P308 in [17] that there exists a solution ψ_L to (82). Furthermore, writing $\psi_L^- = \min\{\psi_L, 0\}$ and $\psi_L^+ = \max\{\psi_L, 0\}$, by the proof of Theorem 3.7 in [17],

$$\inf_{\partial\Omega_L} \psi_L^- - C\sup_{\Omega_L}\left|\frac{\tilde{\mathcal{F}}}{\lambda}\right| \le \psi_L \le \sup_{\partial\Omega_L} \psi_L^+ + C\sup_{\Omega_L}\left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|, \tag{84}$$

where $C = e^d - 1$ with $d = \sup\{f_2(x_1) - f_1(x_1)\}$. Thus,

$$-C\delta^{1-2\gamma} \le \psi_k \le m + C\delta^{1-2\gamma} \text{ for } k \text{ sufficiently large.}$$

Moreover, one can get some nice estimates for ψ_k. This follows from the techniques developed in Chapter 12 in [17]. Using the specific form of estimate (12.14) in P299 in [17] and Remark (4) on global estimate for quasiconformal mappings in P300 in [17], then one can improve the estimate in Line 7 in P304 in [17] to the following more precise form

$$[u]_{1,\alpha} \le C(\gamma, \Omega)\left(1 + |Du|_0 + \left|\frac{f}{\lambda}\right|_0\right), \tag{85}$$

actually, $C(\gamma, \Omega)$ depends only on the diamΩ and C^2 norm of $\partial\Omega$. Here we use notations and symbols in (85) as those in Chapter 12 in [17].

Note that although the estimate (85) is derived with zero boundary conditions, it holds in the case that the boundary value is constant in each connect component of boundary. Indeed, first, it holds for the case that the boundary value is a constant. Then one can generalize the estimate to the case that boundary value is constant in each connected component of the boundary, since all estimates are obtained through localization.

Applying the estimate (85) to the problem (82) shows that, there exists $\mu = \mu(\Lambda/\lambda) > 0$, such that for any $x^0 \in \bar{\Omega}_L$, and for ψ_k with $k \ge 4L$, one has

$$[\psi_k]_{1,\mu;B_1(x^0)\cap\Omega_L} \le C(\Lambda/\lambda, |f_i|_2)\left(1 + |D\psi_k|_{0;B_1(x^0)\cap\Omega_L} + \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right). \tag{86}$$

This, together with interpolation inequality and (84), yields

$$\begin{aligned}\|\psi_k\|_{1;B_1(x^0)\cap\Omega_L} \le& \eta[\psi_k]_{1,\mu;B_1(x^0)\cap\Omega_L} + C_\eta|\psi_k|_0 \\ \le& \eta C(\Lambda/\lambda, |f_i|_2)\left(1 + |D\psi_k|_{0;B_1(x^0)\cap\Omega_L} + \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right) + C_\eta\left(m + C\left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right),\end{aligned}$$

where C, appeared in last term, is the same as that in (84). Taking η_0 sufficiently small so that $\eta C(\Lambda/\lambda, |f_i|_2) \le 1/2$ if $\eta \le \eta_0$, then one has

$$\|\psi_k\|_{1;B_1(x^0)\cap\Omega_L} \le \eta C(\Lambda/\lambda, |f_i|_2)\left(1 + \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right) + C_\eta\left(m + C\left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right). \tag{87}$$

Thus, the Hölder estimate (86) becomes

$$\begin{aligned}\|\psi_k\|_{1,\mu,B_1(x^0)\cap\Omega_L} \le& \|\psi_k\|_{1;B_1(x^0)\cap\Omega_L} + [\psi_k]_{1,\mu;B_1(x^0)\cap\Omega_L} \\ \le& (1 + C(\Lambda/\lambda, |f_i|_2))\|\psi_k\|_{1;B_1(x^0)\cap\Omega_L} + C(\Lambda/\lambda, |f_i|_2)\left(1 + \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right) \\ \le& C(\Lambda/\lambda, |f_i|_2)\left(\eta_0 C(\Lambda/\lambda, |f_i|_2)\left(1 + \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right) + C_{\eta_0}\left(m + C\left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right)\right) \\ &+ C(\Lambda/\lambda, |f_i|_2)\left(1 + \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right) \\ \le& C(\Lambda/\lambda, |f_i|_2))\left(1 + m + \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right).\end{aligned} \tag{88}$$

Note that, for any x, $y \in \bar{\Omega}_L$,

$$\frac{|\nabla\psi_k(x) - \nabla\psi_k(y)|}{|x-y|^\mu} \le \begin{cases} \|\psi_k\|_{1,\mu;B_1(x)\cap\Omega_L}, & \text{if } y \in B_1(x)\cap\bar{\Omega}_L, \\ 2\|\psi_k\|_{1;\Omega_L}, & \text{if } y \notin B_1(x)\cap\bar{\Omega}_L. \end{cases}$$

This, together with (87) and (88), yields the following Hölder estimate

$$[\psi_k]_{1,\mu;\Omega_L} = \sup_{x,y\in\Omega_L}\frac{|\nabla\psi_k(x) - \nabla\psi_k(y)|}{|x-y|^\mu} \le C(\Lambda/\lambda, |f_i|_2)\left(1 + m + \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right). \tag{89}$$

Furthermore, it follows from (88), the Schauder estimate (Theorem 6.2 and Lemma 6.5 in [17]), and the bootstrap argument that

$$\|\psi_k\|_{2,\alpha;B_{1/2}(x^0)\cap\Omega_L} \le C\left(\Lambda/\lambda, |f_i|_{C^{2,\alpha}}, m, \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right).$$

Similar to the argument for (89), one has

$$\|\psi_k\|_{2,\alpha;\Omega_L} \leq C\left(\Lambda/\lambda, |f_i|_{C^{2,\alpha}}, m, \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right). \tag{90}$$

Hence, using Arzela-Ascoli lemma and a diagonal procedure, we see that there exists a sequence ψ_{k_l} such that

$$\psi_{k_l} \to \psi \text{ in } C^{2,\beta}(K) \text{ for any compact set } K \subset \bar{\Omega} \text{ and } \beta < \alpha.$$

Furthermore, ψ satisfies the problem

$$\begin{cases} \tilde{A}_{ij}(D\psi, \psi)\partial_{ij}\psi = \tilde{\mathcal{F}}(\nabla\psi, \psi) \text{ in } \Omega, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)} m \text{ on } \partial\Omega, \end{cases}$$

and the estimate

$$\|\psi\|_{1,\Omega} \leq \eta C(\gamma, |f_i|_2)\left(1 + \left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right) + C_\eta\left(m + C\left|\frac{\tilde{\mathcal{F}}}{\lambda}\right|_0\right),$$

where $\eta \in (0, \eta_0)$. Thanks to estimate (83), one has

$$\|\psi\|_{1,\Omega} \leq \eta C(\gamma, |f_i|_2)(1 + C\delta^{1-2\gamma}) + C_\eta(m + C\delta^{1-2\gamma}), \tag{91}$$

where C depends only on $\bar{\delta}_0$, $\bar{m}$, Λ and λ.

Obviously, there exist $\eta_1 \in (0, \eta_0)$ and $\delta_1 \in (0, \bar{\delta}_0]$ such that

$$\eta_1 C(\gamma, |f_i|_2)(1 + C\bar{\delta}_0^{1-2\gamma}) \leq \sqrt{(\Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0)/2},$$

$$C_{\eta_1}(2\delta_1^{\gamma/2} + C\delta_1^{1-2\gamma}) \leq \sqrt{(\Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0)/2}.$$

Therefore, for any $\delta \in (0, \delta_1)$ and $m \in (\delta^\gamma, 2\delta_1^{\gamma/2})$, the solution ψ satisfies

$$|\nabla\psi|^2 \leq \Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0. \tag{92}$$

Now (78) follows from (91) and (92).

Furthermore, (89) and (90) yield the following higher order estimates

$$\|\psi\|_{1,\mu;\bar{\Omega}} \leq C(\Lambda/\lambda, |f_i|_2)\left(1 + m + \left|\frac{\mathcal{F}}{\lambda}\right|_0\right), \tag{93}$$

$$\|\psi\|_{2,\alpha;\bar{\Omega}} \leq C\left(\Lambda/\lambda, |f_i|_{C^{2,\alpha}}, m, \left|\frac{\mathcal{F}}{\lambda}\right|_0\right). \tag{94}$$

This finishes the proof of the Proposition. □

4 Far Fields Behavior, Existence and Uniqueness of Boundary Value Problem for the Stream Function

In this section, we will study far fields behavior of the solution to (77). It will be shown that the flows induced by the solutions to (77) satisfy asymptotic behavior (17)-(18). This also yields that solutions to (77) satisfy (61). Combining (61) and (92), we can remove both extension and truncation appeared in (77). Therefore, these solutions solve problem (67). Furthermore, the asymptotic behavior is crucial for our formulation for the problem since the stream function formulation is consistent with the original formulation of the problem for the Euler system in the infinitely long nozzle, as long as the flow induced by a solution to (67) satisfies (17)-(18) and (28). Finally, the uniqueness of the solutions will be a consequence of the asymptotic behavior. To study the solution in its far fields, we will use a blow up argument and an energy estimate.

For $x_1 \leq n$, define $\psi^{(n)}(x_1, x_2) = \psi(x_1 - n, x_2)\chi_{\{f_1(x_1-n)<x_2<f_2(x_1-n)\}}$. For any compact set $K \Subset (-\infty, \infty) \times (0, 1)$, it follows from (94) that

$$\|\psi^{(n)}\|_{C^{2,\alpha}(K)} \leq C \text{ for } n \text{ sufficiently large.}$$

Therefore, by Arzela-Ascoli lemma and a diagonal procedure, there exists a subsequence, $\psi^{(n_k)}$, such that

$$\psi^{(n_k)} \to \bar{\psi} \text{ in } C^{2,\beta}(K) \tag{95}$$

for any $K \Subset (-\infty, \infty) \times (0, 1)$, for any $\beta \in (0, \alpha)$. Furthermore, it follows from (4)-(7) and (94) that $\bar{\psi} = 0$ on $x_2 = 0$ and $\bar{\psi} = m$ on $x_2 = 1$. So, $\bar{\psi}$ satisfies

$$\begin{cases} \text{div}\left(\frac{\nabla\bar{\psi}}{\tilde{H}(|\nabla\bar{\psi}|^2,\bar{\psi})}\right) = \tilde{F}'(\bar{\psi})\tilde{F}(\bar{\psi})\tilde{H}(|\nabla\bar{\psi}|^2, \bar{\psi}) \text{ in } D, \\ \bar{\psi} = 0 \text{ on } x_2 = 0, \ \bar{\psi} = m \text{ on } x_2 = 1, \end{cases} \tag{96}$$

where $D = (-\infty, \infty) \times (0, 1)$. Moreover, by (78), one has

$$|\bar{\psi}| \leq C(\varepsilon_0, \delta) \text{ and } |\nabla\bar{\psi}|^2 \leq \Sigma^2(\underline{B} - \varepsilon_0) - 2\varepsilon_0. \tag{97}$$

Thus, by the similar argument in Section 3, on any compact set $E \subset (-\infty, \infty) \times [0, 1]$,

$$\|\bar{\psi}\|_{C^{1,\mu}(E)} \leq C(\varepsilon, \delta).$$

Moreover, it follows from the Schauder estimate for second order uniformly elliptic equations that

$$\|\bar{\psi}\|_{C^{2,\alpha}(E)} \leq C(\varepsilon, \delta). \tag{98}$$

Therefore, $\bar{\psi} \in C^{2,\alpha}(\bar{D})$. In fact, we have the following stronger results

Lemma 5 *There exists $\delta_2 \in (0, \bar{\delta}_0]$ such that if*

(i). $\|B'\|_{C^{0,1}([0,1])} = \delta \leq \delta_2$,

(ii). $m \in (\delta^\gamma, \bar{m})$, *where $\bar{m}$ is defined in (56) in Section 2,*

(iii). there exists $\epsilon \leq \varepsilon_0$ such that $\bar{\psi}$ satisfies

$$|\bar{\psi}| \leq C(\epsilon, \delta) \text{ and } |\nabla\bar{\psi}|^2 - \Sigma^2(\tilde{\mathcal{B}}(\bar{\psi})) \leq -\epsilon, \tag{99}$$

and solves the problem (96), where $\tilde{\mathcal{B}}$ is defined in Section 3,

then $\bar{\psi}$ is independent of x_1, moreover,

$$\bar{\psi}(x_1, x_2) = \bar{\psi}(x_2) = \int_0^{x_2} \rho_0 u_0(s) ds, \tag{100}$$

where ρ_0 and u_0 are uniquely determined by B and m as in Section 2.

Proof: The proof is divided into two steps. First, it will be shown that $\bar{\psi}$ is independent of x_1. Then we will prove that $\bar{\psi}$ is of explicit form (100).

Step 1. Set $w = \bar{\psi}_{x_1}$. Differentiating the equation in (96) with respect to x_1 yields

$$\begin{aligned} &\partial_i \left(\frac{\tilde{A}_{ij}(D\bar{\psi},\bar{\psi})}{\tilde{H}^2(|\nabla\bar{\psi}|^2,\bar{\psi})} \partial_j w \right) - \partial_i \left(\frac{\tilde{H}_2(|\nabla\bar{\psi}|^2,\bar{\psi})\partial_i\bar{\psi}}{\tilde{H}^2(|\nabla\bar{\psi}|^2,\bar{\psi})} w \right) \\ &= \tilde{\Theta}(|\nabla\bar{\psi}|^2, \bar{\psi}) w + \tilde{\vartheta}(|\nabla\bar{\psi}|^2, \bar{\psi}) \partial_i\bar{\psi}\partial_i w, \end{aligned} \tag{101}$$

where $\tilde{A}_{ij}$, $\tilde{\Theta}$ and $\tilde{\vartheta}$ are defined as

$$\tilde{A}_{ij}(q, z) = \tilde{H}(|q|^2, z)\delta_{ij} - 2\tilde{H}_1(|q|^2, z) q_i q_j, \tag{102}$$

$$\tilde{\Theta}(s, z) = (\tilde{F}''(z)\tilde{F}(z) + (\tilde{F}'(z))^2)\tilde{H}(s, z) + \tilde{F}'(z)\tilde{F}(z)\tilde{H}_2(s, z), \tag{103}$$

$$\tilde{\vartheta}(s, z) = 2\tilde{F}(z)\tilde{F}'(z)\tilde{H}_1(s, z), \tag{104}$$

for any $q \in \mathbb{R}^2$, $s \geq 0$, and $z \in \mathbb{R}$, here $\tilde{F}'' \in L^\infty(\mathbb{R}^1)$ since $\|\tilde{F}'\|_{C^{0,1}(\mathbb{R}^1)} \leq C\delta^{1-3\gamma}$. It follows from (99) that there exists a constant Λ depending only on ϵ such that

$$|\tilde{A}_{ij}(D\bar{\psi}, \bar{\psi})| \leq \Lambda(\epsilon).$$

Although it is unknown whether $\bar{\psi} \in C^3(D)$, the equation (101) holds in weak sense. Moreover, w satisfies the boundary conditions

$$w = 0 \text{ on } x_2 = 0, 1.$$

Let η be a C_0^∞ function satisfying

$$\eta = 1 \text{ for } |s| < l, \ \ \eta = 0 \text{ for } |s| > l + 1, \ \text{ and } |\eta'(s)| \leq 2. \tag{105}$$

Now multiplying $\eta^2(x_1)w$ on both sides of (101) and integrating it over D yield

$$\begin{aligned} &\iint_D \frac{\tilde{A}_{ij}(D\bar{\psi}, \bar{\psi})}{\tilde{H}^2(|\nabla\bar{\psi}|^2, \bar{\psi})}\partial_j w \partial_i(\eta^2 w) - \frac{\tilde{H}_2(|\nabla\bar{\psi}|^2, \bar{\psi})\partial_i\bar{\psi}}{\tilde{H}^2(|\nabla\bar{\psi}|^2, \bar{\psi})} w\partial_i(\eta^2 w) dx_1 dx_2 \\ = \ & -\iint_D \tilde{\Theta}(|\nabla\bar{\psi}|^2, \bar{\psi})\eta^2 w^2 dx_1 dx_2 + \tilde{\vartheta}(|\nabla\bar{\psi}|^2, \bar{\psi})\partial_i\bar{\psi}\partial_i w \eta^2 w dx_1 dx_2. \end{aligned}$$

Substituting the explicit forms of A_{ij}, $\tilde{H}_1(|\nabla\bar{\psi}|^2, \bar{\psi})$ and $\tilde{H}_2(|\nabla\bar{\psi}|^2, \bar{\psi})$ into the above equality and noting that $\bar{\psi}$ satisfies (99), one may get

$$
\begin{aligned}
& \iint_D \frac{\eta^2|\nabla w|^2}{\tilde{H}(|\nabla\bar{\psi}|^2,\bar{\psi})} dx_1 dx_2 \\
= \quad & \iint_D \frac{2\tilde{H}_1(|\nabla\bar{\psi}|^2,\bar{\psi})}{\tilde{H}^2(|\nabla\bar{\psi}|^2,\bar{\psi})} |\nabla\psi\cdot\nabla w|^2\eta^2 dx_1 dx_2 \\
& - \iint_D 2\frac{\tilde{A}_{ij}(D\bar{\psi},\bar{\psi})}{\tilde{H}^2(|\nabla\bar{\psi}|^2,\bar{\psi})} \partial_j w \partial_i \eta \eta w dx_1 dx_2 \\
& + \iint_D \frac{\tilde{H}_2(|\nabla\bar{\psi}|^2,\bar{\psi})\partial_i\bar{\psi}}{\tilde{H}^2(|\nabla\bar{\psi}|^2,\bar{\psi})} (\eta^2 w \partial_i w + 2\eta\partial_i\eta w^2) dx_1 dx_2 \\
& - \iint_D (\tilde{F}''(\bar{\psi})\tilde{F}(\bar{\psi}) + (\tilde{F}'(\bar{\psi}))^2)\tilde{H}(|\nabla\bar{\psi}|^2,\bar{\psi})\eta^2 w^2 dx_1 dx_2 \\
& - \iint_D \tilde{F}'(\bar{\psi})\tilde{F}(\bar{\psi})\tilde{H}_2(|\nabla\bar{\psi}|^2,\bar{\psi})\eta^2 w^2 dx_1 dx_2 \\
& - \iint_D 2\tilde{F}'(\bar{\psi})\tilde{F}(\bar{\psi})\tilde{H}_1(|\nabla\bar{\psi}|^2,\bar{\psi})\eta^2 w \nabla\bar{\psi}\cdot\nabla w dx_1 dx_2 \\
= \quad & - \iint_D \frac{|\nabla\bar{\psi}\cdot\nabla w|^2\eta^2}{\tilde{H}(|\nabla\bar{\psi}|^2,\bar{\psi})(\tilde{H}^2(|\nabla\bar{\psi}|^2,\bar{\psi})c^2 - |\nabla\bar{\psi}|^2)} dx_1 dx_2 \\
& - \iint_D 2\frac{\tilde{A}_{ij}(D\bar{\psi},\bar{\psi})}{\tilde{H}^2(|\nabla\bar{\psi}|^2,\bar{\psi})} \partial_j w \partial_i \eta \eta w dx_1 dx_2 \\
& + \iint_D \frac{2\tilde{H}_2(|\nabla\bar{\psi}|^2,\bar{\psi})\nabla\bar{\psi}\cdot\nabla\eta}{\tilde{H}^2(|\nabla\bar{\psi}|^2,\bar{\psi})} \eta w^2 dx_1 dx_2 \\
& + \iint_D \frac{2\tilde{F}(\bar{\psi})\tilde{F}'(\bar{\psi})\tilde{H}\nabla\bar{\psi}\cdot\nabla w}{\tilde{H}^2(|\nabla\bar{\psi}|^2,\bar{\psi})c^2 - |\nabla\bar{\psi}|^2} \eta^2 w dx_1 dx_2 \\
& - \iint_D \left(\tilde{F}''(\bar{\psi})\tilde{F}(\bar{\psi}) + (\tilde{F}'(\bar{\psi}))^2\right) \tilde{H}(|\nabla\bar{\psi}|^2,\bar{\psi})\eta^2 w^2 dx_1 dx_2 \\
& - \iint_D \frac{(\tilde{F}(\psi)\tilde{F}'(\psi))^2\tilde{H}^3(|\nabla\bar{\psi}|^2,\bar{\psi})}{\tilde{H}^2(|\nabla\bar{\psi}|^2,\bar{\psi})c^2 - |\nabla\bar{\psi}|^2} \eta^2 w^2 dx_1 dx_2,
\end{aligned}
$$

which can be written as

$$\iint_D \frac{\eta^2|\nabla w|^2}{\tilde{H}(|\nabla\bar{\psi}|^2,\bar{\psi})} dx_1 dx_2 = \sum_{i=1}^{6} I_i. \tag{106}$$

First, it is easy to see that $I_1 + I_4 + I_6 \le 0$. Second, due to (71), one has

$$|I_5| \le C\delta^{1-3\gamma} \int_{-l-1}^{l+1}\int_0^1 w^2 dx_1 dx_2. \tag{107}$$

Finally, since $\tilde{H} \leq \bar{\varrho}(\bar{B})$, thus if δ_2 is sufficiently small, one gets from above that

$$\begin{aligned}
&\int_{-l}^{l} dx_1 \int_0^1 |\nabla w|^2 dx_2 \\
\leq\ & C(\bar{B},\epsilon)\left(\int_{-l-1}^{-l} dx_1 + \int_{l}^{l+1} dx_1\right)\int_0^1 |\nabla w|^2 + |\nabla w w| + w^2 dx_2 \\
&+C(\bar{B})\delta^{1-3\gamma}\int_{-l}^{l}\int_0^1 w^2 dx_1 dx_2 \\
\leq\ & C(\bar{B},\epsilon)\left(\int_{-l-1}^{-l} dx_1 + \int_{l}^{l+1} dx_1\right)\int_0^1 |\nabla w|^2 + w^2 dx_2 + C(\bar{B})\delta^{1-3\gamma}\int_{-l}^{l}\int_0^1 w^2 dx_1 dx_2.
\end{aligned}$$

Notice that $w = 0$ on $x_2 = 0$. It follows from Poincare inequality that

$$\int_0^1 w^2 dx_2 \leq \int_0^1 |\nabla w|^2 dx_2. \tag{108}$$

Therefore, there exists a constant C independent of l such that

$$\int_{-l}^{l}\int_0^1 |\nabla w|^2 dx_1 dx_2 \leq C\left(\int_{-l-1}^{-l} dx_1 + \int_{l}^{l+1} dx_1\right)\int_0^1 |\nabla w|^2 dx_2 \tag{109}$$

for large l. It follows from (98) that

$$\left(\int_{-l-1}^{-l} dx_1 + \int_{l}^{l+1} dx_1\right)\int_0^1 |\nabla w|^2 dx_2 \leq C$$

for some uniform constant C independent of l. Therefore,

$$\int_{-l}^{l} dx_1 \int_0^1 |\nabla w|^2 dx_2 \leq C$$

for some constant C. Taking $l \to \infty$ yields

$$\int_{-\infty}^{\infty} dx_1 \int_0^1 |\nabla w|^2 dx_2 \leq C.$$

Hence

$$\left(\int_{-l-1}^{-l} dx_1 + \int_{l}^{l+1} dx_1\right)\int_0^1 |\nabla w|^2 dx_2 \to 0 \text{ as } l \to \infty. \tag{110}$$

Taking the limit $l \to \infty$ in (109), one has

$$\int_{-\infty}^{\infty}\int_0^1 |\nabla w|^2 dx_1 dx_2 = 0.$$

So

$$w = 0.$$

Therefore, $\bar{\psi} = \bar{\psi}(x_2)$. Thus $\bar{\psi}$ solves the following boundary value problem

$$\begin{cases} \frac{d}{dx_2}\left(\frac{\bar{\psi}'}{\tilde{H}((\bar{\psi}')^2,\bar{\psi})}\right) = \tilde{F}'(\psi)\tilde{F}(\psi)\tilde{H}((\bar{\psi}')^2,\bar{\psi}), \\ \bar{\psi}(0) = 0, \ \bar{\psi}(1) = m. \end{cases} \tag{111}$$

Step 2. Uniqueness of the solution to the boundary value problem (111).

Suppose that there are two solutions $\bar{\psi}_1$ and $\bar{\psi}_2$ to (111). Let $\bar{\phi} = \bar{\psi}_1 - \bar{\psi}_2$. Then $\bar{\phi}$ satisfies

$$\begin{cases} (\bar{a}\bar{\phi}' + \bar{b}\bar{\phi})' = \bar{c}\bar{\phi}' + \bar{d}\bar{\phi}, \\ \bar{\phi}(0) = \bar{\phi}(1) = 0, \end{cases} \tag{112}$$

where

$$\begin{aligned} \bar{a} &= \int_0^1 \frac{\tilde{H}(|\tilde{\psi}|^2,\tilde{\psi}) - 2\tilde{H}_1(|\tilde{\psi}'|^2,\tilde{\psi})|\tilde{\psi}'|^2}{\tilde{H}^2(|\tilde{\psi}'|,\tilde{\psi})}ds, \ \bar{b} = \int_0^1 \frac{-\tilde{H}_2(|\tilde{\psi}'|^2,\tilde{\psi})\tilde{\psi}'}{\tilde{H}^2(|\tilde{\psi}'|,\tilde{\psi})}ds, \\ \bar{c} &= \int_0^1 \tilde{\vartheta}(|\tilde{\psi}'|^2,\tilde{\psi}')\tilde{\psi}'ds, \ \bar{d} = \int_0^1 \tilde{\Theta}(|\tilde{\psi}'|^2,\tilde{\psi}')ds, \end{aligned}$$

with $\tilde{\psi} = s\bar{\psi}_1 + (1-s)\bar{\psi}_2$, where $\tilde{\Theta}$ and $\tilde{\vartheta}$ are defined in (103) and (104) respectively. Multiplying $\bar{\phi}$ on both sides of the equation in (112), and integrating it over $[0,1]$, we have

$$\int_0^1 \frac{|\bar{\phi}'|^2}{\tilde{H}(|\tilde{\psi}'|^2,\tilde{\psi})}dx_2 \le -\int_0^1 \left((\tilde{F}'(\tilde{\psi}))^2 + \tilde{F}(\tilde{\psi})\tilde{F}''(\tilde{\psi})\right)\tilde{H}(|\tilde{\psi}'|^2,\tilde{\psi})\bar{\phi}^2 dx_2.$$

Note that $\|\tilde{F}'\|_{C^{0,1}(\mathbb{R}^1)} \le C\delta^{1-3\gamma}$, thanks to the smallness of δ and the Poincare inequality, one has

$$\int_0^1 |\bar{\phi}'|^2 \le 0.$$

Therefore, $\bar{\phi} = 0$. So the solution to (111) is unique. On the other hand, by the definition of $\tilde{H}$ and $\tilde{F}$, one knows that the boundary value problem (111) has a solution

$$\bar{\psi} = \bar{\psi}(x_2) = \int_0^{x_2} \rho_0 u_0(s)ds.$$

This finishes the proof of the Lemma. □

It follows from Lemma 5 and (95) that the flow induced by the stream function satisfies (17) and (18).

The asymptotic behavior in the downstream can be obtained by a similar argument.

An important direct consequence of this far fields behavior is a better maximum estimate for the stream function.

Proposition 6 *If* $\|B'\|_{C^{0,1}([0,1])} = \delta \leq \min\{\delta_1, \delta_2\}$, $B'(0) \leq 0$, $B'(1) \geq 0$ *and* $m \in (\delta^\gamma, \bar{m})$, *then the solution established in Proposition 4 satisfies (61).*

Proof: It follows from Proposition 5 that

$$\psi(x_1, x_2) \to \int_0^{x_2} \rho_0 u_0(s) ds \text{ uniformly as } x_1 \to -\infty,$$
$$\psi(x_1, x_2) \to \int_a^{x_2} \rho_1 u_1(s) ds \text{ uniformly as } x_1 \to +\infty.$$

Therefore, for any $\epsilon > 0$, there exists $L > 0$ such that

$$-\epsilon \leq \psi(x_1, x_2) < m + \epsilon \text{ if } |x_1| \geq L. \tag{113}$$

Note that $\tilde{F}'(\psi) \geq 0$ in the domain $\{\psi \geq m\}$, thus

$$\tilde{A}_{ij}(D\psi, \psi)\partial_{ij}\psi \geq 0, \text{ in the domain } \{\psi \geq m\} \cap \{|x_1| \leq L\},$$

where $\tilde{A}_{ij}$ is defined in (76). By maximum principle, one has

$$-\epsilon \leq \psi(x_1, x_2) \leq m + \epsilon \text{ in } \{\psi \geq m\} \cap \{|x_1| \leq L\}.$$

Since $\tilde{F}'(\psi) \leq 0$ in the domain $\{\psi \leq 0\}$, thus, similarly, one can show that

$$-\epsilon \leq \psi(x_1, x_2) \leq m + \epsilon \text{ in } \{\psi \leq 0\} \cap \{|x_1| \leq L\}.$$

Combining these estimates with (113), it yields

$$-\epsilon \leq \psi(x_1, x_2) \leq m + \epsilon \text{ in } \Omega.$$

Since ϵ is arbitrary, one has

$$0 \leq \psi(x_1, x_2) \leq m \text{ in } \Omega.$$

This finishes the proof of the Proposition. □

It follows from estimates (61) and (78) that the solutions established in Proposition 4 are solutions to (67) when the assumptions of Proposition 6 are satisfied.

In fact, one can also use energy estimates to show that uniformly subsonic solution to (67) is unique.

Proposition 7 *Let the boundary of Ω satisfy (4)-(7). Then there exists $\delta_3 \in (0, \bar{\delta}_0]$ such that if*

(i). $\|B'\|_{C^{0,1}([0,1])} = \delta \leq \delta_3$,

(ii). $m \in (\delta^\gamma, \bar{m})$,

then there exists at most one solution ψ to (67) satisfying

$$0 \leq \psi \leq m,\ |\nabla\psi|^2 - \Sigma^2(\mathcal{B}(\psi)) \leq -\epsilon \text{ for some } \epsilon > 0, \tag{114}$$

where H and F are defined by B and m as in Section 2, and $\mathcal{B}(\psi) = h(\rho_0) + \frac{F^2(\psi)}{2}$,

Proof: Let ψ_1 and ψ_2 be two solutions to (67). Set $\psi = \psi_1 - \psi_2$. Then ψ satisfies

$$\begin{cases} \partial_i(a_{ij}\partial_j\psi) + \partial_i(b_i\psi) = c_i\partial_i\psi + d\psi, \\ \psi = 0 \text{ on } S_1 \bigcup S_2, \end{cases} \tag{115}$$

where

$$\begin{aligned} a_{ij} &= \int_0^1 \frac{A_{ij}(D\tilde{\psi}, \tilde{\psi})}{H^2(|\nabla\tilde{\psi}|, \tilde{\psi})} ds,\ b_i = \int_0^1 \frac{-H_2(|\nabla\tilde{\psi}|^2, \tilde{\psi})\partial_i\tilde{\psi}}{H^2(|\nabla\tilde{\psi}|, \tilde{\psi})} ds, \\ c_i &= \int_0^1 \vartheta(|\nabla\tilde{\psi}|^2, \tilde{\psi})\partial_i\tilde{\psi} ds,\ d = \int_0^1 \Theta(|\nabla\tilde{\psi}|^2, \tilde{\psi}) ds, \end{aligned}$$

here $\tilde{\psi} = s\psi_1 + (1-s)\psi_2$, A_{ij}, Θ and ϑ are defined similar to (102), (103) and (104), respectively except we replace $\tilde{F}$ and $\tilde{H}$ by F and H.

Multiplying $\eta^2\psi^+$ on both sides of equation in (115), where η is defined in (105) and $\psi^+(x) = \max\{\psi(x), 0\}$, then similar to the proof of Lemma 5, one has

$$\iint_{\Omega\cap\{|x_1|\leq l\}\cap\{\psi\geq 0\}} |\nabla\psi|^2 dx_1 dx_2 \leq C(\underline{B}, \epsilon) \iint_{\Omega\cap\{l\leq|x_1|\leq l+1\}\cap\{\psi\geq 0\}} |\nabla\psi|^2 dx_1 dx_2.$$

It follows from Lemma 5 that ψ_1 and ψ_2 have the same far fields behavior, thus $|\psi|$ and $|\nabla\psi| \to 0$ as $|x_1| \to \infty$. Thus

$$\iint_{\Omega\cap\{\psi\geq 0\}} |\nabla\psi|^2 = 0,$$

so $\psi \leq 0$. Similarly, one can show that $\psi \geq 0$. Therefore, $\psi = 0$. This finishes the proof of the Proposition. □

5 Refined Estimates for the Boundary Value Problem for Stream Functions

In this section, we will derive some refined estimates for solutions to the problem (67). Combining these refined estimates with the estimates obtained in section 3 and section 4, one will get a solution to the Euler equations (1)-(3), with the boundary condition (8) and the constrains (9) and (13). More precisely, it will be shown that ψ_{x_2} is always positive, therefore, $u = \psi_{x_2}/\rho = \psi_{x_2}/H(|\nabla\psi|^2, \psi)$ satisfies (15). This positivity of the horizontal velocity and the asymptotic behavior yield that (ρ, u, v) induced by ψ satisfies the original Euler equations, the boundary conditions and the constrains on mass flux and Bernoulli's constant.

Lemma 8 *Let the boundary of Ω satisfy (4)-(7). Then there exists $\delta_4 \in (0, \bar{\delta}_0]$ such that if*

(i). $\|B'\|_{C^{0,1}([0,1])} = \delta \leq \delta_4$,

(ii). $m \in (\delta^\gamma, \bar{m})$,

(iii). ψ satisfies (114) and solves (67) with H and F defined by B and m as in Section 2,

then ψ satisfies

$$0 < \psi < m \ in\ \Omega, \tag{116}$$

and

$$\psi_{x_2} > 0 \ in\ \bar{\Omega}. \tag{117}$$

Proof: It follows from (114) that ψ achieves its minimum on S_1 and maximum on S_2, hence

$$\psi_{x_2} \geq 0 \text{ on } \partial\Omega. \tag{118}$$

On the other hand, $U = \psi_{x_2}$ satisfies

$$\begin{aligned} &\partial_i \left(\tfrac{A_{ij}(D\psi,\psi)}{H^2(|\nabla\psi|^2,\psi)} \partial_j U \right) - \partial_i \left(\tfrac{H_2(|\nabla\psi|^2,\psi)\partial_i\psi}{H^2(|\nabla\psi|^2,\psi)} U \right) \\ &= \Theta(|\nabla\psi|^2,\psi)U + \vartheta(|\nabla\psi|^2,\psi)\partial_i\psi\partial_i U \end{aligned} \tag{119}$$

in the weak sense, where A_{ij}, Θ and ϑ are defined similar to (102), (103) and (104) respectively except we replace $\tilde{F}$ and $\tilde{H}$ by F and H. We first claim that

$$U \geq 0 \text{ in } \Omega. \tag{120}$$

Indeed, it follows from Lemma 5 that $U(x_1, x_2) > 0$ when $|x_1| > l$ for some l sufficiently large. Multiplying (119) by $U^- = \min\{U, 0\}$, and using (118), one may get that

$$\begin{aligned} &\iint_{\{U\leq 0\}} \frac{|\nabla U|^2}{H(|\nabla\psi|^2,\psi)} dx_1 dx_2 \\ = \ &\iint_{\{U\leq 0\}} \frac{2H_1(|\nabla\psi|^2,\psi)}{H^2(|\nabla\psi|^2,\psi)} |\nabla\psi\cdot\nabla U|^2 dx_1 dx_2 \\ &+ \iint_{\{U\leq 0\}} \frac{H_2(|\nabla\psi|^2,\psi)\partial_i\psi}{H^2(|\nabla\psi|^2,\psi)} U\partial_i U dx_1 dx_2 \\ &- \iint_{\{U\leq 0\}} \Theta(|\nabla\psi|^2,\psi)U^2 dx_1 dx_2 - \iint_D \vartheta(|\nabla\psi|^2,\psi) U\nabla\psi\cdot\nabla U dx_1 dx_2 \\ \leq \ & -\iint_{\{U\leq 0\}} (F''(\psi)F(\psi) + (F'(\psi))^2) H(|\nabla\psi|^2,\psi) U^2 dx_1 dx_2 \\ \leq \ & C\delta^{1-3\gamma} \iint_{\{U<0\}} U^2 dx_1 dx_2. \end{aligned}$$

Define $K_{x_1} = \{x_2 | f_1(x_1) \leq x_2 \leq f_2(x_1), U(x_1,x_2) < 0\}$, then K_{x_1} is an open set for each x_1. Let $K_{x_1} = \cup_{i\in\mathscr{S}} I^i_{x_1}$, where $I^i_{x_1}$ are connected components of K_{x_1}. For each $x_2 \in I^i_{x_1}$,

$$U(x_1,x_2) = \int_{\min I^i_{x_1}}^{x_2} U(x_1,s) ds.$$

Therefore,

$$
\begin{aligned}
&\iint_{\{U<0\}} U^2(x_1,x_2)dx_1dx_2 \\
=&\int_{-l}^{l} dx_1 \Sigma_{i\in\mathscr{S}} \int_{I^i_{x_1}} U^2(x_1,x_2)dx_2 \\
=&\int_{-l}^{l} dx_1 \Sigma_{i\in\mathscr{S}} \int_{I^i_{x_1}} \left(\int_{\min I^i_{x_1}}^{x_2} \partial_{x_2}U(x_1,s)ds\right)^2 dx_2 \\
\le&\int_{-l}^{l} dx_1 \Sigma_{i\in\mathscr{S}} \int_{I^i_{x_1}} \int_{\min I^i_{x_1}}^{\max I^i_{x_1}} (\partial_{x_2}U(x_1,s))^2 ds (\max I^i_{x_1} - \min I^i_{x_1}) dx_2 \\
=&\int_{-l}^{l} dx_1 \Sigma_{i\in\mathscr{S}} (\max I^i_{x_1} - \min I^i_{x_1})^2 \int_{\min I^i_{x_1}}^{\max I^i_{x_1}} (\partial_{x_2}U(x_1,s))^2 ds \\
\le&\max_{x_1\in\mathbb{R}} |f_2(x_1)-f_1(x_1)|^2 \int_{-l}^{l} dx_1 \Sigma_{i\in\mathscr{S}} \int_{\min I^i_{x_1}}^{\max I^i_{x_1}} (\partial_{x_2}U(x_1,s))^2 ds \\
\le&\max_{x_1\in\mathbb{R}} |f_2(x_1)-f_1(x_1)|^2 \iint_{\{U<0\}} |\nabla U|^2 dx_1dx_2
\end{aligned}
$$

Hence,

$$\iint_{\{U\le 0\}} \frac{|\nabla U|^2}{H(|\nabla\psi|^2,\psi)} dx_1dx_2 \le C\delta^{1-3\gamma} \iint_{\{U\le 0\}} |\nabla U|^2 dx_1dx_2,$$

which implies

$$\iint_{\{U\le 0\}} |\nabla U|^2 dx_1dx_2 \le 0,$$

so (120) must hold.

Now, we use an argument similar to the proof of Lemma 1 in §9.5.2 in [11] to show that

$$\psi_{x_2} = U > 0 \text{ in } \Omega \tag{121}$$

holds for any weak solutions U to (119).

Indeed, let $\tilde{U} = e^{-\sigma x_2}U$. Then $\tilde{U}$ is a nonnegative weak solution to

$$\partial_i\left(\frac{A_{ij}}{H^2}e^{\sigma x_2}\partial_j\tilde{U}\right) + \left(\frac{A_{i2}}{H^2}\sigma - \frac{H_2\partial_i\psi}{H^2} - \vartheta(|\nabla\psi|^2,\psi)\partial_i\psi\right)e^{\sigma x_2}\partial_i\tilde{U} + Ge^{\sigma x_2}\tilde{U} = 0,$$

where A_{ij} and ϑ are defined in (102) and (104), and

$$G = \frac{A_{22}}{H^2}\sigma^2 + \left(\partial_i\left(\frac{A_{i2}}{H^2}\right) - \frac{H_2\partial_2\psi}{H^2} - \vartheta(D\psi,\psi)\partial_2\psi\right)\sigma - \partial_i\left(\frac{H_2\partial_i\psi}{H^2}\right) - \Theta(|\nabla\psi|^2,\psi)$$

with Θ defined in (103). Choose $\sigma > 0$ sufficiently large. Then $G > 0$. Thus

$$\partial_i \left(\frac{A_{ij}}{H^2} e^{\sigma x_2} \partial_j \tilde{U} \right) + \left(\frac{A_{i2}}{H^2} \sigma - \frac{H_2 \partial_i \psi}{H^2} - \vartheta(|\nabla \psi|^2, \psi) \partial_i \psi \right) e^{\sigma x_2} \partial_i \tilde{U} \leq 0.$$

It follows from Theorem 8.19 in [17] that (121) holds.

Now, (116) follows directly from (114) and (121).

Since $\psi = m$ on S_2, if $F'(m) > 0$, then for any $(x_1^0, f_2(x_1^0)) \in S_2$, there exists a small disk $\mathcal{N} \subset \Omega$ satisfying $\bar{\mathcal{N}} \cap \bar{\Omega} = (x_1^0, f_2(x_1^0))$ such that $F'(\psi) \geq 0$ in $\mathcal{N}$, therefore,

$$A_{ij}(D\psi, \psi) \partial_{ij} \psi \geq 0 \text{ in } \mathcal{N}.$$

Moreover, by (116), $\psi < m$ in $\mathcal{N}$. Thus, by the Hopf Lemma, one has $\psi_{x_2}(x_1^0, f_2(x_1^0)) > 0$.

In the case $F'(m) = 0$, ψ satisfies

$$A_{ij}(D\psi, \psi) \partial_{ij}(\psi - m) + R(\psi - m) = 0,$$

where

$$R = -\frac{F(\psi) H^5 c^2}{H^2(|\nabla \psi|^2, \psi) c^2 - |\nabla \psi|^2} \frac{F'(\psi) - F'(m)}{\psi - m}.$$

It follows from the Hopf lemma that

$$\partial_{x_2} \psi > 0 \text{ in } S_2.$$

Similarly, one can show that $\psi_{x_2}(x_1, f_1(x_1)) > 0$ for any $x_1 \in \mathbb{R}$.

This finishes the proof of the Lemma. □

Choose $\delta_0 = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$, then $\delta_0 > 0$. If $\|B'\|_{C^{0,1}([0,1])} = \delta \leq \delta_0$, for any $m \in (\delta^\gamma, 2\delta_0^{\gamma/2})$, there exists a solution to the problem (67). It follows from Lemma 5 and Lemma 8 that the flow field induced by ψ satisfies (28) and (29), hence Proposition 2 guarantees the existence of Euler flows. Furthermore, Proposition 2 and Proposition 7 imply uniqueness of Euler flows with asymptotic condition (13), mass flux condition (9), (15), and asymptotic behavior (17)-(18).

6 Existence of Critical Mass Flux

So far, we have shown that, for the given Bernoulli's function in the upstream satisfying (14), there exist Euler flows as long as $m \in (\delta^\gamma, 2\delta_0^{\gamma/2})$. In this section, we will increase m as large as possible.

Proposition 9 *Let Ω satisfy (4)-(7) and B satisfy (14) and (21). Then there exists $\hat{m} \le \bar{m}$ such that if $m \in (\delta^\gamma, \hat{m})$, there exists a unique ψ which satisfies*

$$0 < \psi < m \text{ in } \Omega, \text{ and } M(m) = \sup_{\bar{\Omega}} \left(|\nabla\psi|^2 - \Sigma^2(\mathcal{B}(\psi)) \right) < 0, \tag{122}$$

and solves (67), where $\mathcal{B}(\psi) = h(\rho_0) + F^2(\psi)/2$. Furthermore, either $M(m) \to 0$ as $m \to \hat{m}$, or there does not exist $\sigma > 0$ such that (67) has solutions for all $m \in (\hat{m}, \hat{m}+\sigma)$ and

$$\sup_{m \in (\hat{m}, \hat{m}+\sigma)} M(m) < 0. \tag{123}$$

Proof: The basic idea of the proof for Proposition is quite similar to that in [2, 31].

For the given Bernoulli's function B in the upstream satisfying (14) and any $m \in (\delta^\gamma, \bar{m})$, one can define ρ_0 and $u_0(x_2)$, and therefore $F(\psi)$ as in Section 2. Note that ρ_0 and F depend on m by definition, thus in this section, we will denote them by $\rho_0(m)$ and $F(\psi; m)$ respectively.

When B satisfies (21), one has

$$F'(m) = F'(0) = 0.$$

Thus $\tilde{F}'$, the extension of F' is Section 3, has the following simple form

$$\tilde{F}'(s) = \begin{cases} F'(s), & \text{if } 0 \le s \le m, \\ 0, & \text{if } s < 0 \text{ or } s > m. \end{cases} \tag{124}$$

Set $\tilde{F}(s) = \int_0^s \tilde{F}'(s)ds$. Then it is easy to check that

$$B_0 < \underline{B} \le h(\rho_0) + \frac{\tilde{F}^2(s)}{2} \le \bar{B} \text{ and } \|\tilde{F}'\|_{C^{0,1}(\mathbb{R}^1)} \le C\delta^{1-2\gamma}. \tag{125}$$

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence of positive numbers such that $\varepsilon_1 \le \varepsilon_0/4$ and $\varepsilon_n \downarrow 0$. One can truncate H associated with ε_n as follows

$$\tilde{H}^{(n)}(|\nabla\psi|^2, \psi; m) = J(\tilde{\Delta}_n(|\nabla\psi|^2, \psi; m), \tilde{\mathcal{B}}_n(\psi; m)). \tag{126}$$

To give a clear explanation of this definition, we first introduce a sequence of smooth increasing functions ζ_n such that

$$\zeta_n(s) = \begin{cases} s, & \text{if } s < -2\varepsilon_n, \\ -\varepsilon_n, & \text{if } s \ge -\varepsilon_n. \end{cases}$$

Now one can define

$$\tilde{\Delta}_n(|\nabla\psi|^2, \psi; m) = \zeta_n(|\nabla\psi|^2 - \Sigma^2(\tilde{\mathcal{B}}_n(\psi; m))) + \Sigma^2(\tilde{\mathcal{B}}_n(\psi; m)),$$

where

$$\tilde{\mathcal{B}}_n(\psi; m) = h(\rho_0(m)) + \frac{\tilde{F}^2(\psi; m)}{2}.$$

It is easy to see that there exist two positive constants $\lambda(n)$ and $\Lambda(n)$ such that

$$\lambda(n)|\xi|^2 \le \tilde{A}_{ij}^{(n)}(q, z; m)\xi_i\xi_j \le \Lambda(n)|\xi|^2$$

for any $z \in \mathbb{R}^1$, $q \in \mathbb{R}^2$ and $\xi \in \mathbb{R}^2$, where

$$\tilde{A}_{ij}^{(n)}(q, z; m) = \tilde{H}^{(n)}(|q|^2, z; m)\delta_{ij} - 2\tilde{H}_1^{(n)}(|q|^2, z; m)q_iq_j.$$

Thus it follows from the argument in Section 3 that for any $m \in (\delta^\gamma, \bar{m})$, there exists a solution $\psi^{(n)}(x; m)$ to the problem

$$\begin{cases} \tilde{A}_{ij}^{(n)}(D\psi, \psi; m)\partial_{ij}\psi = \tilde{\mathcal{F}}_n(D\psi, \psi; m) \text{ in } \Omega, \\ \psi = \frac{x_2 - f_1(x_1)}{f_2(x_1) - f_1(x_1)}m \text{ on } \partial\Omega, \end{cases} \tag{127}$$

where

$$\tilde{\mathcal{F}}_n = \tilde{F}\tilde{F}'\tilde{H}^{(n)}\left(\frac{((\tilde{H}^{(n)})^2 + \Sigma\Sigma'(\zeta_n' - 1))\tilde{\Delta}_n}{(\tilde{H}^{(n)})^2c^2 - \tilde{\Delta}_n} + (\tilde{H}^{(n)})^2\right),$$

where we ignore some obvious independent variables in the definition of $\tilde{\mathcal{F}}_n$. Moreover, if

$$|\nabla\psi^{(n)}|^2 - \tilde{\mathcal{B}}(\psi^{(n)}; m) \le -2\varepsilon_n, \tag{128}$$

then $\zeta'_n = 1$. Similar to Section 3, one has

$$0 \le \psi^{(n)}(x;m) \le m.$$

Since $\tilde{F}$ satisfies (125) independent of ε_n, one can estimate I_5 in (106) as that in (107). Furthermore, it follows from the same arguments in Lemma 5 that the solution to (127) satisfying (128) has far fields behavior as (100). In addition, by Proposition 7, such a solution is unique among the class of solutions satisfying (100).

Note that in general, we do not know uniqueness of solutions to problem (127). Set

$$S_n(m) = \{\psi^{(n)}(x;m) | \psi^{(n)}(x;m) \text{ solves the problem (127)}\}. \tag{129}$$

Define

$$M_n(m) = \inf_{\psi^{(n)} \in S_n(m)} \sup_{\bar{\Omega}} (|\nabla \psi^{(n)}(x;m)|^2 - \Sigma^2(\tilde{\mathcal{B}}(\psi^{(n)};m))), \tag{130}$$

and

$$T_n = \{s | \delta_0^\gamma \le s \le \bar{m}, M_n(m) \le -4\varepsilon_n \text{ if } m \in (\delta^\gamma, s)\}.$$

It follows from Proposition 4, Lemma 5 and Proposition 6 that $[\delta_0^\gamma, 2\delta_0^{\gamma/2}] \subset T_n$, therefore, T_n is not an empty set. Define $m_n = \sup T_n$.

The sequence $\{m_n\}$ has some nice properties.

First, $M_n(m)$ is left continuous for $m \in (\delta^\gamma, m_n]$. Indeed, let $\{m_n^{(k)}\} \subset (\delta^\gamma, m_n)$ and $m_n^{(k)} \uparrow m$. Since $M_n(m_n^{(k)}) \le -4\varepsilon_n$, one has

$$\|\psi^{(n)}(x; m_n^{(k)})\|_{C^{2,\alpha}(\bar{\Omega})} \le C.$$

Therefore, there exists a subsequence $\psi^{(n)}(x; m_n^{(k_l)})$ such that $\psi^{(n)}(x; m_n^{(k_l)}) \to \psi$, moreover, ψ solves (127). Thus $M_n(m) \le \lim M_n(m_n^{(k_l)})$. So $M_n(m) \le -4\varepsilon_n$. Note that all these solutions satisfy the far fields behavior as (100), by uniqueness of solutions in this class, $M_n(m) = \lim M_n(m_n^{(k)})$.

Second, $m_n < \bar{m}$. Suppose not, by the definition of m_n, $\bar{m} \in T_n$. It follows from the left continuity of M_n, $M_n(\bar{m}) \le -4\varepsilon_n$. Thus by means of the proof of Lemma 5, $\psi^{(n)}(x;\bar{m})$

has far field behavior as in (100). However, it follows from the definition of $\bar{m}$ that

$$
\begin{aligned}
& \sup_{x\in\bar{\Omega}}\left(|\nabla\psi^{(n)}(x;\bar{m})|^2-\Sigma^2(\tilde{\mathcal{B}}_n(\psi^{(n)}(x;\bar{m})))\right) \\
\geq\ & \sup_{x_2\in[0,1])}\max\{(|\rho_0(\bar{m})u_0(x_2;\bar{m})|^2-\Sigma^2(B(x_2))),(|\rho_1(\bar{m})u_1(y(x_2);\bar{m})|^2-\Sigma^2(B(x_2)))\} \\
=\ & 0,
\end{aligned}
$$

where $y=y(s)$ is the function defined in (49). Thus $M_n(\bar{m})\geq 0$. This is a contradiction. Therefore $m_n<\bar{m}$.

Finally, $\{m_n\}$ is an increasing sequence. This follows from the definition of $\{m_n\}$ directly.

Define $\hat{m}=\lim_{n\to\infty}m_n$. Based on previous properties of $\{m_n\}$, $\hat{m}$ is well-defined and $\hat{m}\leq\bar{m}$.

Note that for any $m\in(\delta^\gamma,\hat{m})$, there exists $m_n>m$, therefore $M_n(m)\leq-4\varepsilon_n$. Thus $\psi=\psi^{(n)}(x;m)$ solves (67) and

$$
\sup_{\bar{\Omega}}(|\nabla\psi|^2-\Sigma^2(\mathcal{B}(\psi)))=M_n(m)\leq-4\varepsilon_n.
$$

If $\sup_{m\in(\delta^\gamma,\hat{m})}M(m)<0$, then there exists n such that $\sup_{m\in(\delta^\gamma,\hat{m})}M(m)<-4\varepsilon_n$. As the same as the proof for the left continuity of $M_n(m)$ on $(\delta^\gamma,m_n]$, $M_n(\hat{m})\leq-4\varepsilon$. Suppose that there exists $\sigma>0$ such that (67) always has a solution ψ for $m\in(\hat{m},\hat{m}+\sigma)$, and

$$
\sup_{m\in(\hat{m},\hat{m}+\sigma)}M(m)=\sup_{m\in(\hat{m},\hat{m}+\sigma)}\sup_{\bar{\Omega}}(|\nabla\psi|^2-\Sigma^2(\mathcal{B}(\psi)))<0. \tag{131}
$$

Then there exists $k>0$ such that

$$
\sup_{m\in(\hat{m},\hat{m}+\sigma)}M(m)=\sup_{m\in(\hat{m},\hat{m}+\sigma)}\sup_{\bar{\Omega}}(|\nabla\psi|^2-\Sigma^2(\mathcal{B}(\psi)))\leq-4\varepsilon_{n+k}.
$$

This yields that $m_{n+k}\geq\hat{m}+\sigma$. So there is a contradiction. The contradiction implies that either $M(m)\to 0$, or there does not exist $\sigma>0$ such that (67) has solution for all $m\in(\hat{m},\hat{m}+\sigma)$ and (123) holds.

This finishes the proof of the Proposition. □

It follows from Lemma 5, Lemma 8 and Proposition 9 that if B satisfies (14) and $m\in(\delta^\gamma,\hat{m})$, then there exists an Euler flow through the nozzle. Collecting all results obtained together, we complete the proof of Theorem 1.

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