Pointwise Stability of Contact Discontinuity for Viscous Conservation Laws with General Perturbations*

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Abstract

The large time asymptotic behavior towards viscous contact waves for a class of systems of viscous conservation laws is studied in this paper for general initial perturbations. The high order deviation of the viscous solutions from the leading order ansatz is estimated pointwisely via the approximate Green function approach. The structural constraint on the left eigenvector belonging to the principal linearly degenerate family used in [13] is removed so that our results hold, in particular, for the one-dimensional compressible Navier-Stokes equations of gas dynamics in both Lagrangian and Eulerian coordinates.

1 Introduction

The purpose of this paper is to study the large time asymptotic behavior toward viscous contact waves for the solutions of viscous conservation laws for general perturbations, by giving the detailed pointwise behavior of solutions via the approximate Green function approach. This problem was studied by Liu and Xin ([13]), under some structural constraints on both the left and right eigenvectors of the linearly degenerate family belonging to which the contact discontinuity is under investigation. As pointed out in [13], the asymptotic behavior toward the viscous contact wave for solutions to viscous conservation laws without those structural constraints is an open problem. In this paper, we show that the same results in [13] still hold without the constraint on the left eigenvector. Consider the following system of viscous conservation laws

$$\partial_t u + \partial_x f(u) = \partial_x^2 u, \qquad x \in \mathbb{R}^1,\ t > 0,\ u \in \mathbb{R}^n, \tag{1.1}$$

where the flux $f(u) \in \mathbb{R}^n$ is assumed to be smooth. We will study the large time asymptotic behavior of solutions to Cauchy problem (1.1) with the initial data

$$u(x, t = 0) = u_0(x) \to u_\pm \ \text{ as } x \to \pm\infty. \tag{1.2}$$

The corresponding inviscid system

$$\partial_t u + \partial_x f(u) = 0, \qquad x \in \mathbb{R}^1,\ t > 0,\ u \in \mathbb{R}^n, \tag{1.3}$$

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is assumed to be strictly hyperbolic, i.e., the Jacobian matrix $f'(u)$ has real eigenvalues $\lambda_i(u)$ with corresponding right and left eigenvectors $r_i(u)$ and $l_i(u)$ such that

$$\lambda_1(u) < \cdots < \lambda_n(u),\ f'(u)r_i(u) = \lambda_i(u)r_i(u),\ l_i(u)f'(u) = \lambda_i(u)l_i(u),$$

for all u under consideration. Define the matrices L, R and Λ as

$$L(u) \equiv (l_1(u), \cdots, l_n(u))^t,\ R(u) \equiv (r_1(u), \cdots, r_n(u)),$$
$$\Lambda(u) \equiv diag(\lambda_1(u), \cdots, \lambda_n(u)),$$

where $(\cdots)^t$ denotes the transpose. Then we have

$$L(u)f'(u)R(u) = \Lambda,\ L(u)R(u) = Id.$$

The i-th characteristic field is called genuinely nonlinear (or linearly degenerate), if $\nabla\lambda_i(u) \cdot r_i(u) \neq 0$ (or $\nabla\lambda_i(u) \cdot r_i(u) = 0$) for all u under consideration (cf. [10] and [18]). Suppose that there exists a $p \in [1, n]$, such that the p-th field is linearly degenerate, and the triple (u_-, u_+, s) forms a p-contact discontinuity, i.e.,

$$f(u_-) - f(u_+) = s(u_- - u_+),\ s = \lambda_p(u_-) = \lambda_p(u_+).$$

As in [13], the p-th viscous contact wave for (1.1) corresponding to the p-contact discontinuity (u_-, u_+, s) is constructed as follows. Set

$$C_p(u_-) = \{\ u \mid u = u(\rho),\ \frac{du}{d\rho} = r_p(u(\rho)),\ u(\rho = \rho_-) = u_-\ \}.$$

Thus the p-th contact wave curve through u_-, $C_p(u_-)$, is the integral curve associated with the vector field $r_p(u)$ in the state space with the nonsingular parameter ρ. The parameter ρ is chosen to satisfy

$$\begin{cases} u(\rho_-) = u_-,\ u(\rho_+) = u_+; \\ \begin{cases} \partial_t\rho + s\partial_x\rho = \partial_x^2\rho, & x \in \mathbb{R}^1,\ t > -1, \\ \rho(x, t = -1) = \begin{cases} \rho_-, & x < 0, \\ \rho_+, & x > 0. \end{cases} \end{cases} \end{cases} \tag{1.4}$$

The p-th viscous contact wave $\bar{u}(x, t)$ is defined as

$$\bar{u}(x, t) \equiv u(\rho(x, t)) \in C_p(u_-). \tag{1.5}$$

Then it holds that,

$$\partial_t\bar{u} + \partial_x f(\bar{u}) - \partial_x^2\bar{u} = (\nabla r_p \cdot r_p)(\bar{u})(\partial_x\rho)^2.$$

With the following structural constraints on both the left and right eigenvectors,

$$(\nabla l_p \cdot r_p)(u) \equiv 0,\ \text{for}\ \ u \in C_p(u_-), \tag{1.6}$$

and

$$(\nabla r_p \cdot r_p)(u) \equiv 0, \text{ for } u \in C_p(u_-). \tag{1.7}$$

it was shown in [13] that the viscous contact wave $\bar{u}$ is nonlinearly stable for the Cauchy problem (1.1) and (1.2) provided the strength of the contact discontinuity, $|u_+ - u_-|$, is suitably small. Moreover, the leading order asymptotic ansatz of the solution to (1.1) and (1.2) was constructed in [13] as the superposition of the viscous contact wave (with a proper shift) in the p-th field and nonlinear (or linear) diffusion waves in the transversal fields. Also, the high order deviation of the solutions to (1.1) and (1.2) from its leading order asymptotic ansatz was estimated pointwisely and the optimal rate of convergence (in time) was obtained in [13], under both the constraints (1.6) and (1.7). However, it should be noted that for the one-dimensional compressible Navier-Stokes equations of gas dynamics, the constraint on the right eigenvector (1.7) holds true in both Lagrangian coordinates and Eulerian coordinates (cf. [13] and [25]), but the constraint on the left eigenvector (1.6) is only satisfied by the compressible Navier-Stokes equations in Lagrangian coordinates, but not in Eulerian coordinates (see [25] for instance). The main purpose of this paper is to improve the results obtained in [13] by removing the constraint on the left eigenvector (1.6). The significance of this improvement is that it sheds light on the nonlinear stability of planar contact waves for the **multi-dimensional** compressible Navier-Stokes equations. This is because, as well- known, that the Lagrangian coordinates system is very inconvenient to use in multi-dimensions. In [25], the nonlinear stability of superposition of shock waves and contact discontinuities is obtained for $n \times n$ viscous conservation laws solely under the constraint on the right eigenvector (1.7), by using the weighted energy method. For the initial data considered in [25], the Riemann solution to (1.3) with the Riemann data $u_\pm$ (which are the limits of the initial data $u_0(x)$ as $x \to \pm\infty$) consists of shock waves and contact discontinuities. Each wave has a weak but non-zero strength. In this case, the initial perturbation of superposition of viscous shock waves and contact waves only produces the translations of those waves, but not diffusion waves. Also, no decay rate is given in [25]. In this paper, we consider a generic perturbation of a viscous contact wave. As observed in [21] and [13], a generic perturbation produces not only a translation in viscous contact wave, but also diffusion waves in the transversal families. We decompose the solution as the sum of the viscous contact wave, diffusion waves and high order error terms and use the approximate Green function approach to give detailed pointwise estimates of the solution, which yields the optimal decay rate. The same decay rate is obtained as in [13], with the constraint on both the left and right eigenvectors.

The problem of nonlinear stability of elementary waves, such as shock waves, rarefaction waves and contact discontinuities for viscous conservation laws is a fundamental one in understanding the large time asymptotic equivalence between the hyperbolic conservation laws and viscous conservation laws. This problem has been extensively studied for the shock waves and rarefaction waves (cf. [2], [3], [9], [11], [12], [14], [16], [17], [19], [22], [23] and the references therein). However, the problem on the nonlinear stability of contact discontinuities is more subtle due to its degeneracy. For the equations of polytropic gases with the artificial (uniform) viscosity in Lagrangian coordinates, the viscous contact wave is introduced by Xin ([21]), which approximates the contact discontinuity on any finite time interval. For this viscous contact wave, nonlinear stability is proved for small generic perturbations and the detailed asymptotic behavior of solutions is shown ([21]. Later on, for the one-dimensional compressible Navier-Stokes equations of polytropic gas in Lagrangian coordinates, the nonlinear stability of the viscous contact waves is proved by Huang, Matsumura and Xin ([5]) for the perturbations with zero excessive mass ([5]) and further by Huang, Xin and Yang ([8]) for general perturbations ([8]), both with the decay rate of $(1+t)^{-1/4}$ in L^∞-norm by using the energy method. The nonlinear stability of a superposition of viscous contact waves with rarefaction waves for 1-d compressible Navier-Stokes equations in Lagrangian coordinates is proved by Huang, Li, and Matsumura ([4]) by using the energy method. The decay rate is not given in [4]. In the present paper, by using the Green function approach, we obtain the optimal decay rate in L^p-norm for any $1 < p \leq +\infty$, especially we obtain that the decay rate in L^∞-norm is $(1+t)^{-1/2}$. Furthermore, it should be noted the above mentioned systems for polytropic gases for which the nonlinear stability of the viscous contact wave is proved are all in Lagrangian coordinates, in which the constraints both on the left and right eigenvectors (1.6) and (1.7) are satisfied. However, as already mentioned, for those systems in Eulerian coordinates, only the constraint on the right eigenvector (1.7), but not the constraint on the left eigenvector (1.6), holds. It should be noted that the nonlinear stability of contact waves is also proved in [6] for the Jin-Xin relaxation model with the decay rate of $(1+t)^{-1/4}$ L^∞-norm by using the energy method.

The rest of this paper is organized as follows. In Section 2, we describe the leading order asymptotic ansatz and state the main theorem. The equations for the antiderivatives of the high order error terms are derived and some estimates on nonlinear terms are given in Section 3. Exact and approximate Green functions for the principal and transversal fields are introduced in Section 4. In Sections 5 and 6, we derive the *a priori* estimates for the principal and transversal fields, respectively. Based on those *a priori* estimates, the main theorem is proved in Section 7.

2 Main Results

In this section, we describe the leading order time asymptotic ansatz, which is the superposition of the viscous contact wave and diffusion waves.

Under the structural condition 1.7, the viscous contact wave $\bar{u}(x,t)$ introduced in (1.5) is an exact solution to equation (1.1).

For $\delta = |u_+ - u_-|$ small, $r_1(u_-), \cdots, r_{p-1}(u_-), u_+ - u_-, r_{p+1}(u_+), \cdots, r_n(u_+)$

form a basis in $\mathbb{R}^n$. So the initial excessive mass can be decomposed as follows:

$$\int_{\mathbb{R}^1} (u_0(x) - \bar{u}(x,0))dx = x_0(u_+ - u_-) + \sum_{i\neq p} m_i r_i \tag{2.1}$$

with uniquely determined x_0 and m_i $(i \neq p)$. Here and thereafter, we use the following notations:

$$\begin{aligned} &u_i = u_- \text{ for } i < p,\ u_i = u_+ \text{ for } i > p,\\ &\lambda_i \equiv \lambda_i(u_i),\ l_i \equiv l_i(u_i),\ r_i \equiv r_i(u_i),\ \alpha_i \equiv (\nabla\lambda_i \cdot r_i)(u_i), \text{ for } i \neq p,\\ &\lambda_p = \lambda_p(u_\pm) = s, \end{aligned} \tag{2.2}$$

unless otherwise mentioned. A generic perturbation of the viscous contact wave should also introduce waves in the transversal characteristic fields [21, 13]. These are the nonlinear (or linear) diffusion waves introduced by Liu [11], which are governed by the converted Burgers equations

$$\partial_t\theta_i + \partial_x(\lambda_i\theta_i) + \partial_x(\frac{1}{2}\alpha_i\theta_i^2) = \partial_x^2\theta_i, \quad i \neq p. \tag{2.3}$$

The masses carried by the diffusion waves are

$$m_i = \int_{\mathbb{R}^1} \theta_i(x,t)dx, \quad i \neq p. \tag{2.4}$$

By Hopf-Cole transform, the solution to (2.3) and (2.4) has an explicit form

$$\theta_i(x,t) = \frac{1}{\sqrt{1+t}}\chi_i(\frac{x-\lambda_i(1+t)}{\sqrt{4(1+t)}}) \tag{2.5}$$

with

$$\chi_i(y) = \frac{(\alpha_i)^{-1}(\exp(\alpha_i m_i/2) - 1)\exp(-y^2)}{\sqrt{\pi} + (\exp(\alpha_i m_i/2) - 1)\int_y^{+\infty}\exp(-\xi^2)}d\xi \quad (\text{if } \alpha_i \neq 0)$$

or

$$\chi_i(y) = \frac{m_i}{\sqrt{4\pi}}\exp(-y^2) \quad (\text{if } \alpha_i = 0).$$

We now define the leading order time asymptotic ansatz for the solution to (1.1), (1.2) as a superposition of a shift viscous contact wave with diffusion waves in the transversal fields, i.e.,

$$u^a(x,t) = \bar{u}(x+x_0,t) + \theta(x,t) \equiv \bar{u}(x+x_0,t) + \sum_{i\neq p}\theta_i(x,t)r_i. \tag{2.6}$$

Since

$$\begin{aligned} \int_{\mathbb{R}^1}(u_0(x) - u^a(x,0))dx = &\int_{\mathbb{R}^1}(u_0(x) - \bar{u}(x,0))dx + \int_{\mathbb{R}^1}(\bar{u} - u^a)(x,0)dx \\ =&x_0(u_+ - u_-) + \sum_{i\neq p} m_i r_i - \int_{\mathbb{R}^1}(\bar{u}(x+x_0,0) \\ &- \bar{u}(x,0))dx - \int_{\mathbb{R}^1}\sum_{i\neq p}\theta_i(x,0)r_i dx = 0, \end{aligned}$$

it follows from the conservation laws for the viscous contact wave $\bar{u}(x,t)$, the diffusion waves $\theta_i(x,t)$, and the solution $u(x,t)$ to (1.1), (1.2) that

$$\int_{-\infty}^{\infty}(u(x,t)-u^a(x,t))dx=0, \quad \text{for all } t\geq 0. \tag{2.7}$$

The main result in this paper is the following:

Theorem 2.1 *Suppose that the system* (1.3) *is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate. Furthermore, suppose that* (u_-,u_+,s) *is a p-contact discontinuity and the structural condition* (1.7) *holds. Let* $\bar{u}(x,t)$ *be the viscous contact wave as defined by* (1.5), *the initial excessive mass be decomposed as in* (2.1), $u^a(x,t)$ *be a superposition of a shifted viscous contact wave with transversal diffusion waves as constructed in* (2.6). *Then there exists suitable small positive constants* δ_1 *and* δ_2 *such that if*

$$u_0-\bar{u}\in H^1(\mathbb{R}^1), \tag{2.8}$$

$$\delta\equiv|u_+-u_-|\leq\delta_1, \tag{2.9}$$

$$\begin{aligned}&\int_{\mathbb{R}^1}(1+|x|)|u_0-\bar{u}|dx+\int_{\mathbb{R}^1}(1+x^2)|u_0-\bar{u}|^2dx\\&\quad+\|(1+|x|^\alpha)(u_0-\bar{u})\|_{L^\infty_x(\mathbb{R}^1)}\leq\delta_2^2, \quad \text{for } 1<\alpha<5/4,\end{aligned} \tag{2.10}$$

then the Cauchy problem (1.1), (1.2) *admits a unique global (in time) solution* $u(x,t)$ *with the following properties:*

(1) $u-u^a\in C([0,+\infty);H^1(\mathbb{R}^1))$, $u-\bar{u}\in C([0,+\infty);H^1(\mathbb{R}^1))$;

(2) for all $x\in\mathbb{R}^1$, $t\geq 0$,

$$\begin{aligned}u(x,t)&=u^a(x,t)+\partial_x(\sum_{i=1}^n v_i(x,t)r_i(\bar{u}))\\&=\bar{u}(x+x_0,t)+\theta(x,t)+\sum_{i=1}^n\{(\partial_x v_i)r_i(\bar{u})+v_i\nabla r_i(\bar{u})(\partial_x\bar{u})\}\end{aligned} \tag{2.11}$$

with

$$|v_i|=O(1)(\delta_1+\delta_2)(|x-\lambda_i t|^2+(1+t))^{-1/4}, \tag{2.12}$$

$$|\partial_x v_i|=O(1)(\delta_1+\delta_2)(|x-\lambda_i t|^2+(1+t))^{-1/4}(1+t)^{-1/2}, \tag{2.13}$$

and

$$|\partial_x v_i|=O(1)(\delta_1+\delta_2)|x-\lambda_i t|^{-\alpha}, \quad \text{for } |x-\lambda_i t|\geq c_0(1+t), \tag{2.14}$$

where $c_0\geq 4\max_{i,x,t}|\lambda_i(\bar{u}(x,t))|$ *is a positive constant;*

(3) as an immediate consequence of (2),

$$\|u(\cdot,t)-\bar{u}(\cdot+x_0,t)\|_{L^1(\mathbb{R}^1)} \quad \textit{is uniformly bounded in time}, \tag{2.15}$$

$$\|u(\cdot,t)-\bar{u}(\cdot+x_0,t)\|_{L^p(\mathbb{R}^1)}=(1+t)^{-1/2+1/(2p)}, \quad \textit{for all } 1<p\leq\infty. \tag{2.16}$$

Remark: As an immediate consequence of (2), it hold that

$$\|u(\cdot,t)-u^a(\cdot,t)\|_{L^p(\mathbb{R}^1)} = \begin{cases} (1+t)^{-1+1/p}, & 1<p<2 \\ (1+t)^{-1}\ln(1+t), & p=2 \\ (1+t)^{-3/4+1/(2p)}, & 2<p\leq\infty. \end{cases} \tag{2.17}$$

These rates of convergence are optimal.

3 Integral error equations and the characteristic decomposition

Let $u^a(x,t)$ be the leading asymptotic ansatz constructed in (2.6). Without loss of generality, we assume that the speed of the p-contact discontinuity and the shift in the center of viscous contact wave are zero, i.e., $s=0,\ x_0=0$. It follows from the fact that $(\nabla\lambda_i\cdot r_i)(u)=(l_i f''(r_i,r_i))(u)$ and (3) of Lemma 3.1 that

$$
\begin{aligned}
&\partial_t u^a+\partial_x f(u^a)-\partial_x^2 u^a\\
=&\partial_x\left\{f(u^a)-f(\bar u)-\sum_{i\neq p}(\lambda_i\theta_i+\frac{1}{2}\alpha_i\theta_i^2)r_i\right\}\\
=&\partial_x\left\{\sum_{i\neq p}(f'(\bar u)-f'(u_i))r_i\theta_i+\frac{1}{2}\sum_{i,j\neq p,\ i\neq j}f''(\bar u)(r_i,r_j)\theta_i\theta_j\right.\\
&\left.+\frac{1}{2}\sum_{i\neq p}(f''(\bar u)(r_i,r_i)-\alpha_i r_i)\theta_i^2+O(1)|\theta|^3\right\} \qquad (3.1)\\
=&\partial_x\left\{\frac{1}{2}\sum_{i\neq p}\left[\sum_{k=1}^{n}l_k(u_i)f''(u_i)(r_i,r_i)r_k(u_i)-l_i f''(u_i)(r_i,r_i)r_i\right]\theta_i^2\right.\\
&\left.+O(1)|\theta|^3+e\right\}\\
=&\partial_x\left\{\frac{1}{2}\sum_{i\neq p}\sum_{k\neq i}(l_k f''(r_i,r_i)r_k)(u_i)\theta_i^2+O(1)|\theta|^3+e\right\}\equiv\partial_x\epsilon.
\end{aligned}
$$

Here and thereafter, $e=e(x,t)$ represents the exponential error term,

$$
e(x,t)\equiv O(1)(\delta m_0+m_0^2)\exp(-c(t+|x|)) \qquad (3.2)
$$

for some suitable positive constants $O(1)$, c, and $m_0\equiv\sum_{i\neq p}|m_i|$. Before going further, we list some basic properties of the viscous contact wave and diffusion waves in the following lemma. The proof is clear, based on the known properties of the solutions to heat equation and Burgers equation, and the strict hyperbolicity.

Lemma 3.1 *The viscous contact wave $\bar u(x,t)$ and diffusion waves $\theta_i(x,t)$ satisfy the following properties:*

$$
\begin{aligned}
(1)\quad &|\partial_x\bar u(x,t)|=O(1)|\partial_x\rho|=O(1)\delta(1+t)^{-1/2}\exp\{-\frac{x^2}{4(1+t)}\},\\
&|\partial_x^2\bar u(x,t)|=O(1)\delta(1+t)^{-1}\exp\{-\frac{x^2}{8(1+t)}\};
\end{aligned}
$$

(2) $|\bar{u}(x,t)-u_-| \le O(1)\delta \exp\{-\frac{x^2}{8(1+t)}\}, \quad x<0,$

$|\bar{u}(x,t)-u_+| \le O(1)\delta \exp\{-\frac{x^2}{8(1+t)}\}, \quad x>0;$

(3) $|\bar{u}(x,t)-u_i||\theta_i| = O(1)\delta|m_i|\exp(-c(t+|x|)) \qquad$ for $i \neq p$,

$|\theta_j(x,t)\theta_k(x,t)| = O(1)|m_j m_k|\exp(-c(t+|x|)) \quad$ for $j,k \neq p,\ j\neq k$.

Here $\delta = |u_+ - u_-|$ and m_i ($i \neq p$) are determined by (2.1).

Suppose now that $u(x,t)$ is a solution to the Cauchy problem (1.1), (1.2). We can decompose $u(x,t)$ as:

$$u(x,t) = u^a(x,t) + \bar{\omega}(x,t).$$

It follows from (1.1) and (3.1) that $\bar{\omega}(x,t)$ solves

$$\begin{cases} \partial_t\bar{\omega} + \partial_x(f(u^a+\bar{\omega}) - f(u^a)) - \partial_x^2\bar{\omega} = -\partial_x\epsilon, \\ \bar{\omega}(x,t=0) = u_0(x) - u^a(x,0). \end{cases} \tag{3.3}$$

Note that (2.7) enables us to introduce the following anti-derivative variable

$$\omega(x,t) \equiv \int_{-\infty}^{x} \bar{\omega}(y,t)dy = \int_{-\infty}^{x}(u-u^a)(y,t)dy. \tag{3.4}$$

Substituting (3.4) into (3.3) and integrating the resulting system with respect to space variable x, one has the integrated error equations for $\omega(x,t)$,

$$\partial_t\omega + f'(\bar{u})\partial_x\omega - \partial_x^2\omega = E, \tag{3.5}$$

$$\omega(x,0) = \omega_0(x) = \int_{-\infty}^{x}(u_0(y) - u^a(y,0))dy, \tag{3.6}$$

where

$$E = -\epsilon - f^{''}(\bar{u})(\theta,\partial_x\omega) - \frac{1}{2}f^{''}(\bar{u})(\partial_x\omega,\partial_x\omega) + O(1)(|\theta|^3 + |\partial_x\omega|^3)$$

Note that

$$\left|\int_0^{\pm\infty}\left|\int_x^{\pm\infty}(u-\bar{u}-\theta)(y,0)dy\right|^2 dx\right| \le 4\left|\int_0^{\pm\infty}|(u-\bar{u}-\theta)(y,0)|^2 y^2 dy\right|,$$

$$\left|\int_0^{\pm\infty}\left|\int_x^{\pm\infty}(u-\bar{u}-\theta)(y,0)dy\right| dx\right| \le \left|\int_0^{\pm\infty}|(u-\bar{u}-\theta)(y,0)|\,|y|dy\right|,$$

$$(1+|x|)^{1/2}\left|\int_x^{\pm\infty}(u-\bar{u}-\theta)(y,0)dy\right| \le \int_{-\infty}^{+\infty}(1+|y|)^2\,|(u-\bar{u}-\theta)(y,0)|^2\,dy,$$

which can be verified easily by using the Hölder inequality and change of the order of integrations. It follows from some simple inequalities, the explicit forms of the viscous contact wave and diffusion waves (Lemma 3.1), and the assumptions (2.8)-(2.10) that

$$\omega(x,0) = \omega_0(x) \in H^2(\mathbb{R}^1) \cap L^1(\mathbb{R}^1), \tag{3.7}$$

$$\begin{aligned} \|\omega_0\|^2_{H^2(\mathbb{R}^1)} + \|\omega_0\|_{L^1(\mathbb{R}^1)} + &\|(1+|x|)^{1/2}\omega_0\|_{L^\infty(\mathbb{R}^1)} \\ &+ \|(1+|x|)^\alpha \omega_0'\|_{L^\infty(\mathbb{R}^1)} \le C\delta_2^2, \end{aligned} \tag{3.8}$$

for some positive constant C, and δ_2, α given in Theorem 2.1. We diagonalize system (3.5) by introducing a new variable

$$v(x,t) \equiv L(\bar u)\omega(x,t). \tag{3.9}$$

Then $\omega(x,t) = R(\bar u)v(x,t)$, and system (3.5) becomes

$$\partial_t v + \Lambda(\bar u)\partial_x v - \partial_x^2 v = e_1 + e_2, \tag{3.10}$$

where

$$\begin{aligned} &e_1 = -L(\bar u)(\partial_t R(\bar u) + f'(\bar u)\partial_x R(\bar u) - \partial_x^2 R(\bar u))v + 2L(\bar u)\partial_x R(\bar u)\partial_x v, \\ &e_2 = L(\bar u)E. \end{aligned}$$

We conclude this section by listing some simple estimates on the right hand side of (3.10), which will be used later.

Lemma 3.2 *The coupling and nonlinear terms in* (3.10) *admit the following estimates*

$$\begin{aligned} (e_1+e_2)_p = &-\frac{1}{2}\sum_{j\neq p}(l_p f''(r_j,r_j))(u_j)\theta_j^2 + O(1)\Bigg\{\sum_{j\neq p}[(\partial_x\rho)^2(|v_j| + v_j^2) \\ &+ |\partial_x\rho|(|\partial_x v_j| + |\theta||v_j| + |v_j||\partial_x v|) + (\partial_x v_j)^2 \\ &+ |\partial_x v_p||\partial_x v_j|] + |\theta||\partial_x v| + |\theta|^3 + |\partial_x\omega|^3 + e\}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} (e_1+e_2)_i = &-\frac{1}{2}\sum_{j\neq i,p}(l_i f''(r_j,r_j))(u_j)\theta_j^2 - \sum_{j\neq p}[\lambda_i l_i(\nabla r_j\cdot r_p)](\bar u)(\partial_x\rho)v_j \\ &+ O(1)\{\sum_{j\neq p}[(\partial_x\rho)^2(|v_j| + v_j^2) + |\partial_x\rho|(|\partial_x v_j| + |\theta||v_j| + |v_j||\partial_x v|) \\ &+ (\partial_x v_j)^2 + |\partial_x v_p||\partial_x v_j|] + |\theta||\partial_x v| + |\theta|^3 + |\partial_x\omega|^3 + e\},\ i\neq p. \end{aligned} \tag{3.12}$$

Remark: With the constrain on the left eigenvector (1.6) as in [13], $(e_1)_p = (e_2)_p = 0$.

Proof of Lemma 3.2. This lemma follows from direct computations by using the structure of viscous contact wave and diffusion waves, and the structural condition (1.7). Indeed, for all $i = 1, \cdots, n$,

$$
\begin{aligned}
&(e_1)_i \\
=& -\sum_j \{[l_i(\partial_t r_j + \lambda_i \partial_x r_j - \partial_x^2 r_j)](\bar{u})v_j - 2l_i(\bar{u})(\partial_x r_j(\bar{u}))(\partial_x v_j)\} \\
=& \sum_j \{-[l_i(\nabla r_j \cdot r_p)](\bar{u})(\partial_t \rho + \lambda_i(\bar{u})\partial_x \rho - \partial_x^2 \rho)v_j \\
&+ [l_i \nabla(\nabla r_j \cdot r_p)\cdot r_p](\bar{u})(\partial_x \rho)^2 v_j + 2[l_i(\nabla r_j \cdot r_p)](\bar{u})(\partial_x \rho)(\partial_x v_j)\} \\
=& \sum_j \{-[l_i(\nabla r_j \cdot r_p)](\bar{u})\lambda_i(\bar{u})(\partial_x \rho)v_j + [l_i \nabla(\nabla r_j \cdot r_p)\cdot r_p](\bar{u})(\partial_x \rho)^2 v_j \\
&+ 2[l_i(\nabla r_j \cdot r_p)](\bar{u})(\partial_x \rho)(\partial_x v_j)\},
\end{aligned}
\tag{3.13}
$$

where we have used the equation for $\rho(x,t)$ in (1.4) in the last equality. Note that the structural condition (1.7) ensures $\lambda_p(\bar{u}) = 0$, so

$$
(e_1)_p = O(1)\sum_{j\neq p}\{(\partial_x \rho)^2|v_j| + |\partial_x \rho||\partial_x v_j|\}, \tag{3.14}
$$

$$
(e_1)_i = \sum_{j\neq p}\{-[\lambda_i l_i(\nabla r_j \cdot r_p)](\bar{u})(\partial_x \rho)v_j + O(1)[(\partial_x \rho)^2|v_j| + |\partial_x \rho||\partial_x v_j|]\},\ i \neq p. \tag{3.15}
$$

$$
\begin{aligned}
(e_2)_i =& -l_i(\bar{u})\epsilon - l_i(\bar{u})f''(\bar{u})(\theta, \partial_x \omega) - \frac{1}{2}l_i(\bar{u})f''(\bar{u})(\partial_x \omega, \partial_x \omega) \\
&+ O(1)(|\theta|^3 + |\partial_x \omega|^3).
\end{aligned}
\tag{3.16}
$$

We compute these three terms one by one as follows:

$$
\begin{aligned}
l_i(\bar{u})\epsilon =& l_i(u_j)\epsilon + (l_i(\bar{u}) - l_i(u_j))\epsilon \\
=& \frac{1}{2}\sum_{j\neq p}\sum_{k\neq j}(l_k f''(r_j, r_j))(u_j)\theta_j^2 \delta_{ik} + O(1)|\theta|^3 + e \\
=& \frac{1}{2}\sum_{j\neq p, i}(l_i f''(r_j, r_j))(u_j)\theta_j^2 + O(1)|\theta|^3 + e;
\end{aligned}
\tag{3.17}
$$

Since

$$
\partial_x \omega = (\partial_x R(\bar{u}))v + R(\bar{u})\partial_x v = \sum_j \{(\partial_x r_j(\bar{u}))v_j + r_j(\bar{u})(\partial_x v_j)\}, \tag{3.18}
$$

then

$$
\begin{aligned}
l_i(\bar{u})f''(\bar{u})(\theta, \partial_x \omega) =& \sum_{k\neq p}\sum_j \{l_i(\bar{u})f''(\bar{u})(r_k, \partial_x r_j(\bar{u}))\theta_k v_j \\
&+ l_i(\bar{u})f''(\bar{u})(r_k, r_j(\bar{u}))\theta_k(\partial_x v_j)\} \\
=& O(1)\{\sum_{j\neq p}|\partial_x \rho||\theta||v_j| + |\theta||\partial_x v|\};
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
&l_i(\bar{u})f''(\bar{u})(\partial_x\omega, \partial_x\omega) \\
=&\sum_{j,k}\{[l_i f''(\partial_x r_j, \partial_x r_k)](\bar{u})v_j v_k + 2[l_i f''(\partial_x r_j, r_k)](\bar{u})v_j(\partial_x v_k) \\
&+ [l_i f''(r_j, r_k)](\bar{u})(\partial_x v_j)(\partial_x v_k)\} \\
=&O(1)\sum_{j\neq p}\{(\partial_x\rho)^2 v_j^2 + |\partial_x\rho||v_j||\partial_x v| + (\partial_x v_j)^2 + |\partial_x v_p||\partial_x v_j|\},
\end{aligned}
\tag{3.20}
$$

where we have used the fact $\partial_x r_p(\bar{u}) = (\nabla r_p \cdot r_p)(\bar{u})(\partial_x\rho) = 0$ and $[l_i f''(r_p, r_p)](\bar{u}) = 0$ for $i = 1, \cdots, n$, due to

$$
[l_j f''(r_i, r_k)](u) = (\nabla\lambda_i \cdot r_k)(u)\delta_{ij} + [(\lambda_i - \lambda_j)l_j(\nabla r_i \cdot r_k)](u). \tag{3.21}
$$

4 Green functions for principal and transversal fields

We now begin our proof of the theorem by analyzing the asymptotic behavior of solution to (3.10). As it is well known, the accurate pointwise behavior of solutions to parabolic system is achieved by using the parametrix method (cf.[1, 12, 21, 19, 13, 15, 24]). We can construct the approximate Green functions by approximating scalar dual waves corresponding only to the decoupled scalar operators on the left hand side of (3.10) as in [13]. Now that the linear operator on the left hand side of (3.10) is diagonal, and each component has a dual of the form

$$
-\partial_t\bar{\eta}_i - \partial_x(\lambda_i(x,t)\bar{\eta}_i) - \partial_x^2\bar{\eta}_i = 0, \quad i = 1, 2, \cdots, n. \tag{4.1}
$$

Here and from now on, we use the notation

$$
\lambda_i(x,t) \equiv \lambda_i(\bar{u}(x,t)).
$$

Thus the fundamental solution matrix to the linear operator $\partial_t v + \Lambda(\bar{u})\partial_x v - \partial_x^2 v$ is a diagonal matrix with (i,i)-component given by $\bar{\eta}_i(x,t;y,T)$ which solves (4.1) on $\mathbb{R}^1 \times (0,T)$ with data

$$
\bar{\eta}_i(x, t = T; y; T) = \delta(x-y) \tag{4.2}
$$

for any given $T > 0$ and $y \in \mathbb{R}^1$. In the case $i = p$, $\lambda_p(\bar{u}(x,0)) = s = 0$, so the solution to (4.1)-(4.2) is given uniquely by the heat kernel

$$
\bar{\eta}_p(x,t;y,T) = \frac{1}{\sqrt{4\pi(T-t)}}\exp\left\{-\frac{(x-y)^2}{4(T-t)}\right\}. \tag{4.3}
$$

For $i \neq p$, we do not have an explicit solution to (4.1)-(4.2) in general. We thus define an approximate scalar dual $\eta_i(x,t;y,T)$ as follows:

$$
\eta_i(x,t;y,T) = \frac{1}{\sqrt{4\pi(T-t)}}\exp\left\{-\frac{(m_i(x,t;y,T))^2}{4(T-t)}\right\} \tag{4.4}
$$

for $x \in \mathbb{R}^1$, $y \in \mathbb{R}^1$, and $t < T$, where

$$m_i(x,t;y,T) = \lambda_i(x,t) \left(\int_y^x \frac{d\xi}{\lambda_i(\xi,t)} + (T-t) \right). \tag{4.5}$$

It follows that the Green functions for the principal field $\eta_p(x,t;y,T) \equiv \bar{\eta}_p(x,t;y,T)$, and for the transversal fields $\eta_i(x,t;y,T)$ satisfy the following properties:

$$\eta_i(x,t=T;y,T) = \delta(x-y), \quad i = 1,2,\cdots,n, \tag{4.6}$$

and

$$\partial_t \eta_i + \partial_x(\lambda_i(x,t)\eta_i) + \partial_x^2 \eta_i = E_i, \tag{4.7}$$

with the error terms $E_p = 0$ and for $i \neq p$,

$$\begin{aligned} E_i =& (\partial_x \lambda_i)\eta_i - \frac{m_i^2}{2(T-t)} \left[\partial_x \lambda_i + \frac{(\partial_x \lambda_i)^2}{\lambda_i^2} \left(1 - \frac{m_i^2}{2(T-t)} \right) \right. \\ & \left. + \frac{\partial_t \lambda_i + \partial_x^2 \lambda_i}{\lambda_i} \right] \eta_i - \frac{m_i}{2(T-t)} \left[\frac{\partial_x \lambda_i}{\lambda_i} \left(3 - \frac{m_i^2}{T-t} \right) + \lambda_i(\partial_t n_i) \right] \eta_i. \end{aligned} \tag{4.8}$$

Here $\lambda_i = \lambda_i(x,t)$, $m_i = m_i(x,t;y,T)$, and $n_i = n_i(x,t;y) = \int_y^x \frac{d\xi}{\lambda_i(\xi,t)}$.

Taking the scalar product of (3.10) with $\eta = (\eta_1, \cdots, \eta_n)$ and integrating the resulting equation, we obtain the following integral representation for the solution to (3.10) by using (4.6)-(4.7),

$$\begin{aligned} v_i(y,T) =& \int v_i(x,0)\eta_i(x,t=0;y,T)dx \\ & + \int_0^T \int v_i(x,t) E_i(x,t;y,T) dx dt \\ & + \int_0^T \int (e_1 + e_2)_i(x,t)\eta_i(x,t;y,T) dx dt, \ i = 1, \cdots, n. \end{aligned} \tag{4.9}$$

We also need to estimate the derivative of $v(x,t)$. Thus differentiating (4.9) with respect to y yields

$$\begin{aligned} \partial_y v_i(y,T) =& \int v_i(x,0)\partial_y \eta_i(x,t=0;y,T)dx \\ & + \int_0^T \int v_i(x,t) \partial_y E_i(x,t;y,T) dx dt \\ & + \int_0^T \int (e_1 + e_2)_i(x,t)\partial_y \eta_i(x,t;y,T) dx dt, \ i = 1, \cdots, n. \end{aligned} \tag{4.10}$$

Here and from now on, by $\int dx$, we always mean the $\int_{\mathbb{R}^1} dx$ unless otherwise stated. For simplicity of presentation, we will use the following notations:

$$\begin{aligned} d_i(x,t) &\equiv ((x - \lambda_i(1+t))^2 + (1+t))^{-1/4}, \\ K(x,t) &\equiv (|x|^{2\alpha - 1}\chi(x,t) + (1+t))^{-1/2}, \\ H_i(x,t) &\equiv K(x - \lambda_i(1+t), t), \end{aligned} \tag{4.11}$$

where $\chi(x,t)$ is a smooth function satisfying

$$\chi(x,t) = 1, \text{ for } |x| \geq c_0(1+t);\ \chi(x,t) = 0, \text{ for } |x| \leq c_0(1+t)/2.$$

Set also

$$M(t) = \sup_{0\leq\tau\leq t} \max_{1\leq i\leq n} \{\|(v_i d_i^{-1})(\cdot,\tau)\|_{L^\infty} + \|((\partial_x v_i)(d_i H_i)^{-1})(\cdot,\tau)\|_{L^\infty}\}. \quad (4.12)$$

Thus on the time interval the solution exists, one has that

$$\begin{aligned} &|v_i(x,t)| \leq M(t) d_i(x,t), \\ &|\partial_x v_i(x,t)| \leq M(t)(d_i H_i)(x,t), \end{aligned} \quad (4.13)$$

for all $x \in \mathbb{R}^1$, $i = 1, \cdots, n$. As long as one can obtain a priori bound on $M(t)$ independent of the time, the standard continuity argument yields the global (in time) solution, and (4.13) gives the desired pointwise asymptotic form (2.12)-(2.14). Thus, we only need to derive the following a priori estimate

$$M(t) \leq \epsilon_0 \quad (4.14)$$

where ϵ_0 is a small positive constant depending on the initial data and wave strength. The technical estimates for the bound on $M(t)$ are derived in the next two sections.

5 A priori estimate I- on principal waves

We start with the estimate on the wave in the principal field, $v_p(x,t)$ and $\partial_x v_p(x,t)$. This case is easier than the ones for the waves in the transversal fields due to the fact that η_p is exact.

5.1 Estimate on $v_p(y,T)$

It follows from (3.11), (4.9) and the Cauchy-Schwartz inequality that for any $y \in \mathbb{R}^1$, $T > 0$

$$\begin{aligned} |v_p(y,T)| \leq &\int |v_p(x,0)||\eta_p(x,0;y,T)|dx + O(1)\int_0^T \int \{|\theta|^2 \\ &+ \sum_{j\neq p} \left[(\partial_x\rho)^2(|v_j| + v_j^2) + |\partial_x\rho||\partial_x v_j|\right] + |\partial_x v|^2 + |\theta|^3 \\ &+ |\partial_x\rho|^3|v|^3 + |\partial_x v|^3 + e\}(x,t)\eta_p(x,t;y,T)dxdt. \end{aligned} \quad (5.1)$$

Using (2.5), (4.11), (4.13), (1) of Lemma 3.1 and the smallness of δ, m_0, $M(T)$, we have

$$
\begin{aligned}
&|v_p(y,T)| \le \int |v_p(x,0)||\eta_p(x,0;y,T)|dx \\
&+ O(1)\int_0^T \int |\theta(x,t)|^2 \eta_p(x,t;y,T)dxdt \\
&+ O(1)M(T)\int_0^T \int (1+t)^{-3/4}|\partial_x \rho(x,t)|\eta_p(x,t;y,T)dxdt \\
&+ O(1)M^2(T)\sum_{i=1}^n \int_0^T \int (d_i^2 H_i^2)(x,t)\eta_p(x,t;y,T)dxdt \\
&+ O(1)\int_0^T \int e(x,t)\eta_p(x,t;y,T)dxdt \\
&\equiv \sum_{i=1}^5 I_i.
\end{aligned}
\tag{5.2}
$$

Lemma 5.1 *For suitably small δ_1, δ_2 and ϵ_0, one has that*

$$
|v_p(y,T)| \le O(1)(\delta_2 + \delta_1^2 + M^2(T))d_p(y,T) \tag{5.3}
$$

for all $(y,T) \in \mathbb{R}^1 \times (0,\infty)$.

Proof For I_1,

$$
I_1 \le \|v_p(\cdot,0)\|_{L^1}\|\eta_p(\cdot,0;y,T)\|_{L^\infty} \le O(1)T^{-1/2}\delta_2^2, \tag{5.4}
$$

and when $y \ge \sqrt{1+T}$ (the case $y \le -\sqrt{1+T}$ is similar), one has

$$
\begin{aligned}
I_1 =& \left\{\int_{-\infty}^{y/2} + \int_{y/2}^{\infty}\right\}|v_p(x,0)|\eta_p(x,0;y,T)dx \\
\le& \|v_p(\cdot,0)\|_{L^2}\left(\int_{-\infty}^{y/2} \frac{1}{4\pi T}\exp\left\{-\frac{(x-y)^2}{2T}\right\}dx\right)^{1/2} \\
&+ \|v_p(\cdot,0)(1+|\cdot|)^{1/2}\| \int_{y/2}^{\infty}(1+|x|)^{-1/2}\eta_p(x,0;y,T)dx \\
=& \frac{O(1)\delta_2}{T^{1/4}}\exp\left\{-\frac{y^2}{16T}\right\} + \frac{O(1)\delta_2^2}{(1+|y|)^{1/2}}.
\end{aligned}
\tag{5.5}
$$

So

$$
I_1 \le O(1)\delta_2 d_p(y,T). \tag{5.6}
$$

Next, we consider

$$
\begin{aligned}
I_2 &= O(1)\int_0^T\int |\theta(x,t)|^2\eta_p(x,t;y,T)dxdt \\
&\equiv O(1)\sum_{i\neq p}\int_0^T\int |\theta_i(x,t)|^2\eta_p(x,t;y,T)dxdt \\
&\le O(1)m_0^2\sum_{i\neq p}\int_0^T \frac{1}{\sqrt{(1+T)(1+t)}}\exp\left\{-\frac{(y-\lambda_i(1+t))^2}{4(1+T)}\right\}dt,
\end{aligned}
\tag{5.7}
$$

where we have used the following formula

$$
\begin{aligned}
&\exp\left\{-\frac{(x-\lambda_i(1+t))^2}{4\mu_1(1+t)}\right\}\exp\left\{-\frac{(x-y+\lambda_j(T-t))^2}{4\mu_2(T-t)}\right\} \\
=&\exp\left\{-\frac{\mu_1(1+t)+\mu_2(T-t)}{4\mu_1\mu_2(1+t)(T-t)}[x-\lambda_i(1+t) \right. \\
&\left. -\frac{\mu_1(1+t)}{\mu_1(1+t)+\mu_2(T-t)}(y-\lambda_j(T-t)-\lambda_i(1+t))]^2\right\} \\
&\times\exp\left\{-\frac{[y-\lambda_i(1+t)-\lambda_j(T-t)]^2}{4[\mu_1(1+t)+\mu_2(T-t)]}\right\}
\end{aligned}
\tag{5.8}
$$

for some given positive constants μ_1 and μ_2. For the integral

$$
I_2^i \equiv \int_0^T \frac{1}{\sqrt{(1+T)(1+t)}}\exp\left\{-\frac{(y-\lambda_i(1+t))^2}{C(1+T)}\right\}dt,
\tag{5.9}
$$

we will treat the case $i>p$ only, since the case $i<p$ is similar.

When $|y|\le\sqrt{1+T}$,

$$
\begin{aligned}
I_2^i &\le O(1)\int_0^T \frac{1}{\sqrt{(1+T)(1+t)}}\exp\left\{-\frac{\lambda_i^2(1+t)^2}{2C(1+T)}\right\}dt \\
&\le O(1)(1+T)^{-1/4}.
\end{aligned}
$$

When $y<-\sqrt{1+T}$,

$$
\begin{aligned}
I_2^i &\le \int_0^T \frac{1}{\sqrt{(1+T)(1+t)}}\exp\left\{-\frac{y^2}{C(1+T)}\right\}\exp\left\{-\frac{\lambda_i^2(1+t)^2}{C(1+T)}\right\}dt \\
&\le \frac{O(1)}{(1+T)^{1/4}}\exp\left\{-\frac{y^2}{C(1+T)}\right\}.
\end{aligned}
$$

When $y > \sqrt{1+T}$,

$$
\begin{aligned}
I_2^i = & \int_0^{\frac{y}{2\lambda_i}-1} \frac{1}{\sqrt{(1+T)(1+t)}} \exp\left\{-\frac{(y-\lambda_i(1+t))^2}{C(1+T)}\right\} dt \\
& + \int_{\frac{y}{2\lambda_i}-1}^{T} \frac{1}{\sqrt{(1+T)(1+t)}} \exp\left\{-\frac{(y-\lambda_i(1+t))^2}{C(1+T)}\right\} dt \\
\leq & \int_0^{\frac{y}{2\lambda_i}-1} \frac{1}{\sqrt{(1+T)(1+t)}} \exp\left\{-\frac{y^2/4+(y/2-\lambda_i(1+t))^2}{C(1+T)}\right\} dt \\
& + \int_{\frac{y}{2\lambda_i}-1}^{T} \frac{1}{\sqrt{(1+T)(y/(2\lambda_i))}} \exp\left\{-\frac{(y-\lambda_i(1+t))^2}{C(1+T)}\right\} dt \\
\leq & \frac{O(1)}{(1+T)^{1/4}} \exp\left\{-\frac{y^2}{4C(1+T)}\right\} + \frac{O(1)}{\sqrt{y}}.
\end{aligned}
$$

Hence,

$$
d_p^{-1}(y,T) I_2^i \leq O(1), \quad \text{for} \quad i \neq p. \tag{5.10}
$$

It follows from (5.7) and (5.10) that

$$
d_p^{-1}(y,T) I_2 \leq O(1) m_0^2. \tag{5.11}
$$

Similarly,

$$
\begin{aligned}
I_3 = & O(1)M(T) \int_0^T \int (1+t)^{-3/4} |\partial_x \rho(x,t)| \eta_p(x,t;y,T) dx dt \\
\leq & O(1)\delta M(T) \int_0^T \frac{1}{(1+T)^{1/2}(1+t)^{3/4}} \exp\left\{-\frac{y^2}{4(1+T)}\right\} dt \\
\leq & O(1)\delta M(T) \frac{1}{(1+T)^{1/4}} \exp\left\{-\frac{y^2}{4(1+T)}\right\} \\
\leq & O(1)\delta M(T) d_p(y,T).
\end{aligned} \tag{5.12}
$$

To estimate the term I_4, we consider the integral

$$
\begin{aligned}
I_4^i \equiv & \int_0^T \int (d_i^2 H_i^2)(x,t) \eta_p(x,t;y,T) dx dt \\
= & \int_0^T \int_{|x-\lambda_i(1+t)| \leq \sqrt{1+t}} (d_i^2 H_i^2)(x,t) \eta_p(x,t;y,T) dx dt \\
& + \int_0^T \int_{|x-\lambda_i(1+t)| > \sqrt{1+t}} (d_i^2 H_i^2)(x,t) \eta_p(x,t;y,T) dx dt \\
\equiv & I_{41}^i + I_{42}^i
\end{aligned} \tag{5.13}
$$

For the case $i > p$, $\lambda_i > 0$,

$$
\begin{aligned}
I_{41}^i \le{} & O(1)\int_0^T \int\limits_{|x-\lambda_i(1+t)|\le\sqrt{1+t}} \frac{1}{(1+t)^{3/2}(T-t)^{1/2}} \\
& \times \exp\left\{-\frac{(x-\lambda_i(1+t))^2}{4(1+t)}\right\} \exp\left\{-\frac{(x-y)^2}{4(T-t)}\right\} dxdt \\
\le{} & O(1)\int_0^T \frac{1}{(1+T)^{1/2}(1+t)} \exp\left\{-\frac{(y-\lambda_i(1+t))^2}{4(1+T)}\right\} dt \\
\le{} & O(1) I_2^i \le O(1) d_p(y,T).
\end{aligned} \tag{5.14}
$$

Rewrite the integral I_{42}^i as

$$
I_{42}^i = O(1)\int_0^T \int\limits_{|z|>\sqrt{1+t}} \frac{1}{|z|(1+t)(T-t)^{1/2}} \exp\left\{-\frac{(z-\bar{y})^2}{4(T-t)}\right\} dzdt, \tag{5.15}
$$

where $\bar{y} = y - \lambda_i(1+t)$. Note that

$$
\begin{aligned}
I_{42}^i \le{} & O(1)\int_0^T \frac{dt}{(1+t)(T-t)^{1/2}} \int\limits_{|z|>\sqrt{1+t}} \frac{1}{|z|} \exp\left\{-\frac{(z-\bar{y})^2}{4(T-t)}\right\} dz \\
\le{} & O(1)\int_0^T \frac{dt}{(1+t)(T-t)^{1/2}} \left\{ \left(\int\limits_{|z|>\sqrt{1+t}} \frac{1}{|z|^2} dz \right)^{1/2} \right. \\
& \left. \times \left(\int\limits_{|z|>\sqrt{1+t}} \exp\left\{-\frac{(z-\bar{y})^2}{2(T-t)}\right\} dz \right)^{1/2} \right\} \\
\le{} & O(1)\int_0^T \frac{1}{(1+t)^{5/4}(T-t)^{1/4}} dt \le O(1) T^{-1/4}.
\end{aligned} \tag{5.16}
$$

It remains to bound I_{42}^i for $|y| \ge 2\sqrt{1+T}$. When $y \le -2\sqrt{1+T}$, then $\bar{y} = y - \lambda_i(1+t) \le y \le -2\sqrt{1+T}$, so

$$
\begin{aligned}
I_{42}^i ={} & O(1)\int_0^T \frac{dt}{(1+t)(T-t)^{1/2}} \left\{ \int_{-\infty}^{\bar{y}/2} \frac{1}{|z|} \exp\left\{-\frac{(z-\bar{y})^2}{4(T-t)}\right\} dz \right. \\
& \left. + \left\{ \int_{\bar{y}/2}^{-\sqrt{1+t}} + \int_{\sqrt{1+t}}^{\infty} \right\} \frac{1}{|z|} \exp\left\{-\frac{(z-\bar{y})^2}{4(T-t)}\right\} dz \right\} \\
\le{} & O(1)\int_0^T \left\{ \frac{1}{(1+t)|\bar{y}|} + \frac{1}{(1+t)^{5/4}(T-t)^{1/4}} \exp\left\{-\frac{\bar{y}^2}{CT}\right\} \right\} dt \\
\le{} & O(1)|y|^{-1}\ln(1+T) + O(1)T^{-1/4}\exp\left\{-\frac{y^2}{CT}\right\}.
\end{aligned} \tag{5.17}
$$

For $y \geq 2\sqrt{1+T}$, set $J_1 = [0,T] \cap \{t : \bar{y} < 2\sqrt{1+t}\}$ and $J_2 = [0,T] \cap \{t : \bar{y} \geq 2\sqrt{1+t}\}$. Then $[0,T] = J_1 + J_2$. Note that

$$
\begin{aligned}
&\int_{J_1} \frac{dt}{(1+t)(T-t)^{1/2}} \int\limits_{|z|>\sqrt{1+t}} \frac{1}{|z|} \exp\left\{-\frac{(z-\bar{y})^2}{4(T-t)}\right\} dz \\
\leq &O(1) \int_{J_1} \frac{1}{(1+t)^{5/4}(T-t)^{1/4}} dt \\
\leq &O(1) \int_0^T \frac{1}{|y|^{1/2}(1+t)^{3/4}(T-t)^{1/4}} dt \leq O(1)|y|^{-1/2},
\end{aligned} \tag{5.18}
$$

and

$$
\begin{aligned}
&\int_{J_2} \frac{dt}{(1+t)(T-t)^{1/2}} \left\{\left\{\int_{-\infty}^{-\sqrt{1+t}} + \int_{\sqrt{1+t}}^{\bar{y}/2}\right\} \frac{1}{|z|} \exp\left\{-\frac{(z-\bar{y})^2}{4(T-t)}\right\} dz \right. \\
&\qquad \left. + \int_{\bar{y}/2}^{+\infty} \frac{1}{|z|} \exp\left\{-\frac{(z-\bar{y})^2}{4(T-t)}\right\} dz \right\} \\
\leq &O(1) \int_{J_2} \left\{\frac{1}{(1+t)^{5/4}(T-t)^{1/4}} \exp\left\{-\frac{\bar{y}^2}{CT}\right\} + \frac{1}{|\bar{y}|(1+t)}\right\} dt \\
\leq &O(1) \int_0^{\frac{y}{2\lambda_i}-1} \left\{\frac{1}{(1+t)^{5/4}(T-t)^{1/4}} \exp\left\{-\frac{y^2}{4CT}\right\} + \frac{1}{|y|(1+t)}\right\} dt \\
&\quad + O(1) \int_{\frac{y}{2\lambda_i}-1}^{T} \left\{|y|^{-1/2}(1+t)^{-3/4}(T-t)^{-1/4} + (1+t)^{-3/2}\right\} dt \\
\leq &O(1) \left\{T^{-1/4} \exp\left\{-\frac{y^2}{4CT}\right\} + |y|^{-1} \ln|y|\right\} + O(1)|y|^{-1/2}.
\end{aligned} \tag{5.19}
$$

It follows from (5.14) and (5.16)-(5.19) that

$$
I_4^i \leq O(1) d_p(y,T), \quad \text{for} \quad i > p. \tag{5.20}
$$

The same arguments yield that

$$
I_4^i \leq O(1) d_p(y,T), \quad \text{for} \quad i < p. \tag{5.21}
$$

For the case $i = p$, $\lambda_p = 0$, the estimate is similar but much easier than the one for $i > p$. Indeed, we have

$$
\begin{aligned}
I_{41}^p &\leq O(1) \int_0^T \frac{1}{(1+T)^{1/2}(1+t)} \exp\left\{-\frac{y^2}{4(1+T)}\right\} dt \\
&\leq O(1)(1+T)^{-1/2} \ln(1+T) \exp\left\{-\frac{y^2}{4(1+T)}\right\} \\
&\leq O(1) d_p(y,T),
\end{aligned} \tag{5.22}
$$

and

$$
I_{42}^p \leq O(1) T^{-1/4}. \tag{5.23}
$$

To bound I_{42}^p for $|y| \geq 2\sqrt{1+T}$, we only deal with $y \leq -2\sqrt{1+T}$, since $y \geq 2\sqrt{1+T}$ is similar.

$$
\begin{aligned}
I_{42}^p = O(1)\int_0^T \frac{dt}{(1+t)(T-t)^{1/2}} &\left\{\int_{-\infty}^{y/2} \frac{1}{|z|}\exp\left\{-\frac{(z-y)^2}{4(T-t)}\right\}dz\right.\\
&\left.+\left\{\int_{y/2}^{-\sqrt{1+t}} + \int_{\sqrt{1+t}}^{\infty}\right\}\frac{1}{|z|}\exp\left\{-\frac{(z-y)^2}{4(T-t)}\right\}dz\right\}\\
&\leq O(1)\int_0^T \left\{\frac{1}{(1+t)|y|} + \frac{1}{(1+t)^{5/4}(T-t)^{1/4}}\exp\left\{-\frac{y^2}{CT}\right\}\right\}dt\\
&\leq O(1)|y|^{-1}\ln(1+T) + O(1)T^{-1/4}\exp\left\{-\frac{y^2}{CT}\right\}.
\end{aligned}
\tag{5.24}
$$

It follows from (5.22)-(5.24) that

$$
I_4^p \leq O(1)d_p(y,T). \tag{5.25}
$$

From (5.20), (5.21) and (5.25) , we obtain

$$
I_4 = O(1)M^2(T)\sum_{i=1}^n I_4^i \leq O(1)M^2(T)d_p(y,T). \tag{5.26}
$$

For I_5,

$$
\begin{aligned}
I_5 &= O(1)\int_0^T\int(\delta m_0 + m_0^2)\exp\{-c(t+|x|)\}\eta_p(x,t;y,T)dxdt\\
&\leq \begin{cases} O(1)(\delta m_0 + m_0^2)(1+T)^{-1/2}, & |y| \leq \sqrt{1+T}\\ O(1)(\delta m_0 + m_0^2)\left(T^{-1/2}\exp\{-y^2/(CT)\} + \exp\{-C_1|y|\}\right), & \text{otherwise.}\end{cases}
\end{aligned}
$$

Thus,

$$
I_5 \leq O(1)(\delta m_0 + m_0^2)d_p(y,T). \tag{5.27}
$$

From (2.10), we can see that $|m_0| \leq O(1)\delta_2$, so we finish the proof of this lemma.

5.2 Estimate on $\partial_y v_p(y,T)$

We now turn to the estimate for the derivative $\partial_y v_p(y,T)$. It follows from (3.11), (4.10) and the Cauchy-Schwartz inequality that for any $y \in \mathbb{R}^1$, $T > 0$

$$
\begin{aligned}
|\partial_y v_p(y,T)| \leq & \left| \int v_p(x,0)\partial_y \eta_p(x,0;y,T)dx \right| \\
& - \sum_{j\neq p} \frac{1}{2}(l_p f''(r_j,r_j))(u_j) \int_0^T \int \theta_j^2(\partial_y \eta_p(x,t;y,T))dxdt \\
& + O(1)\int_0^T \int \{\sum_{j\neq p}[(\partial_x\rho)^2(|v_j| + v_j^2) + |\partial_x \rho||\partial_x v_j| \\
& + |\theta|^2|v_j|] + |\partial_x v|^2 + |\theta||\partial_x v| + |\theta|^3 + |\partial_x\rho|^3|v|^3 \\
& + |\partial_x v|^3 + e\}|\partial_y\eta_p(x,t;y,T)|dxdt.
\end{aligned}
\tag{5.28}
$$

By (2.5), (4.11), (4.13), (1) of Lemma 3.1, the smallness of δ, m_0, $M(T)$ and the fact $\partial_y\eta_p(x,t;y,T) = -\partial_x\eta_p(x,t;y,T)$, we have

$$
\begin{aligned}
|\partial_y v_p(y,T)| \leq & \left| \int v_p(x,0)\partial_y\eta_p(x,0;y,T)dx \right| \\
& + O(1)\int_0^{T/2}\int |\theta(x,t)|^2|\partial_y\eta_p(x,t;y,T)|dxdt \\
& + O(1)\sum_{i\neq p}\int_{T/2}^T \int |[\theta_i(\partial_x\theta_i)](x,t)|\eta_p(x,t;y,T)dxdt \\
& + O(1)M(T)\int_0^T\int (1+t)^{-3/4}|\partial_x\rho(x,t)||\partial_y\eta_p(x,t;y,T)|dxdt \\
& + O(1)(M(T)+m_0^2)\int_0^T\int (1+t)^{-3/4}|\theta(x,t)||\partial_y\eta_p(x,t;y,T)|dxdt \\
& + O(1)\sum_{i=1}^n M^2(T)\int_0^T\int (d_i^2H_i^2)(x,t)|\partial_y\eta_p(x,t;y,T)|dxdt \\
& + O(1)\int_0^T\int e(x,t)|\partial_y\eta_p(x,t;y,T)|dxdt \\
\equiv & \sum_{i=6}^{12} I_i.
\end{aligned}
\tag{5.29}
$$

Lemma 5.2 *For all $(y,T) \in \mathbb{R}^1 \times (0,\infty)$, it holds that*

$$|\partial_y v_p(y,T)| \leq O(1)(\delta_2 + \delta_1^2 + M^2(T))(d_pH_p)(y,T) \tag{5.30}$$

provided that δ_1, δ_2 and ϵ_0 are suitably small.

Proof Note that

$$|\partial_y\eta_p(x,t;y,T)| = |\frac{y-x}{2(T-t)}\eta_p(x,t;y,T)| = \frac{O(1)}{T-t}\exp\left\{-\frac{(x-y)^2}{4\mu(T-t)}\right\} \tag{5.31}$$

for some constant $\mu > 1$. For I_6, since

$$I_6 \le \frac{O(1)}{\sqrt{T}} \int |v_p(x,0)| \frac{1}{\sqrt{T}} \exp\left\{ -\frac{(x-y)^2}{4\mu(T-t)} \right\} dx,$$

so the analysis for I_1 gives

$$I_6 \le O(1)\delta_2 T^{-1/2} d_p(y,T). \tag{5.32}$$

When $y > c_0(1+T)/2$ (the case $y < -c_0(1+T)/2$ is similar), one has

$$\begin{aligned}
I_6 =& \left| \int (\partial_x v_p)(x,0)\eta_p(x,0;y,T)dx \right| \\
=& \left\{ \int_{-\infty}^{y/2} + \int_{y/2}^{\infty} \right\} |\partial_x v_p(x,0)| \eta_p(x,0;y,T)dx \\
\le& O(1)T^{-1/2} \exp\left\{ -y^2/(16T) \right\} \|\partial_x v_p(\cdot,0)\|_{L^1} \\
&+ \|\partial_x v_p(\cdot,0)(1+|\cdot|)^\alpha\|_{L^\infty} \int_{y/2}^{\infty} (1+|x|)^{-\alpha} \eta_p(x,0;y,T)dx \\
=& \frac{O(1)\delta_2^2}{T^{1/2}} \exp\left\{ -\frac{y^2}{16T} \right\} + \frac{O(1)\delta_2^2}{(1+|y|)^\alpha}.
\end{aligned} \tag{5.33}$$

It follows from (5.32)-(5.33) that

$$I_6 \le O(1)\delta_2 (d_p H_p)(y,T). \tag{5.34}$$

For I_9,

$$\begin{aligned}
I_9 =& O(1)M(T) \int_0^T \int (1+t)^{-3/4} |\partial_x \rho(x,t)| |\partial_y \eta_p(x,t;y,T)| dxdt \\
=& O(1)\delta M(T) \int_0^T \int \frac{1}{(1+t)^{5/4}(T-t)} \exp\left\{ -\frac{x^2}{4(1+t)} \right\} \\
&\qquad \times \exp\left\{ -\frac{(x-y)^2}{4\mu(T-t)} \right\} dxdt \\
\le& O(1)\delta M(T) \int_0^T \frac{1}{(1+T)^{1/2}(1+t)^{3/4}(T-t)^{1/2}} \exp\left\{ -\frac{y^2}{C(1+T)} \right\} dt \\
\le& O(1)\delta M(T) \frac{1}{(1+T)^{1/2}T^{1/4}} \exp\left\{ -\frac{y^2}{C(1+T)} \right\} \\
\le& O(1)\delta M(T)(d_p H_p)(y,T).
\end{aligned} \tag{5.35}$$

Next, we estimate I_{10}. Note that for $i \neq p$,

$$
\begin{aligned}
&\int_0^T \int (1+t)^{-3/4} |\theta_i(x,t)| |\partial_y \eta_p(x,t;y,T)| dx dt \\
=&O(1) m_0 \int_0^T \int \frac{1}{(1+t)^{5/4}(T-t)} \exp\left\{ -\frac{(x-\lambda_i(1+t))^2}{4(1+t)} \right\} \\
&\qquad \times \exp\left\{ -\frac{(x-y)^2}{4\mu(T-t)} \right\} dx dt \\
=&O(1) m_0 \int_0^T \frac{1}{\sqrt{(1+T)(T-t)}(1+t)^{3/4}} \exp\left\{ -\frac{(y-\lambda_i(1+t))^2}{C(1+T)} \right\} dt \\
\equiv&O(1) m_0 I_{10}^i.
\end{aligned} \tag{5.36}
$$

We will only treat the case $i > p$, since $i < p$ can be dealt similarly.

$$
I_{10}^i \leq \int \frac{1}{(1+T)^{1/2}(1+t)^{3/4}(T-t)^{1/2}} dt \leq \frac{O(1)}{(1+T)^{1/2} T^{1/4}}. \tag{5.37}
$$

For $y < -\sqrt{1+T}$,

$$
\begin{aligned}
I_{10}^i &\leq \int \frac{1}{(1+T)^{1/2}(1+t)^{3/4}(T-t)^{1/2}} \exp\left\{ -\frac{y^2}{C(1+T)} \right\} dt \\
&\leq \frac{O(1)}{(1+T)^{1/2} T^{1/4}} \exp\left\{ -\frac{y^2}{C(1+T)} \right\} = \frac{O(1)}{|y|^{3/2}}.
\end{aligned} \tag{5.38}
$$

It remains to estimate I_{10}^i for $y > \sqrt{1+T}$. If $y/(2\lambda_i) - 1 < 0$, then $1 < 2\lambda_i / y$,

$$
I_{10}^i \leq \frac{O(1)}{(1+T)^{1/2} T^{1/4}} (\frac{2\lambda_i}{y})^{3/2} = \frac{O(1)}{|y|^{3/2}}. \tag{5.39}
$$

If $0 \leq y/(2\lambda_i) - 1 \leq T/2$, then $y \leq c_0(1+T)/2$,

$$
\begin{aligned}
I_{10}^i =& \left\{ \int_0^{\frac{y}{2\lambda_i}-1} + \int_{\frac{y}{2\lambda_i}-1}^{\frac{T}{2}} + \int_{\frac{T}{2}}^{T} \right\} \frac{1}{(1+T)^{1/2}(1+t)^{3/4}(T-t)^{1/2}} \\
&\qquad \times \exp\left\{ -\frac{(y-\lambda_i(1+t))^2}{C(1+T)} \right\} dt \\
=& \int_0^{\frac{y}{2\lambda_i}-1} \frac{1}{(1+T)^{1/2}(1+t)^{3/4}(T-t)^{1/2}} \exp\left\{ -\frac{y^2}{4C(1+T)} \right\} dt \\
&+ \int_{\frac{y}{2\lambda_i}-1}^{\frac{T}{2}} \frac{O(1)}{(1+T)^{1/2}|y|^{3/4} T^{1/2}} \exp\left\{ -\frac{(y-\lambda_i(1+t))^2}{C(1+T)} \right\} dt \\
&+ \int_{\frac{T}{2}}^{T} \frac{O(1)}{(1+T)^{1/2}(1+T)^{3/4}(T-t)^{1/2}} \exp\left\{ -\frac{(y-\lambda_i(1+t))^2}{C(1+T)} \right\} dt \\
=& \frac{O(1)}{(1+T)^{1/2} T^{1/4}} \exp\left\{ -\frac{y^2}{4C(1+T)} \right\} + \frac{O(1)}{T^{1/2}|y|^{3/4}} + \frac{O(1)}{(1+T)^{3/4} T^{1/4}}.
\end{aligned} \tag{5.40}
$$

If $T/2 < y/(2\lambda_i) - 1 \le T$, then $y \le c_0(1+T)/2$,

$$
\begin{aligned}
I_{10}^i = & \left\{ \int_0^{\frac{T}{2}} + \int_{\frac{T}{2}}^{T} \right\} \frac{1}{(1+T)^{1/2}(1+t)^{3/4}(T-t)^{1/2}} \\
& \times \exp\left\{ -\frac{(y-\lambda_i(1+t))^2}{C(1+T)} \right\} dt \\
\le & \frac{O(1)}{(1+T)^{1/2}T^{1/4}} \exp\left\{ -\frac{y^2}{4C(1+T)} \right\} + \frac{O(1)}{(1+T)^{3/4}T^{1/4}}.
\end{aligned}
\tag{5.41}
$$

If $y/(2\lambda_i) - 1 > T$, then $\lambda_i(1+T) < y/2$,

$$
\begin{aligned}
I_{10}^i \le & \int_0^T \frac{1}{(1+T)^{1/2}(1+t)^{3/4}(T-t)^{1/2}} \exp\left\{ -\frac{y^2}{4C(1+T)} \right\} dt \\
\le & \frac{O(1)}{(1+T)^{1/2}T^{1/4}} \exp\left\{ -\frac{y^2}{4C(1+T)} \right\}.
\end{aligned}
\tag{5.42}
$$

It follows from (5.37)-(5.42) that

$$
(d_p^{-1}H_p^{-1})(y,T)I_{10}^i \le O(1).
$$

This together with (5.36) gives that

$$
(d_p^{-1}H_p^{-1})(y,T)I_{10} \le O(1)m_0(M(T)+m_0^2). \tag{5.43}
$$

For I_7,

$$
\begin{aligned}
I_7 = & O(1)\int_0^{\frac{T}{2}} \int |\theta(x,t)|^2 |\partial_y \eta_p(x,t;y,T)| dxdt \\
= & O(1)m_0^2 \sum_{i \ne p} \int_0^{\frac{T}{2}} \int \frac{1}{(1+t)(T-t)} \exp\left\{ -\frac{(x-\lambda_i(1+t))^2}{4(1+t)} \right\} \\
& \times \exp\left\{ -\frac{(x-y)^2}{4\mu(T-t)} \right\} dxdt \\
\le & O(1)m_0^2 \sum_{i \ne p} \int_0^{\frac{T}{2}} \frac{1}{\sqrt{(1+T)(1+t)(T-t)}} \exp\left\{ -\frac{(y-\lambda_i(1+t))^2}{C(1+T)} \right\} dt \\
\le & \frac{O(1)m_0^2}{T^{1/2}} \sum_{i \ne p} \int_0^{\frac{T}{2}} \frac{1}{\sqrt{(1+T)(1+t)}} \exp\left\{ -\frac{(y-\lambda_i(1+t))^2}{C(1+T)} \right\} dt \\
\le & O(1)m_0^2 (d_p H_p)(y,T).
\end{aligned}
\tag{5.44}
$$

Here we obtain the last inequality by the same arguments as for I_2^i and I_{10}^i. Exactly the same analysis gives that

$$
\begin{aligned}
I_8 = O(1)m_0^2 \sum_{i\neq p} \int_{\frac{T}{2}}^{T} \int \frac{1}{(1+t)^{3/2}(T-t)^{1/2}} \exp\left\{-\frac{(x-\lambda_i(1+t))^2}{C(1+t)}\right\} \\
\times \exp\left\{-\frac{(x-y)^2}{4(T-t)}\right\} dxdt \\
= O(1)m_0^2 \sum_{i\neq p} \int_{\frac{T}{2}}^{T} \frac{1}{(1+T)^{1/2}(1+t)} \exp\left\{-\frac{(y-\lambda_i(1+t))^2}{C'(1+T)}\right\} dt \\
\leq O(1)m_0^2 (d_p H_p)(y,T).
\end{aligned}
\tag{5.45}
$$

To estimate the term I_{11}, we will work on the following term,

$$
\begin{aligned}
I_{11}^i &\equiv \int_0^T \int (d_i^2 H_i^2)(x,t) |\partial_y \eta_p(x,t;y,T)| dxdt \\
&= \int_0^T \int_{|x-\lambda_i(1+t)|\leq\sqrt{1+t}} (d_i^2 H_i^2)(x,t) |\partial_y \eta_p(x,t;y,T)| dxdt \\
&\quad + \int_0^T \int_{|x-\lambda_i(1+t)|>\sqrt{1+t}} (d_i^2 H_i^2)(x,t) |\partial_y \eta_p(x,t;y,T)| dxdt \\
&\equiv I_{11}^{i1} + I_{11}^{i2}.
\end{aligned}
$$

As before, in the case $i > p$, $\lambda_i > 0$, one can get,

$$
\begin{aligned}
I_{11}^{i1} \leq O(1) \int_0^T \int_{|x-\lambda_i(1+t)|\leq\sqrt{1+t}} \frac{1}{(1+t)^{3/2}(T-t)} \\
\times \exp\left\{-\frac{(x-\lambda_i(1+t))^2}{4(1+t)}\right\} \exp\left\{-\frac{(x-y)^2}{4\mu(T-t)}\right\} dxdt \\
\leq O(1) \int_0^T \frac{1}{(1+T)^{1/2}(1+t)(T-t)^{1/2}} \exp\left\{-\frac{(y-\lambda_i(1+t))^2}{4\mu(1+T)}\right\} dt \\
\leq O(1)(d_p H_p)(y,T).
\end{aligned}
\tag{5.46}
$$

The last inequality is obtained by the same calculation as for I_{10}^i. Rewrite the integral I_{11}^{i2} as

$$
\begin{aligned}
I_{11}^{i2} = O(1) \int_0^T \int_{|z|>\sqrt{1+t}} \frac{1}{|z|[|z|^{2\alpha-1}\chi(z,t) + (1+t)](T-t)} \\
\times \exp\left\{-\frac{(z-\bar{y})^2}{4\mu(T-t)}\right\} dzdt,
\end{aligned}
$$

with $\bar{y} = y - \lambda_i(1+t)$. Note that

$$\begin{aligned} I_{11}^{i2} &\le O(1)\int_0^T \frac{dt}{(1+t)(T-t)} \int\limits_{|z|>\sqrt{1+t}} \frac{1}{|z|}\exp\left\{-\frac{(z-\bar{y})^2}{4\mu(T-t)}\right\}dz \\ &\le O(1)\int_0^T \frac{1}{(1+t)^{5/4}(T-t)^{3/4}}dt \le O(1)T^{-3/4}. \end{aligned} \tag{5.47}$$

It remains to bound I_{11}^{i2} for $|y| \ge 2\sqrt{1+T}$. For $y \le -2\sqrt{1+T}$, then $\bar{y} = y - \lambda_i(1+t) \le y \le -2\sqrt{1+T}$, so

$$\begin{aligned} &\int_0^T \int_{-\infty}^{\bar{y}/2} \frac{1}{|z|[|z|^{2\alpha-1}\chi(z,t)+(1+t)](T-t)}\exp\left\{-\frac{(z-\bar{y})^2}{4\mu(T-t)}\right\}dzdt \\ \le& \int_0^{\frac{|y|}{2c_0-\lambda_i}-1} \frac{1}{|\bar{y}|(|\bar{y}|+(1+t))(T-t)^{1/2}}dt \\ &\quad + \int_{\frac{|y|}{2c_0-\lambda_i}-1}^T \frac{1}{|\bar{y}|(1+t)(T-t)^{1/2}}dt \\ \le& \frac{1}{|\bar{y}|^{3/2}}\left\{\int_0^T \frac{1}{\sqrt{(1+t)(T-t)}}dt + \int_0^T \frac{1}{\sqrt{|y|(1+t)(T-t)}}dt\right\} \\ \le& O(1)|y|^{-3/2}, \end{aligned} \tag{5.48}$$

and

$$\begin{aligned} &\int_0^T \left\{\int_{\bar{y}/2}^{-\sqrt{1+t}} + \int_{\sqrt{1+t}}^{\infty}\right\} \frac{1}{|z|(1+t)(T-t)}\exp\left\{-\frac{(z-\bar{y})^2}{4\mu(T-t)}\right\}dzdt \\ \le& O(1)\int_0^T \frac{1}{(1+t)^{5/4}(T-t)^{3/4}}\exp\left\{-\frac{\bar{y}^2}{16\mu T}\right\} \\ \le& \frac{O(1)}{T^{3/4}}\exp\left\{-\frac{y^2}{CT}\right\}. \end{aligned} \tag{5.49}$$

For $y \ge 2\sqrt{1+T}$, set $J_1 = [0,T]\cap\{t : \bar{y} < 2\sqrt{1+t}\}$ and $J_2 = [0,T]\cap\{t :\ \bar{y} \ge 2\sqrt{1+t}\}$, then $[0,T] = J_1 + J_2$. Note that

$$\begin{aligned} &\int_{J_1} \int\limits_{|z|>\sqrt{1+t}} \frac{1}{|z|[|z|^{2\alpha-1}\chi(z,t)+(1+t)](T-t)}\exp\left\{-\frac{(z-\bar{y})^2}{4\mu(T-t)}\right\}dzdt \\ &\le O(1)\int_{J_1} \frac{1}{(1+t)^{3/2}(T-t)^{1/2}}dt \\ &= O(1)\left\{\int_0^{y/(2\lambda_i)-1} \frac{1}{\bar{y}^{3/2}(1+t)^{3/4}(T-t)^{1/2}}dt\right. \\ &\qquad \left. + \int_{y/(2\lambda_i)-1}^{T/2} \frac{1}{(1+t)^{3/2}T^{1/2}}dt + \int_{T/2}^T \frac{1}{(1+T)^{3/2}(T-t)^{1/2}}\right\} \\ &\le O(1)\left(|y|^{-3/2} + T^{-1/2}y^{-1/2} + T^{-1}\right), \end{aligned} \tag{5.50}$$

and for J_2

$$
\begin{aligned}
&\int_{J_2}\left\{\int_{-\infty}^{-\sqrt(1+t)}+\int_{\sqrt(1+t)}^{\bar{y}/2}\right\}\frac{1}{|z|[|z|^{2\alpha-1}\chi(z,t)+(1+t)](T-t)}\\
&\qquad\qquad\times\exp\left\{-\frac{(z-\bar{y})^2}{4\mu(T-t)}\right\}dzdt\\
\leq&O(1)\int_0^{y/(2\lambda_i)-1}\frac{1}{(1+t)^{5/4}(T-t)^{3/4}}\exp\left\{-\frac{\bar{y}^2}{CT}\right\}dt\\
&+O(1)\left\{\int_{y/(2\lambda_i)-1}^{T/2}+\int_{T/2}^{T}\right\}\frac{1}{(1+t)^{3/2})(T-t)^{1/2}}dt\\
\leq&O(1)\left(\frac{1}{T^{3/4}}\exp\left\{-\frac{y^2}{4CT}\right\}+T^{-1/2}y^{-1/2}+T^{-1}\right),
\end{aligned}
\tag{5.51}
$$

and

$$
\begin{aligned}
&\int_{J_2}\int_{\bar{y}/2}^{\infty}\frac{1}{|z|[|z|^{2\alpha-1}\chi(z,t)+(1+t)](T-t)}\exp\left\{-\frac{(z-\bar{y})^2}{4\mu(T-t)}\right\}dzdt\\
\leq&O(1)\int_0^{y/(2c_0+\lambda_i)-1}\frac{1}{|\bar{y}|[|\bar{y}|+(1+t)](T-t)^{1/2}}dt\\
&+O(1)\int_{y/(2c_0+\lambda_i)-1}^{y/(2\lambda_i)-1}\frac{1}{|\bar{y}|(1+t)(T-t)^{1/2}}dt\\
&+O(1)\left\{\int_{y/(2\lambda_i)-1}^{T/2}+\int_{T/2}^{T}\right\}\frac{1}{(1+t)^{3/2}(T-t)^{1/2}}dt\\
\leq&O(1)\left(\frac{1}{|y|^{3/2}}\int_0^T\frac{1}{\sqrt{(1+t)(T-t)}}dt+\frac{1}{|y|^{3/2}}+\frac{1}{\sqrt{T|y|}}+\frac{1}{T}\right).
\end{aligned}
\tag{5.52}
$$

It follows from (5.46), (5.47)-(5.52) that

$$
I_{11}^i\leq O(1)(d_pH_p)(y,T)\quad\text{for}\quad i>p. \tag{5.53}
$$

Similar arguments yield that

$$
I_{11}^i\leq O(1)(d_pH_p)(y,T)\quad\text{for}\quad i<p. \tag{5.54}
$$

For the case $i = p$, $\lambda_p = 0$, the estimate is similar but much easier than the one for $i > p$. Indeed, we have

$$
\begin{aligned}
I_{11}^{p1} &\le O(1)\int_0^T \int_{|x|\le\sqrt{1+t}} \frac{1}{(1+t)^{3/2}(T-t)} \\
&\qquad \times \exp\left\{-\frac{x^2}{4(1+t)}\right\}\exp\left\{-\frac{(x-y)^2}{4\mu(T-t)}\right\}dxdt \\
&\le O(1)\int_0^T \frac{1}{(1+T)^{1/2}(1+t)(T-t)^{1/2}}\exp\left\{-\frac{y^2}{4\mu(1+T)}\right\}dt \\
&\le \frac{O(1)}{\sqrt{T(1+T)}}\exp\left\{-\frac{y^2}{4\mu(1+T)}\right\} \le O(1)(d_pH_p)(y,T),
\end{aligned}
\tag{5.55}
$$

and

$$
I_{11}^{p2} \le O(1)T^{-3/4}. \tag{5.56}
$$

To bound I_{11}^{p2} for $|y| \ge 2\sqrt{1+T}$, we only deal with $y \le -2\sqrt{1+T}$, since $y \ge 2\sqrt{1+T}$ is similar.

$$
\begin{aligned}
&\int_0^T\int_{-\infty}^{y/2}\frac{1}{|z|[|z|^{2\alpha-1}\chi(z,t)+(1+t)](T-t)}\exp\left\{-\frac{(z-y)^2}{4\mu(T-t)}\right\}dzdt \\
\le &\int_0^{\frac{|y|}{2c_0}-1}\frac{1}{|y|(|y|+(1+t))(T-t)^{1/2}}dt \\
&\qquad +\int_{\frac{|y|}{2c_0}-1}^T\frac{1}{|y|(1+t)(T-t)^{1/2}}dt \\
\le &O(1)|y|^{-3/2},
\end{aligned}
\tag{5.57}
$$

and

$$
\begin{aligned}
&\int_0^T\left\{\int_{y/2}^{-\sqrt{1+t}}+\int_{\sqrt{1+t}}^{\infty}\right\}\frac{1}{|z|(1+t)(T-t)}\exp\left\{-\frac{(z-y)^2}{4\mu(T-t)}\right\}dzdt \\
\le &O(1)\int_0^T\frac{1}{(1+t)^{5/4}(T-t)^{3/4}}\exp\left\{-\frac{y^2}{16\mu T}\right\} \\
\le &\frac{O(1)}{T^{3/4}}\exp\left\{-\frac{y^2}{16\mu T}\right\}.
\end{aligned}
\tag{5.58}
$$

It follows from (5.55)-(5.58) that

$$
I_{11}^{p} \le O(1)(d_pH_p)(y,T). \tag{5.59}
$$

This together with (5.53) and (5.54) shows

$$
I_{11} = O(1)M^2(T)\sum_{i=1}^{n} I_{11}^i \le O(1)M^2(T)(d_pH_p)(y,T). \tag{5.60}
$$

The term I_{12} can be estimated trivially as for I_5, so we omit here. Also, we have $m_0 \le O(1)\delta_2$ by (2.10). Now, we get the estimate for the derivative $\partial_y v_p(y,T)$.

6 A priori estimate II- on transversal waves

In this section, we give the estimates on the waves moving in the transversal directions. In [13], those *a priori* estimates for $|v_i(y,T)|$ and $|\partial_y v_i(y,T)|$ with $i \neq p$ are derived by the elaborate analysis of the properties of the error terms E_i ($i \neq p$) defined by (4.8) and the Green functions for transversal fields η_i ($i \neq p$) defined by (4.4)-(4.5), such as, the asymptotic behaviors of E_i (cf. Lemma 3.2 in [13]), the essential support and effective propagation speed of η_i (cf. Lemma 3.3 in [13]).

Since the constraint on the left eigenvector (1.6) has only effect on the principal wave, but not on the transversal waves. This can be verified easily by comparing Lemma 3.2 with Lemma 3.1 in [13]. Therefore, the *a priori* estimates on transversal waves are the same no matter whether the constraint on the left eigenvector (1.6) is imposed. So we list the following two lemmas about the estimates on the transversal waves without proofs. The interested reader may refer to [13] for the proofs.

Lemma 6.1 *For suitably small δ_1, δ_2 and ϵ_0, one has that*

$$|v_i(y,T)| \leq O(1)(\delta_2 + \delta_1^2 + M^2(T))d_i(y,T) \tag{6.1}$$

for all $(y,T) \in \mathbb{R}^1 \times (0,\infty)$ and $i \neq p$.

Lemma 6.2 *For all $(y,T) \in \mathbb{R}^1 \times (0,\infty)$ and $i \neq p$, it holds that*

$$|\partial_y v_i(y,T)| \leq O(1)(\delta_2 + \delta_1^2 + M^2(T))(d_i H_i)(y,T) \tag{6.2}$$

provided that δ_1, δ_2 and ϵ_0 are suitably small.

This completes the a priori estimates on the transversal waves.

7 Proof of the stability theorem

With the a priori estimates derived in the previous two sections, we can now complete the proof of our main theorem easily. It follows from Lemma 5.1, 5.2, 6.1, 6.2 and (4.11)-(4.12) that

$$M(T) \leq O(1)(\delta_2 + \delta_1^2 + M^2(T))$$

provided that δ_1, δ_2 and ϵ_0 are suitably small. We then have

$$M(T) \leq O(1)(\delta_1 + \delta_2).$$

Since (3.8) implies that

$$M(0) \leq O(1)(\delta_1 + \delta_2),$$

so by continuity we have for suitably small δ_1 and δ_2,

$$M(t) \leq O(1)(\delta_1 + \delta_2), \quad \text{for} \quad t \geq 0. \tag{7.1}$$

Consequently, we have shown

Proposition 7.1 *(A priori estimate) Let T be any positive constant and $M(T)$ be defined by* (4.12)*. Suppose that $\omega(x,t)$ is a smooth solution to the Cauchy problem* (3.5)-(3.8) *defined on $[0,T]$ and $v(x,t)$ is the corresponding characteristic variable defined by* (3.9)*. Then there exist positive constant δ_0 and C such that if $\delta_1+\delta_2 \leq \delta_0$, then for all $t \in [0,T]$,*

$$M(t) \leq C(\delta_1+\delta_2)$$

and $|v_i| = C(\delta_1+\delta_2)(|x-\lambda_i t|^2+(1+t))^{-1/4}$,

$$|\partial_x v_i| = C(\delta_1+\delta_2)(|x-\lambda_i t|^2+(1+t))^{-1/4}(1+t)^{-1/2},$$

$$|\partial_x v_i| = C(\delta_1+\delta_2)|x-\lambda_i t|^{-\alpha}, \quad \text{for } |x-\lambda_i t| \geq c_0(1+t).$$

Now, the theorem 2.1 follows from Proposition 7.1 by the standard argument for parabolic equations. Thus the proof of Theorem 2.1 is complete.

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